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On the Davey Stewartson and degenerate Zakharov systems

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Abstract

The purpose of this work is the study of the well-posedness of the initial value problem (IVP) associated to two systems: The first one is the Davey-Stewartson, where we prove global well posedness in some Lorents spaces and consequently we find self-similar solutions. The second system is the Degenerated Zakharov, where we prove local well posedness in the sobolev space $H^3(\mathbb{R}^3)$ improving a result from Linares, Ponce and Saut.

Contents

Introduction

In this thesis we are concerned with the well-posedness of the IVP associated to two dispersive systems.

The first system is the Davey-Stewartson

$$\begin{cases} i\partial_t u + \delta\partial_{x_1}^2 u + \sum_{j=2}^n \partial_{x_j}^2 u = \chi|u|^\alpha u + bu\partial_{x_1}\varphi, \\ \partial_{x_1}^2\varphi + m\partial_{x_2}^2\varphi + \sum_{j=3}^n \partial_{x_j}^2\varphi = \partial_{x_1}(|u|^\alpha), \end{cases} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R} \text{ and } n = 2 \text{ or } 3, \quad (1)$$

where the exponent α is such that $\frac{4(n+1)}{n(n+2)} < \alpha < \frac{4(n+1)}{n^2}$, $n = 2, 3$, the parameters χ, b are constants in $\mathbb{R}^+$ and δ and m are real positive.

The Davey-Stewartson systems are the $2D$ generalization of the cubic $1D$ Schrödinger equation $i\partial_t u + \Delta u = |u|^2 u$ and model the evolution of weakly nonlinear water waves that travel predominantly in one direction, but in which the amplitude is modulated slowly in two horizontal directions.

The system (1), $n = 2$, $\alpha = 2$, was first derived by Davey and Stewartson ([DS]) in the context of water waves, but their analysis did not take account of the effect of surface tension (or capillarity). This effect was later included by Djordjevic and Redekopp [DR] who have shown that the parameter m can become negative when capillary effects are important. Independently, Ablowitz and Haberman [AH] obtained a particular form of (1), $n = 2$, as an example of completely integrable model also generalizing the two-dimensional nonlinear Schrödinger equation.

There has been a lot of work in the literature (see for instance [GS], [H1], [LP1], [Oh], [Oz]) concerning different issues regarding the Davey-Stewartson systems including

solvability of the initial and initial-boundary value problems, blow-up solutions, existence of periodic solutions.

In [GS], Ghidaglia and Saut studied the existence of solutions of IVP (1), $n = 2$, $\alpha = 2$. They classified the system as elliptic-elliptic, elliptic-hyperbolic, hyperbolic-elliptic and hyperbolic-hyperbolic, according to respective sign of $(\delta, m) : (+, +), (+, -), (-, +), (-, -)$. The particular cases $(\delta, \chi, b, m) = (1, -1, -2, -1)$ (elliptic-hyperbolic) and $(\delta, \chi, b, m) = (-1, -2, 1, 1)$ and $(-1, 2, -1, 1)$ (hyperbolic-elliptic) are know as $DSI, DSII$ defocusing and $DSII$ focusing, respectively. For these particular cases, the inverse scattering techniques has led to remarkable issues including: the existence of solitons (Anker and Freeman [AnFr], Ablowitz and Fokas [AF], Fokas and Santini [FS]); solvability of the cauchy problem(Beals and Coifman [BC], Fokas and Santini [FS] and their bibliografy).

For the elliptic-elliptic and hyperbolic-elliptic cases, Ghidaglia and Saut [GS] reduced the system (1), $n = 2$, to the nonlinear cubic Schrödinger equation with a nonlocal nonlinear term, i.e.

$$i\partial_t u + \delta \partial_{x_1}^2 u + \partial_{x_2}^2 u = \chi |u|^2 u + H(u),$$

where $H(u) = (\Delta^{-1} \partial_x^2 |u|^2) u$. They showed local well-posedness for data in L^2, H^1 and H^2 using Strichartz estimates (see Theorem 1.7) and the continuity properties of the operator Δ^{-1}.

The remaining cases, elliptic-hyperbolic and hyperbolic-hyperbolic, were treated by Linares and Ponce [LP1], Hayashi [H1], [H2], Chihara [Ch], Hayashi and Hirata [HH1], [HH2], Hayashi and Saut [HS].

In the elliptic-hyperbolic case $(\delta, m) = (1, -1)$, after a rotation in the $x_1 x_2$ plane, the system (1), $n = 2$, can be written as

$$\begin{cases} i\partial_t u + \Delta u &= (\chi + \frac{b}{2})|u|^2 u - \frac{b}{4}(\int_{x_1}^{\infty} \partial_{x_2} |u|^2 dx'_1 \int_{x_2}^{\infty} \partial_{x_1} |u|^2 dx'_2)u \\ &\quad + \frac{b}{\sqrt{2}}((\partial_{x_1} \varphi_1) + (\partial_{x_2} \varphi_2))u, \\ u(x, 0) &= u_0(x), \end{cases} \quad (2)$$

where φ is assumed to satisfies the radiation condition

$$\lim_{x_2 \to \infty} \varphi(x,t) = \varphi_1(x_1,t), \quad \lim_{x_1 \to \infty} \varphi(x,t) = \varphi_2(x_2,t).$$

In the hyperbolic-hyperbolic case $(\delta, m) = (-1, -1)$, after a rotation in the $x_1 x_2$ plane, the system (1), $n = 2$, can be written as

$$\begin{cases} i\partial_t u - 2\partial_{x_1}\partial_{x_2}u &= (\chi + \frac{b}{2})|u|^2 u - \frac{b}{4}(\int_{x_1}^{\infty} \partial_{x_2}|u|^2 dx'_1 \int_{x_2}^{\infty} \partial_{x_1}|u|^2 dx'_2)u \\ &\quad + \frac{b}{\sqrt{2}}((\partial_{x_1}\varphi_1) + (\partial_{x_2}\varphi_2))u, \\ u(x,0) &= u_0(x). \end{cases} \tag{3}$$

In these cases $L^p - L^q$ time decay estimates of the Schrödinger group $e^{it\Delta}$ (problem (2)) or $e^{-2it\partial_{x_1}\partial_{x_2}}$ (problem (3)) cannot be applied. The difficulty of the problems (2) and (3) arises from the facts that the nonlinear terms contain derivatives of the unknown function and that $\int_{x_j}^{\infty} \partial_{x_k}|u|^2 dx'_j$ does not decay when $|x_j| \to \infty$ where $j \neq k$ $(j, k = 1, 2)$.

Linares and Ponce [LP1] proved local well-posedness for the IVP (2) under smallness assumption on data in $H^{m,0} \cap H^{6,6}$, $m \geqslant 12$, $\varphi_1 = \varphi_2 \equiv 0$ and local well-posedness for the IVP (3) under smallness assumption on data in $H^{6,0} \cap H^{3,2}$, $\varphi_1 = \varphi_2 \equiv 0$. They used smoothing effect of Kato's type associated to the groups $e^{it\Delta}$ and $e^{-2it\partial_{x_1}\partial_{x_2}}$ respectively. Using pseudo-differential operators Chihara [Ch] obtained a local result for small data in $u_0 \in H^{m,0}$ for the IVP (2). Also for the IVP (2) Hayashi and Hirata proved local result (see [HH2]) in the usual Sobolev space $H^{5/2,0}$ for small data in L^2 norm and global result (see [HH1]) for small data in $H^{3,0} \cap H^{0,3}$.

Hayashi [H1] showed local well-posedness for small data in $H^{m,0} \cap H^{0,l}, m, l > 1$, to the IVP (2) and local well-posedness for small data in $H^{\delta,0} \cap H^{0,\delta}, \delta > 1$, to the IVP (3). Using the parabolic regularized equation of the IVP (2), Hayashi [H2] proved local existence and uniqueness without the smallness conditions on the data which were assumed in previous works [Ch], [H1], [HH2], [LP1]. In [HS], Hayashi and Saut proved local existence solutions

in analytic function space to the IVP (2). The global existence of small solutions to (2) was also given in [HS] when the data are real analytic and satisfy the exponential decay condition.

Here we will concentrate in the elliptic-elliptic case.

We can reduce the IVP (1) to the nonlinear Schrödinger equation (see Section 2.1 to more details)

$$\begin{cases} i\partial_t u + \delta \partial_{x_1}^2 u + \sum_{j=2}^n \partial_{x_j}^2 u = \chi |u|^\alpha u + bu E(|u|^\alpha), \\ u(x,0) = u_0(x), \end{cases} \qquad \forall\, x \in \mathbb{R}^n,\, t \in \mathbb{R}, \qquad (4)$$

where

$$\widehat{E(f)}(\xi) = \frac{\xi_1^2}{\xi_1^2 + m\xi_2^2 + \sum_{j=3}^n \xi_j^2} \hat{f}(\xi) = p(\xi)\hat{f}(\xi).$$

Using Strichartz estimates to the Schrödinger equation we deduce some inequalities that will be the key to run the fixed point argument and prove well posedness in some weak L^p spaces.

Now observe that if $u(x,t)$ satisfies

$$iu_t + \delta \partial_{x_1}^2 u + \sum_{j=2}^n \partial_{x_j}^2 u = \chi |u|^\alpha u + bu E(|u|^\alpha),$$

then also does $\beta^{\alpha/2} u(\beta x, \beta^2 t)$, for all $\beta > 0$.

Therefore it is natural to ask whether solutions $u(x,t)$ of (1) exist and satisfy, for $\beta > 0$:

$$u(x,t) = \beta^{2/\alpha} u(\beta x, \beta^2 t).$$

Such solutions are called self-similar solutions of the equation (4). Formally:

Definition 0.1. *$u(x, t)$ is said to be a self-similar solution to the Schrödinger equation in* (4) *if*

$$u(x,t) = u_\beta(x,t) = \beta^{2/\alpha} u(\beta x, \beta^2 t),\ \forall\ \beta > 0.$$

So, supposing local well posedness and u a self-similar solution we must have

$$u(x,0) = u_\beta(x,0), \ \forall \ \beta > 0,$$

i.e.,

$$u_0(x) = \beta^{2/\alpha} u(\beta x).$$

In other words, $u_0(x)$ is homogeneous with degree $-2/\alpha$ and every initial data that gives a self-similar solution must verify this property. Unfortunately, those functions do not belong to the usual spaces where strong solutions exists, such as the Sobolev spaces $H^s(\mathbb{R}^n)$. We shall therefore replace them by other functional spaces that allow homogeneous functions.

There are many motivations to find self-similar solutions. One of then is that they can give a good description of the large time behaviour for solutions of dispersive equations. For example, Escobedo and Kavian [EK] proved that on $\mathbb{R}^n$, for $1 < p < 1 + 2/n$, solutions to $\partial_t u - \Delta u + |u|^{p-1} u = 0$ behave like a self-similar solution as $t \to \infty$.

The idea of constructing self-similar solutions by solving the initial value problem for homogeneous data was first used by Giga and Miyakawa [GM], for the Navier Stokes equation in vorticity form. The idea of [GM] was used latter by Cannone and Planchon [CP], Planchon [P] (for the Navier-Stokes equation); Kwak [K], Snoussi, Tayachi and Weissler [STW] (for nonlinear parabolic problems); Kavian and Weissler [KW], Pecher [Pe], Ribaud and Youssfi [RY2] (for the nonlinear wave equation); Cazenave and Weissler [CW1],[CW2], Ribaud and Youssfi [RY1], Furioli [F], Cazenave, Vega and Vilela [CVeVi] (for the nonlinear Schrödinger equation).

In [CP] Canone and Planchon constructed self- similar solutions for three-dimensional incompressible Navier stokes equation in Besov spaces. In [P], Planchon proved that the IVP for semi-linear wave equations is well-posed in the Besov spaces $\dot{B}_2^{s_p,\infty}(\mathbb{R}^n)$, where the nonlinearity is of type u^p, with $p \in \mathbb{N}$ and $s_p = \frac{n}{2} - \frac{2}{p} > \frac{1}{2}$. This result allowed to obtain

self-similar solutions.

Kawak [K] proved existence and uniqueness of non-negative solutions to the semilinear heat equation

$$\partial_t u = \Delta u + F(u), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \tag{5}$$

where $F(u) = -|u|^{p-1}u$, in the range $1 < p < 1 + 2/n$, with initial data $u(x,0) = a|x|^{-2(p-1)}, x \neq 0$ for $a > 0$. It was proved that maximal and minimal solutions are self-similar with the form

$$W_a(x,t) = t^{-1/(p-1)} g_a(|x|/t^{1/2}),$$

where $g = g_a$ satisfies

$$g'' + (\frac{r}{2} + \frac{n-1}{r})g' + (\frac{1}{p-1})g - g^p = 0,$$

$$g \geqslant 0, \quad g'(0) = 0, \quad g(0) > 0, \quad \lim_{r \to \infty} r^{2/(p-1)} g(r) = 0.$$

Snoussi, Tayachi, and Weissler [STW] consider the nonlinear heat equation (5) with $F(u) = a|u|^{p-1}u + f(u)$, where $a \in \mathbb{R}$, $p > 1 + (2/n)$ and f satisfies certain growth conditions. In order to treat a more general nonlinear term, they extended the methods used in [CW1] and proved the existence of global solutions for small initial data with respect to a norm wich is related to the structure of the equation. Moreover, some of those global solutions are asymptotic for large time to self-similar solutions of the single power heat equation, i.e., with $f \equiv 0$.

The existence of self-similar solutions for the nonlinear wave equation (6)

$$\begin{cases} \partial_t^2 u - \Delta u = \gamma |u|^\alpha u \\ u(x,0) = f(x), \quad \partial_t u(x,0) = g(x). \end{cases} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \gamma \in \mathbb{R} \text{ and } \alpha > 0 \tag{6}$$

has been first proved by Kavian and Weissler [KW] in the radially symetric case, i.e., for (f,g) of the form

$$f(x) = c_1 |x|^{-2/(\alpha-1)}, \quad g(x) = c_2 |x|^{-\frac{\alpha+1}{\alpha-1}}.$$

They proved the existence of radially simetric self-similar solutions for subcritical and critical values of α i.e. for $\alpha \leqslant \alpha^*(n)$ where $\alpha^*(n)$ is given by

$$\alpha^*(n) = \frac{n+2}{n-2}, \quad \alpha^*(1) = \alpha^*(2) = \infty.$$

Pecher [Pe] considered the cauchy problem for the semilinear wave equation (6) in three dimensions and showed the existence of self-similar solutions to homogeneous singular data of the type

$$f(x) = \epsilon_1 |x|^{-2/\alpha}, \quad g(x) = \epsilon_2 |x|^{-\frac{2}{\alpha}-1},$$

where ϵ_1 and ϵ_2 are small. The self-similar solutions were compared to certain weak solutions $u \in L^\infty(0, \infty; \dot{H}^{1,2}(\mathbb{R}^3)), u' \in L^\infty(0, \infty; L^2(\mathbb{R}^3))$ wich existence were already proved in many early papers. These weak solutions were shown then to behave asymptotically as $t \to \infty$ like the self-similar solutions with the same data constructed before in the sense that their difference tends to zero as $t \to \infty$ faster than either of them separately.

Finally Ribaud and Youssfi [RY2] improved the study of self-similar solutions to the equation (6) for all dimension $n \geqslant 2$.

In [CW1] Cazenave and Weissler proved the existence of global solutions, including self-similar solutions, to the nonlinear Schrödinger equation (7) (NLS) using norms analogous to those used in [CP].

$$i\partial_t u + \Delta u = \gamma |u|^\alpha u, \qquad \alpha > 0, \ \gamma \in \mathbb{R}, \ (x,t) \in \mathbb{R}^n \times [0, \infty). \tag{7}$$

In [CW2], Cazenave and Weissler proved the existence of a class of self-similar solutions to the equation (7), with higher regularity than the solutions constructed in [CW1]. The results are valid for a range of α which differs from, but overlaps with, the range of α considered in [CW1].

Ribaud and Youssfi [RY1] improved the results in [CW1] and [CW2]. They obtained new global existence results for the (NLS) equation (7) with small initial data which allowed to prove that there exists a large class of self-similar solutions.

Furioli [F] improved the result by Ribaud and Youssfi [RY1] on the existence of self-similar solutions for the nonlinear Schrödinger equations (7) extending of available nonlinearities $\alpha + 1$ to α smaller than 1.

Also Cazenave, Vega and Vilela ([CVeVi]) studied the global Cauchy problem for the equation (7). Using a generalization of the Strichartz's estimates for the Schrödinger equation (see Theorem 1.7) they showed that, under some restrictions on α, if the initial value is sufficiently small in some weak L^p space then there exists a global solution. This result provided a common framework to the classical H^s solutions and to self-similar solutions. As we already mentioned we use their ideas in our work. From the condition $m > 0$ we are allowed to reduce the Davey-Stewartson system (1) to the Schrödinger equation in (4). Now comparing both Schrödinger equations in (4) and (7) we observe that we have the nonlocal term $buE(|u|^2)$ to treat. The main ingredient to do that will be an interpolation theorem and the generalization of the Strichartz's estimates for the Schrödinger equation derivated in [CVeVi]. As a consequence, we prove that the Cauchy problem (4) is globally well posed in the sense of distribution for $n = 2$ and 3. The existence of self-similar solutions will then be a direct consequence of the global well posedness.

The second system is the degenerated Zakharov system

$$\begin{cases} i(\partial_t E + i\partial_z E) + \Delta_\perp E &= nE, \\ \partial_t^2 n - \Delta_\perp n &= \Delta_\perp(|E|^2), \\ E(x,y,z,0) &= E_0(x,y,z), \qquad \forall\, (x,y,z) \in \mathbb{R}^3,\, t > 0, \\ n(x,y,z,0) &= n_0(x,y,z), \\ \partial_t n(x,y,z,0) &= n_1(x,y,z), \end{cases} \qquad (8)$$

where $\Delta_\perp = \partial_x^2 + \partial_y^2$, E is a complex valued function, and n is a real valued function.

The system (8) describes the laser propagation when the paraxial approximation is used and the effect of the group velocity is negligible. We use the term degenerate in the sense that there is no dispersive term in the space variable z in the first equation.

The IVP (8) is one variation of the following system:

$$\begin{cases} i\partial_t E + \Delta E &= nE, \\ \partial_t^2 n - \Delta n &= \Delta(|E|^2), \\ E(0) &= E_0, \qquad \forall\, t > 0, \\ n(0) &= n_0, \\ \partial_t n(0) &= n_1. \end{cases} \tag{9}$$

The system (9) was introduced in [Z] to describe the long wave Lungmuir turbulence in a plasma.

In [CC], Colin and Colin posed the question of the well-posedness of the IVP (8). In [LiPoS], Linares, Ponce and Saut answered this question showing the local well-posedness result of the IVP system (8) in a suitable Sobolev spaces (see explanations below). The results proved in [LiPoS] extended previous ones for the Zakharov system (9), where transversal dispersion is taken into account (see [OT], [GTV] and references therein). However, the system (8) is quite different from the classical Zakharov system (9) since the cauchy problem for the periodic data exhibits strong instabilities of the Hadamard type implying ill-posedness (see [CM]).

Since our result is an improvement of the local well posedness result in [LiPoS] we now explain with more details their ideas.

At first we reduce the IVP (8) into an IVP associated to a single equation, that is,

$$\begin{cases} i(\partial_t E + \partial_z E) + \Delta_\perp E &= nE, \\ E(x,y,z,0) &= E_0(x,y,z), \end{cases} \qquad \forall\, (x,y,z) \in \mathbb{R}^3,\ t > 0, \tag{10}$$

where

$$n(t) = N'(t)n_0 + N(t)n_1 + \int_0^t N(t-t')\Delta_\perp(|E(t')|^2)dt',$$

with

$$N(t)f = (-\Delta_\perp)^{-1/2}\sin((-\Delta_\perp)^{1/2}t)f, \tag{11}$$

and

$$N'(t)f = \cos((\Delta_\perp)^{1/2}t)f, \tag{12}$$

where $(-\Delta_\perp)^{1/2} f = ((\xi_1^2 + \xi_2^2)^{1/2} \hat{f})^\vee$.

Then we consider the integral equivalent formulation of IVP (10), that is,

$$
\begin{aligned}
E(t) =& \mathcal{E}(t)E_0 + \int_0^t \mathcal{E}(t-t')(N'(t')n_0 + N(t')n_1)E(t')dt' \\
&+ \int \mathcal{E}(t-t')(\int_0^{t'} N(t'-s)\Delta_\perp(|E(s)|^2)ds)E(t')dt',
\end{aligned}
\tag{13}
$$

where

$$
\mathcal{E}(t)E_0 = (e^{-it(\xi_1^2 + \xi_2^2 + \xi_3)} \widehat{E_0}(\xi_1, \xi_2, \xi_3))^\vee
\tag{14}
$$

is the solution of the linear problem associated to (10).

Observing that the linear equation in (10) is almost a linear Schrödinger equation (but not quite due to the propagation on the $z-$direction), [LiPoS] proved similar smoothing effects for the operator $\mathcal{E}(t)$ as those of the Schrödinger propagator.

Using these results for the operator $\mathcal{E}(t)$ and properties of the wave operators $N(t)$ and $N'(t)$, they proved that the integral operator (13) is a contraction in the following space:

$$
C([0,T] : \widetilde{H}^{2j+1}(\mathbb{R}^3)), \ j \geqslant 2,
$$

where

$$
\widetilde{H}^{2j+1}(\mathbb{R}^3) = \{f \in H^{2j+1}(\mathbb{R}^3), D_x^{1/2}\partial^\alpha f, \ D_y^{1/2}\partial^\alpha f \in L^2(\mathbb{R}^3), \ |\alpha| \leqslant 2j+1, \ j \in \mathbb{N}\}, \tag{15}
$$

$$
\widehat{\partial_x^\alpha f}(\xi) = (2\pi i\xi)^\alpha \hat{f}(\xi), \tag{16}
$$

$$
D_x^{1/2}f = (|\xi_1|^{1/2}\hat{f})^\vee \text{ and } D_y^{1/2}f = (|\xi_2|^{1/2}\hat{f})^\vee. \tag{17}
$$

Now we state the theorem proved in [LiPoS]:

Theorem 0.2. *For initial data (E_0, n_0, n_1) in $\widetilde{H}^{2j+1}(\mathbb{R}^3) \times H^{2j+1}(\mathbb{R}^3) \times H^{2j}(\mathbb{R}^3)$ and $\partial_z n_1 \in H^{2j}(\mathbb{R}^3)$, $j \in \mathbb{N}$, $j \geqslant 2$, there exist $T > 0$ and a unique solution E of the integral equation*

(13) *such that*

$$E \in C([0,T] : \widetilde{H}^{2j+1}(\mathbb{R}^3)), \tag{18}$$

$$\sum_{|\alpha| \leqslant 2j+1} \|D_x^{1/2} \partial^\alpha E\|_{L_x^\infty L_{yzT}^2} < \infty, \tag{19}$$
and

$$\sum_{|\alpha| \leqslant 2j+1} \|D_y^{1/2} \partial^\alpha E\|_{L_y^\infty L_{xzT}^2} < \infty. \tag{20}$$

Moreover, for $T' \in (0,T)$, *the map* $(E_0, n_0, n_1) \mapsto E(t)$ *from* $\widetilde{H}^{2j+1}(\mathbb{R}^3) \times H^{2j+1}(\mathbb{R}^3) \times H^{2j}(\mathbb{R}^3)$ *into the class defined by* (18)-(20) *is Lipschitz.*

From (18)-(20) *one also has that*

$$n \in C([0,T] : H^{2j+1}(\mathbb{R}^3)).$$

Proof. We refer to [LiPoS] for a proof of this theorem. $\qquad\square$

In the present work, we intend to improve Theorem 0.2. To do so, we establish the following maximal function type estimates for the solution of the linear IVP associated with the system (10):

$$\|\mathcal{E}(t)E_0\|_{L_x^2 L_{yzT}^\infty} \leqslant c(T,s)\|E_0\|_{H^s(\mathbb{R}^3)}, \ s > 3/2. \tag{21}$$

The argument to prove (21) follows the ideas in [KZ], where they obtained a L_x^4-maximal function estimates for solutions of the linear problem associated to the modified Kadomttsev-Petviashvili (KP) equation. The estimate (21) improves the following one

$$\|\mathcal{E}(t)E_0\|_{L_x^2 L_{yzT}^\infty} \leqslant c(1+T)\|E_0\|_{H^4(\mathbb{R}^3)}$$

obtained in [LiPoS], using just Sobolev embedding.

This estimate enable us to improve Theorem 0.2.

Finally, we stablish a conection between the two problems. To do that, we consider the Zakharov-Rubenchik system

$$\begin{cases} i(\partial_t\psi + v_g\partial_z\psi) + \frac{w''}{2}\partial_z^2\psi + \frac{v_g}{2k_0}\Delta_\perp\psi = (q|\psi|^2 + \beta\rho + \alpha\partial_z\varphi)\psi, \\ \partial_t\rho + \rho_{00}\Delta\varphi + \alpha\partial_z|\psi|^2 = 0, \\ \partial\varphi + \frac{c_s^2}{\rho_{00}}\rho + \beta|\varphi|^2 = 0. \end{cases} \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \quad (22)$$

where ψ denotes the complex amplitude of the carrying wave whose wave number k and frequency w are related by the dispersion relation $w = w(k)$. $v_g = w'(k)$ is the group velocity of the carrying wave. The functions ρ and φ denote the density fluctuation and the hydrodynamic potential respectively. The parameters q and α, measure the self-interaction of the carrying wave and the Doppler shift respectively. $c_s = \sqrt{p'(\rho_{00})}$ is the sound velocity and $\beta = \frac{\partial w(k_0)}{\partial \rho}$ is related to the enthalpy.

According to Zakharov and Kutnetsov [ZK] if we proceed formally from (22), we can obtain limits system of Zakharov and Davey-Stewartson type. In fact, the two limits systems are

$$\begin{cases} i(\partial_t\psi + v_g\partial_z\psi) + \frac{w''}{2}\partial_z^2\psi + \frac{v_g}{2k_0}\Delta_\perp\psi = \beta\rho\psi, \\ \partial_t\rho - c_s^2\Delta\rho = \rho_{00}\beta\Delta|\psi|^2, \end{cases} \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \quad (23)$$

and

$$\begin{cases} i\partial_t\psi + \frac{w''}{2}\partial_z^2\psi + \frac{v_g}{2k_0}\Delta_\perp\psi = q|\psi|^2\psi + \frac{\alpha c_s^2}{\rho_{00}v_g}\rho\psi, \\ \partial_t\rho - c_s^2\Delta\rho = \rho_{00}\beta\Delta|\psi|^2, \end{cases} \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R},$$

respectively. We notice that doing the following change of variable in the system (23)

$$\tilde{z} = z - v_g t,$$

we obtain the Zakharov system in (9)

$$\begin{cases} i\partial_t\psi + \frac{w''}{2}\partial_z^2\psi + \frac{v_g}{2k_0}\Delta_\perp\psi = \beta\rho\psi, \\ (\partial_t - v_g\partial_z)^2\rho - c_s^2\Delta\rho = \rho_{00}\beta\Delta|\psi|^2. \end{cases} \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R},$$

In the first chapter we describe the notations, define the functional spaces we will work as well functions, distributions and operators that appear in the next chapters. Also we give some well known results that will be used along this work.

Next, in the second chapter, we prove that the Cauchy problem (1) is globally well-posed in some Lorentz space and find self-similar solutions.

Hereafter, we refer to the expression " well-posedness " in the following sense:

Definition 0.3. *Let $(X; \|\cdot\|)$ be a Banach space. We will say that the Cauchy problem (4) is locally well posed in X if for all $u_0 \in X$ there is $T = T(\|u_0\|) > 0$ and a unique solution u to (4) such that*

1. *$u \in C([-T, T] : X)$ and*

2. *$F : X \to C([-T, T] : X), \ F(u_0) = u$ is continuous.*

If 1 and 2 hold for any $T > 0$, we say that (4) is globally well posed in X.

Finally, in the third chapter, we prove that the IVP (8) is locally well-posed in the Sobolev space H^3, improving Theorem 0.2.

Chapter 1

Preliminaries

1.1   Notations

1. We will use the standard multi-index notation. A multi-index $\beta = (\beta_1, \ldots, \beta_n)$ is a n-tuple of nonnegative integers. Given $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we define the symbols

$$x^\beta := x_1^\beta \ldots x_n^\beta,$$

and the order of β

$$|\beta| := \sum_{i=1}^{n} \beta_i.$$

For multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ we define

Partial order:

$$\beta \leqslant \alpha \Leftrightarrow \beta_i \leqslant \alpha_i \quad \forall i \in 1, \ldots, n$$

and Binomial coefficient

$$\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}.$$

2. For a complex number z with $Re z > 0$ define

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

$\Gamma(z)$ is called the gamma function.

3. Given $f : \mathbb{R}^n \to \mathbb{C}$, $\bar{f}$ will denote the complex conjugate of f.

4. The characteristic function of an interval $I \subset \mathbb{R}$ is defined as

$$\chi_I(x) = \begin{cases} 1 \text{ if } x \in I \\ 0 \text{ if } x \notin I. \end{cases}$$

1.2 Functional Spaces, the Fourier Transform and the Hilbert Transform

The Lebesgue spaces

Let $1 \leqslant p \leqslant \infty$. We define $L^p(\mathbb{R}^n)$ as the set of all measurable functions from $\mathbb{R}^n$ to $\mathbb{C}$ such that

$$\|f\|_{L^p} := \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} < \infty \text{ for } p < \infty,$$

and

$$\|f\|_{L^\infty} := \inf\{c > 0; |f(x)| \leqslant c \text{ for almost every } x\}.$$

The mixed "space-time" Lebesgue spaces are defined by

$$L_t^q L_x^p := \{u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C} \text{ measurable } ; \ \|u\|_{L_t^q L_x^p} < \infty\},$$

where

$$\|u\|_{L_t^q L_x^p} := \left(\int_{\mathbb{R}} \|u(\cdot, t)\|_{L^p(\mathbb{R}^n)}^q dt \right)^{1/q}. \tag{1.1}$$

The Fourier Transform

The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$, denoted by $\hat{f}$, is defined as:

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx, \quad \text{for } \xi \in \mathbb{R}^n,$$

where $x \cdot \xi = x_1 \xi_1 + \cdots + x_n \xi_n$.

$\check{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{2\pi i x \cdot \xi} dx$ is the inverse of the Fourier transform. Throughout this work, the symbol $\hat{\ }$ denote the Fourier transform in the space variable.

An important propertie of the Fourier transform in the Lebesgue space L^2 is given in the following theorem

Theorem 1.1 (Plancherel). *Let $f \in L^2$. Then $\hat{f} \in L^2$ and*

$$\|\hat{f}\|_{L^2} = \|f\|_{L^2}.$$

Proof. We refer to [G] for a proof of this theorem. □

The Schwartz class

The Schwartz class denoted by $S(\mathbb{R}^n)$, is defined as

$$S(\mathbb{R}^n) = \{\varphi \in C^\infty(\mathbb{R}^n); \|f\|_{\nu\beta} := \|x^\nu \partial_x^\beta f\|_{L^\infty} < \infty \text{ for any } \nu, \beta \in (\mathbb{N})^n\}.$$

The topology in $S(\mathbb{R}^n)$ is that induced by the family of semi-norms $\{\|\cdot\|_{\nu\beta}\}_{(\nu,\beta) \in \mathbb{N}^{2n}}$.

The next lemma establishes a relationship between the Fourier transform and the function space $S(\mathbb{R}^n)$:

Lemma 1.2. *The Fourier transform is a homeomorphism from $S(\mathbb{R}^n)$ onto itself.*

Proof. We refer to [G] for a proof of this lemma. □

Finally we list some properties of Fourier Transform in Schwartz class :

(1) For $f \in S(\mathbb{R}^n)$, we can use Fourier Transform to define derivatives as:

  (a) derivatives for multiindices

$$\partial^\alpha f = ((2\pi i \xi)^\alpha \hat{f})^\vee, \quad \alpha \in \mathbb{N}^n,$$

  and

  (b) fractional derivatives, i.e.,

$$D^l f = ((2\pi |\xi|)^l \hat{f})^\vee, \quad l \in \mathbb{R}.$$

(2) Let f and $g \in S(\mathbb{R}^n)$. Then we have that $f * g \in S(\mathbb{R}^n)$ and

$$\widehat{fg} = \hat{f} * \hat{g}. \tag{1.2}$$

(3) If $\tau_h f(x) = f(x - h)$ denotes the translation by $h \in \mathbb{R}^n$, then

$$(\tau_{-h}\hat{f})(\xi) = (e^{-\widehat{2\pi i(h \cdot x)}}f)(\xi). \tag{1.3}$$

The Hilbert Transform

For $\varphi \in S(\mathbb{R}^n)$ we define its Hilbert transform $H(\varphi)$ by

$$\widehat{H(\varphi)}(\xi) = -i sgn(\xi)\hat{\varphi}(\xi).$$

It follows direct from the definition that

$$H(H(\varphi)) = -\varphi. \tag{1.4}$$

By Plancherel we can extend the Hilbert transform as an isometry in $L^2(\mathbb{R}^n)$, i.e.,

$$\|H(\varphi)\|_{L^2} = \|\varphi\|_{L^2}. \tag{1.5}$$

Using Hilbert transform we can stablish the following relationship between ∂_x and D_x:

Lemma 1.3. *Given* $\varphi \in S(\mathbb{R})$ *we have that*

$$\partial_x \varphi = D_x^{1/2} \tilde{D}_x^{1/2} \varphi,$$

where $\tilde{D}_x^{1/2} = -2\pi H D_x^{1/2}$.

Proof. By the definition of H we see that

$$H\partial_x\varphi = 2\pi D_x\varphi. \tag{1.6}$$

In fact

$$\widehat{H\partial_x\varphi} = -isgn(\xi)2\pi i\xi\hat{\varphi} = 2\pi|\xi|\hat{\varphi} = 2\pi\widehat{D_x\varphi}.$$

Now by properties (1.4) and (1.6) we obtain

$$\partial_x\varphi = -H(H(\partial_x\varphi)) = -2\pi H(D_x\varphi) = -2\pi H(D_x^{1/2}D_x^{1/2}\varphi) = \tilde{D}_x^{1/2}D_x^{1/2}\varphi.$$

$$\square$$

Tempered Distributions

We say that a linear functional $\Psi : S(\mathbb{R}^n) \to \mathbb{C}$ defines a tempered distribution if Ψ is continuous. We denote $S'(\mathbb{R}^n)$ as the set of all tempered distributions. We will use the symbol $\langle \cdot, \phi \rangle$ to denote the value of Ψ on $\phi \in S(\mathbb{R}^n)$.

Examples of Tempered distributions

Given $f \in L^p(\mathbb{R}^n), 1 \leqslant p \leqslant \infty$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$:

1. We associate with f the distribution T_f whose value on $\phi \in S(\mathbb{R}^n)$ is given by

$$T_f(\phi) = \langle f, \phi \rangle := \int_{\mathbb{R}^n} f(x)\phi(x)dx.$$

2. We can define the diferential of f in the distribution sense as:

$$\partial_x^\beta f : S(\mathbb{R}^n) \to \mathbb{C}$$

$$\langle \partial_x^\beta f, \phi \rangle = (-1)^{|\beta|} \langle f, \partial_x^\beta \phi \rangle,$$

where $\partial_x^\beta \phi = \partial_{x_1}^{\beta_1} \ldots \partial_{x_n}^{\beta_n} \phi$.

3. Let $T \in S'(\mathbb{R}^n)$. We define the Fourier transform $\hat{T}$ of a tempered distribution T by

$$\langle \hat{T}, \phi \rangle = \langle T, \hat{\phi} \rangle \quad \forall \phi \in S(\mathbb{R}^n).$$

Also, we have the following extension of lemma 1.2 in $S'(\mathbb{R}^n)$:

Lemma 1.4. *The Fourier transform is a isomorphism from $S'(\mathbb{R}^n)$ into itself.*

Proof. We refer to [LP2] for a proof of this lemma. $\qquad \square$

From Lemma 1.4 we can get the following computation related with the fundamental solution of the evolution Schrödinger equation:

$$\left(e^{-\widehat{4\pi^2 it|x|^2}}\right)(\xi) = \lim_{\substack{\epsilon \to 0^+ \\ S'(\mathbb{R}^n)}} \left(e^{-\widehat{4\pi^2(\epsilon + it)|x|^2}}\right)(\xi) = \frac{e^{i|\xi|^2/4t}}{(4\pi it)^{n/2}} \tag{1.7}$$

The Sobolev spaces

We will also use the fractional Sobolev spaces. Let $s \in \mathbb{R}$, then

$$H^s(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) \ : \ (1 + |\xi|^2)^{s/2}\widehat{f}(\xi) \in L^2(\mathbb{R}^n)\}$$

with the norm

$$\|f\|_{H^s} := \|(1 + |\xi|^2)^{s/2}\widehat{f}(\xi)\|_{L^2},$$

and its homogeneous version

$$\dot{H}^s(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) \ : \ |\xi|^s\widehat{f}(\xi) \in L^2(\mathbb{R}^n)\}$$

with the norm

$$\|f\|_{\dot{H}^s} := \||\xi|^s\widehat{f}(\xi)\|_{L^2}. \tag{1.8}$$

The weighted Sobolev spaces denoted by $H^{m,l}(\mathbb{R}^n)$ are defined as follows:

$$H^{m,l}(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n); \|f\|_{H^{m,l}} = \|(1 - \Delta)^{m/2}(1 + |x|^2)^{l/2}f\|_{L^2} < \infty\},$$

where $(1 - \Delta)^{m/2}f = ((1 - |\xi|^2)^{m/2}\hat{f})^{\vee}$.

Analogously we define the weighted homogeneous Sobolev spaces $\dot{H}^{m,l}(\mathbb{R}^n)$:

$$\dot{H}^{m,l}(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n); \|f\|_{\dot{H}^{m,l}} = \|(1 - \Delta)^{m/2}|x|^l f\|_{L^2} < \infty\}.$$

The Lorentz spaces

The next spaces were introduced by Lorentz ([L1], [L2]) and generalizes the L^p spaces:

Definition 1.5. *The Lorentz space $L^{pq}(\mathbb{R}^n), 1 \leqslant p, q \leqslant \infty$, is defined as follows:*

$L^{pq}(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} \ measurable \ ; \|f\|_{L^{pq}(\mathbb{R}^n)} := \left(\int_0^\infty \left(t^{1/p}f^*(t)\right)^q \frac{1}{t}dt\right)^{1/q} < \infty\} \ for \ q < \infty$

and

$L^{p\infty}(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} \ measurable \ ; \|f\|_{L^{p\infty}(\mathbb{R}^n)} := \sup_{\lambda > 0} \lambda\alpha(\lambda, f)^{1/p} < \infty\}$

where

$$f^*(t) = \inf_{\lambda > 0}\{\alpha(\lambda, f) \leqslant t\},$$

$$\alpha(\lambda, f) = \mu(\{x \in \mathbb{R}^n; |f(x)| > \lambda\}),$$

and

$$\mu = Lebesgue \ measure.$$

The function $\alpha(\lambda, f)$ is called distribution function.

In Chapter 2 the $L^{p\infty}$ spaces will be particularly relevant in our analysis. They are also called weak L^p spaces. For more information about Lorentz spaces we refer to [BeL].

1.3 Important Theorems

Now we present some facts in Lorentz spaces. The next theorem establishes a relationship between Lorentz Spaces $L^{p\infty}$ and L^q spaces:

Theorem 1.6 (Interpolation's theorem). *Given $0 < p_0 < p_1 \leqslant \infty$, then for all p, q and θ such that $p_0 < q \leqslant \infty$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $0 < \theta < 1$ we have :*

$$(L^{p_0}, L^{p_1})_{\theta,q} = L^{pq} \quad with \quad \|f\|_{(L^{p_0}, L^{p_1})_{\theta,q}} = \|f\|_{L^{pq}},$$

where

$$(L^{p_0}, L^{p_1})_{\theta,q} = \{a \ Lebesgue \ measurable; \|a\|_{(L^{p_0}, L^{p_1})_{\theta,q}} := \left(\int_0^\infty t^{-\theta} k(t, a)^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty\}, q < \infty,$$

$$(L^{p_0}, L^{p_1})_{\theta,\infty} = \{a \ Lebesgue \ measurable; \|a\|_{(L^{p_0}, L^{p_1})_{\theta,\infty}} := \sup_{t>0} t^{-\theta} k(t, a) < \infty\}$$

and

$$k(t, a) = \inf_{a = a_0 + a_1} (\|a_0\|_{L^{p_0}} + t\|a_1\|_{L^{p_1}}).$$

Proof. We refer to [BeL] for a proof of this theorem. □

Another relationship between Lorentz Spaces and L^p spaces is given by the following decomposition:

Let $1 \leqslant p_1 < p < p_2 < \infty$. Then

$$L^{p\infty} = L^{p_1} + L^{p_2}. \tag{1.9}$$

The next theorem is a generalization of the classical Strichartz estimates for the Schrödinger equation. The proof is based on ideas developed by Keel and Tao [KT].

Theorem 1.7. *Consider* $r, \tilde{r}, q$ *and* $\tilde{q}$ *such that*

$$2 < r, \tilde{r} \leqslant \infty, \quad \frac{1}{\tilde{r}'} - \frac{1}{r} < \frac{2}{n},$$

$$\frac{1}{\tilde{q}'} - \frac{1}{q} + \frac{n}{2}\left(\frac{1}{\tilde{r}'} - \frac{1}{r}\right) = 1, \tag{1.10}$$

$$\begin{cases} r, \tilde{r} \neq \infty & if \quad n = 2, \\ \dfrac{n-2}{n}\left(1 - \dfrac{1}{\tilde{r}'}\right) \leqslant \dfrac{1}{r} \leqslant \dfrac{n}{n-2}\left(1 - \dfrac{1}{\tilde{r}'}\right) & if \quad n \geqslant 3, \end{cases} \tag{1.11}$$

and

$$\begin{cases} 0 < \dfrac{1}{q} \leqslant \dfrac{1}{\tilde{q}'} < 1 - \dfrac{n}{2}\left(\dfrac{1}{\tilde{r}'} + \dfrac{1}{r} - 1\right) & if \quad \dfrac{1}{\tilde{r}'} + \dfrac{1}{r} \geqslant 1, \\ -\dfrac{n}{2}\left(\dfrac{1}{\tilde{r}'} + \dfrac{1}{r} - 1\right) < \dfrac{1}{q} \leqslant \dfrac{1}{\tilde{q}'} < 1 & if \quad \dfrac{1}{\tilde{r}'} + \dfrac{1}{r} < 1. \end{cases} \tag{1.12}$$

Then we have the following inequalities:

$$\| \int_0^t e^{i(t-\tau)\Delta} F(\cdot, \tau) d\tau \|_{L_t^q L_x^r} \leqslant c \| F \|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}, \tag{1.13}$$

$$\| \int_{-\infty}^t e^{i(t-\tau)\Delta} F(\cdot, \tau) d\tau \|_{L_t^q L_x^r} \leqslant c \| F \|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}, \tag{1.14}$$

$$\| \int_{-\infty}^{+\infty} e^{i(t-\tau)\Delta} F(\cdot, \tau) d\tau \|_{L_t^q L_x^r} \leqslant c \| F \|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}. \tag{1.15}$$

Proof. We refer to [McV] for a proof of this theorem. □

Now we turn our attention to some inequalities on the Lorentz spaces semi-norm $\| \cdot \|_{L^{p\infty}(\mathbb{R}^{n+1})}$ that we will use in Chapter 2:

Lemma 1.8. *Let $1 < p, q, r < \infty$. The following estimates hold:*

$$|f| < |g| \Rightarrow \|f\|_{L^{p\infty}(\mathbb{R}^n)} \leqslant \|g\|_{L^{p\infty}(\mathbb{R}^n)}, \tag{1.16}$$

$$\|f + g\|_{L^{p\infty}(\mathbb{R}^n)} \leqslant 2\left(\|f\|_{L^{p\infty}(\mathbb{R}^n)} + \|g\|_{L^{p\infty}(\mathbb{R}^n)}\right). \tag{1.17}$$

Moreover if $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, then

$$\|fg\|_{L^{r\infty}(\mathbb{R}^n)} \leqslant \|f\|_{L^{p\infty}(\mathbb{R}^n)}\|g\|_{L^{q\infty}(\mathbb{R}^n)}. \tag{1.18}$$

Proof. The inequality (1.16) follows directly from Definition 1.5.

The property (1.17) follows from Definition 1.5 observing that

$$\mu(\{x, |f(x)| > \frac{\lambda}{2}\}) + \mu(\{x, |g(x)| > \frac{\lambda}{2}\}) \geqslant \mu(\{x, |(f + g)(x)| > \lambda\}).$$

For the proof of inequality (1.18) we refer to [O]. $\qquad\square$

Remark 1.9. *From inequality (1.17) in Lemma 1.8 we see that $\|\cdot\|_{L^{p\infty}}$ is not a norm, but a quasi-norm, i.e., it only satisfies a quasi-triangular inequality. On the other hand, the spaces $\|\cdot\|_{L^{p\infty}}$ are complete with respect to their quasi-norm and they are therefore quasi Banach spaces. Moreover, for $p > 1$ it is possible to replace this quasi-norm to a norm such that $\|\cdot\|_{L^{p\infty}}$ become Banach. We refer to [G] for more details.*

The next result will be important in Section 2.3 to find self-similar solutions:

Proposition 1.10. *Let $\varphi(x) = |x|^{-p}$ where $0 < \operatorname{Re} p < n$. Then $e^{it\Delta}\varphi$ is given by the*

explicit formula below for $x \neq 0$ and $t > 0$:

$$
e^{it\Delta}\varphi(x) = |x|^{-p} \sum_{k=0}^{m} A_k(a,b) e^{k\pi i/2} \left(\frac{|x|^2}{4t} \right)^{-k}
$$

$$
+ |x|^{-p} A_{m+1}(a,b) \left(\frac{|x|^2}{4t} \right)^{-m-1} \frac{(m+1)e^{aki/2}}{\Gamma(m+2-b)}
$$

$$
\times \int_0^{\infty} \int_0^1 (1-s)^m \left(-i - \frac{4ts\tau}{|x|^2} \right)^{-a-m-1} e^{-\tau} \tau^{m+1-b} ds\, d\tau
$$

$$
+ e^{i|x|^2/4t} |x|^{-n+p} (4t)^{\frac{n}{2}-p} \sum_{k=0}^{l} B_k(b,a) e^{-(n+2k)\pi i/4} \left(\frac{|x|^2}{4t} \right)^{-k}
$$

$$
+ e^{i|x|^2/4t} |x|^{-n+p} (4t)^{\frac{n}{2}-p} B_{l+1}(b,a) \left(\frac{|x|^2}{4t} \right)^{-l-1} \frac{(l+1)e^{aki/2}}{\Gamma(l+2-b)}
$$

$$
\times \int_0^{\infty} \int_0^1 (1-s)^l \left(-i - \frac{4ts\tau}{|x|^2} \right)^{-b-l-1} e^{-\tau} \tau^{l+1-a} ds\, d\tau,
$$

where $a = p/2, b = (n-p)/2, m, l \in \mathbb{N}$ such that $m + 2 > \operatorname{Re} b$ and $l + 2 > \operatorname{Re} a$

and

$$
A_k(a,b) = \frac{\Gamma(a+k)\Gamma(k+1-b)}{\Gamma(a)\Gamma(1-b)k!}, \quad B_k(b,a) = \frac{\Gamma(b+k)\Gamma(k+1-a)}{\Gamma(a)\Gamma(1-a)k!}
$$

where Γ denotes the gamma function.

Proof. We refer to [CW1] for a proof of this proposition. $\qquad \square$

Theorem 1.11 (Dominated Convergence theorem)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space and $\{f_n\}$ be a sequence of $\mathcal{A}$-measurable functions from $(X, \mathcal{A}, \mu)$ to $\mathbb{R}$. Let g be a $\mathcal{A}$-measurable nonnegative integrable function on $(X, \mathcal{A}, \mu)$. Suppose that for each s in X,*

i. $|f_n(s)| \leqslant g(s)$ for all $n \geqslant 1$ and

ii. $\lim_{n \to \infty} f_n(s) = f(s)$.

Then

$$
\lim_{n \to \infty} \int_X f_n(x) d\mu = \int_X f(x) d\mu.
$$

Proof. We refer to [I] for a proof of this theorem. □

Theorem 1.12 (Lebesgue Differentiation Theorem). *If $f \in L^1_{loc}(\mathbb{R}^n)$ then*

$$lim_{r \to 0^+} \frac{1}{|B_r|} \int_{B_r} f(x - y) dy = f(x) \ a.e.,$$

where $B_r = \{y \in \mathbb{R}^n; |y - x| < r\}$ and $L^1_{loc}(\mathbb{R}^n) = \{f; \chi_{B_r} f \in L^1\}$.

Proof. We refer to [G] for a proof of this lemma. □

In Chapter 3 we will need the following results:

Lemma 1.13. *If a and b are nonnegative real numbers and p and q are real positive numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ then we have*

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. We consider an auxiliary function ϕ defined for all positive real numbers t by

$$\phi(t) = \frac{t^p}{p} + \frac{t^{-q}}{q}.$$

It is easy to see that

$$\phi'(t) = t^{p-1} - t^{-q-1} \Rightarrow 1 \text{ is the only critical value of } \phi \text{ (in the domain of } \phi\text{)} \qquad (1.19)$$

and

$$\lim_{t \to 0} \phi(t) = \lim_{t \to \infty} \phi(t) = \infty. \qquad (1.20)$$

From (1.19) and (1.20) we have

$$\frac{t^p}{p} + \frac{t^{-q}}{q} = \phi(t) \geqslant \phi(1) = 1. \qquad (1.21)$$

The result follows taking $t = \frac{a^{1/q}}{b^{1/p}}$ where a and b are positive numbers. □

Lemma 1.14 (Holder's inequality). *Let* $1 \leqslant p, q \leqslant \infty$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Given* $f \in L^p, g \in L^q$ *we have* $fg \in L^1$ *and*

$$\|fg\|_{L^1} \leqslant \|f\|_{L^p} \|g\|_{L^q}.$$

Proof. We refer to [I]. $\qquad\square$

Lemma 1.15 (Young's inequality). *Let* $1 \leqslant p, q, r \leqslant \infty$ *satisfy* $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}$. *Then for all* $f \in L^p(\mathbb{R}^n)$ *and all* $g \in L^r(\mathbb{R}^n)$ *we have*

$$\|f * g\|_{L^q} \leqslant \|g\|_{L^r} \|f\|_{L^p}.$$

Proof. We refer to [G]. $\qquad\square$

Lemma 1.16 (Leibiniz' Rule). *Let* $\alpha \in \mathbb{N}^n$ *be a multi-index and* $f, g \in C^{|\alpha|}(\mathbb{R}^n)$. *Then*

$$\partial_x^\alpha(fg) = \sum_{\substack{\beta \in \mathbb{N}^n \\ \beta \leqslant \alpha}} \binom{\alpha}{\beta} \partial_x^{\alpha-\beta} f \partial_x^\beta g. \tag{1.22}$$

Proof. The identity (1.22) is deduced by repeated aplication of the one dimension Leibiniz rule

$$\partial_x^m(fg) = \sum_{k=0}^m \binom{m}{k} \partial_x^k f \partial_x^{m-k} g,$$

and induction. $\qquad\square$

Lemma 1.17 (Fractionary Leibiniz rule).

$$\|D_x^{1/2}(fg)\|_{L_x^2} \leqslant \|D_x^{1/2} f\|_{L_x^4} \|g\|_{L_x^4} + \|f\|_{L_x^\infty} \|D_x^{1/2} g\|_{L_x^2}, \ x \in \mathbb{R}. \tag{1.23}$$

Proof. We refer to [KPV] for a proof of this lemma. $\qquad\square$

Theorem 1.18 (Fubini's Theorem). *Let* $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{C}$ *be such that*

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} |f(x,y)| dx dy < \infty.$$

Then

$$\int_{\mathbb{R}^n} (\int_{\mathbb{R}^m} f(x,y) dx) dy = \int_{\mathbb{R}^m} (\int_{\mathbb{R}^n} f(x,y) dy) dx = \int_{\mathbb{R}^n \times \mathbb{R}^m} f(x,y) d(x,y).$$

Proof. We refer to [I] for a proof of this theorem.   □

Lemma 1.19 (Minkowski integral inequality). *If* $1 \leqslant p \leqslant \infty$*, then*

$$\left(\int_{\mathbb{R}^n} |\int_{\mathbb{R}^m} f(x,y)dx|^p dy\right)^{1/p} \leqslant \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} |f(x,y)|^p dy\right)^{1/p} dx.$$

Proof. Define $A(y) = \int_{\mathbb{R}^m} |f(x,y)|dx$ and let $1 \leqslant q \leqslant \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then by Fubini's theorem and Holder's inequality we have

$$\begin{aligned}
\int_{\mathbb{R}^n} A(y)^p dy &= \int_{\mathbb{R}^n} A(y)A(y)^{p/q}dy \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} |f(x,y)|dx\right)A(y)^{p/q}dy \\
&= \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} |f(x,y)|A(y)^{p/q}dy\right)dx \\
&\leqslant \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} |f(x,y)|^p dy\right)^{1/p}\left(\int_{\mathbb{R}^n} A(y)^p dy\right)^{1/q}dx \\
&= \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} |f(x,y)|^p dy\right)^{1/p}dx\left(\int_{\mathbb{R}^n} A(y)^p dy\right)^{1/q}.
\end{aligned}$$

Therefore

$$\left(\int_{\mathbb{R}^n} A(y)^p dy\right)^{1-1/q} \leqslant \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} |f(x,y)|^p dy\right)^{1/p}dx. \tag{1.24}$$

Finally from (1.24) and the definition of $A(y)$ we conclude

$$\left(\int_{\mathbb{R}^n} |\int_{\mathbb{R}^m} f(x,y)dx|^p dy\right)^{1/p} \leqslant \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} |f(x,y)|dx\right)^p dy\right)^{1/p} \leqslant \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} |f(x,y)|^p dy\right)^{1/p}dx.$$

                                                                                □

The next theorem gives a description of the Sobolev space H^k without using the Fourier transform whenever $k \in \mathbb{Z}^+$.

Theorem 1.20. *If k is a positive integer, then $H^k(\mathbb{R}^n)$ coincides with the space of functions $f \in L^2(\mathbb{R}^n)$ whose derivatives (in the distribution sense) $\partial_x^\alpha f$ belong to $L^2(\mathbb{R}^n)$ for every $\alpha \in (\mathbb{Z}^+)^n$ with $|\alpha| = \alpha_1 + \ldots + \alpha_n \leqslant k$,*

where $\widehat{\partial_x^\alpha f}(\xi) = (2\pi i\xi)^\alpha \hat{f}(\xi)$.

In this case the norms $\|f\|_{H^k}$ and $\sum_{|\alpha| \leqslant k} \|\partial_x^\alpha f\|_{L^2}$ are equivalents.

Proof. We refer To [LP2] for a proof of this theorem.   □

Theorem 1.21. *If $s > n/2$, then $H^s(\mathbb{R}^3)$ is an algebra with respect to the product of functions. That is, if f, $g \in H^s(\mathbb{R}^3)$, then $fg \in H^s(\mathbb{R}^3)$ with*

$$\|fg\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_{H^s}.$$

Proof. We refer to [LP2] for a proof of this theorem.   □

We also need the following embbeding results:

Theorem 1.22. *If $s > n/2$, then $H^s(\mathbb{R}^n)$ is continuously embedded in $C^\infty(\mathbb{R}^n)$, i.e.,*

$$\|f\|_{L^\infty} \lesssim c_s\|f\|_{H^s}.$$

Proof. We refer to [LP2] for a proof of this theorem.   □

Theorem 1.23. *If $s \in (0, n/2)$, then $H^s(\mathbb{R}^n)$ is continuously embedded in $L^p(\mathbb{R}^n)$ with $p = 2n/(n-2s)$. Moreover*

$$\|f\|_{L^p} \lesssim \|D^s f\|_{L^2} \lesssim \|f\|_{H^s},$$

where $D^l f = ((2\pi|\xi|^l)\hat{f})^\vee$.

Proof. We refer To [LP2] for a proof of this theorem..   □

Before stating the next result, due to Ginibre and Velo, we give some notations.

For any vector space $\mathcal{D}$, we denote by $\mathcal{D}^*$ its algebraic dual, by $\mathcal{L}(\mathcal{D}, X)$ the space of linear maps from $\mathcal{D}$ to some other vector space X, and by $\langle \psi, f \rangle_\mathcal{D}$ the pairing between $\mathcal{D}^*$ and $\mathcal{D}$ (with $f \in \mathcal{D}$ and $\psi \in \mathcal{D}^*$). If X is a Banach space, $\|\cdot\|_X$ will denote the norm in X.

Lemma 1.24. *Let $\mathcal{H}$ be a Hilbert space, X a Banach space, X^* the dual of X, and $\mathcal{D}$ a vector space densely contained in X. Let $T_1 \in \mathcal{L}(\mathcal{D}, \mathcal{H})$ and let $T_1^* \in \mathcal{L}(\mathcal{H}, \mathcal{D}^*)$ be its adjoint, defined by*

$$\langle T_1^* v, f \rangle_\mathcal{D} = \langle v, T_1 f \rangle, \quad \forall f \in \mathcal{D}, \ \forall v \in \mathcal{H},$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathcal{H}$. Then the following three conditions are equivalent.

(1) There exists a, $0 \leqslant a < \infty$ such that for all $f \in \mathcal{D}$,

$$\|T_1 f\|_{\mathcal{H}} \leqslant \|f\|_X.$$

(2) $\mathcal{R}(T_1^*) = \{T_1^*(v); \ \forall v \in \mathcal{H}\} \subset X^*$, and there is a , $0 \leqslant a < \infty$, such that for all $v \in \mathcal{H}$

$$\|T_1^* v\|_{X*} \leqslant a\|v\|_{\mathcal{H}}.$$

(3) $\mathcal{R}(T_1^* T_1) \subset X^*$ and there exists a, $0 \leqslant a < \infty$, such that for all $f \in D$,

$$\|T_1^* T_1 f\|_{X*} \leqslant a^2 \|f\|_X.$$

The constant a is the same in all three parts. If one of (all) those conditions is (are) satisfied, the operators T_1 and $T_1 T_1^*$ extend by continuity to bounded operators from X to H and from X to X^* respectively.

Proof. We refer to [GV] for a proof of this lemma. $\qquad \square$

Next, we state a result in interpolation of operators in mixed Lebesgue spaces of type $L_z^q L_{xy}^p$. It will be usefull to generate Strichartz estimates to the degenerated Zakharov system in the third chapter .

Theorem 1.25 (Riez-Thorin)**.** *Let $p_0 \neq p_1$ and $q_0 \neq q_1$. Let T be a bounded linear operator from $L_z^2 L_{xy}^{p_0}$ to $L_z^2 L_{xy}^{q_0}$ with norm M_0 and from $L_z^2 L_{xy}^{p_1}$ to $L_z^2 L_{xy}^{q_1}$ with norm M_1. Then T is bounded from $L_z^2 L_{xy}^{p_\theta}$ to $L_z^2 L_{xy}^{q_\theta}$ with norm M_θ such that*

$$M_\theta \leqslant M_0^{1-\theta} M_1^{\theta},$$

with

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \ \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Proof. We refer to [LP2] for a proof of this theorem. $\qquad \square$

In fact, the Riez-Thorin theorem appears in literature to L^p spaces but the proof of Theorem 1.25 is basically the same.

Theorem 1.26 (Hardy-Littlewood-Sobolev). *Let* $0 < \alpha < n$, $1 < p < q < \infty$, *with* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Then the Riez potential defined as*

$$I_\alpha f(x) = c_\alpha \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{(n-\alpha)}} dy,$$

is a bounded linear operator from $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$, *i.e.,*

$$\|I_\alpha(f)\|_{L^q} \leqslant c_{p,\alpha,n}\|f\|_{L^p}.$$

Proof. We refer to [LP2] for a proof of this theorem. $\qquad\square$

Chapter 2

Well Posedness for the Davey Stewartson System on Weak L^p

2.1 Introduction

Here we shall study the Cauchy problem for the Davey Stewartson systems:

$$\begin{cases} i\partial_t u + \delta\partial_{x_1}^2 u + \sum_{j=2}^n \partial_{x_j}^2 u = \chi|u|^\alpha u + bu\partial_{x_1}\varphi, \\ \partial_{x_1}^2\varphi + m\partial_{x_2}^2\varphi + \sum_{j=3}^n \partial_{x_j}^2\varphi = \partial_{x_1}(|u|^\alpha). \end{cases} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R} \text{ and } n = 2 \text{ or } 3, \quad (2.1)$$

where $u = u(x,t)$ is a complex-valued function and $\phi = \phi(x,t)$ is a real-valued function.

The exponent α is such that $\frac{4(n+1)}{n(n+2)} < \alpha < \frac{4(n+1)}{n^2}$, $n = 2,3$, the parameters χ and b are constants in $\mathbb{R}$, δ and m are real positive and we can consider δ, χ normalized in such a way that $|\delta| = |\chi| = 1$.

Solving the second equation in (2.1), we can express φ in terms of u and get

$$\partial_{x_1}\varphi = E(|u|^2),$$

where the operator $E = E_m$ is defined in Fourier variables by

$$\widehat{E(f)}(\xi) = \frac{\xi_1^2}{\xi_1^2 + m\xi_2^2 + \sum_{j=3}^n \xi_j^2}\hat{f}(\xi) = p(\xi)\hat{f}(\xi). \quad (2.2)$$

We can therefore reduce (2.1) to the nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \delta\partial_{x_1}^2 u + \sum_{j=2}^n \partial_{x_j}^2 u = \chi|u|^\alpha u + buE(|u|^\alpha), \\ u(x,0) = u_0(x). \end{cases} \qquad \forall\, x \in \mathbb{R}^n,\, t \in \mathbb{R}, \qquad (2.3)$$

We consider the equation (2.3) in its integral form

$$u(t) = U(t)u_0 + i\int_0^t U(t-s)(\chi|u|^\alpha u + buE(|u|^\alpha))(s)ds, \qquad (2.4)$$

where $U(t)u_0$ defined as

$$\widehat{U(t)u_0}(\xi) = e^{-it\psi(\xi)}\widehat{u_0}(\xi), \qquad (2.5)$$

$$\psi(\xi) = 4\pi^2\delta\xi_1^2 + 4\pi^2\sum_{j=2}^n \xi_j^2, \quad n = 2 \text{ or } 3, \qquad (2.6)$$

is the solution of the linear problem

$$\begin{cases} i\partial_t u + \delta\partial_{x_1}^2 u + \sum_{j=2}^n \partial_{x_j}^2 u = 0, \\ u(x,0) = u_0(x). \end{cases} \qquad \forall\, x \in \mathbb{R}^n,\, t \in \mathbb{R}, \qquad (2.7)$$

Remark 2.1. *Note that*

1. *$U(t)$ is the unitary group associated to the linear Schrödinger equation (2.7) (see, for example, [LP2] and references there in).*

2. *It follows directly from Lemma 1.2 that $U(t)(S'(\mathbb{R}^n)) \subset S'(\mathbb{R}^n)$.*

3. *The function $\alpha(\lambda, f)$ in Definition 1.5 has the following invariance with respect to the group $U(t)u_0(\xi) = (e^{-it\psi(\xi)}\widehat{u_0})^\vee(\xi)$:*

$$\alpha(\lambda, U(t+\tau)\varphi) = \int \chi_{\{(y,t)\in\mathbb{R}^n\times\mathbb{R};\,|(e^{-i(t+\tau)\psi(y)}\widehat{\varphi})^\vee|>\lambda\}}(x,s)dxds$$

$$= \int \chi_{\{(y,t)\in\mathbb{R}^n\times\mathbb{R};\,|(e^{-it\psi(y)}\widehat{\varphi})^\vee|>\lambda\}}(x,s)dxds = \alpha(\lambda, U(t)\varphi).$$

The last identity gives us the following result: $\forall \; \varphi \in S'(\mathbb{R}^n)$ *and* $\tau \in \mathbb{R}$:

$$\|U(t)\varphi\|_{L^{p\infty}(\mathbb{R}^{n+1})} = \|U(t+\tau)\varphi\|_{L^{p\infty}(\mathbb{R}^{n+1})}. \qquad (2.8)$$

In Section 2.2 we prove that the Cauchy problem (2.3) is globally well posed in the sense of distribution for $n = 2$ and 3 and in Section 2.3 we find self-similar solutions for (2.3).

2.2 Global Well Posedness

In this section we prove global well posedness for IVP (2.3) in the subspace $Y \subset S'(\mathbb{R}^n)$ where:

$$Y = \{\varphi \in S'(\mathbb{R}^n) : U(t)\varphi \in L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})\},$$

and

$$\|\varphi\|_Y = \|U(t)\varphi\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}.$$

Observe that by the identity (2.8), $U(t)$ is an isometry in Y, i.e. $\|\varphi\|_Y = \|U(t)\varphi\|_Y$.

We first state some properties to the operator E, defined in (2.2), that will be usefull to our main purpose:

The following result was proved by Xiangking (see [X]): given $1 < q < \infty$, E is strong (q, q), i.e. is a bounded operator from $L^q(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, and $\|E\| \leqslant 1$. It means that the following inequality holds for $1 < q < \infty$:

$$\|E(f)\|_{L^q(\mathbb{R}^n)} \leqslant \|f\|_{L^q(\mathbb{R}^n)}. \qquad (2.9)$$

Lemma 2.2. *The operator* $E : L^p(\mathbb{R}^n) \subset S'(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \subset S'(\mathbb{R}^n)$ *defined in (2.2) is injective for* $1 \leqslant p \leqslant \infty$ *and* $n \geqslant 2$.

Proof. By lemma 1.4 and the fact that E is a linear operator is enough to prove that

$$\widehat{E(f)} = 0 \text{ in } S'(\mathbb{R}^n) \Rightarrow f = 0 \quad \text{Lebesgue-qtp.}$$

$\widehat{E(f)} = 0$ in $S'(\mathbb{R}^n)$ means that

$$\langle f, (\widehat{p(\xi)\phi}) \rangle = 0 \quad \forall \phi \in S(\mathbb{R}^n).$$

By lemma 1.2 is enough to prove that

$$\langle f, p(\xi)\phi \rangle = \int_{\mathbb{R}^n} f(\xi)p(\xi)\phi(\xi)d\xi = 0 \quad \forall \phi \in S(\mathbb{R}^n). \tag{2.10}$$

Now we recall the following result: given $g \in L^q(\mathbb{R}^n)$, if it is true that

$$\int_{\mathbb{R}^n} g(x)\phi(x)dx = 0 \quad \forall \phi \in S(\mathbb{R}^n), \tag{2.11}$$

then we must have $g = 0$ Lebesgue-qtp.

From (2.10) and (2.11) we have $p(\xi)\phi(\xi) = 0$ Lebesgue-qtp. Since $p(\xi) = \frac{\xi_1^2}{\xi_1^2 + m\xi_2^2 + \sum_{j=3}^n \xi_j^2}$ only vanishes in a set of null measure in $\mathbb{R}^n$, $n \geqslant 2$, we must have $\phi(\xi) = 0$ Lebesgue-qtp. $\qquad\square$

Proposition 2.3. *Consider $F : \mathbb{R}_x^n \times \mathbb{R}_t \to \mathbb{C}$. Then for $1 < p < \infty$:*

$$\|E(F)\|_{L^{p\infty}(\mathbb{R}^{n+1})} = \left\| \left(p(\xi)\widehat{F}(\xi, t) \right)^{\vee} \right\|_{L^{p\infty}(\mathbb{R}^{n+1})} \leqslant \|F\|_{L^{p\infty}(\mathbb{R}^{n+1})}.$$

Proof. We first observe that the inequality (2.9) can be extended to the mixed "space-time" Lebesgue spaces:

$$\|E(F)\|_{L_{tx}^q} = \|E(F)\|_{L_t^q L_x^q} = \|\|E(F)\|_{L_x^q}\|_{L_t^q} \leqslant \|\|F\|_{L_x^q}\|_{L_t^q} = \|F\|_{L_t^q L_x^q} = \|F\|_{L_{tx}^q}. \tag{2.12}$$

The inequality (2.12) and Theorem 1.6 will give us the result.

In fact, fix $1 < p < \infty$. Take $1 < p_0, p_1 < \infty$ and $0 < \theta < 1$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. By Theorem 1.6 we have $\|E(F)\|_{L^{p\infty}(\mathbb{R}^{n+1})} = \|E(F)\|_{(L^{p_0}, L^{p_1})_{\theta\infty}}$.

If

$$F = f_0 + f_1 \in L^{p_0}(\mathbb{R}^{n+1}) + L^{p_1}(\mathbb{R}^{n+1}),$$

then

$$E(F) = E(f_0) + E(f_1) \in L^{p_0}(\mathbb{R}^{n+1}) + L^{p_1}(\mathbb{R}^{n+1}),$$

and

$$\|E(f_j)\|_{L^{p_j}(\mathbb{R}^{n+1})} \leqslant \|f_j\|_{L^{p_j}(\mathbb{R}^{n+1})}, \ j = 0, \ 1.$$

So

$$
\begin{aligned}
K(t, E(F)) &= \inf_{E(F)=F_0+F_1} \left(\|F_0\|_{L^{p_0}(\mathbb{R}^n)} + t\|F_1\|_{L^{p_1}(\mathbb{R}^n)}\right) \\
&\leqslant \inf_{E(F)=E(f_0)+E(f_1)} \left(\|E(f_0)\|_{L^{p_0}(\mathbb{R}^n)} + t\|E(f_1)\|_{L^{p_1}(\mathbb{R}^n)}\right) \\
&\leqslant \inf_{E(F)=E(f_0)+E(f_1)} \left(\|f_0\|_{L^{p_0}(\mathbb{R}^n)} + t\|f_1\|_{L^{p_1}(\mathbb{R}^n)}\right).
\end{aligned}
$$

Since E is injective (lemma 2.2), $E(F) = E(f_0) + E(f_1) \Rightarrow F = f_0 + f_1$ Lebesgue a.e.

Then

$$K(t, E(F)) \leqslant \inf_{F=f_0+f_1} \left(\|f_0\|_{L^{p_0}(\mathbb{R}^n)} + t\|f_1\|_{L^{p_1}(\mathbb{R}^n)}\right) = K(t, f).$$

Using Theorem 1.6 once more we obtain the result. $\qquad\square$

Observe that from the idea of this proof we could obtain a more general result:

Lemma 2.4. *Let $A : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ be a linear, bounded and injective operator. Then A is bounded from $L^{p\infty}(\mathbb{R}^n)$ to $L^{q\infty}(\mathbb{R}^n)$.*

Next we define two operators and derive some properties about then.

1. Denote by G the following integral operator:

$$G(F)(x,t) = \int_0^t U(t-s)F(\cdot, s)(x)ds, \qquad (2.13)$$

where $U(t)$ group defined in (2.5).

2. Given $f \in S(\mathbb{R}^n)$ we define the operator T as follows:

$$(Tf)(x,t) = (U(t)\hat{f})^{\vee}(x).$$

Observe that by Strichartz estimates (see Theorem 1.7 with the group $U(t)$ instead of $e^{it\Delta}$), if (q, r) is an admissible pair, i.e., $\frac{1}{q} + \frac{n}{2r} = \frac{n}{4}$, we have

$$T : L^2(\mathbb{R}^n) \to L_t^q L_x^r.$$

The dual of the operator T is, as usual, denoted by T^* and given by:

$$T^* : L_t^{q'} L_x^{r'} \to L^2(\mathbb{R}^n)$$

$$(T^*F)(x) = \int_{+\infty}^{-\infty} U(-t)F(\cdot, t)(x)dt.$$

Finally we note that the integral operator that appears in the inequality (1.15) ($e^{it\Delta}$ replaced by $U(t)$) is exactly the composition of T and T^*:

$$(TT^*F)(x, t) = \int_{-\infty}^{+\infty} U(t - \tau)F(x, \tau)d\tau. \tag{2.14}$$

The following properties holds to the operators G and TT^*:

Proposition 2.5. *Let* $1 \leqslant p,\ r < \infty$ *such that*

$$\frac{1}{p} - \frac{1}{r} = \frac{2}{n + 2},$$

and

$$\frac{2(n + 1)}{n} < r < \frac{2(n + 1)(n + 2)}{n^2}.$$

Then the following inequalities holds:

$$\|G(F)\|_{L^{r\infty}(\mathbb{R}^{n+1})} \leqslant c\|F\|_{L^{p\infty}(\mathbb{R}^{n+1})}, \tag{2.15}$$

$$\|TT^*(F)\|_{L^{r\infty}(\mathbb{R}^{n+1})} \leqslant c\|F\|_{L^{p\infty}(\mathbb{R}^{n+1})}. \tag{2.16}$$

Proof. To prove properties (2.15) and (2.16) we need Theorem 1.7 (with $U(t)$ instead of $e^{it\Delta}$) and the interpolation theorem. In fact taking $r = q$ and $\tilde{r}' = \tilde{q}' =: p$ in Theorem 1.7, the hypothesis (1.10) becomes

$$\frac{1}{p} - \frac{1}{r} = \frac{2}{n + 2},$$

and the inequalities (1.13) and (1.15) becomes respectively

$$\|GF\|_{L^r(\mathbb{R}^{n+1})} \leqslant c\|F\|_{L^p(\mathbb{R}^{n+1})}, \tag{2.17}$$

and

$$\|TT^*F\|_{L^r(\mathbb{R}^{n+1})} \leqslant c\|F\|_{L^p(\mathbb{R}^{n+1})}. \tag{2.18}$$

The restriction $\frac{2(n+1)}{n} < r < \frac{2(n+1)(n+2)}{n^2}$ comes from hypothesis (1.11).

The result follows applying lemma 2.4 to inequalities (2.17) and (2.18). $\qquad\qquad\square$

The next theorem is the main result of this chapter. It proves that taking "small" initial data in the space Y, the integral equation (2.4) has a unique solution in

$$B(0, 3\delta_1) = \{f \in L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1}); \|f\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} < 3\delta_1\}.$$

This result allows us to prove well posedness of equation (2.3) and to find self-similar solutions for $\delta > 0$ (see next section).

Theorem 2.6. *There exists a $\delta_1 > 0$ such that given $\frac{4(n+1)}{n(n+2)} < \alpha < \frac{4(n+1)}{n^2}$ and $u_0 \in Y$ with $\|u_0\|_Y < \delta_1$ then there exists a unique $u \in B(0, 3\delta_1) \subset L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})$ solution of (2.4) such that $\|u\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} < 3\delta_1$ for some $\delta_1 > 0$.*

Proof. Consider the following operator

$$(\Phi u)(t) = U(t)u_0 - iG(\chi|u|^\alpha u + buE(|u|^\alpha))(t),$$

G as in (2.13).

We want to use the Picard fixed point theorem to find a solution of $u = \Phi(u)$ in

$$B(0, 3\delta_1) \subset L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1}).$$

Note that $\left(B(0, 3\delta_1), \|\cdot\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\right)$ is a complete metric space.

We must prove that:

(1). $\Phi(B(0, 3\delta_1) \subset B(0, 3\delta_1)$

(2). $\|\Phi(u) - \Phi(v)\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} \leqslant c\|u - v\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}, \ 0 < c < 1.$

To prove 1 take $u \in B(0, 3\delta_1)$.

By property (1.17) in Lemma 1.8 it follows that

$$\|\Phi(u)\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} \leqslant 2\left(\|U(t)u_0\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} + \|iG\left(\chi|u|^{\alpha}u + buE(|u|^{\alpha})\right)\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\right).$$

By hypothesis $\|U(t)u_0\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} < \delta_1$.

Using inequalities (2.15) from Proposition 2.5, (1.17) from Lemma 1.8 and the fact that $|\chi| = 1$ we get for the second term

$$\|iG(\chi|u|^{\alpha}u + buE(|u|^{\alpha}))\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} \leqslant 2c\left(\||u|^{\alpha}u\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})} + \|buE(|u|^{\alpha})\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})}\right).$$

By property (1.18) from Lemma 1.8 we have

$$\||u|^{\alpha}u\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})} \leqslant \|u\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}^{\alpha+1}.$$

Applying Proposition 2.3 and inequality (1.18) we get

$$\|buE(|u|^{\alpha})\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})} \leqslant |b|\|u\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\|E(|u|^{\alpha})\|_{L^{\frac{n+2}{2}\infty}(\mathbb{R}^{n+1})}$$

$$\leqslant |b|\|u\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\|u(t)\|^{\alpha}_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}.$$

Using that $u \in B(0, 3\delta_1)$ we have

$$\|\Phi(u)\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} < 2\delta_1 + 4c(3\delta_1)^{\alpha+1} + 4c|b|(3\delta_1)^{\alpha+1} < 3\delta_1.$$

Now we prove 2, i.e., that Φ is a contraction in $B(0, 3\delta_1)$. Take $u, v \in B(0, 3\delta_1)$:

$$\Phi(u) - \Phi(v) = iG(\chi(|v|^{\alpha}v - |u|^{\alpha}u)) + iG\big(b(vE(|v|^{\alpha}) - uE(|u|^{\alpha}))\big).$$

By properties (1.17) and (2.15) we get

$$\|\Phi(u) - \Phi(v)\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}$$

$$\leqslant 2c\left(\||v|^{\alpha}v - |u|^{\alpha}u\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})} + |b|\|vE(|v|^{\alpha}) - uE(|u|^{\alpha})\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})}\right)$$

$$\leqslant 2c\left(\|v(|v|^{\alpha} - |u|^{\alpha})\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})} + \||u|^{\alpha}(u - v)\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})}\right)$$

$$+ 2c|b|\left(\|E(|v|^{\alpha})(v - u)\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})} + \|u(E(|v|^{\alpha}) - E(|u|^{\alpha}))\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})}\right).$$

Applying inequality (1.18) and Proposition 2.3 we obtain

$$\|\Phi(u) - \Phi(v)\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}$$

$$\leqslant 2c\left(\|v\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\||v|^{\alpha} - |u|^{\alpha}\|_{L^{\frac{(n+2)}{2}\infty}(\mathbb{R}^{n+1})} + \||u|^{\alpha}\|_{L^{\frac{(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\|u - v\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\right)$$

$$+ 2c|b|\left(\|E(|v|^{\alpha})\|_{L^{\frac{(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\|u - v\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\right.$$

$$\left. + \|u\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\|E(|v|^{\alpha}) - E(|u|^{\alpha})\|_{L^{\frac{(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\right)$$

$$\leqslant 2c\left(\|v\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\||v|^{\alpha} - |u|^{\alpha}\|_{L^{\frac{(n+2)}{2}\infty}(\mathbb{R}^{n+1})} + \|u\|^{\alpha}_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\|u - v\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\right)$$

$$+ 2c|b|\left(\|v\|^{\alpha}_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\|u - v\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} + \|u\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\||v|^{\alpha} - |u|^{\alpha}\|_{L^{\frac{(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\right).$$

Now we set

$$g(u) = |u|^{\alpha}.$$

It follows by the Mean Value Theorem that

$$|g(u) - g(v)| \leqslant c(\alpha)(|u|^{\alpha-1} - |v|^{\alpha-1})|u - v|.$$

This property and Lemma 1.8 imply that

$$\||v|^{\alpha} - |u|^{\alpha}\|_{L^{\frac{(n+2)}{2}\infty}(\mathbb{R}^{n+1})}$$

$$\leqslant c(\alpha)\left(\||u|^{\alpha-1}|u - v|\|_{L^{\frac{(n+2)}{2}\infty}(\mathbb{R}^{n+1})} + \||v|^{\alpha-1}|u - v|\|_{L^{\frac{(n+2)}{2)}\infty}(\mathbb{R}^{n+1})}\right)$$

$$\leqslant c(\alpha)\left(\|u\|^{\alpha-1}_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\|u - v\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} + \|v\|^{\alpha-1}_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\|u - v\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\right).$$

By virtue of the last inequality and the hypothesis $u,\ v \in B(0, 3\delta_1)$ we get

$$\|\Phi(u) - \Phi(v)\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}$$

$$\leqslant 2c(3\delta_1)\left(2(3\delta_1)^{\alpha-1}\|u - v\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\right) + 2c(3\delta_1)^{\alpha}\|u - v\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}$$

$$+ 2c|b|(3\delta_1)^{\alpha}\|u - v\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} + 2c|b|(3\delta_1)\left(2(3\delta_1)^{\alpha-1}\|u - v\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\right),$$

and finally

$$\|\Phi(u) - \Phi(v)\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} \leqslant \delta_1^\alpha(c_1 + c_2|b|)\|v - u\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}.$$

Again taking $0 < \delta_1 \ll 1$ we get the contraction. $\qquad\qquad\square$

Remark 2.7. *Since Strichartz estimates still holds (up to endpoints) to the unitary group $U(t)$ defined in (2.5) with $\delta < 0$ in (2.6), and since we do not use the endpoints in the Proposition 2.5, we conclude that Theorem 2.6 holds for the nonelliptic Schrodinger problem, i.e., the IVP (2.7) with $\delta < 0$.*

The next proposition shows that giving any initial data in Y and assuming the existence of a solution u to the integral equation (2.4) we have that u is the solution (in the weak sense) of the differential equation (2.3). We emphasize that Theorem 2.6 provides the existence of solutions to the equation (2.4) under the assumption of small initial data.

Proposition 2.8. *Given $\frac{4(n+1)}{n(n+2)} < \alpha < \frac{4(n+1)}{n^2}$, $u_0 \in Y$ and let $u \in L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})$ be the solution of (2.4). It follows that $t \in \mathbb{R} \to u(t) \in S'(\mathbb{R}^n)$ is continuous and $u(0) = u_0$. In particular, u is a solution of (2.3). Moreover $u(t_0) \in Y \; \forall \; t_0 \in \mathbb{R}$. In addition, there exist $u_\pm$ such that $\|U(t)u_\pm\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} < \infty$ and $U(-t)u(t) \to u_\pm$ in $S'(\mathbb{R}^n)$ as $t \to \pm\infty$.*

Proof. By hypothesis $u \in L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})$. So by the inequality (1.18) in Lemma 1.8 and Proposition 2.3

$$|u|^\alpha u \text{ and } uE(|u|^\alpha) \in L^{\frac{\alpha(n+2)}{2(\alpha+1)}}(\mathbb{R}^{n+1}).$$

Now we can use the decomposition in (1.9) and write

$$|u|^\alpha u = f_1 + f_2, \tag{2.19}$$

and

$$uE(|u|^\alpha) = f_3 + f_4, \tag{2.20}$$

where $f_j \in L^{p_j}(\mathbb{R}^{n+1})$ for some $1 \leqslant p_1 < \frac{\alpha(n+2)}{2(\alpha+1)} < p_2 < \infty$ and $1 \leqslant p_3 < \frac{\alpha(n+2)}{2(\alpha+1)} < p_4 < \infty$.

Replacing (2.19) and (2.20) in (2.4) we get

$$u(t) = U(t)u_0 + i\chi G(f_1)(t) + i\chi G(f_2)(t) + ibG(f_3)(t) + ibG(f_4)(t). \tag{2.21}$$

Observe that from the decomposition (2.21) and remark 2.1 we have that $u(t) \in S'(\mathbb{R}^n)$.

Now, if we take $\phi \in S(\mathbb{R}^n)$ then $U(t)\phi \in C(\mathbb{R} : S(\mathbb{R}^n))$ and also $G(\phi)(t) \in C(\mathbb{R} : S(\mathbb{R}^n))$. By duality we can extend $U(t)$ to $S'(\mathbb{R}^n)$ and get $U(t)\phi \in C(\mathbb{R} : S'(\mathbb{R}^n))$ for $\phi \in S'(\mathbb{R}^n)$.

Using Dominated Convergence Theorem (Theorem 1.11) we have $G(\phi)(t) \in C(\mathbb{R} : S'(\mathbb{R}^n))$ for $\phi \in S'(\mathbb{R}^n)$ and by (2.21)

$$u(t) \in C(\mathbb{R} : S'(\mathbb{R}^n)). \tag{2.22}$$

Making $t \to 0$ in (2.21) we get $u(0) = u_0$.

Now we prove that $u(t)$ satisfies the equation

$$iu_t + \delta u_{x_1 x_1} + \sum_{j=2}^{n} u_{x_j x_j} = \chi |u|^\alpha u + buE(|u|^\alpha),$$

in $S'(\mathbb{R}^n)$ for all $t \in \mathbb{R}$:

Define $F(u) := \chi |u|^\alpha u + buE(|u|^\alpha)$. We must prove that $\forall \phi \in S(\mathbb{R}^n)$

$$i\lim_{h \to 0} \langle \frac{u(t+h) - u(t)}{h}, \phi \rangle = \langle -(\delta \partial_{x_1 x_1} + \sum_{j=2}^{n} \partial_{x_j x_j})u(t) + F(u)(t), \phi \rangle, \tag{2.23}$$

where $\langle f, g \rangle = \int_{\mathbb{R}^n} f(x)g(x)dx$.

Note that by (2.19), (2.20) and (2.22) we have

$$F(u)(t) \in C(\mathbb{R}, S'(\mathbb{R}^n)).$$

Using the integral equation (2.4) and the definition of the operator G in (2.13) we have the following expression for $u(t)$

$$u(t) = U(t)u_0 + iG(Fu)(t). \tag{2.24}$$

Thus

$$u(t+h) - u(t) = \left(\frac{U(t+h) - U(t)}{h} \right) u_0 + i \left(\frac{G(Fu)(t+h)}{h} - \frac{G(Fu)(t)}{h} \right).$$

Without loss of generality we can suppose $h > 0$.

Now, taking $\phi \in S(\mathbb{R}^n)$ we have that

$$\left\langle \left(\frac{U(t+h) - U(t)}{h} \right) u_0, \phi \right\rangle = -\left\langle u_0, \left(\frac{U(-t-h) - U(-t)}{-h} \right) \phi \right\rangle \xrightarrow[h \to 0]{}$$

$$- \left\langle u_0, i(\delta \partial_{\xi_1 \xi_1} + \sum_{j=2}^{n} \partial_{\xi_j \xi_j}) U(-t)\phi \right\rangle = \left\langle i(\delta \partial_{\xi_1 \xi_1} + \sum_{j=2}^{n} \partial_{\xi_j \xi_j}) U(t) u_0, \phi \right\rangle.$$

By group properties and the definition of G and F we have

$$\frac{1}{h} \langle G(Fu)(t+h), \phi \rangle - \frac{1}{h} \langle G(Fu)(t), \phi \rangle =$$

$$= \frac{1}{h} \left\langle \int_0^{t+h} U(t - t' + h) Fu(t') dt', \phi \right\rangle - \frac{1}{h} \left\langle \int_0^{t} U(t - t') Fu(t') dt', \phi \right\rangle$$

$$= \frac{1}{h} \left\langle U(h) \int_0^{t+h} U(t - t') Fu(t') dt', \phi \right\rangle - \frac{1}{h} \left\langle \int_0^{t} U(t - t') Fu(t') dt', \phi \right\rangle$$

$$= \left\langle \left(\frac{U(h) - Id}{h} \right) \int_0^{t+h} U(t - t') Fu(t') dt', \phi \right\rangle +$$

$$+ \frac{1}{h} \left\langle \int_0^{t+h} U(t - t') Fu(t') dt', \phi \right\rangle - \frac{1}{h} \left\langle \int_0^{t} U(t - t') Fu(t') dt', \phi \right\rangle$$

$$= - \int_0^{t+h} \left\langle U(t - t') Fu(t'), \left(\frac{U(-h) - Id}{-h} \right) \phi \right\rangle dt' + \frac{1}{h} \int_t^{t+h} \langle U(t - t') Fu(t'), \phi \rangle dt'.$$

Next, since $F(u)(t) \in C(\mathbb{R}, S'(\mathbb{R}^n))$, we can use the Lebesgue dominated convergence Theorem (Theorem 1.11) and the Lebesgue differentiation Theorem (Theorem 1.12) to obtain

$$\frac{1}{h} \langle G(Fu)(t+h), \phi \rangle - \frac{1}{h} \langle G(Fu)(t), \phi \rangle$$

$$\xrightarrow[h \to 0]{} - \int_0^{t} \left\langle U(t - t') Fu(t'), i(\delta \partial_{\xi_1 \xi_1} + \sum_{j=2}^{n} \partial_{\xi_j \xi_j}) \phi \right\rangle dt' + \langle Fu(t), \phi \rangle$$

$$= \int_0^{t} \left\langle i(\delta \partial_{\xi_1 \xi_1} + \sum_{j=2}^{n} \partial_{\xi_j \xi_j}) U(t - t') Fu(t'), \phi \right\rangle dt' + \langle Fu(t), \phi \rangle$$

$$= \left\langle i(\delta \partial_{\xi_1 \xi_1} + \sum_{j=2}^{n} \partial_{\xi_j \xi_j}) \int_0^{t} U(t - t') Fu(t') dt', \phi \right\rangle + \langle Fu(t), \phi \rangle.$$

Therefore

$$\frac{1}{h}\langle u(t+h) - u(t), \phi\rangle = \tag{2.25}$$

$$= \left\langle \left(\frac{U(t+h) - U(t)}{h}\right) u_0, \phi\right\rangle + i\left\langle \left(\frac{G(Fu)(t+h)}{h} - \frac{G(Fu)(t)}{h}\right), \phi\right\rangle \xrightarrow[h\to 0]{}$$

$$\left\langle i(\delta\partial_{\xi_1\xi_1} + \sum_{j=2}^{n} \partial_{\xi_j\xi_j})U(t)u_0, \phi\right\rangle - \left\langle (\delta\partial_{\xi_1\xi_1} + \sum_{j=2}^{n} \partial_{\xi_j\xi_j})G(Fu)(t), \phi\right\rangle + i\langle F(u), \phi\rangle.$$

From (2.24) and (2.25) we have (2.23).

To prove $\|u(t_0)\|_Y < \infty$, take $r = \frac{\alpha(n+2)}{2}$ on the inequality (2.16) of Proposition 2.5.

Then we have $\|TT^*F\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} \lesssim \|F\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})}$.

Hence

$$\left\|U(t)\int U(-s)F(s)ds\right\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} \lesssim \|F\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})}. \tag{2.26}$$

From inequality (2.26) and identity (2.8), $\forall\, t_0 \in \mathbb{R}$ we get

$$\left\|U(t)\int U(t_0 - s)F(s)ds\right\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} \lesssim \|F\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})}.$$

Now taking $\chi_{(0,t_0)}F$ instead of F in the last inequality we have

$$\left\|U(t)\int_0^{t_0} U(t_0 - s)F(s)ds\right\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} \lesssim \|F\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})},$$

i.e.,

$$\|U(t)G(F)(t_0)\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} \lesssim \|F\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})}. \tag{2.27}$$

Now taking $t = t_0$ in the integral equation (2.4) and applying $U(t)$ we have

$$U(t)u(t_0) = U(t+t_0)u_0 + iU(t)G(\chi|u|^\alpha u + buE(|u|^\alpha))(t_0).$$

By the properties (1.17) and (2.8) we obtain

$$\|U(t)u(t_0)\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} \leqslant 2\bigg(\|U(t)u_0\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}$$

$$+ \|U(t)G(\chi|u|^\alpha u + buE(|u|^\alpha))(t_0)\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\bigg).$$

Using inequality (2.27) and the same arguments as in Theorem 2.6 we get

$$\|U(t)u(t_0)\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} \leqslant 2 \left(\|U(t)u_0\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} + \|(\chi|u|^\alpha u + buE(|u|^\alpha))\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})} \right)$$

$$\leqslant 2\|U(t)u_0\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} + 4\|(\chi|u|^\alpha u\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})} + 4|b|\|uE(|u|^\alpha)\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})}$$

$$\leqslant 2\|U(t)u_0\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} + 4\|u\|^{\alpha+1}_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} + 4|b|\|u\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\|E(|u|^\alpha)\|_{L^{\frac{(n+2)}{2}\infty}(\mathbb{R}^{n+1})}$$

$$\leqslant 2\|U(t)u_0\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} + 4\|u\|^{\alpha+1}_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} + 4|b|\|u\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})}\|u\|^\alpha_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} < \infty.$$

Finally, to prove the last statement of the theorem we set

$$u_+ = u_0 + i \int_0^\infty U(-\tau)(\chi|u|^\alpha u + buE(|u|^\alpha))(\tau)d\tau.$$

It follows from inequalities (1.17) and (2.26) that:

$$\|U(t)u_+\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} \leqslant 2 \left(\|U(t)u_0\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} + \|(\chi|u|^\alpha u + buE(|u|^\alpha))\|_{L^{\frac{\alpha(n+2)}{2(\alpha+1)}\infty}(\mathbb{R}^{n+1})} \right) < \infty.$$

We deduce from the decompositions in (2.19) and (2.20) that

$$U(-t)u(t) - u_+ = \int_t^\infty U(-\tau)(\chi|u|^\alpha u + buE(|u|^\alpha))(\tau)d\tau \to 0 \text{ in } S'(\mathbb{R}^n) \text{ as } t \to \infty.$$

The result for $t \to -\infty$ is proved similarly.

$\square$

2.3 Self-similar solutions

In this section we find self-similar solutions to (2.3). Without lost of generality we can suppose $\delta = 1$, so our equation becomes:

$$\begin{cases} iu_t + \Delta u &= \chi|u|^\alpha u + buE(|u|^\alpha), \\ u(x,0) &= u_0(x). \end{cases} \qquad \forall\, x \in \mathbb{R}^n, n = 2, 3,\, t \in \mathbb{R}, \qquad (2.28)$$

We already know that a self-similar solution must have an homogeneous initial condition with degree $-2/\alpha$. So the idea is to prove that $u_0(x) = \epsilon|x|^{-2/\alpha} \in Y$ where $0 < \epsilon \ll 1$.

Then by Theorem 2.6 and Proposition 2.8 we have existence and uniqueness for equation (2.28) in Y. Since $u(x,t)$ and $\beta^{2/\alpha}u(\beta x, \beta^2 t)$ are both solutions, we must have $u = u_\beta$ and therefore self-similar solutions in Y.

To prove that $u_0 \in Y$, we consider the homogeneous problem with initial condition $u_0(x) = |x|^{-2/\alpha}$:

$$\begin{cases} iu_t + \Delta u &= 0, \\ u(x,0) &= |x|^{-2/\alpha}. \end{cases} \qquad \forall\, x \in \mathbb{R}^n, n = 2, 3,\ t \in \mathbb{R}. \qquad (2.29)$$

We know that the solution to the equation (2.29) is given by

$$u(x,t) = U(t)u_0(x),$$

where $U(t) = e^{it\Delta}$.

Since $u_\beta(x,t) = \beta^{2/\alpha}u(\beta x, \beta^2 t)$, $\beta > 0$ is also a solution, we must have

$$\beta^{2/\alpha}u(\beta x, \beta^2 t) = U(t)u_0(x) = u(x,t).$$

Taking $\beta = 1/\sqrt{t}$ we get

$$u(x,t) = t^{-1/\alpha}f(x/\sqrt{t}), \qquad (2.30)$$

where $f(x) = u(x,1)$.

By Proposition 1.10 we have that for $\alpha > 2/n$

$$|f(x)| \leqslant c(1 + |x|)^{-\sigma} \text{ where } \sigma = \begin{cases} 2/\alpha; & \alpha \geqslant 4/n \\ n - 2/\alpha; & \alpha < 4/n \end{cases} \qquad (2.31)$$

Next, we calculate $\alpha(\lambda, u) = |\{(x,t); |u(x,t)| > \lambda\}|$.

By (2.30) and (2.31)

$$\alpha(\lambda, u) \leqslant \int_{\{(x,t); |t^{-1/\alpha}\left(1+\frac{|x|}{\sqrt{t}}\right)^{-\sigma}|>\lambda\}} d(x,t) \leqslant \int_{\{(x,t); 0 \leqslant t < \lambda^{-\alpha} \text{ and } |x| < t^{1/2}[(t\lambda^\alpha)^{-1/\alpha\sigma}-1]\}} d(x,t)$$

$$\leqslant c\lambda^{-n/}\int_0^{\lambda^{-\alpha}} t^{\frac{n}{2}-\frac{n}{\sigma\alpha}}[1 - (t\lambda^\alpha)^{\frac{1}{\sigma\alpha}}]^n dt \leqslant \lambda^{\frac{-\alpha(n+2)}{2}}.$$

Therefore $\|U(\cdot)u_0\|_{L^{\frac{\alpha(n+2)}{2}\infty}(\mathbb{R}^{n+1})} \leqslant c$.

Choosing $0 < \epsilon \ll 1$ and taking the initial condition $u_0(x) = \epsilon|x|^{-2/\alpha}$ we conclude the result.

Chapter 3

On a Degenerate Zakharov System

3.1   Introduction

We consider the initial value problem associated to the degenerate Zakharov system

$$\begin{cases} i(\partial_t E + \partial_z E) + \Delta_\perp E &= nE, \\ \partial_t^2 n - \Delta_\perp n &= \Delta_\perp(|E|^2), \\ E(x, y, z, 0) &= E_0(x, y, z), \quad \forall\, (x, y, z) \in \mathbb{R}^3,\ t > 0, \\ n(x, y, z, 0) &= n_0(x, y, z), \\ \partial_t n(x, y, z, 0) &= n_1(x, y, z). \end{cases} \tag{3.1}$$

where $\Delta_\perp = \partial_x^2 + \partial_y^2$, E is a complex valued function, and n is a real valued function.

The system (3.1) describes the laser propagation when the paraxial approximation is used and the effect of the group velocity is negligible.

We now state the main result of this chapter:

Theorem 3.1. *For initial data* (E_0, n_0, n_1) *in* $\widetilde{H}^3(\mathbb{R}^3) \times H^3(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$ *and* $\partial_z n_1 \in$ $H^2(\mathbb{R}^3)$, *there exist* $T > 0$ *and a unique solution* E *of the integral equation* (13) *such that*

$$E \in C([0, T] : \widetilde{H}^3(\mathbb{R}^3)), \tag{3.2}$$

$$\sum_{|\alpha| \leqslant 3} \|D_x^{1/2} \partial^\alpha E\|_{L_x^\infty L_{yzT}^2} < \infty, \tag{3.3}$$

and

$$\sum_{|\alpha| \leqslant 3} \|D_y^{1/2} \partial^\alpha E\|_{L_y^\infty L_{xzT}^2} < \infty. \tag{3.4}$$

where $\widetilde{H}^3(\mathbb{R}^3)$ was defined in (15).

Moreover, there exists a neighborhood $\tilde{V}$ of $(E_0, n_0, n_1) \in \widetilde{H}^3(\mathbb{R}^3) \times H^3(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$ such that the map $\mathcal{F} : (E_0, n_0, n_1) \mapsto E(t)$ from $\tilde{V}$ into the class defined by (3.2) is smooth.

One also has that

$$n \in C([0, T] : H^3(\mathbb{R}^3)).$$

To prove Theorem 3.1, we combine smoothing effects and the L_x^2-maximal function estimate (21) to apply the contraction principle.

It turns out, however, to be a hard task to reach all Sobolev indices $s > 3/2$ (see explanation below), and our local well-posedness result is given in H^3.

In Section 3.2 we recall the linear estimates proved by Linares, Ponce and Saut ([LiPoS]), prove the L_x^2-maximal estimate (21) and also we prove Strichartz estimates in mixed Lebesgue spaces $L_T^q L_z^2 L_{xy}^p$ for p and q satisfying a certain condition. Observe that in the z-direction we have the Lebesgue space with fixed index 2. It happens because we do not have dispersion in this direction. Unfortunately these Strichartz estimates are not suficient to reach $s > 3/2$. In Section 3.3 we establish some estimates involving the nonlinear term that allow us to simplify the exposition of the proof of the main result. In this section it will appear clearly why we also could not reach the value $s = 2$ even if the maximal function estimate (21) attain this value. Finally in Section 3.4 we combine smoothing effects and property (21) to apply the contraction principle and proof Theorem 3.1.

Throughout this chapter H_{xyz}^3 and L_{xyz}^2 will always be denoted by H^3 and L^2. H_x^3 denotes the Sobolev space H^3 just in the spatial variable x and so on.

3.2 Linear Estimates

At first we recall the smoothing properties of solutions of the associated linear problems. We refer the reader to [LiPoS] for more details.

Also, we prove the maximal function estimate (21). We don't know if this estimate is sharp or not. In fact, following ideas from kenig and Ziesler for the KPI equation we show that (21) does not work to $s < 1$. There is still a gap between 1 and $3/2$.

Consider the linear problem:

$$\begin{cases} \partial_t E + \partial_z E - i\Delta_\perp E = 0, \\ E(x, y, z, 0) = E_0(x, y, z). \end{cases} \quad \forall\, (x, y, z) \in \mathbb{R}^3,\ t > 0, \tag{3.5}$$

where $\Delta_\perp = \partial_x^2 + \partial_y^2$.

The solution of the linear IVP (3.5) is given by the unitary group $\mathcal{E}(t) : H^s \to H^s$ such that

$$E(t) = \mathcal{E}(t)E_0 = \big(e^{-it(\xi_1^2 + \xi_2^2 + \xi_3)}\widehat{E_0}(\xi_1, \xi_2, \xi_3)\big)^\vee. \tag{3.6}$$

Proposition 3.2. *The solution of the linear problem* (3.5) *satisfies*

$$\|D_x^{1/2}\mathcal{E}(t)f\|_{L_x^\infty L_{yzT}^2} \leqslant c\|f\|_{L_{xyz}^2}, \tag{3.7}$$

$$\|D_x^{1/2}\int_0^t \mathcal{E}(t - t')G(t')dt'\|_{L_T^\infty L_{xyz}^2} \leqslant c\|G\|_{L_x^1 L_{yzT}^2}, \tag{3.8}$$

and

$$\|\partial_x \int_0^t \mathcal{E}(t - t')G(t')dt'\|_{L_x^\infty L_{yzT}^2} \leqslant c\|G\|_{L_x^1 L_{yzT}^2}. \tag{3.9}$$

These estimates hold exchanging x and y. Here $D_x^{1/2}f = (2\pi|\xi_1|^{1/2}\hat{f})^\vee$.

Proof. We refer to [LiPoS] for a proof of this proposition. $\qquad\square$

Now we state and prove the inequality (21). We will need the following lemma:

Lemma 3.3. *Van der Corput*

Let $k \in \mathbb{Z}^+$ *and* $|\phi^k(x)| \geqslant \lambda > 0$ *for any* $x \in [a, b]$ *with* $\phi'(x)$ *monotonic in the case* $k = 1$. *Then*

$$|\int_a^b e^{i\phi(x)}f(x)dx| \leqslant c_k\lambda^{-1/k}(\|f\|_\infty + \|f'\|_1).$$

Proof. We refer to [LP2] for a proof of this lemma. □

The next proposition is the key to the improvement of the Theorem 3.1. The main idea is to use the dispersion in the first two variables (that is where lemma 3.3 will be important) and in the third variable where we do not have dispersion, we use Sobolev embedding. As we alredy said, we do not know if the estimate (3.10) is sharp or not.

Proposition 3.4. *For $s > 3/2$, and $T \geqslant 0$ we have*

$$\|\mathcal{E}(t)E_0\|_{L_x^2 L_{yzT}^\infty} \leqslant c(T,s)\|E_0\|_{H^s}, \ \forall E_0 \in H^s. \tag{3.10}$$

The same estimate holds exchanging x and y.

The proof of Proposition 3.4 is a direct consequence of the next lemma, as we shall see later:

Lemma 3.5. *For every $T > 0$ and $k \geqslant 0$ there exists a constant $c(T) \geqslant 0$ and a function $H_{k,T}(\alpha) > 0$ such that*

$$\int_0^{+\infty} H_{k,T}(\alpha)d\alpha \leqslant c(T)2^{3k}, \tag{3.11}$$

and

$$\left| \int \int \int e^{(i(-t\xi_1^2 - t\xi_2^2 - t\xi_3 + x\xi_1 + y\xi_2 + z\xi_3))} \psi_1(\xi_1)\psi_2(\xi_2)\psi_3(\xi_3)d\xi_1 d\xi_2 d\xi_3 \right| \leqslant H_{k,T}(|x|), \tag{3.12}$$

for $|t| \leqslant T$ and $(x, y, z) \in \mathbb{R}^3$ where $\psi_j(\xi_j) = \mu(2^{k+1} - |\xi_j|)$, and μ denotes a infinitely differentiable function in $\mathbb{R}$ such that $\mu = 1$ for $x \geqslant 1$ and $\mu = 0$ for $x \leqslant 0$.

Proof. Denote by $J(t, x, y, z)$ the left-hand side in Eq.(3.12). We can rewrite $J(t, x, y, z)$ in the following way:

$$J(t,x,y,z) = \int e^{i(-t\xi_1^2 + x\xi_1)}\psi_1(\xi_1)d\xi_1 \int e^{i(-t\xi_2^2 + y\xi_2)}\psi_2(\xi_2)d\xi_2 \int e^{i(-t\xi_3 + z\xi_3)}\psi_3(\xi_3)d\xi_3.$$

Denoting by

$$J_1 = \int e^{i\varphi_1(\xi_1)}\psi_1(\xi_1)d\xi_1,$$

where $\varphi_1(\xi_1) = (-t\xi_1^2 + x\xi_1)$,

$$J_2 = \int e^{i(-t\xi_2^2 + y\xi_2)} \psi_2(\xi_2) d\xi_2,$$

and

$$J_3 = \int e^{i(-t\xi_3 + z\xi_3)} \psi_3(\xi_3) d\xi_3,$$

we have $|J| \leqslant |J_1\|J_2\|J_3|$.

Following Faminskii's ideas (see [Fa]), we split $\mathbb{R}$ in the following cases:

For $|x| < 1$ we use the support of ψ_j, $j = 1, 2, 3$ and get $|J| \leqslant c2^{3k}$.

Next, supose that $|x| \geqslant \max\{1, 2^3 2^k t\}$. In this case $|x| \geqslant 4|\xi_1|t$ and so $|\varphi_1'(\xi_1)| \geqslant |x|/2$. Using integration by parts twice we get:

$$J_1 = \int e^{i\varphi_1} \left(\frac{1}{\varphi_1'} \left(\frac{\psi_1}{\varphi_1'}\right)'\right)' d\xi_1.$$

Now by the support of ψ_1 and the inequalities $|\varphi_1'(\xi_1)| \geqslant |x|/2$ and $|x|^{-1} \leqslant 1$ we have:

$$|J_1| \leqslant c(T) \int_{\{|\xi_1| \leqslant 2^{k+1}\}} \frac{1}{|x|^2} d\xi_1 \leqslant c(T) 2^k |x|^{-2}.$$

Then $|J| \leqslant 2^{3k} c(T)|x|^{-2}$, by the support of ψ_2 and ψ_3.

It remains the case $1 \leqslant |x| \leqslant 2^3 2^k t$. Observe that in this case $t \geqslant 2^{-k-3} > 0$ and $t^{-2} \leqslant c|x|^{-2} 2^{2k}$.

Here we use Lemma 3.3 for J_1 and J_2:

Since $|\varphi_1''(\xi_1)| = 2t > 0$ then by Van der Corput $|J_1| \leqslant ct^{-1/2}$.

Again by Van der Corput $|J_2| \leqslant ct^{-1/2}$.

So by the support of ψ_3 we have $|J| \leqslant ct^{-1} 2^k \leqslant cTt^{-2} 2^k \leqslant c2^{3k}|x|^{-2}$.

Finally we define

$$H_{k,T}(\alpha) = \begin{cases} c2^{3k} & \text{for } 0 \leqslant \alpha < 1, \\ c(T)2^{3k}\alpha^{-2} & \text{for } 1 \geqslant \alpha, \end{cases}$$

and this function satisfies (3.11) and (3.12). □

Remark 3.6. *Observe that Lemma 3.5 still works if we change ψ_j by $\psi_j\mu(|\xi_j| - 2^k + 1)$, $j = 1, 2$ or 3.*

Now we turn to the proof of Proposition 3.4:

Proof. Using the same notation as in Lemma 3.5, i.e., $\psi_j = \mu(2^{k+1} - |\xi_j|)$, $j = 1, 2, 3$, we introduce the sequence $\bar{\psi}_k$ as follows:

$$\bar{\psi}_0(\xi_1, \xi_2, \xi_3) = \mu(2 - |\xi_1|)\mu(2 - |\xi_2|)\mu(2 - |\xi_3|),$$

and for $k \geqslant 1$,

$$\bar{\psi}_k(\xi_1, \xi_2, \xi_3) = \psi_1\psi_2\psi_3\mu(|\xi_1| - 2^k + 1) + \psi_1\psi_2\psi_3\mu(|\xi_2| - 2^k + 1) + \psi_1\psi_2\psi_3\mu(|\xi_3| - 2^k + 1).$$

Observe that $\sum_{k \geqslant 0} \bar{\psi}_k = 1$.

Now we define the operator $\widehat{B_k f}(\xi) = \bar{\psi}_k^{1/2}(\xi)\hat{f}(\xi)$, where $\xi = (\xi_1, \xi_2, \xi_3)$.

Then

$$\|B_k f\|_{L^2} \leqslant c2^{-ks}\|f\|_{H^s}, \tag{3.13}$$

$$\widehat{B_k^2 f} = \bar{\psi}_k \hat{f}, \tag{3.14}$$

$$\sum_{k \geqslant 0} E(t)B_k^2 E_0 = E(t)E_0, \tag{3.15}$$

and

$$|\int_{-T}^{T} (E(t - \tau)(B_k^2 g(\tau, \cdot, \cdot, \cdot)))(x, y, z)d\tau| \leqslant \tag{3.16}$$

$$c(H_{k,T}(|\cdot|) * \int_{-T}^{T}\int\int |g(\tau, \cdot, y, z)|d\tau dy dz)(x),$$

for $|t| \leqslant T$ and $g \in C_0^\infty(\mathbb{R}^4)$.

In fact, by the support of $\bar{\psi}_k$

$$
\begin{aligned}
\|B_k f\|_{L^2}^2 =& \|\bar{\psi}_k^{1/2} \hat{f}\|_{L^2}^2 \\
=& \int\int\int \bar{\psi}_k(\xi_1, \xi_2, \xi_3)|\hat{f}(\xi_1, \xi_2, \xi_3)|^2 d\xi_1 d\xi_2 d\xi_3 \\
=& \int_{2^k-1}^{2^{k+1}} \int\int \psi_1\psi_2\psi_3\mu(|\xi_1| - 2^k + 1)|\hat{f}(\xi_1, \xi_2, \xi_3)|^2 d\xi_1 d\xi_2 d\xi_3 \\
&+ \int\int_{2^k-1}^{2^{k+1}}\int \psi_1\psi_2\psi_3\mu(|\xi_2| - 2^k + 1)|\hat{f}(\xi_1, \xi_2, \xi_3)|^2 d\xi_1 d\xi_2 d\xi_3 \\
&+ \int\int\int_{2^k-1}^{2^{k+1}} \psi_1\psi_2\psi_3\mu(|\xi_3| - 2^k + 1)|\hat{f}(\xi_1, \xi_2, \xi_3)|^2 d\xi_1 d\xi_2 d\xi_3.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|B_k f\|_{L^2}^2 =& \int_{2^k-1}^{2^{k+1}} \int\int \frac{|\xi_1|^{2s}}{|\xi_1|^{2s}} \psi_1\psi_2\psi_3\mu(|\xi_1| - 2^k + 1)|\hat{f}(\xi_1, \xi_2, \xi_3)|^2 d\xi_1 d\xi_2 d\xi_3 \\
&+ \int\int_{2^k-1}^{2^{k+1}}\int \frac{|\xi_2|^{2s}}{|\xi_2|^{2s}} \psi_1\psi_2\psi_3\mu(|\xi_2| - 2^k + 1)|\hat{f}(\xi_1, \xi_2, \xi_3)|^2 d\xi_1 d\xi_2 d\xi_3 \\
&+ \int\int\int_{2^k-1}^{2^{k+1}} \frac{|\xi_3|^{2s}}{|\xi_3|^{2s}} \psi_1\psi_2\psi_3\mu(|\xi_3| - 2^k + 1)|\hat{f}(\xi_1, \xi_2, \xi_3)|^2 d\xi_1 d\xi_2 d\xi_3 \\
\leqslant& \int_{2^k-1}^{2^{k+1}} \int\int |\xi_1|^{2s}2^{-2ks} \psi_1\psi_2\psi_3\mu(|\xi_1| - 2^k + 1)|\hat{f}(\xi_1, \xi_2, \xi_3)|^2 d\xi_1 d\xi_2 d\xi_3 \\
&+ \int\int_{2^k-1}^{2^{k+1}}\int |\xi_2|^{2s}2^{-2ks} \psi_1\psi_2\psi_3\mu(|\xi_2| - 2^k + 1)|\hat{f}(\xi_1, \xi_2, \xi_3)|^2 d\xi_1 d\xi_2 d\xi_3 \\
&+ \int\int\int_{2^k-1}^{2^{k+1}} |\xi_3|^{2s}2^{-2ks} \psi_1\psi_2\psi_3\mu(|\xi_3| - 2^k + 1)|\hat{f}(\xi_1, \xi_2, \xi_3)|^2 d\xi_1 d\xi_2 d\xi_3 \\
\leqslant& 2^{-2ks} \int (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi.
\end{aligned}
$$

Then (3.14) follows directly from the definition of B_k.

To prove inequality (3.15) we use property (3.14) and the property $\sum_{k\geqslant 0} \bar{\psi}_k = 1$:

$$
\sum_{k\geqslant 0} E(t)B_k^2 E_0 = E(t)\sum_{k\geqslant 0} B_k^2 E_0 = E(t)\sum_{k\geqslant 0}(\bar{\psi}_k\widehat{E_0})^\vee = E(t)(\sum_{k\geqslant 0}\bar{\psi}_k\widehat{E_0})^\vee = E(t)E_0.
$$

By (3.14) we have

$$|\int_{-T}^{T}(E(t-\tau)B_k^2 g(\tau,\xi))(x,y,z)d\tau| =$$

$$= |\int_{-T}^{T}(e^{i(t-\tau)(\xi_1^2+\xi_2^2+\xi_3)}\widehat{B_k^2 g(\tau,\xi)})^\vee(x,y,z)d\tau|$$

$$= |\int_{-T}^{T}(e^{i(t-\tau)(\xi_1^2+\xi_2^2+\xi_3)}\bar{\psi}_k(\xi)g(\tau,\xi))^\vee(x,y,z)d\tau|$$

$$\leqslant |\int_{-T}^{T}(e^{i(t-\tau)(\xi_1^2+\xi_2^2+\xi_3)}\psi_1\psi_2\psi_3\mu(|\xi_1|-2^k+1)g(\tau,\xi))^\vee(x,y,z)d\tau|$$

$$+ |\int_{-T}^{T}(e^{i(t-\tau)(\xi_1^2+\xi_2^2+\xi_3)}\psi_1\psi_2\psi_3\mu(|\xi_2|-2^k+1)g(\tau,\xi))^\vee(x,y,z)d\tau|$$

$$+ |\int_{-T}^{T}(e^{i(t-\tau)(\xi_1^2+\xi_2^2+\xi_3)}\psi_1\psi_2\psi_3\mu(|\xi_3|-2^k+1)g(\tau,\xi))^\vee(x,y,z)d\tau|.$$

Finally using the inequality (3.12) (with ψ_j replaced by $\psi_j\mu(|\xi_j|-2^k+1)$) we obtain

$$|\int_{-T}^{T}(E(t-\tau)B_k^2 g(\tau,\xi))(x,y,z)d\tau|$$

$$\leqslant \int_{-T}^{T}(|(e^{i(t-\tau)(\xi_1^2+\xi_2^2+\xi_3)}\psi_1\psi_2\psi(2^{k+1}-|\xi_3|)\mu(|\xi_1|-2^k+1))^\vee| * |g(\tau,\xi)|)(x,y,z)d\tau$$

$$+ \int_{-T}^{T}(|(e^{i(t-\tau)(\xi_1^2+\xi_2^2+\xi_3)}\psi_1\psi_2\psi_3\mu(|\xi_2|-2^k+1))^\vee| * |g(\tau,\xi)|)(x,y,z)d\tau$$

$$+ \int_{-T}^{T}(|(e^{i(t-\tau)(\xi_1^2+\xi_2^2+\xi_3)}\psi_1\psi_2\psi_3\mu(|\xi_3|-2^k+1))^\vee| * |g(\tau,\xi)|)(x,y,z)d\tau$$

$$\leqslant c\int_{-T}^{T}(H_{k,T}(|\xi_1|) * |g(\tau,\xi)|)(x,y,z)d\tau = c(H_{k,T}(|\cdot|) * \int_{-T}^{T}\int\int |g(\tau,\cdot,y,z)|d\tau dy dz)(x),$$

which gives us (3.16).

Now defining

$$A_k : L^1([-T,T];L^2(\mathbb{R}^3)) \to L^2(\mathbb{R}^3), \ A_k g = \int \chi_{[-T,T]}(\tau)E(-\tau)B_k g(\tau,\xi_1,\xi_2,\xi_3)d\tau$$

and

$$X = L_x^2(\mathbb{R}; L_{tyz}^1([-T,T]\times\mathbb{R}^2)),$$

we have

$$A_k^* : L^2(\mathbb{R}^3) \to L^\infty([-T,T]; L^2(\mathbb{R}^3)), A_k^* h = E(t)B_k h,$$

and

$$X^* = L_x^2(\mathbb{R}; L_{tyz}^\infty([-T,T] \times \mathbb{R}^2)). \tag{3.17}$$

By (3.16), Young's inequality (Lemma 1.15) and inequality (3.11) we can apply an argument due to Stein-Tomas and conclude

$$\begin{aligned}
\|A_k^* A_k g\|_{X*} &= \|\int_{-T}^T (E(t-\tau)B_k^2 g(\tau, \xi, \xi, \xi))d\tau\|_{X*} \\
&= (\int \sup_{|t|\leqslant T} \sup_{y,z} |\int_{-T}^T E(t-\tau)B_k^2 g(\tau, x, y, z)d\tau|^2 dx)^{1/2} \\
&\leqslant c(\int (H_{k,T}(|\xi|) * \int_{-T}^T \int\int |g(\tau, \xi, y, z)|d\tau dy dz)^2(x)dx)^{1/2} \\
&\leqslant c(\int (H_{k,T}(|\xi|) * \|g(\xi)\|_{L_{\tau yz}^1})^2(x)dx)^{1/2} \\
&\leqslant c(\int H_{k,T}(|x|)dx)\|g\|_{L_x^2 L_{\tau yz}^1} \leqslant c(T)2^{3k}\|g\|_X \;\; \forall g \in C_0^\infty(\mathbb{R}^4).
\end{aligned}$$

Therefore by Lemma 1.24 we have

$$\|A_k^* h\|_{X*} \leqslant (c(T)2^{3k})^{1/2}\|h\|_{L^2}, \; \forall \; h \in L^2(\mathbb{R}^3).$$

So by the last inequality and (3.13) we get

$$\|E(t)B_k^2 E_0\|_{X*} = \|A_k^* B_k E_0\|_{X*} \leqslant c(T)^{1/2}2^{3k/2}\|B_k E_0\|_{L^2} \leqslant c(T)^{1/2}2^{-k(s-3/2)}\|E_0\|_{H^s}. \tag{3.18}$$

Thus by (3.15), Holder's inequality and (3.18) we obtain

$$\begin{aligned}
\|E(t)E_0\|_{X*} &= \|\sum_{k\geqslant 0} E(t)B_k^2 E_0\|_{X*} \leqslant \sum_{k\geqslant 0} 2^{-\epsilon k}\|2^{\epsilon k}E(t)B_k^2 E_0\|_{X*} \leqslant c(\epsilon)(\sum_{k\geqslant 0} \|2^{\epsilon k}E(t)B_k^2 E_0\|_{X*}^2)^{1/2} \\
&\leqslant c(\epsilon)(\sum_{k\geqslant 0} c(T)2^{2k\epsilon - 2k(s-3/2)}\|E_0\|_{H^s}^2)^{1/2} \leqslant c(\epsilon, T)\|E_0\|_{H^s}(\sum_{k\geqslant 0} 2^{2k\epsilon - 2k(s-3/2)})^{1/2} \\
&\leqslant c(\epsilon, T)\|E_0\|_{H^s},
\end{aligned}$$

if $0 < \epsilon < 2s - 3$.

From (3.17) and the last inequality we obtain the result. □

Now, following ideas from Kenig and Ziesler for the K-P equation (see[KZ]), we show that (3.10) does not work for $s < 1$. The main idea is to suppose that inequality (3.10) holds and then to construct a certain function E_0. Then using inequality (3.10) and change of variables we must have $s \geqslant 1$.

Proposition 3.7. *For each $s < 1$ there exists E_0 such that*

$$\|E(t)E_0\|_{L_x^2 L_{yzT}^\infty} \geqslant c(T,s)\|E_0\|_{H^s}.$$

Proof. If (3.10) is true, then we can choose $\hat{E}_0(\xi) = \hat{\theta}(\frac{\xi}{2^k})$ where $k \in \mathbb{N}$ and $\hat{\theta} \in C_0^\infty$ such that

$$\hat{\theta}(\xi) = \begin{cases} 1 & \text{on} & \{\xi \in \mathbb{R}^3; 1 \leqslant |\xi| \leqslant 2\}, \\ 0 & \text{on} & \{\xi \in \mathbb{R}^3; |\xi| \leqslant 1/2\} \cup \{\xi \in \mathbb{R}^3; |\xi| \geqslant 4\}. \end{cases}$$

So by change of variables

$$\begin{aligned} \|E_0\|_{H^s} &= \Big(\int_{\{\frac{1}{2} \leqslant \frac{|\xi|}{2^k} \leqslant 4\}} (1+|\xi|^2)^s |\hat{\theta}(\frac{\xi}{2^k})|^2 d\xi\Big)^{\frac{1}{2}} = \Big(\int_{\{\frac{1}{2} \leqslant |\xi| \leqslant 4\}} (1+2^{2k}|\xi|^2)^s |\hat{\theta}(\xi)|^2 2^{3k} d\xi\Big)^{\frac{1}{2}} \\ &\leqslant 2^{3k/2+ks} \Big(\int_{\{\frac{1}{2} \leqslant |\xi| \leqslant 4\}} (2^{-2k}+|\xi|^2)^s |\hat{\theta}(\xi)|^2 d\xi\Big)^{\frac{1}{2}} \leqslant 2^{3k/2+ks} c(s) \Big(\int_{\{\frac{1}{2} \leqslant |\xi| \leqslant 4\}} |\hat{\theta}(\xi)|^2 d\xi\Big)^{\frac{1}{2}} \\ &\leqslant 2^{3k/2+ks} c(s). \end{aligned}$$

Next, we calculate $E(t)E_0$. Again by changing variables we get

$$\begin{aligned} (E(t)E_0)(x,y,z) &= \int_{\{\frac{1}{2} \leqslant \frac{|\xi|}{2^k} \leqslant 4\}} e^{i(x\xi_1+y\xi_2+z\xi_3+t(\xi_1^2+\xi_2^2+\xi_3))} \hat{\theta}(\frac{\xi}{2^k}) d\xi \\ &= 2^{3k} \int_{\{\frac{1}{2} \leqslant |\xi| \leqslant 4\}} e^{i(x2^k\xi_1+y2^k\xi_2+z2^k\xi_3+t(2^{2k}\xi_1^2+2^{2k}\xi_2^2+2^k\xi_3))} \hat{\theta}(\xi) d\xi \\ &= 2^{3k} \int_{\{\frac{1}{2} \leqslant |\xi| \leqslant 4\}} e^{ix\bar{\xi}} e^{is} \hat{\theta}(\xi) d\xi, \end{aligned}$$

where $\bar{\xi} = 2^k \xi_1$, $s = y2^k\xi_2 + z2^k\xi_3 + t(2^{2k}\xi_1^2 + 2^{2k}\xi_2^2 + 2^k\xi_3)$.

Now, by Taylor's expansion

$$
\begin{aligned}
|(E(t)E_0)(x,y,z)| &= 2^{3k}|\int_{\{\frac{1}{2}\leqslant|\xi|\leqslant 4\}} e^{ix\bar{\xi}}e^{is}\hat{\theta}(\xi)d\xi| \\
&= 2^{3k}|\int_{\{\frac{1}{2}\leqslant|\xi|\leqslant 4\}} (\cos(x\bar{\xi})+i\sin(x\bar{\xi}))(\cos(s)+i\sin(s))\hat{\theta}(\xi)d\xi| \\
&\geqslant 2^{3k}|\int_{\{\frac{1}{2}\leqslant|\xi|\leqslant 4\}} (\cos(x\bar{\xi})\cos(s)-\sin(x\bar{\xi})\sin(s))\hat{\theta}(\xi)d\xi| \\
&\geqslant 2^{3k}|\int_{\{\frac{1}{2}\leqslant|\xi|\leqslant 4\}} [(1-\tfrac{(x\bar{\xi})^2}{2}+r(x\bar{\xi}))(1-\tfrac{s^2}{2}+r(s))+ \\
&\qquad\qquad -((x\bar{\xi})^2-r_1(x\bar{\xi}))(s-r_1(s))]\hat{\theta}(\xi)d\xi| \\
&\geqslant 2^{3k}|\int_{\{\frac{1}{2}\leqslant|\xi|\leqslant 4\}} [1-\eta(x,s,\bar{\xi})+\rho(x,s,\bar{\xi})]\hat{\theta}(\xi)d\xi|,
\end{aligned}
$$

where

$$
\eta(x,s,\bar{\xi}) = \frac{(x\bar{\xi})^2}{2}+\frac{s^2}{2}+\frac{s^2 r(x\bar{\xi})}{2}+\frac{(x\bar{\xi})^2 r(s)}{2}+sx\bar{\xi}+r_1(x\bar{\xi})r_1(s),
$$

$$
\rho(x,s,\bar{\xi}) = r(x\bar{\xi})+\frac{s^2(x\bar{\xi})^2}{2}+r(s)+r(x\bar{\xi})r(s)+x\bar{\xi}r_1(s)+sr_1(x\bar{\xi}),
$$

$$
r(\alpha) = (\alpha)^4-(\alpha)^6+(\alpha)^8-\ldots
$$

and

$$
r_1(\alpha) = (\alpha)^3-(\alpha)^5+(\alpha)^7-\ldots.
$$

If we choose $0 < \delta \ll 1$ and take $|x| \leqslant \delta 2^{-k}$, y, $z \simeq \delta 2^{-k}$, $t \simeq \delta 2^{-2k}$, then s, $x\bar{\xi} \simeq O(\delta)$, $0 < r(s)$, $r_1(s)$, $r(x\bar{\xi})$, $r_1(x\bar{\xi}) \ll 1$ and $1-\eta(x,s,\bar{\xi}) > c > 0$. So

$$
\|(E(t)E_0)(x,y,z)| \geqslant c2^{3k}|\int_{\{\frac{1}{2}\leqslant|\xi|\leqslant 4\}} A\cdot\hat{\theta}(\xi)d\xi| \geqslant c2^{3k}|\int_{\{1\leqslant|\xi|\leqslant 2\}} A\cdot 1\ d\xi| \geqslant c2^{3k}.
$$

Then

$$
\|(E(t)E_0)\|_{L_x^2 L_{yzT}^\infty} \geqslant (\int_{|x|\leqslant\delta 2^{-k}} (\sup_{\substack{t\simeq\delta 2^{-2k} \\ y,z\simeq\delta 2^{-k}}} |E(t)E_0|)^2 dx)^{1/2} \geqslant 2^{3k}2^{-k/2} = 2^{5k/2}.
$$

Finally we have

$$
c\ 2^{5k/2} \leqslant \|(E(t)E_0)\|_{L_x^2 L_{yzT}^\infty} \leqslant \|E_0\|_{H^s} \leqslant 2^{3k/2+ks}\ \forall k\in\mathbb{N},
$$

which implies $s \geqslant 1$. $\qquad\square$

Now we establish Strichartz estimates to the linear problem (3.5). Before that, we give some notations and stablish a lemma that we will need.

We denote by τ^{x_3} the translation in the third variable, i.e.,

$$(\tau_h^{x_3} f)(x, y, z) = f(x, y, z - h).$$

Given $f \in L^2(\mathbb{R}^3)$ we denote by $\widehat{f}^{\,x_1}$ the Fourier transform of f in the first variable:

$$\widehat{f}^{\,x_1}(x, y, z) = \int_{\mathbb{R}} e^{-2\pi i x \xi_1} f(\xi_1, y, z) d\xi_1.$$

Analogously we define $\widehat{f}^{\,x_2}, \widehat{f}^{\,x_3}$ and $\widehat{f}^{\,x_1 x_2}$.

Lemma 3.8. *If $t \neq 0, \frac{1}{p} + \frac{1}{p'} = 1$ and $p' \in [1, 2]$, then the group $\mathcal{E}(t)$ defined in (3.6) is a continuous linear operator from $L^{p'}(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3)$ and*

$$\|\mathcal{E}(t)f\|_{L_z^2 L_{xy}^p} \leqslant \frac{c}{|t|^{(\frac{1}{p'} - \frac{1}{p})}} \|f\|_{L_z^2 L_{xy}^{p'}}.$$

Proof. From Theorem 1.1 we have that

$$\|\mathcal{E}(t)f\|_{L_z^2 L_{xy}^2} = \|\mathcal{E}(t)f\|_{L^2} = \|e^{-it(\xi_1^2 + \xi_2^2 + \xi_3)} \hat{f}\|_{L^2} = \|\hat{f}\|_{L^2} = \|f\|_{L_z^2 L_{xy}^2}. \tag{3.19}$$

Using property (1.2) and equality (1.7) we obtain

$$
\begin{aligned}
(\mathcal{E}(t)f)(x, y, z) &= \left(e^{-it(\xi_1^2 + \xi_2^2 + \xi_3)} \widehat{f}(\xi_1, \xi_2, \xi_3)\right)^{\vee}(x, y, z) \\
&= \left(e^{-it\xi_3}\left(e^{-it(\xi_1^2 + \xi_2^2)} \widehat{f}(\xi_1, \xi_2, \xi_3)\right)^{\vee_{x_1 x_2}}(x, y, \cdot)\right)^{\vee_{x_3}}(\cdot, \cdot, z) \\
&= \left(e^{-it\xi_3}\left((e^{-it(\xi_1^2 + \xi_2^2)})^{\vee_{x_1 x_2}} *_{x_1 x_2} \widehat{f}^{\,x_3}(\xi_1, \xi_2, \xi_3)\right)(x, y, \cdot)\right)^{\vee_{x_3}}(\cdot, \cdot, z) \\
&= \left(e^{-it\xi_3}\left(\frac{e^{i(\xi_1^2 + \xi_2^2)/4|t|}}{4\pi|t|} *_{x_1 x_2} \widehat{f}^{\,x_3}(\xi_1, \xi_2, \xi_3)(x, y, \cdot)\right)\right)^{\vee_{x_3}}(\cdot, \cdot, z) \\
&= \left(e^{-it\xi_3} g(x, y, \cdot)\right)^{\vee_{x_3}}(\cdot, \cdot, z)
\end{aligned}
$$

where

$$g(x, y, \cdot) = \left(\frac{e^{i(\xi_1^2 + \xi_2^2)/4|t|}}{4\pi t} *_{x_1 x_2} \widehat{f}^{\,x_3}(\xi_1, \xi_2, \xi_3)\right)(x, y, \cdot),$$

and $*_{x_1 x_2}$ is the convolution in the first two variables, i.e.,

$$(f *_{x_1 x_2} g)(x, y, z) = \int_{\mathbb{R}^2} f(x - x_1, y - x_2, z) g(x_1, x_2, z) dx_1 dx_2.$$

By property (1.3) and Young's inequality we have

$$
\begin{aligned}
\|\mathcal{E}(t)f(\cdot, \cdot, z)\|_{L_z^2 L_{xy}^\infty} &= \|(\tau_{-t/2\pi}^{x_3} \breve{g}^{x_3})(\cdot, \cdot, z)\|_{L_z^2 L_{xy}^\infty} \\
&= \|\tau_{-t/2\pi}^{x_3} (\frac{e^{i(\xi_1^2 + \xi_2^2)/4|t|}}{4\pi t} *_{x_1 x_2} f(\xi_1, \xi_2, \xi_3))(\cdot, \cdot, z)\|_{L_z^2 L_{xy}^\infty} \\
&= \|(\frac{e^{i(\xi_1^2 + \xi_2^2)/4|t|}}{4\pi t} *_{x_1 x_2} \tau_{-t/2\pi}^{x_3} f(\xi_1, \xi_2, \xi_3))(\cdot, \cdot, z)\|_{L_z^2 L_{xy}^\infty} \quad (3.20) \\
&\leqslant \|\|\frac{e^{i(\xi_1^2 + \xi_2^2)/4|t|}}{4\pi t}\|_{L_{xy}^\infty} \|\tau_{-t/2\pi}^{x_3} f(\cdot, \cdot, z)\|_{L_{xy}^1}\|_{L_z^2} \leqslant c\frac{1}{|t|} \|\tau_{-t/2\pi}^{x_3} f(\cdot, \cdot, z)\|_{L_z^2 L_{xy}^1}
\end{aligned}
$$

Interpolation (Theorem 1.25) between (3.19) and (3.20) yields the result. $\square$

Now we are able to prove Strichartz estimates. We notice that our result do not cover the endpoit $(p, q) = (\infty, 2)$.

Proposition 3.9. *The unitary group* $\{\mathcal{E}(t)\}_{-\infty}^{+\infty}$ *defined in* (3.6) *satisfies*

$$\|\mathcal{E}(t)f\|_{L_t^q L_z^2 L_{xy}^p} \leqslant c\|f\|_{L_{xyz}^2}, \quad (3.21)$$

$$\|\int_{\mathbb{R}} \mathcal{E}(t - t')g(\cdot, t')dt'\|_{L_t^q L_z^2 L_{xy}^p} \leqslant c\|g\|_{L_t^{q'} L_z^2 L_{xy}^{p'}} \quad (3.22)$$

and

$$\|\int_{\mathbb{R}} \mathcal{E}(t)g(\cdot, t)dt\|_{L_{xyz}^2} \leqslant c\|g\|_{L_t^{q'} L_z^2 L_{xy}^{p'}}, \quad (3.23)$$

where

$$\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1, \quad \frac{2}{q} = 1 - \frac{2}{p} \quad and \quad p = \frac{2}{\theta}, \ \theta \in (0, 1].$$

Proof. At first we prove that (3.22) implies (3.23) and that (3.23) implies (3.21). In fact, if inequality (3.22) holds we use an argument due to P. Tomas and Theorem 1.18 to get

$$
\begin{aligned}
\|\int_{\mathbb{R}} \mathcal{E}(t)g(\cdot,\cdot,\cdot,t)dt\|_{L_{xyz}^2}^2 &= \int_{\mathbb{R}^3} (\int_{\mathbb{R}} \mathcal{E}(t)g(\cdot,\cdot,\cdot,t)dt)(\overline{\int_{\mathbb{R}} \mathcal{E}(t')g(\cdot,\cdot,\cdot,t')dt'})d(x,y,z) \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{R}} \mathcal{E}(t)g(\cdot,\cdot,\cdot,t)(\int_{\mathbb{R}} \mathcal{E}(-t')\overline{g(\cdot,\cdot,\cdot,t')}dt')dt \; d(x,y,z) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^3} g(\cdot,\cdot,\cdot,t)(\int_{\mathbb{R}} \mathcal{E}(t-t')\overline{g(\cdot,\cdot,\cdot,t')}dt')d(x,y,z) \; dt.
\end{aligned}
$$

By Lemma 1.14 we obtain

$$
\begin{aligned}
\|\int_{\mathbb{R}} \mathcal{E}(t)g(\cdot,\cdot,\cdot,t)dt\|_{L_{xyz}^2}^2 &= \int_{\mathbb{R}} \int_{\mathbb{R}} \|g(\cdot,\cdot,z,t)\|_{L_{xy}^{p'}} \|\int_{\mathbb{R}} \mathcal{E}(t-t')\overline{g(\cdot,\cdot,z,t')}dt'\|_{L_{xy}^p} dz \; dt \\
&\leqslant \int_{\mathbb{R}} \|g(t)\|_{L_z^2 L_{xy}^{p'}} \|\int_{\mathbb{R}} \mathcal{E}(t-t')\overline{g(t')}dt'\|_{L_z^2 L_{xy}^p} dt \\
&\leqslant \|g\|_{L_t^{q'} L_z^2 L_{xy}^{p'}} \|\int_{\mathbb{R}} \mathcal{E}(\cdot - t')\overline{g(t')}dt'\|_{L_t^q L_z^2 L_{xy}^p}.
\end{aligned}
$$

Finally using (3.22) we conclude

$$
\|\int_{\mathbb{R}} \mathcal{E}(t)g(\cdot,\cdot,\cdot,t)dt\|_{L_{xyz}^2}^2 \leqslant c\|g\|_{L_t^{q'} L_z^2 L_{xy}^{p'}}^2,
$$

which implies (3.23).

Now, suppose that (3.23) holds. Using duality we have that

$$
\|\mathcal{E}(t)f\|_{L_T^q L_z^2 L_{xy}^p} = sup\{|\int_{\mathbb{R}} \int_{\mathbb{R}^3} \mathcal{E}(t)f(x,y,z)w(x,y,z,t)d(x,y,z) \; dt|; \|w\|_{L_t^{q'} L_z^2 L_{xy}^{p'}} = 1\}.
$$

By Lemma 1.14, Theorem 1.18 and (3.23) we obtain

$$
\begin{aligned}
|\int_{\mathbb{R}} \int_{\mathbb{R}^3} \mathcal{E}(t)f(x,y,z)w(x,y,z,t)d(x,y,z) \; dt| &\leqslant |\int_{\mathbb{R}^3} f(x,y,z)(\int_{\mathbb{R}} \mathcal{E}(-t)w(x,y,z,t)dt)d(x,y,z)| \\
&\leqslant c\|f\|_{L_{xyz}^2} \|\int_{\mathbb{R}} \mathcal{E}(-t)w(x,y,z,t)dt\|_{L_{xyz}^2} \\
&\leqslant c\|f\|_{L_{xyz}^2} \|w\|_{L_t^{q'} L_z^2 L_{xy}^{p'}} \leqslant c\|f\|_{L_{xyz}^2},
\end{aligned}
$$

and we have (3.21).

Therefore the problem is reduced to proof (3.22). Minkowski's inequality (Lemma 1.19) and Lemma 3.8 give

$$\|\int_{\mathbb{R}} \mathcal{E}(t-t')g(\cdot,t')dt'\|_{L_z^2 L_{xy}^p} \leqslant \int_{\mathbb{R}} \|\mathcal{E}(t-t')g(\cdot,t')\|_{L_z^2 L_{xy}^p}dt' \leqslant c\int_{\mathbb{R}} |t-t'|^\alpha \|g(\cdot,t')\|_{L_z^2 L_{xy}^{p'}}dt',$$

where $\alpha = -(\frac{1}{p'} - \frac{1}{p})$.

Theorem 1.26 (Hardy-Littlewood-Sobolev) and the last inequality imply

$$\|\int_{\mathbb{R}} \mathcal{E}(t-t')g(\cdot,t')dt'\|_{L_t^q L_z^2 L_{xy}^p} = \|\|\int_{\mathbb{R}} \mathcal{E}(t-t')g(\cdot,t')dt'\|_{L_z^2 L_{xy}^p}\|_{L_t^q}$$

$$\leqslant c\|\int_{\mathbb{R}} |t-t'|^\alpha \|g(\cdot,t')\|_{L_z^2 L_{xy}^{p'}}dt'\|_{L_t^q}$$

$$\leqslant c\|g\|_{L_t^{q'} L_z^2 L_{xy}^{p'}}.$$

$\square$

Next we establish some estimates associated to solutions of the linear problem

$$\begin{cases} \partial_t^2 n + \Delta_\perp n = 0 \\ n(x,0) = n_0(x) \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \\ \partial_t n(x,0) = n_1(x), \end{cases} \tag{3.24}$$

where $\Delta_\perp = \partial_x^2 + \partial_y^2$. The solution of the problem (3.24) can be written as

$$n(x,t) = N'(t)n_0 + N(t)n_1, \tag{3.25}$$

where $N(t)$ and $N'(t)$ where defined in (11) and (12).

Lemma 3.10. *For $f \in L^2(\mathbb{R}^3)$ we have*

$$\|N(t)f\|_{L^2(\mathbb{R}^3)} \leqslant |t|\|f\|_{L^2(\mathbb{R}^3)}, \tag{3.26}$$

$$\|N'(t)f\|_{L^2(\mathbb{R}^3)} \leqslant \|f\|_{L^2(\mathbb{R}^3)}, \tag{3.27}$$

and

$$\|(-\Delta_\perp)^{1/2}N(t)f\|_{L^2(\mathbb{R}^3)} \leqslant \|f\|_{L^2(\mathbb{R}^3)}. \tag{3.28}$$

Proof. We refer to [LiPoS] and references therein for a proof of this lemma. □

Lemma 3.11.

$$\|N'(t)n_0\|_{L_x^2 L_{yzT}^\infty} \leqslant \|n_0\|_{H^2(\mathbb{R}^3)},$$
(3.29)

and

$$\|N(t)n_1\|_{L_x^2 L_{yzT}^\infty} \leqslant T\|n_1\|_{H^2(\mathbb{R}^3)}.$$
(3.30)

These estimates hold exchanging x and y.

Proof. We refer to [LiPoS] and references therein for a proof of this lemma. □

Also we need the following result:

Lemma 3.12.

$$\sum_{|\alpha|\leqslant 3}\|N(t)\partial^\alpha f\|_{L_{xyz}^2} \leqslant c\|f\|_{H^2(\mathbb{R}^3)} + c|t|\|\partial_z f\|_{H^2(\mathbb{R}^3)}.$$

Proof. Fix $|\alpha| = |(\alpha_1, \alpha_2, \alpha_3)| = 3$. By Theorem 1.1 we have

$$\|N(t)\partial^\alpha f\|_{L_{xyz}^2} = \||x_1|^{\alpha_1}|x_2|^{\alpha_2}\partial_z^{\alpha_3}\widehat{N(t)}f\|_{L_{xyz}^2}.$$

Now we split in two cases. The first one is $a := \alpha_1 + \alpha_2 = 1, 2$ or 3. Observe that in this case we must have $\alpha_3 \leqslant 2$. By Lemma 1.13 we get $|x_1|^{\alpha_1}|x_2|^{\alpha_2} \leqslant c(|x_1| + |x_2|)^a$.

So, using Plancherel once more we have

$$\|N(t)\partial^\alpha f\|_{L_{xyz}^2} \leqslant c\|(|x_1| + |x_2|)^a\partial_z^{\alpha_3}\widehat{N(t)}f\|_{L_{xyz}^2} \leqslant c\|(-\Delta_\perp)^{\frac{1}{2}}(-\Delta_\perp)^{\frac{a-1}{2}}\partial_z^{\alpha_3}N(t)f\|_{L_{xyz}^2}.$$

By Lemma 3.10 we obtain

$$\|N(t)\partial^\alpha f\|_{L_{xyz}^2} \leqslant c\|(-\Delta_\perp)^{\frac{a-1}{2}}\partial_z^{\alpha_3}f\|_{L_{xyz}^2} \leqslant c\|(\partial_x^{a-1} + \partial_y^{a-1})\partial_z^{\alpha_3}f\|_{L_{xyz}^2}.$$

Using that $(a-1) + \alpha_3 = 2$ and Theorem 1.20 we obtain

$$\|N(t)\partial^\alpha f\|_{L^2_{xyz}} \leqslant c\|f\|_{H^2(\mathbb{R}^3)}.$$

The second case is $\alpha_1 + \alpha_2 = 0$. In this case we must have $\alpha_3 = 3$. Then by Lemma 3.10 and Theorem 1.20 we conclude

$$\|N(t)\partial^\alpha f\|_{L^2_{xyz}} \leqslant c\|N(t)\partial_z^3 f\|_{L^2_{xyz}} \leqslant c|t|\|\partial_z^3 f\|_{L^2_{xyz}} \leqslant c|t|\|\partial_z f\|_{H^2(\mathbb{R}^3)}.$$

$\square$

3.3   Nonlinear Estimates

In this section we will find estimates for the nonlinear terms in our analysis.

We recall the integral formulation of the IVP (10):

$$E(t) = \mathcal{E}(t)E_0 + \int_0^t \mathcal{E}(t-t')(N'(t')n_0 + N(t')n_1)E(t')dt'$$
$$+ \int \mathcal{E}(t-t')(\int_0^{t'} N(t'-s)\Delta_\perp(|E(s)|^2)ds)E(t')dt'.$$

We can rewrite this expression as

$$E(t) = \mathcal{E}(t)E_0 + \int_0^t \mathcal{E}(t-t')(EF)(t')dt' + \int_0^t \mathcal{E}(t-t')(EH)(t')dt', \qquad (3.31)$$

where

$$F(t) = N'(t)n_0 + N(t)n_1, \qquad (3.32)$$

and

$$H(t) = \int_0^t N(t-t')\Delta_\perp(|E|^2)(t')dt'. \qquad (3.33)$$

In the next lemma we treat of the nonlinearity H in the Sobolev norm $\|\cdot\|_{H^3}$. In the proof of this lemma will be clear why the Sobolev index $s = 2$ could not be reached.

Lemma 3.13.

$$\sum_{|\alpha|\leqslant 3}\|\partial^\alpha H\|_{L^2_{xyzT}} + \|H\|_{L^2_x L^\infty_{yzT}} + \|H\|_{L^2_y L^\infty_{xzT}} \leqslant \tag{3.34}$$

$$\leqslant cT\|E\|^2_{L^\infty_T H^3} + cT^{1/2}\sum_{\substack{|\alpha_1|\leqslant 3 \\ |\alpha_2|\leqslant 1}}\|\partial_x\partial^{\alpha_1}E\|_{L^\infty_x L^2_{yzT}}\|\partial^{\alpha_2}E\|_{L^2_x L^\infty_{yzT}}$$

$$+ cT^{1/2}\sum_{\substack{|\alpha_1|\leqslant 3 \\ |\alpha_2|\leqslant 1}}\|\partial_y\partial^{\alpha_1}E\|_{L^\infty_y L^2_{xzT}}\|\partial^{\alpha_2}E\|_{L^2_y L^\infty_{xzT}}.$$

Proof. Using the definition of H in (3.33) and the inequality (3.28) we have

$$\sum_{|\alpha|\leqslant 3}\|\partial^\alpha H\|_{L^2_{xyzT}} \leqslant c\sum_{|\alpha|\leqslant 3}\int_0^T \|(-\Delta_\perp)^{1/2}N(t'-s)(-\Delta_\perp)^{1/2}\partial^\alpha(|E|^2)(s)\|_{L^2_{xyzT}}ds$$

$$\leqslant c\sum_{|\alpha|\leqslant 3}\int_0^T \|(-\Delta_\perp)^{1/2}\partial^\alpha(|E|^2)(s)\|_{L^2_{xyzT}}ds \tag{3.35}$$

$$\leqslant cT^{1/2}\sum_{|\alpha|\leqslant 3}\|\partial_x\partial^\alpha(E\bar{E})\|_{L^2_{xyzT}} + cT^{1/2}\sum_{|\alpha|\leqslant 3}\|\partial_y\partial^\alpha(E\bar{E})(s)\|_{L^2_{xyzT}}.$$

Fix $|\alpha| = 3$.

Now by Leibini'z Rule (Lemma 1.16) we have

$$\|\partial_x\partial^\alpha(E\bar{E})\|_{L^2_{xyzT}} \leqslant c\sum_{\beta_1+\beta_2\leqslant\alpha}\|\partial_x(\partial^{\beta_1}E\partial^{\beta_2}\bar{E})\|_{L^2_{xyzT}}$$

$$\leqslant c\sum_{\beta_1+\beta_2\leqslant\alpha}\left(\|\partial_x\partial^{\beta_1}E\partial^{\beta_2}\bar{E}\|_{L^2_{xyzT}} + \|\partial^{\beta_1}E\partial_x\partial^{\beta_2}\bar{E}\|_{L^2_{xyzT}}\right).$$

The idea is to split the sum $\sum_{\beta_1+\beta_2\leqslant\alpha}$ in three cases depending on the value of $|\beta_2|$. For $|\beta_2| = 0,1$ we use Lemma 1.14 twice and get

$$\sum_{\substack{\beta_1+\beta_2\leqslant\alpha \\ |\beta_2|\in\{0,1\}}}\|\partial_x\partial^{\beta_1}E\partial^{\beta_2}\bar{E}\|_{L^2_{xyzT}} \leqslant \sum_{\substack{\beta_1+\beta_2\leqslant\alpha \\ |\beta_2|\in\{0,1\}}}\|\partial_x\partial^{\beta_1}E\|_{L^\infty_x L^2_{yzT}}\|\partial^{\beta_2}\bar{E}\|_{L^2_x L^\infty_{yzT}}$$

$$\leqslant \sum_{\substack{|\beta_1|\leqslant 3 \\ |\beta_2|\leqslant 1}}\|\partial_x\partial^{\beta_1}E\|_{L^\infty_x L^2_{yzT}}\|\partial^{\beta_2}E\|_{L^2_x L^\infty_{yzT}}.$$

We notice that the previous arguments used to treat the cases $|\beta_2| = 0,1$ generated the new norm $\sum_{|\beta_2|\leqslant 1}\|\partial^{\beta_2}\cdot\|_{L^2_x L^\infty_{yzT}}$. To treat this new norm it will be essencial the maximal

function estimate (3.10). If we tried to treat the case $|\beta_2| = 2$ using the same arguments it would appear one more derivative,i.e., $\sum_{|\beta_2| \leqslant 2} \|\partial^{\beta_2} \cdot\|_{L^2_x L^\infty_{yzT}}$ and our maximal function estimate would not be enough. So we must use another argument.

For $|\beta_2| = 2$ we use Lemma 1.14 and Theorems 1.23 and 1.20 and obtain

$$
\begin{aligned}
\sum_{\substack{\beta_1 + \beta_2 \leqslant \alpha \\ |\beta_2| = 2}} \|\partial_x \partial^{\beta_1} E \partial^{\beta_2} \bar{E}\|_{L^2_{xyzT}} &\leqslant \sum_{\substack{\beta_1 + \beta_2 \leqslant \alpha \\ |\beta_2| = 2}} \|\|\partial_x \partial^{\beta_1} E\|_{L^4_{xyz}} \|\partial^{\beta_2} \bar{E}\|_{L^4_{xyz}}\|_{L^2_T} \\
&\leqslant \sum_{\substack{\beta_1 + \beta_2 \leqslant \alpha \\ |\beta_2| = 2}} \|\|\partial_x D^{3/4} \partial^{\beta_1} E\|_{L^2_{xyz}} \|D^{3/4} \partial^{\beta_2} \bar{E}\|_{L^2_{xyz}}\|_{L^2_T} \\
&\leqslant \sum_{\substack{\beta_1 + \beta_2 \leqslant \alpha \\ |\beta_2| = 2}} \|\|D^{1+3/4+|\beta_1|} E\|_{L^2_{xyz}} \|D^{3/4+|\beta_2|} \bar{E}\|_{L^2_{xyz}}\|_{L^2_T} \\
&\leqslant c\|E\|^2_{L^2_T H^3} \leqslant cT^{1/2}\|E\|^2_{L^\infty_T H^3}.
\end{aligned}
$$

In the last argument we can see why it was not possible to reach the Sobolev index $s = 2$. In fact, the difficult case is $|\beta_2| = 2$ (wich implies $|\beta_1| = 0$ since we are in the case $s = 2$). So, using the last argument we would have the following inequality

$$
\sum_{\substack{|\beta_1| = 0 \\ |\beta_2| = 2}} \|\partial_x \partial^{\beta_1} E \partial^{\beta_2} \bar{E}\|_{L^2_{xyzT}} \leqslant \sum_{\substack{|\beta_1| = 0 \\ |\beta_2| = 2}} \|\|\partial_x D^{3/4} E\|_{L^2_{xyz}} \|D^{3/4} \partial^{\beta_2} \bar{E}\|_{L^2_{xyz}}\|_{L^2_T} \leqslant \|\|E\|_{H^2} \|E\|_{H^3}\|_{L^2_T},
$$

therefore it does not work. At this point it would be usefull if we had Strichartz estimates with endpoits like

$$
\|\mathcal{E}(t)f\|_{L^q_t L^\infty} \leqslant \|f\|_{L^2},
$$

but unfortunately, this is not the case. The Strichartz estimates we got are not usefull.

Finally for $|\beta_2| = 3$, we use again Lemma 1.14 and Theorems 1.22 and 1.20 to deduce

$$\sum_{\substack{\beta_1 + \beta_2 \leqslant \alpha \\ |\beta_2| = 3}} \|(\partial_x \partial^{\beta_1} E)\partial^{\beta_2} \bar{E}\|_{L^2_{xyzT}} \leqslant \sum_{|\beta_2| = 3} \|\|\partial_x E\|_{L^\infty_{xyz}} \|\partial^{\beta_2} \bar{E}\|_{L^2_{xyz}} \|_{L^2_T}$$

$$\leqslant \sum_{|\beta_2| = 3} \|\|\partial_x E\|_{H^2} \|\partial^{\beta_2} E\|_{L^2_{xyz}} \|_{L^2_T}$$

$$\leqslant \sum_{|\beta_2| \leqslant 3} \|\|\partial^{\beta_2} E\|^2_{L^2_{xyz}} \|_{L^2_T}$$

$$\leqslant c \|\|E\|^2_{H^3} \|_{L^2_T} \leqslant c T^{1/2} \|E\|^2_{L^\infty_T H^3}.$$

Analogously we estimate $\displaystyle\sum_{\beta_1 + \beta_2 \leqslant \alpha} \|\partial^{\beta_1} E \partial_x \partial^{\beta_2} \bar{E}\|_{L^2_{xyzT}}$.

Therefore

$$\sum_{|\alpha| \leqslant 3} \|\partial_x \partial^\alpha (E\bar{E})(s)\|_{L^2_{xyzT}} \leqslant c T^{1/2} \sum_{\substack{|\beta_1| \leqslant 3 \\ |\beta_2| \leqslant 1}} \|\partial_x \partial^{\beta_1} E\|_{L^\infty_x L^2_{yzT}} \|\partial^{\beta_2} E\|_{L^2_x L^\infty_{yzT}} \tag{3.36}$$

$$+ c T \|E\|^2_{L^\infty_T H^3}.$$

By similar arguments we obtain

$$\sum_{|\alpha| \leqslant 3} \|\partial_y \partial^\alpha (E\bar{E})(s)\|_{L^2_T L^2} \leqslant c T^{1/2} \sum_{\substack{|\beta_1| \leqslant 3 \\ |\beta_2| \leqslant 1}} \|\partial_y \partial^{\beta_1} E\|_{L^\infty_y L^2_{xzT}} \|\partial^{\beta_2} E\|_{L^2_y L^\infty_{xzT}} \tag{3.37}$$

$$+ c T \|E\|^2_{L^\infty_T H^3}.$$

Replacing inequalities (3.36) and (3.37) in (3.35) we get the first result.

Now, we use Lemma 3.11, Lemma 1.14 and Theorem 1.20 to obtain

$$\|H\|_{L^2_x L^\infty_{yzT}} \leqslant \int_0^T \|N(t - t')\Delta_\perp(|E|^2)(t')\|_{L^2_x L^\infty_{yzT}} dt' \leqslant T \int_0^T \|\Delta_\perp(|E|^2)(t')\|_{H^2} dt'$$

$$\leqslant T^{3/2} \|\Delta_\perp(|E|^2)\|_{L^2_T H^2} \leqslant T^{3/2} \sum_{|\beta| \leqslant 2} \|\Delta_\perp \partial^\beta |E|^2\|_{L^2_T L^2}$$

$$\leqslant T^{3/2} \sum_{|\beta| \leqslant 2} \left(\|\partial^2_x \partial^\beta |E|^2\|_{L^2_{xyzT}} + \|\partial^2_y \partial^\beta |E|^2\|_{L^2_{xyzT}} \right)$$

$$\leqslant T^{3/2} \sum_{|\alpha| \leqslant 3} \left(\|\partial_x \partial^\alpha |E|^2\|_{L^2_{xyzT}} + \|\partial_y \partial^\alpha |E|^2\|_{L^2_{xyzT}} \right).$$

Hence the previous arguments can be applied to obtain the result. $\qquad\square$

Lemma 3.14.

$$\sum_{|\alpha|\leqslant 3}\|\partial^\alpha(EF)\|_{L^2_{xyzT}} \leqslant c\|E\|_{L^\infty_T H^3}\big(T^{1/2}\|n_0\|_{H^3} + cT^{1/2}\|n_1\|_{H^2} + cT^{3/2}\|\partial_z n_1\|_{H^2}\big). \qquad (3.38)$$

and

$$\begin{aligned}
\sum_{|\alpha|\leqslant 3}\|\partial^\alpha(EH)\|_{L^2_{xyzT}} \leqslant{}& cT^{1/2}\|E\|_{L^\infty_T H^3}\sum_{\substack{|\alpha_1|\leqslant 3\\|\alpha_2|\leqslant 1}}\|\partial_x\partial^{\alpha_1}E\|_{L^\infty_x L^2_{yzT}}\|\partial^{\alpha_2}E\|_{L^2_x L^\infty_{yzT}}\\
&+ cT^{1/2}\|E\|_{L^\infty_T H^3}\sum_{\substack{|\alpha_1|\leqslant 3\\|\alpha_2|\leqslant 1}}\|\partial_y\partial^{\alpha_1}E\|_{L^\infty_y L^2_{xzT}}\|\partial^{\alpha_2}E\|_{L^2_y L^\infty_{xzT}} \qquad (3.39)\\
&+ cT\|E\|^3_{L^\infty_T H^3}.
\end{aligned}$$

Proof. To obtain the estimate (3.38) we use the Lemma 1.16 to yield

$$\|\partial^\alpha(EF)(t')\|_{L^2_{xyzT}} \leqslant c\sum_{\beta_1+\beta_2\leqslant\alpha}\|\partial^{\beta_1}E\partial^{\beta_2}F\|_{L^2_{xyzT}}. \qquad (3.40)$$

Fix $|\alpha| = 3$.

For $|\beta_2| = 0,1$ we use Lemma 1.14, Theorems 1.22 and 1.20 and the definition of F in (3.32) to obtain

$$\begin{aligned}
\sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|\in\{0,1\}}}\|\partial^{\beta_1}E\partial^{\beta_2}F\|_{L^2_{xyzT}} \leqslant{}& \sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|\in\{0,1\}}}\big\|\|\partial^{\beta_1}E\|_{L^2}\|\partial^{\beta_2}F\|_{L^\infty}\big\|_{L^2_T} \leqslant c\sum_{|\beta_2|\in\{0,1\}}\big\|\|E\|_{H^3}\|\partial^{\beta_2}F\|_{H^2}\big\|_{L^2_T}\\
\leqslant{}& c\|E\|_{L^\infty_T H^3}\sum_{|\alpha_1|\leqslant 3}\big\|\|\partial^{\alpha_1}F\|_{L^2}\big\|_{L^2_T}\\
\leqslant{}& c\|E\|_{L^\infty_T H^3}\sum_{|\alpha_1|\leqslant 3}\big(\|N'(t)\partial^{\alpha_1}n_0\|_{L^2_{xyzT}} + \|N(t)\partial^{\alpha_1}n_1\|_{L^2_{xyzT}}\big).
\end{aligned}$$

By Lemma 3.10 and Lemma 3.12 we have

$$\begin{aligned}
\sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|\in\{0,1\}}}\|\partial^{\beta_1}E\partial^{\beta_2}F\|_{L^2_{xyzT}} \leqslant{}& c\|E\|_{L^\infty_T H^3}\sum_{|\alpha_1|\leqslant 3}\big(\|\partial^{\alpha_1}n_0\|_{L^2_T L^2} + T^{1/2}\|n_1\|_{H^2} + T^{3/2}\|\partial_z n_1\|_{H^2}\big)\\
\leqslant{}& c\|E\|_{L^\infty_T H^3}\sum_{|\beta_2|\leqslant 3}\big(T^{1/2}\|\partial^{\beta_2}n_0\|_{L^2} + T^{1/2}\|n_1\|_{H^2} + T^{3/2}\|\partial_z n_1\|_{H^2}\big).
\end{aligned}$$

Therefore by Theorem 1.20 we obtain

$$\sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|\in\{0,1\}}}\|\partial^{\beta_1}E\partial^{\beta_2}F\|_{L^2_{xyzT}}\leqslant c\|E\|_{L^\infty_T H^3}\big(T^{1/2}\|n_0\|_{H^3}+T^{1/2}\|n_1\|_{H^2}+T^{3/2}\|\partial_z n_1\|_{H^2}\big). \quad (3.41)$$

For $|\beta_2|=2,3$ we use the same arguments, i.e., Lemma 1.14, Lemma 3.10, Lemma 3.12 and Theorem 1.22 to conclude

$$\begin{aligned}
\sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|\in\{2,3\}}}\|\partial^{\beta_1}E\partial^{\beta_2}F\|_{L^2_{xyzT}} &\leqslant \sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|\in\{2,3\}}}\big\|\|\partial^{\beta_1}E\|_{L^\infty}\|\partial^{\beta_2}F\|_{L^2}\big\|_{L^2_T}\leqslant c\sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|\in\{2,3\}}}\big\|\|\partial^{\beta_1}E\|_{H^2}\|\partial^{\beta_2}F\|_{L^2}\big\|_{L^2_T}\\
&\leqslant c\sum_{\substack{|\alpha_1|\leqslant 3\\|\beta_2|\in\{2,3\}}}\big\|\|\partial^{\alpha_1}E\|_{L^2}\big(\|N'(t)\partial^{\beta_2}n_0\|_{L^2}+\|N(t)\partial^{\beta_2}n_1\|_{L^2}\big)\big\|_{L^2_T}\\
&\leqslant c\|E\|_{L^\infty_T H^3}\sum_{|\alpha_2|\leqslant 3}\big(\|\partial^{\alpha_2}n_0\|_{L^2_T L^2}+T^{1/2}\|n_1\|_{H^2}+T^{3/2}\|\partial_z n_1\|_{H^2}\big)\\
&\leqslant c\|E\|_{L^\infty_T H^3}\sum_{|\alpha_2|\leqslant 3}\big(T^{1/2}\|\partial^{\alpha_2}n_0\|_{L^2}+T^{1/2}\|n_1\|_{H^2}+T^{3/2}\|\partial_z n_1\|_{H^2}\big).
\end{aligned}$$

By Theorem 1.20 we have

$$\sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|\in\{2,3\}}}\|\partial^{\beta_1}E\partial^{\beta_2}F\|_{L^2_{xyzT}}\leqslant c\|E\|_{L^\infty_T H^3}\big(T^{1/2}\|n_0\|_{H^3}+T^{1/2}\|n_1\|_{H^2}+T^{3/2}\|\partial_z n_1\|_{H^2}\big). \quad (3.42)$$

From inequalities (3.41) and (3.42) we obtain (3.38).

To prove (3.39) we use the Lemma 1.16 and get

$$\|\partial^\alpha(EH)\|_{L^2_{xyzT}}\leqslant c\sum_{\beta_1+\beta_2\leqslant\alpha}\|\partial^{\beta_1}E\partial^{\beta_2}H\|_{L^2_{xyzT}}.$$

Fix $|\alpha|=3$.

For $|\beta_2|=0,1$ we use Lemma 1.14 and Theorems 1.22 and 1.20 to obtain

$$\begin{aligned}
\sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|\in\{0,1\}}}\|\partial^{\beta_1}E\partial^{\beta_2}H\|_{L^2_{xyzT}} &\leqslant \sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|\in\{0,1\}}}\big\|\|\partial^{\beta_1}E\|_{L^2}\|\partial^{\beta_2}H\|_{L^\infty}\big\|_{L^2_T}\leqslant c\sum_{|\beta_2|\in\{0,1\}}\big\|\|E\|_{H^3}\|\partial^{\beta_2}H\|_{H^2}\big\|_{L^2_T}\\
&\leqslant c\|E\|_{L^\infty_T H^3}\sum_{|\alpha_2|\leqslant 3}\|\partial^{\alpha_2}H\|_{L^2_T L^2}.
\end{aligned}$$

Using Lemma 3.13 we have

$$\sum_{\substack{\beta_1+\beta_2\leqslant\alpha \\ |\beta_2|\in\{0,1\}}} \|\partial^{\beta_1} E \partial^{\beta_2} H\|_{L^2_{xyzT}} \leqslant cT^{1/2}c\|E\|_{L^\infty_T H^3} \sum_{\substack{|\alpha_1|\leqslant 3 \\ |\alpha_2|\leqslant 1}} \|\partial_x\partial^{\alpha_1} E\|_{L^\infty_x L^2_{yzT}}\|\partial^{\alpha_2} E\|_{L^2_x L^\infty_{yzT}}$$

$$+ cT^{1/2}c\|E\|_{L^\infty_T H^3} \sum_{\substack{|\alpha_1|\leqslant 3 \\ |\alpha_2|\leqslant 1}} \|\partial_y\partial^{\alpha_1} E\|_{L^\infty_y L^2_{xzT}}\|\partial^{\alpha_2} E\|_{L^2_y L^\infty_{xzT}}$$

$$+ cT\|E\|^3_{L^\infty_T H^3}.$$

For $|\beta_2| = 2,3$ we use Lemma 1.14 and Theorem 1.22 to get

$$\sum_{\substack{\beta_1+\beta_2\leqslant\alpha \\ |\beta_2|\in\{2,3\}}} \|\partial^{\beta_1} E \partial^{\beta_2} H\|_{L^2_{xyzT}} \leqslant \sum_{\substack{\beta_1+\beta_2\leqslant\alpha \\ |\beta_2|\in\{2,3\}}} \|\|\partial^{\beta_1} E\|_{L^\infty}\|\partial^{\beta_2} H\|_{L^2}\|_{L^2_T} \leqslant c \sum_{\substack{\beta_1+\beta_2\leqslant\alpha \\ |\beta_2|\in\{2,3\}}} \|\|\partial^{\beta_1} E\|_{H^2}\|\partial^{\beta_2} H\|_{L^2}\|_{L^2_T}$$

$$\leqslant c\|E\|_{L^\infty_T H^3} \sum_{|\beta_2|\in\{2,3\}} \|\partial^{\beta_2} H\|_{L^2_T L^2}.$$

Again Lemma 3.13 yields to

$$\sum_{\substack{\beta_1+\beta_2\leqslant\alpha \\ |\beta_2|\in\{2,3\}}} \|\partial^{\beta_1} E \partial^{\beta_2} H\|_{L^2_{xyzT}} \leqslant cT^{1/2}c\|E\|_{L^\infty_T H^3} \sum_{\substack{|\alpha|\leqslant 3 \\ \beta_1+\beta_2\leqslant\alpha \\ |\beta_2|\in\{0,1\}}} \|\partial_x\partial^{\beta_1} E\|_{L^\infty_x L^2_{yzT}}\|\partial^{\beta_2} E\|_{L^2_x L^\infty_{yzT}}$$

$$+ cT^{1/2}c\|E\|_{L^\infty_T H^3} \sum_{\substack{|\alpha|\leqslant 3 \\ \beta_1+\beta_2\leqslant\alpha \\ |\beta_2|\in\{0,1\}}} \|\partial_y\partial^{\beta_1} E\|_{L^\infty_y L^2_{xzT}}\|\partial^{\beta_2} E\|_{L^2_y L^\infty_{xzT}}$$

$$+ cT\|E\|^3_{L^\infty_T H^3}.$$

$\square$

Lemma 3.15.

$$\sum_{|\alpha|\leqslant 3} \|D_x^{1/2}\int_0^t \mathcal{E}(t-t')\partial^\alpha(EF)(t')dt'\|_{L^\infty_T L^2_{xyz}} + \sum_{|\alpha|\leqslant 3} \|\partial_x\int_0^t \mathcal{E}(t-t')\partial^\alpha(EF)(t')dt'\|_{L^\infty_x L^2_{yzT}} \leqslant$$

$$\leqslant cT^{1/2}\|E\|_{L^\infty_T H^3}\big(\|n_0\|_{H^3}(1+T) + \|n_1\|_{H^2}(T+T^{1/2}) + \|\partial_z n_1\|_{H^2}T^{3/2}\big) +$$

$$+ c\sum_{|\beta_1|\leqslant 1} \|\partial^{\beta_1} E\|_{L^2_x L^\infty_{yzT}}\big(T^{1/2}\|n_0\|_{H^3} + T^{1/2}\|n_1\|_{H^2(\mathbb{R}^3)} + T^{3/2}\|\partial_z n_1\|_{H^2(\mathbb{R}^3)}\big).$$

These estimates holds exchanging x and y.

Proof. Fix $|\alpha| = 3$.

By Lemma 1.16 we have

$$\|D_x^{1/2} \int_0^t \mathcal{E}(t - t')\partial^\alpha(EF)(t')dt'\|_{L_T^\infty L_{xyz}^2} + \|\partial_x \int_0^t \mathcal{E}(t - t')\partial^\alpha(EF)(t')dt'\|_{L_x^\infty L_{yzT}^2} \leqslant$$

$$\sum_{\beta_1 + \beta_2 \leqslant \alpha} \left(\|D_x^{1/2} \int_0^t \mathcal{E}(t - t')(\partial^{\beta_1} E \partial^{\beta_2} F)(t')dt'\|_{L_T^\infty L_{xyz}^2} + \|\partial_x \int_0^t \mathcal{E}(t - t')(\partial^{\beta_1} E \partial^{\beta_2} F)(t')dt'\|_{L_x^\infty L_{yzT}^2} \right).$$

Then we split the sum in three cases, depending on the value of $|\beta_2|$.

For $|\beta_2| = 0$ we use Proposition 3.2 to get

$$\sum_{\substack{\beta_1 + \beta_2 \leqslant \alpha \\ |\beta_2| = 0}} \left(\|D_x^{1/2} \int_0^t \mathcal{E}(t - t')(\partial^{\beta_1} E \partial^{\beta_2} F)(t')dt'\|_{L_T^\infty L_{xyz}^2} + \|\partial_x \int_0^t \mathcal{E}(t - t')(\partial^{\beta_1} E \partial^{\beta_2} F)(t')dt'\|_{L_x^\infty L_{yzT}^2} \right) \leqslant$$

$$\leqslant \sum_{|\beta_1| = 3} \|\partial^{\beta_1} EF\|_{L_x^1 L_{yzT}^2}.$$

Using Lemma 1.14 twice, Theorem 1.20 and the properties of the operator $N(t)$ (Lemma 3.11) we obtain

$$\sum_{|\beta_1| = 3} \|\partial^{\beta_1} EF\|_{L_x^1 L_{yzT}^2} \leqslant \sum_{|\beta_1| = 3} \|\partial^{\beta_1} E\|_{L_{xyzT}^2} \|F\|_{L_x^2 L_{yzT}^\infty}$$

$$\leqslant cT^{1/2}\|E\|_{L_T^2 H^3} \left(\|N'(t)n_0\|_{L_x^2 L_{yzT}^\infty} + \|N(t)n_1\|_{L_x^2 L_{yzT}^\infty} \right)$$

$$\leqslant cT^{1/2}\|E\|_{L_T^\infty H^3} \left(\|n_0\|_{H^2} + T\|n_1\|_{H^2} \right).$$

For $|\beta_2| = 1$ we use Proposition 3.2 and group properties to get

$$\sum_{\substack{\beta_1 + \beta_2 \leqslant \alpha \\ |\beta_2| = 1}} \left(\|D_x^{1/2} \int_0^t \mathcal{E}(t - t')(\partial^{\beta_1} E \partial^{\beta_2} F)(t')dt'\|_{L_T^\infty L_{xyz}^2} + \|\partial_x \int_0^t \mathcal{E}(t - t')(\partial^{\beta_1} E \partial^{\beta_2} F)(t')dt'\|_{L_x^\infty L_{yzT}^2} \right) \leqslant$$

$$\leqslant c \sum_{\substack{\beta_1 + \beta_2 \leqslant \alpha \\ |\beta_2| = 1}} \int_0^T \left(\|\mathcal{E}(t - t')D_x^{1/2}(\partial^{\beta_1} E \partial^{\beta_2} F)(t')\|_{L_T^\infty L_{xyz}^2} + \|D_x^{1/2}\mathcal{E}(t - t')D_x^{1/2}(\partial^{\beta_1} E \partial^{\beta_2} F)(t')\|_{L_x^\infty L_{yzT}^2} \right)dt'$$

$$\leqslant c \sum_{\substack{\beta_1 + \beta_2 \leqslant \alpha \\ |\beta_2| = 1}} \int_0^T \|D_x^{1/2}(\partial^{\beta_1} E \partial^{\beta_2} F)(t')\|_{L_{xyz}^2} dt'.$$

Using Lemma 1.16 and Lemma 1.14 we conclude

$$\sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|=1}}\big(\|D_x^{1/2}\int_0^t\mathcal{E}(t-t')(\partial^{\beta_1}E\partial^{\beta_2}F)(t')dt'\|_{L_T^\infty L_{xyz}^2}+\|\partial_x\int_0^t\mathcal{E}(t-t')(\partial^{\beta_1}E\partial^{\beta_2}F)(t')dt'\|_{L_x^\infty L_{yzT}^2}\big)\leqslant$$

$$\leqslant c\sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|=1}}\int_0^T\big(\|\|D_x^{1/2}\partial^{\beta_1}E(t')\|_{L_x^2}\|\partial^{\beta_2}F(t')\|_{L_x^\infty}\|_{L_{yz}^2}+\|\|\partial^{\beta_1}E(t')\|_{L_x^4}\|D_x^{1/2}\partial^{\beta_2}F(t')\|_{L_x^4}\|_{L_{yz}^2}\big)dt'$$

$$\leqslant c\sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|=1}}\int_0^T\big(\|D_x^{1/2}\partial^{\beta_1}E(t')\|_{L_{xyz}^2}\|\partial^{\beta_2}F(t')\|_{L_{xyz}^\infty}+\|\partial^{\beta_1}E(t')\|_{L_{xyz}^4}\|D_x^{1/2}\partial^{\beta_2}F(t')\|_{L_{xyz}^4}\big)dt'.$$

By Theorem 1.22, Theorem 1.23 and Theorem 1.20 we have

$$\sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|=1}}\big(\|D_x^{1/2}\int_0^t\mathcal{E}(t-t')(\partial^{\beta_1}E\partial^{\beta_2}F)(t')dt'\|_{L_T^\infty L_{xyz}^2}+\|\partial_x\int_0^t\mathcal{E}(t-t')(\partial^{\beta_1}E\partial^{\beta_2}F)(t')dt'\|_{L_x^\infty L_{yzT}^2}\big)\leqslant$$

$$\leqslant c\sum_{|\beta_2|=1}\int_0^T\|E(t')\|_{H^3}\|\partial^{\beta_2}F(t')\|_{H^2}dt'+c\sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|=1}}\int_0^T\|D^{3/4}\partial^{\beta_1}E(t')\|_{L_{xyz}^2}\|D^{3/4}D_x^{1/2}\partial^{\beta_2}F(t')\|_{L_{xyz}^2}dt'$$

$$\leqslant c\sum_{|\alpha_2|\leqslant3}\|E\|_{L_T^\infty H^3}\int_0^T\|\partial^{\alpha_2}F(t')\|_{L_{xyz}^2}dt'+\sum_{|\alpha_2|\leqslant3}\int_0^T\|E(t')\|_{H^3}\|\partial^{\alpha_2}F(t')\|_{L_{xyz}^2}dt'$$

$$\leqslant c\sum_{|\alpha_2|\leqslant3}\|E\|_{L_T^\infty H^3}\int_0^T\big(\|N'(t')\partial^{\alpha_2}n_0\|_{L_{xyz}^2}+\|N(t')\partial^{\alpha_2}n_1\|_{L_{xyz}^2}\big)dt'.$$

Finally using Lemma 3.12 and Lemma 3.10 we obtain

$$\sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|=1}}\big(\|D_x^{1/2}\int_0^t\mathcal{E}(t-t')(\partial^{\beta_1}E\partial^{\beta_2}F)(t')dt'\|_{L_T^\infty L_{xyz}^2}+\|\partial_x\int_0^t\mathcal{E}(t-t')(\partial^{\beta_1}E\partial^{\beta_2}F)(t')dt'\|_{L_x^\infty L_{yzT}^2}\big)\leqslant$$

$$\leqslant c\sum_{|\alpha_2|\leqslant3}\|E\|_{L_T^\infty H^3}\int_0^T\big(\|\partial^{\alpha_2}n_0\|_{L_{xyz}^2}+\|n_1\|_{H^2(\mathbb{R}^3)}+|t|\|\partial_zn_1\|_{H^2(\mathbb{R}^3)}\big)dt'$$

$$\leqslant c\|E\|_{L_T^\infty H^3}\big(T\|n_0\|_{H^3(\mathbb{R}^3)}+T\|n_1\|_{H^2(\mathbb{R}^3)}+T^2\|\partial_zn_1\|_{H^2(\mathbb{R}^3)}\big).$$

For $|\beta_2|=2,3$ we use Proposition 3.2 to get

$$\sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|\in\{2,3\}}}\big(\|D_x^{1/2}\int_0^t\mathcal{E}(t-t')(\partial^{\beta_1}E\partial^{\beta_2}F)(t')dt'\|_{L_T^\infty L_{xyz}^2}+\|\partial_x\int_0^t\mathcal{E}(t-t')(\partial^{\beta_1}E\partial^{\beta_2}F)(t')dt'\|_{L_x^\infty L_{yzT}^2}\big)$$

$$\leqslant c\sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|\in\{2,3\}}}\|\partial^{\beta_1}E\partial^{\beta_2}F\|_{L_x^1 L_{yzT}^2}.$$

Using Lemma 1.14 twice, Lemma 3.12, Theorem 1.20 and Lemma 3.10 we get

$$\sum_{\substack{\beta_1+\beta_2\leqslant\alpha \\ |\beta_2|\in\{2,3\}}} \|\partial^{\beta_1}E\partial^{\beta_2}F\|_{L^1_xL^2_{yzT}} \leqslant$$

$$\leqslant c \sum_{\substack{\beta_1+\beta_2\leqslant\alpha \\ |\beta_2|\in\{2,3\}}} \|\partial^{\beta_1}E\|_{L^2_xL^\infty_{yzT}}\|\partial^{\beta_2}F\|_{L^2_{xyzT}}$$

$$\leqslant c \sum_{\substack{\beta_1+\beta_2\leqslant\alpha \\ |\beta_2|\in\{2,3\}}} \|\partial^{\beta_1}E\|_{L^2_xL^\infty_{yzT}}\left(\|N'(t)\partial^{\beta_2}n_0\|_{L^2_TL^2_{xyz}} + \|N(t)\partial^{\beta_2}n_1\|_{L^2_TL^2_{xyz}}\right)$$

$$\leqslant c \sum_{\substack{\beta_1+\beta_2\leqslant\alpha \\ |\beta_2|\in\{2,3\}}} \|\partial^{\beta_1}E\|_{L^2_xL^\infty_{yzT}}\left(\|\partial^{\beta_2}n_0\|_{L^2_TL^2} + T^{1/2}\|n_1\|_{H^2(\mathbb{R}^3)} + T^{3/2}\|\partial_z n_1\|_{H^2(\mathbb{R}^3)}\right)$$

$$\leqslant c\left(T^{1/2}\|n_0\|_{H^3} + T^{1/2}\|n_1\|_{H^2(\mathbb{R}^3)} + T^{3/2}\|\partial_z n_1\|_{H^2(\mathbb{R}^3)}\right) \sum_{|\beta_1|\leqslant 1} \|\partial^{\beta_1}E\|_{L^2_xL^\infty_{yzT}}.$$

$$\square$$

Lemma 3.16.

$$\sum_{|\alpha|\leqslant 3}\|D_x^{1/2}\int_0^t\mathcal{E}(t-t')\partial^\alpha(EH)(t')dt'\|_{L^\infty_TL^2_{xyz}} + \sum_{|\alpha|\leqslant 3}\|\partial_x\int_0^t\mathcal{E}(t-t')\partial^\alpha(EH)(t')dt'\|_{L^\infty_xL^2_{yzT}} \leqslant$$

$$\leqslant c\Big(\sum_{\substack{|\alpha_1|\leqslant 3 \\ |\alpha_2|\leqslant 1}}\|\partial_x\partial^{\alpha_1}E\|_{L^\infty_xL^2_{yzT}}\|\partial^{\alpha_2}E\|_{L^2_xL^\infty_{yzT}} + \sum_{\substack{|\alpha_1|\leqslant 3 \\ |\alpha_2|\leqslant 1}}\|\partial_y\partial^{\alpha_1}E\|_{L^\infty_yL^2_{xzT}}\|\partial^{\alpha_2}E\|_{L^2_yL^\infty_{xzT}}\Big)\times$$

$$\times\left(T + T^{1/2}\sum_{|\alpha_2|\leqslant 1}\|\partial^{\alpha_2}E\|_{L^2_xL^\infty_{yzT}} + \|E\|_{L^\infty_TH^3}\right) + cT\|E\|^2_{L^\infty_TH^3}\times\left(T^{1/2}\|E\|_{L^\infty_TH^3} + \sum_{|\alpha_2|\leqslant 1}\|\partial^{\alpha_2}E\|_{L^2_xL^\infty_{yzT}}\right).$$

These estimates holds exchanging x and y.

Proof. By Lemma 1.16 we have

$$\|\partial_x\int_0^t\mathcal{E}(t-t')\partial^\alpha(EH)(t')dt'\|_{L^\infty_xL^2_{yzT}} + \|D_x^{1/2}\int_0^t\mathcal{E}(t-t')\partial^\alpha(EH)(t')dt'\|_{L^\infty_TL^2_{xyz}}$$

$$\leqslant \sum_{\beta_1+\beta_2\leqslant\alpha}\left(\|\partial_x\int_0^t\mathcal{E}(t-t')(\partial^{\beta_1}E\partial^{\beta_2}H)(t')dt'\|_{L^\infty_xL^2_{yzT}} + \|D_x^{1/2}\int_0^t\mathcal{E}(t-t')(\partial^{\beta_1}E\partial^{\beta_2}H)(t')dt'\|_{L^\infty_TL^2_{xyz}}\right).$$

Fix $|\alpha| = 3$.

For $|\beta_2| = 0$ we use Proposition 3.2 to obtain

$$\sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|=0}} \left(\|\partial_x \int_0^t \mathcal{E}(t-t')(\partial^{\beta_1}E\partial^{\beta_2}H)(t')dt'\|_{L_x^\infty L_{yzT}^2} + \|D_x^{1/2} \int_0^t \mathcal{E}(t-t')(\partial^{\beta_1}E\partial^{\beta_2}H)(t')dt'\|_{L_T^\infty L_{xyz}^2} \right) \leqslant$$

$$\leqslant \sum_{|\beta_1|\leqslant 3} \|(\partial^{\beta_1}EH)(t')dt'\|_{L_x^1 L_{yzT}^2}.$$

Now using Lemma 1.14 twice, Theorem 1.20 and Lemma 3.13 we obtain

$$\sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|=0}} \left(\|\partial_x \int_0^t \mathcal{E}(t-t')(\partial^{\beta_1}E\partial^{\beta_2}H)(t')dt'\|_{L_x^\infty L_{yzT}^2} + \|D_x^{1/2} \int_0^t \mathcal{E}(t-t')(\partial^{\beta_1}E\partial^{\beta_2}H)(t')dt'\|_{L_T^\infty L_{xyz}^2} \right)$$

$$\leqslant c \sum_{|\beta_1|\leqslant 3} \|\partial^{\beta_1}E\|_{L_{xyzT}^2}\|H\|_{L_x^2 L_{yzT}^\infty} \leqslant cT\|E\|_{L_T^\infty H^3} \Big(\sum_{\substack{|\alpha_1|\leqslant 3\\|\alpha_2|\leqslant 1}} \|\partial_x\partial^{\alpha_1}E\|_{L_x^\infty L_{yzT}^2}\|\partial^{\alpha_2}E\|_{L_x^2 L_{yzT}^\infty}$$

$$\tag{3.43}$$

$$+ \sum_{\substack{|\alpha_1|\leqslant 3\\|\alpha_2|\leqslant 1}} \|\partial_y\partial^{\alpha_1}E\|_{L_y^\infty L_{xzT}^2}\|\partial^{\alpha_2}E\|_{L_y^2 L_{xzT}^\infty} \Big) + cT^{3/2}\|E\|_{L_T^\infty H^3}^3.$$

For $|\beta_2| = 1$ we use Lemma 1.3, Proposition 3.2 and group properties to get

$$\sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|=1}} \left(\|D_x^{1/2} \int_0^t \mathcal{E}(t-t')(\partial^{\beta_1}E\partial^{\beta_2}H)(t')dt'\|_{L_T^\infty L_{xyz}^2} + \|\partial_x \int_0^t \mathcal{E}(t-t')(\partial^{\beta_1}E\partial^{\beta_2}H)(t')dt'\|_{L_x^\infty L_{yzT}^2} \right) \leqslant$$

$$\leqslant \sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|=1}} \int_0^T \left(\|\mathcal{E}(t-t')D_x^{1/2}(\partial^{\beta_1}E\partial^{\beta_2}H)(t')\|_{L_T^\infty L_{xyz}^2} + \|D_x^{1/2}\mathcal{E}(t-t')\tilde{D}_x^{1/2}(\partial^{\beta_1}E\partial^{\beta_2}H)(t')\|_{L_x^\infty L_{yzT}^2} \right)dt'$$

$$\leqslant \sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|=1}} \int_0^T \|D_x^{1/2}(\partial^{\beta_1}E\partial^{\beta_2}H)(t')\|_{L_{xyz}^2}dt'.$$

By Lemma 1.17 we obtain

$$\sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|=1}} \int_0^T \|D_x^{1/2}(\partial^{\beta_1}E\partial^{\beta_2}H)(t')\|_{L_{xyz}^2}dt' \leqslant$$

$$\leqslant c \sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|=1}} \int_0^T \|\partial^{\beta_1}E(t')\|_{L_{xyz}^4}\|D_x^{1/2}\partial^{\beta_2}H(t')\|_{L_{xyz}^4}dt' + \sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|=1}} \int_0^T \|D_x^{1/2}\partial^{\beta_1}E(t')\|_{L_{xyz}^2}\|\partial^{\beta_2}H(t')\|_{L^\infty}dt'.$$

Using Theorems 1.22, 1.23 and Theorem 1.20 we get

$$\sum_{\substack{\beta_1+\beta_2\leqslant\alpha \\ |\beta_2|=1}} \int_0^T \|D_x^{1/2}(\partial^{\beta_1}E\partial^{\beta_2}H)(t')\|_{L^2_{xyz}}dt' \leqslant$$

$$\leqslant \sum_{\substack{\beta_1+\beta_2\leqslant\alpha \\ |\beta_2|=1}} \int_0^T \|D^{3/4}\partial^{\beta_1}E(t')\|_{L^2_{xyz}}\|D^{3/4}D_x^{1/2}\partial^{\beta_2}H(t')\|_{L^2_{xyz}}dt'$$

$$+ \sum_{\substack{\beta_1+\beta_2\leqslant\alpha \\ |\beta_2|=1}} \int_0^T \|D_x^{1/2}\partial^{\beta_1}E(t')\|_{L^2_{xyz}}\|\partial^{\beta_2}H(t')\|_{H^2}dt'$$

$$\leqslant \sum_{|\alpha_1|\leqslant 3} \int_0^T \|E(t')\|_{H^3}\|\partial^{\alpha_1}H(t')\|_{L^2_{xyz}}dt'$$

$$+ T^{1/2}\|E\|_{L^\infty_T H^3} \sum_{|\alpha_1|\leqslant 3} \|\partial^{\alpha_1}H(t')\|_{L^2_{xyzT}}.$$

Finally by Lemma 3.13 we have

$$\sum_{\substack{\beta_1+\beta_2\leqslant\alpha \\ |\beta_2|=1}} \left(\|D_x^{1/2}\int_0^t \mathcal{E}(t-t')(\partial^{\beta_1}E\partial^{\beta_2}H)(t')dt'\|_{L^\infty_T L^2_{xyz}} + \|\partial_x\int_0^t \mathcal{E}(t-t')(\partial^{\beta_1}E\partial^{\beta_2}H)(t')dt'\|_{L^\infty_x L^2_{yzT}}\right) \leqslant$$

$$\leqslant cT\|E\|_{L^\infty_T H^3}\Big(\sum_{\substack{|\alpha_1|\leqslant 3 \\ |\alpha_2|\leqslant 1}} \|\partial_x\partial^{\alpha_1}E\|_{L^\infty_x L^2_{yzT}}\|\partial^{\alpha_2}E\|_{L^2_x L^\infty_{yzT}} \tag{3.44}$$

$$+ \sum_{\substack{|\alpha_1|\leqslant 3 \\ |\alpha_2|\leqslant 1}} \|\partial_y\partial^{\alpha_1}E\|_{L^\infty_y L^2_{xzT}}\|\partial^{\alpha_2}E\|_{L^2_y L^\infty_{xzT}}\Big) + cT^{3/2}\|E\|^3_{L^\infty_T H^3}\Big).$$

For $|\beta_2|=2,3$ we use Proposition 3.2 to get

$$\sum_{\substack{\beta_1+\beta_2\leqslant\alpha \\ |\beta_2|\in\{2,3\}}} \left(\|D_x^{1/2}\int_0^t \mathcal{E}(t-t')(\partial^{\beta_1}E\partial^{\beta_2}H)(t')dt'\|_{L^\infty_T L^2_{xyz}} + \|\partial_x\int_0^t \mathcal{E}(t-t')(\partial^{\beta_1}E\partial^{\beta_2}H)(t')dt'\|_{L^\infty_x L^2_{yzT}}\right)$$

$$\leqslant \sum_{\substack{\beta_1+\beta_2\leqslant\alpha \\ |\beta_2|\in\{2,3\}}} \|\partial^{\beta_1}E\partial^{\beta_2}H\|_{L^1_x L^2_{yzT}}.$$

Next, we use Lemma 1.14 twice and Lemma 3.13 to obtain

$$\sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|\in\{2,3\}}}\big(\|D_x^{1/2}\int_0^t\mathcal{E}(t-t')(\partial^{\beta_1}E\partial^{\beta_2}H)(t')dt'\|_{L_T^\infty L_{xyz}^2}+\|\partial_x\int_0^t\mathcal{E}(t-t')(\partial^{\beta_1}E\partial^{\beta_2}H)(t')dt'\|_{L_x^\infty L_{yzT}^2}\big)$$

$$\leqslant\sum_{\substack{\beta_1+\beta_2\leqslant\alpha\\|\beta_2|\in\{2,3\}}}\|\partial^{\beta_1}E\|_{L_x^2 L_{yzT}^\infty}\|\partial^{\beta_2}H\|_{L_{xyzT}^2}\tag{3.45}$$

$$\leqslant cT^{1/2}\sum_{|\beta_1|\leqslant 1}\|\partial^{\beta_1}E\|_{L_x^2 L_{yzT}^\infty}\Big(\sum_{\substack{|\alpha_1|\leqslant 3\\|\alpha_2|\leqslant 1}}\|\partial_x\partial^{\alpha_1}E\|_{L_x^\infty L_{yzT}^2}\|\partial^{\alpha_2}E\|_{L_x^2 L_{yzT}^\infty}$$

$$+\sum_{\substack{|\alpha_1|\leqslant 3\\|\alpha_2|\leqslant 1}}\|\partial_y\partial^{\alpha_1}E\|_{L_y^\infty L_{xzT}^2}\|\partial^{\alpha_2}E\|_{L_y^2 L_{xzT}^\infty}+T^{1/2}\|E\|_{L_T^\infty H^3}^2\Big).$$

The result follows from inequalyties (3.43), (3.44) and (3.45). □

3.4   Proof of Theorem 3.1

We define

$$X_{a,T}=\{E\in C([0,T]:\widetilde{H}^3(\mathbb{R}^3)):\|E\|\leqslant a\},$$

where

$$\|E\|:=\|E\|_{L_T^\infty H^3(\mathbb{R}^3)}+\sum_{|\alpha|\leqslant 3}\Big(\|D_x^{1/2}\partial^\alpha E\|_{L_T^\infty L_{xyz}^2}+\|D_y^{1/2}\partial^\alpha E\|_{L_T^\infty L_{xyz}^2}\Big)\tag{3.46}$$

$$+\sum_{|\alpha|\leqslant 3}\Big(\|\partial_x\partial^\alpha E\|_{L_x^\infty L_{yzT}^2}+\|\partial_y\partial^\alpha E\|_{L_y^\infty L_{xzT}^2}\Big)+\sum_{|\alpha|\leqslant 1}\Big(\|\partial^\alpha E\|_{L_x^2 L_{yzT}^\infty}+\|\partial^\alpha E\|_{L_y^2 L_{xzT}^\infty}\Big),$$

$$\tag{3.47}$$

and $D_x^{1/2}$ and $D_y^{1/2}$ were defined in section 3.1.

We also define the integral operator on $X_{a,T}$,

$$\Psi(E)(t)=\mathcal{E}(t)E_0+\int_0^t\mathcal{E}(t-t')E(t')(N'(t')n_0+N(t')n_1)dt'$$

$$+\int_0^t\mathcal{E}(t-t')E(t')\left(\int_0^{t'}N(t'-s)\Delta_\perp(|E|^2)(s)ds\right)dt'\tag{3.48}$$

$$=\mathcal{E}(t)E_0+\int_0^t\mathcal{E}(t-t')(EF)(t')dt'+\int_0^t\mathcal{E}(t-t')(EH)(t')dt',$$

where F and H were defined in (3.32) and (3.33), respectively.

We will show that for appropriate a and T the operator $\Psi(\cdot)$ defines a contraction on $X_{a,T}$.

We start by estimating the $H^3(\mathbb{R}^3)$-norm of $\Psi(E)$.

Let $E \in X_{a,T}$. By Fubini's Theorem (Theorem 1.18), Minkowski's inequality (Lemma 1.19), Theorem 1.21 and group properties we have

$$\|\Psi(E)(t)\|_{H^3} \leqslant \|E_0\|_{H^3} + \|E\|_{L_T^\infty H^3} \int_0^T \|F(t')\|_{H^3} dt' + \|E\|_{L_T^\infty H^3} \int_0^T \|H(t')\|_{H^3} dt'. \quad (3.49)$$

From Theorem 1.20, Lemma 3.10 and Lemma 3.12 we have

$$\int_0^T \|F(t')\|_{H^3} dt' \leqslant \sum_{|\alpha|\leqslant 3} \int_0^T \|\partial^\alpha F(t')\|_{L^2} dt' \leqslant \sum_{|\alpha|\leqslant 3} \int_0^T \left(\|N'(t')\partial^\alpha n_0\|_{L^2} + \|N(t')\partial^\alpha n_1\|_{L^2} \right) dt'$$

$$\leqslant \sum_{|\alpha|\leqslant 3} \int_0^T \left(\|\partial^\alpha n_0\|_{L^2} + c\|n_1\|_{H^2(\mathbb{R}^3)} + c|t|\|\partial_z n_1\|_{H^2(\mathbb{R}^3)} \right) dt' \quad (3.50)$$

$$\leqslant cT\|n_0\|_{H^3} + cT\|n_1\|_{H^2(\mathbb{R}^3)} + cT^2\|\partial_z n_1\|_{H^2(\mathbb{R}^3)}.$$

Next, from Theorem 1.20 and Lemmas 1.14 and 3.13 we deduce

$$\int_0^T \|H(t')\|_{H^3} dt' \leqslant \sum_{|\alpha|\leqslant 3} \int_0^T \|\partial^\alpha H(t')\|_{L^2} dt' \leqslant T \sum_{|\alpha|\leqslant 3} \|\partial^\alpha H\|_{L_{xyzT}^2}$$

$$\leqslant cT^{3/2} \sum_{\substack{|\alpha_1|\leqslant 3 \\ |\alpha_2|\leqslant 1}} \|\partial_x \partial^{\alpha_1} E\|_{L_x^\infty L_{yzT}^2} \|\partial^{\alpha_2} E\|_{L_x^2 L_{yzT}^\infty} \quad (3.51)$$

$$+ cT^{3/2} \sum_{\substack{|\alpha_1|\leqslant 3 \\ |\alpha_2|\leqslant 1}} \|\partial_y \partial^{\alpha_1} E\|_{L_y^\infty L_{xzT}^2} \|\partial^{\alpha_2} E\|_{L_y^2 L_{xzT}^\infty}$$

$$+ cT^2 \|E\|_{L_T^\infty H^3}^2.$$

Therefore

$$\int_0^T \|H(t')\|_{H^3} dt' \leqslant c(T^{3/2} + T^2)\|E\|^2. \quad (3.52)$$

Finally

$$\|\Psi(E)(t)\|_{L_T^\infty H^3} \leqslant c\|E_0\|_{H^3} + c\|E\|(T\|n_0\|_{H^3} + T\|n_1\|_{H^2(\mathbb{R}^3)} + T^2\|\partial_z n_1\|_{H^2(\mathbb{R}^3)}) +$$

$$+ c(T^{3/2} + T^2)\|E\|^3.$$

Next, we calculate the norms

$$\sum_{|\alpha|\leqslant 1}\|\partial^\alpha\cdot\|_{L_x^2 L_{yzT}^\infty},\quad \sum_{|\alpha|\leqslant 3}\|\partial_x\partial^\alpha\cdot\|_{L_x^\infty L_{yzT}^2},\quad \sum_{|\alpha|\leqslant 1}\|\partial^\alpha\cdot\|_{L_y^2 L_{xzT}^\infty},\quad \sum_{|\alpha|\leqslant 3}\|\partial_y\partial^\alpha\cdot\|_{L_y^\infty L_{xzT}^2}.$$

By simetry is enough to estimate the two first norms.

Using the definition of Ψ in (3.48) and Proposition 3.4 we have

$$\sum_{|\alpha|\leqslant 1}\|\partial^\alpha\Psi(E)\|_{L_x^2 L_{yzT}^\infty} \leqslant c\sum_{|\alpha|\leqslant 1}\Big(\|\mathcal{E}(t)\partial^\alpha E_0\|_{L_x^2 L_{yzT}^\infty} + \int_0^T\|\mathcal{E}(t-t')\partial^\alpha(EF)(t')\|_{L_x^2 L_{yzT}^\infty}dt'$$

$$+ \int_0^T\|\mathcal{E}(t-t')\partial^\alpha(EH)(t')\|_{L_x^2 L_{yzT}^\infty}dt'\Big)$$

$$\leqslant c\sum_{|\alpha|\leqslant 1}\Big(\|\partial^\alpha E_0\|_{H^2} + \int_0^T\|\partial^\alpha(EF)(t')\|_{H^2}dt'$$

$$+ \int_0^T\|\partial^\alpha(EH)(t')\|_{H^2}dt'\Big).$$

By Theorem 1.20 we obtain

$$\sum_{|\alpha|\leqslant 1}\|\partial^\alpha\Psi(E)\|_{L_x^2 L_{yzT}^\infty} \leqslant c\|E_0\|_{H^3} + c\sum_{|\beta|\leqslant 3}\Big(\int_0^T\|\partial^\beta(EF)(t')\|_{L^2}dt' + \int_0^T\|\partial^\beta(EH)(t')\|_{L^2}dt'\Big). \tag{3.53}$$

Using Lemmas 1.14 and 3.14 we get

$$\sum_{|\beta|\leqslant 3}\int_0^T\|\partial^\beta(EF)(t')\|_{L^2}dt' \leqslant cT^{1/2}\sum_{|\alpha|\leqslant 3}\|\partial^\alpha(EF)\|_{L_{xyzT}^2} \tag{3.54}$$

$$\leqslant cT^{1/2}\|E\|_{L_T^\infty H^3}\big(T^{1/2}\|n_0\|_{H^3} + T^{1/2}\|n_1\|_{H^2} + T^{3/2}\|\partial_z n_1\|_{H^2}\big).$$

Therefore

$$\sum_{|\beta|\leqslant 3}\int_0^T\|\partial^\beta(EF)(t')\|_{L^2}dt' \leqslant cT^{1/2}\|E\|\big(T^{1/2}\|n_0\|_{H^3} + T^{1/2}\|n_1\|_{H^2} + T^{3/2}\|\partial_z n_1\|_{H^2}\big). \tag{3.55}$$

Applying Lemmas 1.14 and 3.14 we have

$$\sum_{|\beta|\leqslant 3}\int_0^T \|\partial^\beta(EH)(t')\|_{L^2}dt' \leqslant cT^{1/2}\sum_{|\beta|\leqslant 3}\|\partial^\beta(EH)(t')\|_{L^2_{xyzT}} \tag{3.56}$$

$$\leqslant cT\|E\|_{L^\infty_T H^3}\sum_{\substack{|\alpha_1|\leqslant 3\\ |\alpha_2|\leqslant 1}}\|\partial_x\partial^{\alpha_1}E\|_{L^\infty_x L^2_{yzT}}\|\partial^{\alpha_2}E\|_{L^2_x L^\infty_{yzT}}$$

$$+ cT\|E\|_{L^\infty_T H^3}\sum_{\substack{|\alpha_1|\leqslant 3\\ |\alpha_2|\leqslant 1}}\|\partial_y\partial^{\alpha_1}E\|_{L^\infty_y L^2_{xzT}}\|\partial^{\alpha_2}E\|_{L^2_y L^\infty_{xzT}}$$

$$+ cT^{3/2}\|E\|_{L^\infty_T H^3}^3.$$

So

$$\sum_{|\alpha|\leqslant 3}\int_0^T\|\partial^\alpha(EH)(t')\|_{L^2}dt' \leqslant c(T+T^{3/2})\|E\|^3. \tag{3.57}$$

Combining (3.53), (3.55) and (3.57) it follows that

$$\sum_{|\alpha|\leqslant 1}\|\partial^\alpha\Psi(E)\|_{L^2_x L^\infty_{yzT}} \leqslant c\|E_0\|_{H^3} + T^{1/2}\|E\|\big(T^{1/2}\|n_0\|_{H^3}+T^{1/2}\|n_1\|_{H^2}+T^{3/2}\|\partial_z n_1\|_{H^2}\big)+$$

$$+ c(T+T^{3/2})\|E\|^3.$$

To calculate the next norm we use the definition of Ψ in (3.48), Lemma 1.3 and Proposition 3.2 to get

$$\sum_{|\alpha|\leqslant 3}\|\partial_x\partial^\alpha\Psi(E)\|_{L^\infty_x L^2_{yzT}} \leqslant \sum_{|\alpha|\leqslant 3}\big(\|D_x^{1/2}\tilde{D}_x^{1/2}\mathcal{E}(t)\partial^\alpha E_0\|_{L^\infty_x L^2_{yzT}} + \|\partial_x\int_0^t\mathcal{E}(t-t')\partial^\alpha(EF)(t')dt'\|_{L^\infty_x L^2_{yzT}}$$

$$+ \|\partial_x\int_0^t\mathcal{E}(t-t')\partial^\alpha(EH)(t')dt'\|_{L^\infty_x L^2_{yzT}}\big)$$

$$\leqslant \sum_{|\alpha|\leqslant 3}\big(\|\tilde{D}_x^{1/2}\partial^\alpha E_0\|_{L^2} + \|\partial_x\int_0^t\mathcal{E}(t-t')\partial^\alpha(EF)(t')dt'\|_{L^\infty_x L^2_{yzT}}$$

$$+ \|\partial_x\int_0^t\mathcal{E}(t-t')\partial^\alpha(EH)(t')dt'\|_{L^\infty_x L^2_{yzT}}\big).$$

By propertie (1.5) and Lemmas 3.15 and 3.16 we have

$$\sum_{|\alpha|\leqslant 3} \|\partial_x \partial^\alpha \Psi(E)\|_{L_x^\infty L_{yzT}^2} \leqslant$$

$$\leqslant \sum_{|\alpha|\leqslant 3} \|D_x^{1/2}\partial^\alpha E_0\|_{L^2} + cT^{1/2}\|E\|_{L_T^\infty H^3}\big((1+T)\|n_0\|_{H^3} + (T+T^{1/2})\|n_1\|_{H^2} +$$

$$+ T^{3/2}\|\partial_z n_1\|_{H^2}\big) + c\sum_{|\beta_1|\leqslant 1} \|\partial^{\beta_1} E\|_{L_x^2 L_{yzT}^\infty}\big(T^{1/2}\|n_0\|_{H^3} + T^{1/2}\|n_1\|_{H^2} + T^{3/2}\|\partial_z n_1\|_{H^2}\big) +$$

$$+ c\Big(\sum_{\substack{|\alpha_1|\leqslant 3 \\ |\alpha_2|\leqslant 1}} \|\partial_x \partial^{\alpha_1} E\|_{L_x^\infty L_{yzT}^2}\|\partial^{\alpha_2} E\|_{L_x^2 L_{yzT}^\infty} + \sum_{\substack{|\alpha_1|\leqslant 3 \\ |\alpha_2|\leqslant 1}} \|\partial_y \partial^{\alpha_1} E\|_{L_y^\infty L_{xzT}^2}\|\partial^{\alpha_2} E\|_{L_y^2 L_{xzT}^\infty}\Big)\times$$

$$\times \big(T + T^{1/2}\sum_{|\alpha_2|\leqslant 1} \|\partial^{\alpha_2} E\|_{L_x^2 L_{yzT}^\infty} + \|E\|_{L_T^\infty H^3}\big) + cT\|E\|_{L_T^\infty H^3}^2 \times$$

$$\times \big(T^{1/2}\|E\|_{L_T^\infty H^3} + \sum_{|\alpha_2|\leqslant 1} \|\partial^{\alpha_2} E\|_{L_x^2 L_{yzT}^\infty}\big).$$

Finally by definition (3.46) we have

$$\sum_{|\alpha|\leqslant 3} \|\partial_x \partial^\alpha \Psi(E)\|_{L_x^\infty L_{yzT}^2} \leqslant \sum_{|\alpha|\leqslant 3} \|D_x^{1/2}\partial^\alpha E_0\|_{L^2} + cT^{1/2}\|E\|\big(\|n_0\|_{H^3}(2+T)+$$

$$+ \|n_1\|_{H^2}(1+T+T^{1/2}) + \|\partial_z n_1\|_{H^2}(T+T^{3/2})\big)+ \qquad (3.58)$$

$$+ T\|E\|^2 + (T^{1/2}+T+T^{3/2})\|E\|^3.$$

It remains to calculate the norms $\sum_{|\alpha|\leqslant 3} \|D_x^{1/2}\partial^\alpha \cdot\|_{L_T^\infty L_{xyz}^2}$ and $\sum_{|\alpha|\leqslant 3} \|D_y^{1/2}\partial^\alpha \cdot\|_{L_T^\infty L_{xyz}^2}$. Again by simetry we calculate just the first one.

$$\sum_{|\alpha|\leqslant 3} \|D_x^{1/2}\partial^\alpha \Psi(E)\|_{L_T^\infty L_{xyz}^2} \leqslant \sum_{|\alpha|\leqslant 3} \|D_x^{1/2}\partial^\alpha E_0\|_{L_{xyz}^2} + \sum_{|\alpha|\leqslant 3} \|D_x^{1/2}\int_0^t \mathcal{E}(t-t')\partial^\alpha(EF)(t')dt'\|_{L_T^\infty L_{xyz}^2} +$$

$$+ \sum_{|\alpha|\leqslant 3} \|D_x^{1/2}\int_0^t \mathcal{E}(t-t')\partial^\alpha(EH)(t')dt'\|_{L_T^\infty L_{xyz}^2} = \qquad (3.59)$$

$$= \|D_x^{1/2}\partial^\alpha E_0\|_{L_{xyz}^2} + I + II.$$

By Lemma 3.15 we have

$$I \leqslant c(T^{1/2}+1)\|E\|_{L_T^\infty H^3}\big(\|n_0\|_{H^3}(1+T)+T\|n_1\|_{H^2}+T^2\|\partial_z n_1\|_{H^2}\big)+ \qquad (3.60)$$

$$+ c\sum_{|\beta_1|\leqslant 1}\|\partial^{\beta_1}E\|_{L_x^2 L_{yzT}^\infty}\big(\|n_0\|_{H^3}+T^{1/2}\|n_1\|_{H^2(\mathbb{R}^3)}+T^{3/2}\|\partial_z n_1\|_{H^2(\mathbb{R}^3)}\big).$$

By Lemma 3.16 we obtain

$$II \leqslant c\Big(\sum_{\substack{|\alpha_1|\leqslant 3 \\ |\alpha_2|\leqslant 1}}\|\partial_x\partial^{\alpha_1}E\|_{L_x^\infty L_{yzT}^2}\|\partial^{\alpha_2}E\|_{L_x^2 L_{yzT}^\infty}+\sum_{\substack{|\alpha_1|\leqslant 3 \\ |\alpha_2|\leqslant 1}}\|\partial_y\partial^{\alpha_1}E\|_{L_y^\infty L_{xzT}^2}\|\partial^{\alpha_2}E\|_{L_y^2 L_{xzT}^\infty}\Big)\times$$

$$\times\Big(T+T^{1/2}\sum_{|\alpha_2|\leqslant 1}\|\partial^{\alpha_2}E\|_{L_x^2 L_{yzT}^\infty}+\|E\|_{L_T^\infty H^3}\Big)+cT\|E\|_{L_T^\infty H^3}^2\times \qquad (3.61)$$

$$\times\Big(T^{1/2}\|E\|_{L_T^\infty H^3}+\sum_{|\alpha_2|\leqslant 1}\|\partial^{\alpha_2}E\|_{L_x^2 L_{yzT}^\infty}\Big).$$

From (3.59), (3.60) and (3.61) we get

$$\sum_{|\alpha|\leqslant 3}\|D_x^{1/2}\partial^\alpha\Psi(E)\|_{L_T^\infty L_{xyz}^2}\leqslant\sum_{|\alpha|\leqslant 3}\|D_x^{1/2}\partial^\alpha E_0\|_{L_{xyz}^2}+\|E\|\big(\|n_0\|_{H^3}(2+2T+T^{3/2})+ \qquad (3.62)$$

$$+(T^{1/2}+T+T^{3/2})\|n_1\|_{H^2}+(T^{3/2}+T^2+T^{5/2})\|\partial_z n_1\|_{H^2}\big)+$$

$$+\|E\|^2 T+\|E\|^3(1+T^{1/2}+T+T^{3/2}).$$

Hence, by apropriate choices of $a = a(\|E_0\|_{\tilde{H}^3},T)$ and T (T sufficiently small depending on $\|n_0\|_{\tilde{H}^3}$, $\|n_1\|_{\tilde{H}^2}$ and $\|\partial_z n_1\|_{\tilde{H}^2}$), we see that Ψ maps $X_{a,T}$ into $X_{a,T}$. Now if $E, W \in X_{a,T}$,

$$(\Psi(E)-\Psi(W))(t) = \int_0^t \mathcal{E}(t-t')((E-W)F)(t')dt' + \int_0^t \mathcal{E}(t-t')((E-W)H)(t')dt'.$$

The same arguments as in (3.49)-(3.52) shows that

$$\|(\Psi(E)-\Psi(W))(t)\|_{H^3} \leqslant \|E-W\|_{L_T^\infty H^3}\int_0^T\|F(t')\|_{H^3}dt'+\|E-W\|_{L_T^\infty H^3}\int_0^T\|H(t')\|_{H^3}dt'$$

$$\leqslant \|E-W\|c(T\|n_0\|_{H^3}+T\|n_1\|_{H^2}+T^2\|\partial_z n_1\|_{H^2}+(T^2+T^{3/2})a^2).$$

To calculate the next norm we use Proposition 3.4 to obtain

$$\sum_{|\alpha|\leqslant 1}\|\partial^\alpha(\Psi(E)-\Psi(W))\|_{L^2_x L^\infty_{yzT}}\leqslant\int_0^t\|\mathcal{E}(t-t')\partial^\alpha((E-W)F)(t')\|_{L^2_x L^\infty_{yzT}}dt'$$

$$+\int_0^t\|\mathcal{E}(t-t')\partial^\alpha((E-W)H)(t')\|_{L^2_x L^\infty_{yzT}}dt'$$

$$\leqslant\int_0^t\|\partial^\alpha((E-W)F)(t')\|_{H^2}dt'$$

$$+\int_0^t\|\partial^\alpha((E-W)H)(t')\|_{H^2}dt'.$$

Next we combine the arguments used in (3.53)-(3.56) to conclude

$$\sum_{|\alpha|\leqslant 1}\|\partial^\alpha(\Psi(E)-\Psi(W))\|_{L^2_x L^\infty_{yzT}}\leqslant cT^{1/2}\|E-W\|(T^{1/2}\|n_0\|_{H^3}+T^{1/2}\|n_1\|_{H^2}+T^{3/2}\|\partial_z n_1\|_{H^2})+$$

$$+cT\|E-W\|a^2+cT^{3/2}\|E-W\|a^2.$$

Now we calculate the next two norms.

$$\sum_{|\alpha|\leqslant 3}\|\partial_x\partial^\alpha(\Psi(E)-\Psi(W))\|_{L^\infty_x L^2_{yzT}}\leqslant\sum_{|\alpha|\leqslant 3}\|\partial_x\int_0^t\mathcal{E}(t-t')\partial^\alpha((E-W)F)(t')dt'\|_{L^\infty_x L^2_{yzT}}+$$

$$+\sum_{|\alpha|\leqslant 3}\|\partial_x\int_0^t\mathcal{E}(t-t')\partial^\alpha((E-W)H)(t')dt'\|_{L^\infty_x L^2_{yzT}},$$

and

$$\sum_{|\alpha|\leqslant 3}\|D_x^{1/2}\partial^\alpha(\Psi(E)-\Psi(W))\|_{L^\infty_T L^2_{xyz}}\leqslant\sum_{|\alpha|\leqslant 3}\|D_x^{1/2}\int_0^t\mathcal{E}(t-t')\partial^\alpha((E-W)F)(t')dt'\|_{L^\infty_T L^2_{xyz}}+$$

$$+\sum_{|\alpha|\leqslant 3}\|D_x^{1/2}\int_0^t\mathcal{E}(t-t')\partial^\alpha((E-W)H)(t')dt'\|_{L^\infty_T L^2_{xyz}}.$$

Following the same ideas as in Lemma 3.15 and Lemma 3.16 we conclude that

$$\sum_{|\alpha|\leqslant 3}\left(\|\partial_x\partial^\alpha(\Psi(E)-\Psi(W))\|_{L^\infty_x L^2_{yzT}}+\|D_x^{1/2}\partial^\alpha(\Psi(E)-\Psi(W))\|_{L^\infty_T L^2_{xyz}}\right)\leqslant$$

$$\leqslant c(T^{1/2}+1)\|E-W\|\left((2+T)\|n_0\|_{H^3}+(T+T^{1/2})\|n_1\|_{H^2}+(T^2+T^{3/2})\|\partial_z n_1\|_{H^2}\right)+$$

$$+\|E-W\|a^2(1+T^2)(T+T^{1/2}).$$

By simetry we obtain the results to the norms $\sum_{|\alpha|\leqslant 1}\|\partial^\alpha \cdot \|_{L_y^2 L_{xzT}^\infty}$, $\sum_{|\alpha|\leqslant 3}\|\partial_y\partial^\alpha \cdot \|_{L_y^\infty L_{xzT}^2}$ and $\sum_{|\alpha|\leqslant 3}\|D_y^{1/2}\partial^\alpha \cdot \|_{L_T^\infty L_{xyz}^2}$. Finally we can choose $a = a(\|E_0\|_{\tilde{H}^3}, T)$ and T (T sufficiently small depending on $\|n_0\|_{\tilde{H}^3}$, $\|n_1\|_{\tilde{H}^2}$ and $\|\partial_z n_1\|_{\tilde{H}^2}$) that satisfies

$$\Psi(X_{a,T}) \subset X_{a,T}, \tag{3.63}$$

and

$$\Psi \text{ is a contraction.} \tag{3.64}$$

It remains to prove $E \in C([0,T],\tilde{H}^3)$ and the uniqueness in $\tilde{H}^3$. It is enough to prove the continuity in $t = 0$. So Taking a small $T_0 > 0$ we must show that

$$\|E(t) - E_0\|_{H^3} \to 0, \ t \to 0, \tag{3.65}$$

$$\|D_x^{1/2}\partial^\alpha(E(t) - E_0)\|_{L^2} \to 0, \ t \to 0, \text{ for each } |\alpha| \leqslant 3, \tag{3.66}$$

and

$$\|D_y^{1/2}\partial^\alpha(E(t) - E_0)\|_{L^2} \to 0, \ t \to 0, \text{ for each } |\alpha| \leqslant 3, \tag{3.67}$$

where $t \in [0,T_0)$.

Since E satisfies the integral equation

$$E(t) = \mathcal{E}(t)E_0 + \int_0^t \mathcal{E}(t-t')(EF)(t')dt' + \int_0^t \mathcal{E}(t-t')(EH)(t')dt', \tag{3.68}$$

we have that

$$E(t) - E_0 = \mathcal{E}(t)E_0 - E_0 + \int_0^t \mathcal{E}(t-t')(EF)(t')dt' + \int_0^t \mathcal{E}(t-t')(EH)(t')dt'. \tag{3.69}$$

Then

$$\|E(t) - E_0\|_{H^3} \leqslant \|\mathcal{E}(t)E_0 - E_0\|_{H^3} + \int_0^t \|\mathcal{E}(t-t')(EF)(t')\|_{H^3}dt' + \int_0^t \|\mathcal{E}(t-t')(EH)(t')\|_{H^3}dt'.$$

By group properties we get

$$\|\mathcal{E}(t)E_0 - E_0\|_{H^3} \to 0, \ t \to 0.$$

Using Lemma 3.14 we have the hyphothesis of Theorem 1.11 and therefore

$$\int_0^t \|\mathcal{E}(t - t')(EF)(t')\|_{H^3} dt' = \sum_{|\alpha| \leqslant 3} \int_0^t \|\partial^\alpha(EF)(t')\|_{L^2} dt' \to 0, \ t \to 0.$$

Again by Lemma 3.14 and Theorem 1.11 we have

$$\int_0^t \|\mathcal{E}(t - t')(EH)(t')\|_{H^3} dt' = \sum_{|\alpha| \leqslant 3} \int_0^t \|\partial^\alpha(EH)(t')\|_{L^2} dt' \to 0, \ t \to 0,$$

which proves (3.65).

To prove (3.66) we take $|\alpha| = 3$ and use the expression (3.69) to obtain

$$\|D^{1/2}\partial^\alpha(E(t) - E_0)\|_{L^2} \leqslant \|D^{1/2}\partial^\alpha(\mathcal{E}(t)E_0 - E_0)\|_{L^2} + \|D^{1/2}\partial^\alpha \int_0^t \mathcal{E}(t - t')(EF)(t')dt'\|_{L^2}$$

$$\tag{3.70}$$

$$+ \|D^{1/2}\partial^\alpha \int_0^t \mathcal{E}(t - t')(EH)(t')dt'\|_{L^2}.$$

Using group properties we have

$$\|D^{1/2}\partial^\alpha(\mathcal{E}(t)E_0 - E_0)\|_{L^2} = \|\mathcal{E}(t)D^{1/2}\partial^\alpha E_0 - D^{1/2}\partial^\alpha E_0\|_{L^2} \to 0, \ t \to 0. \tag{3.71}$$

Now, following the arguments in the proof of Lemma 3.15 we get

$$\|D^{1/2}\partial^\alpha \int_0^t \mathcal{E}(t - t')(EF)(t')dt'\|_{L^2} \leqslant T_0^{1/2}a\big(\|n_0\|_{H^3}(2 + T_0^{1/2}) + \|n_1\|_{H^2}(1 + T_0^{1/2} + T_0)$$

$$\tag{3.72}$$

$$+ (T_0 + T_0^{3/2})\|\partial_z n_1\|_{H^2}\big).$$

Again, we can follow the argumentes in the proof of Lemma 3.16 and obtain

$$\|D^{1/2}\partial^\alpha \int_0^t \mathcal{E}(t - t')(EH)(t')dt'\|_{L^2} \leqslant T_0^{1/2}a(1 + T_0^{1/2} + T_0). \tag{3.73}$$

Since we can take T_0 as small as we want, the result follows from (3.70)-(3.73).

The proof of propertie (3.67) is similar to (3.66).

To prove the uniqueness we consider $W(t)$ other solution of the integral equation (3.68) in some interval $[0, T_1] \subset [0, T]$ such that

$$W \in X_{a_1, T_1} \text{ with } a_1 > a, \tag{3.74}$$

and

$$W \in C([0, T_1], \tilde{H}^3(\mathbb{R}^3)). \tag{3.75}$$

From properties (3.74) and (3.75) we have that exists $T_2 < T_1$ such that

$$\sup_{t \in [0, T_2]} \|W(t)\|_{\tilde{H}^3} \leqslant a.$$

Also, using the same arguments as in (3.53), (3.54) and (3.56) we conclude that exists $T_3 < T_2$ such that

$$\sum_{|\alpha| \leqslant 1} \|\partial^\alpha W\|_{L_x^2 L_{yzT}^\infty} \leqslant c\|E_0\|_{\tilde{H}^3} + T_3^{3/2} a(\|n_0\|_{H^3} + \|n_1\|_{H^2} + T_3\|\partial_z n_1\|_{H^2}) < a.$$

Similarly:

Exists $T_4 < T_3$ such that

$$\sum_{|\alpha| \leqslant 1} \|\partial^\alpha W\|_{L_y^2 L_{xzT}^\infty} < a,$$

exists $T_5 < T_4$ such that

$$\sum_{|\alpha| \leqslant 3} \|\partial_x \partial^\alpha W\|_{L_x^\infty L_{yzT}^2} < a,$$

and exists $T_6 < T_5$ such that

$$\sum_{|\alpha| \leqslant 3} \|\partial_y \partial^\alpha W\|_{L_y^\infty L_{xzT}^2} < a.$$

Therefore we have $W \in X_{a,T_6}$. By the uniqueness of the solutions in $X_{a,T}$ we must have

$$W(t) = E(t), \ t \in [0, T_6].$$

Applying again this process with data $E(T_6)$, we can extend the solution W to the interval $[0, T]$ and obtain $W(t) = E(t), \ t \in [0, T]$.

We make two observations about this extension to the interval $[0, T]$:

1. Since $[0, T]$ is a compact set, the process will have only a finite number of iteractions.

2. If there exists $\tilde{T} < T$ such that

$$\{T_j\} \to \tilde{T}, j \to \infty, \tag{3.76}$$

then we must have $\tilde{T} = 0$ (or we could reapply the process and get $\tilde{T}_1 > \tilde{T}$ what would be a contradiction with (3.76)). Since we have

$$T_{j+1} \simeq \frac{1}{\|W(T_j)\|_{\tilde{H}^3}},$$

then $\tilde{T} = 0$ would imply $\|W(T_j)\|_{\tilde{H}^3} \to \infty, \ j \to \infty$, what is a contradiction.

To prove (3.3) (and by simetry (3.4)) we use the integral equation (3.68), Proposition 3.2 and Lemma 3.14 to obtain

$$\sum_{|\alpha| \leqslant 3} \|D_x^{1/2} \partial^\alpha(E)\|_{L_x^\infty L_{yzT}^2} \leqslant \|E_0\|_{H^3} + T^{1/2}\|\partial^\alpha(EF)(t')dt'\|_{L_{xyzT}^2} + T^{1/2}\|\partial^\alpha(EH)(t')dt'\|_{L_{xyzT}^2}$$

$$\leqslant \|E_0\|_{H^3} + c\|E\|_{L_T^\infty H^3}\left(T^{1/2}\|n_0\|_{H^3} + cT^{1/2}\|n_1\|_{H^2} + cT^{3/2}\|\partial_z n_1\|_{H^2}\right) +$$

$$+ cT^{1/2}\|E\|_{L_T^\infty H^3} \sum_{\substack{|\alpha_1| \leqslant 3 \\ |\alpha_2| \leqslant 1}} \|\partial_x \partial^{\alpha_1} E\|_{L_x^\infty L_{yzT}^2}\|\partial^{\alpha_2} E\|_{L_x^2 L_{yzT}^\infty}$$

$$+ cT^{1/2}\|E\|_{L_T^\infty H^3} \sum_{\substack{|\alpha_1| \leqslant 3 \\ |\alpha_2| \leqslant 1}} \|\partial_y \partial^{\alpha_1} E\|_{L_y^\infty L_{xzT}^2}\|\partial^{\alpha_2} E\|_{L_y^2 L_{xzT}^\infty}$$

$$+ cT\|E\|_{L_T^\infty H^3}^3 < \infty.$$

Now we prove the last part of the theorem. We already know that taking a initial data $W_0 = (E_0, n_0, n_1)$ in the set $Z = \{\tilde{W}_0 = (\tilde{E}_0, \tilde{n}_0, \tilde{n}_1) \in \tilde{H}^3 \times H^3 \times H^2 \; ; \; \partial_z n_1 \in H^3\}$ and choosing a satisfying (3.63) and (3.64) we have a unique solution $E \in X_{a,T}$ of the integral equation (3.68). So, fixing a initial data $W_0 \in Z$, let V be a neighborhood of W_0 in Z and $E(t)$ the corresponding solution in $X_{a,T}$.

Define
$$
\begin{array}{rcl}
H_1: & V \times X_{a,T} & \mapsto \quad X_{a,T} \\
& \big((\tilde{E}_0, \tilde{n}_0, \tilde{n}_1), \tilde{E}(t)\big) & \mapsto \quad \tilde{E}(t) - \Psi_{\tilde{W}_0}(\tilde{E})(t) = \\
& & = \tilde{E}(t) - \big(\mathcal{E}(t)\tilde{E}_0 + \int_0^t \mathcal{E}(t-t')F_1(\tilde{E})(t')dt'\big),
\end{array}
\tag{3.77}
$$
where
$$
F_1(\tilde{E}) = \tilde{E}\big(\tilde{F} + \tilde{H}\big),
$$

$$
\tilde{F} = N(t)\tilde{n}_0 + N'(t)\tilde{n}_1,
\tag{3.78}
$$

and

$$
\tilde{H} = \int_0^t N(t-t')\Delta_\perp(|\tilde{E}|^2)(t')dt'.
\tag{3.79}
$$

Thus H_1 is smooth, $H_1(W_0, E(t)) = 0$, and

$$
D_W H_1(W_0, E(t))W(t) = W(t) + \int_0^t \mathcal{E}(t-t')D_W F_1(t')dt'.
$$

Using the same arguments before it is easy to see that

$$
\|W\| c(a,T) \leqslant \|D_W H_1(W_0, E(t))W(t)\| \leqslant \|W\| c_1(a,T),
$$

where a and T must satisfies (3.63) and (3.64).

Then $D_W H_1(W_0, E(t)) : X_{a,T} \to X_{a,T}$ is one-to-one and onto. Thus by the implicit function Theorem exists $h : \tilde{V} \to X_{a,T}$ smooth $(V \subset \tilde{V})$ such that

$$
H(\tilde{W}_0, h(\tilde{W}_0)) = 0 \;\; \forall \; \tilde{W}_0 \in \tilde{V} \subset V,
$$

and

$$
h(\tilde{W}_0) = \mathcal{E}(t)\tilde{E}_0 + \int_0^t \mathcal{E}(t-t')F_1(h(\tilde{W}_0))(t')dt'
$$

is a solution of the integral equation (3.68) with data $\tilde{W}_0$ (instead of W_0).

Conclusion

In conclusion, we point out some open problems connected with this work:

- In the second chapter we proved global well-posedness and self-similar solutions to the Davey-Stewartson system in the elliptic-elliptic case. To the hyperbolic-elliptic case we could get just the global well posedness (see Remark 2.7). Do we have self-similar solution in this case?

- Also in the second chapter, we don't know any results about ill-posedness of the Davey-Stewartson system.

- In the third chapter there are several questions still unanswered that concerns to: conservation laws (we only know in L^2), ill-posedness, the sobolev indices $s \neq 3, 5, 7, 9, \ldots$. The Strichartz estimates (Proposition 3.9) will certainly be usefull in future works to improve the Sobolev indices.

- There is still a gap between 1 and 3/2 in the propositions 3.4 and 3.7 (maximal function estimate to the homogeneous Zakharov equation).

Bibliography

[AH]   J.M. Ablowitz, R. Haberman, Nonlinear evolution equations in two and three dimensions, Phys. Rev. Lett. **35** (1975), 1185-1188.

[AF]   M.J. Ablowitz, A.S. Fokas, On the inverse scattering transform of multidimensional nonlinear equations related to first-order systems in the plane, J. Math. Phys. **5** (1984), 2494-2505.

[AnFr] D. Anker, N.C. Freeman, On the soliton solutions of the Davey-Stewartson equation for long waves , Proc. R. Soc. A **360** (1978), 529-540.

[BC]   R. Beals, R.R. Coifman, The spectral problem for the Davey-Stewartson and Ishimori hierarchies, Proc. Conf. on Nonlinear Evolution Equations: Integrability and Spectral Methods, Machester, U. K., 1988.

[BeL]   J. Bergh, J. Lofstrom, Interpolation Spaces. An introduction, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[Ch]   H. Chihara, The initial value problem for the elliptic-hyperbolic Davey-Stewartson equation, J. Math. Kyoto Univ. **39** (1999), 41-66.

[CC]   M. Colin, M. Colin, On a quasilinear Zakharov system describing laser-plasma interactions, Differential Integral Equations **17** (2004), 297-330.

[CM]   T. Colin, G. Métivier, Instabilities in Zakharov equations for lazer propagation in
         a plasma, Phase space analysis of partial differential equations, Progr. Nonlinear
         Differential Equations Appl. **69** (2006), 297-330, Birkhäuser Boston, MA.

[CP]   M. Cannone, F. Planchon, Self-similar solution for the Navier-Stokes equations in
         $\mathbb{R}^3$, Comm. PDE **21** (1996), 179-193.

[CVeVi] T. Cazenave, L. Vega, M.C. Vilela, A note on the nonlinear Schrödinger equation
         in weak L^p spaces, Comm. Contemporary Math. **3** (2001), 153-162.

[CW1]   T. Cazenave, F.B. Weissler, Asymptotically self-similar global solutions of the
         nonlinear Schrödinger and heat equations, Math. Z. **228** (1998), 83-120.

[CW2]   T. Cazenave, F.B. Weissler, More self-similar solutions of the nonlinear
         Schrödinger equation, NoDEA Nonlinear Differential Equations Appl. **5** (1998),
         355-365.

[DR]   V.D. Djordjevic, L.G. Redekopp, On two-dimensional packets of capillary-gravity
         waves, J. Fluid Mech **79** (1977), 703-714.

[DS]   A. Davey, K. Stewartson, On three dimensional packets of surface waves, Proc.
         Roy. Londan Soc. A **338** (1974), 101-110.

[EK]   M. Escobedo, O. Kavian, Asymptotic behaviour of positive solutions of a non-
         linear heat equation, Houston J. Math. **14**, no 1 (1988), 39-50.

[F]   G. Furioli, On the existence of self-similar solutions of the nonlinear Schrödinger
         equation with power nonlinearity between 1 and 2, Differential Integral Equations
         14 , no 10 (2001), 1259-1266.

[Fa]   A.V. Faminskii, The cauchy problem for the Zakharov-Kuznetsov equation. Dif-
         ferential Equations **31**, no 6 (1995), 1002-1012.

[FS]   A.S. Fokas, P.M. Santini, Recursion operators and bi-Hamiltonian structures in
          multidimensions I, II, Commun. Math. Phys. **115**, no 3 (1988), 375-419, **116**, no
          3 (1988), 449-474.

[G]   L. Grafakos, Classical and Modern Fourier Analysis, Pearson Education, Inc.,
          Upper Saddle River, NJ, 2004, 931pp.

[GM]   Y. Giga, T. Miyakawa, Navier-Stokes flow in $\mathbb{R}^3$ with measures as initial vorticity
          and Morrey spaces, Comm. Partial Differential Equations **14** (1989), 577-618.

[GS]   J.M. Ghidaglia, J.C. Saut, On the initial problem for the Davey-Stewartson sys-
          tems, Nonlinearity **3** (1990), 475-506.

[GV]   J. Ginibre, G. Velo, Smoothing Properties and Retarded Estimates for some Dis-
          persive Evolution Equations, Comm. Math. Phys. **144**, no 1 (1992), 163-188.

[GTV]   J. Ginibre, Y. Tsutsumi, G. Velo, On the cauchy problem for the Zakharov system,
          J. Funct. Anal. **151**, no 2 (1997), 384-436.

[H1]   N. Hayashi, Local existence in time of small solutions to the Davey-Stewartson
          system, Annales de l'I.H.P. Physique Theorique **65** (1996), 313-366.

[H2]   N. Hayashi, Local existence in time of solutions to the elliptic-hyperbolic
          Davey-Stewartson system without smallness condition on the data, J.Analysé
          Mathématique **73** (1997), 133-164.

[HH1]   N. Hayashi, H. Hirata, Global existence and asymptotic behaviour of small solu-
          tions to the elliptic-hyperbolic Davey-Stewartson system, Nonlinearity **9** (1996),
          1387-1409.

[HH2]   N. Hayashi, H. Hirata, Local existence in time of small solutions to the elliptic-hyperbolic Davey-Stewartson system in the usual Sobolev space, Proc. Edinburgh Math. Soc. **40** (1997), 563-581.

[HS]   N. Hayashi, J.C. Saut, Global existence of small solutions to the Davey-Stewartson and the Ishimori systems, Differential Integral Equations **8** (1995), 1657-1675.

[I]   C. Isnard, Introdução à medida e integração, Projeto Euclides, IMPA, 2007.

[K]   M. Kawak, A semilinear heat equation with singular initial data, Proc. Royal Soc. Edinburgh Sect. A **128** (1998), 745-758.

[KT]   M. Keel, T. Tao, Endpoint Strichartz estimates, Amer. J. Math. **120**, no 5 (1998), 955–980.

[KW]   O. Kavian, F. Weissler, Finite energy self-similar solutions of a nonlinear wave equation, Comm. PDE **15** (1990), 1381-1420.

[KZ]   C.E. Kenig, S.N. Ziesler, Maximal function estimate with aplications to a modified Kadomtsev-Petviashvili equation, Comm. Pure Appl. Anal. **4** (2005), 45-91.

[KPV]   C.E. Kenig, G. Ponce, L. Vega, Well-posedness and scattering results for the generalizad Korteweg-de Vries equation via the contraction principle, Comm. Pure Appl. Math. **46** (1993), 527-620.

[L1]   G.G. Lorentz, Some new function spaces, Ann. Math. **51** (1950), 37-55.

[L2]   G.G. Lorentz, On the theory of spaces Λ, Pac. J. Math. **1** (1951), 411-429.

[LP1]   F. Linares, G. Ponce, On the Davey-Stewartson systems, Ann. Inst. Henry Poincaré **10**, no 5 (1993), 523-548.

[LP2] F. Linares, G. Ponce, Introduction to Nonlinear Dispersive Equation, Springer, New York, 2009, 256pp.

[LiPoS] F. Linares, G. Ponce, J.C. Saut, On a degenerate Zakharov System. Bull. Braz. Math. Soc. **36**, no 1 (2005), 1-23.

[McV] M.C. Vilela, Las estimaciones de Strichartz bilineales en el contexto de la ecuación de Schrödinger, Ph.D.thesis, Universidad del Pais Vasco (2003), Bilbao.

[O]   R. O'neil, Convolution Operators and $L^{p,q}$ Spaces, Duke Math. J. **30** (1963), 129-142.

[Oh]   M. Ohta, Stability of standing waves for the generalized Davey-Stewartson system, J. Dynam. Differential Equations **6**, no 2 (1994), 325–334.

[OT]   T. Ozawa, Y. Tsutsumi, Existence and smoothing effect of solutions for the Zakharov equations, Publ. Res. Inst. Math. Sci. **28**, no 3 (1992)

[Oz]   T. Ozawa, Exact blow-up solutions to the Cauchy problem for the Davey-Stewartson systems, Proc. Roy. Soc. London Ser. A **436** (1992), no 1897, 345–349.

[P]   F. Planchon, Self-Similar solutions and semilinear wave equations in Besov spaces, J. Math. Pures Appl. **79**, no 8 (2000), 809-820.

[Pe]   H. Pecher, Self-similar and asymptotically self-similar solutions of nonlinear wave equations, Math. Ann. **316** (2000), 259-281.

[RY1] F. Ribaud, A. Youssfi, Regular and self-similar solutions of nonlinear Schrödinger equations, J. Math Pures Appl. **77** (1998), 1065-1079.

[RY2] F. Ribaud, A. Youssfi, Global solutions and self-similar solutions of semilinear wave equation, Math. Z. **239** (2002), 231-262.

[STW]   S. Snoussi, S. Tayachi, F.B. Weissler, Asymptoticaly self-similar global solutions of a general semilinear heat equation, Math. Ann. **321** (2001), 131-155.

[X]   Z. Xiangking, Self-Similar solutions to a generalized Davey-Stewartson system, Adv. Math. (China) **36**, no 5 (2007), 579–585.

[Z]   V.E. Zakharov, Collapse of Langmuir Waves, Sov. Phys. JETP **35** (1972), 908-914.

[ZK]   V.E. Zakharov, E.A. Kutnetsov, Hamilton formalism for systems of hydrodynamic type, Math. Physics Review, Sov. Sci. Rev. **4** (1984), 167-220.