Two Cauchy Problems Associated to the Brinkman Flow

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Contents

Abstract

In this work we deal with two Cauchy problems associated to the Brinkman Flow, which models fluid viscosity in certain types of porous media. In the first of them, we study the local and global well-posedness in Sobolev Spaces $H^s(\mathbb{R}^n), s > \frac{n}{2} + 1$; using Kato's Theory for Quasilinear Equations and Parabolic Regularization.

Moreover, we study the same problem with Bore-Like initial conditions, and we establish local solutions in $H^s(\mathbb{R}), s > \frac{3}{2}$, and a L^2-global estimate of that solution.

Key words: *Brinkman Flow, Kato's Quasilinear Theory, Parabolic Regularization, Comparison Principle and Bore-Like initial condition.*

Resumo

Neste trabalho trataremos dois problemas de Cauchy associados ao Fluxo de Brinkman, que modela o fluxo viscoso em certos tipo de meios porosos. No primeiro deles, estudamos a boa colocação, tanto local quanto global nos espaços de Sobolev usuais, com $s > \frac{n}{2} + 1$, usando a Teoria de Kato para as equações Quasilineares e o Método de Regularização Parabólica.

Além disso, estudaremos o mesmo problema com condições iniciais tipo *Bore-Like*, estabelecemos soluções locais em $H^s(\mathbb{R}), s > \frac{3}{2}$ e estimativa global da solução obtida em $L^2(\mathbb{R})$.

Palavras Chave: *Fluxo de Brinkman, Teoria Quasilinear de Kato, Regularização Parabólica, Principio de Comparação e condições iniciais tipo Bore-Like.*

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Chapter 1

Introduction.

1.1 The Brinkman Flow equation (BFE)

In most problems of slow incompressible fluid flow in porous media, it is assumed that the macroscopic velocity and pressure are related by Darcy's law:

$$\nabla P(\rho) = -\left(\frac{\mu}{k}\right)\vec{\mathbf{v}} \tag{1.1}$$

where μ, k and ρ are respectively, the fluid's viscosity, the porous medium's permeability and the fluid's density.

The Brinkman Flow equation involves modifying the usual Darcy's law by the addition of a standard viscosity term whose coefficient is usually identified with the pure fluid viscosity ([38]). Specifically, Brinkman modified (1.1) to the form

$$\nabla P(\rho) = -\left(\frac{\mu}{k}\right)\vec{\mathbf{v}} + \mu_{eff}\Delta\vec{\mathbf{v}} \tag{1.2}$$

Combining the continuity equation or conservation equation of the flow, i.e.,

$$\phi\partial_t\rho + \operatorname{div}(\rho\,\vec{\mathbf{v}}) = 0 \tag{1.3}$$

and the Brinkman's law (1.2), we obtain the Brinkman Flow equation

$$\phi\partial_t\rho = \operatorname{div}\left(\rho\left(\frac{\mu}{k} - \mu_{eff}\Delta\right)^{-1}\nabla P(\rho)\right) \tag{1.4}$$

where ϕ is the porosity of the medium.

In this work we are interested in the properties of the real valued solutions to the Cauchy Problem associated to the Brinkman Flow ([13],[42]). Namely,

$$\begin{cases} \phi\partial_t\rho = \text{div}\left(\rho\left(\frac{\mu}{k} - \mu_{eff}\Delta\right)^{-1}\nabla P\left(\rho\right)\right) + F(t,\rho),\ x \in \mathbb{R}^n,\ t \in (0,T_0] \\ \rho(0) = \rho_0 \end{cases} \tag{1.5}$$

This models fluid flows in certain types of porous media. Here μ , k, and μ_{eff} denote the fluid viscosity, the porous media permeability and the pure fluid viscosity, respectively, while ρ is the fluid's density, v its velocity, P is the pressure, F is an external mass flow rate, and ϕ is the porosity of the medium.

In what follows, to simplify the notation, we will choose all the coefficients in (1.5) to be equal to 1. At the moment we want to consider only the mathematical structure of the system. At a later stage, the constants should be put back in, and various limiting cases should be studied. Thus our problem becomes:

$$\begin{cases} \partial_t\rho = \text{div}\left(\rho\left(1-\Delta\right)^{-1}\nabla P\left(\rho\right)\right) + F\left(t,\rho\right),\ t \in (0,T_0] \\ \rho\left(0\right) = \rho_0. \end{cases} \tag{1.6}$$

Then we solve (1.6), and compute $\vec{\mathbf{v}}$ using the simplified Brinkman's condition $\vec{\mathbf{v}} = -\left(1-\Delta\right)^{-1}\nabla P\left(\rho\right)$. Of course, the following compatibility condition must be satisfied:

$$\vec{\mathbf{v}}_0 = -\left(1-\Delta\right)^{-1}\nabla P\left(\rho_0\right). \tag{1.7}$$

This work is organized as follows:

In Chapter 2, we analyze the local well-posedness of (1.6) with Kato's Quasilinear Theory ([18], [20], [26], [28]). We will prove that (1.6) is locally well-posed in the sense described if $s > \frac{n}{2} + 1$.

In ([2]) the authors proved that the problem is well-posed in the one dimensional case. As immediately consequence of Kato's Method they obtained continuous dependence of the solution with respect to the initial conditions.

Chapter 3, is dedicated to the study of the the same problem in the context of parabolic regularization in order to obtain global well-posedness . We will prove global estimates (in the cases of case $F(t,\rho) = 0, P(\rho) = \rho^{2k}, k = 1, 2, ...$) using the Comparison Principle mentioned in Section 3.2.

Parabolic Regularization is also applied in the study of other equations such as The Benjamin-Ono and the Korteweg-de Vries ([16]),[17]). This leads to local and global results. The global estimates are obtained with the help of the conserved quantities associated to these equations (note that these equations are Hamiltonian Systems ([22],[39]).

In ([2]), such global results for the Brinkman Equation are obtained without using additional information of the equation, because for $n = 1$, $(1 - \Delta)^{-1}$ has a bounded Kernel. In our case we need to use a Comparison Principle for the solutions to obtain the global estimates in $H^s(\mathbb{R}^n)$, $n > 1$.

Since the Brinkman equation is not invariant under translations, we obtain continuity of the solution from the right. However, combining this method with Kato's Theory we are able to guarantee the continuity of the unique solution that we constructed.

Finally, in Chapter 4, we study the (BFE) with $F(t,\rho) = 0, P(\rho) = \rho^2$ with Bore-Like Initial conditions on the real line. We proved that (1.6) has a unique local(in time) solution in $C_+([0, \tilde{T}_s], H^s(\mathbb{R})), s > \frac{3}{2}$. We also obtain a global L^2-estimate for the solution.

The real reason for introducing Bore-Like initial conditions is that they model certain travelling waves that occur in nature. Such a wave is formed when a large river, like the Amazon, flows into the sea at times of exceptionally high tide. The wave moves upstream two or three times as fast as the normal tidal current, with a shape which corresponds roughly to the one described by the conditions that characterize the Bore-Like initial datum.The problem with bores is that they have a infinite mass, and Sobolev Spaces Methods, in principle, cannot be applied.

The Brinkman Equation can also be used, for example, to study the viscous fluid between two parallels plates packed with regular square arrays of cylinders ([40], [50]).

Other problems with this type of initial condition were studied in ([19], [21]). In this articles global results were determined with the help of the associated conserved quantities which allows one to obtain global estimates H^s(this typically makes use of the fact that the equation in question are Hamiltonian Systems).

We conclude this chapter with preliminaries and notations.

1.2 Some Preliminaries

The initial value problem associated to the Brinkman Flow equation (1.6), corresponds to general problems of the form:

$$\begin{cases} \partial_t u = G(t,u) \in X \\ u(0) = u_0 \in Y \end{cases} \tag{1.8}$$

Here X and Y are Banach spaces and $G : (0, T_0] \times Y \to X$ is continuous with respect to the relevant topologies. In practice one often takes X and Y to be Sobolev Spaces type L^2.
We will say that (1.8) is *locally well-posed* or, that the solutions of (1.8) define a *dynamical system*, if the following conditions are satisfied:

- (LWP-I) *Existence and Persistence*: There exists $T > 0$ and a function $u \in C([0,T], Y)$ satisfying the differential equation in (1.8), with the time derivative computed with respect to the norm of X and such that $u(0) = u_0$, i.e.

$$\lim_{h\to 0} \left\| \frac{u(t+h) - u(t)}{h} - G(t, u(t)) \right\|_X = 0 \tag{1.9}$$

- (LWP-II) *Uniqueness*: There is at most one solution to the problem at hand

- (LWP-III) *Continuous dependence*: The map $u_0 \to u(t)$ is continuous with respect to the appropriate topologies. More precisely, if $(u_0)_n \to u_0$ in Y, then for any $T' \in [0,T), u_n$, the solution corresponding to

$(u_0)_n$, can be extended (if necessary) to $[0, T']$ for all n sufficiently large and

$$\lim_{n\to\infty} \sup_{[0,T']} \|u_n(t) - u(t)\|_Y = 0 \tag{1.10}$$

In the case that T can be taken arbitrarily large, we will say that (1.8) is *globally well-posed*. If any of those conditions is not satisfied, then (1.8) is *ill-posed*. It deserves remark that any of the above conditions can fail, including persistence.

Finally, we will introduce some notations and definitions that we will be used throughout this work.
Let $s \in \mathbb{R}$, the Sobolev Space type L^2, denoted as $H^s(\mathbb{R}^n)$ is determined as

$$H^s(\mathbb{R}^n) = \{f \in S'(\mathbb{R}^n) : (1+\xi^2)^{\frac{s}{2}} \hat{f}(\xi) \in L^2(\mathbb{R}^n)\}$$

where $S'(\mathbb{R}^n)$ represents the set of temperate distributions, and $\hat{f}$ is the Fourier Transform of f, defined as

$$\hat{f}(\xi) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(x) e^{-ix\xi}\, dx$$

The space $H^s(\mathbb{R}^n)$, is a Hilbert Space with respect to the inner product

$$\langle f, g\rangle_s = \int_{\mathbb{R}^n} (1+\xi^2)^{\frac{s}{2}} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi$$

It is easy to see that if $s \geq r$ then $H^s(\mathbb{R}^n) \hookrightarrow H^r(\mathbb{R}^n)$ where the inclusion is continuous and dense. In particular, if $s \geq 0$ we are dealing with L^2 functions. As s increases things get better and better: if $k \geq 0$ is an integer, $f \in H^k(\mathbb{R}^n)$ if and only if $\partial^\alpha f \in L^2$ for all multi-indexes α such that $|\alpha| \leq k$.

According to the Sobolev's Lemma, if $f \in H^s(\mathbb{R}^n)$ with $s > \frac{n}{2}$, then $f \in C_\infty(\mathbb{R}^n)$, the set of all continuous functions that tend to zero at infinity, and f satisfies

$$\|f\|_{L^\infty} \leq C_s(s,n)\|f\|_s \tag{1.11}$$

In this case $H^s(\mathbb{R}^n)$ is a Banach Algebra with respect to the usual multiplication of functions. In particular,

$$\|fg\|_s \leq C_s(s,n)\|f\|_s\|g\|_s \tag{1.12}$$

where $C(s,n)$ is a constant depending only s and n.

Other notations that we will be used are:

$\mathbb{R}$ - the real number
$\| \bullet \|_X$ - the norm in a Banach Space X
$B(X,Y)$ - the space of all bounded linear operators from X to Y
$\| \bullet \|_{B(X,Y)}$ - the operator norm in $B(X,Y)$
$\partial_x = \frac{\partial}{\partial x}$, $\partial_t = \frac{\partial}{\partial t}$
$D(A)$ - the domain of an operator A
$R(A)$ - the range of an operator A
$S(\mathbb{R}^n)$ - the Schwartz space of rapidly decreasing C^∞ functions
$L^p = L^p(\mathbb{R}^n), 1 \leq p \leq \infty$
$L^p_s = L^p_s(\mathbb{R}^n) = J^{-s}L^p(\mathbb{R}^n)$ with norm $\| \bullet \|_{L^p_s} = \| \bullet \|_{s,p}$
$C(I,X)$ - the space of continuous functions on an interval I into a Banach Space X. If I is compact, it is a Banach Spaces with a supremum norm.
$C_+(I,X)$ - the space of functions on an interval I, such that there all continuous on the right side.
$C^1_+(I,X)$ - the space of functions defined on an interval I, with derivative belonging to $C_+(I,X)$
$C_w(I,X)$ - the space of all weakly continuous functions from I to X
$A \lesssim B$ - there exist a constant $c > 0$ such that $A \leq cB$
$\rightarrow$ - strong convergence
$\rightharpoonup$ - weak convergence

Chapter 2

Local Theory

This chapter will make use of Kato's Theory in order to obtain local existence results to problem in question.

2.1 Kato's Theory for Quasilinear Partial Differential Equations.

We consider the Cauchy Problem for the quasi-linear equation, that is:

$$\begin{cases} \partial_t u = G(t,u) = -A(u)u + F(t,u) \in X, \quad t \in (0, T_0] \\ u(0) = u_0 \in Y \end{cases} \tag{2.1}$$

Here $A(u)$ is a linear operator depending of u, and u_0 is the initial value.

We will need the following assumptions.

(K1) X is a reflexive Banach space. There is another reflexive space $Y \hookrightarrow X$ and there exists an isomorphism S from Y onto X such that $\|S\varphi\|_X = \|\varphi\|_Y, \forall \phi \in Y$.
(K2) The linear operator $A(u) \in G(X, 1, \beta)$ for $u \in W \subset Y$, where W is an open ball in Y and β is the real number. In other words, for each $u \in W$, $-A(u)$ generates a C^0 semigroup such that

$$\|e^{-sA(u)}\|_{B(X)} \leq e^{\beta s}, s \in [0, \infty), u \in W \tag{2.2}$$

(K3) For each $u \in W$ we have

$$SA(u)S^{-1} = A(u) + B(u) \tag{2.3}$$

where $B(u) = [S, A(u)]S^{-1} \in B(X)$ is uniformly bounded ,that is, there is a constant μ_B such that

$$\|B(u)\|_{B(X)} \leq \mu_B \tag{2.4}$$

(K4) $Y \subset D(A(u))$ (so that $A(u)|_Y \in B(Y,X)$ by the Closed Graph Theorem).
The maps $u \in W \mapsto A(u)$ is Lipschitz continuous in the sense:

$$\|A(u) - A(v)\|_{B(Y,X)} \leq \mu_A \|u - v\|_X \tag{2.5}$$

(K5) The function F satisfies:

a) For each $u \in W$, the maps $t \in (0, T_0] \to F(t,u) \in X$ is continuous.

b) For each $t \in (0, T_0]$, the maps $u \in W \to F(t,u) \in X$ is Lipschitz continuous in this topology, that is, there is a constant μ_F such that:

$$\|F(t,u) - F(t,v)\|_X \leq \mu_F \|u - v\|_X \tag{2.6}$$

`Remarks about K2`. In many cases $A(u)$ is defined for all $u \in Y$, so that, W may be chosen as an arbitrary ball centered in zero.
It is well known that given a C^0 semigroup $U(s)$, there are constants $M > 0$ and $\beta \in \mathbb{R}$ and a unique closed operator A (satisfying the conditions of the Hille-Yosida-Philips Theorem, see [43, Vol.II]) such that $U(s)$ is generated by $-A$, i.e. ,$U(s) = e^{-sA}$. Moreover $\|U(s)\|_{B(X)} \leq Me^{-s\beta}$. Conversely, given a closed operator A satisfying the conditions of the Hille-Yosida-Philips Theorem, there are $M > 0, \beta \in \mathbb{R}$ and a C^0 semigroup $U(s)$ satisfying $\|U(s)\|_{B(X)} \leq Me^{-s\beta}$. The collection of all such A's will be denoted by $G(X, M, \beta)$.
If $A \in G(X, 1, 0)$, that is, if $(-A)$ generates a contraction semigroup, we say that A is *maximally accretive(or m-accretive)*. If $A \in G(X, 1, \beta)$, that is, $\|U(s)\|_{B(X)} \leq Me^{-s\beta}$,$A$ is said to be *quasi maximally accretive(or quasi m-accretive)*.

`Remark about K3,K4`. The relation (2.3) should be satisfied in the strict sense, including the domain relation. Thus $x \in X$ is in $D(A(u))$ if and only

if $S^{-1}x \in D(A(u))$ with $A(u)S^{-1}x \in Y$
This strict equality is not as restrictive as it looks: in some sense $S^{-1}x$ is "better behaved" than an arbitrary element of $D(A(u))$. Consider KdV for example, and let $A(u) = \partial_x^3 + u\partial_x, Y = D(A(u)) = H^s(\mathbb{R}), s > \frac{3}{2}, X = L^2(\mathbb{R})$ and $S = (1 - \partial_x^2)^{\frac{s}{2}}$. Then if $\phi \in D(A(u))$, it follows that $S^{-1}\phi \in H^{2s}(\mathbb{R}) \hookrightarrow H^s(\mathbb{R}) = D(A(u))$.

This results where extended to $G(X, M, \beta)$ in [31],[32] and [33] . The non-Reflexive Spaces case is treated in [35],[37] and [45].

Theorem 2.1.1. (Abstract Local Theory for Quasilinear Equations). *Assume that **K1** − **K5** are satisfied. Then there exist $T \in (0, T_0]$ and a unique $u \in C([0, T]; Y)$ such that (2.1) is satisfied with the derivative taken with respect to the norm of X.*

This theory is studied in [18],[20],[26],[27] and [35]

2.2 Existence and Uniqueness of the BFE

In this section we will apply Kato's Theory for the Brinkman Flow equation (BFE).

Theorem 2.2.1. (Existence and Uniqueness)

Let $\vec{\Theta}(\rho) = J^{-2}\nabla P(\rho), J = (1 - \Delta)^{\frac{1}{2}}$. Define

$$A(\rho)f = -\operatorname{div}(f\, J^{-2}\nabla P(\rho)) = -\operatorname{div}(f\vec{\Theta}(\rho)), \tag{2.7}$$

so that the PDE in (1.6) can be written as

$$\partial_t\rho + A(\rho)\rho = F(t, \rho). \tag{2.8}$$

Let $\rho_0 \in H^s(\mathbb{R}^n)$, $s > \frac{n}{2} + 1$ and assume that P and F satisfy the following assumptions:

a) *P maps $H^s(\mathbb{R}^n)$ into itself, $P(0) = 0$ and is Lipschitz in the following senses:*

$$\|P(\rho) - P(\widetilde{\rho})\|_s \le L_s(\|\rho\|_s, \|\widetilde{\rho}\|_s)\|\rho - \widetilde{\rho}\|_s \tag{2.9}$$

$$\|P(\rho) - P(\widetilde{\rho})\| \le \widetilde{L_s}(\|\rho\|_s, \|\widetilde{\rho}\|_s)\|\rho - \widetilde{\rho}\| \tag{2.10}$$

where $L_s, \widetilde{L_s} : [0, \infty) \times [0, \infty) \to [0, \infty)$ are continuous and monotone non-decreasing with respect to each of its arguments

b) *$F:(0,T_0]\times H^s(\mathbb{R}^n)\longrightarrow H^s(\mathbb{R}^n)$, $F(t,0)=0$ and satisfies the following Lipschitz conditions:*

$$\|F(t,\rho)-F(t,\widetilde{\rho})\|_s\le M_s(\|\rho\|_s,\|\widetilde{\rho}\|_s)\|\rho-\widetilde{\rho}\|_s \tag{2.11}$$

$$\|F(t,\rho)-F(t,\widetilde{\rho})\|\le \widetilde{M_s}(\|\rho\|_s,\|\widetilde{\rho}\|_s)\|\rho-\widetilde{\rho}\| \tag{2.12}$$

where $M_s,\widetilde{M_s}:[0,\infty)\times[0,\infty)\to[0,\infty)$ are continuous and monotone non-decreasing with respect to each of its arguments

c) *For each $\rho\in W$, the map $t\in(0,T_0]\to F(t,\rho)$ is continuous respect to the topology of X.*

Then there exists $T\in(0,T_0]$ and unique $\rho\in C([0,T],H^s)$ such that (1.6) is satisfied with the derivative taken with respect to the norm of H^{s-1}.

Proof. We will verify the assumptions according of abstract theory.

(K1) We take the basic spaces $X=L^2(\mathbb{R}^n), Y=H^s(\mathbb{R}^n)$ and we prove that $S=(1-\Delta)^{\frac{s}{2}}=J^s$ is an isomorphism from Y to X.
Let $f\in Y$, applying Parseval Identity we have:

$$\|S(f)\|=\|J^s(f)\|=\|(1+\xi^2)^{\frac{s}{2}}\hat{f}(\xi)\|=\|f\|_s \tag{2.13}$$

(K2) Since X is a Hilbert space, it is sufficient to prove that $A(\rho)$ is maximally accretive in X. (See [27],[41] and [43, Vol. II]).

$$\big\langle A(\rho)f,f\big\rangle\ge-\beta\|f\|^2,\forall f\in D(A(\rho))=Y;\rho\in W\subset Y \tag{2.14}$$

Integrating by parts and Sobolev Lemma, implies

$$\begin{aligned}\big\langle A(\rho)f,f\big\rangle&=\big\langle-\operatorname{div}(f\,\vec{\Theta}(\rho)),f\big\rangle=-\sum_{i=1}^n\int f\partial_{x_i}(f\,\Theta_i(\rho))\,dx \\ &=\sum_{i=1}^n\int f\partial_{x_i}f\,\Theta_i(\rho)\,dx=\frac{1}{2}\sum_{i=1}^n\int\partial_{x_i}(f^2)\Theta_i(\rho)\,dx \\ &=-\frac{1}{2}\sum_{i=1}^n\int f^2\partial_{x_i}\Theta_i(\rho)\,dx=-\frac{1}{2}\int(\operatorname{div}\vec{\Theta}(\rho))f^2\,dx \\ &\ge-\underbrace{\frac{\|\operatorname{div}\vec{\Theta}(\rho)\|_{L^\infty}}{2}}_{\beta}\|f\|^2\end{aligned} \tag{2.15}$$

$$Rg(A(\rho)+\lambda) = X = L^2(\mathbb{R}^n), \forall \lambda > \beta$$

The fact that $A(\rho)$ is a closed operator combined with the inequality (2.14) shows that $(A(\rho)+\lambda)$ has closed range for all $\lambda > \beta$
Thus it suffices to show that $(A(\rho)+\lambda)$ has dense range for $\lambda > \beta$. For this, is sufficient to prove that $R(A(\rho)+\lambda)^{\perp} = \{0\}$, because $A(\rho)$ is a linear operator.

Let $g \in L^2(\mathbb{R}^n) : \langle (A(\rho)+\lambda)f, g\rangle = 0, \forall f \in D(A(\rho)) = H^s(\mathbb{R}^n)$

Integrating by parts, yields

$$\begin{aligned}\langle (A(\rho)+\lambda)f, g\rangle = 0 &\Rightarrow \langle A(\rho)f, g\rangle + \langle \lambda f, g\rangle = 0 \\ &\Rightarrow \langle f, \nabla g\, \vec{\Theta}(\rho)\rangle + \langle \lambda f, g\rangle = 0 \\ &\Rightarrow \langle f, \nabla g\, \vec{\Theta}(\rho) + \lambda g\rangle = 0,\ \forall f \in D(A(\rho)) = H^s(\mathbb{R}^n) \\ &\Rightarrow \nabla g\, \vec{\Theta}(\rho) + \lambda g = 0\end{aligned} \tag{2.16}$$

Therefore, multiplying by g, integrating by parts, and using (2.14) we have:

$$\begin{aligned}g\nabla g\, \vec{\Theta}(\rho) + \lambda g^2 = 0 &\Rightarrow \frac{1}{2}\int \nabla(g^2)\, \vec{\Theta}(\rho)\, dx + \lambda \|g\|^2 = 0 \\ &\Rightarrow \underbrace{-\frac{1}{2}\int g^2 \mathrm{div}\, \vec{\Theta}(\rho)\, dx}_{=\langle A(\rho)g, g\rangle} + \lambda\|g\|^2 = 0 \\ &\Rightarrow \langle A(\rho)g, g\rangle + \lambda\|g\|^2 = 0 \\ &\Rightarrow 0 \geq -\beta\|g\|^2 + \lambda\|g\|^2 = (\lambda-\beta)\|g\|^2 \\ &\Rightarrow g = 0\end{aligned} \tag{2.17}$$

(K3) Let $W = \{\rho \in H^s(\mathbb{R}^n) : \|\rho\|_s \leq R\}$.

$B(\rho) = [S, A(\rho)]S^{-1} \in B(L^2) \Leftrightarrow [S, A(\rho)] \in B(H^s, L^2), \|[S, A(\rho)]\|_{B(H^s,L^2)} \leq \mu_B$

Let $f \in H^s(\mathbb{R}^n)$

$$
\begin{aligned}
[S, A(\rho)]f &= SA(\rho)f - A(\rho)Sf = -J^s \mathrm{div}\,(f\vec{\Theta}(\rho)) + \mathrm{div}\,((J^s f)\vec{\Theta}(\rho)) \\
&= -J^s \big[\sum_{i=1}^{n} \partial_{x_i}(f\Theta_i(\rho))\big] + \sum_{i=1}^{n} \partial_{x_i}((J^s f)\Theta_i(\rho)) \\
&= -\sum_{i=1}^{n} \big[J^s(\partial_{x_i}\Theta_i(\rho)f) - \partial_{x_i}\Theta_i(\rho)(J^s f)\big] \\
&\quad -\sum_{i=1}^{n} \big[J^s(\Theta_i(\rho)\partial_{x_i}f) - \Theta_i(\rho)\partial_{x_i}(J^s f)\big] \\
&= \underbrace{-\sum_{i=1}^{n} \big[J^s, \partial_{x_i}\Theta_i(\rho)\big]f}_{A} \underbrace{-\sum_{i=1}^{n} \big[J^s, \Theta_i(\rho)\big]\partial_{x_i}f}_{B} \qquad (2.18)
\end{aligned}
$$

Using Lemma A.1.1 in Appendix,$\|J^{-2}\partial_{x_i}\|_{B(H^s,H^{s+1})} \le 1$ and (2.9)

$$
\begin{aligned}
\|A\| &\le \sum_{i=1}^{n} \big\|[J^s, \partial_{x_i}\Theta_i(\rho)]f\big\| \le c\sum_{i=1}^{n} \|\nabla\partial_{x_i}\Theta_i(\rho)\|_{s-1}\|f\|_{s-1} \qquad (2.19) \\
&\le c\sqrt{n}\|f\|_{s-1}\sum_{i=1}^{n} \|\partial_{x_i}\Theta_i(\rho)\|_s \le c\sqrt{n}\|f\|_s \sum_{i=1}^{n} \|\Theta_i(\rho)\|_{s+1} \\
&\le c\sqrt{n}\|f\|_s \sum_{i=1}^{n} \|J^{-2}\partial_{x_i}\|_{B(H^s,H^{s+1})}\|P(\rho)\|_s \le c\,n\sqrt{n}\|f\|_s\|P(\rho) - P(0)\|_s \\
&\le c\,n\sqrt{n}\,L_s(\|\rho\|_s, 0)\|\rho\|_s\|f\|_s \le \mu(R)\|f\|_s
\end{aligned}
$$

$$
\begin{aligned}
\|B\| &\le \sum_{i=1}^{n} \big\|[J^s, \Theta_i(\rho)]\partial_{x_i}f\big\| \le c\sum_{i=1}^{n} \|\nabla\Theta_i(\rho)\|_{s-1}\|\partial_{x_i}f\|_{s-1} \qquad (2.20) \\
&\le c\,\sqrt{n}\|f\|_s \sum_{i=1}^{n} \|\Theta_i(\rho)\|_{s+1} \le ...... \le c\,n\sqrt{n}\,L_s(\|\rho\|_s, 0)\|\rho\|_s\|f\|_s \\
&\le \mu(R)\|f\|_s
\end{aligned}
$$

Then

$$\|[S, A(\rho)]f\| \le \|A\| + \|B\| \le 2\mu(R)\|f\| \Rightarrow \|[S, A(\rho)]\|_{B(H^s,L^2)} \le 2\mu(R) = \mu_B(R)$$

(K4) $D(A(\rho)) = H^s(\mathbb{R}^n)$; $\|A(\rho) - A(\widetilde{\rho})\|_{B(H^s,L^2)} \leqq \mu_A \|\rho - \widetilde{\rho}\|_{L^2}$
Let $f \in H^s(\mathbb{R}^n)$.

$$\begin{aligned}\|(A(\rho) - A(\tilde{\rho}))f\| &= \|\mathrm{div}\,(f\vec{\Theta}(\tilde{\rho})) - \mathrm{div}\,(f\vec{\Theta}(\rho))\| \\ &\leq \sum_{i=1}^{n} \|\partial_{x_i}[f(\Theta_i(\rho) - \Theta_i(\tilde{\rho}))]\| \\ &= \sum_{i=1}^{n} \|(\partial_{x_i} f)(\Theta_i(\rho) - \Theta_i(\tilde{\rho})) + f\partial_{x_i}(\Theta_i(\rho) - \Theta_i(\tilde{\rho}))\| \\ &\leq \sum_{i=1}^{n} \underbrace{\|(\partial_{x_i} f)(\Theta_i(\rho) - \Theta_i(\tilde{\rho}))\|}_{C} + \sum_{i=1}^{n} \underbrace{\|f\partial_{x_i}(\Theta_i(\rho) - \Theta_i(\tilde{\rho}))\|}_{D}\end{aligned} \tag{2.21}$$

Using Sobolev Lemma, $\|J^{-2}\partial_{x_i}\|_{B(L^2,H^1)} \leq 1$ and (2.10).

$$\begin{aligned}C &\leq \|\partial_{x_i} f\|_{L^\infty} \|\Theta_i(\rho) - \Theta_i(\tilde{\rho})\| \\ &\lesssim \|\partial_{x_i} f\|_{s-1} \|\Theta_i(\rho) - \Theta_i(\tilde{\rho})\|_1 \\ &\lesssim \|f\|_s \|J^{-2}\partial_{x_i}\|_{B(L^2;H^1)} \|\|P(\rho) - P(\tilde{\rho})\| \\ &\lesssim \|f\|_s \widetilde{L_s}(\|\rho\|_s, \|\tilde{\rho}\|_s) \|\rho - \tilde{\rho}\|\end{aligned} \tag{2.22}$$

$$\begin{aligned}D &\leq \|f\|_{L^\infty} \|\partial_{x_i}(\Theta_i(\rho) - \Theta_i(\tilde{\rho}))\| \\ &\lesssim \|f\|_s \|(\Theta_i(\rho) - \Theta_i(\tilde{\rho}))\|_1 \\ &\lesssim \|f\|_s \|J^{-2}\partial_{x_i}\|_{B(L^2;H^1)} \|\|P(\rho) - P(\tilde{\rho})\| \\ &\lesssim \|f\|_s \widetilde{L_s}(\|\rho\|_s, \|\tilde{\rho}\|_s) \|\rho - \tilde{\rho}\|\end{aligned} \tag{2.23}$$

Then

$$\begin{aligned}\|(A(\rho) - A(\tilde{\rho}))f\| &\lesssim 2\|f\|_s \widetilde{L_s}(\|\rho\|_s, \|\tilde{\rho}\|_s) \|\rho - \tilde{\rho}\| \\ &\lesssim \mu_A(R) \|\rho - \tilde{\rho}\| \|f\|_s \\ &\Rightarrow \|A(\rho) - A(\widetilde{\rho})\|_{B(H^s,L^2)} \leq \mu_A(R) \|\rho - \widetilde{\rho}\|\end{aligned} \tag{2.24}$$

(K5) This assumptions is satisfied due to the conditions about F in Theorem 2.2.1 b). □

2.3 Continuous Dependence of the initial data

To formulate the continuous dependence of the solution u on the data, we consider the sequence of value initial problems for $n \in \mathbb{N} \cup \{\infty\}$:

$$\begin{cases} \partial_t u^n + A^n(u^n)u^n = F^n(t, u^n) \in X, \, t \in (0, T_0] \\ u^n(0) = u_0^n \in Y \end{cases} \tag{2.25}$$

Theorem 2.3.1. *Assume that **K1** – **K5** hold for all equations in the sequence, with the same X, Y, S, W, and that the corresponding constants μ_B, μ_A, μ_F can be chosen independently of n.*
Assume also that:

a) *The maps $u \in W \to B(u) = [S, A(u)]S^{-1} \in B(X)$ is Lipschitz continuous in the sense:*

$$\|B(u) - B(v)\|_{B(X)} \leq \lambda_B \|u - v\|_Y \tag{2.26}$$

b) *The function $F : [0, T_0] \times W \to Y$ is bounded,i.e.,there exists a constant $\lambda_F \geq 0$ such that:*

$$\|F(t, u)\|_Y \leq \lambda_F \tag{2.27}$$

c) *For each $t \in (0, T_0]$, the maps $u \in W \to F(t, u) \in Y$ is Lipschitz continuous in this topology,i.e, there is a constant λ_{FF} such that:*

$$\|F(t, u) - F(t, v)\|_Y \leq \lambda_{FF} \|u - v\|_Y \tag{2.28}$$

d) *$A^n(u) \to A(u)$ strongly in $B(Y, X)$*

e) *$B^n(u) \to B(u)$ strongly in $B(X)$*

f) *$F^n(t, u) \to F(t, u)$ strongly in Y.*

If $u_0, u_0^n \in W$ and $u_0^n \to u_0$ in the topology of Y, then there is $0 < T'' \leq T_0$, such that there are unique solutions $u^n \in C([0, T''], W) \cap C^1([0, T''], X)$ with $u^n(0) = u_0^n$ to (2.25) and a unique solution u in the same class. Moreover, we have: $u^n(t) \to u(t)$ in Y, uniformly in $t \in [0, T'']$.

See [18],[26] and [35].

Consider the following sequence of initial value problems for the BFE.

$$\begin{cases} \partial_t \rho^n - \operatorname{div}[\rho^n(1 - \Delta)^{-1} \nabla P^n(\rho)] = F^n(t, \rho^n), \quad x \in \mathbb{R}^n, 0 < t \leq T \\ \rho^n(0) = (\rho_0)^n \end{cases} \tag{2.29}$$

Consider the same basic spaces $X = L^2(\mathbb{R}^n), Y = H^s(\mathbb{R}^n), W \subset Y, S = (1 - \Delta)^{\frac{s}{2}} = J^s$

Theorem 2.3.2. (Continuous Dependence). *In addition to the assumptions in Theorem 2.2.1, assume that:*
The sequences P^n *and* F^n *satisfies*

a) $P^n(\rho) \overset{H^s}{\overbrace{\longrightarrow}} P(\rho)$, $n \to \infty$, $\rho \in W$

b) $F^n(t,\rho) \overset{H^s}{\overbrace{\longrightarrow}} F(t,\rho)$, $n \to \infty$, $(t,\rho) \in (0, T_0] \times W$

If $\rho_0, \rho_0^n \in W$ *and* $\rho_0^n \to \rho_0$ *in the topology of* Y*, then there is* $0 < T'' \leq T_0$*, such that there are unique solutions* $\rho^n \in C([0,T'']; W) \cap C^1([0,T'']; H^{s-1})$ *with* $\rho^n(0) = \rho_0^n$ *to (2.29) and a unique solution* $\rho(t)$ *in the same class. Moreover, we have:* $\rho^n(t) \to \rho(t)$ *in* Y*, uniformly in* $t \in [0, T'']$.

Proof. We will verify the assumptions of abstract theorem. The assumptions about F, are satisfied immediately, by conditions imposed in Theorem 2.2.1(b) and this Theorem (2.3.2 (b)).

a) $B(\rho)$ is Lipschitz continuous in the sense:

$$\|B(\rho) - B(\widetilde{\rho})\|_{B(L^2)} \leq \lambda_B \|\rho - \widetilde{\rho}\|_s \tag{2.30}$$

Firstly, we prove that: $\|A(\rho) - A(\tilde{\rho})\|_{B(H^s, L^2)} \leq n L_s(\|\rho\|_s, \|\tilde{\rho}\|_s)\|\rho - \tilde{\rho}\|_s$
Let $f \in H^s(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$.

$$\begin{aligned}
\|(A(\rho) - A(\tilde{\rho}))f\| &= \|\mathrm{div}\,(f.\vec{\Theta}(\tilde{\rho})) - \mathrm{div}\,(f.\vec{\Theta}(\rho))\| \\
&\leq \sum_{i=1}^{n} \|\partial_{x_i}[f(\Theta_i(\rho) - \Theta_i(\tilde{\rho}))]\| \\
&\leq \sum_{i=1}^{n} \|f[\Theta_i(\rho) - \Theta_i(\tilde{\rho})]\|_1 \leq \sum_{i=1}^{n} \|f[\Theta_i(\rho) - \Theta_i(\tilde{\rho})]\|_s \\
&\leq \|f\|_s \sum_{i=1}^{n} \|\Theta_i(\rho) - \Theta_i(\tilde{\rho})\|_{s+1} \\
&\leq \|f\|_s \sum_{i=1}^{n} \|J^{-2}\partial_{x_i}\|_{B(H^s, H^{s+1})} \|P(\rho) - P(\tilde{\rho})\|_s \\
&\leq n\, L_s(\|\rho\|_s, \|\tilde{\rho}\|_s)\|\rho - \tilde{\rho}\|_s \|f\|_s
\end{aligned} \tag{2.31}$$

Therefore, we have

$$\|A(\rho) - A(\tilde{\rho})\|_{B(H^s, L^2)} \leq n\, L_s(\|\rho\|_s, \|\tilde{\rho}\|_s)\|\rho - \tilde{\rho}\|_s \tag{2.32}$$

The result follows of the assumptions of $A(\rho)$ and the domain conditions described in Remark about `K3,K4`, in Section 2.1

d) Let $f \in H^s(\mathbb{R}^n)$

$$
\begin{aligned}
A^n(\rho)f &= -\text{div}\,[f.J^{-2}\nabla P^n(\rho)] \\
&= -\sum_{i=1}^{n} \partial_{x_i}[fJ^{-2}\partial_{x_i}P^n(\rho)] \\
&\overbrace{\longrightarrow \dots\dots \longrightarrow}^{H^{s-1}} -\sum_{i=1}^{n} \partial_{x_i}[fJ^{-2}\partial_{x_i}P(\rho)] \\
&= -\text{div}\,[fJ^{-2}\nabla P(\rho)] \\
&= A(\rho)f \\
&\Longrightarrow A^n(\rho) \underbrace{\longrightarrow}_{B(H^s,H^{s-1})} A(\rho)
\end{aligned}
\tag{2.33}
$$

e) Let $f \in L^2(\mathbb{R}^n)$

$$
\begin{aligned}
B^n(\rho)f &= [S, A^n(\rho)]S^{-1}f \\
&\overbrace{\longrightarrow \dots \longrightarrow}^{L^2} [S, A(\rho)]S^{-1}f \\
&= B(\rho)f \\
&\Longrightarrow B^n(\rho) \underbrace{\longrightarrow}_{B(L^2)} B(\rho)
\end{aligned}
\tag{2.34}
$$

Therefore, continuous dependence follows. □

`Remark`. Note that although we took $X = L^2(\mathbb{R}^n)$, in the sections 2.2 and 2.3, the differential equation in (2.8) implies that $\partial_t \rho \in H^{s-1}$.

Chapter 3

Global Theory

This Chapter will make use of Parabolic Regularization technique and Comparison Principle, to obtain the H^s $(s > \frac{n}{2} + 1)$ global estimate for solutions of BFE.

3.1 Parabolic Regularization of the BFE

3.1.1 Existence and uniqueness of Regularized BFE

In this section we begin the analysis of the problem:

$$\begin{cases} \partial_t \rho_\mu = \mu \Delta \rho_\mu + \operatorname{div}\left[\rho_\mu J^{-2} \nabla P(\rho_\mu)\right] + F(t, \rho_\mu) \in H^{s-2}(\mathbb{R}^n), & t \in I = (0, T_0] \\ \rho_\mu(0) = \rho_0 \in H^s(\mathbb{R}^n) \end{cases} \tag{3.1}$$

where $\mu > 0$ and the time derivative is computed in the norm of H^{s-2}.
The nonlinearity $\tilde{F}(t, \rho) = \operatorname{div}\left[\rho J^{-2} \nabla P(\rho)\right] + F(t, \rho)$ has the following properties:

Lemma 3.1.1. *Let $s > \frac{n}{2} + 1$ be fixed, P, F satisfy (2.9)-(2.12) as in Theorem 2.2.1. Then $\tilde{F}(t, \rho)$ is a continuous map from $I \times H^s$ to H^{s-1} and satisfies the estimates*

$$\|\tilde{F}(t, \rho) - \tilde{F}(t, \tilde{\rho})\|_{s-1} \leq \gamma(\|\rho\|_s, \|\tilde{\rho}\|_s)\|\rho - \tilde{\rho}\|_s \tag{3.2}$$

$$\left\langle \rho - \tilde{\rho}, \tilde{F}(t, \rho) - \tilde{F}(t, \tilde{\rho}) \right\rangle \leq L_0(\|\rho\|_s, \|\tilde{\rho}\|_s)\|\rho - \tilde{\rho}\|^2 \tag{3.3}$$

for all $\rho, \tilde{\rho} \in H^s$, where $\gamma, L_0 : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are continuous functions, monotone nondecreasing with respect to each of their arguments.

Proof.

$$\begin{aligned}
&\left\|\tilde{F}(t,\rho) - \tilde{F}(t,\tilde{\rho})\right\|_{s-1} \\
&= \left\|\operatorname{div}(\rho\,\vec{\Theta}(\rho)) - \operatorname{div}(\tilde{\rho}\,\vec{\Theta}(\tilde{\rho})) + F(t,\rho) - F(t,\tilde{\rho})\right\|_{s-1} \\
&\leq \left\|\operatorname{div}\left[\rho\,\vec{\Theta}(\rho) - \tilde{\rho}\,\vec{\Theta}(\tilde{\rho})\right]\right\|_{s-1} + \left\|F(t,\rho) - F(t,\tilde{\rho})\right\|_{s-1} \\
&\leq \underbrace{\left\|\operatorname{div}\left[\rho(\vec{\Theta}(\rho) - \vec{\Theta}(\tilde{\rho}))\right]\right\|_{s-1}}_{=F1} + \underbrace{\left\|\operatorname{div}\left[(\rho-\tilde{\rho})\vec{\Theta}(\tilde{\rho})\right]\right\|_{s-1}}_{=F2} \\
&\quad + \left\|F(t,\rho) - F(t,\tilde{\rho})\right\|_{s-1}
\end{aligned} \tag{3.4}$$

Applying the fact that $H^s(\mathbb{R}^n), s > \frac{n}{2}$ is a Banach Algebra, $\|J^{-2}\partial_{x_i}\|_{B(H^s,H^{s+1})} \leq 1, \forall i$ and (2.9) we have:

$$\begin{aligned}
F1 &= \Big\| \sum_{i=1}^{n} \partial_{x_i}\left[\rho(\Theta_i(\rho) - \Theta_i(\tilde{\rho}))\right]\Big\|_{s-1} \leq \sum_{i=1}^{n} \left\|\rho\left[\Theta_i(\rho) - \Theta_i(\tilde{\rho})\right]\right\|_s \\
&\lesssim \|\rho\|_s \sum_{i=1}^{n} \left\|\Theta_i(\rho) - \Theta_i(\tilde{\rho})\right\|_s \lesssim \|\rho\|_s \sum_{i=1}^{n} \left\|\Theta_i(\rho) - \Theta_i(\tilde{\rho})\right\|_{s+1} \\
&\lesssim \|\rho\|_s \sum_{i=1}^{n} \left\|P(\rho) - P(\tilde{\rho})\right\|_s \lesssim n\|\rho\|_s L_s(\|\rho\|_s, \|\tilde{\rho}\|_s)\|\rho - \tilde{\rho}\|_s
\end{aligned} \tag{3.5}$$

$$F2 = \left\| \sum_{i=1}^{n} \partial_{x_i}\left[(\rho-\tilde{\rho})\Theta_i(\tilde{\rho})\right]\right\|_{s-1} \leq \sum_{i=1}^{n} \left\|\partial_{x_i}\left[(\rho-\tilde{\rho})\Theta_i(\tilde{\rho})\right]\right\|_{s-1} \tag{3.6}$$

$$\lesssim \|\rho - \tilde{\rho}\|_s \sum_{i=1}^{n} \|\Theta_i(\tilde{\rho})\|_{s+1} \lesssim \|\rho - \tilde{\rho}\|_s \sum_{i=1}^{n} \left\|P(\tilde{\rho})\right\|_s$$

$$\lesssim nL_s(\|\tilde{\rho}\|_s, 0)\|\rho - \tilde{\rho}\|_s\|\tilde{\rho}\|_s \tag{3.7}$$

Finally, using (2.11), we have

$$\left\|\tilde{F}(t,\rho) - \tilde{F}(t,\tilde{\rho})\right\|_{s-1} \leq \gamma(\|\rho\|_s, \|\tilde{\rho}\|_s)\|\rho - \tilde{\rho}\|_s \tag{3.8}$$

with

$$\gamma(\|\rho\|_s, \|\tilde{\rho}\|_s) = n\Big[\|\rho\|_s L_s(\|\rho\|_s, \|\tilde{\rho}\|_s) + \|\tilde{\rho}\|_s L_s(\|\tilde{\rho}\|_s, 0)\Big] + M_s(\|\rho\|_s, \|\tilde{\rho}\|_s) \tag{3.9}$$

The continuity of $\tilde{F}$ is a consequence of (3.8). As $I \times H^s(\mathbb{R}^n)$ is complete,we prove that $\tilde{F}$ is sequentially continuous. Let $t_n \to t, \rho_n \to \rho$ in $H^s \Rightarrow$ $\exists \tau > 0,\ M > 0 : \|t_n\| \leq \tau\ ; \|\rho_n\| \leq N$ for large n. Then

$$\begin{aligned} \left\|\tilde{F}(t_n,\rho_n) - \tilde{F}(t,\rho)\right\|_{s-1} &\leq \gamma(\|\rho_n\|_s, \|\rho\|_s)\|\rho_n - \rho\|_s \\ &\leq C(N)\|\rho_n - \rho\|_s \longrightarrow 0 \text{ as } n \to \infty \end{aligned} \tag{3.10}$$

where $C(N)$ is a constant depending of N.
On the other hand, estimating (3.3)

$$\begin{aligned} &\left\langle \rho - \tilde{\rho}, \tilde{F}(t,\rho) - \tilde{F}(t,\tilde{\rho}) \right\rangle \\ &= \underbrace{\left\langle \rho - \tilde{\rho}, \operatorname{div}\left[\rho\,\vec{\Theta}(\rho) - \tilde{\rho}\,\vec{\Theta}(\tilde{\rho})\right] \right\rangle}_{=C1} + \underbrace{\left\langle \rho - \tilde{\rho}, F(t,\rho) - F(t,\tilde{\rho}) \right\rangle}_{=C2} \end{aligned} \tag{3.11}$$

Integrating by parts

$$\begin{aligned} C1 &= \sum_{i=1}^{n} \left\langle \rho - \tilde{\rho}, \partial_{x_i}\left[\rho\,\Theta_i(\rho) - \tilde{\rho}\,\Theta_i(\tilde{\rho})\right] \right\rangle \\ &= -\sum_{i=1}^{n} \left\langle \partial_{x_i}(\rho - \tilde{\rho}), \rho\,\Theta_i(\rho) - \tilde{\rho}\,\Theta_i(\tilde{\rho}) \right\rangle \\ &= \underbrace{-\sum_{i=1}^{n} \left\langle \partial_{x_i}(\rho - \tilde{\rho}), \rho(\Theta_i(\rho) - \Theta_i(\tilde{\rho})) \right\rangle}_{=C11} \underbrace{-\sum_{i=1}^{n} \left\langle \partial_{x_i}(\rho - \tilde{\rho}), (\rho - \tilde{\rho})\Theta_i(\tilde{\rho}) \right\rangle}_{=C12} \end{aligned} \tag{3.12}$$

$$\begin{aligned} C11 &= -\sum_{i=1}^{n} \left\langle \partial_{x_i}(\rho - \tilde{\rho}), \rho J^{-2}\partial_{x_i}(P(\rho) - P(\tilde{\rho})) \right\rangle \\ &= \sum_{i=1}^{n} \left\langle (\rho - \tilde{\rho}), \partial_{x_i}\left[\rho J^{-2}\partial_{x_i}(P(\rho) - P(\tilde{\rho}))\right] \right\rangle \\ &= \underbrace{\sum_{i=1}^{n} \left\langle (\rho - \tilde{\rho}), (\partial_{x_i}\rho) J^{-2}\partial_{x_i}(P(\rho) - P(\tilde{\rho})) \right\rangle}_{=C111} \\ &\quad + \underbrace{\sum_{i=1}^{n} \left\langle (\rho - \tilde{\rho}), \rho J^{-2}\partial_{x_i}^2(P(\rho) - P(\tilde{\rho})) \right\rangle}_{C112} \end{aligned} \tag{3.13}$$

Applying Cauchy-Schwartz inequality, Sobolev's Lemma, (2.10)and the inequalities $\|J^{-2}\partial_{x_i}\|_{B(L^2,H^1)} \leq 1, \|J^{-2}\partial^2_{x_i}\|_{B(L^2)} \leq 1$,

$$\begin{aligned} C111 &\leq \sum_{i=1}^{n} \big\|\rho-\tilde\rho\big\|\big\|(\partial_{x_i}\rho)J^{-2}\partial_{x_i}(P(\rho)-P(\tilde\rho))\big\| \\ &\leq \sum_{i=1}^{n} \big\|\rho-\tilde\rho\big\|\big\|(\partial_{x_i}\rho)\big\|_{L^\infty}\big\|J^{-2}\partial_{x_i}\big\|_{B(L^2,H^1)}\big\|P(\rho)-P(\tilde\rho)\big\| \\ &\lesssim \sum_{i=1}^{n} \big\|\rho-\tilde\rho\big\|\big\|\rho\big\|_s\big\|J^{-2}\partial_{x_i}\big\|_{B(L^2,H^1)}\widetilde{L_s}(\|\rho\|_s,\|\tilde\rho\|_s)\|\rho-\tilde\rho\| \\ &\lesssim n\|\rho\|_s\widetilde{L_s}(\|\rho\|_s,\|\tilde\rho\|_s)\|\rho-\tilde\rho\|^2 \end{aligned} \tag{3.14}$$

$$\begin{aligned} C112 &\leq \sum_{i=1}^{n} \big\|\rho-\tilde\rho\big\|\big\|\rho J^{-2}\partial^2_{x_i}(P(\rho)-P(\tilde\rho))\big\| \\ &\leq \sum_{i=1}^{n} \big\|\rho-\tilde\rho\big\|\big\|\rho\big\|_{L^\infty}\big\|J^{-2}\partial^2_{x_i}\big\|_{B(L^2)}\big\|P(\rho)-P(\tilde\rho)\big\| \\ &\lesssim \sum_{i=1}^{n} \big\|\rho-\tilde\rho\big\|\big\|\rho\big\|_s\big\|J^{-2}\partial^2_{x_i}\big\|_{B(L^2)}\widetilde{L_s}(\|\rho\|_s,\|\tilde\rho\|_s)\|\rho-\tilde\rho\| \\ &\lesssim n\|\rho\|_s\widetilde{L_s}(\|\rho\|_s,\|\tilde\rho\|_s)\|\rho-\tilde\rho\|^2 \end{aligned} \tag{3.15}$$

Integrating by parts, using Sobolev's Lemma, positivity of the pressure and $\|J^{-2}\|_{B(H^s,H^{s+2})} \leq 1$, we obtain

$$\begin{aligned} C12 &= -\sum_{i=1}^{n}\Big\langle(\rho-\tilde\rho)\partial_{x_i}(\rho-\tilde\rho),\Theta_i(\tilde\rho)\Big\rangle = -\frac{1}{2}\sum_{i=1}^{n}\Big\langle\partial_{x_i}(\rho-\tilde\rho)^2,J^{-2}\partial_{x_i}P(\tilde\rho)\Big\rangle \\ &= \frac{1}{2}\Big\langle(\rho-\tilde\rho)^2,J^{-2}\Delta P(\tilde\rho)\Big\rangle = \frac{1}{2}\Big[\Big\langle(\rho-\tilde\rho)^2,J^{-2}P(\tilde\rho)\Big\rangle - \Big\langle(\rho-\tilde\rho)^2,P(\tilde\rho)\Big\rangle\Big] \\ &\leq \frac{1}{2}\Big\langle(\rho-\tilde\rho)^2,J^{-2}P(\tilde\rho)\Big\rangle \leq \frac{1}{2}\|J^{-2}P(\tilde\rho)\|_{L^\infty}\|\rho-\tilde\rho\|^2 \\ &\lesssim \frac{1}{2}\|J^{-2}\|_{B(H^s,H^{s+2})}\|P(\tilde\rho)\|_s\|\rho-\tilde\rho\|^2 \lesssim \frac{1}{2}L_s(\|\tilde\rho\|_s,0)\|\tilde\rho\|_s\|\rho-\tilde\rho\|^2 \end{aligned} \tag{3.16}$$

Substituting $\big(3.14\big),\big(3.15\big)$ in $\big(3.13\big)$, and $\big(3.16\big)$ in $\big(3.12\big)$; we have:

$$C1 = \Big\langle\rho-\tilde\rho,\operatorname{div}\big[\rho\,\vec\Theta(\rho)-\tilde\rho\,\vec\Theta(\tilde\rho)\big]\Big\rangle \lesssim \widetilde{C}(\|\rho\|_s,\|\tilde\rho\|_s)\|\rho-\tilde\rho\|^2 \tag{3.17}$$

where

$$\widetilde{C}(\|\rho\|_s, \|\tilde{\rho}\|_s) = 2\,n\|\rho\|_s\widetilde{L_s}(\|\rho\|_s, \|\tilde{\rho}\|_s) + \frac{1}{2}\|\tilde{\rho}\|_s L_s(\|\tilde{\rho}\|_s, 0) \tag{3.18}$$

Consider (2.12) in the last term in (3.11) we have:

$$\begin{aligned} C2 &\leq \left|\left\langle \rho - \tilde{\rho}, F(t,\rho) - F(t,\tilde{\rho})\right\rangle\right| \leq \|\rho - \tilde{\rho}\|\|F(t,\rho) - F(t,\tilde{\rho})\| \\ &\leq \widetilde{M_s}(\|\rho\|_s, \|\tilde{\rho}\|_s)\|\rho - \tilde{\rho}\|^2 \end{aligned} \tag{3.19}$$

Finally, substituting (3.17) and (3.19) in (3.11) we have

$$\left\langle \rho - \tilde{\rho}, \tilde{F}(t,\rho) - \tilde{F}(t,\tilde{\rho})\right\rangle \leq L_0(\|\rho\|_s, \|\tilde{\rho}\|_s)\|\rho - \tilde{\rho}\|^2 \tag{3.20}$$

with

$$L_0(\|\rho\|_s, \|\tilde{\rho}\|_s) = \widetilde{C}(\|\rho\|_s, \|\tilde{\rho}\|_s) + \widetilde{M_s}(\|\rho\|_s, \|\tilde{\rho}\|_s) \tag{3.21}$$

This finishes the proof. □

In what follows, we consider the linear part of (3.1) and prove certain properties of the semigroup ([41]) associated to it.

Applying the Fourier transform to linear part, we have:

$$\widehat{\partial_t\rho_\mu}(\xi) = \partial_t\widehat{\rho_\mu}(\xi) = -\mu\xi^2\widehat{\rho_\mu}(\xi) \Rightarrow \widehat{\rho_\mu}(\xi) = e^{-\mu t\xi^2}\widehat{\rho_0}(\xi) \tag{3.22}$$

Then

$$\rho_\mu(t) = U_\mu(t)\rho_0 = e^{\mu t\Delta}\rho_0 := (e^{-\mu t\xi^2}\widehat{\rho_0}(\xi))^\vee \tag{3.23}$$

In the next lemma, we will show the smoothing properties of semigroup $U_\mu(t)$

Lemma 3.1.2. *Let $\lambda \in [0,\infty)$; $s \in \mathbb{R}$.*

a) *$U_\mu(t) \in B(H^s(\mathbb{R}^n), H^{s+\lambda}(\mathbb{R}^n)), \forall t > 0$ and satisfies:*

$$\|U_\mu(t)(\varphi)\|_{s+\lambda} \leq K_\lambda\Big[1 + \Big(\frac{1}{2\mu t}\Big)^\lambda\Big]^{\frac{1}{2}}\|\varphi\|_s \tag{3.24}$$

where $g_\mu(t) = K_\lambda\Big[1+\Big(\frac{1}{2\mu t}\Big)^\lambda\Big]^{\frac{1}{2}} \in L^1_{loc}([0,\infty))$ if $\lambda < 2$, K_λ is a constant depending only on λ.

b) *The maps $t \in (0,\infty) \to U_\mu(t)\varphi$ is continuous with respect to the topology of $H^{s+\lambda}(\mathbb{R}^n)$*

Proof. See [16],[17] and [18]. □

We are now in position to start the analysis of (3.1). The first step is to obtain a convenient integral equation that can be used to solve it.

Theorem 3.1.1. *Problem (3.1) is equivalent to the integral equation*

$$\rho_\mu(t) = e^{\mu t\Delta}\rho_0 + \int_0^t e^{\mu(t-t')\Delta}\tilde{F}(t',\rho_\mu(t'))\,dt' \tag{3.25}$$

More precisely, if $\rho_\mu \in C([0,T_0],H^s)$ is a solution of (3.1) then ρ_μ satisfies (3.25). Conversely, if $\rho_\mu \in C([0,T_0],H^s)$ solves (3.25) then $\rho_\mu \in C^1([0,T_0],H^{s-2})$ and satisfies (3.1).

Proof. In the first step we prove that if $\rho_\mu \in C([0,T],H^s)$ is a solution of (3.1) then ρ_μ satisfies (3.25)

Using the method of variation of parameters and Fourier Transform, we propose the solution to the nonlinear transformed equation is in the form $\hat{\rho_\mu}(t) = C(t)e^{-\mu t\xi^2}$. Then, we substitute it in the transformed partial differential equation, to obtain

$$C(t) = \int_0^t \widehat{\tilde{F}}(t',\rho_\mu(t'))\,e^{\mu t'\xi^2}\,dt' + c_0 \tag{3.26}$$

where c_0 is a constant of integration
Using the initial condition ρ_0, then $c_0 = C(0) = \hat{\rho}_0$
Therefore, applying inverse Fourier Transform, follows the integral equation for a solution.

On the other hand, we prove that a solution of the integral equation satisfies a partial differential equation.

We define

$$v(t) = \int_0^t e^{\mu(t-t')\Delta}\tilde{F}(t',\rho_\mu(t'))\,dt'\ ;t \in [0,T_0] \tag{3.27}$$

$$
\begin{aligned}
&\frac{v(t+h)-v(t)}{h} = \\
&\frac{1}{h}\Big\{ \int_0^{t+h} e^{\mu(t+h-t')\Delta}\tilde{F}(t',\rho_\mu(t'))\,dt' - \int_0^{t} e^{\mu(t-t')\Delta}\tilde{F}(t',\rho_\mu(t'))\,dt' \Big\} = \\
&\frac{1}{h}\Big\{ \int_0^{t} e^{\mu h\Delta}e^{\mu(t-t')\Delta}\tilde{F}(t',\rho_\mu(t'))\,dt' + \int_t^{t+h} e^{\mu h\Delta}e^{\mu(t-t')\Delta}\tilde{F}(t',\rho_\mu(t'))\,dt' - \\
&\qquad \int_0^{t} e^{\mu(t-t')\Delta}\tilde{F}(t',\rho_\mu(t'))\,dt' \Big\} = \\
&\frac{1}{h}\Big(e^{\mu h\Delta}-Id\Big)\int_0^{t} e^{\mu(t-t')\Delta}\tilde{F}(t',\rho_\mu(t'))\,dt' + \frac{1}{h}\int_t^{t+h} e^{\mu(t+h-t')\Delta}\tilde{F}(t',\rho_\mu(t'))\,dt'
\end{aligned} \tag{3.28}
$$

Applying the Bochner Integral Mean Value ([15],[49]) in the final equality with $\eta_h \in [t, t+h]$ follows

$$
\begin{aligned}
&\frac{v(t+h)-v(t)}{h} = \\
&\frac{1}{h}\Big(e^{\mu h\Delta}-Id\Big)\int_0^{t} e^{\mu(t-t')\Delta}\tilde{F}(t',\rho_\mu(t'))\,dt' + e^{\mu(t+h-\eta_h)\Delta}\tilde{F}(\eta_h,\rho_\mu(\eta_h))
\end{aligned} \tag{3.29}
$$

Applying limit as $h \to 0^+$ in (3.29), we obtain

$$
\partial_t^+ v(t) = \mu\Delta\int_0^{t} e^{\mu(t-t')\Delta}\tilde{F}(t',\rho_\mu(t'))\,dt' + \tilde{F}(t,\rho_\mu(t)) \tag{3.30}
$$

Making the same analysis and considering $h < 0$, we have that $\partial_t^- v(t) = \partial_t^+ v(t)$. This implies that there exists $\partial_t v(t) = \partial_t^+ v(t) = \partial_t^- v(t)$
Now, considering the integral equation and the definition of $v(t)$, we have

$$
\partial_t \rho_\mu(t) = \partial_t\big[e^{\mu t\Delta}\rho_0 + v(t)\big] = \partial_t(e^{\mu t\Delta}\rho_0) + \partial_t v(t) \tag{3.31}
$$

Applying properties of continuous semigroup ([41, Chap. 1], [18, App.]) $e^{\mu t\Delta}$ in (3.31), follows the result

$$
\partial_t \rho_\mu(t) = \mu\Delta e^{\mu t\Delta}\rho_0 + \mu\Delta\int_0^{t} e^{\mu(t-t')\Delta}\tilde{F}(t',\rho_\mu(t'))\,dt' + \tilde{F}(t,\rho_\mu(t)) \tag{3.32}
$$

$$
= \mu\Delta\Big[e^{\mu t\Delta}\rho_0 + \int_0^{t} e^{\mu(t-t')\Delta}\tilde{F}(t',\rho_\mu(t'))\,dt'\Big] + \tilde{F}(t,\rho_\mu(t)) \tag{3.33}
$$

$$
= \mu\Delta\rho_\mu(t) + \tilde{F}(t,\rho_\mu(t)) \tag{3.34}
$$

Finally, as the the operator $\mu\Delta : H^s \longrightarrow H^{s-2}$ and $\tilde{F}(t, \rho_\mu(t)) \in H^{s-1} \hookrightarrow H^{s-2}$ we have that $\rho_\mu(t) \in C^1([0, T_0], H^{s-2})$. □

We will prove that the above integral equation, has a unique solution in $C([0, T^\mu]; H^s)$ for any $0 < T^\mu \le T_0$ and for all $\mu > 0$.

Theorem 3.1.2. *Let $\mu > 0$ be fixed and $\rho_0 \in H^s(\mathbb{R}^n)$, $s > \frac{n}{2}$. Then there exists $T^\mu = T(s, \|\rho_0\|_s, \mu)$ and a unique function $\rho_\mu \in C([0, T^\mu], H^s) \cap C((0, T^\mu]; H^\infty)$ satisfying the integral equation (3.25).*

Proof. We have:

$$\rho_\mu(t) = \overbrace{\underbrace{e^{\mu t\Delta}\rho_0}_{\in H^s(\mathbb{R}^n)} + \underbrace{\int_0^t e^{\mu(t-t')\Delta} \underbrace{\left[\operatorname{div}\left[\rho_\mu(1-\Delta)^{-1}\nabla P(\rho_\mu)\right] + F(t', \rho_\mu(t'))\right]}_{\in H^{s-1}(\mathbb{R}^n)} dt'}_{\in V = H^{s-1+\lambda}(\mathbb{R}^n)}}^{\in Y = H^s(\mathbb{R}^n),\ (\lambda=1)} \tag{3.35}$$

Consider the spaces $V = H^{s-1+\lambda}(\mathbb{R}^n), Y = H^s(\mathbb{R}^n)$. Thus, we have that $V \subseteq Y \subseteq X = H^{s-2}(\mathbb{R}^n)$ if $\lambda \ge 1$. In the rest of the proof of theorem, we use $\lambda = 1$ for simplicity.

Consider the map

$$B(v(t)) = U_\mu(t)\rho_0 + \int_0^t U_\mu(t-t')\tilde{F}(t', v(t'))\, dt' \tag{3.36}$$

defined in the complete metric space

$$X_s(T_0) = \left\{v \in C([0, T_0], H^s(\mathbb{R}^n)) : \left\|v(t) - U_\mu(t)\rho_0\right\| \le M,\ \forall t \in [0, T_0]\right\} \tag{3.37}$$

when the topology in the space $X_s(T_0)$ is defined by the sup-norm, that is $d(v, w) = \sup_{t\in[0,T_0]} \|v(t) - w(t)\|_s$, with $v, w \in X_s(T_0)$

We will show that by taking T^μ sufficiently small the map (3.36) is a contraction in $X_s(T_0)$. Once this is established, we will show that this is in fact the only possible solution in $C([0, T^\mu], H^s(\mathbb{R}^n))$.

Let $v(t) \in X_s(T_0)$, it is easy to see that $\|v(t)\|_s \leq M + \|\rho_0\|_s$. The continuity of the semigroup $U_\mu(t)$ and $\tilde{F}(t,\rho)$ implies that $B(v(t)) \in C([0,T_0], H^s(\mathbb{R}^n))$.

On other hand, considering the properties of semigroup $U_\mu(t)$ in Lemma 3.1.2 and $\tilde{F}(t, v(t))$, yields

$$\begin{aligned}
\|B(v(t)) - U_\mu(t)\rho_0\|_s &\leq \int_0^t \|U_\mu(t-t')\tilde{F}(t', v(t'))\|_s \, dt' \\
&\leq \int_0^t g_\mu(t-t')\|\tilde{F}(t', v(t'))\|_{s-1} \, dt' \\
&\leq \int_0^t g_\mu(t-t')\gamma(\|v(t')\|_s, 0)\|v(t')\|_s \, dt' \\
&\leq (M + \|\rho_0\|_s)\gamma(M + \|\rho_0\|_s, 0) \int_0^t g_\mu(r) \, dr \\
&\leq (M + \|\rho_0\|_s)\gamma(M + \|\rho_0\|_s, 0) \int_0^{T_0} g_\mu(r) \, dr
\end{aligned} \tag{3.38}$$

As $g_\mu(r) \in L^1_{loc}([0,\infty))$

$$\gamma(M + \|\rho_0\|_s, 0) \int_0^{T_0} g_\mu(r) \, dr \longrightarrow 0, \text{ as } T_0 \to 0 \tag{3.39}$$

Then

$$\exists\, \tau \in (0, T_0] : \gamma(M + \|\rho_0\|_s, 0) \int_0^{\tau} g_\mu(r) \, dr \leq \frac{M}{M + \|\rho_0\|_s} \leq 1 \tag{3.40}$$

Therefore, we have

$$\exists\, \tau \in (0, T_0] : \|B(v(t)) - U_\mu(t)\rho_0\|_s \leq M \Rightarrow B(v(t)) \in X(\tau) \tag{3.41}$$

Next, we will prove that this map is a contraction: Let $v(t), w(t) \in X(\tau)$

$$\begin{aligned}
\|B(v(t)) - B(w(t))\|_s &\leq \int_0^t \left\| U_\mu(t-t')[\tilde{F}(t', v(t')) - \tilde{F}(t', w(t'))] \right\|_s dt' \\
&\leq \int_0^t g_\mu(t-t') \left\| \tilde{F}(t', v(t')) - \tilde{F}(t', w(t')) \right\|_{s-1} dt' \\
&\leq \int_0^t g_\mu(t-t')\gamma(\|v(t')\|_s, \|w(t')\|_s)\|v(t') - w(t')\|_s \, dt' \\
&\leq \left[\gamma(M + \|\rho_0\|_s, M + \|\rho_0\|_s) \int_0^t g_\mu(r) \, dr\right] d(v,w)
\end{aligned} \tag{3.42}$$

Then

$$d(B(v), B(w)) \leq \left[\gamma(M + \|\rho_0\|_s, M + \|\rho_0\|_s) \int_0^\tau g_\mu(r)\,dr\right] d(v,w) \tag{3.43}$$

Similarly

$$\gamma(M + \|\rho_0\|_s, M + \|\rho_0\|_s) \int_0^\tau g_\mu(r)\,dr \longrightarrow 0, \text{ as } \tau \to 0 \tag{3.44}$$

Then

$$\exists\, T^\mu \in (0,\tau] : \gamma(M + \|\rho_0\|_s, M + \|\rho_0\|_s) \int_0^{T^\mu} g_\mu(r)\,dr = \delta < 1 \tag{3.45}$$

Therefore, we have

$$\exists\, T^\mu \in (0,\tau] : d(B(v), B(w)) \leq \delta\, d(v,w) \tag{3.46}$$

Existence and uniqueness in $X_s(T_0)$ is a usual application of Banach's fixed Point Theorem. This gives us T^μ and $\rho_\mu \in C([0,T^\mu], H^s(\mathbb{R}^n))$. The fact that $\rho_\mu \in C((0,T^\mu], H^\infty(\mathbb{R}^n))$ now follows from the integral equation using a simple bootstrapping argument with $\lambda \in (1,2)$

Next we deal with uniqueness in $C([0,T^\mu], H^s(\mathbb{R}^n))$. This is a immediately consequence of the following weak continuous dependence result (weak in the sense that we consider the same intervals of existence of solutions).

Lemma 3.1.3. *Let $\mu > 0$ and $\rho_\mu, \tilde\rho_\mu$ solutions of (3.1) in $C([0,T^\mu], H^s(\mathbb{R}^n))$ with initial condition data $\rho_0, \tilde\rho_0$ respectively. Then*

$$\|\rho_\mu(t) - \tilde\rho_\mu(t)\|_s \leq e^{\gamma(\widehat{M},\widehat{M})}\|\rho_0 - \tilde\rho_0\|_s \tag{3.47}$$

where $\widehat{M} = \max\left[\sup_{t\in[0,T^\mu]}\|\rho_\mu(t)\|_s, \sup_{t\in[0,T^\mu]}\|\tilde\rho_\mu(t)\|_s\right]$

Proof. Let $\rho_\mu(t), \tilde\rho_\mu(t) \in C((0,T^\mu], H^s(\mathbb{R}^n))$, with initial conditions $\rho_0, \tilde\rho_0$ respectively.

Then

$$\begin{aligned}
&\|\rho_\mu(t) - \tilde\rho_\mu(t)\|_s \\
&\leq \|U_\mu(t)(\rho_0 - \tilde\rho_0)\|_s + \left\| \textstyle\int_0^t U_\mu(t-t')\left[\tilde F(t',\rho_\mu(t')) - \tilde F(t',\tilde\rho_\mu(t'))\right] dt' \right\|_s \\
&\leq \|\rho_0 - \tilde\rho_0\|_s + \textstyle\int_0^t g_\mu(t-t')\|\tilde F(t',\rho_\mu(t')) - \tilde F(t',\tilde\rho_\mu(t')\|_{s-1}\,dt' \\
&\leq \|\rho_0 - \tilde\rho_0\|_s + \textstyle\int_0^t g_\mu(t-t')\gamma(\|\rho_\mu(t')\|_s, \|\tilde\rho_\mu(t')\|_s)\|\rho_\mu(t') - \tilde\rho_\mu(t')\|_s\,dt' \\
&\leq \|\rho_0 - \tilde\rho_0\|_s + \gamma(\widehat{M},\widehat{M}) \textstyle\int_0^t g_\mu(t-t')\|\rho_\mu(t') - \tilde\rho_\mu(t')\|_s\,dt'
\end{aligned} \tag{3.48}$$

Applying Gronwall's inequality in (3.48)

$$\|\rho_\mu(t) - \tilde{\rho}_\mu(t)\|_s \leq e^{\gamma(\widehat{M},\widehat{M})}\|\rho_0 - \tilde{\rho}_0\|_s \tag{3.49}$$

□

Finally, uniqueness of solution in $C([0,T^\mu], H^s(\mathbb{R}^n))$ follows of the above inequality taking the same initial conditions for the solution, i.e.,

$$\|\rho_\mu(t) - \tilde{\rho}_\mu(t)\|_s \leq 0 \Rightarrow \rho_\mu(t) = \tilde{\rho}_\mu(t),\ \forall t \in [0,T^\mu] \tag{3.50}$$

This finishes the proof. □

3.1.2 Existence and uniqueness of BFE

Our aim in this subsection is to establish local existence and uniqueness theorems for the problem (1.6) in certain Sobolev spaces. To do this, one of the fundamental steps is to prove that the solution of (3.1) can be extended to an interval independent of μ, because the main difficulty is that $T^\mu \to 0$ as $\mu \to 0$. Finally we will take the limit when $\mu \to 0$.

Lemma 3.1.4. *Assume that $\mu > 0$ and that P, F satisfy (2.9),(2.10) and (2.11),(2.12) respectively for some fixed $s > \frac{n}{2}$. Then there exists $\tilde{T}_s = \tilde{T}(s, \|\rho_0\|_s)$ independent of $\mu > 0$, such that all solutions $\rho_\mu(t)$ can be extended, if necessary, to $(0, \tilde{T}_s]$ satisfying $\left\|\rho_\mu(t)\right\|_s^2 \leq h(t);\ t \in [0, \tilde{T}_s]$.*

Proof. Considering $\rho = \rho_\mu(t) \in C((0,T^\mu]; H^\infty(\mathbb{R}^n))$, the following calculations are entirely rigorous.

$$\begin{aligned} \partial_t\|\rho\|_s^2 = &\ 2\Big\langle \rho, \partial_t\rho \Big\rangle_s \\ = &\ 2\Big[\underbrace{\Big\langle \rho, \mu\Delta\rho\Big\rangle_s}_{=B1} + \underbrace{\Big\langle \rho, F(t,\rho)\Big\rangle_s}_{=B2} + \underbrace{\Big\langle \rho, \operatorname{div}\big(\rho\,\vec{\Theta}(\rho)\big)\Big\rangle_s}_{=B3}\Big] \end{aligned} \tag{3.51}$$

As $H_0 = -\Delta$ is a self-adjoint and positive operator, in H^s.

$$B1 = -\mu\big\langle \rho, H_0\rho\big\rangle_s = -\mu\overbrace{\big\langle H_0\rho, \rho\big\rangle_s}^{\geq 0} \leq 0 \tag{3.52}$$

Applying Cauchy Schwartz inequality and properties of F

$$\begin{aligned} B2 = \big\langle \rho, F(t,\rho)\big\rangle_s &\leq \big|\big\langle \rho, F(t,\rho)\big\rangle_s\big| \\ &\leq \|\rho\|_s \|F(t,\rho)\|_s = \|\rho\|_s\|F(t,\rho) - \overbrace{F(t,0)}^{=0}\|_s \\ &\leq M_s(\|\rho\|_s, 0)\|\rho\|_s^2 \end{aligned} \tag{3.53}$$

Using the commutator

$$\big[\partial_{x_i} J^s, \Theta_i(\rho)\big]\rho = \partial_{x_i} J^s(\rho\,\Theta_i(\rho)) - \Theta_i(\rho)\partial_{x_i} J^s \rho \tag{3.54}$$

We expand

$$\begin{aligned} B3 &= \Big\langle \rho, \sum_{i=1}^{n} \partial_{x_i}\big[\rho\,\Theta_i(\rho)\big]\Big\rangle_s = \sum_{i=1}^{n}\Big\langle \rho, \partial_{x_i}\big[\rho\,\Theta_i(\rho)\big]\Big\rangle_s \\ &= \sum_{i=1}^{n}\Big\langle J^s\rho, J^s\partial_{x_i}\big[\rho\,\Theta_i(\rho)\big]\Big\rangle = \sum_{i=1}^{n}\Big\langle J^s\rho, \partial_{x_i}J^s\big[\rho\,\Theta_i(\rho)\big]\Big\rangle \\ &= \sum_{i=1}^{n}\Big\langle J^s\rho, \big[\partial_{x_i}J^s, \Theta_i(\rho)\big]\rho\Big\rangle + \sum_{i=1}^{n}\Big\langle J^s\rho, \Theta_i(\rho)\partial_{x_i}(J^s\rho)\Big\rangle \\ &= \underbrace{\Big\langle J^s\rho, \sum_{i=1}^{n}\big[\partial_{x_i}J^s, \Theta_i(\rho)\big]\rho\Big\rangle}_{=B31} + \underbrace{\frac{1}{2}\sum_{i=1}^{n}\Big\langle \partial_{x_i}(J^s\rho)^2, \Theta_i(\rho)\Big\rangle}_{=B32} \end{aligned} \tag{3.55}$$

Thus, using Cauchy Schwartz inequality, Lemma A.1.3 in Appendix and properties of the pressure

$$\begin{aligned} B31 &= \Big\langle J^s\rho, \sum_{i=1}^{n}\big[\partial_{x_i}J^s, (1-\Delta)^{-1}\partial_{x_i}P(\rho)\big]\rho\Big\rangle \\ &\leq \big\|J^s\rho\big\|\Big\|\sum_{i=1}^{n}\big[\partial_{x_i}J^s, \partial_{x_i}(1-\Delta)^{-1}P(\rho)\big]\rho\Big\| \\ &\leq c\|\rho\|_s\Big[\big\|J^2J^{-2}P(\rho)\big\|_{L^\infty}\big\|J^s\rho\big\| + \big\|J^{s+2}J^{-2}P(\rho)\big\|\|\rho\|_{L^\infty}\Big] \\ &\lesssim \|\rho\|_s\Big[\big\|P(\rho)\big\|_s\|\rho\|_s + \big\|P(\rho)\big\|_s\|\rho\|_s\Big] \\ &\lesssim 2\|\rho\|_s^3 L_s(\|\rho\|_s, 0) \end{aligned} \tag{3.56}$$

Integrating by parts, considering that the Resolvent $J^{-2} = (1-\Delta)^{-1}$ preserves positivity ([43, Vol. II]) and Sobolev's Lemma

$$\begin{aligned} B32 &= -\frac{1}{2}\sum_{i=1}^{n}\Big\langle (J^s\rho)^2, \partial_{x_i}(1-\Delta)^{-1}\partial_{x_i}P(\rho)\Big\rangle \qquad (3.57)\\ &= -\frac{1}{2}\Big\langle (J^s\rho)^2, (1-\Delta)^{-1}\Delta P(\rho)\Big\rangle \\ &= -\frac{1}{2}\underbrace{\Big\langle (J^s\rho)^2, (1-\Delta)^{-1}P(\rho)\Big\rangle}_{\geq 0} + \frac{1}{2}\Big\langle (J^s\rho)^2, P(\rho)\Big\rangle \\ &\leq \frac{1}{2}\Big\langle (J^s\rho)^2, P(\rho)\Big\rangle \leq \frac{1}{2}\|P(\rho)\|_{L^\infty}\big\|J^s\rho\big\|^2 \\ &\lesssim \frac{1}{2}\|P(\rho)\|_s\|\rho\|_s^2 \lesssim \frac{1}{2}\|\rho\|_s^3 L_s(\|\rho\|_s, 0) \end{aligned}$$

Summing (3.52), (3.53), (3.56), (3.57) in (3.51), we have the final estimate:

$$\begin{aligned} \partial_t\|\rho_\mu(t)\|_s^2 &\lesssim M_s\big(\|\rho_\mu(t)\|_s, 0\big)\|\rho_\mu(t)\|_s^2 + \|\rho_\mu(t)\|_s^3 L_s\big(\|\rho_\mu(t)\|_s, 0\big) \qquad (3.58)\\ &= M_s\Big((\|\rho_\mu(t)\|_s^2)^{\frac{1}{2}}, 0\Big)\|\rho_\mu(t)\|_s^2 + (\|\rho_\mu(t)\|_s^2)^{\frac{3}{2}} L_s\Big((\|\rho_\mu(t)\|_s^2)^{\frac{1}{2}}, 0\Big) \\ &= G\Big(\|\rho_\mu(t)\|_s^2\Big) \qquad (3.59) \end{aligned}$$

Let $h(t)$ be the maximal solution ([8]) of initial value problem for ordinary differential equation:

$$\begin{cases} \partial_t h(t) = G(h(t)) \\ h(0) = \|\rho_0\|_s^2 \end{cases}$$

Then $\big\|\rho_\mu(t)\big\|_s^2 \leq h(t)$; $t \in [0, \tilde{T}_s]$; $\tilde{T}_s \in [0, T_0)$, whenever both sides are defined. This finishes the proof since $h(t)$ not depends of μ and we can extend $\rho_\mu(t)$ to interval $[0, \tilde{T}_s]$. □

We are now in position to state and prove the main result of this section.

Theorem 3.1.3. *Let $\rho_0 \in H^s(\mathbb{R}^n)$, $s > \frac{n}{2} + 1$. Then there exists $\tilde{T}_s = \tilde{T}(s, \|\rho_0\|_s) > 0$ and unique $\rho \in C_+([0, \tilde{T}_s], H^s(\mathbb{R}^n))$. Moreover $\rho(t)$ satisfies that $\partial_t\rho \in C_+^1([0, \tilde{T}_s], H^{s-1}(\mathbb{R}^n))$, $\|\rho(t)\|_s^2 \leq h(t)$, and the initial value problem (1.6).*

Proof. We choose any such interval as in the preceding theorem, and write $\rho = \rho_\mu(t),\ \tilde\rho = \rho_\nu(t);\ \mu, \nu > 0;\ \rho_\mu(0) = \rho_\nu(0) = \rho_0$. Let $M^2 = \sup_{t\in[0,\tilde T_s]} h(t)$, and note that

$$\begin{aligned}\partial_t\|\rho-\tilde\rho\|^2 &= 2\big\langle \rho-\tilde\rho, \partial_t(\rho-\tilde\rho)\big\rangle \\ &= 2\Big[\big\langle \rho-\tilde\rho, \tilde F(t,\rho)-\tilde F(t,\tilde\rho)\big\rangle + \underbrace{\big\langle \rho-\tilde\rho, \mu\Delta\rho-\nu\Delta\tilde\rho\big\rangle}_{=A}\Big]\end{aligned} \tag{3.60}$$

Integrating by parts and Cauchy Schwartz inequality implies that

$$\begin{aligned} A &= \big\langle \rho-\tilde\rho, \mu\Delta\rho-\nu\Delta\tilde\rho\big\rangle \\ &= \big\langle \rho-\tilde\rho, \mu\Delta\rho-\nu\Delta\rho+\nu\Delta\rho-\nu\Delta\tilde\rho\big\rangle \\ &= \big\langle \rho-\tilde\rho, (\mu-\nu)\Delta\rho\big\rangle - \overbrace{\nu\big\langle \rho-\tilde\rho, H_0(\rho-\tilde\rho)\big\rangle}^{\geq 0} \\ &\leq (\mu-\nu)\big\langle \rho-\tilde\rho, \Delta\rho\big\rangle \leq |\mu-\nu|\big|\big\langle \rho-\tilde\rho, \Delta\rho\big\rangle\big| \\ &= \big|\mu-\nu\big|\Big|\sum_{i=1}^n \Big\langle \partial_{x_i}\rho, \partial_{x_i}(\rho-\tilde\rho\Big\rangle\Big| \\ &\leq \big|\mu-\nu\big|\sum_{i=1}^n \overbrace{\big\|\partial_{x_i}\rho\big\|}^{\leq\|\rho\|_1\leq\|\rho\|_s\leq M}\Big(\overbrace{\big\|\partial_{x_i}\rho\big\|}^{\leq\|\rho\|_1\leq\|\rho\|_s\leq M} + \overbrace{\big\|\partial_{x_i}\tilde\rho\big\|}^{\leq\|\tilde\rho\|_1\leq\|\tilde\rho\|_s\leq M}) \\ &\leq 2nM^2|\mu-\nu| \end{aligned} \tag{3.61}$$

Finally, substituting (3.3), (3.61) in (3.60); we have:

$$\begin{aligned}\partial_t\|\rho-\tilde\rho\|^2 &\leq 4nM^2|\mu-\nu| + 2\,L_0(\|\rho\|_s, \|\tilde\rho\|_s)\|\rho-\tilde\rho\|^2 \\ &\leq 4nM^2|\mu-\nu| + 2\,L_0(M,M)\|\rho-\tilde\rho\|^2\end{aligned} \tag{3.62}$$

Integrating the last estimate from 0 to t:

$$\|\rho_\mu(t)-\rho_\nu(t)\|^2 \leq 4\,n\,M^2\tilde T_s|\mu-\nu| + \int_0^t 2\,L_0(M,M)\|\rho_\mu(\tau)-\rho_\nu(\tau)\|^2 d\tau \tag{3.63}$$

Gronwall's inequality, then shows that

$$\|\rho_\mu(t)-\rho_\nu(t)\|^2 \leq 4\,n\,M^2\tilde T_s|\mu-\nu|\,e^{2\,\tilde T_s L_0(M,M)} \tag{3.64}$$

$$\lim_{\mu\to 0,\,\nu\to 0} \|\rho_\mu(t)-\rho_\nu(t)\|^2 = 0 \Rightarrow \rho_\mu(t) \longrightarrow \rho_\nu(t) \text{ in } L^2,\ t\in[0,\tilde T_s]$$

Now, $\rho_\mu(t)$ is a Cauchy net in the space $L^2(\mathbb{R}^n)$, which is complete. Therefore, there exists $\rho(t) \in C([0,\tilde{T}_s], L^2(\mathbb{R}^n))$ that satisfies

$$\lim_{\mu\to 0} \sup_{[0,\tilde{T}_s]} \|\rho_\mu(t) - \rho(t)\| = 0$$

Thus $t \in [0,\tilde{T}_s] \to \rho_\mu(t)$ is continuous and uniformly bounded in $L^2(\mathbb{R}^n)$.

We claim that $\{\rho_\mu(t)\}_{\mu>0}$ is a weak Cauchy net in $H^s(\mathbb{R}^n)$ uniformly with respect to $t \in [0,\tilde{T}_s]$. Indeed, given $\varphi \in H^s(\mathbb{R}^n)$ and $\epsilon > 0$, choose $\varphi_\epsilon \in S(\mathbb{R}^n)$ such that $\|\varphi - \varphi_\epsilon\|_s < \epsilon$.

$$\begin{aligned}\langle \rho_\mu(t) - \rho_\nu(t), \varphi\rangle_s &= \langle \rho_\mu(t) - \rho_\nu(t), \varphi - \varphi_\epsilon\rangle_s + \langle \rho_\mu(t) - \rho_\nu(t), \varphi_\epsilon\rangle_s \\ &\le |\langle \rho_\mu(t) - \rho_\nu(t), \varphi - \varphi_\epsilon\rangle_s| + |\langle J^s(\rho_\mu(t) - \rho_\nu(t)), J^s\varphi_\epsilon\rangle| \\ &\le \|\rho_\mu(t) - \rho_\nu(t)\|_s \|\varphi - \varphi_\epsilon\|_s + \|\rho_\mu(t) - \rho_\nu(t)\| \|\varphi_\epsilon\|_{2s} \\ &\le 2\, M\, \epsilon + \|\rho_\mu(t) - \rho_\nu(t)\| \|\varphi_\epsilon\|_{2s}\end{aligned} \tag{3.65}$$

So that $\lim_{\mu\to 0,\nu\to 0} \sup_{[0,\tilde{T}_s]} \langle \rho_\mu(t) - \rho_\nu(t), \varphi\rangle_s = 0$ uniformly.

Since $H^s(\mathbb{R}^n)$ is reflexive, it is weakly complete ([15],[49]), and there exists $v(t) \in C_w([0,\tilde{T}_s], H^s(\mathbb{R}^n))$ satisfying

$$\lim_{\mu\to 0} \langle \rho_\mu(t), \varphi\rangle_s = \langle v(t), \varphi\rangle_s \,\forall \varphi \in S(\mathbb{R}^n) \tag{3.66}$$

It is easy to see that $v(t) = \rho(t)\, \forall t \in [0,\tilde{T}_s]$, as a consequence of uniqueness of weakly limit. In particular, $\rho(t)$ is weakly continuous and uniformly bounded by the function $\sqrt{h(t)}$. Indeed,

$$\begin{aligned}\|\rho(t)\|_s &= \sup_{\|\psi\|_s=1} |\langle \rho(t), \psi\rangle_s| = \sup_{\|\psi\|_s=1} \lim_{\mu\to 0} |\langle \rho_\mu(t), \psi\rangle_s| \\ &\le \sup_{\|\psi\|_s=1} \lim_{\mu\to 0} \|\rho_\mu(t)\|_s \|\psi\|_s \le \sqrt{h(t)}\end{aligned} \tag{3.67}$$

It remains to prove that $\rho(t) \in C^1_w([0,\tilde{T}_s], H^{s-1}(\mathbb{R}^n))$. Let $\psi \in H^{s-1}(\mathbb{R}^n)$

$$\langle \rho_\mu(t), \psi\rangle_{s-1} = \langle U_\mu(t)\rho_0, \psi\rangle_{s-1} + \int_0^t \langle \tilde{F}(t', \rho_\mu(t')), \psi\rangle_{s-1}\, dt', \forall t \in [0,\tilde{T}_s] \tag{3.68}$$

Since $\rho_\mu(t) \rightharpoonup \rho(t)$ in $H^s(\mathbb{R}^n)$, it follows that, $\tilde{F}(t,\rho_\mu(t)) \rightharpoonup \tilde{F}(t,\rho(t))$ uniformly in $H^{s-1}(\mathbb{R}^n)$, therefore, taking the limit as $\mu \to 0$ in (3.68) we obtain

$$\langle \rho(t), \psi\rangle_{s-1} = \langle \rho_0, \psi\rangle_{s-1} + \int_0^t \langle \tilde{F}(t',\rho(t')), \psi\rangle_{s-1}\, dt', \forall t \in [0,\tilde{T}_s] \tag{3.69}$$

As the integrand on the right-hand side of (3.69) is a continuous function, from the Fundamental Theorem of Calculus, follows that:

$$\langle \partial_t \rho(t), \psi\rangle_{s-1} = \langle \tilde{F}(t,\rho(t)), \psi\rangle_{s-1}, \forall t \in [0,\tilde{T}_s] \tag{3.70}$$

Since the map $t \in [0,\tilde{T}_s] \longrightarrow \tilde{F}(t,\rho(t))$ is weakly continuous and uniformly bounded, Petti's Theorem ([49, Chap.V]) implies that it is strongly measurable. Thus we may define a Bochner integral

$$\int_0^t \tilde{F}(t',\rho(t'))\, dt' \tag{3.71}$$

Combining this remark with (3.69) we conclude

$$\rho(t) = \rho_0 + \int_0^t \tilde{F}(t',\rho(t'))\, dt' \tag{3.72}$$

Thus $\rho(t) \in AC([0,\tilde{T}_s], H^{s-1}(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n)$. Therefore $\partial_t\rho(t)$ exists almost everywhere in $[0,\tilde{T}_s]$ and is given by

$$\partial_t\rho(t) = \tilde{F}(t,\rho(t)) = \operatorname{div}\left[\rho(t) J^{-2}\nabla P(\rho(t))\right] + F(t,\rho(t)), a.e., \, t \in [0,\tilde{T}_s] \tag{3.73}$$

Next we claim that there is only such function in the class

$$\Omega(\tilde{T}_s) = C([0,\tilde{T}_s], L^2(\mathbb{R}^n)) \cap C_w([0,\tilde{T}_s], H^s(\mathbb{R}^n)) \cap AC([0,\tilde{T}_s], H^{s-1}(\mathbb{R}^n))$$

Let $\rho(t), \eta(t) \in \Omega(\tilde{T}_s)$ with $\rho(0) = \eta(0) = \rho_0$, a calculation similar to that leading to (3.62) implies:

$$\partial_t \|\rho(t) - \eta(t)\|^2 \leq 2\, L_0(M,M) \|\rho(t) - \eta(t)\|^2 \tag{3.74}$$

Integrating from 0 to t:

$$\|\rho(t) - \eta(t)\|^2 \leq \underbrace{\|\rho(0) - \eta(0)\|^2}_{=0} + \int_0^t 2\, L_0(M,M) \|\rho(t') - \eta(t')\|^2 dt' \tag{3.75}$$

Applying Gronwall's lemma in the last estimate,we have:

$$\|\rho(t)-\eta(t)\|^2 \leq 0 \Rightarrow \rho(t)=\eta(t)\in\Omega(\tilde{T}_s) \tag{3.76}$$

It remains to prove that the unique solution in $\Omega(\tilde{T}_s)$ belongs to $C_+([0,\tilde{T}_s],H^s(\mathbb{R}^n))$. Let $\varphi\in H^s(\mathbb{R}^n)$ be such that $\|\varphi\|_s=1$. We have

$$|\langle\rho(t),\varphi\rangle_s| \leq \|\rho(t)\|_s \leq \sqrt{h(t)},\ \forall\varphi\in H^s(\mathbb{R}^n),\forall t\in[0,\tilde{T}_s] \tag{3.77}$$

Therefore

$$\begin{aligned}|\langle\rho_0,\varphi\rangle_s| &= \lim_{t\to0^+}|\langle\rho(t),\varphi\rangle_s| = \liminf_{t\to0^+}|\langle\rho(t),\varphi\rangle_s| \\ &\leq \liminf_{t\to0^+}\|\rho(t)\|_s \leq \limsup_{t\to0^+}\|\rho(t)\|_s \\ &\leq \limsup_{t\to0^+}\sqrt{h(t)} = \|\rho_0\|_s\ \forall\varphi\in H^s(\mathbb{R}^n)\end{aligned} \tag{3.78}$$

Taking the sup over $\|\varphi\|_s=1$ we conclude that

$$\liminf_{t\to0^+}\|\rho(t)\|_s = \limsup_{t\to0^+}\|\rho(t)\|_s = \|\rho_0\|_s \tag{3.79}$$

so that the limit of $\|\rho(t)\|_s$ exists as $t\to0^+$ and

$$\lim_{t\to0^+}\|\rho(t)\|_s = \|\rho_0\|_s \tag{3.80}$$

Since $\rho(t)\rightharpoonup\rho_0$ weakly in $H^s(\mathbb{R}^n)$, it follows that

$$\lim_{t\to0^+}\rho(t)=\rho_0 \tag{3.81}$$

in the norm of $H^s(\mathbb{R}^n)$ □

`Remark` Kato's Theory and uniqueness of solution, implies that $\rho\in C([0,\tilde{T}_s],H^s(\mathbb{R}^n))$, $s>\frac{n}{2}+1$.

3.2 Comparison Principle for the BFE

Consider the initial value problem (BFE) with $F(t,\rho)=0,\ P(\rho)=\rho^{2k}, k=1,2,3.....$

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\,\vec{\mathbf{v}}) = 0,\ x\in\mathbb{R}^n,\ t\in(0,T_0] \\ \qquad\qquad \vec{\mathbf{v}} = -\nabla(-\Delta+1)^{-1}\rho^{2k} = -\vec{\Theta}(\rho) \\ (\rho(0),\ \vec{\mathbf{v}}(0)) = (\rho_0,\ \vec{\mathbf{v}}_0) \end{cases} \tag{3.82}$$

Theorem 3.2.1. (Comparison Principle). *Let* $(\rho,\vec{\mathbf{v}})$ *and* $(\eta,\vec{\mathbf{w}})$ *be solutions of (BFE) with* $P(\rho)=\rho^{2k}, P(\eta)=\eta^{2k}, k=1,2,3....$; *and initial values* $(\rho_0,\vec{v}_0)$ *and* $(\eta_0,\vec{w}_0)$ *respectively. Then*

$$0\le\eta_0(x)\le\rho_0(x)\ \text{in}\,\mathbb{R}^n \Rightarrow 0\le\eta(x,t)\le\rho(x,t)\ \text{in}\,\mathbb{R}^n\times[0,T_0] \tag{3.83}$$

Proof. Let $R(t,y)=\rho(\vec{\phi}(t,y),t); S(t,y)=\eta(\vec{\psi}(t,y),t)$, and $Q(t,y)=R(t,y)-S(t,y)$ where $\vec{\phi}(t,y)$ and $\vec{\psi}(t,y)$ satisfy the following equations respectively.

$$\begin{cases} \dfrac{\partial\vec{\phi}}{\partial t}(t,y) = \vec{\mathbf{v}}(\vec{\phi}(t,y),t) & \vec{\phi}(t,y) = (\phi_1(t,y),\phi_2(t,y),...,\phi_n(t,y)) \\ \vec{\phi}(0,y) = y & v_i = -\partial_{x_i}(1-\Delta)^{-1}\rho^{2k} \end{cases} \tag{3.84}$$

$$\begin{cases} \dfrac{\partial\vec{\psi}}{\partial t}(t,y) = \vec{\mathbf{w}}(\vec{\psi}(t,y),t) & \vec{\psi}(t,y) = (\psi_1(t,y),\psi_2(t,y),...,\psi_n(t,y)) \\ \vec{\psi}(0,y) = y & w_i = -\partial_{x_i}(1-\Delta)^{-1}\eta^{2k} \end{cases} \tag{3.85}$$

Combining (3.82) with (3.84), and (3.82) with (3.85) we have that $R(t)$ and $S(t)$ satisfy the ordinary differential equations, respectively

$$\begin{cases} \dfrac{dR}{dt} = -R\operatorname{div}\vec{\mathbf{v}} & \dfrac{dS}{dt} = -S\operatorname{div}\vec{\mathbf{w}} \\ R(0,y) = \rho_0(y) & S(0,y) = \eta_0(y) \end{cases} \tag{3.86}$$

Solving (3.86), we obtain:

$$R(t) = R(0) \exp\big[-\int_0^t \operatorname{div} \vec{\mathbf{v}}(\vec{\phi}(s,y),s)\, ds \big] \overbrace{\Rightarrow}^{\rho_0(y)\geq 0} R(t) \geq 0 \tag{3.87}$$

Analogously we have that:

$$S(t) = S(0) \exp\big[-\int_0^t \operatorname{div} \vec{\mathbf{w}}(\vec{\psi}(s,y),s) ds \big] \overbrace{\Rightarrow}^{\eta_0(y)\geq 0} S(t) \geq 0 \tag{3.88}$$

On the other hand, differentiating $Q(t)$:

$$\begin{aligned} \frac{dQ}{dt} &= \frac{dR}{dt} - \frac{dS}{dt} = (-\operatorname{div} \vec{\mathbf{v}})R(t) + (\operatorname{div} \vec{\mathbf{w}})S(t) \\ &= -\rho \operatorname{div} \vec{\mathbf{v}} + \eta \operatorname{div} \vec{\mathbf{w}} \\ &= -(\rho - \eta)\operatorname{div} \vec{\mathbf{v}} + \eta(\operatorname{div} \vec{\mathbf{w}} - \operatorname{div} \vec{\mathbf{v}}) \\ &= -Q(t)(\operatorname{div} \vec{\mathbf{v}}) + S(t)(\operatorname{div} \vec{\mathbf{w}} - \operatorname{div} \vec{\mathbf{v}}) \end{aligned} \tag{3.89}$$

Then

$$\begin{aligned} \operatorname{div} \vec{\mathbf{v}} &= -\operatorname{div} \vec{\Theta}(\rho) \\ &= -\operatorname{div} J^{-2}\nabla\rho^{2k} = -\operatorname{div} \nabla J^{-2}\rho^{2k} \\ &= -\Delta(1-\Delta)^{-1}\rho^{2k} = (1-\Delta-1)(1-\Delta)^{-1}\rho^{2k} \\ &= \rho^{2k} - (1-\Delta)^{-1}\rho^{2k} \end{aligned} \tag{3.90}$$

Analogously

$$\operatorname{div} \vec{\mathbf{w}} = ... = \eta^{2k} - (1-\Delta)^{-1}\eta^{2k} \tag{3.91}$$

Substituting (3.90), (3.91) in (3.89):

$$
\begin{aligned}
\frac{dQ}{dt} &= -Q(t)(\operatorname{div}\vec{\mathbf{v}}) + S(t)\Big(\operatorname{div}\vec{\mathbf{w}} - \operatorname{div}\vec{\mathbf{v}}\Big) \qquad (3.92)\\
&= -Q(t)(\operatorname{div}\vec{\mathbf{v}}) + S(t)\Big(\eta^{2k} - (1-\Delta)^{-1}\eta^{2k} - \rho^{2k} + (1-\Delta)^{-1}\rho^{2k}\Big)\\
&= -Q(t)(\operatorname{div}\vec{\mathbf{v}}) - S(t)\Big(\rho^{2k} - \eta^{2k}\Big) + S(t)(1-\Delta)^{-1}\big(\rho^{2k} - \eta^{2k}\Big)\\
&= -Q(t)(\operatorname{div}\vec{\mathbf{v}}) - S(t)\underbrace{(\rho-\eta)}_{Q(t)}\underbrace{\Big(\sum_{i=0}^{2k-1}\rho^{2k-1-i}\eta^{i}\Big)}_{P(\rho,\eta)}\\
&\quad + \underbrace{S(t)(1-\Delta)^{-1}\Big[(\rho-\eta)\Big(\sum_{i=0}^{2k-1}\rho^{2k-1-i}\eta^{i}\Big)\Big]}_{B(t,Q)}
\end{aligned}
$$

As consequence of the above calculation, we have a new ordinary differential equation for $Q(t)$, i.e.:

$$
\begin{cases}
\dfrac{dQ}{dt} = -\big[\operatorname{div}\vec{\mathbf{v}} + S(t)P(R(t),S(t))\big]Q(t) + B(t,Q)\\
Q(0) = \rho_0(y) - \eta_0(y)
\end{cases}
\qquad (3.93)
$$

Applying method of variation of parameters in (3.93), the integral solution is:

$$
Q(t) = U(t,0)Q(0) + \int_0^t U(t,s)B(s,Q(s))\,ds \qquad (3.94)
$$

where

$$
U(t,s) = \exp\big[-\int_s^t[\operatorname{div}(\vec{\mathbf{v}}(\vec{\phi}(\tau,y),\tau)) + S(\tau)P(R(\tau),S(\tau))]d\tau\big]. \qquad (3.95)
$$

In view of conditions for ρ_0 and η_0, we have that $R(t) \geq 0$ and $S(t) \geq 0$. Consider the sequence

$$
\begin{aligned}
Q_{n+1}(t) &= U(t,0)Q(0) + \int_0^t U(t,s)B(s,Q_n(s))ds \qquad n = 1,2,....\\
Q_0(t) &= \rho_0(y) - \eta_0(y)
\end{aligned}
$$

If $Q(0) \geq 0$, then $Q_n(t) \geq 0$, for all n. Thus

$$Q(t) = \rho\big(\vec{\phi}(t,y),t\big) - \eta\big(\vec{\psi}(t,y),t\big) = \lim_{n\to\infty} Q_n(t) \geq 0$$

To complete the proof we need to show the functions $y \in \mathbb{R}^n \to \vec{\phi}(t,y) \in \mathbb{R}^n$ and $y \in \mathbb{R}^n \to \vec{\psi}(t,y) \in \mathbb{R}^n$ are onto.To do this, we analyze in detail the map $y \in \mathbb{R}^n \to \vec{\phi}(t,y) \in \mathbb{R}^n$.
From (3.84), integrating de 0 a t, we obtain :

$$\phi_i(t) - y_i = \int_0^t v_i(\vec{\phi}(s,y),s)\,ds;\ i = 1,2,....$$

Then

$$|\phi_i(t) - y_i| \leq \int_0^t |v_i(\vec{\phi}(s,y),s)|\,ds \leq a_i(\|\rho_0\|_s,t)t\ ,i=1,2,....\ ;\ s>\frac{n}{2}$$

$$y_i - a_i(\|\rho_0\|_s,t) \leq \phi_i(t,y) \leq y_i + a_i(\|\rho_0\|_s,t),\ \forall y_i \in \mathbb{R}$$

Taking $z_i \in \mathbb{R}$; $y_i^{(1)} << 0, y_i^{(2)} >> 0$ such that $z_i \in (y_i^{(1)}, y_i^{(2)})$ we have:

$$y_i^{(1)} + a_i(\|\rho_0\|_s,t) < z_i < y_i^{(2)} - a_i(\|\rho_0\|_s,t)$$

Therefore

$$\phi_i(t,y_i^{(1)}) < z_i < \phi_i(t,y_i^{(2)})$$

The theorem of the average value, for continuous functions ϕ_i implies that exists $y_i \in (y_i^{(1)}, y_i^{(2)})$ and satisfies $\phi_i(t,y_i) = z_i$
An analogous argument, proves that the map $y \in \mathbb{R}^n \longrightarrow \vec{\psi}(t,y)$ is onto. □

3.3 Global estimates in $H^s(\mathbb{R}^n)$, $s > \frac{n}{2}+1$

In this section we obtain the global H^s-estimate for the solution of Brinkman Flow equation. This will be a consequence of global-well posedness of the regularized problem.

Theorem 3.3.1. (Global Solution). *Let $s > \frac{n}{2}+1$, $P(\rho) = \rho^{2k}$, $F \equiv 0$ and $\rho_0 \in H^s(\mathbb{R}^n)$ with $0 \leq \rho_0(x) \leq 1$ in $\mathbb{R}^n$. Then (3.82) is globally well-posed in the sense described in Chapter 1 and satisfies $0 \leq \rho(x,t) \leq 1$, $\forall t \geq 0$.*

Proof. From the Comparison Principle follows that $0 \leq \rho(x,t) \leq 1$. Using the regularized initial value problem, with the simplified notations $\rho_\mu(t) \equiv \tilde{\rho}$; $\vec{\mathbf{v}_\mu}(t) \equiv \vec{\mathbf{v}}$.

$$\begin{cases} \partial_t \tilde{\rho} - \mu\Delta\tilde{\rho} + \operatorname{div}[\tilde{\rho}\,\vec{\mathbf{v}}] = 0 \\ \vec{\mathbf{v}} - \Delta\vec{\mathbf{v}} = -\nabla\tilde{\rho}^{2k} \\ (\tilde{\rho}(0), \vec{\mathbf{v}}(0)) = (\tilde{\rho}_0, \vec{\mathbf{v}}_0) \end{cases} \tag{3.96}$$

We have that $\vec{\mathbf{v}} = -(1-\Delta)^{-1}\nabla\tilde{\rho}^{2k} = -J^{-2}\nabla\tilde{\rho}^{2k} = -\vec{\Theta}(\tilde{\rho})$

Applying J^s to regularized equation:

$$\frac{d}{dt}(J^s\tilde{\rho}) - \mu(J^s\Delta\tilde{\rho}) + J^s\operatorname{div}(\tilde{\rho}\,\vec{\mathbf{v}}) = 0 \tag{3.97}$$

Multiplying (3.97) by $J^s\tilde{\rho}$ and integrate over $\mathbb{R}^n$

$$\frac{1}{2}\frac{d}{dt}\int (J^s\tilde{\rho})^2\,dx = \mu\int (J^s\tilde{\rho})J^s(\Delta\tilde{\rho})\,dx - \int (J^s\tilde{\rho})(J^s\operatorname{div}(\tilde{\rho}\,\vec{\mathbf{v}}))\,dx \tag{3.98}$$

$$\frac{1}{2}\frac{d}{dt}\int (J^s\tilde{\rho})^2\,dx = \underbrace{\mu\int (J^s\tilde{\rho})\Delta(J^s\tilde{\rho})\,dx}_{\leq 0} - \sum_{i=1}^{n}\int (J^s\tilde{\rho})\partial_{x_i}J^s(\tilde{\rho}\,v_i)\,dx \tag{3.99}$$

Using the commutator $[\partial_{x_i}J^s, v_i]\tilde{\rho} = \partial_{x_i}J^s(\tilde{\rho}\,v_i) - v_i\partial_{x_i}J^s\tilde{\rho}$, we obtain:

$$\frac{1}{2}\frac{d}{dt}\int (J^s\tilde{\rho})^2\,dx \leq -\sum_{i=1}^{n}\int (J^s\tilde{\rho})[\partial_{x_i}J^s, v_i]\tilde{\rho}\,dx - \sum_{i=1}^{n}\int (J^s\tilde{\rho})v_i\partial_{x_i}J^s\tilde{\rho}\,dx \tag{3.100}$$

Integrating by parts in (3.100)

$$\frac{1}{2}\frac{d}{dt}\int (J^s\tilde{\rho})^2\,dx \leq -\sum_{i=1}^{n}\int (J^s\tilde{\rho})[\partial_{x_i}J^s, v_i]\tilde{\rho}\,dx + \frac{1}{2}\int (J^s\tilde{\rho})^2\operatorname{div}\vec{\mathbf{v}}\,dx \tag{3.101}$$

Using (3.90) in (3.101)

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\textstyle\int (J^s\tilde{\rho})^2\,dx \leq -\sum_{i=1}^{n}\int (J^s\tilde{\rho})[\partial_{x_i}J^s, v_i]\tilde{\rho}\,dx + \tfrac{1}{2}\int (J^s\tilde{\rho})^2\tilde{\rho}^{2k}\,dx \\ &-\tfrac{1}{2}\textstyle\int (J^s\tilde{\rho})^2(1-\Delta)^{-1}\tilde{\rho}^{2k}\,dx \end{aligned} \tag{3.102}$$

From the second equation in (3.96) we have $v_i = -\partial_{x_i}(1-\Delta)^{-1}\tilde{\rho}^{2k}$. Substituting it in (3.102)

$$\begin{aligned} \tfrac{d}{dt}\textstyle\int (J^s\tilde{\rho})^2\,dx &\le \textstyle\int (J^s\tilde{\rho})^2\tilde{\rho}^{2k}\,dx - \overbrace{\int (J^s\tilde{\rho})^2(1-\Delta)^{-1}\tilde{\rho}^{2k}\,dx}^{\ge 0} \\ &+2\textstyle\int (J^s\tilde{\rho})\big(\sum_{i=1}^{n}[\partial_{x_i}J^s, \partial_{x_i}(1-\Delta)^{-1}\tilde{\rho}^{2k}]\tilde{\rho}\big)\,dx \end{aligned} \tag{3.103}$$

Observing that the third term it is non negative, and applying Cauchy Schwartz inequality in the fourth term; we observe:

$$\begin{aligned} \frac{d}{dt}\int (J^s\tilde{\rho})^2\,dx &\le \|\tilde{\rho}^{2k}\|_{L^\infty}\int (J^s\tilde{\rho})^2\,dx \\ &+\ 2\,\|J^s\tilde{\rho}\|\Big\|\sum_{i=1}^{n}\big[\partial_{x_i}J^s, \partial_{x_i}(1-\Delta)^{-1}\tilde{\rho}^{2k}\big]\tilde{\rho}\Big\| \end{aligned} \tag{3.104}$$

Using Lemma A.1.3 of Appendix, in (3.104), with $f = (1-\Delta)^{-1}\tilde{\rho}^{2k}$ and $g = \tilde{\rho}$, we obtain:

$$\frac{d}{dt}\|\tilde{\rho}\|_s^2 \le \|\tilde{\rho}^{2k}\|_{L^\infty}\|\tilde{\rho}\|_s^2 + 2c\|\tilde{\rho}\|_s\Big[\|\tilde{\rho}^{2k}\|_{L^\infty}\|\tilde{\rho}\|_s + \|\tilde{\rho}^{2k}\|_s\|\tilde{\rho}\|_{L^\infty}\Big] \tag{3.105}$$

Applying Corollary A.1.1 in (3.105):

$$\frac{d}{dt}\|\tilde{\rho}\|_s^2 \lesssim \|\tilde{\rho}\|_{L^\infty}^{2k}\|\tilde{\rho}\|_s^2 \tag{3.106}$$

In the following we need calculate $\|\tilde{\rho}\|_{L^\infty}$. Applying the Comparison Principle for ρ and Sobolev Lemma we have

$$\|\tilde{\rho}\|_{L^\infty} \le \|\tilde{\rho}-\rho\|_{L^\infty} + \|\rho\|_{L^\infty} \lesssim 1 + \|\tilde{\rho}-\rho\|_s \tag{3.107}$$

Calculating $\|\tilde{\rho}-\rho\|_s$, with $\|\tilde{\rho}-\rho\|_s = \sup_{\|\varphi\|_s=1}|\langle\tilde{\rho}-\rho,\varphi\rangle_s|$
In the analysis of weak convergence of sequence ρ_μ we saw

$$\begin{aligned} |\langle\rho_\mu(t)-\rho_\nu(t),\varphi\rangle_s| &\le \|\rho_\mu(t)-\rho_\nu(t)\|_s\|\varphi-\varphi_\epsilon\|_s + \|\rho_\mu(t)-\rho_\nu(t)\|\|\varphi_\epsilon\|_{2s} \\ &\le 2M\epsilon + \|\rho_\mu(t)-\rho_\nu(t)\|\|\varphi_\epsilon\|_{2s} \end{aligned} \tag{3.108}$$

Applying limit as $\nu \to 0$ in (3.108)

$$|\langle \rho_\mu(t) - \rho(t), \varphi \rangle_s| \leq 2M\epsilon + \|\rho_\mu(t) - \rho(t)\| \|\varphi_\epsilon\|_{2s} \tag{3.109}$$

Considering that $\|\rho_\mu(t) - \rho_\nu(t)\| \leq 2M\sqrt{n\tilde{T}_s|\mu - \nu|}e^{\tilde{T}_s L_0(M,M)}$ and applying limit as $\nu \to 0$

$$\|\rho_\mu(t) - \rho(t)\| \leq 2M\sqrt{n\,\tilde{T}_s\,\mu}\,e^{\tilde{T}_s L_0(M,M)} = \widetilde{C}(n, M, \tilde{T}_s)\sqrt{\mu} \tag{3.110}$$

Substituting (3.110) in (3.109) and considering that $\|\varphi_\epsilon\|_{2s} \leq \epsilon^{-s}\|\varphi\|_s$ with φ_ϵ constructed as in [21, Lemma 2.6, pg 900], yields

$$|\langle \rho_\mu(t) - \rho(t), \varphi \rangle_s| \leq 2M\epsilon + \widetilde{C}(n, M, \tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s}\|\varphi\|_s \tag{3.111}$$

Then

$$\|\tilde{\rho} - \rho\|_s = \sup_{\|\varphi\|_s = 1} |\langle \tilde{\rho} - \rho, \varphi \rangle_s| \leq 2M\epsilon + \widetilde{C}(n, M, \tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s} \tag{3.112}$$

and

$$\|\tilde{\rho}\|_{L^\infty} \lesssim 1 + 2M\epsilon + \widetilde{C}(n, M, \tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s}, \forall \epsilon > 0 \tag{3.113}$$

Let $r(\tau) = \tau^{2k}$ a non-decreasing function, it follows that:

$$\frac{d}{dt}\|\tilde{\rho}\|_s^2 \lesssim r(1 + 2M\epsilon + \widetilde{C}(n, M, \tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s})\|\tilde{\rho}\|_s^2 \tag{3.114}$$

Integrating from 0 to t in (3.114)

$$\|\tilde{\rho}\|_s^2 \lesssim \|\rho_0\|_s^2 + r(1 + 2M\epsilon + \widetilde{C}(n, M, \tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s}) \int_0^t \|\tilde{\rho}(\tau)\|_s^2\, d\tau \tag{3.115}$$

From Gronwalls inequality in (3.115), follows a priori-estimate in $H^s(\mathbb{R}^n)$; $s > \frac{n}{2} + 1$

$$\|\tilde{\rho}\|_s^2 \lesssim \|\rho_0\|_s^2\; e^{r\left(1 + 2M\epsilon + \widetilde{C}(n,M,\tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s}\right)\tilde{T}_s},\;\forall \tilde{T}_s > 0, \forall \epsilon > 0 \tag{3.116}$$

Finally, applying [49, Theo. 1, pg 120] in (3.116) we obtain the final estimate

$$\begin{aligned} \|\rho(t)\|_s^2 &\leq \liminf_{\mu\to 0} \|\rho_\mu(t)\|_s^2 \\ &\leq \liminf_{\mu\to 0} \|\rho_0\|_s^2 \; e^{r\left(1+2M\epsilon+\widetilde{C}(n,M,\tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s}\right)\tilde{T}_s} \\ &= \lim_{\mu\to 0} \|\rho_0\|_s^2 \; e^{r\left(1+2M\epsilon+\widetilde{C}(n,M,\tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s}\right)\tilde{T}_s} \\ &= \|\rho_0\|_s^2 \; e^{r(1+2M\epsilon)\tilde{T}_s} \, \forall \epsilon > 0 \end{aligned} \tag{3.117}$$

Therefore,applying limit as ϵ tends to zero, follows the final estimate

$$\|\rho(t)\|_s^2 \leq \|\rho_0\|_s^2 \; e^{\tilde{T}_s}, \;\; \forall t \in [0, \tilde{T}_s] \tag{3.118}$$

□

Chapter 4

BFE with Bore-Like Initial Conditions

4.1 Local Theory with Parabolic Regularization

4.1.1 Auxiliar problem for the BFE

In this section we will describe how to deal with certain types of initial data that do not belong to a Sobolev Space. More precisely, we will consider the Cauchy problem for the Brinkman Flow equation with bore-like data and $P(\rho) = \rho^2$, that is:

$$\begin{cases} \partial_t \rho = \partial_x\big(\rho(1-\Delta)^{-1}\partial_x \rho^2\big), x \in \mathbb{R},\ t \in I = (0, T_0] \\ \rho(x,0) = \rho_0(x) \end{cases} \tag{4.1}$$

where $\rho_0 : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the following conditions:

(BL1) $\rho_0(x) \longrightarrow C_\pm$ as $x \to \pm$ where $C_- > C_+ \geq 0$.

(BL2) $\rho_0'(x) \in H^s(\mathbb{R})$, for some $r \geq 0$

(BL3) $\rho_0(x) - C_- \in L^2((-\infty, 0])$ and $\rho_0(x) - C_+ \in L^2([0, +\infty))$

Note that a function ρ_0 with these conditions is necessarily bounded. The real reason for introducing conditions (BL1)-(BL3) is that they model certain

travelling waves that occur in nature. For more information on bores we refer the reader to [6],[19],[20] and references therein.
The problem with bores is that $\|\rho_0\| = \infty$, and Sobolev Space methods, in principle, cannot be applied. Consider the following lemma.

Lemma 4.1.1. *Let ρ_0 satisfy conditions (BL1) and (BL2). Then for each $\tau \in (0,\infty)$ there exists a $\psi_\tau \in C^\infty(\mathbb{R})$ such that $\psi'_\tau \in H^\infty(\mathbb{R})$ and $\phi_\tau = \rho_0 - \psi_\tau \in H^{s+1}(\mathbb{R})$, $s \geq 0$. Moreover*

$$\|\rho_0 - \psi_\tau\| \leq \left(\frac{2\tau}{e}\right)^{\frac{1}{2}} \|\rho_0'\| \qquad \|\psi'_\tau\|_s \leq C(s,\tau)\|\rho_0'\| \tag{4.2}$$

Finally, if ρ_0 satisfies (BL3) then ψ_τ also has this property and $\lim_{x\to\pm\infty} \psi_\tau = C_\pm$.

Proof. See [19],[20],[21]. □

Let τ be fixed, and write $\psi_\tau = \psi$ for simplicity. It is then natural to define $\rho(x,t) = w(x,t) + \psi(x)$, and study the auxiliary initial value problem associated to $w(x,t)$, namely:

$$\begin{cases} \partial_t w = \partial_x\big(wJ^{-2}\partial_x w^2\big) + E(w,\psi) \in H^{s-1}(\mathbb{R}), \quad x \in \mathbb{R},\ t \in I = (0,T_0] \\ w(x,0) = \rho_0(x) - \psi(x) = \phi(x) \in H^s(\mathbb{R});\ s \geq 1 \end{cases} \tag{4.3}$$

where

$$\begin{aligned} E(w,\psi) = {} & \partial_x\big(\psi J^{-2}\partial_x\psi^2\big) + \partial_x\big(wJ^{-2}\partial_x\psi^2\big) + \partial_x\big(\psi J^{-2}\partial_x w^2\big) \\ & + 2\,\partial_x\big(wJ^{-2}\partial_x(w\psi)\big) + 2\,\partial_x\big(\psi J^{-2}\partial_x(w\psi)\big) \end{aligned} \tag{4.4}$$

This is the problem that we will study. Observe that the PDE in (4.3), is a perturbation of (BFE) with five extra terms. We will employ parabolic regularization to show that (4.3) is locally well posed if $s > \frac{3}{2}$.

4.1.2 Parabolic Regularization for the auxiliary problem

In order to solve (4.3), locally in time, we will introduce an artificial viscosity μ, solve the regularized problem and then take limits as the viscosity tends to zero. A little more precisely, we consider Cauchy problem.

$$\begin{cases} \partial_t w_\mu = \mu\partial_x^2 w_\mu + \tilde{E}(w_\mu,\psi) \in H^{s-2}(\mathbb{R}), \quad x \in \mathbb{R},\ t \in I = (0,T_0] \\ w_\mu(x,0) = \phi(x) \in H^s(\mathbb{R});\ s \geq 1 \end{cases} \tag{4.5}$$

where

$$\tilde{E}(w_\mu, \psi) = \partial_x\big(w_\mu J^{-2}\partial_x w_\mu^2\big) + E(w_\mu, \psi) \tag{4.6}$$

Using the method of variation of parameters we can show that (4.5) is equivalent to the integral equation:

$$w_\mu(t) = U_\mu(t)\phi + \int_0^t U_\mu(t-t')\tilde{E}(w_\mu(t'), \psi)\, dt' \tag{4.7}$$

This equivalence is in fact rigorous in view of lemma 3.1.2 which asserts that $t \longrightarrow U_\mu(t) = e^{\mu t \partial_x^2}$, the semigroup generated by $\mu\partial_x^2$, is infinitely smoothing. Its necessary to prove a result analogous to Lemma 3.1.1 in order to show the properties of the nonlinear part $\tilde{E}(w, \psi)$

Lemma 4.1.2. *Let $s > \frac{3}{2}$ be fixed. Then $\tilde{E}(w, \psi)$ is a continuous map from H^s to H^{s-1} and satisfies the estimates*

$$\|\tilde{E}(w,\psi) - \tilde{E}(\tilde{w},\psi)\|_{s-1} \le \beta(\|w\|_s, \|\tilde{w}\|_s, \psi)\|w - \tilde{w}\|_s \tag{4.8}$$

$$\Big\langle w - \tilde{w}, \tilde{E}(w,\psi) - \tilde{E}(\tilde{w},\psi)\Big\rangle \le Q_0(\|w\|_s, \|\tilde{w}\|_s, \psi)\|w - \tilde{w}\|^2 \tag{4.9}$$

for all $w, \tilde{w} \in H^s$, where $\beta, Q_0 : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are continuous functions, monotone nondecreasing with respect to each of their arguments.

Proof. We have

$$\begin{aligned}\|\tilde{E}(w,\psi) - \tilde{E}(\tilde{w},\psi)\|_{s-1} \le{}& \underbrace{\|\partial_x\big(wJ^{-2}\partial_x w^2\big) - \partial_x\big(\tilde{w}J^{-2}\partial_x\tilde{w}^2\big)\|_{s-1}}_{=A} \\ &+ \underbrace{\|\partial_x\big((w-\tilde{w})J^{-2}\partial_x\psi^2\big)\|_{s-1}}_{=B} + \underbrace{\|\partial_x\big(\psi J^{-2}\partial_x(w^2 - \tilde{w}^2)\big)\|_{s-1}}_{=C} \\ &+2\underbrace{\|\partial_x\big(\psi J^{-2}\partial_x(\psi(w-\tilde{w}))\big)\|_{s-1}}_{=D} + 2\underbrace{\|\partial_x\big(wJ^{-2}\partial_x(w\psi)\big) - \partial_x\big(\tilde{w}J^{-2}\partial_x(\tilde{w}\psi)\big)\|_{s-1}}_{=E}\end{aligned} \tag{4.10}$$

The first term (A), is exactly the nonlinear part of the (BFE), and that estimate was made in the Chapter 3, in n-dimensional case, that is

$$A \lesssim (\|w\|_s + \|\tilde{w}\|_s)^2\|w - \tilde{w}\|_s \tag{4.11}$$

Since $H^s, s > \frac{3}{2} > \frac{1}{2}$ is a Banach algebra, using the Cauchy-Schwartz inequality , $\|J^{-2}\|_{B(H^s,H^{s+2})} \leq 1, \|J^{-2}\partial_x\|_{B(H^s,H^{s+1})} \leq 1$, Lemma A.1.7 and Corollary A.1.2 in Appendix, we have

$$\begin{aligned} B &= 2\,\|\partial_x\big((w-\tilde{w})J^{-2}(\psi\psi')\big)\|_{s-1} \lesssim \|w-\tilde{w}\|_s\|J^{-2}(\psi\psi')\|_s \\ &\lesssim \|w-\tilde{w}\|_s\|J^{-2}(\psi\psi')\|_{s+2} \lesssim \|w-\tilde{w}\|_s\|J^{-2}\|_{B(H^s,H^{s+2})}\|\psi\psi'\|_s \\ &\lesssim (\|\psi\|_{L^\infty} + \|\psi'\|_s)^2\|w-\tilde{w}\|_s \end{aligned} \tag{4.12}$$

$$\begin{aligned} C &\leq \|\psi J^{-2}\partial_x(w^2-\tilde{w}^2)\|_s \lesssim \|J^{-2}\partial_x(w^2-\tilde{w}^2)\|_s(\|\psi\|_{L^\infty} + \|\psi'\|_s) \\ &\lesssim \|J^{-2}\partial_x\|_{B(H^s,H^{s+1})}\|w^2-\tilde{w}^2\|_s(\|\psi\|_{L^\infty} + \|\psi'\|_s) \\ &\lesssim (\|w\|_s + \|\tilde{w}\|_s)(\|\psi\|_{L^\infty} + \|\psi'\|_s)\|w-\tilde{w}\|_s \end{aligned} \tag{4.13}$$

$$\begin{aligned} D &\leq \|\psi J^{-2}\partial_x(\psi(w-\tilde{w}))\|_s \lesssim \|J^{-2}\partial_x(\psi(w-\tilde{w}))\|_s(\|\psi\|_{L^\infty} + \|\psi'\|_s) \\ &\lesssim \|J^{-2}\partial_x\|_{B(H^s,H^{s+1})}\|\psi(w-\tilde{w})\|_s(\|\psi\|_{L^\infty} + \|\psi'\|_s) \\ &\lesssim (\|\psi\|_{L^\infty} + \|\psi'\|_s)^2\|w-\tilde{w}\|_s \end{aligned} \tag{4.14}$$

$$\begin{aligned} E &\leq \|\partial_x\big(wJ^{-2}\partial_x(\psi(w-\tilde{w}))\big)\|_{s-1} + \|\partial_x\big((w-\tilde{w})J^{-2}\partial_x(\psi\tilde{w})\big)\|_{s-1} \\ &\leq \|wJ^{-2}\partial_x(\psi(w-\tilde{w}))\|_s + \|(w-\tilde{w})J^{-2}\partial_x(\psi\tilde{w})\|_s \\ &\lesssim \|w\|_s\|J^{-2}\partial_x(\psi(w-\tilde{w}))\|_s + \|w-\tilde{w}\|_s\|J^{-2}\partial_x(\psi\tilde{w})\|_s \\ &\lesssim \|w\|_s\|J^{-2}\partial_x\|_{B(H^s,H^{s+1})}\|\psi(w-\tilde{w})\|_s + \|w-\tilde{w}\|_s\|J^{-2}\partial_x\|_{B(H^s,H^{s+1})}\|\psi\tilde{w}\|_s \\ &\lesssim \|w\|_s\|w-\tilde{w}\|_s(\|\psi\|_{L^\infty} + \|\psi'\|_s) + \|w-\tilde{w}\|_s\|\tilde{w}\|_s(\|\psi\|_{L^\infty} + \|\psi'\|_s) \\ &\lesssim (\|w\|_s + \|\tilde{w}\|_s)(\|\psi\|_{L^\infty} + \|\psi'\|_s)\|w-\tilde{w}\|_s \end{aligned} \tag{4.15}$$

Taking

$$\tilde{C}(s,\psi) = \max\{(\|\psi\|_{L^\infty} + \|\psi'\|_s)^2, (\|\psi\|_{L^\infty} + \|\psi'\|_s)\} \tag{4.16}$$

and substituting (4.11) to (4.15) in (4.10)

$$\|\tilde{E}(w,\psi) - \tilde{E}(w,\psi)\|_{s-1} \leq \beta(\|w\|_s, \|\tilde{w}\|_s, \psi)\|w-\tilde{w}\|_s \tag{4.17}$$

where

$$\beta(\|w\|_s, \|\tilde{w}\|_s, \psi) = (\|w\|_s + \|\tilde{w}\|_s)^2 + 3\,\tilde{C}(s,\psi)(\|w\|_s + \|\tilde{w}\|_s) + 3\,\tilde{C}(s,\psi) \tag{4.18}$$

Finally, we will estimate (4.9)

$$\begin{aligned}
&\Big\langle w-\tilde{w}, \tilde{E}(w,\psi)-\tilde{E}(\tilde{w},\psi)\Big\rangle \\
&\qquad = \underbrace{\Big\langle w-\tilde{w}, \partial_x\big(wJ^{-2}\partial_x w^2\big)-\partial_x\big(\tilde{w}J^{-2}\partial_x\tilde{w}^2\big)\Big\rangle}_{=I} \\
&\quad + \underbrace{\Big\langle w-\tilde{w}, \partial_x\big((w-\tilde{w})J^{-2}\partial_x\psi^2\big)\Big\rangle}_{=II} + \underbrace{\Big\langle w-\tilde{w}, \partial_x\big(\psi J^{-2}\partial_x(w^2-\tilde{w}^2)\big)\Big\rangle}_{=III} \\
&\qquad + 2\underbrace{\Big\langle w-\tilde{w}, \partial_x\big(wJ^{-2}\partial_x(w\psi)\big)-\partial_x\big(\tilde{w}J^{-2}\partial_x(\tilde{w}\psi)\big)\Big\rangle}_{=IV} \\
&\qquad\qquad + 2\underbrace{\Big\langle w-\tilde{w}, \partial_x\big(\psi J^{-2}\partial_x(\psi(w-\tilde{w})\big)\Big\rangle}_{=V} \qquad (4.19)
\end{aligned}$$

The first term in (4.19) was estimated in Chapter 3, by

$$I \lesssim (\|w\|_s+\|\tilde{w}\|_s)^2\|w-\tilde{w}\|^2 \qquad (4.20)$$

Integration by parts, Cauchy-Schwartz inequality, Sobolev Lemma, Lemma A.1.7, Corollary A.1.2 in Appendix ,$\|J^{-2}\partial_x\|_{B(H^s,H^{s+1})} \leq 1$ and $\|J^{-2}\partial_x^2\|_{B(H^s)} \leq 1$ leads the other following estimates in (4.19)

$$\begin{aligned}
II &= 2\Big\langle w-\tilde{w}, \partial_x\big((w-\tilde{w})J^{-2}(\psi\psi')\big)\Big\rangle \qquad (4.21)\\
&= -2\Big\langle (w-\tilde{w})J^{-2}(\psi\psi'), \partial_x(w-\tilde{w})\Big\rangle \\
&= -\Big\langle \partial_x(w-\tilde{w})^2, J^{-2}(\psi\psi')\Big\rangle = \Big\langle (w-\tilde{w})^2, J^{-2}\partial_x(\psi\psi')\Big\rangle \\
&\leq \|w-\tilde{w}\|^2\|J^{-2}\partial_x(\psi\psi')\|_{L^\infty} \lesssim \|w-\tilde{w}\|^2\|J^{-2}\partial_x(\psi\psi')\|_{s+1} \\
&\lesssim \|w-\tilde{w}\|^2\|J^{-2}\partial_x\|_{B(H^s,H^{s+1})}\|\psi\psi'\|_s \lesssim (\|\psi\|_{L^\infty}+\|\psi'\|_s)^2\|w-\tilde{w}\|^2
\end{aligned}$$

$$\begin{aligned}
III &\leq \|w-\tilde{w}\|\|\partial_x\big(\psi J^{-2}\partial_x(w^2-\tilde{w}^2)\big)\| \qquad (4.22)\\
&\leq \|w-\tilde{w}\|\|\psi J^{-2}\partial_x(w^2-\tilde{w}^2)\|_1 \\
&\lesssim \|w-\tilde{w}\|\|J^{-2}\partial_x(w^2-\tilde{w}^2)\|_1(\|\psi\|_{L^\infty}+\|\psi'\|_1) \\
&\lesssim \|w-\tilde{w}\|\|J^{-2}\partial_x\|_{B(L^2,H^1)}\|(w-\tilde{w})(w+\tilde{w})\|(\|\psi\|_{L^\infty}+\|\psi'\|_1) \\
&\lesssim \|w-\tilde{w}\|^2\|w+\tilde{w}\|_{L^\infty}(\|\psi\|_{L^\infty}+\|\psi'\|_1) \\
&\lesssim (\|w\|_s+\|\tilde{w}\|_s)(\|\psi\|_{L^\infty}+\|\psi'\|_s)\|w-\tilde{w}\|^2
\end{aligned}$$

$$
\begin{aligned}
IV &= \Big\langle w-\tilde{w}, \partial_x\big(wJ^{-2}\partial_x((w-\tilde{w})\psi)\big)\Big\rangle \\
&\quad + \Big\langle w-\tilde{w}, \partial_x\big((w-\tilde{w})J^{-2}\partial_x(\tilde{w}\psi)\big)\Big\rangle \\
&= \Big\langle w-\tilde{w}, \partial_x\big(wJ^{-2}\partial_x((w-\tilde{w})\psi)\big)\Big\rangle + \tfrac{1}{2}\Big\langle (w-\tilde{w})^2, J^{-2}\partial_x^2(\tilde{w}\psi)\Big\rangle \\
&\le \|w-\tilde{w}\|\|\partial_x\big(wJ^{-2}\partial_x((w-\tilde{w})\psi)\big)\| + \tfrac{1}{2}\|J^{-2}\partial_x^2(\tilde{w}\psi)\|_{L^\infty}\|w-\tilde{w}\|^2 \\
&\lesssim \|w-\tilde{w}\|\|wJ^{-2}\partial_x((w-\tilde{w})\psi)\|_1 + \tfrac{1}{2}\|J^{-2}\partial_x^2(\tilde{w}\psi)\|_s\|w-\tilde{w}\|^2 \\
&\lesssim \|w-\tilde{w}\|\|w\|_1\|J^{-2}\partial_x((w-\tilde{w})\psi)\|_1 + \tfrac{1}{2}\|J^{-2}\partial_x^2\|_{B(H^s)}\|\tilde{w}\psi\|_s\|w-\tilde{w}\|^2 \\
&\lesssim \|w-\tilde{w}\|^2\|w\|_s\|J^{-2}\partial_x\|_{B(L^2,H^1)}\|\psi\|_{L^\infty} + \tfrac{1}{2}\|\tilde{w}\|_s(\|\psi\|_{L^\infty}+\|\psi'\|_s)\|w-\tilde{w}\|^2 \\
&\lesssim (\|w\|_s + \tfrac{1}{2}\|\tilde{w}\|_s)(\|\psi\|_{L^\infty}+\|\psi'\|_s)\|w-\tilde{w}\|^2
\end{aligned}
\tag{4.23}
$$

$$
\begin{aligned}
V &\le \|w-\tilde{w}\|\|\partial_x\big(\psi J^{-2}\partial_x(\psi(w-\tilde{w})\big)\| \\
&\le \|w-\tilde{w}\|\|\psi J^{-2}\partial_x(\psi(w-\tilde{w})\|_1 \\
&\lesssim \|w-\tilde{w}\|(\|\psi\|_{L^\infty}+\|\psi'\|_1)\|J^{-2}\partial_x(\psi(w-\tilde{w}))\|_1 \\
&\lesssim \|w-\tilde{w}\|(\|\psi\|_{L^\infty}+\|\psi'\|_1)\|J^{-2}\partial_x\|_{B(L^2,H^1)}\|\psi(w-\tilde{w})\| \\
&\lesssim \|w-\tilde{w}\|^2(\|\psi\|_{L^\infty}+\|\psi'\|_1)\|\psi\|_{L^\infty} \\
&\lesssim (\|\psi\|_{L^\infty}+\|\psi'\|_s)^2\|w-\tilde{w}\|^2
\end{aligned}
\tag{4.24}
$$

Then, substituting (4.20)-(4.24) in (4.19), follows the result, i.e.,

$$
\Big\langle w-\tilde{w}, \tilde{E}(w,\psi)-\tilde{E}(\tilde{w},\psi)\Big\rangle \le Q_0(\|w\|_s, \|\tilde{w}\|_s, \psi)\|w-\tilde{w}\|^2 \tag{4.25}
$$

where

$$
Q_0(\|w\|_s, \|\tilde{w}\|_s, \psi) = (\|w\|_s+\|\tilde{w}\|_s)^2 + 3\,\tilde{C}(s,\psi)(\|w\|_s+\|\tilde{w}\|_s) + 3\,\tilde{C}(s,\psi) \tag{4.26}
$$

This finishes the proof. □

Now, introducing the complete metric space

$$
X_s(T_0) = \big\{v \in C([0,T_0], H^s(\mathbb{R})) : \big\|v(t) - U_\mu(t)\phi\big\| \le M,\ \forall t \in [0,T_0]\big\} \tag{4.27}
$$

and the mapping

$$
B(v(t)) = U_\mu(t)\phi + \int_0^t U_\mu(t-t')\tilde{E}(v(t'),\psi)\,dt' \tag{4.28}
$$

and combining these definitions with lemmas 3.1.2 and 4.1.2 a), it is not difficult to prove the following Theorem.

Theorem 4.1.1. *Let $\mu > 0$ be fixed and $\phi \in H^s(\mathbb{R})$, $s \geq 1$. Then there exists $T^\mu = T(s, \|\phi\|_s, \mu) > 0$ and a unique solution w_μ of IVP (4.5) satisfying*

$$w_\mu \in C([0, T^\mu], H^s(\mathbb{R})) \cap C((0, T^\mu], H^\infty(\mathbb{R}))$$

Proof. It is similar to the proof of theorem 3.1.2 in section 3.1.1 □

The next step is to study the limit $w = \lim_{\mu \to 0} w_\mu$. First we must show that w_μ can be extended to an interval of time independent of μ.

Lemma 4.1.3. *Let $s > \frac{3}{2}$ and $w_\mu \in C([0, T^\mu], H^s(\mathbb{R}))$ be the solution of the IVP (4.5) with $\mu > 0$. Then $w_\mu(t)$ can be extended to an interval $[0, \tilde{T}_s]$ where $\tilde{T}_s = \tilde{T}(s, \|\phi\|_s)$ is independent of μ. Moreover, there exists $h(t) \in C([0, \tilde{T}_s], \mathbb{R})$ such that:*

$$\|w_\mu(t)\|_s^2 \leq h(t) \qquad h(0) = \|\phi\|_s^2 \tag{4.29}$$

Proof. Since $w_\mu(t) \in H^\infty(\mathbb{R})$, $\forall t \in (0, T^\mu]$, we may safely differentiate $\|w_\mu(t)\|_s^2$ with respect to t to get

$$\begin{aligned} \partial_t \|w_\mu(t)\|_s^2 &= 2 \left\langle w_\mu(t), \partial_t w_\mu(t) \right\rangle_s \\ &= 2 \left\langle w_\mu(t), \mu \partial_x^2 w_\mu(t) \right\rangle_s + 2 \left\langle w_\mu(t), \tilde{E}(w_\mu(t), \psi) \right\rangle_s \end{aligned} \tag{4.30}$$

Since $H_0 = -\partial_x^2$ is a self-adjoint and positive operator in H^s we have

$$\left\langle w_\mu(t), \mu \partial_x^2 w_\mu(t) \right\rangle_s = -\mu \left\langle w_\mu(t), H_0 w_\mu(t) \right\rangle_s \leq 0 \tag{4.31}$$

Expanding the other term

$$\begin{aligned} \left\langle w_\mu, \tilde{E}(w_\mu, \psi) \right\rangle_s &= \left\langle w_\mu, \partial_x \big(w_\mu J^{-2} \partial_x w_\mu^2 \big) \right\rangle_s + \left\langle w_\mu, \partial_x \big(\psi J^{-2} \partial_x \psi^2 \big) \right\rangle_s \\ &+ \left\langle w_\mu, \partial_x \big(w_\mu J^{-2} \partial_x \psi^2 \big) \right\rangle_s + \left\langle w_\mu, \partial_x \big(\psi J^{-2} \partial_x w_\mu^2 \big) \right\rangle_s \\ &+ \left\langle w_\mu, \partial_x \big(w_\mu J^{-2} \partial_x (w_\mu \psi) \big) \right\rangle_s + \left\langle w_\mu, \partial_x \big(\psi J^{-2} \partial_x (w_\mu \psi) \big) \right\rangle_s \end{aligned} \tag{4.32}$$

We now estimate each term in (4.32):

The first term in $\tilde{E}(w_\mu, \psi)$ is the similar to the case treated in chapter 3, where we know that:

$$\Big\langle w_\mu(t), \partial_x\big(w_\mu J^{-2}\partial_x w_\mu^2\big)\Big\rangle_s \lesssim \|w_\mu\|_s^4 \tag{4.33}$$

Using Cauchy Schwartz inequality, Lemmas A.1.5, A.1.7, any operators estimates and Corollary A.1.2 we will estimate the others terms

$$\begin{aligned}
&\Big\langle w_\mu, \partial_x\big(\psi J^{-2}\partial_x\psi^2\big)\Big\rangle_s \leq 2\Big|\Big\langle w_\mu, \partial_x\big(\psi J^{-2}(\psi\psi')\big)\Big\rangle_s\Big| \\
&\leq 2\|w_\mu\|_s\|\partial_x\big(\psi J^{-2}(\psi\psi')\big)\|_s \leq 2\|w_\mu\|_s\|\psi J^{-2}(\psi\psi')\|_{s+1} \\
&\lesssim \|w_\mu\|_s\|J^{-2}(\psi\psi')\|_{s+1}(\|\psi\|_{L^\infty} + \|\psi'\|_{s+1}) \\
&\lesssim \|w_\mu\|_s\|J^{-2}\|_{B(H^s,H^{s+2})}\|\psi\psi'\|_s(\|\psi\|_{L^\infty} + \|\psi'\|_{s+1}) \\
&\lesssim \|w_\mu\|_s(\|\psi\|_{L^\infty} + \|\psi'\|_s)^2(\|\psi\|_{L^\infty} + \|\psi'\|_{s+1}) \\
&\lesssim (\|\psi\|_{L^\infty} + \|\psi'\|_{s+1})^3\|w_\mu\|_s
\end{aligned} \tag{4.34}$$

$$\begin{aligned}
&\Big\langle w_\mu, \partial_x\big(w_\mu J^{-2}\partial_x\psi^2\big)\Big\rangle_s = 2\Big\langle w_\mu, \partial_x\big(w_\mu J^{-2}(\psi\psi')\big)\Big\rangle_s \\
&= 2\Big[\Big\langle w_\mu, (\partial_x w_\mu)J^{-2}(\psi\psi')\Big\rangle_s + \Big\langle w_\mu, (w_\mu J^{-2}\partial_x(\psi\psi')\Big\rangle_s\Big] \\
&\lesssim \Big|\Big\langle J^{-2}(\psi\psi')(\partial_x w_\mu), w_\mu\Big\rangle_s\Big| + \Big|\Big\langle w_\mu, w_\mu J^{-2}\partial_x(\psi\psi')\Big\rangle_s\Big| \\
&\lesssim 2c\|J^{-2}\partial_x(\psi\psi')\|_{s-1}\|w_\mu\|_s^2 + \|w_\mu\|_s\|w_\mu J^{-2}\partial_x(\psi\psi')\|_s \\
&\lesssim \|\psi\psi'\|_s\|w_\mu\|_s^2 \\
&\lesssim (\|\psi\|_{L^\infty} + \|\psi'\|_s)^2\|w_\mu\|_s^2
\end{aligned} \tag{4.35}$$

$$\begin{aligned}
&\Big\langle w_\mu, \partial_x\big(\psi J^{-2}\partial_x w_\mu^2\big)\Big\rangle_s \leq \Big|\Big\langle w_\mu, \partial_x\big(\psi J^{-2}\partial_x w_\mu^2\big)\Big\rangle_s\Big| \\
&\leq \|w_\mu\|_s\|\partial_x\big(\psi J^{-2}\partial_x w_\mu^2\big)\|_s \leq \|w_\mu\|_s\|\psi J^{-2}\partial_x w_\mu^2\|_{s+1} \\
&\lesssim \|w_\mu\|_s\|J^{-2}\partial_x w_\mu^2\|_{s+1}(\|\psi\|_{L^\infty} + \|\psi'\|_{s+1}) \\
&\lesssim (\|\psi\|_{L^\infty} + \|\psi'\|_{s+1})\|w_\mu\|_s^3
\end{aligned} \tag{4.36}$$

$$\begin{aligned}
&\Big\langle w_\mu, \partial_x\big(w_\mu J^{-2}\partial_x(w_\mu\psi)\big)\Big\rangle_s \\
&= \Big\langle w_\mu, (\partial_x w_\mu)J^{-2}\partial_x(w_\mu\psi)\Big\rangle_s + \Big\langle w_\mu, w_\mu J^{-2}\partial_x^2(w_\mu\psi)\Big\rangle_s \\
&\le \Big|\Big\langle w_\mu, (\partial_x w_\mu)J^{-2}\partial_x(w_\mu\psi)\Big\rangle_s\Big| + \Big|\Big\langle w_\mu, w_\mu J^{-2}\partial_x^2(w_\mu\psi)\Big\rangle_s\Big| \\
&\lesssim \|w_\mu\psi\|_s\|w_\mu\|_s^2 + \|w_\mu\|_s\|w_\mu J^{-2}\partial_x^2(w_\mu\psi)\|_s \\
&\lesssim \|w_\mu\psi\|_s\|w_\mu\|_s^2 + \|w_\mu\|_s^2\|J^{-2}\partial_x^2(w_\mu\psi)\|_s \\
&\lesssim \|w_\mu\psi\|_s\|w_\mu\|_s^2 \\
&\lesssim (\|\psi\|_{L^\infty} + \|\psi'\|_s)\|w_\mu\|_s^3
\end{aligned} \tag{4.37}$$

$$\begin{aligned}
&\Big\langle w_\mu, \partial_x\big(\psi J^{-2}\partial_x(w_\mu\psi)\big)\Big\rangle_s \le \Big|\Big\langle w_\mu, \partial_x\big(\psi J^{-2}\partial_x(w_\mu\psi)\big)\Big\rangle_s\Big| \\
&\le \|w_\mu\|_s\|\partial_x\big(\psi J^{-2}\partial_x(w_\mu\psi)\big)\|_s \\
&\le \|w_\mu\|_s\|\psi J^{-2}\partial_x(w_\mu\psi)\|_{s+1} \\
&\lesssim \|w_\mu\|_s\|J^{-2}\partial_x(w_\mu\psi)\|_{s+1}(\|\psi\|_{L^\infty} + \|\psi'\|_{s+1}) \\
&\lesssim \|w_\mu\|_s\|w_\mu\psi\|_s(\|\psi\|_{L^\infty} + \|\psi'\|_{s+1}) \\
&\lesssim \|w_\mu\|_s^2(\|\psi\|_{L^\infty} + \|\psi'\|_{s+1})^2
\end{aligned} \tag{4.38}$$

Substituting (4.31),(4.33)-(4.38) in (4.30) we have:

$$\begin{aligned}
\partial_t\|w_\mu(t)\|_s^2 &\lesssim \widetilde{K}(s,\psi)\Big(\|w_\mu(t)\|_s^4 + \|w_\mu(t)\|_s^3 + \|w_\mu(t)\|_s^2 + \|w_\mu(t)\|_s\Big) \\
&= \widetilde{K}(s,\psi)\Big((\|w_\mu(t)\|_s^2)^2 + (\|w_\mu(t)\|_s^2)^{\frac{3}{2}} + \|w_\mu(t)\|_s^2 + (\|w_\mu(t)\|_s^2)^{\frac{1}{2}}\Big) \\
&= G(\|w_\mu(t)\|_s^2)
\end{aligned} \tag{4.39}$$

with

$$\widetilde{K}(s,\psi) = \max\{6K(s,\psi), 2\}$$

and

$$K(s,\psi) = \max\{(\|\psi\|_{L^\infty} + \|\psi'\|_{s+1})^3, \|\psi\|_{L^\infty} + \|\psi'\|_{s+1})^2, (\|\psi\|_{L^\infty} + \|\psi'\|_{s+1})\} \tag{4.40}$$

Considering $h = h(t)$ maximal solution for:

$$\begin{cases} \partial_t h(t) = G(h(t)) = \widetilde{K}\big[h^2 + h^{\frac{3}{2}} + h + h^{\frac{1}{2}}\big],\ t \in (0, T^*) \\ \quad h(0) = \|\phi\|_s^2 \end{cases} \tag{4.41}$$

Then for $\tilde{T}_s \in [0, T^*)$, we have:

$$\|w_\mu(t)\|_s^2 \le h(t),\ t \in [0, \tilde{T}_s] \tag{4.42}$$

Since $h(t)$ does not depend on μ, it follows that all solutions can be extended to some interval $[0, \tilde{T}_s]$ with $\tilde{T}_s < T^*$, where T^* is the maximal time of the solution $h(t)$. □

Now we are ready to establish the following result:

Theorem 4.1.2. *Let $\phi \in H^s(\mathbb{R}),\ s > \frac{3}{2}$. Then there exists $\tilde{T}_s = T(s, \|\phi\|_s)$ and a unique $w \in C_+([0, \tilde{T}_s], H^s(\mathbb{R})) \cap C^1_+([0, \tilde{T}_s], H^{s-1}(\mathbb{R}))$ that solves the PDE in (4.3) and satisfies*

$$\|w(t)\|_s^2 \leq h(t),\ t \in [0, \tilde{T}_s] \tag{4.43}$$

Proof. Let $\mu > 0, \nu > 0$ and w_μ, w_ν solutions obtained in Theorem 4.1.1, with the same initial condition.
In the following we will prove that $\{w_\mu\}$ is a Cauchy sequence in $L^2(\mathbb{R})$ space.

Estimating $\partial_t \|w_\mu(t) - w_\nu(t)\|^2$

$$\begin{aligned}\partial_t \|w_\mu(t) - w_\nu(t)\|^2 &= 2\Big\langle w_\mu(t) - w_\nu(t), \partial_t(w_\mu(t) - w_\nu(t))\Big\rangle \\ &= 2\Big\langle w_\mu(t) - w_\nu(t), \mu\partial_x^2 w_\mu(t) - \nu\partial_x^2 w_\nu(t)\Big\rangle \\ &\quad + 2\Big\langle w_\mu(t) - w_\nu(t), \tilde{E}(w_\mu(t), \psi) - \tilde{E}(w_\nu(t), \psi)\Big\rangle\end{aligned} \tag{4.44}$$

The first in (4.44) is similar to the case treated in chapter 3, where we know that:

$$\Big\langle w_\mu - w_\nu, \mu\partial_x^2 w_\mu - \nu\partial_x^2 w_\nu\Big\rangle \leq 2M^2|\mu - \nu| \tag{4.45}$$

and the second term was estimated in Lemma 4.1.2
Therefore substituting(4.45),(4.9), in (4.44) we have

$$\partial_t \|w_\mu(t) - w_\nu(t)\|^2 \leq 4M^2|\mu - \nu| + 2\, Q_0(M, M, \psi)\|w_\mu(t) - w_\nu(t)\|^2 \tag{4.46}$$

Integrating (4.46) from 0 to t

$$\|w_\mu(t) - w_\nu(t)\|^2 \leq 4M^2|\mu - \nu|t + \int_0^t 2\, Q_0(M, M, \psi)\|w_\mu(\tau) - w_\nu(\tau)\|^2\, d\tau \tag{4.47}$$

Applying Gronwall's Inequality in (4.47)

$$\begin{aligned}\|w_\mu(t) - w_\nu(t)\|^2 &\leq 4M^2\tilde{T}_s|\mu - \nu|\, e^{\int_0^t 2\, Q_0(M,M,\psi)\, d\tau} \\ &\leq 4M^2\tilde{T}_s|\mu - \nu|\, e^{2\, Q_0(M,M,\psi)\tilde{T}},\ t \in (0, \tilde{T}_s)\end{aligned} \tag{4.48}$$

Applying limit as $\mu \to 0, \nu \to 0$

$$\lim_{\mu\to 0,\nu\to 0} \|w_\mu(t) - w_\nu(t)\|^2 = 0, ie, \rho_\mu(t) \overbrace{\to}^{L^2} \rho_\nu(t)\, t \in [0, \tilde{T}_s] \tag{4.49}$$

Thus, $\{w_\mu(t)\}_{\mu>0}$ is a Cauchy sequence in the space $L^2(\mathbb{R}^n)$, that is complete. Therefore, there exists $w(t) \in C([0, \tilde{T}_s], L^2(\mathbb{R}^n))$ that satisfies

$$\lim_{\mu\to 0} \sup_{[0,\tilde{T}_s]} \|w_\mu(t) - w(t)\| = 0$$

Thus $t \in [0, \tilde{T}_s] \to w_\mu(t)$ is continuous and uniformly bounded in $L^2(\mathbb{R}^n)$. The remainder of the proof is similar to that of Theorem 3.1.3 □

Once this is done, the next corollary follows

Corollary 4.1.1. *Let w be the solution of IVP (4.3) given in Theorem 4.1.2. Then $\rho = w + \psi$ is the unique solution of IVP (4.1) satisfying $\rho - \psi \in C_+([0, \tilde{T}_s], H^s(\mathbb{R})),\ s > \frac{3}{2}$.*

4.2 L^2-Global Estimate

Theorem 4.2.1. *Let $s > \frac{3}{2}$, $P(\rho) = \rho^2$, $F \equiv 0$ and the initial Bore-Like condition ρ_0 with $0 \le \rho_0(x) \le 1$. Then, the solution ρ is globally well-posed in the sense that: $0 \le \rho(x,t) \le 1$ and $\rho - \psi \in C([0,T], L^2)$, $\forall\, T > 0$.*

Proof. As a immediate consequence of the Comparison Principle, we have that $0 \le \rho(x) \le 1$
Consider the regularized auxiliary Brinkman Flow equation

$$\begin{cases} \partial_t w_\mu = \mu \partial_x^2 w_\mu + \partial_x\big(w_\mu J^{-2} \partial_x w_\mu^2\big) + E(w_\mu, \psi) \in H^{s-1}(\mathbb{R}), \quad x \in \mathbb{R} \\ w_\mu(x,0) = \rho_0(x) - \psi(x) = \phi(x) \in H^s(\mathbb{R});\ s \ge 1 \end{cases} \tag{4.50}$$

where

$$\begin{aligned} E(w_\mu, \psi) = {} & \partial_x\big(\psi J^{-2}\partial_x \psi^2\big) + \partial_x\big(w_\mu J^{-2}\partial_x \psi^2\big) + \partial_x\big(\psi J^{-2}\partial_x w_\mu^2\big) \\ & + 2\partial_x\big(w_\mu J^{-2}\partial_x(w_\mu\psi)\big) + 2\partial_x\big(\psi J^{-2}\partial_x(w_\mu\psi)\big) \end{aligned} \tag{4.51}$$

Multiplying the equation by w_μ and integrating over $\mathbb{R}$

$$\frac{1}{2}\partial_t \int w_\mu^2\, dx = I1 + I2 + I3 + I4 + I5 + I6 + I7 \tag{4.52}$$

where

$$I1 = \mu \int w_\mu \partial_x^2 w_\mu\, dx,\ \ I2 = \int w_\mu \partial_x\big(w_\mu J^{-2}\partial_x w_\mu^2\big)\, dx$$

$$I3 = \int w_\mu \partial_x\big(\psi J^{-2}\partial_x \psi^2\big)\, dx,\ \ I4 = \int w_\mu \partial_x\big(w_\mu J^{-2}\partial_x \psi^2\big)\, dx$$

$$I5 = \int w_\mu \partial_x\big(\psi J^{-2}\partial_x w_\mu^2\big)\, dx,\ \ I6 = 2\int w_\mu \partial_x\big(w_\mu J^{-2}\partial_x (w_\mu \psi)\big)\, dx$$

and

$$I7 = 2\int w_\mu \partial_x\big(\psi J^{-2}\partial_x (w_\mu \psi)\big)\, dx$$

In what follows we estimate each integral:

Since $H_0 = -\partial_x^2$ is a self-adjoint and positive operator, we have

$$I1 = \mu\big\langle w_\mu, -H_0 w_\mu\big\rangle = -\mu\big\langle H_0 w_\mu, w_\mu\big\rangle \le 0 \tag{4.53}$$

Integrating by parts the second integral

$$\begin{aligned} I2 = \int w_\mu \partial_x\big(w_\mu J^{-2}\partial_x w_\mu^2\big)\, dx &= -\frac{1}{2}\int \partial_x(w_\mu^2) J^{-2}\partial_x(w_\mu^2) \\ &= -\frac{\|\partial_x(w_\mu^2)\|_{-1}}{2} \le 0 \end{aligned} \tag{4.54}$$

Using the Cauchy-Schwartz inequality, Lemma A.1.7, Corollary A.1.2 in appendix, and that $\|J^{-2}\|_{B(H^s,H^{s+2})} \le 1$, we have

$$\begin{aligned} I3 &= 2\int w_\mu \partial_x\big(\psi J^{-2}(\psi\psi')\big)\, dx \le 2\big|\big\langle w_\mu, \partial_x\big(\psi J^{-2}(\psi\psi')\big)\big\rangle\big| \\ &\le 2\|w_\mu\|\|\partial_x\big(\psi J^{-2}(\psi\psi')\big)\|_{s+1} \le 2\|w_\mu\|\|\psi J^{-2}(\psi\psi')\|_{s+2} \\ &\lesssim 2\|w_\mu\|\|J^{-2}(\psi\psi')\|_{s+2}(\|\psi\|_{L^\infty} + \|\psi'\|_{s+2}) \lesssim 2\|w_\mu\|\|\psi\psi'\|_s(\|\psi\|_{L^\infty} + \|\psi'\|_{s+2}) \\ &\lesssim 2\|w_\mu\|(\|\psi\|_{L^\infty} + \|\psi'\|_s)^2(\|\psi\|_{L^\infty} + \|\psi'\|_{s+2}) \\ &\lesssim (1 + \|w_\mu\|^2)(\|\psi\|_{L^\infty} + \|\psi'\|_{s+2})^3 \\ &\lesssim (\|\psi\|_{L^\infty} + \|\psi'\|_{s+2})^3 + (\|\psi\|_{L^\infty} + \|\psi'\|_{s+2})^3\|w_\mu\|^2 \end{aligned} \tag{4.55}$$

Combining integration by parts, Sobolev's Lemma ([1]), Corollary A.1.2 in appendix A and the fact that $\|J^{-2}\partial_x\|_{B(H^s,H^{s+1})} \leq 1$, we obtain

$$\begin{aligned} I4 &= 2\int w_\mu \partial_x\big(w_\mu J^{-2}\partial_x(\psi\psi')\big)\,dx = -\int \partial_x(w_\mu^2)J^{-2}(\psi\psi')\,dx \\ &= \int w_\mu^2 J^{-2}\partial_x(\psi\psi')\,dx \leq \|J^{-2}\partial_x(\psi\psi')\|_{L^\infty}\|w_\mu\|^2 \\ &\lesssim \|J^{-2}\partial_x(\psi\psi')\|_{s+1}\|w_\mu\|^2 \lesssim \|\psi\psi'\|_s\|w_\mu\|^2 \\ &\lesssim (\|\psi\|_{L^\infty} + \|\psi'\|_s)^2\|w_\mu\|^2 \end{aligned} \tag{4.56}$$

Applying Cauchy-Schwartz inequality, $\|J^{-2}\partial_x\|_{B(L^2,H^1)} \leq 1$ and Lemma A.1.7, it follows that

$$\begin{aligned} I5 &= \int w_\mu \partial_x\big(\psi J^{-2}\partial_x w_\mu^2\big)\,dx \leq \big|\big\langle w_\mu, \partial_x\big(\psi J^{-2}\partial_x w_\mu^2\big)\big\rangle\big| \\ &\leq \|w_\mu\|\|\partial_x\big(\psi J^{-2}\partial_x w_\mu^2\big)\| \leq \|w_\mu\|\|\psi J^{-2}\partial_x w_\mu^2\|_1 \\ &\lesssim \|w_\mu\|\|J^{-2}\partial_x w_\mu^2\|_1(\|\psi\|_{L^\infty} + \|\psi'\|_1) \lesssim \|w_\mu\|\|w_\mu^2\|(\|\psi\|_{L^\infty} + \|\psi'\|_s) \\ &\lesssim \|w_\mu\|^2\|w_\mu\|_{L^\infty}(\|\psi\|_{L^\infty} + \|\psi'\|_s) \end{aligned} \tag{4.57}$$

Using integration by parts, $\|J^{-2}\partial_x^2\|_{B(L^2)} \leq 1$ and Cauchy-Schwartz inequality, we have

$$\begin{aligned} I6 &= 2\int w_\mu \partial_x\big(w_\mu J^{-2}\partial_x(w_\mu\psi)\big)\,dx = -\int \partial_x(w_\mu^2)J^{-2}\partial_x(w_\mu\psi)\,dx \\ &= \int w_\mu^2 J^{-2}\partial_x^2(w_\mu\psi)\,dx \leq \big|\big\langle w_\mu^2, J^{-2}\partial_x^2(w_\mu\psi)\big\rangle\big| \\ &\leq \|w_\mu^2\|\|J^{-2}\partial_x^2(w_\mu\psi)\| \leq \|w_\mu\|\|w_\mu\|_{L^\infty}\|w_\mu\psi\| \\ &\leq \|w_\mu\|_{L^\infty}\|\psi\|_{L^\infty}\|w_\mu\|^2 \end{aligned} \tag{4.58}$$

In the last integral, we used the Cauchy-Schwartz inequality, $\|J^{-2}\partial_x\|_{B(L^2,H^1)} \leq 1$ and Lemma A.1.7.

$$\begin{aligned} I7 &= 2\int w_\mu \partial_x\big(\psi J^{-2}\partial_x(w_\mu\psi)\big)\,dx \leq 2\,\big|\big\langle w_\mu, \partial_x\big(\psi J^{-2}\partial_x(w_\mu\psi)\big)\big\rangle\big| \\ &\lesssim \|w_\mu\|\|\partial_x\big(\psi J^{-2}\partial_x(w_\mu\psi)\big)\| \leq \|w_\mu\|\|\psi J^{-2}\partial_x(w_\mu\psi)\|_1 \\ &\lesssim \|w_\mu\|\|J^{-2}\partial_x(w_\mu\psi)\|_1(\|\psi\|_{L^\infty} + \|\psi'\|_1) \lesssim \|w_\mu\|\|w_\mu\psi\|(\|\psi\|_{L^\infty} + \|\psi'\|_s) \\ &\lesssim \|w_\mu\|^2\|\psi\|_{L^\infty}(\|\psi\|_{L^\infty} + \|\psi'\|_s) \lesssim \|w_\mu\|^2(\|\psi\|_{L^\infty} + \|\psi'\|_s)^2 \end{aligned} \tag{4.59}$$

In estimates $I5$ and $I6$, we need to estimate $\|w_\mu\|_{L^\infty}$. The Comparison Principle for ρ implies that

$$\begin{aligned}\|w_\mu\|_{L^\infty} &\le \|w_\mu - w\|_{L^\infty} + \|w\|_{L^\infty} \lesssim \|w_\mu - w\|_s + \|\rho - \psi\|_{L^\infty} \\ &\lesssim \|w_\mu - w\|_s + \|\rho\|_{L^\infty} + \|\psi\|_{L^\infty} \lesssim 1 + \|w_\mu - w\|_s + \|\psi\|_{L^\infty}\end{aligned} \tag{4.60}$$

In order, to estimate $\|w_\mu - w\|_s$,we use arguments similar to those employed in the proof of H^s- global estimates of Brinkman Equation, in section 3.3. We have,

$$\|w_\mu - w\|_s \le 2M\epsilon + \widetilde{C}(M, \tilde{T}_s, \psi)\sqrt{\mu}\,\epsilon^{-s} \tag{4.61}$$

where

$$\widetilde{C}(M, \tilde{T}_s, \psi) = 2\,M\sqrt{\tilde{T}_s}\,e^{Q_0(M,M,\psi)\tilde{T}_s}$$

Then

$$\|w_\mu\|_{L^\infty} \le 2M\epsilon + \widetilde{C}(M, \tilde{T}_s, \psi)\sqrt{\mu}\,\epsilon^{-s} + 1 + \|\psi\|_{L^\infty} \tag{4.62}$$

Substituting (4.62) in $I5, I6$

$$I5 \lesssim (\|\psi\|_{L^\infty} + \|\psi'\|_s)\big[2M\epsilon + \widetilde{C}(M, \tilde{T}_s, \psi)\sqrt{\mu}\,\epsilon^{-s} + 1 + \|\psi\|_{L^\infty}\big]\|w_\mu\|^2 \tag{4.63}$$

$$I6 \le \|\psi\|_{L^\infty}\big[2M\epsilon + \widetilde{C}(M, \tilde{T}_s, \psi)\sqrt{\mu}\,\epsilon^{-s} + 1 + \|\psi\|_{L^\infty}\big]\|w_\mu\|^2 \tag{4.64}$$

On the other hand, substituting (4.53),(4.54),(4.55),(4.56),(4.63),(4.64) and (4.59) in (4.52), we have

$$\partial_t\|w_\mu\|^2 \lesssim (\|\psi\|_{L^\infty} + \|\psi'\|_{s+2})^3 + G(\psi, M, \tilde{T}_s, \mu, \epsilon)\|w_\mu\|^2 \tag{4.65}$$

where

$$\begin{aligned}G(\psi, M, \tilde{T}_s, \mu, \epsilon) &= (\|\psi\|_{L^\infty} + \|\psi'\|_{s+2})^3 + 2\,(\|\psi\|_{L^\infty} + \|\psi'\|_s)^2 \\ &\quad + (\|\psi\|_{L^\infty} + \|\psi'\|_s)\big[2M\epsilon + \widetilde{C}(M, \tilde{T}_s, \psi)\sqrt{\mu}\,\epsilon^{-s} + 1 + \|\psi\|_{L^\infty}\big] \\ &\quad + \|\psi\|_{L^\infty}\big[2M\epsilon + \widetilde{C}(M, \tilde{T}_s, \psi)\sqrt{\mu}\,\epsilon^{-s} + 1 + \|\psi\|_{L^\infty}\big]\end{aligned} \tag{4.66}$$

Integrating (4.65) from 0 to t

$$\|w_\mu(t)\|^2 \le \Big(\|\phi\|^2 + (\|\psi\|_{L^\infty} + \|\psi'\|_{s+2})^3\,\tilde{T}_s\Big) + G(\psi, M, \tilde{T}_s, \mu, \epsilon)\int_0^t \|w_\mu(\tau)\|^2\,d\tau \tag{4.67}$$

Gronwall's Inequality in (4.67) implies

$$\|w_\mu(t)\|^2 \leq \Big(\|\phi\|^2 + (\|\psi\|_{L^\infty} + \|\psi'\|_{s+2})^3\,\tilde{T}_s\Big)\, e^{G(\psi,M,\tilde{T}_s,\mu,\epsilon)\tilde{T}_s},\ \forall\,\epsilon > 0 \quad (4.68)$$

Therefore, as we know that the sequence w_μ converges strongly in L^2, we may apply limit as μ tends to zero, in (4.68). Then, it follows that

$$\|w(t)\|^2 \leq \Big(\|\phi\|^2 + (\|\psi\|_{L^\infty} + \|\psi'\|_{s+2})^3\,\tilde{T}_s\Big)\, e^{\widetilde{G}(\psi,M,\epsilon)\tilde{T}_s},\ \forall\,\epsilon > 0 \qquad (4.69)$$

with

$$\begin{aligned}\widetilde{G}(\psi, M, \epsilon) = {} & (\|\psi\|_{L^\infty} + \|\psi'\|_{s+2})^3 + 2\,(\|\psi\|_{L^\infty} + \|\psi'\|_s)^2 \\ & + (\|\psi\|_{L^\infty} + \|\psi'\|_s)\big[2M\epsilon + 1 + \|\psi\|_{L^\infty}\big] \\ & + \|\psi\|_{L^\infty}\big[2M\epsilon + 1 + \|\psi\|_{L^\infty}\big]\end{aligned} \qquad (4.70)$$

As ϵ is sufficiently small, we take the limit as $\epsilon \to 0$ to get

$$\|w(t)\|^2 \leq \Big(\|\phi\|^2 + (\|\psi\|_{L^\infty} + \|\psi'\|_{s+2})^3\,\tilde{T}_s\Big)\, e^{K(\psi)\tilde{T}_s} \qquad (4.71)$$

where

$$K(\psi) = (\|\psi\|_{L^\infty}+\|\psi'\|_{s+2})^3+2\,(\|\psi\|_{L^\infty}+\|\psi'\|_s)^2+(2\|\psi\|_{L^\infty}+\|\psi'\|_s)\big[1+\|\psi\|_{L^\infty}\big] \qquad (4.72)$$

Finally, the L^2 global estimate for $\rho - \psi$ follows

$$\|\rho - \psi)\|^2 \leq \Big(\|\rho_0 - \psi\|^2 + (\|\psi\|_{L^\infty} + \|\psi'\|_{s+2})^3\,\tilde{T}_s\Big)\, e^{K(\psi)\tilde{T}_s} \qquad (4.73)$$

□

Appendix A

Appendix

A.1 Some inequalities

In what follows we consider some inequalities. In most cases these functions are assumed to be in $S(\mathbb{R}^n)$, but the resulting inequalities may be extended as usual to more general functions by continuity. In addition to the Sobolev norm $\|\bullet\|_s$, we occasionally use the Sobolev seminorm $\|\bullet\|_{[s]}$ given by

$$\|f\|_{[s]}^2 = \int\limits_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 \, d\xi, \; s > -\frac{n}{2}$$

Lemma A.1.1. *Let* $J = (1-\Delta)^{\frac{1}{2}}, s > \frac{n}{2} + 1$. *Then*

$$\left\| [J^s, M_f] g \right\| \leq c \|\nabla f\|_{s-1} \|g\|_{s-1} \tag{A.1}$$

Proof. See [31, Appendix, pág. 122]. □

Lemma A.1.2. *Let* $\sigma \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n - (0,0))$ *satisfy*

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)| \leq C_{\alpha,\beta} (|\xi| + |\eta|)^{-|\alpha| - |\beta|} \tag{A.2}$$

for $(\xi, \eta) \neq (0,0)$ *and any* $\alpha, \beta \in (\mathbb{Z}^+)^n$
If $\sigma(D)$ *denotes the bilinear operator*

$$\sigma(D)(f,g)(x) = \iint e^{i\lfloor x, \xi+\eta \rfloor} \sigma(\xi,\eta) \hat{f}(\xi) \hat{g}(\eta) \, d\xi d\eta \tag{A.3}$$

Then

$$\|\sigma(D)(f,g)\| \leq C \|f\|_{L^\infty} \|g\|_{L^p}, \; p \in (1, \infty) \tag{A.4}$$

Proof. See [9, pág. 154]. □

Lemma A.1.3. *If $s > 0$ and $1 < p < \infty$, then*

$$\Big\| \sum_{k=1}^{n} \big[\partial_{x_k} J^s(g\partial_{x_k} f) - \partial_{x_k} f(\partial_{x_k} J^s g)\big] \Big\|_{L^p} \leq c\Big(\|J^2 f\|_{L^\infty} \|J^s g\|_{L^p} + \|J^{s+2} f\|_{L^p} \|g\|_{L^\infty} \Big) \tag{A.5}$$

`Remark:` In this Lemma we use $\lfloor \bullet, \bullet \rfloor$ to denote the Euclidean scalar product in $\mathbb{R}^n$.

Proof. The proof of this Lemma is similar to that of Lemma X1 in [34], is based on the following result due to R.R.Coifman and Y. Meyer (Lemma A.1.2)
Applying the Inversion Fourier Transform, properties of Fourier Transform, Fubini's Theorem and a change of variable, we start from the formula

$$\begin{aligned}
&\big(\partial_{x_k} J^s(g\partial_{x_k} f) - \partial_{x_k} f(\partial_{x_k} J^s g)\big)(x) \\
&= c \int \Big(\partial_{x_k} J^s(g\partial_{x_k} f) - \partial_{x_k} f(\partial_{x_k} J^s g)\Big)^{\wedge}(\xi)\, e^{ix\xi}\, d\xi \\
&= c \iint \big[\xi_k \eta_k (1+|\eta|^2)^{\frac{s}{2}} - (\xi_k + \eta_k)(1+|\xi+\eta|^2)^{\frac{s}{2}} \xi_k\big]\, e^{ix(\xi+\eta)} \hat{f}(\xi) \hat{g}(\eta)\, d\xi d\eta
\end{aligned} \tag{A.6}$$

Then

$$\begin{aligned}
&\textstyle\sum_{k=1}^{n} \big(\partial_{x_k} J^s(g\partial_{x_k} f) - \partial_{x_k} f(\partial_{x_k} J^s g)\big)(x) \\
&= c \iint e^{i\lfloor x, \xi+\eta \rfloor} \big[\lfloor \xi, \eta \rfloor (1+|\eta|^2)^{\frac{s}{2}} - \lfloor \xi, \xi+\eta \rfloor (1+|\xi+\eta|^2)^{\frac{s}{2}}\big] \hat{f}(\xi) \hat{g}(\eta)\, d\xi d\eta \\
&= -c \iint e^{i\lfloor x, \xi+\eta \rfloor} \big[\lfloor \xi, \xi+\eta \rfloor (1+|\xi+\eta|^2)^{\frac{s}{2}} - \lfloor \xi, \eta \rfloor (1+|\eta|^2)^{\frac{s}{2}}\big] \hat{f}(\xi) \hat{g}(\eta)\, d\xi d\eta \\
&= -c \textstyle\sum_{j=1}^{3} \sigma_j(D)(f,g)(x)
\end{aligned} \tag{A.7}$$

where

$$\sigma_j(\xi,\eta) = \Big\{ \lfloor \xi, \xi+\eta \rfloor (1+|\xi+\eta|^2)^{\frac{s}{2}} - \lfloor \xi, \eta \rfloor (1+|\eta|^2)^{\frac{s}{2}} \Big\} \Phi_j\Big(\frac{|\xi|}{|\eta|}\Big) \tag{A.8}$$

and the Φ_j are functions on $\mathbb{R}$ with the following properties

$$\begin{gathered}
0 \leq \Phi_j \leq 1, \quad j = 1,\, 2,\, 3, \\
\Phi_1 + \Phi_2 + \Phi_3 = 1 \ \text{ on } \ [0,\, \infty), \\
\operatorname{supp} \Phi_1 \subset [-1/3,\, 1/3], \ \ \operatorname{supp} \Phi_2 \subset [1/4,\, 4], \ \ \operatorname{supp} \Phi_3 \subset [3,\, \infty).
\end{gathered} \tag{A.9}$$

`Step 1` First we consider $\sigma_1(D)(f,g)$. We write

$$\sigma_1(\xi,\eta) = \Big\{\lfloor \xi,\xi+\eta\rfloor(1+|\xi+\eta|^2)^{\frac{s}{2}} - \lfloor \xi,\eta\rfloor(1+|\eta|^2)^{\frac{s}{2}}\Big\}\Phi_1\Big(\frac{|\xi|}{|\eta|}\Big) \tag{A.10}$$

$$= (1+|\eta|^2)^{\frac{s}{2}}\Big\{\lfloor \xi,\xi+\eta\rfloor\Big(\frac{1+|\xi+\eta|^2}{1+|\eta|^2}\Big)^{\frac{s}{2}} - \lfloor \xi,\eta\rfloor\Big\}\Phi_1\Big(\frac{|\xi|}{|\eta|}\Big)$$

$$= (1+|\eta|^2)^{\frac{s}{2}}\Big\{\lfloor \xi,\xi+\eta\rfloor\Big(1+(1+|\eta|^2)^{-1}\lfloor \xi,\xi+2\eta\rfloor\Big)^{\frac{s}{2}} - \lfloor \xi,\eta\rfloor\Big\}\Phi_1\Big(\frac{|\xi|}{|\eta|}\Big)$$

Using the generalized Newton's Binomial Theorem, we have

$$\sigma_1(\xi,\eta) = \sigma_{1,1}(\xi,\eta) + \sigma_{1,2}(\xi,\eta) \tag{A.11}$$

with

$$\sigma_{1,1}(\xi,\eta) = (1+|\eta|^2)^{\frac{s}{2}}|\xi|^2\Phi_1\Big(\frac{|\xi|}{|\eta|}\Big) \tag{A.12}$$

and

$$\sigma_{1,2}(\xi,\eta) = \Big\{\sum_{r=1}^{n}\lfloor \xi,\xi+\eta\rfloor C_r(1+|\eta|^2)^{\frac{s}{2}-r}\lfloor \xi,\xi+2\eta\rfloor^r\Phi_1\Big(\frac{|\xi|}{|\eta|}\Big)\Big\} \tag{A.13}$$

If we multiply (A.11) by $\hat{f}(\xi)\hat{g}(\eta)$, we obtain

$$\sigma_1(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta) = \sigma_{1,1,0}(\xi,\eta)\widehat{J^2f}(\xi)\widehat{J^sg}(\eta) + \Big(\sum_{r=1}^{\infty}\sigma_{1,2,r}(\xi,\eta)\Big)\widehat{J^2f}(\xi)\widehat{J^sg}(\eta) \tag{A.14}$$

where

$$\sigma_{1,1,0}(\xi,\eta) = \frac{|\xi|^2}{1+|\xi|^2}\Phi_1\Big(\frac{|\xi|}{|\eta|}\Big) \tag{A.15}$$

and

$$\sigma_{1,2,r}(\xi,\eta) = C_r\frac{\lfloor \xi,\xi+\eta\rfloor}{1+|\xi|^2}(1+|\eta|^2)^{-r}\lfloor \xi,\xi+2\eta\rfloor^r\Phi_1\Big(\frac{|\xi|}{|\eta|}\Big) \tag{A.16}$$

Thus

$$\sigma_1(D)(f,g)(x) = \iint\limits_{\frac{|\xi|}{|\eta|}\le\frac{1}{3}} e^{i\lfloor x,\xi+\eta\rfloor}\sigma_1(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)\,d\xi d\eta \tag{A.17}$$

$$= \sigma_{1,1,0}(D)(J^2f,J^sg)(x) + \sigma_{1,2}(D)(J^2f,J^sg)(x)$$

with

$$\sigma_{1,1,0}(D)(J^2 f, J^s g)(x) = \iint\limits_{\frac{|\xi|}{|\eta|}\le\frac{1}{3}} e^{i\lfloor x,\xi+\eta\rfloor}\sigma_{1,1,0}(\xi,\eta)\widehat{J^2 f}(\xi)\widehat{J^s g}(\eta)\, d\xi d\eta \tag{A.18}$$

and

$$\sigma_{1,2}(D)(J^2 f, J^s g)(x) = \iint\limits_{\frac{|\xi|}{|\eta|}\le\frac{1}{3}} e^{i\lfloor x,\xi+\eta\rfloor}\Big(\sum_{r=1}^{\infty}\sigma_{1,2,r}(\xi,\eta)\Big)\widehat{J^2 f}(\xi)\widehat{J^s g}(\eta)\, d\xi d\eta \tag{A.19}$$

As is easily seen $\sigma_{1,1,0}(\xi,\eta), \sigma_{1,2,r}(\xi,\eta)$ satisfy (A.2). Since the series defined by $\sigma_{1,2}(\xi,\eta)$ converges (due fact that $|\xi| \le \frac{|\eta|}{3}$ for $\Phi_1 \neq 0$), it follows from estimate (A.3) in Lemma A.1.2 that

$$\|\sigma_1(D)(f,g)\|_{L^p} \le c\|J^2 f\|_{L^\infty}\|J^s g\|_{L^p} \tag{A.20}$$

`Step 2` Next we consider $\sigma_3(D)(f,g)$. Here we write $\sigma_3(\xi,\eta) = \sigma_{3,1}(\xi,\eta) - \sigma_{3,2}(\xi,\eta)$, where

$$\sigma_{3,1}(\xi,\eta) = \Big(\lfloor\xi,\xi+\eta\rfloor(1+|\xi+\eta|^2)^{\frac{s}{2}} - 1\Big)\Phi_3\Big(\frac{|\xi|}{|\eta|}\Big) \tag{A.21}$$

and

$$\sigma_{3,2}(\xi,\eta) = \Big(\lfloor\xi,\eta\rfloor(1+|\eta|^2)^{\frac{s}{2}} - 1\Big)\Phi_3\Big(\frac{|\xi|}{|\eta|}\Big) \tag{A.22}$$

Now

$$\begin{aligned}\sigma_3(\xi,\eta)\hat f(\xi)\hat g(\eta) &= \sigma_{3,1,1}(\xi,\eta)\widehat{J^{s+2} f}(\xi)\hat g(\eta) - \sigma_{3,2,1}(\xi,\eta)\widehat{J^2 f}(\xi)\widehat{J^s g}(\eta) \\ &+ \sigma_{3,2,2}(\xi,\eta)\widehat{J^2 f}(\xi)\widehat{J^s g}(\eta)\end{aligned} \tag{A.23}$$

with

$$\sigma_{3,1,1}(\xi,\eta) = (1+|\xi|^2)^{-\frac{s+2}{2}}\Big(\lfloor\xi,\xi+\eta\rfloor(1+|\xi+\eta|^2)^{\frac{s}{2}} - 1\Big)\Phi_3\Big(\frac{|\xi|}{|\eta|}\Big) \tag{A.24}$$

$$\sigma_{3,2,1}(\xi,\eta) = \frac{\lfloor\xi,\eta\rfloor}{1+|\xi|^2}\Phi_3\Big(\frac{|\xi|}{|\eta|}\Big) \tag{A.25}$$

$$\sigma_{3,2,2}(\xi,\eta) = \frac{(1+|\eta|^2)^{-\frac{s}{2}}}{1+|\xi|^2}\Phi_3\Big(\frac{|\xi|}{|\eta|}\Big) \tag{A.26}$$

Thus

$$\begin{aligned} \sigma_3(D)(f,g)(x) &= \iint_{\frac{|\xi|}{|\eta|}\geq 3} e^{i\lfloor x,\xi+\eta\rfloor}\sigma_3(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)\,d\xi d\eta \\ &= \sigma_{3,1,1}(D)(J^{s+2}f,g)(x) - \sigma_{3,2,1}(D)(J^2f,J^sg)(x) + \sigma_{3,2,2}(D)(J^2f,J^sg)(x) \end{aligned} \tag{A.27}$$

where

$$\sigma_{3,1,1}(D)(J^{s+2}f,g)(x) = \iint_{\frac{|\xi|}{|\eta|}\geq 3} e^{i\lfloor x,\xi+\eta\rfloor}\sigma_{3,1,1}(\xi,\eta)\widehat{J^{s+2}f}(\xi)\hat{g}(\eta)\,d\xi d\eta \tag{A.28}$$

$$\sigma_{3,2,1}(D)(J^2f,J^sg)(x) = \iint_{\frac{|\xi|}{|\eta|}\geq 3} e^{i\lfloor x,\xi+\eta\rfloor}\sigma_{3,2,1}(\xi,\eta)\widehat{J^2f}(\xi)\widehat{J^sg}(\eta)\,d\xi d\eta \tag{A.29}$$

$$\sigma_{3,2,2}(D)(J^2f,J^sg)(x) = \iint_{\frac{|\xi|}{|\eta|}\geq 3} e^{i\lfloor x,\xi+\eta\rfloor}\sigma_{3,2,2}(\xi,\eta)\widehat{J^2f}(\xi)\widehat{J^sg}(\eta)\,d\xi d\eta \tag{A.30}$$

Since (A.24),(A.25),(A.26) satisfy (A.2) with $|\xi|\geq 3|\eta|$ for $\Phi_3\Big(\frac{|\xi|}{|\eta|}\Big)\neq 0$, we obtain

$$\begin{aligned} &\|\sigma_3(D)(f,g)(x)\|_{L^p} \\ &\leq \|\sigma_{3,1,1}(D)(J^{s+2}f,g)(x)\|_{L^p} + \|\sigma_{3,2,1}(D)(J^2f,J^sg)(x)\|_{L^p} \\ &\quad + \|\sigma_{3,2,2}(D)(J^2f,J^sg)(x)\|_{L^p} \\ &\leq c\|J^{s+2}f\|_{L^p}\|g\|_{L^\infty} + c\|J^2f\|_{L^\infty}\|J^sg\|_{L^p} \end{aligned} \tag{A.31}$$

`Step 3` Estimating $\sigma_2(f,g)(x)$ is more complicated, due to the fact $\xi+\eta$ may vanish in the domain of integration, so that any negative power of $1+|\xi+\eta|^2$ will not satisfy (A.2)
We write $\sigma_2(\xi,\eta)$ in the form $\sigma_2(\xi,\eta) = \sigma_{2,1}(\xi,\eta) - \sigma_{2,2}(\xi,\eta)$ with

$$\sigma_{2,1}(\xi,\eta) = \lfloor\xi,\xi+\eta\rfloor(1+|\xi+\eta|^2)^{\frac{s}{2}}\Phi_2\Big(\frac{|\xi|}{|\eta|}\Big) \tag{A.32}$$

$$\sigma_{2,2}(\xi,\eta) = \lfloor\xi,\eta\rfloor(1+|\eta|^2)^{\frac{s}{2}}\Phi_2\Big(\frac{|\xi|}{|\eta|}\Big) \tag{A.33}$$

The term $\sigma_{2,2}(D)(f,g)$ can be easily handled. Indeed, since

$$\sigma_{2,2}(\xi,\eta)\hat{f}(\xi)\hat{g} = \sigma_{2,2,1}(\xi,\eta)\widehat{J^{s+2}f}(\xi)\hat{g}(\eta) \tag{A.34}$$

where

$$\sigma_{2,2,1}(\xi,\eta) = \frac{\lfloor \xi,\eta \rfloor}{1+|\xi|^2}(1+|\xi|^2)^{-\frac{s}{2}}(1+|\eta|^2)^{\frac{s}{2}}\Phi_2\Big(\frac{|\xi|}{|\eta|}\Big) \tag{A.35}$$

we have

$$\begin{aligned} \sigma_{2,2}(D)(f,g)(x) &= \iint\limits_{\frac{1}{4}\le\frac{|\xi|}{|\eta|}\le 4} e^{i\lfloor x,\xi+\eta\rfloor}\sigma_{2,2,1}(\xi,\eta)\widehat{J^{s+2}f}(\xi)\hat{g}(\eta)\,d\xi d\eta \\ &= \sigma_{2,2,1}(D)(J^{s+2}f,g)(x) \end{aligned} \tag{A.36}$$

As (A.35) satisfies condition (A.2), it follows that

$$\|\sigma_{2,2}(D)(f,g)(x)\|_{L^p} = \|\sigma_{2,2,1}(D)(J^{s+2}f,g)(x)\|_{L^p} \le c\|J^{s+2}f\|_{L^p}\|g\|_{L^\infty} \tag{A.37}$$

The same method can be used to estimate $\|\sigma_{2,1}(D)(f,g)(x)$ if s is so large that $s \ge k = k(n,p)$(see [34, Remark X3 for the definition of k]); then there are no negative powers of $1+|\xi+\eta|^2$ to estimate.
But, this method fails if s is not so large. To avoid this difficulty, we shall use complex interpolation by extending $\sigma_{2,1}(\xi,\eta) \equiv \sigma^s_{2,1}(\xi,\eta)$ to complex values of s with $0 \le \Re(s) \le k$, condition (A.2) and maximum principle in complex function theory (see [34])
Finally, we obtain

$$\|\sigma_{2,1}(D)(f,g)(x)\|_{L^p} \le c\|J^{s+2}f\|_{L^p}\|g\|_{L^\infty} \tag{A.38}$$

`Step 4` Finally, collecting the results (A.20),(A.31),(A.37) and (A.38); we have proved the Lemma. □

Lemma A.1.4. *If $s > 0$ and $1 < p < \infty$, then $L^p_s \cap L^\infty$ is a Banach Algebra. Moreover*

$$\|fg\|_{s,p} \le c(\|f\|_{L^\infty}\|g\|_{L^p} + \|f\|_{L^p}\|g\|_{L^\infty}) \tag{A.39}$$

Proof. See [34] □

Corollary A.1.1. *Let* $f \in H^s = L^2_s, s > \frac{n}{2}, k = 1, 2, ...$ *Then*

$$\|f^{2k}\|_s \lesssim \|f\|^{2k-1}_{L^\infty}\|f\|_s \tag{A.40}$$

Proof. Since $f \in H^s, s > \frac{n}{2}$, Sobolev's Lemma implies that $f \in L^\infty$. Then $f \in L^2_s \cap L^\infty, s > \frac{n}{2} > 0$

(Case k=1) Applying the above Lemma we obtain

$$\|f^2\|_s \leq 2c\|f\|_{L^\infty}\|f\| \leq 2c\|f\|_{L^\infty}\|f\|_s \Rightarrow \|f^2\|_s \lesssim \|f\|_{L^\infty}\|f\|_s \tag{A.41}$$

(Case k=2) The above Lemma and the case k=1, implies

$$\begin{aligned}\|f^4\|_s = \|f^2.f^2\|_s &\leq 2c\|f^2\|_{L^\infty}\|f^2\| \leq 2c\|f\|^2_{L^\infty}\|f^2\|_s \\ &\leq 4c^2\|f\|^3_{L^\infty}\|f\|_s\end{aligned} \tag{A.42}$$

Then

$$\|f^4\|_s \lesssim \|f\|^3_{L^\infty}\|f\|_s \tag{A.43}$$

(Case k=k) Suppose as induction hypothesis that

$$\|f^{2k}\|_s \lesssim \|f\|^{2k-1}_{L^\infty}\|f\|_s \tag{A.44}$$

(Case k+1) By above Lemma and induction hypothesis

$$\|f^{2(k+1)}\|_s = \|f^{2k}f^2\|_s \lesssim (\|f^{2k}\|_{L^\infty}\|f^2\|_s + \|f^2\|_{L^\infty}\|f^{2k}\|_s) \tag{A.45}$$

$$\lesssim 2\,\|f\|^{2k+1}_{L^\infty}\|f\|_s \tag{A.46}$$

Then

$$\|f^{2(k+1)}\|_s \lesssim \|f\|^{2k+1}_{L^\infty}\|f\|_s \tag{A.47}$$

Finally, the result is proved. □

Lemma A.1.5. *Let* $s > \frac{n}{2} + 1, t \geq 1, f, g \in S(\mathbb{R}^n)$. *Then there exists* $C = C(s, n, t) > 0$ *such that*

$$\left|\left\langle fDg, g\right\rangle_t\right| \leq C\left[\|\nabla f\|_{s-1}\|g\|^2_t + \|\nabla f\|_{t-1}\|g\|_t\|g\|_s\right] \tag{A.48}$$

with $D = \partial^\alpha, |\alpha| = 1$

Proof. See [31] □

Lemma A.1.6.

$$\|fg\|_{[s]} \leq c(\|f\|_{L^\infty}\|g\|_{[s]} + \|f\|_{[s]}\|g\|_{L^\infty}), s \geq 0 \tag{A.49}$$

Proof. See [14],[31]. □

Lemma A.1.7. *Let $f \in H^s, \psi \in L^\infty, \psi' \in H^\infty$. Then $f\psi \in H^s$ and satisfies:*

$$\|f\psi\|_s \lesssim \|f\|_s(\|\psi\|_{L^\infty} + \|\psi'\|_s) \tag{A.50}$$

Proof. We define an equivalent norm to $\|\bullet\|_s^2$ by $|||\bullet|||_s^2 = \|\bullet\|^2 + \|\bullet\|_{[s]}^2$
Applying (A.49) to calculate $\|f\psi\|_{[s]}$ we have

$$\begin{aligned}\|f\psi\|_{[s]} = \|D^s(f\psi)\| &\lesssim \big(\|f\|_{L^\infty}\|\psi\|_{[s]} + \|f\|_{[s]}\|\psi\|_{L^\infty}\big) \\ &\lesssim \|f\|_s\big(\|\psi'\|_s + \|\psi\|_{L^\infty}\big)\end{aligned} \tag{A.51}$$

Then, substituting (A.51) in the equivalent norm to $f\psi$

$$\begin{aligned}|||f\psi|||_s^2 = \|f\psi\|^2 + \|f\psi\|_{[s]}^2 &\lesssim \|f\|_s^2\|\psi\|_{L^\infty}^2 + \|f\|_s^2\big(\|\psi'\|_s + \|\psi\|_{L^\infty}\big)^2 \\ &\lesssim \|f\|_s^2\big(2\|\psi\|_{L^\infty}^2 + 2\|\psi'\|_s\|\psi\|_{L^\infty} + \|\psi'\|_s^2\big) \\ &\lesssim \|f\|_s^2\big(\|\psi'\|_s + \|\psi\|_{L^\infty}\big)^2\end{aligned} \tag{A.52}$$

Finally, using the relation between equivalents norms, the desired result is obtained. □

Corollary A.1.2. *Let $\psi \in L^\infty$ with $\psi' \in H^\infty$. Then $\psi\psi' \in H^s$ and satisfies:*

$$\|\psi\psi'\|_s \lesssim (\|\psi\|_{L^\infty} + \|\psi'\|_s)^2 \tag{A.53}$$

Proof. This result is a direct consequence of the previous Lemma.
Let $\psi' \in H^\infty \subset H^s$
Using the estimate (A.50), we get $\|\psi\psi'\|_s$

$$\|\psi\psi'\|_s \lesssim \|\psi'\|_s(\|\psi\|_{L^\infty} + \|\psi'\|_s) \lesssim (\|\psi\|_{L^\infty} + \|\psi'\|_s)^2 \tag{A.54}$$

□

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