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Subsonic and Supersonic Limits for the Zakharov-Rubenchik System

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“*A mi amada, irremplazable y dedicada esposa, Katia*”

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“*Los seres humanos no nacen para siempre el día en que sus madres los alumbran, sino que la vida los obliga a parirse a sí mismos una y otra vez.*”

Gabriel García Márquez.

Notations

$C^k(\Omega)$ denotes the class of functions defined on Ω with k continuous derivatives,

$C^k_\infty(\Omega)$ denotes the class of functions in $C^k(\Omega)$ vanishing at infinity,

$C^\infty_c(\Omega)$ denotes the space of C^∞-functions with compact support in Ω,

$\mathcal{S}$ is the Schwartz class of C^∞-functions decaying at infinity,

$L^p(\Omega)$ denotes the Lebesgue space with norm denoted by $\|\cdot\|_{L^p}$, $1 \le p \le +\infty$,

$L^p_{loc}(\Omega) := \{u \in L^p(K)\,\text{for each}\, K \subset \bar{K} \subset \Omega :\ \bar{K}\,\text{is compact}\}$,

X' denotes the dual space of the normed space X,

$\mathcal{S}'$ denotes the tempered dsitribution space of linear continuous functionals on $\mathcal{S}$,

$L^p(\Omega, X)$ denotes the L^p-space of functions which take values in X,

$W^{k,p}, H^s$ are the classical Sobolev spaces,

$H^{s+} := H^{s+\delta}, \delta > 0$ small enough,

$\widehat{u}$ is the Fourier transform of the distribution u,

$\partial^\alpha := \partial x^{\alpha_1}_{x_1} \cdots \partial x^{\alpha_n}_{x_n}$, if $\alpha = (\alpha_1, \ldots, \alpha_n)$,

$\Delta := \sum_{i=1}^n \partial^2_{x_i}$ is the Laplacian operator,

$\nabla = (\partial_{x_1}, \ldots, \partial_{x_n})$ is the gradient vector,

$\Box := \partial^2_t - \Delta$ is the wave operator,

$\dfrac{\delta J}{\delta u}$ is the variational derivative of the functional $J[u]$,

v_g, c_s are the group and sound velocities respectively, $M = \dfrac{v_g}{c_s}$ is the Mach number,

$a \lesssim b$ means $a \leq Cb$ for some constant C that may change from line to line,

$\rightharpoonup$ ($\underset{*}{\rightharpoonup}$) denotes (star) weak convergence,

$\hookrightarrow$ ($\underset{c}{\hookrightarrow}$) denotes (compact) continuos embedding,

T_m denotes the multiplier transformation with multiplier m.

Abstract

We study the asymptotic behavior of solutions of Zakharov-Rubenchik system when appropriate parameters tend to zero. Namely, we establish weak and strong convergence results of these solutions to solutions of Zakharov system (supersonic limit) or Davey-Stewartson system (subsonic limit).

Contents

Introduction

In 1972, V. Zakharov and A. Rubenchik [41] derived a system of equations which describes the interaction of a spectrally narrow high-frequency wave packet with a low-frequency oscillations of acoustic type, in a conservative medium on $\mathbb{R}^n$ $(n = 2, 3)$ characterized by a Hamiltonian $\mathcal{H}$. The universal system obtained by them is modeled by the coupled equations

$$\begin{cases} i(\partial_t \psi + v_g \partial_z \psi) + \dfrac{w''}{2} \partial_z^2 \psi + \dfrac{v_g}{2k_0} \Delta_\perp \psi = (q|\psi|^2 + \beta\rho + \alpha \partial_z \varphi)\psi, \\ \partial_t \rho + \rho_{00} \Delta\varphi + \alpha \partial_z |\psi|^2 = 0, \\ \partial_t \varphi + \dfrac{c_s^2}{\rho_{00}} \rho + \beta |\psi|^2 = 0, \end{cases} \tag{ZR$_0$}$$

where $\psi = \psi(\mathbf{x}, t)$ denotes the complex amplitude of the (high frequency (HF)) carrying wave whose wave number k and frequency w are related by the dispersion relation $w = w(k)$, $v_g = w'(k)$ is the group velocity of the carrying wave, which according to [41] and [26] is in the direction of the z-axis, that is, $\mathbf{v_g} = (0, 0, v_g)$. The functions ρ and φ denote the density fluctuation and the hydrodynamic potential respectively, the parameters q, α, measure the self-interaction of the carrying wave and the Doppler shift respectively, $c_s = \sqrt{p'(\rho_{00})}$ is the sound velocity (p is the pressure), $\beta = \frac{\partial w(k_0)}{\partial \rho} \sim w/\rho_{00}$, the center of the HF packet is at k_0 ($k \sim k_0$) and the energy of the HF packet narrow is $\varepsilon_0 \approx \int w(k_0)|\psi|^2 dx$.

The term $\delta w(k_0) := \beta\rho + \alpha \partial_z \varphi$ in the first equation in (ZR$_0$) represents the variation of $w(k_0)$ because of the presence of an acoustic wave, where a good approximation of α is given by

$$\alpha = \begin{cases} \dfrac{w}{v_p} = k_0, \quad \text{or} \\ \dfrac{w}{v_g} = \dfrac{c_s}{\lambda_0 v_g}. \end{cases}$$

We use the notation $\mathbf{x} = (x, y, z)$ if $n = 3$ and $\mathbf{x} = (x, z)$ if $n = 2$, $\Delta_\perp = \partial_x^2 + \partial_y^2$ if $n = 3$ and $\Delta_\perp = \partial_x^2$ if $n = 2$.

The Hamiltonian for the system is

$$\mathcal{H} = \int \Big[w_0|\psi|^2 + \frac{i}{2}(\bar{\psi}\partial_z\psi - \psi\partial_z\bar{\psi})v_g + \frac{w''}{2}|\partial_z\psi|^2 + \frac{v_g}{2k_0}|\nabla_\perp\psi|^2 + \frac{q}{2}|\psi|^4$$
$$+ \frac{\rho_{00}}{2}|\nabla\varphi|^2 + \frac{c_s^2}{2\rho_{00}}\rho^2 + |\psi|^2(\beta\rho + \alpha\partial_z\varphi)\Big]d\mathbf{x},$$

which is the conserved energy of (ZR_0). In fact, this is (see [26]) a combination of the energy for the first equation and the energy

$$\mathcal{H} = \int \Big[\frac{\rho}{2}|\nabla\varphi|^2 + \varepsilon(\rho)\Big]d\mathbf{x}$$

of the equations

$$\begin{cases} \partial_t\rho + \nabla\cdot(\rho\nabla\varphi) = 0, \\ \partial_t\varphi + \dfrac{1}{2}|\nabla\varphi|^2 + \omega(\rho) = 0, \end{cases} \qquad \text{(Euler's equations)}$$

where ρ now is the mass density, $\omega(\rho) = \partial\varepsilon/\partial\rho$ is the enthalpy and $\varepsilon(\rho)$ is the internal energy density.

The variables ρ and φ form a pair of canonically conjugate variables, they verify the Hamilton equations

$$\frac{\delta\mathcal{H}}{\delta\varphi} = \frac{\partial\rho}{\partial t}, \qquad \frac{\delta\mathcal{H}}{\delta\rho} = -\frac{\partial\varphi}{\partial t}.$$

An equivalent system to (ZR_0) was first deduced by Benney and Roskes [4] (see also [39]) in the context of gravity waves (waves generated in a fluid medium or at the interface between two media, which has the restoring force of gravity or buoyancy). They obtained the system for the evolution of a train of waves propagation in the z-direction expressed in a frame of reference moving with group velocity v_g.

A modified ZR_0 system was proposed by Zakharov and Kuznetsov [26] for certain special regimes, depending on the level of nonlinearity $q|\psi|^2$ in the subsonic regime $v_g < c_s$. They proposed the system

$$\begin{cases} i(\partial_t\psi + v_g\partial_z\psi) + \dfrac{w''}{2}\partial_z^2\psi + \dfrac{v_g}{2k_0}\Delta_\perp\psi = (q|\psi|^2 + \beta\rho + \alpha\partial_z\varphi)\psi, \\ -v_g\partial_z\rho + \rho_{00}\Delta\varphi + \alpha\partial_z|\psi|^2 = 0, \\ -v_g\partial_z\varphi + \dfrac{c_s^2}{\rho_{00}}\rho + \beta|\psi|^2 = 0 \end{cases} \qquad (ZRK_0)$$

in that case.

The importance of the systems (ZR_0) and (ZRK_0) can not be overlooked. For instance, they do contain as specific limits the well known Zakharov and Davey-Stewartson systems. They are thus richer than those simpler models and should capture more of the original dynamics.

The systems (ZR_0) and (ZRK_0) can be transformed in a non-dimensional form (see [39] to know the rescaling in the variables and Sections 1.1 to see the new parameters) as

$$\begin{cases} i\partial_t\psi + \epsilon\partial_z^2\psi + \sigma_1\Delta_\perp\psi = (q|\psi|^2 + \mathtt{W}(\rho + \alpha D\partial_z\varphi))\psi, \\ \partial_t\rho + \sigma_2\partial_z\rho = -\Delta\varphi - \alpha D\partial_z|\psi|^2, \\ \partial_t\varphi + \sigma_2\partial_z\varphi = -\dfrac{1}{M^2}\rho - |\psi|^2, \end{cases} \tag{ZR$_1$}$$

and

$$\begin{cases} i\partial_t\psi + \epsilon\partial_z^2\psi + \sigma_1\Delta_\perp\psi = (q|\psi|^2 + \mathtt{W}(\rho + \alpha D\partial_z\varphi))\psi, \\ -\sigma_2\partial_z\rho = \Delta\varphi + \alpha D\partial_z|\psi|^2, \\ -\sigma_2\partial_z\varphi = \dfrac{1}{M^2}\rho + |\psi|^2, \end{cases} \tag{ZRK$_1$}$$

respectively.

We can decouple the last two equations in (ZR_1) and (ZRK_1) to obtain

$$\begin{cases} i\partial_t\psi + \epsilon\partial_z^2\psi + \sigma_1\Delta_\perp\psi = (q|\psi|^2 + \mathtt{W}(\rho + \alpha D\partial_z\varphi))\psi, \\ \partial_t^2\rho - \dfrac{1}{M^2}\Delta\rho = \Delta|\psi|^2 - \alpha D\partial_t\partial_z|\psi|^2, \\ \partial_t^2\varphi - \dfrac{1}{M^2}\Delta\varphi = \dfrac{\alpha D}{M^2}\partial_z|\psi|^2 - \partial_t|\psi|^2, \end{cases} \tag{ZR}$$

if $\sigma_2 = 0$ for (ZR_1), and

$$\begin{cases} i\partial_t\psi + \epsilon\partial_z^2\psi + \sigma_1\Delta_\perp\psi = \Big(q + \dfrac{\alpha\beta}{v_g}\Big)|\psi|^2\psi + \Big(\dfrac{\beta^2\rho_{00}}{v_g^2} + \dfrac{\alpha\gamma}{v_g^3}\Big)\rho\psi \\ \Delta(\rho + M^2|\psi|^2) = M^2\partial_z^2\rho + \dfrac{\alpha v_g^3}{\rho_{00}\gamma}\partial_z^2|\psi|^2, \end{cases} \tag{ZRK}$$

for (ZRK_1), where we have used that $M = |v_g|/c_s, \mathtt{W} = \beta^2\rho_{00}/v_g^2, D = |v_g|/\beta\rho_{00}$ and chosen $\sigma_2 = sgn(v_g)$ and $\gamma = \beta c_s^2 = \beta v_g^2/M^2$ constant.

Formally, when $q, \alpha \to 0$ in (ZR) we obtain

$$\begin{cases} i\partial_t\psi + \epsilon\partial_z^2\psi + \sigma_1\Delta_\perp\psi = \mathtt{W}\rho\psi, \\ \partial_t^2\rho - \dfrac{1}{M^2}\Delta\rho = \Delta|\psi|^2, \end{cases} \tag{Z}$$

and when $M, \beta \to 0$ in (ZRK) we obtain

$$\begin{cases} i\partial_t \psi + \epsilon \partial_z^2 \psi + \sigma_1 \Delta_\perp \psi = q|\psi|^2\psi + \dfrac{\alpha\gamma}{v_g^3}\rho\psi \\ \Delta\rho = \dfrac{\alpha v_g^3}{\rho_{00}\gamma}\partial_z^2|\psi|^2. \end{cases} \tag{DS}$$

One is also interested in introducing explicitly the ion sound velocity in the system (Z) by replacing $\Box_M = \partial_t^2 - \frac{1}{M}\Delta$ by $\Box_{M_c} = c^{-2}\partial_t^2 - \frac{1}{M}\Delta$ and considering the limit $c \to \infty$. In that limit the system (Z) reduces formally to the well know nonlinear Schrödinger equation

$$i\partial_t \psi + \epsilon \partial_z^2 \psi + \sigma_1 \Delta_\perp \psi = -M^2 \mathtt{W} |\psi|^2 \psi, \tag{S}$$

which gives a description of a system (or a quantum state of a physical system) evolving with time. Approximate solutions to the time-independent Schrödinger equation are commonly used to calculate the energy levels and other properties of atoms and molecules.

The system (DS) is the Davey-Stewartson system, it models the evolution of weakly nonlinear water waves that travel predominantly in one direction, but in which the wave amplitude is modulated slowly in two horizontal directions (see [11]). It is a model for an inviscid, incompressible homogeneous fluid whose motion is irrotational (potential flow). In the case $\epsilon > 0, \sigma_1 = 1$ and $n = 2$, Ghidaglia and Saut [17] classified the (DS) system as elliptic-elliptic type. In this case they reduced the DS system to a Schrödinger equation with a nonlocal nonlinear term by solving the second Poisson-like equation via a Hörmander-Mikhlin multiplier type (see Section 3.1) and established local existence and uniqueness for initial data in L^2, H^1 and H^2 (and global for small data in L^2) as well as blow-up results. In [36] Ozawa found exact blow-up solutions (see Chapter 9 in [28] for more results concerning this model).

The system (Z) is called the Zakharov system and it was introduced in [50] to describe the long wave Langmuir turbulence in a plasma. In the case $\sigma_1 = \epsilon = 1$, Schochet and Weinstein [46] obtained a local existence result and uniqueness result with time interval $[0, T]$ independent of the ionic speed of sound c. This allowed them to prove that solutions of (Z) converge to solutions of (S) as $c \to \infty$. For small amplitude solutions were obtained rates of this convergence by Added and Added in [1]. Ozawa and Tsutsumi [38] found optimal rates of convergence of solutions of (Z) to solutions of (S). Kenig, Ponce and Vega [25] proved that the IVP for (Z) is locally well posed uniformly on the parameters in appropriate Sobolev spaces. The best local well posedness results in this case are due to Ginibre, Tsutsumi and Velo [19]. By using a method developed by Bourgain [5] they showed local well posedness for the Z system for initial data in $H^k(\mathbb{R}^n) \times H^l(\mathbb{R}^n) \times H^{l-1}(\mathbb{R}^n)$ provided

$$\begin{cases} l \leq k \leq l+1 \quad \text{if} \quad n \geq 2, \quad \text{or} \\ l \geq 0, \ 2k-(l+1) \geq 0 \quad \text{if} \quad n=2,3. \end{cases}$$

In the case $\epsilon = \sigma_1 = M = \mathtt{W} = 1$, they split ρ into its positive and negative frequency parts according to

$$\rho_{\pm} = \rho \pm i\theta^{-1}\partial_t \rho,$$

where $\theta = (-\Delta)^{1/2}$. Then Z system was rewritten as

$$\begin{cases} i\partial_t \psi + \Delta\psi = \frac{1}{2}(\rho_+ + \rho_-)\psi, \\ (i\partial_t \mp \theta)\rho_{\pm} = \mp\theta^{-1}\Box\rho = \pm\theta|\psi|^2, \end{cases}$$

and they considered the case of a single equation

$$i\partial_t V = \phi(-i\nabla)V + f(V),$$

where ϕ is a real function (or real symmetric matrix valued function) defined in $\mathbb{R}^n$ and f some nonlinear function. Eventually V was replaced by (ψ, ρ_+, ρ_-) and ϕ was the diagonal matrix with entries $(\xi^2, |\xi|, -|\xi|)$. So, they studied the integral equation

$$V(t) = S(t)V_0 - i\int_0^t S(t-t')f(V(t'))dt'$$

where $V(0) = V_0$ is the initial data and $S(t) = e^{-it\phi(-i\nabla)}$ is the unitary group that solves the underlying linear equation.

In order to eliminate spurious infrared divergences, at the expense of breaking the dilation invariance for low momenta, a trivial modification of the system (Z) can be perform (see [19]) rewriting the wave equation as a Klein-Gordon equation

$$(\Box + 1)\rho = \Delta|\psi|^2 + \rho$$

and defining the positive and negative parts of ρ as

$$\rho_{\pm} = \rho \pm i(\theta_1)^{-1}\partial_t \rho,$$

where $\theta_1 = \sqrt{1-\Delta}$.

Another results about existence of solution of the Z system can be found in [44], [37] and [3]. Concerning blow-up results for solutions of the Z system, Glangetas and Merle [16], [32], proved the existence of blow-up solutions in dimension two.

The general Klein-Gordon equation is

$$\frac{\hbar^2}{2mc^2}\partial_t^2\rho - \frac{\hbar^2}{2m}\Delta\rho + \frac{mc^2}{2}\rho + f(\rho) = 0 \tag{KG}$$

where $\rho = \rho(t,x) : \mathbb{R}^{1+n} \to \mathbb{C}$, $f(\rho) = \lambda|\rho|^\gamma\rho$ with $\gamma > 0$ and $\lambda \in \mathbb{R}$, c is the speed of light, $\hbar$ is the Planck constant, and $m > 0$ is the mass of particle.
Rescaling t, x, ρ, λ and c, we can normalize the other constants as $\hbar = m = 2$.

Substituting $\Psi = e^{-ic^2t}\rho$, we obtain from (KG) the equation

$$c^{-2}\partial_t^2\Psi + 2i\partial_t\Psi - \Delta\Psi + f(\Psi) = 0. \tag{1}$$

Then if $c \to \infty$ we have (formally) the Schrödinger equation

$$2i\partial_t\Psi - \Delta\Psi + f(\Psi) = 0. \tag{2}$$

With regard to the convergence of solutions of (1) to solutions of (2), Tsutsumi [47] proved L^2 convergence assuming H^2 convergence of the initial data in the case where $n \leq 2$ and $\gamma \leq 2$. L^q convergence was shown for $2 \leq q < 2n/(n-2)$ by Najman [33] and Machihara [29] assuming H^1 boundedness and L^2 convergence of the initial data under the assumption $n \leq 3$ and some restrictive assumption on γ. In [30] Machihara, Nakanishi and Ozawa proved that any finite energy solution of Klein-Gordon equation converges to the corresponding solution of the nonlinear Schrödinger equation in the energy space, i.e., H^1 convergence was proved assuming H^1 convergence of the initial data, for any n and any $0 < \gamma < 4/(n-2)$.

The first step to study the limit (ZR) to (Z) is to establish local (global) well-posedness for the IVP. This will give us an idea what spaces the solutions exist and where we can expect to prove the before mentioned convergence. In this regards Ponce and Saut [39] reduced the ZR system to the Schrödinger equation

$$\begin{cases} i\partial_t\psi + \mathcal{L}\psi = H(\rho_0, \varphi_0, \psi), \\ \psi(x,0) = \psi_0, \end{cases} \tag{3}$$

or its integral equation version

$$\psi(t) = e^{it\mathcal{L}}\psi_0 - i\int_0^t e^{i(t-t')\mathcal{L}}H(\rho_0, \varphi_0, \psi)(t')dt', \tag{4}$$

with

$$\mathcal{L} = \epsilon\partial_z^2 + \sigma_1\Delta_\perp, \quad H(\rho_0, \varphi_0, \psi) = (q|\psi|^2 + \mathtt{W}(\rho + D\partial_z\varphi))\psi \tag{5}$$

and ρ, φ given from

$$\rho(t) = U'(t)\rho_0 + U(t)\rho_1 + \int_0^t U(t-t')F_1(\psi)(t')dt', \tag{6}$$

$$\varphi(t) = U'(t)\varphi_0 + U(t)\varphi_1 + \int_0^t U(t-t')F_2(\psi)(t')dt', \tag{7}$$

where

$$\begin{cases} F_1(\psi) = \Delta|\psi|^2 - \alpha D\partial_t\partial_z|\psi|^2, \\ F_2(\psi) = \dfrac{\alpha D}{M^2}\partial_z|\psi|^2 - \partial_t|\psi|^2, \end{cases}$$

and

$$\begin{cases} U(t)f = M(-\Delta)^{-1/2}\sin(M^{-1}(-\Delta)^{1/2}t)f, \\ U'(t)f = \cos(M^{-1}(-\Delta)^{1/2}t)f, \end{cases}$$

define the group of the wave equation.

Then they provide the following result

Theorem 0.1 *Let $s > n/2$, $n = 2, 3$. Then given $(\psi_0, \rho_0, \varphi_0) \in H^s \times H^{s-1/2} \times H^{s+1/2}(\mathbb{R}^n)$, there exist $T = T(\|\psi_0\|_{H^s}, \|\rho_0\|_{H^{s-1/2}}, \|\varphi_0\|_{H^{s+1/2}}) > 0$ and a unique solution $\psi(\cdot)$ of the integral equation* (3)-(4) *such that*

$$\psi \in C([0,T]; H^s(\mathbb{R}^n)) \tag{8}$$

with

$$\|(1-\Delta)^{s/2+1/4}\|_{\ell^\infty_\mu L^2_T L^2_x} < \infty. \tag{9}$$

Moreover, the map $(\psi_0, \rho_0, \varphi_0) \mapsto \psi(t)$ from $H^s \times H^{s-1/2} \times H^{s+1/2}$ into the class (1.30)-(1.31) *is locally Lipschitz, and one has that*

$$(\rho, \varphi) \in C([0,T]; H^{s-1/2}(\mathbb{R}^n) \times H^{s+1/2}(\mathbb{R}^n)). \tag{10}$$

The main ingredient to prove Theorem 0.1 is the smoothing effect of Kato's type, that is, solutions of the linear problem

$$\begin{cases} i\partial_t\psi + \mathcal{L}\psi = 0, \\ \psi(x,0) = \psi_0, \end{cases} \tag{11}$$

satisfy

$$\|D_x^{1/2}e^{it\mathcal{L}}f\|_{\ell^\infty_\mu L^2_T L^2_x} \le c\|f\|_{L^2}. \tag{12}$$

where $\|\cdot\|_{\ell^\infty_\mu L^p_T L^q_x}$ is defined by

$$\|F\|_{\ell^\infty_\mu L^2_T L^2_x} = \sup_{\mu\in\mathbb{Z}^n}\Big(\int_0^T\int_{Q_\mu}|F(x,t)|^2 dxdt\Big)^{1/2},$$

and $\{Q_\mu\}_{\mu\in\mathbb{Z}^n}$ is a family of unit cubes parallel to the coordinates axis with disjoint interiors covering $\mathbb{R}^n$.

Estimates of this type combined with the linear properties of the linear wave equation (see Theorem 1.4 below) and the contraction mapping principle yield the local result.

They also obtained existence of a global weak solution for initial data in $H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$. For that it was first proved that the energy functional

$$\begin{aligned}\mathcal{H} =&\mathtt{W}^{-1}\int\Big[|\psi|^2+\epsilon|\partial_z\psi|^2+\sigma_1|\nabla_\perp\psi|^2+\frac{q}{2}|\psi|^4+\mathtt{W}\rho|\psi|^2+\sigma_2\mathtt{W}\rho\partial_z\varphi\\&+\frac{\mathtt{W}}{2}|\nabla\varphi|^2+\frac{\mathtt{W}}{2M^2}\rho^2+\alpha D\mathtt{W}|\psi|^2\partial_z\varphi\Big]d\mathbf{x},\end{aligned}$$

is constant in time. Then under the hypothesis of positive energy they used a classical compactness method to obtain the result.

To study the limit (ZRK) to (DS) we need to know about the existence of solutions of the ZRK system. In fact this system belongs to a wider class of equations (see Chapter 4 for more details), the Zakharov-Schulman equations or ZS system

$$\begin{cases} i\partial_t u+\mathcal{L}_1 u = uv,\\ \mathcal{L}_2 v=\mathcal{L}_3(|u|^2),\\ u(x,0)=u_0,\end{cases}\tag{ZS}$$

where $u:\mathbb{R}^n\times[0,\infty)\to\mathbb{C}$, $v:\mathbb{R}^n\times[0,\infty)\to\mathbb{R}$ and

$$\mathcal{L}_k=\sum_{i,j=1}^n a^k_{i,j}\partial^2_{x_ix_j},\quad k=1,2,3,$$

with $a^k_{i,j}=a^k_{j,i}$ real constants.

This system model the interaction of small amplitude high frequency waves with acoustic types waves [42].

The system above can be rewrited as

$$\begin{cases} i\partial_t u+\mathcal{L}_1 u = u\mathcal{L}_2^{-1}\mathcal{L}_3(|u|^2),\\ u(x,0)=u_0.\end{cases}\tag{13}$$

In the case where the quadratic forms associated to the operators $\mathcal{L}_1, \mathcal{L}_2$ are non-degenerate with an appropriate radiation condition on the behavior of v at infinity when $\mathcal{L}_2$ is non-elliptic, Kenig-Ponce-Vega [25] proved the following result

Theorem 0.2 *Let $n > 1$. There exist $s > 0$, $m \in \mathbb{Z}^+$, and $\delta > 0$ such that for any*

$$u_0 \in H^s \cap H^{s_0}(|x|^m dx) := Y_{s,m}, \quad s_0 = [(s+3)/2],$$

with $\|u_0\|_{Y_{s,m}} \leq \delta$, there exist $T = T(\|u_0\|_{Y_{s,m}}) > 0$ (with $T(\theta) \to \infty$ as $\theta \to 0$) and a unique solution $u(\cdot, t)$ of the IVP (13) *satisfying*

$$u \in C([0,T]; H^s) \cap C([0,T]; H^{s_0}(|x|^m dx)), \tag{14}$$

$$D_x^{s+1/2} u \in \ell_\mu^\infty(L^2(Q_\mu \times [0,T])), \tag{15}$$

and

$$\partial_x^\beta u \in \ell_\mu^1(L^\infty(Q_\mu \times [0,T]) \cap \ell_\mu^2(Q_\mu \times [0,T])), \quad |\beta| \leq s_0. \tag{16}$$

For any $T' < T$ there exist a neighborhood $\mathcal{O}$ of u_0 in $Y_{s,m}$ such that the map $\tilde{u}_0 \to \tilde{u}$ from $\mathcal{O}$ into the class defined in (14)-(16) *with T' instead of T is Lipschitz.*
Moreover, if $u_0 \in Y_{s',m'}$ with $s' \geq s$ and $[s_0'] \geq m' \geq m$, then the above results hold with s', m' instead of s, m in the same time interval $[0,T]$.

In the case where $\mathcal{L}_2$ is elliptic, recently Oliveira-Panthee-Silva in [35] proved that the IVP (ZS) (or (13)) is locally well-posed for given initial data in H^s, $s \geq n/4$, for $n = 2, 3$ (see Theorem 3.1 below).

In this work we study the asymptotic behavior of the solutions of Zakharov-Rubenchik system in the forms (ZR) and (ZRK) when appropriate parameters tend to zero. Namely, we establish weak and strong convergence results of these solutions to solutions of Zakharov system and Davey-Stewartson system, respectively.

In the case of the weak convergence of solutions of ZR system, we follow the same argument as that in [1]. For the strong convergence we use some ideas from [33] and [29].

Let us now describe the content of this work.

In the first chapter we present the non-dimensional form of ZR_0 system and obtain the system (ZR). Some preliminary results for the (ZR) are presented. We deal with this system next and the integral equation versions are presented joint to Strichartz estimates for the group associated.

In the second chapter we present the main results related to limit for the Zakharov-Rubenchik system, Eq. (ZR). The weak and strong convergence of the solutions are established when $v_g \to +\infty, (M \to +\infty)$, and when both, v_g and $c_s \to \infty$. In fact, if $(\psi_\alpha, \rho_\alpha, \varphi_\alpha)$ is the solution of ZR system and (ψ, ρ) is the solution of Z system, then

$$(\psi_\alpha, \rho_\alpha) \rightharpoonup (\psi, \rho) \quad \text{(weakly star)} \quad \text{in} \quad L^\infty((0, +\infty); H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)), \quad n = 2, 3,$$

$$(\psi_\alpha, \rho_\alpha) \to (\psi, \rho) \quad \text{(strongly)} \quad \text{in} \quad L^\infty((0, T); L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)), \quad n = 2, 3$$

and

$$(\psi_\alpha, \rho_\alpha) \to (\psi, \rho) \quad \text{(strongly)} \quad \text{in} \quad L^\infty((0, T); H^{5/2}(\mathbb{R}^3) \times L^2(\mathbb{R}^3))$$

for some $T > 0$, as $\alpha \to 0$ (because of $\alpha \sim 1/v_g$ and $v_g \to +\infty$).

The proof of the weak limit is a classical argument in the theory of compactness, whose main ingredient is the Aubin-Lions Theorem and the Ascoli Theorem. Strong limits are conveniently treated by decomposing the nonlinearities and using the Strichartz estimates associated with the group of the Schrödinger equation and the wave group.

In the third chapter we deal with a modified Zakharov-Rubenchik system, the (ZRK_0) system. We reduce this to the non-dimensional form (ZRK_1) and the more simple form (ZRK). We regard this system as a Zakharov-Schulman system and reduce it to a Schrödinger equation type with a nonlocal nonlinear term, next we establish the local theory according Ghidaglia-Saut's work in [17] and Oliveira-Panthee-Silva's work in [35]. In the final section we derive some conservation laws of the modified system. In fact, we establish that the amounts

$$\mathcal{M}(\psi) = \int_{\mathbb{R}^2} |\psi|^2 dxdz,$$

and

$$\mathcal{E}(\psi) = \int_{\mathbb{R}^2} \big(|\psi_x|^2 + |\psi_z|^2 + \frac{c_1}{2}|\psi|^4 + c_2 S(\psi)\big) dxdz,$$

are constants of motion, where

$$S(\psi) = \int_0^t [\rho(|\psi|^2)_{t'}](t')dt'.$$

In the fourth chapter we present the main result related to limit for the modified Zakharov-Rubenchik system, Eq. (ZRK). Although we have conserved quantities, which contain terms that would like to control as was done with solutions of (ZR), the presence of the term $S(\psi)$ in $\mathcal{E}$ creates a difficulty. However, the strong convergence of the solutions is established when $c_s \to +\infty$, $(M \to 0)$, and when both, v_g and $c_s \to \infty$ in two dimensions. In fact, if (ψ_β, ρ_β) is the solution of ZRK system and (ψ, ρ) is the solution of DS system, then

$$(\psi_\beta, \rho_\beta) \to (\psi, \rho) \quad \text{(strongly)} \quad \text{in} \quad L^\infty((0,T); L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)),$$

for some $T > 0$, as $\beta \to 0$ (because of $\beta \sim 1/c_s^2$ and $c_s \to +\infty$).

The main tools in the proof of this result are the Strichartz estimates and the following result about the behavior of the multipliers defined by the equations of Poisson type in (ZRK) and (DS).

Proposition 0.1 *Let us consider the multipliers*

$$\widehat{\mathsf{T}_{Q_M} f} = Q_M \widehat{f}, \qquad Q_M(\xi) = \frac{M^2|\xi|^2 - \dfrac{\alpha v_g^3}{\rho_{00}\gamma}\xi_2^2}{M^2\xi_2^2 - |\xi|^2}. \tag{17}$$

and

$$\widehat{\mathsf{T}_{Q_0} f} = Q_0 \widehat{f}, \qquad Q_0(\xi) = \frac{\alpha v_g^3 \xi_2^2}{\rho_{00}\gamma|\xi|^2}. \tag{18}$$

Then

1. *Q_M is uniformly bounded for any $0 < M^2 \le 1/2$ and $\xi \neq 0$, that is,*

$$\exists\, C > 0 : \|Q_M\|_{L^\infty} \le C, \quad C \quad \textit{is independent of} \quad M,$$

2. $\lim_{M \to 0} \|\mathsf{T}_{(Q_M - Q_0)}(f)\|_{L^2} = 0 \quad$ *if* $\quad f \in L^2$.

Proof. See below Section 3. ■

Finally, the appendix deals with some function spaces and the Lions-Aubin and Ascoli theorems are stated because of their usefulness in the demonstration of weak convergence results. We introduce also the multiplier definition and the important theorem of Hörmander-Mikhlin. Next the functional derivative is introduced. It permits to verify the Hamilton equations for the models.

Chapter 1

The Zakharov Rubenchik System

1.1 Preliminary Results

The Zakharov-Rubenchik system for the interaction of a spectrally narrow high-frequency wave packet with a low-frequency oscillations of acoustic type is

$$\begin{cases} i(\partial_t\psi + v_g\partial_z\psi) + \dfrac{w''}{2}\partial_z^2\psi + \dfrac{v_g}{2k_0}\Delta_\perp\psi = (q|\psi|^2 + \beta\rho + \alpha\partial_z\varphi)\psi, \\ \partial_t\rho + \rho_{00}\Delta\varphi + \alpha\partial_z|\psi|^2 = 0, \\ \partial_t\varphi + \dfrac{c_s^2}{\rho_{00}}\rho + \beta|\psi|^2 = 0, \end{cases} \tag{ZR$_0$}$$

where $\psi = \psi(\mathbf{x}, t)$ denotes the complex amplitude of the (high frequency (HF)) carrying wave whose wave number k and frequency w are related by the dispersion relation $w = w(k)$, $v_g = w'(k)$ is the group velocity of the carrying wave, which according to [41] and [26] is in the direction of the z-axis, that is, $\mathbf{v_g} = (0, 0, v_g)$. The functions ρ and φ denote the density fluctuation and the hydrodynamic potential respectively, the parameters q, α, measure the self-interaction of the carrying wave and the Doppler shift respectively, $c_s = \sqrt{p'(\rho_{00})}$ is the sound velocity (p is the pressure), $\beta = \frac{\partial w(k_0)}{\partial \rho} \sim w/\rho_{00}$, the center of the HF packet is at k_0 ($k \sim k_0$) and the energy of the HF packet narrow is $\varepsilon_0 \approx \int w(k_0)|\psi|^2 dx$.

We use the notation $\mathbf{x} = (x, y, z)$ if $n = 3$ and $\mathbf{x} = (x, z)$ if $n = 2$, $\Delta_\perp = \partial_x^2 + \partial_y^2$ if $n = 3$ and $\Delta_\perp = \partial_x^2$ if $n = 2$.

After some transformations (see [40], [39]) we can rewrite the ZR$_0$ system in a non-dimensional form as

$$i\partial_t\psi + \epsilon\partial_z^2\psi + \sigma_1\Delta_\perp\psi = (\sigma|\psi|^2 + W(\rho + \alpha D\partial_z\varphi))\psi, \tag{1.1}$$

$$\partial_t\rho + \sigma_2\partial_z\rho = -\Delta\varphi - \alpha D\partial_z|\psi|^2, \tag{1.2}$$

$$\partial_t\varphi + \sigma_2\partial_z\varphi = -\frac{1}{M^2}\rho - |\psi|^2, \tag{1.3}$$

where $M = |v_g|/c_s$ (Mach number), $\epsilon = w''|k|/|v_g|$, $W = \beta^2\rho_{00}/|q|v_g^2$, $D = |v_g|/\beta\rho_{00}$, $\sigma_1 = sgn(k_0v_g) = k_0v_g/|k_0v_g|$, $\sigma = sgn(q) = q/|q|$, and $\sigma_2 = -sgn(v_g) = -v_g/|v_g|$.

Concerning well-posedness results for the Cauchy problem associated to (1.1)-(1.3), Ponce and Saut [39] proved that this problem is locally well-posed in $H^s(\mathbb{R}^n) \times H^{s-1/2}(\mathbb{R}^n) \times H^{s+1/2}(\mathbb{R}^n)$ if $s > n/2$ and $n = 2, 3$. About the energy, we have the

Proposition 1.1 *The equations* (1.1)-(1.3) *conserve the energy*

$$\begin{aligned}\mathcal{H} =& W^{-1}\int\Big[|\psi|^2 + \epsilon|\partial_z\psi|^2 + \sigma_1|\nabla_\perp\psi|^2 + \frac{\sigma}{2}|\psi|^4 + W\rho|\psi|^2 + \sigma_2 W\rho\partial_z\varphi \\ &+ \frac{W}{2}|\nabla\varphi|^2 + \frac{W}{2M^2}\rho^2 + \alpha DW|\psi|^2\partial_z\varphi\Big]d\mathbf{x},\end{aligned}$$

that is

$$\frac{d\mathcal{H}}{dt} = 0. \tag{1.4}$$

Moreover

$$\frac{\delta\mathcal{H}}{\delta\varphi} = \frac{\partial\rho}{\partial t}\ , \qquad \frac{\delta\mathcal{H}}{\delta\rho} = -\frac{\partial\varphi}{\partial t}. \tag{1.5}$$

Proof. See [39] for the proof of (1.4). Equality (1.5) are functional derivatives of $\mathcal{H}$. These can be obtained from a direct calculation (see Appendix below for the variational derivative). ■

The conservation law (1.4) allowed Ponce and Saut to prove in [39] the existence of global weak solutions for the equations (1.1)-(1.3) with initial data in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and for some range of the parameters involved. More precisely they proved the

Theorem 1.1 *Assume that* $\epsilon > 0, \sigma_1 = \sigma = 1$ *and that the quadratic form*

$$\mathcal{Q}(x,y,z) = \frac{W}{2M^2}x^2 + \frac{W}{2}y^2 + \frac{1}{2}z^2 + \sigma_2 Wxy + \alpha DWyz + Wxz, \tag{1.6}$$

is positive definite. Then for initial data $(\psi_0, \varphi_0, \rho_0) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, *there exists a global weak solution of* (1.1)-(1.3) *such that*

$$\psi, \varphi \in L^\infty(\mathbb{R}_+; H^1(\mathbb{R}^n)), \;\; \rho \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^n)) \tag{1.7}$$

$$\partial_t \psi, \partial_t \rho \in L^\infty(\mathbb{R}_+; H^{-1}(\mathbb{R}^n)), \;\; \partial_t \varphi \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^n)). \tag{1.8}$$

Moreover, if (1.10) *is not positive definite, the same conclusion above still holds if* $(\psi_0, \varphi_0, \rho_0)$ *are small enough in* $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

Because of the discontinuity of the nonlinear term $\sigma_2|\psi|^2$ when $q \to 0$, we introduce the following changes of variables that will permit us go over to the limit:

$$\psi = \sqrt{|q|}\widetilde{\psi}, \;\; \rho = |q|\widetilde{\rho}, \;\; \varphi = |q|\widetilde{\varphi}. \tag{1.9}$$

Then the equations (1.1)-(1.3) are transformed, after dropping the tilde symbols, in

$$\begin{cases} i\partial_t \psi + \epsilon \partial_z^2 \psi + \sigma_1 \Delta_\perp \psi = (q|\psi|^2 + \mathtt{W}(\rho + \alpha D \partial_z \varphi))\psi, \\ \partial_t \rho + \sigma_2 \partial_z \rho = -\Delta \varphi - \alpha D \partial_z |\psi|^2, \\ \partial_t \varphi + \sigma_2 \partial_z \varphi = -\dfrac{1}{M^2}\rho - |\psi|^2, \end{cases} \tag{ZR$_1$}$$

where $\mathtt{W} = |q|W = \beta^2 \rho_{00}/v_g^2$.

Remark 1.1 *The well-posedness results of [39] and Proposition 1.1 are true for any parameters* $\sigma, \sigma_1, \sigma_2$ *(not necessarily* ± 1*) and any* $W > 0$. *Theorem 1.1 still holds if* σ_1, σ *are any positive numbers. The latter is just to guarantee (with the respective positive defined form* $\mathcal{Q}$*) the uniform bound of the solutions in the energy space. Therefore, the initial value problem associated to* (ZR$_1$) *is locally well-posed in* $H^s(\mathbb{R}^n) \times H^{s-1/2}(\mathbb{R}^n) \times H^{s+1/2}(\mathbb{R}^n)$ *for any* σ_1, q, σ_2 *and* $s > n/2$. *Theorem 1.1 is also true with* $q > 0$ *instead of* σ *and* $\mathtt{W} > 0$ *instead of* W. *More precisely,*

Theorem 1.2 *Assume that* $\epsilon > 0, \sigma_1 = 1, q > 0$ *and that the quadratic form*

$$\mathcal{Q}(x,y,z) = \frac{\mathtt{W}}{2M^2}x^2 + \frac{\mathtt{W}}{2}y^2 + \frac{q}{2}z^2 + \sigma_2 \mathtt{W}xy + \alpha D\mathtt{W}yz + \mathtt{W}xz, \tag{1.10}$$

is positive definite. Then for initial data $(\psi_0, \varphi_0, \rho_0) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, *there exists a global weak solution of* (ZR_1) *such that*

$$\psi, \varphi \in L^\infty(\mathbb{R}_+; H^1(\mathbb{R}^n)),\ \ \rho \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^n)) \tag{1.11}$$

$$\partial_t\psi, \partial_t\rho \in L^\infty(\mathbb{R}_+; H^{-1}(\mathbb{R}^n)),\ \ \partial_t\varphi \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^n)). \tag{1.12}$$

Moreover, if (1.10) *is not positive definite, the same conclusion above still holds if* $(\psi_0, \varphi_0, \rho_0)$ *are small enough in* $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

1.2 The Integral Equation Version

Recall that σ_1, q, σ_2 can be any parameters in (ZR_1) (see Remark 1.1 above). So we consider $\sigma_2 = 0$.

We are going to rewrite the system (ZR_1) in a convenient form by decoupling the last two equations. Then we apply the operator ∂_t to them. One gets

$$\begin{cases} i\partial_t\psi + \epsilon\partial_z^2\psi + \sigma_1\Delta_\perp\psi = (q|\psi|^2 + \mathtt{W}(\rho + \alpha D\partial_z\varphi))\psi, \\ \partial_t^2\rho - \dfrac{1}{M^2}\Delta\rho = \Delta|\psi|^2 - \alpha D\partial_t\partial_z|\psi|^2, \\ \partial_t^2\varphi - \dfrac{1}{M^2}\Delta\varphi = \dfrac{\alpha D}{M^2}\partial_z|\psi|^2 - \partial_t|\psi|^2. \end{cases} \tag{ZR}$$

We are interested in the behavior of the solutions of the system (ZR) when $q, \alpha \to 0$.

We expect the solutions of initial value problems associated to converge toward the solutions of systems

$$\begin{cases} i\partial_t\psi + \epsilon\partial_z^2\psi + \sigma_1\Delta_\perp\psi = \mathtt{W}\rho\psi, \\ \partial_t^2\rho - \dfrac{1}{M^2}\Delta\rho = \Delta|\psi|^2. \end{cases} \tag{Z}$$

We will deal with integral equation versions of the systems (ZR) and (Z).

Introducing the following notations

$$\begin{cases} \mathcal{L} = \epsilon\partial_z^2 + \sigma_1\Delta_\perp \\ \Box_M = \partial_t^2 - \dfrac{1}{M^2}\Delta, \\ F_1(\psi) = \Delta|\psi|^2 - \alpha D\partial_t\partial_z|\psi|^2, \\ F_2(\psi) = \dfrac{\alpha D}{M^2}\partial_z|\psi|^2 - \partial_t|\psi|^2, \end{cases}$$

the IVP associated to the system (ZR) can be expressed as

$$\begin{cases} i\partial_t\psi + \mathcal{L}\psi = (q|\psi|^2 + \mathtt{W}(\rho + \alpha D\partial_z\varphi))\psi, \\ \Box_M\rho = F_1(\psi), \\ \Box_M\varphi = F_2(\psi), \\ \psi(x,0) = \psi_0(x), \\ \rho(x,0) = \rho_0(x), \partial_t\rho(x,0) = \rho_1(x) = -(\Delta\varphi_0 + \alpha D\partial_z|\psi_0|^2)(x), \\ \varphi(x,0) = \varphi_0(x), \partial_t\varphi(x,0) = \varphi_1(x) = -(\dfrac{1}{M^2}\rho_0 + |\psi_0|^2)(x). \end{cases} \tag{NS$_0$}$$

The second and third equations are acoustic wave equations with nonlinear terms F_1 and F_2 depending only on ψ. They satisfy the integral equations (see [39])

$$\rho(t) = U'(t)\rho_0 + U(t)\rho_1 + \int_0^t U(t-t')F_1(\psi)(t')dt', \tag{1.13}$$

$$\varphi(t) = U'(t)\varphi_0 + U(t)\varphi_1 + \int_0^t U(t-t')F_2(\psi)(t')dt', \tag{1.14}$$

where

$$\begin{cases} U(t)f = M(-\Delta)^{-1/2}\sin(M^{-1}(-\Delta)^{1/2}t)f, \\ U'(t)f = \cos(M^{-1}(-\Delta)^{1/2}t)f, \end{cases}$$

define the group of the wave equation with the following estimates

$$\|U(t)f\|_{L^2} \leq |t|\,\|f\|_{L^2}, \quad \|\nabla_x U(t)f\|_{L^2} \leq M\|f\|_{L^2}, \tag{1.15}$$

$$\|U'(t)f\|_{L^2} \leq \|f\|_{L^2}. \tag{1.16}$$

These are consequence of the properties of the functions $\sin(\cdot)$ and $\cos(\cdot)$, and the Plancherel identity.

The terms F_1, F_2 involve derivatives in the t-variable of ψ, but we can remove them by using the following formulas (which follows by integration by parts)

$$\int_0^t U(t-t')\partial_t G(t')dt' = -U(t)G(0) + \int_0^t U'(t-t')G(t')dt', \tag{1.17}$$

$$\int_0^t U'(t-t')\partial_t G(t')dt' = -U'(t)G(0) + \frac{1}{M^2}\int_0^t U(t-t')\Delta G(t')dt'. \tag{1.18}$$

The system (NS_0) is a nonlinear Schrödinger equation type formally equivalent to the IVP associated to the system (ZR) and we rewrite it as

$$\begin{cases} i\partial_t\psi + \mathcal{L}\psi = H(\rho_0, \varphi_0, \psi), \\ \psi(x,0) = \psi_0, \end{cases} \tag{1.19}$$

with $\alpha \sim c_s/v_g$, or its integral equation version

$$\psi(t) = e^{it\mathcal{L}}\psi_0 - i\int_0^t e^{i(t-t')\mathcal{L}} H(\rho_0, \varphi_0, \psi)(t')dt', \tag{1.20}$$

with

$$H(\rho_0, \varphi_0, \psi) = (q|\psi|^2 + \mathtt{W}(\rho + D\partial_z\varphi))\psi \tag{1.21}$$

and ρ, φ given from (1.13), (1.14).

Remark 1.2 *The integral formulation* (1.20) *joint to* (1.13) *and* (1.14) *are the general versions of the solutions obtained by Ponce-Saut in [39] for the ZR system. They obtained the well-posedness result applying a fixed point argument on a space convenient by using more refined estimates involving the smoothing effects of the group associated to both, Schrödinger and Wave equations. More precisely, they used that*

Theorem 1.3 *The group* $\{e^{it\mathcal{L}}\}_{t=-\infty}^{\infty}$ *satisfies:*

$$\|D_x^{1/2} e^{it\mathcal{L}} f\|_{\ell_\mu^\infty L_T^2 L_x^2} \le c\|f\|_{L^2}, \tag{1.22}$$

$$\sup_{0\le t\le T} \|D_x^{1/2}\int_0^t e^{i(t-t')\mathcal{L}} G(t')dt'\|_{L^2} \le c\|G\|_{\ell_\mu^1 L_T^2 L_x^2}, \tag{1.23}$$

$$\|\nabla_x \int_0^t e^{i(t-t')\mathcal{L}} G(t')dt'\|_{\ell_\mu^\infty L_T^2 L_x^2} \le c\|G\|_{\ell_\mu^1 L_T^2 L_x^2} \tag{1.24}$$

where

$$\widehat{D_x^{1/2} f} = (|\xi|^{1/2}\widehat{f}), \quad \mathcal{L} = \epsilon\partial_z^2 + \Delta_\perp \quad \textit{and}$$

c is a constant independent of T.

Theorem 1.4 *In the 3-dimensional case one has that*

$$\|U'(t)f\|_{\ell^2_\mu L^\infty_T L^2_x} \leq c(1+TM)^3\|f\|_{L^2}, \tag{1.25}$$
$$\|U(t)\nabla_x f\|_{\ell^2_\mu L^\infty_T L^2_x} \leq cM(1+TM)^3\|f\|_{L^2}, \tag{1.26}$$
$$\|U(t)f\|_{\ell^2_\mu L^\infty_T L^2_x} \leq cT(1+TM)^3\|f\|_{L^2}, \tag{1.27}$$
$$\|\nabla_x \int_0^t U(t-t')h(t')dt'\|_{\ell^2_\mu L^\infty_T L^2_x} \leq cM(1+TM)^3\|h\|_{\ell^2_\mu L^1_T L^2_x}, \tag{1.28}$$
$$\|\int_0^t U'(t-t')h(t')dt'\|_{\ell^2_\mu L^\infty_T L^2_x} \leq c(1+TM)^3\|h\|_{\ell^2_\mu L^1_T L^2_x}. \tag{1.29}$$

Here $\|\cdot\|_{\ell^r_\mu L^p_T L^q_x}$ *is defined by*

$$\|F\|_{\ell^r_\mu L^p_T L^q_x} = (\sum_{\mu\in\mathbb{Z}^n}(\int_0^T(\int_{Q_\mu}|F(x,t)|^q dx)^{p/q}dt)^{r/p})^{1/r},$$

where $\{Q_\mu\}_{\mu\in\mathbb{Z}^n}$ *is a family of unit cubes parallel to the coordinates axis with disjoint interiors covering* $\mathbb{R}^n$*.*

In the case $n=3, s=2+1/2$ *and for* $(\psi_0,\rho_0,\varphi_0)\in H^s\times H^{s-1/2}\times H^{s+1/2}$ *fixed, they defined the operator*

$$\Phi(\omega)(t) = e^{it\mathcal{L}}\psi_0 - i\int_0^t H(\rho_0,\varphi_0,\omega)(t')dt',$$

with ω *in the function space* X^a_T*, where* $\omega\in X^a_T$ *if*

$$\omega:\mathbb{R}^n\times[0,T]\to\mathbb{C},\ \omega\in C([0,T]:H^s(\mathbb{R}^n))$$

and

$$|||\omega|||_T = \sup_{0\leq t\leq T}\|\omega(t)\|_{H^s} + \sum_{|\lambda|=3}\|\partial^\lambda_x\omega\|_{\ell^\infty_\mu L^2_T L^2_x} \leq a.$$

So for appropriate values of the parameters $a, T>0$, $\Phi(\cdot)$ *defines a contraction in* X^a_T*.*

The general case $s>n/2$ *follows by combining the argument above with the calculus of inequalities involving fractional derivatives deduced in [24].*

Then they obtained that

Theorem 1.5 *Let $s > n/2$, $n = 2, 3$. Then given $(\psi_0, \rho_0, \varphi_0) \in H^s \times H^{s-1/2} \times H^{s+1/2}(\mathbb{R}^n)$, there exist $T = T(\|\psi_0\|_{H^s}, \|\rho_0\|_{H^{s-1/2}}, \|\varphi_0\|_{H^{s+1/2}}) > 0$ and a unique solution $\psi(\cdot)$ of the integral equation (3)-(4) such that*

$$\psi \in C([0,T]; H^s(\mathbb{R}^n)) \tag{1.30}$$

with

$$\|(1-\Delta)^{s/2+1/4}\|_{\ell^\infty_\mu L^2_T L^2_x} < \infty. \tag{1.31}$$

Moreover, the map $(\psi_0, \rho_0, \varphi_0) \mapsto \psi(t)$ from $H^s \times H^{s-1/2} \times H^{s+1/2}$ into the class (1.30)-(1.31) is locally Lipschitz, and one has that

$$(\rho, \varphi) \in C([0,T]; H^{s-1/2}(\mathbb{R}^n) \times H^{s+1/2}(\mathbb{R}^n)). \tag{1.32}$$

The estimates involving the smoothing effect of Kato's type, in its homogeneous and inhomogeneous versions, associated to the unitary group $\{e^{it\mathcal{L}} : t \in \mathbb{R}\}$ were established by Constantin-Saut [8], Ponce-Saut [39], Kenig-Ponce-Vega [25] and other authors.

The estimates associated to group of the wave equation (Theorem 1.4) were established by Ponce and Saut in [39] by combining the standard energy estimates and the finite propagation speed of the solution.

From the proof of Theorem 1.5 in [39], the life-time

$$T = T(\alpha, q, \beta, M, ||\psi_0||_{H^s}, ||\rho_0||_{H^{s-1/2}}, ||\varphi_0||_{H^{s+1/2}}) > 0,$$

for solutions of ZR system, is a continuous decreasing function, indeed,

$$T < \frac{1}{6[1 + |q|a^2 + 4(1 + |\mathtt{W}| + |\alpha D|)(1+M)^4 a^2]} \qquad \text{and} \tag{1.33}$$

$$T^{1/2} < \frac{1}{6[1 + 2(|\mathtt{W}| + |\alpha D|)(1+M)^4(\|\rho_0\| + \|\rho_1\| + \|\varphi_0\| + \|\varphi_1\| + \|\psi_0\|^2)]}, \tag{1.34}$$

with

$$a = 2\|\psi_0\|_{H^s},\ \|\psi_0\|^2 = \|\psi_0\|^2_{H^{s-1/2}} \tag{1.35}$$

and

$$\|\rho_0\| = \|\rho_0\|_{H^{s-1/2}}, \|\rho_1\| = \|\rho_1\|_{H^{s-3/2}}, \|\varphi_0\| = \|\varphi_0\|_{H^{s+1/2}}, \|\varphi_1\| = \|\varphi_1\|_{H^{s-1/2}}. \tag{1.36}$$

Furthermore, T does not vanish when $M \to +\infty$ (or $\alpha D \to +\infty$) if we choose conveniently the data. For instance, with $\|\psi_0\| \le 1/M^2$ we have

$$(1+M)^4 a^2 \le 4(1 + 1/M)^4,$$

so

$$\frac{1}{6[1+|q|a^2+4(1+|\mathtt{W}|+|\alpha D|)4(1+1/M)^4]} < \frac{1}{6[1+|q|a^2+4(1+|\mathtt{W}|+|\alpha D|)(1+M)^4a^2]}.$$

These properties of T will be essential in Chapter 2, for example to control the growth of terms of the form TM^4 when $M \to +\infty$.

Now let us consider the system (ZR) with $q = \alpha = 0$, that is,

$$\begin{cases} i\partial_t\psi + \mathcal{L}\psi = \mathtt{W}\rho\psi, \\ \Box_M\rho = \Delta|\psi|^2, \end{cases} \tag{$Z_\perp$}$$

or

$$\begin{cases} i\partial_t\psi + \Delta\psi = \mathtt{W}\rho\psi, \\ \Box_M\rho = \Delta|\psi|^2, \end{cases} \tag{Z}$$

if $\epsilon = \sigma_1 = 1$.

In this case we have

$$\rho(t) = U'(t)\rho_0 + U(t)\rho_1 + \int_0^t U(t-t')F(\psi)(t')dt', \tag{1.37}$$

with $F(\psi) = \Delta|\psi|^2$. Therefore the IVP associated to ($Z_\perp$) is formally equivalent to

$$\begin{cases} i\partial_t\psi + \mathcal{L}\psi = I(\rho_0,\rho_1,\psi), \\ \psi(x,0) = \psi_0, \end{cases} \tag{NS_Z}$$

or

$$\psi(t) = e^{it\mathcal{L}}\psi_0 - i\int_0^t e^{i(t-t')\mathcal{L}} I(\rho_0,\rho_1,\psi)(t')dt', \tag{1.38}$$

where

$$I(\rho_0,\rho_1,\psi) = \mathtt{W}\rho\psi \tag{1.39}$$

and ρ given from (1.37).

By compatibility with the known results for the limit systems (Z) and (DS), we are going to consider the elliptic case, this is, we assume $\mathcal{L} = \Delta$ and present the Strichartz estimates of the group $\{e^{it\Delta}\}_{t=-\infty}^{\infty}$.

Let I any time interval. Then we have

Theorem 1.6 *The group* $\{e^{it\Delta}\}_{t=-\infty}^{\infty}$ *satisfies:*

$$\|e^{i(\cdot)\Delta}\|_{L^q(I,L^p(\mathbb{R}^n))} \leq c\|f\|_{L^2(\mathbb{R}^n)}, \tag{1.40}$$

$$\|\int_{-\infty}^{\infty} e^{i(\cdot-t')\Delta} g(\cdot,t')dt'\|_{L^q(I,L^p(\mathbb{R}^n))} \leq c\|g\|_{L^{q'}(I,L^{p'}(\mathbb{R}^n))}, \tag{1.41}$$

$$\|\int_{-\infty}^{\infty} e^{it\Delta} g(\cdot,t)dt\|_{L^2_x} \leq c\|g\|_{L^{q'}(I,L^{p'}(\mathbb{R}^n))}, \tag{1.42}$$

with

$$2 \leq p < \theta(n), \quad \frac{2}{q} = \frac{n}{2} - \frac{n}{p}, \quad \theta(n) = \infty \quad \text{if} \quad n = 1, 2, \quad \theta(n) = \frac{2n}{n-2} \quad \text{if} \quad n \geq 3 \tag{A}$$

and $c = c(p, n)$ *a constant that depends only on* p *and* n*.*

Proof. See for example [45] or [28] and references therein. ■

Corollary 1.1 *Let* $(p_0, q_0), (p_1, q_1) \in \mathbb{R}^2$ *satisfying the condition (A). Then for all* $T > 0$ *we have*

$$\|\int_0^t e^{i(\cdot-t')\Delta} g(\cdot,t')dt'\|_{L^{q_1}(0,T;L^{p_1}(\mathbb{R}^n))} \leq c\|g\|_{L^{q_0'}(0,T;L^{p_0'}(\mathbb{R}^n))} \tag{1.43}$$

with $c = c(n, p_0, p_1)$*.*

Proof. See [28]. ■

Chapter 2

Supersonic Regime Results

2.1 Weak Convergence Result

Our first result concerning the asymptotic behavior of the solutions of the system

$$\begin{cases} i\partial_t\psi + \epsilon\partial_z^2\psi + \sigma_1\Delta_\perp\psi = (q|\psi|^2 + \mathtt{W}(\rho + \alpha D\partial_z\varphi))\psi, \\ \partial_t\rho = -\Delta\varphi - \alpha D\partial_z|\psi|^2, \\ \partial_t\varphi = -\dfrac{1}{M^2}\rho - |\psi|^2, \end{cases} \tag{ZR$_1$ with $\sigma_2 = 0$}$$

or decoupling the las two equations, of the system

$$\begin{cases} i\partial_t\psi + \epsilon\partial_z^2\psi + \sigma_1\Delta_\perp\psi = (q|\psi|^2 + \mathtt{W}(\rho + \alpha D\partial_z\varphi))\psi, \\ \partial_t^2\rho - \dfrac{1}{M^2}\Delta\rho = \Delta|\psi|^2 - \alpha D\partial_t\partial_z|\psi|^2, \\ \partial_t^2\varphi - \dfrac{1}{M^2}\Delta\varphi = \dfrac{\alpha D}{M^2}\partial_z|\psi|^2 - \partial_t|\psi|^2, \end{cases} \tag{ZR}$$

when q, α tend to zero, can be stated as follows.

Theorem 2.1 *Under the hypotheses of Theorem 1.2, let* $(\psi_{q\alpha}, \rho_{q\alpha}, \varphi_{q\alpha})$ *be any solution of* (ZR)*. Then as* $q, \alpha \to 0^+$, $(\psi_{q\alpha}, \rho_{q\alpha}, \varphi_{q\alpha})$ *converge to* (ψ, ρ, φ) *in* $L^\infty(\mathbb{R}_+; H^1(\mathbb{R}^n)) \times L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^n)) \times L^\infty(\mathbb{R}_+; H^1(\mathbb{R}^n))$ *weak star, where* (ψ, ρ) *is the unique solution of the Zakharov system*

$$\begin{cases} i\partial_t\psi + \epsilon\partial_z^2\psi + \sigma_1\Delta_\perp\psi - \mathtt{W}\rho\psi = 0, \\ \partial_t^2\rho - \dfrac{1}{M^2}\Delta\rho = \Delta|\psi|^2, \end{cases} \tag{$Z_\perp$}$$

with initial data in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times H^{-1}(\mathbb{R}^n)$.

Moreover, $(\partial_t\psi_{q\alpha}, \partial_t\rho_{q\alpha}, \partial_t\varphi_{q\alpha})$ *converge to* $(\partial_t\psi, \partial_t\rho, \partial_t\varphi)$ *in* $L^\infty(\mathbb{R}_+; H^{-1}(\mathbb{R}^n)) \times L^\infty(\mathbb{R}_+; H^{-1}(\mathbb{R}^n)) \times L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^n))$ *weak star, and additionally,* $\partial_t\rho = -\Delta\varphi$.

Proof.

We shall use a classical compactness method and we will follow the same ideas of H. Added and S. Added in [1] for the weak limit of Zakharov system.

Present the proof of the theorem in three steps, but before it is worth recalling the conserved energy functional (in the case $\sigma_2 = 0$)

$$\begin{aligned} \mathcal{H}(t) = & \mathtt{W}^{-1} \int \Big[|\psi|^2 + \epsilon|\partial_z\psi|^2 + \sigma_1|\nabla_\perp\psi|^2 + \frac{\mathtt{W}}{2M^2}\rho^2 + \frac{\mathtt{W}}{2}|\nabla\varphi|^2 + \frac{q}{2}|\psi|^4 \\ & + \alpha D\mathtt{W}|\psi|^2\partial_z\varphi + \mathtt{W}\rho|\psi|^2 \Big] d\mathbf{x}. \end{aligned}$$

We have the positive definite quadratic form

$$\mathcal{Q}(x, y, z) = \frac{\mathtt{W}}{2M^2}x^2 + \frac{\mathtt{W}}{2}y^2 + \frac{q}{2}z^2 + \alpha D\mathtt{W}yz + \mathtt{W}xz. \tag{2.1}$$

Hence

$$\mathcal{Q}(0, \partial_z\varphi, |\psi|^2) = \frac{\mathtt{W}}{2}|\partial_z\varphi|^2 + \frac{q}{2}|\psi|^4 + \alpha D\mathtt{W}|\psi|^2\partial_z\varphi > 0 \tag{2.2}$$

Recall that $\mathtt{W}, D > 0$ and by Theorem 1.2 $\epsilon > 0$ and $\sigma_1 = 1$. In addition, $\rho > 0$ because it is the mass density.

As we are considering $q, \alpha \to 0^+$, then we assume $q, \alpha > 0$.

In these condition we have

Step 1. By the conservation of energy $\mathcal{H}$, the norms $\|\psi_{q\alpha}\|_{L^\infty((0,\infty);H^1(\mathbb{R}^n))}$ and $\|\rho_{q\alpha}\|_{L^\infty((0,\infty);L^2(\mathbb{R}^n))}$ are bounded uniformly in q and α. So $\||\psi_{q\alpha}|^2\|_{L^\infty((0,\infty);L^2(\mathbb{R}^n))}$ is bounded uniformly by the Sobolev embedding.

We can obtain a uniform bound for $\|\nabla \varphi_{q\alpha}\|_{L^\infty((0,\infty);L^2(\mathbb{R}^n))}$ by combining the Cauchy-Schwarz inequality in the energy $\mathcal{H}$ with the norms above, and choosing α small enough.

Note also that if we multiply the third equation in (ZR$_1$) by φ we get that

$$\begin{aligned}
\frac{1}{2}\partial_t \|\varphi\|^2_{L^2_x} &\le \frac{1}{M^2}\int |\rho\varphi| + \int |\psi|^2|\varphi| + C \\
&\le \|\varphi\|_{L^2_x}\Big(\frac{1}{M^2}\|\rho\|_{L^2_x} + \|\psi\|^2_{L^4_x}\Big) + C \\
&\le \frac{1}{2}\|\varphi\|^2_{L^2_x} + \frac{1}{2}\Big(\frac{1}{M^2}\|\rho\|_{L^2_x} + \|\psi\|^2_{L^4_x}\Big)^2 + C. \qquad (2.3)
\end{aligned}$$

Hence

$$\partial_t \|\varphi\|^2_{L^2_x} \le \|\varphi\|^2_{L^2_x} + C, \qquad (2.4)$$

thereupon the Gronwall inequality implies that $\|\varphi_{q\alpha}\|_{L^\infty((0,\infty);L^2(\mathbb{R}^n))}$ is uniformly bounded, so $\|\varphi_{q\alpha}\|_{L^\infty((0,\infty);H^1(\mathbb{R}^n))}$ is uniformly bounded.

For simplicity we set $q = \alpha$, and the associated solutions $(\psi_\alpha, \rho_\alpha, \varphi_\alpha)$. So, some subsequence of $(\psi_\alpha, \rho_\alpha, \varphi_\alpha)$, also labeled by α, and $|\psi_\alpha|^2$ have a weak limit (ψ, ρ, φ) and Γ respectively. More precisely

$$\begin{aligned}
&\psi_\alpha \underset{*}{\rightharpoonup} \psi \quad \text{in} \quad L^\infty((0,\infty);H^1(\mathbb{R}^n)), && (2.5)\\
&\rho_\alpha \underset{*}{\rightharpoonup} \rho \quad \text{in} \quad L^\infty((0,\infty);L^2(\mathbb{R}^n)), && (2.6)\\
&\varphi_\alpha \underset{*}{\rightharpoonup} \varphi \quad \text{in} \quad L^\infty((0,\infty);H^1(\mathbb{R}^n)), && (2.7)\\
&|\psi_\alpha|^2 \underset{*}{\rightharpoonup} \Gamma \quad \text{in} \quad L^\infty((0,\infty);L^2(\mathbb{R}^n)), && (2.8)
\end{aligned}$$

thereby (see Proposition A.1 below),

$$\begin{aligned}
&(\epsilon\partial_z^2\psi_\alpha + \sigma_1\Delta_\perp\psi_\alpha) \underset{*}{\rightharpoonup} (\epsilon\partial_z^2\psi + \sigma_1\Delta_\perp\psi) \quad \text{in} \quad L^\infty((0,\infty);H^{-1}(\mathbb{R}^n)), && (2.9)\\
&\Delta\rho_\alpha \underset{*}{\rightharpoonup} \Delta\rho \quad \text{in} \quad L^\infty((0,\infty);H^{-2}(\mathbb{R}^n)), && (2.10)\\
&\Delta\varphi_\alpha \underset{*}{\rightharpoonup} \Delta\varphi \quad \text{in} \quad L^\infty((0,\infty);H^{-1}(\mathbb{R}^n)), && (2.11)\\
&\Delta|\psi_\alpha|^2 \underset{*}{\rightharpoonup} \Delta\Gamma \quad \text{in} \quad L^\infty((0,\infty);H^{-2}(\mathbb{R}^n)), && (2.12)\\
&\partial_z|\psi_\alpha|^2 \underset{*}{\rightharpoonup} \partial_z\Gamma \quad \text{in} \quad L^\infty((0,\infty);H^{-1}(\mathbb{R}^n)). && (2.13)
\end{aligned}$$

Let us note that the map

$$\begin{aligned}
H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) &\to H^{-1}(\mathbb{R}^n) \qquad (2.14)\\
(u,v) &\mapsto uv
\end{aligned}$$

is continuous. So it can be assumed that $\rho_\alpha\psi_\alpha,\ \psi_\alpha\partial_z\varphi_\alpha,\ |\psi_\alpha|^2\psi_\alpha$ have a $*$-weak limit in $L^\infty((0,\infty);H^{-1}(\mathbb{R}^n))$.

Let

$$\rho_\alpha\psi_\alpha \underset{*}{\rightharpoonup} \Lambda \quad \text{in} \quad L^\infty((0,\infty);H^{-1}(\mathbb{R}^n)). \tag{2.15}$$

Therefore, taking into account (2.5)-(2.15), the equations in (ZR) imply that

$$\partial_t\psi_\alpha \underset{*}{\rightharpoonup} \partial_t\psi \quad \text{in} \quad L^\infty((0,\infty);H^{-1}(\mathbb{R}^n)), \tag{2.16}$$

$$\partial_t\rho_\alpha \underset{*}{\rightharpoonup} \partial_t\rho \quad \text{in} \quad L^\infty((0,\infty);H^{-1}(\mathbb{R}^n)), \tag{2.17}$$

$$\partial_t^2\rho_\alpha \underset{*}{\rightharpoonup} \partial_t^2\rho \quad \text{in} \quad L^\infty((0,\infty);H^{-2}(\mathbb{R}^n)), \tag{2.18}$$

$$\partial_t\varphi_\alpha \underset{*}{\rightharpoonup} \partial_t\varphi \quad \text{in} \quad L^\infty((0,\infty);L^2(\mathbb{R}^n)), \tag{2.19}$$

with $\partial_t\rho = -\Delta\varphi$ and

$$\begin{cases} i\partial_t\psi + \epsilon\partial_z^2\psi + \sigma_1\Delta_\perp\psi = \mathtt{W}\Lambda, \\ \partial_t^2\rho - \dfrac{1}{M^2}\Delta\rho = \Delta\Gamma, \end{cases}$$

in the distribution sense (actually in $L^\infty((0,\infty);H^{-1}(\mathbb{R}^n)) \times L^\infty((0,\infty);H^{-2}(\mathbb{R}^n))$).

The proof of the theorem will be complete if we establish that

$$\Lambda = \rho\psi \text{ and } \Gamma = |\psi|^2. \tag{2.20}$$

Let us consider, for the remainder of the proof, any finite time interval $[0,T]$.

Step 2. Let us show that $\Gamma = |\psi|^2$.

Let $\Omega \subset \mathbb{R}^n$ be any bounded sub-domain, $B_0 = H^1(\Omega), B = L^4(\Omega), B_1 = H^{-1}(\Omega)$ and consider $\psi_\alpha|_\Omega$.

By Rellich-Kondrachov's theorem we know that $B_0 \underset{c}{\hookrightarrow} B$, and we have

$$\psi_\alpha \in L^\infty((0,\infty);H^{-1}(\mathbb{R}^n)) \hookrightarrow L^2((0,T);H^1(\mathbb{R}^n)) \tag{2.21}$$

and

$$\partial_t\psi_\alpha \in L^\infty((0,\infty);H^{-1}(\mathbb{R}^n)) \hookrightarrow L^2((0,T);H^{-1}(\mathbb{R}^n)), \tag{2.22}$$

so,

$$\psi_\alpha \in \Big\{V \in L^2((0,T);B_0), \partial_t V \in L^2((0,T),B_1)\Big\} \underset{c}{\hookrightarrow} L^2((0,T);B), \tag{2.23}$$

with compact embedding due to Lions-Aubin's Theorem.

Therefore, some subsequence of $\psi_\alpha|_\Omega$ (also labeled by α) converges strongly to $\psi|_\Omega$ in $L^2([0,T];L^4(\Omega))$. So we can assume that

$$\psi_\alpha \underset{\alpha\to 0}{\longrightarrow} \psi \text{ strongly in } L^2([0,T];L^4_{loc}(\mathbb{R}^n)), \tag{2.24}$$

hence

$$\psi_\alpha \underset{\alpha\to 0}{\longrightarrow} \psi \text{ strongly in } L^2([0,T];L^2_{loc}(\mathbb{R}^n)), \tag{2.25}$$

and thus,

$$\psi_\alpha \underset{\alpha\to 0}{\longrightarrow} \psi \ a.e \text{ in } (t,x) \in [0,T]\times\mathbb{R}^n, \tag{2.26}$$

thereupon

$$|\psi_\alpha|^2 \underset{\alpha\to 0}{\longrightarrow} |\psi|^2 \ a.e. \text{ in } (t,x) \in [0,T]\times\mathbb{R}^n. \tag{2.27}$$

Since $|\psi_\alpha|^2 \in L^\infty((0,\infty);L^2(\mathbb{R}^n)) \hookrightarrow L^2([0,T];L^2(\mathbb{R}^n))$ is bounded uniformly in α, then by Lemma A.1

$$|\psi_\alpha|^2 \rightharpoonup |\psi|^2 \text{ in } L^2([0,T];L^2(\mathbb{R}^n)), \tag{2.28}$$

that is,

$$|\psi_\alpha|^2 \underset{*}{\rightharpoonup} |\psi|^2 \text{ in } L^2([0,T];L^2(\mathbb{R}^n)) \tag{2.29}$$

by reflexivity. Then $\Gamma = |\psi|^2$.

Step 3. Let us show that $\Lambda = \rho\psi$.

We shall prove that $\rho_\alpha\psi_\alpha \underset{*}{\rightharpoonup} \rho\psi$ in $L^2([0,T];H^{-1}(\mathbb{R}^n))$, so the embedding

$$L^\infty((0,\infty);H^{-1}(\mathbb{R}^n)) \hookrightarrow L^2([0,T];H^{-1}(\mathbb{R}^n))$$

and the convergence (2.15) will ensure that $\Lambda = \rho\psi$.

Let ϕ be some test function in $L^2([0,T];H^1(\mathbb{R}^n))$ vanishing out of a compact set $\Omega\subset\mathbb{R}^n$. Then

$$\begin{aligned}\langle \rho_\alpha\psi_\alpha - \rho\psi, \phi\rangle &= \int_0^T\int_{-\Omega}(\rho_\alpha\psi_\alpha-\rho\psi)\phi\ dx\ dt\\ &= \int_0^T\int_\Omega \rho_\alpha(\psi_\alpha-\psi)\phi\ dx\ dt + \int_0^T\int_\Omega(\rho_\alpha-\rho)\psi\phi\ dx\ dt\\ &=: I_{1\alpha}(\phi)+I_{2\alpha}(\phi).\end{aligned} \tag{2.30}$$

We have

$$|I_{1\alpha}(\phi)| \leqq \|\rho_\alpha\|_{L^\infty((0,\infty);L^2(\mathbb{R}^n))}\|\phi\|_{L^2([0,T];L^4(\Omega))}\|\psi_\alpha-\psi\|_{L^2([0,T];L^4(\Omega))} \to 0 \tag{2.31}$$

because of (2.24).

Let us note that $\psi\phi \in L^1([0,T]; L^2(\mathbb{R}^n))$. In fact,

$$\|\psi\phi\|_{L^1([0,T];L^2(\mathbb{R}^n))} \leqq \|\psi\|_{L^2([0,T];L^4(\mathbb{R}^n))}\|\phi\|_{L^2([0,T];L^4(\mathbb{R}^n))} < \infty. \tag{2.32}$$

Therefore, using that $\rho_\alpha \underset{*}{\rightharpoonup} \rho$ in $L^\infty((0,T); L^2(\mathbb{R}^n))$, it follows that $|I_{2\alpha}(\phi)| \to 0$. Thus

$$\langle \rho_\alpha\psi_\alpha - \rho\psi, \phi\rangle \underset{\alpha\to 0}{\longrightarrow} 0 \quad \text{for all } \phi \text{ test.} \tag{2.33}$$

But, $\{\rho_\alpha\psi_\alpha - \rho\psi\}_{\alpha>0} \subset (L^2([0,T]; H^1))' \subset C(L^2([0,T]; H^1); \mathbb{C})$ is a sequence (family) pointwise bounded and equicontinuous, because of

$$|\langle \rho_\alpha\psi_\alpha - \rho\psi, \phi\rangle| \leqq \|\rho_\alpha\psi_\alpha - \rho\psi\|_{L^2([0,T];H^{-1})}\|\phi\|_{L^2([0,T];H^1)} < C \tag{2.34}$$

and

$$|\langle \rho_\alpha\psi_\alpha - \rho\psi, \phi_1 - \phi_2\rangle| \leqq \|\rho_\alpha\psi_\alpha - \rho\psi\|_{L^2([0,T];H^{-1})}\|\phi_1 - \phi_2\|_{L^2([0,T];H^1)}, \tag{2.35}$$

so Ascoli's theorem implies that some subsequence of $\rho_\alpha\psi_\alpha - \rho\psi$ converges pointwise to a continuous function, and the convergence is uniform on each compact subset of $L^2([0,T]; H^1)$. Therefore

$$\rho_\alpha\psi_\alpha - \rho\psi \underset{*}{\rightharpoonup} 0 \quad \text{in} \quad L^2([0,T]; H^{-1}(\mathbb{R}^n)). \tag{2.36}$$

■

2.2 Strong Convergence Result

We will deal with the integral form of systems

$$\begin{cases} i\partial_t\psi + \Delta\psi = (q|\psi|^2 + \mathtt{W}(\rho + \alpha D\partial_z\varphi))\psi, \\ \partial_t^2\rho - \dfrac{1}{M^2}\Delta\rho = \Delta|\psi|^2 - \alpha D\partial_t\partial_z|\psi|^2, \\ \partial_t^2\varphi - \dfrac{1}{M^2}\Delta\varphi = \dfrac{\alpha D}{M^2}\partial_z|\psi|^2 - \partial_t|\psi|^2, \end{cases} \tag{ZR}$$

and

$$\begin{cases} i\partial_t\psi + \Delta\psi = \mathtt{W}\rho\psi, \\ \partial_t^2\rho - \dfrac{1}{M^2}\Delta\rho = \Delta|\psi|^2. \end{cases} \tag{Z}$$

But let us recall that the IVP is associated to the system of Zakharov-Rubenchik

$$\begin{cases} i\partial_t\psi + \Delta\psi = (q|\psi|^2 + \mathtt{W}(\rho + \alpha D\partial_z\varphi))\psi, \\ \partial_t\rho = -\Delta\varphi - \alpha D\partial_z|\psi|^2, \\ \partial_t\varphi = -\dfrac{1}{M^2}\rho - |\psi|^2, \end{cases} \tag{ZR$_1$}$$

and

$$\begin{cases} \psi(x,0) = \psi_0(x), \quad \rho(x,0) = \rho_0(x), \quad \varphi(x,0) = \varphi_0(x) \\ \partial_t\rho(x,0) = \rho_1(x) = -(\Delta\varphi_0 + \alpha D\partial_z|\psi_0|^2)(x), \quad \partial_t\varphi(x,0) = \varphi_1(x) = -(\dfrac{1}{M^2}\rho_0 + |\psi_0|^2)(x). \end{cases}$$

We will denote by $(\psi_\alpha, \rho_\alpha, \varphi_\alpha)$ the solution of the IVP associated to (ZR) (in fact to (ZR$_1$)) with initial data $(\psi_{\alpha 0}, \rho_{\alpha 0}, \varphi_{\alpha 0})$. Since $v_g \gg c_s$ should imply a Doppler shift negligible $\alpha \approx 0$, then we are going to consider $q = \alpha = c_s(\lambda_0 v_g)^{-1}$, $\beta = w\rho_{00}^{-1} = c_s(\lambda_0\rho_{00})^{-1}$ and make $v_g \to +\infty$.

Let us denote $T^* = T^*_{\alpha,q,\beta,M} = T^*_{\alpha,\beta}$ the life-time of the solution of (ZR) such that it does not vanish when $v_g \to +\infty$, and T_0 the life-time of the solution of (Z), which can be chosen independent of v_g, c_s for small data.

The arguments we use to obtain our strong convergence results are inspired in the ideas of the works of Najman [33] and Machihara [29] regarding the non-relativistic limit of the nonlinear Klein-Gordon equation.

Next we have one of our main results.

Theorem 2.2 *We assume that* $q = \alpha, n = 2, 3$,

$$\psi_{\alpha 0} \in H^4, \rho_{\alpha 0} \in H^{7/2}, \varphi_{\alpha 0} \in H^{9/2}, \tag{2.37}$$

$$\psi_0 \in H^4, \rho_0 \in H^{7/2}, \varphi_0 \in H^{9/2}, \tag{2.38}$$

$$\sup_{\alpha>0}\|\psi_{\alpha 0}\|_{H^4} < \infty, \tag{2.39}$$

$$\|\psi_0\|_{H^{5/2}} \le \inf_{\alpha>0}\|\psi_{\alpha 0}\|_{H^{5/2}},\ \|\rho_0\|_{H^2} \le \inf_{\alpha>0}\|\rho_{\alpha 0}\|_{H^2},\ \|\varphi_0\|_{H^3} \le \inf_{\alpha>0}\|\varphi_{\alpha 0}\|_{H^3}, \tag{2.40}$$

$$\lim_{\alpha\to 0}\|\psi_{\alpha 0} - \psi_0\|_{H^3} = \lim_{\alpha\to 0}\|\rho_{\alpha 0} - \rho_0\|_{H^2} = \lim_{\alpha\to 0}\|\varphi_{\alpha 0} - \varphi_0\|_{H^4} = 0. \tag{2.41}$$

Then we have

$$i)\lim_{\alpha\to 0}T^*_{\alpha,\beta}(\|\psi_{\alpha 0}\|_{H^{5/2}}, \|\rho_{\alpha 0}\|_{H^2}, \|\varphi_{\alpha 0}\|_{H^3}) = T^*_{0,\beta}(\|\psi_0\|_{H^{5/2}}, \|\rho_0\|_{H^2}, \|\varphi_0\|_{H^3}), \tag{2.42}$$

$$ii)\lim_{\alpha\to 0}\|\rho_{\alpha 1} + \Delta\varphi_0\|_{H^2} = 0, \tag{2.43}$$

$$iii)\lim_{\alpha\to 0}\|\psi_\alpha - \psi\|_{L^\infty(0,T;L^2)} = \lim_{\alpha\to 0}\|\rho_\alpha - \rho\|_{L^\infty(0,T;L^2)} = 0 \quad \text{if} \quad T \le T_m, \tag{2.44}$$

where $T_m = \min\{T_0, T^*_{0,\beta}\}$ *and* (ψ, ρ) *is the solution of* (Z) *with initial data* (ψ_0, ρ_0, ρ_1), $\rho_1 := -\Delta\varphi_0$.

Proof.

The equality (2.42) and (2.43) are consequences of the hypotheses (2.41) because of $T^* = T^*_{\alpha,\beta}$ is a continuous decreasing function of the initial data and the parameters, and

$$\rho_{\alpha 1} = -(\Delta\varphi_{\alpha 0} + \alpha D \partial_z |\psi_{\alpha 0}|^2).$$

The hypotheses (2.40) ensures that $T^*_{\alpha,\beta}$ is increasing when $\alpha \to 0$.

For the convergence (2.44) we use the integral versions of (ZR) and (Z), thus

$$\psi_\alpha(t) = e^{it\Delta}\psi_{\alpha 0} - i\int_0^t e^{i(t-t')\Delta} H(\rho_{\alpha 0}, \varphi_{\alpha 0}, \psi_\alpha)(t')dt', \tag{2.45}$$

with

$$H(\rho_{\alpha 0}, \varphi_{\alpha 0}, \psi_\alpha) = (q|\psi_\alpha|^2 + \mathtt{W}(\rho_\alpha + \alpha D\partial_z \varphi_\alpha))\psi_\alpha, \tag{2.46}$$

$$\rho_\alpha(t) = U'(t)\rho_{\alpha 0} + U(t)\rho_{\alpha 1} + \int_0^t U(t-t')F_1(\psi_\alpha)(t')dt', \tag{2.47}$$

$$\varphi_\alpha(t) = U'(t)\varphi_{\alpha 0} + U(t)\varphi_{\alpha 1} + \int_0^t U(t-t')F_2(\psi_\alpha)(t')dt' \tag{2.48}$$

and

$$\psi(t) = e^{it\Delta}\psi_0 - i\int_0^t e^{i(t-t')\Delta} I(\rho_0, \rho_1, \psi)(t')dt', \tag{2.49}$$

with

$$I(\rho_0, \rho_1, \psi) = \mathtt{W}\rho\psi, \tag{2.50}$$

$$\rho(t) = U'(t)\rho_0 + U(t)\rho_1 + \int_0^t U(t-t')F(\psi)(t')dt', \tag{2.51}$$

for $0 < t < T \leq T_m$.

Let us recall that

$$F(\psi) = \Delta|\psi|^2 \tag{2.52}$$

$$F_1(\psi) = \Delta|\psi|^2 - \alpha D\partial_t\partial_z|\psi|^2, \tag{2.53}$$

$$F_2(\psi) = \frac{\alpha D}{M^2}\partial_z|\psi|^2 - \partial_t|\psi|^2. \tag{2.54}$$

Then

$$(\psi_\alpha - \psi)(t) = e^{it\Delta}(\psi_{\alpha 0} - \psi_0) - i\int_0^t e^{i(t-t')\Delta}(H - I)(t')dt', \tag{2.55}$$

$$(\rho_\alpha - \rho)(t) = U'(t)(\rho_{\alpha 0} - \rho_0) + U(t)(\rho_{\alpha 1} - \rho_1) + \int_0^t U(t-t')(F_1 - F)(t')dt'. \tag{2.56}$$

By using Theorem 1.6 and Corollary 1.1 we have

$$\begin{aligned}\|\psi_\alpha - \psi\|_{L^\infty(0,T;L^2)} &\leq c\|\psi_{\alpha 0} - \psi_0\|_{L^2} \\ &\quad + \|\int_0^t e^{i(t-t')\Delta}(H - I)(t')dt'\|_{L^\infty(0,T;L^2)} \\ &\lesssim \|\psi_{\alpha 0} - \psi_0\|_{L^2} + \|H - I\|_{L^1(0,T;L^2)},\end{aligned} \tag{2.57}$$

and from the energy estimates (1.15)-(1.16) we have

$$\begin{aligned}\|\rho_\alpha - \rho\|_{L^2_x} \leq& \|\rho_{\alpha 0} - \rho_0\|_{L^2_x} + |t|\,\|\rho_{\alpha 1} - \rho_1\|_{L^2_x} \\ &+ \|\int_0^t U(t-t')(F_1 - F)(t')dt'\|_{L^2_x}.\end{aligned} \tag{2.58}$$

The nonlinear terms $H - I$ and $\int_0^t U(t-t')(F_1 - F)(t')dt'$ can be written as

$$H - I = N_1 + N_2 + N_3 + N_4 + N_5 + N_6 + N_7 \tag{2.59}$$

$$\int_0^t U(t-t')(F_1 - F)(t')dt' = N_{11} + N_{22} \tag{2.60}$$

where

$$N_1 = q|\psi_\alpha|^2\psi_\alpha, \tag{2.61}$$

$$N_2 = \mathtt{W}\alpha D\psi_\alpha\partial_z\varphi_\alpha, \tag{2.62}$$

$$N_3 = \mathtt{W}\psi_\alpha(U'(t)(\rho_{\alpha 0} - \rho_0) + U(t)(\rho_{\alpha 1} - \rho_1)), \tag{2.63}$$

$$N_4 = \mathtt{W}(\psi_\alpha - \psi)(U'(t)\rho_0 + U(t)\rho_1), \tag{2.64}$$

$$N_5 = -\mathtt{W}\alpha D\psi_\alpha\int_0^t U(t-t')\partial_t\partial_z|\psi_\alpha|^2dt', \tag{2.65}$$

$$N_6 = \mathtt{W}\psi_\alpha\int_0^t U(t-t')\Delta(|\psi_\alpha|^2 - |\psi|^2)dt', \tag{2.66}$$

$$N_7 = \mathtt{W}(\psi_\alpha - \psi)\int_0^t U(t-t')\Delta|\psi|^2dt' \tag{2.67}$$

$$N_{11} = -\alpha D\int_0^t U(t-t')\partial_t\partial_z|\psi_\alpha|^2dt', \tag{2.68}$$

$$N_{22} = \int_0^t U(t-t')\Delta(|\psi_\alpha|^2 - |\psi|^2)dt'. \tag{2.69}$$

The embedding theorems of Section A.1, the standard energy estimates (1.15) and (1.16), and integration by parts (formulas (1.17) and (1.18)) are useful for estimate $||H - I||_{L^2_x}$. For instance, we use the inequality

$$\|\psi_\alpha\|_{L^\infty_x} \leq \|\psi_\alpha\|_{H^{1+}}$$

in two dimensions. In three dimensions we can put $\|\psi_\alpha\|_{H^{3/2+}}$ instead of $\|\psi_\alpha\|_{H^{1+}}$ without any change in the demonstration.

Thus,

$$\|N_1\|_{L^2_x} = |q|\, \|\psi_\alpha\|^3_{L^6_x} \lesssim |q|\, \|\psi_\alpha\|^3_{H^1}, \tag{2.70}$$

$$\|N_2\|_{L^2_x} \leq |\mathtt{W}\alpha D|\, \|\psi_\alpha\|_{L^\infty_x} \|\partial_z \varphi_\alpha\|_{L^2_x} \leq |\mathtt{W}\alpha D|\, \|\psi_\alpha\|_{H^{1+}} \|\varphi_\alpha\|_{H^1}. \tag{2.71}$$

Next, we estimate $\|\varphi_\alpha\|_{H^1}$.

$$\begin{aligned}\|\varphi_\alpha\|_{H^1} &\leq \|U'(t)\varphi_{\alpha 0}\|_{H^1} + \|U(t)\varphi_{\alpha 1}\|_{H^1} + \|\int_0^t U(t-t')F_2(\psi_\alpha)(t')dt'\|_{H^1} \\ &\leq \|\varphi_{\alpha 0}\|_{H^1} + |t|\, \|\varphi_{\alpha 1}\|_{H^1} + \|\int_0^t U(t-t')F_2(\psi_\alpha)(t')dt'\|_{H^1},\end{aligned} \tag{2.72}$$

with

$$\begin{aligned}&\|\int_0^t U(t-t')F_2(\psi_\alpha)(t')dt'\|_{H^1} = \|\int_0^t U(t-t')(\frac{\alpha D}{M^2}\partial_z|\psi_\alpha|^2 - \partial_t|\psi_\alpha|^2)(t')dt'\|_{H^1} \\ &\leq \frac{|\alpha D|}{M^2}\int_0^t |t-t'|\, \|\partial_z|\psi_\alpha|^2\|_{H^1}dt' + \|\int_0^t U(t-t')\partial_t|\psi_\alpha|^2 dt'\|_{H^1} \\ &\leq \frac{|\alpha D|}{M^2}\int_0^t |t-t'|\, \||\psi_\alpha|^2\|_{H^2}dt' + \| - U(t)|\psi_{\alpha 0}|^2 + \int_0^t U'(t-t')|\psi_\alpha|^2 dt'\|_{H^1} \\ &\leq \frac{|\alpha D|}{M^2}\int_0^t |t-t'|\, \|\psi_\alpha\|^2_{H^2}dt' + |t|\, \|\psi_{\alpha 0}\|^2_{H^2} + \int_0^t \|\psi_\alpha\|^2_{H^2}dt'\end{aligned} \tag{2.73}$$

and

$$\|\varphi_{\alpha 1}\|_{H^1} = \|\frac{1}{M^2}\rho_{\alpha 0} + |\psi_{\alpha 0}|^2\|_{H^1} \leq \frac{1}{M^2}\|\rho_{\alpha 0}\|_{H^1} + \|\psi_{\alpha 0}\|^2_{H^2}. \tag{2.74}$$

Let us estimate N_3 and N_4. Thus

$$\begin{aligned}\|N_3\|_{L^2_x} &\leq |\mathtt{W}|\|\psi_\alpha\|_{L^\infty_x}\|U'(t)(\rho_{\alpha 0} - \rho_0) + U(t)(\rho_{\alpha 1} - \rho_1)\|_{L^2_x} \\ &\leq |\mathtt{W}|\|\psi_\alpha\|_{H^{1+}}(\|\rho_{\alpha 0} - \rho_0\|_{L^2_x} + |t|\, \|\rho_{\alpha 1} - \rho_1\|_{L^2_x})\end{aligned} \tag{2.75}$$

and

$$\begin{aligned}\|N_4\|_{L^2_x} &\leq |\mathtt{W}|\|\psi_\alpha - \psi\|_{L^\infty_x}\|U'(t)(\rho_0) + U(t)(\rho_1)\|_{L^2_x} \\ &\leq |\mathtt{W}|\|\psi_\alpha - \psi\|_{H^{1+}}(\|\rho_0\|_{L^2_x} + |t|\, \|\rho_1\|_{L^2_x}).\end{aligned} \tag{2.76}$$

For N_5 we have

$$\begin{aligned}\|N_5\|_{L^2_x} &\leq |\mathtt{W}\alpha D|\,\|\psi_\alpha\|_{L^\infty_x}\|\int_0^t U(t-t')\partial_t\partial_z|\psi_\alpha|^2 dt'\|_{L^2_x}\\ &\leq |\mathtt{W}\alpha D|\,\|\psi_\alpha\|_{H^{1+}}\| -U(t)\partial_z|\psi_{\alpha 0}|^2+\int_0^t U'(t-t')\partial_z|\psi_\alpha|^2 dt'\|_{L^2_x}\\ &\leq |\mathtt{W}\alpha D|\,\|\psi_\alpha\|_{H^{1+}}\Big(|t|\,\|\partial_z|\psi_{\alpha 0}|^2\|_{L^2_x}+\int_0^t \|\partial_z|\psi_\alpha|^2\|_{L^2_x}dt'\Big)\\ &\leq |\mathtt{W}\alpha D|\,\|\psi_\alpha\|_{H^{1+}}\Big(|t|\,\|\psi_{\alpha 0}\|^2_{H^2}+\int_0^t \|\psi_\alpha\|^2_{H^2}dt'\Big) \end{aligned}\tag{2.77}$$

and for N_6 we have

$$\begin{aligned}\|N_6\|_{L^2_x} &\leq |\mathtt{W}|\,\|\psi_\alpha\|_{L^\infty_x}\|\int_0^t U(t-t')\Delta(|\psi_\alpha|^2-|\psi|^2)dt'\|_{L^2_x}\\ &\leq |\mathtt{W}|\,\|\psi_\alpha\|_{H^{1+}}\int_0^t |t-t'|\,\|\Delta(|\psi_\alpha|^2-|\psi|^2)\|_{L^2_x}dt'\\ &\leq |\mathtt{W}|\,\|\psi_\alpha\|_{H^{1+}}\int_0^t |t-t'|\big(\||\psi_\alpha|^2\|_{H^2}+\||\psi|^2\|_{H^2}\big)dt'\\ &\leq |\mathtt{W}|\,\|\psi_\alpha\|_{H^{1+}}\int_0^t |t-t'|\big(\|\psi_\alpha\|^2_{H^2}+\|\psi\|^2_{H^2}\big)dt'. \end{aligned}\tag{2.78}$$

Finally we estimates N_7.

$$\begin{aligned}\|N_7\|_{L^2_x} &\leq |\mathtt{W}|\,\|\psi_\alpha-\psi\|_{L^\infty_x}\|\int_0^t U(t-t')\Delta|\psi|^2 dt'\|_{L^2_x}\\ &\leq |\mathtt{W}|\,\|\psi_\alpha-\psi\|_{H^{1+}}\int_0^t |t-t'|\,\|\Delta|\psi|^2\|_{L^2_x}dt'\\ &\leq |\mathtt{W}|\,\|\psi_\alpha-\psi\|_{H^{1+}}\int_0^t |t-t'|\,\|\psi\|^2_{H^2}dt'. \end{aligned}\tag{2.79}$$

Since

$$\mathtt{W}=\beta^2\rho_{00}/v_g^2,\ D=|v_g|/\beta\rho_{00},\ M=|v_g|/c_s \quad \text{with} \quad v_g\to\infty,\tag{2.80}$$

and

$$\|\psi_\alpha\|_{H^s}\lesssim\|\psi_{\alpha 0}\|_{H^s} \quad \text{for all} \quad s>n/2,\tag{2.81}$$

because of fixed point argument in [39], then by combining both of them with the hypotheses (2.41) and the result (2.43) we obtain that

$$\|N_i\|_{L^2_x}\to 0 \quad \text{as} \quad \alpha\to 0, \quad \text{for any} \quad 0<t<T\leq T_m,\ i=1,...,7.\tag{2.82}$$

Note that
$$\mathtt{W}D = \beta/v_g \quad \text{and} \quad D/M^2 = c_s^2/\beta\rho_{00}v_g.$$

Then
$$\|H - I\|_{L^1(0,T;L^2)} \leq T\|H - I\|_{L^\infty(0,T;L^2)} \to 0 \quad \text{as} \quad \alpha \to 0 \tag{2.83}$$
and therefore
$$\|\psi_\alpha - \psi\|_{L^\infty(0,T;L^2)} \lesssim \|\psi_{\alpha 0} - \psi_0\|_{L^2} + \|H - I\|_{L^1(0,T;L^2)} \to 0 \quad \text{as} \quad \alpha \to 0. \tag{2.84}$$
This finished the first part of (iii) (equality (2.44)).

Next, we estimate N_{11} and N_{22}. As N_{11} is similar to N_5, we have
$$\|N_{11}\|_{L^2_x} \leq |\alpha D|\big(|t|\,\|\psi_{\alpha 0}\|^2_{H^2} + \int_0^t \|\psi_\alpha\|^2_{H^2}dt'\big). \tag{2.85}$$

For N_{22} we use that (see (NS_0) and (Z))
$$\Delta\big(|\psi_\alpha|^2 - |\psi|^2\big) = \partial_t^2\rho_\alpha - \frac{1}{M^2}\Delta\rho_\alpha + \alpha D\partial_t\partial_z|\psi_\alpha|^2 - \partial_t^2\rho + \frac{1}{M^2}\Delta\rho, \tag{2.86}$$
hence
$$\|N_{22}\|_{L^2_x} \leq A_{11} + A_{22} + A_{33}, \tag{2.87}$$
with $A_{22} = \|N_{11}\|_{L^2_x}$,
$$\begin{aligned} A_{11} &= \frac{1}{M^2}\|\int_0^t U(t-t')\Delta(\rho_\alpha - \rho)dt'\|_{L^2_x} \\ &\leq \frac{1}{M^2}\int_0^t |t-t'|\,\|\Delta(\rho_\alpha - \rho)\|_{L^2_x}dt' \end{aligned} \tag{2.88}$$
and
$$\begin{aligned} A_{33} &= \|\int_0^t U(t-t')\partial_t^2(\rho_\alpha - \rho)dt'\|_{L^2_x} \\ &\leq \big(|t|\,\|\rho_{\alpha 1} - \rho_1\|_{L^2_x} + \|\rho_{\alpha 0} - \rho_0\|_{L^2_x} + \frac{1}{M^2}\|\int_0^t U(t-t')\Delta(\rho_\alpha - \rho)dt'\|_{L^2_x}\big) \\ &\leq \big(|t|\,\|\rho_{\alpha 1} - \rho_1\|_{L^2_x} + \|\rho_{\alpha 0} - \rho_0\|_{L^2_x} + \frac{1}{M^2}\int_0^t |t-t'|\|\Delta(\rho_\alpha - \rho)\|_{L^2_x}dt'\big) \end{aligned} \tag{2.89}$$

because of the integration by parts formulas (1.17), (1.18).

Now we need estimate the last term in (2.88) and (2.89). Thus,

$$\begin{aligned}\|\Delta(\rho_\alpha-\rho)\|_{L^2_x} &= \|U'(t)\Delta(\rho_{\alpha 0}-\rho_0)+U(t)\Delta(\rho_{\alpha 1}-\rho_1)\\ &\quad +\int_0^t U(t-t')\Delta\big(F_1(\psi_\alpha)-\Delta|\psi|^2\big)dt'\|_{L^2_x}\\ &\le \|\rho_{\alpha 0}-\rho_0\|_{H^2}+|t|\,\|\rho_{\alpha 1}-\rho_1\|_{H^2}\\ &\quad +\|\int_0^t U(t-t')\Delta\big(\Delta|\psi_\alpha|^2-\Delta|\psi|^2\big)dt'\|_{L^2_x}\\ &\quad +|\alpha D|\|\int_0^t U(t-t')\Delta\big(\partial_t\partial_z|\psi|^2\big)dt'\|_{L^2_x}\\ &\le \|\rho_{\alpha 0}-\rho_0\|_{H^2}+|t|\,\|\rho_{\alpha 1}-\rho_1\|_{H^2}\\ &\quad +\|\int_0^t |t-t'|\big(\|\psi_\alpha\|^2_{H^4}+\|\psi\|^2_{H^4}\big)dt'\|_{L^2_x}\\ &\quad +|\alpha D||t|\|\psi_{\alpha 0}\|^2_{H^3}+|\alpha D|\int_0^t \|\psi_\alpha\|^2_{H^3}dt'. \end{aligned} \tag{2.90}$$

It is here that the hypotheses (2.37)-(2.39) play their important role to guarantee bounds for $||\psi_\alpha||_{H^4}$ and $||\psi||_{H^4}$.

The same arguments used to assure the convergence in (2.82) imply that

$$\|\int_0^t U(t-t')(F_1-F)(t')dt'\|_{L^2_x}\le \|N_{11}\|_{L^2_x}+\|N_{22}\|_{L^2_x}\to 0 \quad \text{as} \quad \alpha\to 0, \tag{2.91}$$

for any $0<t<T\le T_m$.

Therefore

$$\begin{aligned}\|\rho_\alpha-\rho\|_{L^\infty(0,T;L^2)} \le & \|\rho_{\alpha 0}-\rho_0\|_{L^2_x}+|T|\,\|\rho_{\alpha 1}-\rho_1\|_{L^2_x}+\\ & \|\int_0^t U(t-t')(F_1-F)(t')dt'\|_{L^\infty(0,T;L^2)}\to 0. \end{aligned}\tag{2.92}$$

This completes the proof. ■

The most natural question about the supersonic limit of (ZR) is whether any solution (with finite energy) converges strongly to a solution of (Z) in the energy space $H^1\times L^2$. In this sense, our best result is a combination of the above theorem of L^2-convergence and the following result of $H^{5/2}$-convergence in three dimensions.

The advantage of working in three spatial dimensions is that we can make use of estimates obtained by Ponce and Saut in [39] (Theorem 1.4 here) for the group associated with the wave equation.

Theorem 2.3 *Under the hypotheses of Theorem 2.2 with $n = 3$, we also have that*

$$\lim_{\alpha\to 0}\|\psi_\alpha - \psi\|_{L^\infty(0,T;H^{2+1/2})} = 0 \tag{2.93}$$

for all $T \le T_m$, T_m as was defined before.

Proof.

Similarly to what was done in the proof of previous theorem, we begin with estimating

$$\|\psi_\alpha - \psi\|_{L^\infty(0,T;H^{2+1/2})} \le \|\psi_{\alpha 0} - \psi_0\|_{H^{2+1/2}} + \|\int_0^t e^{i(t-t')\Delta}(H-I)(t')dt'\|_{L^\infty(0,T;H^{2+1/2})}, \tag{2.94}$$

where $H - I = N_1 + \cdots + N_7$ is like in that argument. Then

$$\begin{aligned}\|\int_0^t e^{i(t-t')\Delta}(H-I)(t')dt'\|_{L^\infty(0,T;H^{2+1/2})} &\le \int_0^T (\|N_1\|_{H^{2+1/2}} + \|N_2\|_{H^{2+1/2}})dt \\ &\quad + \sum_{j=3}^{7}\|\int_0^t e^{i(t-t')\Delta}N_j(t')dt'\|_{L^\infty(0,T;H^{2+1/2})}.\end{aligned} \tag{2.95}$$

But,

$$\int_0^T \|N_1\|_{H^{2+1/2}}dt \le T|q|\sup_{0\le t\le T}\|\psi_\alpha\|^3_{H^{2+1/2}}, \tag{2.96}$$

$$\int_0^T \|N_2\|_{H^{2+1/2}}dt \le T|\mathtt{W}\alpha D|\sup_{0\le t\le T}\|\psi_\alpha\|_{H^{2+1/2}}\|\partial_z\varphi_\alpha\|_{H^{2+1/2}} \tag{2.97}$$

and

$$\begin{aligned}\|\partial_z\varphi_\alpha\|_{H^{2+1/2}} &\le \|\varphi_\alpha\|_{H^{3+1/2}} \\ &\le \|\varphi_{\alpha 0}\|_{H^{3+1/2}} + |t|\|\varphi_{\alpha 1}\|_{H^{3+1/2}} + \|\int_0^t U(t-t')F_2(\psi_\alpha)(t')dt'\|_{H^{3+1/2}}\end{aligned} \tag{2.98}$$

with

$$\|\varphi_{\alpha 1}\|_{H^{3+1/2}} \le \frac{1}{M^2}\|\rho_{\alpha 0}\|_{H^{3+1/2}} + \|\psi_{\alpha 0}\|^2_{H^{3+1/2}} \tag{2.99}$$

and

$$\begin{aligned}\|\int_0^t U(t-t')F_2(\psi_\alpha)(t')dt'\|_{H^{3+1/2}} &\le \frac{|\alpha D|}{M^2}\int_0^t |t-t'|\|\psi_\alpha\|^2_{H^{3+1/2}}dt' + |t|\|\psi_{\alpha 0}\|^2_{H^{3+1/2}} \\ &\quad + \int_0^t \|\psi_\alpha\|^2_{H^{3+1/2}}dt'\end{aligned} \tag{2.100}$$

as was done for φ_α in (2.72).

By using that $(1+|\xi|)^{2+1/2} \sim (1+|\xi|^2|\xi|^{1/2})$ and the estimate (1.23) of Theorem 1.3 we have

$$\begin{aligned}\sum_{j=3}^{7} \| \int_0^t e^{i(t-t')\Delta} N_j(t')dt' \|_{L^\infty(0,T;H^{2+1/2})} &\lesssim \sum_{j=3}^{7} \int_0^T \|N_j\|_{L^2} dt \\ &\quad + \sum_{j=3}^{7} \sum_{|\lambda|=2} \sup_{0<t<T} \|\partial_x^\lambda D_x^{1/2} \int_0^t e^{i(t-t')\Delta} N_j(t')dt'\|_{L^2} \\ &\lesssim \sum_{j=3}^{7} \int_0^T \|N_j\|_{L^2} dt \\ &\quad + \sum_{j=3}^{7} \sum_{|\lambda|=2} \|\partial_x^\lambda N_j\|_{\ell^1_\mu L^2_T L^2_x}. \end{aligned} \tag{2.101}$$

The terms $\|N_j\|_{L^2}, j = 3, \dots, 7$, have been already estimated in the proof of Theorem 2.2 and we saw that $\|N_j\|_{L^2} \to 0$ as $\alpha \to 0$. Then we just need to deal with the terms $\sum_{j=3}^{7} \sum_{|\lambda|=2} \|\partial_x^\lambda N_j\|_{\ell^1_\mu L^2_T L^2_x}$. To do that we use the Leibniz rule for derivatives, the Hölder inequality and the estimates in Theorem 1.4.

(i) For $j = 3$ we have

$$\begin{aligned}&\sum_{|\lambda|=2} \|\partial_x^\lambda N_3\|_{\ell^1_\mu L^2_T L^2_x} = |\mathtt{W}| \sum_{|\lambda|=2} \|\partial_x^\lambda [\psi_\alpha(U'(t)(\rho_{\alpha 0}-\rho_0) + U(t)(\rho_{\alpha 1}-\rho_1))]\|_{\ell^1_\mu L^2_T L^2_x} \\ &\lesssim |\mathtt{W}| \sum_{|\gamma|+|\beta|=2} \|\partial_x^\gamma \psi_\alpha \partial_x^\beta [U'(t)(\rho_{\alpha 0}-\rho_0) + U(t)(\rho_{\alpha 1}-\rho_1)]\|_{\ell^1_\mu L^2_T L^2_x} \\ &\lesssim |\mathtt{W}| \sum_{|\gamma|=2} \|\partial_x^\gamma \psi_\alpha\|_{\ell^2_\mu L^2_T L^\infty_x} \|U'(t)(\rho_{\alpha 0}-\rho_0) + U(t)(\rho_{\alpha 1}-\rho_1)\|_{\ell^2_\mu L^\infty_T L^2_x} \\ &\quad + |\mathtt{W}| \big(\sum_{|\gamma|=1} \|\partial_x^\gamma \psi_\alpha\|_{\ell^2_\mu L^2_T L^\infty_x} \big) \big(\sum_{|\beta|=1} \|\partial_x^\beta [U'(t)(\rho_{\alpha 0}-\rho_0) + U(t)(\rho_{\alpha 1}-\rho_1)]\|_{\ell^2_\mu L^\infty_T L^2_x} \big) \\ &\quad + |\mathtt{W}| \|\psi_\alpha\|_{\ell^2_\mu L^2_T L^\infty_x} \big(\sum_{|\beta|=2} \|\partial_x^\beta [U'(t)(\rho_{\alpha 0}-\rho_0) + U(t)(\rho_{\alpha 1}-\rho_1)]\|_{\ell^2_\mu L^\infty_T L^2_x} \big). \end{aligned} \tag{2.102}$$

Now we use that $\|\cdot\|_{\ell^2_\mu L^2_T L^\infty_x} \lesssim T^{1/2} \sup_{0\le t\le T} \|\cdot\|_{L^2_x}$, so

$$\begin{aligned}\|\partial_x^\gamma \psi_\alpha\|_{\ell^2_\mu L^2_T L^\infty_x} &\lesssim T^{1/2} \sup_{0\le t\le T} \|\partial_x^\gamma \psi_\alpha\|_{L^2_x} \\ &\lesssim T^{1/2} \sup_{0\le t\le T} \|\psi_\alpha\|_{H^2} \\ &\lesssim T^{1/2} \|\psi_{\alpha 0}\|_{H^4} \end{aligned} \tag{2.103}$$

for all $|\gamma| \leq 2$.

For the other pieces in (2.102) we have

$$\begin{aligned} \|\partial_x^\beta [U'(t)(\rho_{\alpha 0} - \rho_0) + U(t)(\rho_{\alpha 1} - \rho_1)]\|_{\ell^2_\mu L^\infty_T L^2_x} &\lesssim (1+TM)^3 \|\partial_x^\beta (\rho_{\alpha 0} - \rho_0)\|_{L^2} \\ &\quad + T(1+TM)^3 \|\partial_x^\beta (\rho_{\alpha 1} - \rho_1)\|_{L^2} \\ &\lesssim (1+TM)^3 \|\rho_{\alpha 0} - \rho_0\|_{H^2} \\ &\quad + T(1+TM)^3 \|\rho_{\alpha 1} - \rho_1\|_{H^2} \end{aligned} \tag{2.104}$$

for all $|\beta| \leq 2$.

(ii) For $j = 4$ the argument is the same because

$$N_4 = \mathtt{W}(\psi_\alpha - \psi)[(U'(t)\rho_0 + U(t)\rho_1]. \tag{2.105}$$

(iii) Let us consider $j = 5$.

In this case we have

$$\begin{aligned} N_5 &= -\mathtt{W}\alpha D\psi_\alpha \int_0^t U(t-t')\partial_t \partial_z |\psi_\alpha|^2 dt' \\ &= -\mathtt{W}\alpha D\psi_\alpha \big(- U(t)\partial_z |\psi_{\alpha 0}|^2 + \int_0^t U'(t-t')\partial_z |\psi_\alpha|^2 \big). \end{aligned} \tag{2.106}$$

We need to estimate $\sum_{|\lambda|=2} \|\partial_x^\lambda N_5\|_{\ell^1_\mu L^2_T L^2_x}$.

Again, combining the Holder inequality and the Leibniz rule we see that we should be concerned only with estimating

$$\|\partial_x^\beta U(t)\partial_z |\psi_{\alpha 0}|^2\|_{\ell^2_\mu L^\infty_T L^2_x} \tag{2.107}$$

and

$$\|\partial_x^\beta \int_0^t U'(t-t')\partial_z |\psi_\alpha|^2 dt'\|_{\ell^2_\mu L^\infty_T L^2_x} \tag{2.108}$$

for all $|\beta| \leq 2$.

So

$$\begin{aligned} \|\partial_x^\beta U(t)\partial_z |\psi_{\alpha 0}|^2\|_{\ell^2_\mu L^\infty_T L^2_x} &\lesssim T(1+TM)^3 \|\partial_x^\beta \partial_z |\psi_{\alpha 0}|^2\|_{L^2} \\ &\lesssim T(1+TM)^3 \|\psi_{\alpha 0}\|_{H^3} \end{aligned} \tag{2.109}$$

and

$$\begin{aligned}
\|\partial_x^\beta \int_0^t U'(t-t')\partial_z|\psi_\alpha|^2 dt'\|_{\ell^2_\mu L^\infty_T L^2_x} &\lesssim (1+TM)^3 \|\partial_x^\beta \partial_z |\psi_\alpha|^2\|_{\ell^2_\mu L^1_T L^2_x} \\
&\lesssim (1+TM)^3 T \|\partial_x^\beta \partial_z |\psi_\alpha|^2\|_{\ell^2_\mu L^2_T L^2_x} \\
&\lesssim (1+TM)^3 T \Big(\int_0^T \int_{\mathbb{R}^3} |\partial_x^\beta \partial_z |\psi_\alpha|^2|^2 dx dt \Big)^{1/2} \\
&\lesssim (1+TM)^3 T \Big(\int_0^T \|\psi_\alpha\|^2_{H^3} dt \Big)^{1/2} \\
&\lesssim (1+TM)^3 T^{3/2} \|\psi_{\alpha 0}\|_{H^4}. \qquad (2.110)
\end{aligned}$$

(iv) For $j = 6$ we have

$$N_6 = \mathtt{W}\psi_\alpha \int_0^t U(t-t')\Delta(|\psi_\alpha|^2 - |\psi|^2) dt'. \qquad (2.111)$$

Similarly as was done before, we only need to estimate

$$\|\partial_x^\beta \int_0^t U(t-t')\Delta(|\psi_\alpha|^2 - |\psi|^2) dt'\|_{\ell^2_\mu L^\infty_T L^2_x}, \quad |\beta| \le 2. \qquad (2.112)$$

We obtain

$$\begin{aligned}
&\|\partial_x^\beta \int_0^t U(t-t')\Delta(|\psi_\alpha|^2 - |\psi|^2) dt'\|_{\ell^2_\mu L^\infty_T L^2_x} \\
&\qquad \lesssim M(1+TM)^3 \|\partial_x^{\beta+1}(|\psi_\alpha|^2 - |\psi|^2)\|_{\ell^2_\mu L^1_T L^2_x} \\
&\qquad \lesssim M(1+TM)^3 T \|\partial_x^{\beta+1}(|\psi_\alpha|^2 - |\psi|^2)\|_{\ell^2_\mu L^2_T L^2_x} \qquad (2.113)
\end{aligned}$$

where $\beta + 1$ is a multi index such that $|\beta + 1| = |\beta| + 1$.

Then

$$\begin{aligned}
&\|\partial_x^\beta \int_0^t U(t-t')\Delta(|\psi_\alpha|^2 - |\psi|^2) dt'\|_{\ell^2_\mu L^\infty_T L^2_x} \\
&\qquad \lesssim M(1+TM)^3 T \Big(\int_0^T \|\psi_\alpha\|^2_{H^3} + \|\psi\|^2_{H^3} dt \Big)^{1/2}. \qquad (2.114)
\end{aligned}$$

(v) For $j = 7$ we have

$$N_7 = \mathtt{W}(\psi_\alpha - \psi) \int_0^t U(t-t')\Delta|\psi|^2 dt', \qquad (2.115)$$

then the estimates are similar to those above, so we omit them.

Recall that $\|\psi_\alpha\|_{H^s} \lesssim \|\psi_{\alpha 0}\|_{H^s}$ for all $s \geq n/2$, and

$$\mathtt{W}, q, \alpha \sim 1/v_g^2, D \sim v_g \quad \text{and} \quad v_g \to \infty. \tag{2.116}$$

This completes the proof. ■

We have consider the supersonic regime for the (ZR) model, i.e., when $M > 1$, where $M := \frac{|v_g|}{c_s}$ is the Mach number defined in the Section 1.1, and we have established the limit behavior of the solutions of (ZR) when $v_g \to \infty$ while c_s remains constant ($M \to \infty$). It is also interesting to think on the case $|v_g| = Mc_s$ considering simultaneously the limits $v_g, c_s \to \infty$.

According to Doppler effect negligible ($\alpha \to 0$ when $v_g > c_s$) we can assume $M > 1$.

Let us recall the initial system and the system limit:

$$\begin{cases} i\partial_t \psi + \epsilon \partial_z^2 \psi + \sigma_1 \Delta_\perp \psi = (q|\psi|^2 + \mathtt{W}(\rho + \alpha D \partial_z \varphi))\psi, \\ \partial_t^2 \rho - \dfrac{1}{M^2} \Delta \rho = \Delta |\psi|^2 - \alpha D \partial_t \partial_z |\psi|^2, \\ \partial_t^2 \varphi - \dfrac{1}{M^2} \Delta \varphi = \dfrac{\alpha D}{M^2} \partial_z |\psi|^2 - \partial_t |\psi|^2, \end{cases} \tag{ZR}$$

and

$$\begin{cases} i\partial_t \psi + \epsilon \partial_z^2 \psi + \sigma_1 \Delta_\perp \psi = \mathtt{W} \rho \psi, \\ \partial_t^2 \rho - \dfrac{1}{M^2} \Delta \rho = \Delta |\psi|^2, \end{cases} \tag{Z}$$

where $\mathtt{W} = \beta^2 \rho_{00}/v_g^2$, $D = |v_g|/\beta \rho_{00}$ and $\gamma = \beta c_s^2$.

Note that the solutions of these systems depend on v_g in consideration.

Now, we are going to consider $\beta = \sqrt{v_g}$, and $\alpha = c_s/v_g = M^{-1}$, thereupon $\mathtt{W} = \rho_{00}/v_g$ and $D = \sqrt{v_g}/\rho_{00}$, so $\mathtt{W}D = 1/\sqrt{v_g}$.

In the case $M = c_s^2$ we have $\alpha D = \dfrac{1}{\rho_{00}\sqrt{c_s}}$.

$T^* = T^*_{c_s}$ will be the minimum life-time of the solution of (ZR) and T_0 the life-time of the solution of (Z).

We have the following result

Theorem 2.4 *Assume* $q=\alpha, M=c_s^2, n=2,3$ *and*

$$\psi_{c_s0}\in H^{4+},\rho_{c_s0}\in H^{7/2+},\varphi_{c_s0}\in H^{9/2+}, \tag{2.117}$$

$$\psi_0\in H^{4+},\rho_0\in H^{7/2+},\varphi_0\in H^{9/2+}, \tag{2.118}$$

$$\sup_{c_s>0}\|\psi_{c_s0}\|_{H^{4+}}<\infty, \tag{2.119}$$

$$\|\psi_0\|_{H^{5/2}}\le\inf_{c_s>0}\|\psi_{c_s0}\|_{H^{3/2}},\ \|\rho_0\|_{H^2}\le\inf_{c_s>0}\|\rho_{c_s0}\|_{H^1},\ \|\varphi_0\|_{H^3}\le\inf_{c_s>0}\|\varphi_{c_s0}\|_{H^2}, \tag{2.120}$$

$$\lim_{c_s\to\infty}\|\psi_{c_s0}-\psi_0\|_{H^3}=\lim_{c_s\to\infty}\|\rho_{c_s0}-\rho_0\|_{H^2}=\lim_{c_s\to\infty}\|\varphi_{c_s0}-\varphi_0\|_{H^4}=0. \tag{2.121}$$

Then we have

$$i)\lim_{c_s\to\infty}T^*_{c_s}(\|\psi_{c_s0}\|_{H^{3/2}},\|\rho_{c_s0}\|_{H^1},\|\varphi_{c_s0}\|_{H^2})=T^*_\infty(\|\psi_0\|_{H^{3/2}},\|\rho_0\|_{H^1},\|\varphi_0\|_{H^2}), \tag{2.122}$$

$$ii)\lim_{c_s\to\infty}\|\rho_{c_s1}+\Delta\varphi_0\|_{H^2}=0, \tag{2.123}$$

$$iii)\lim_{c_s\to\infty}\|\tilde{\psi}_{c_s}-\psi_{c_s}\|_{L^\infty(0,T;L^2)}=\lim_{c_s\to\infty}\|\tilde{\rho}_{c_s}-\rho_{c_s}\|_{L^\infty(0,T;L^2)}=0\quad\text{if}\quad T\le T_m, \tag{2.124}$$

where $T_m=\min\{T_0,T^*_\infty\}$, $(\psi_{c_s},\rho_{c_s},\varphi_{c_s})$ *is the solution of* (ZR) *with initial data* $(\psi_{c_s0},\rho_{c_s0},\varphi_{c_s0})$ *and* $(\tilde{\psi}_{c_s},\tilde{\rho}_{c_s})$ *is the solution of* (Z) *with initial data* (ψ_0,ρ_0,ρ_1), $\rho_1:=-\Delta\varphi_0$.

Proof.

First note that if (2.117)-(2.121) are true then the hypotheses of the Theorem 2.2 are also true. Therefore (2.122) and (2.123) result of that theorem.

The term $\|\tilde{\psi}_{c_s}-\psi_{c_s}\|_{L^\infty(0,T;L^2)}$ was estimated in the proof of Theorem 2.2, in the estimate (2.57) and the nonlinear estimates in (2.70)-(2.79). Again, the assertion (2.82) is true because of (2.81) and our choice of parameters $\alpha,\beta,\mathtt{W},D$. Therefore $\|\tilde{\psi}_{c_s}-\psi_{c_s}\|_{L^\infty(0,T;L^2)}\to 0$ when $c_s\to\infty$.

The term $\|\tilde{\rho}_{c_s}-\rho_{c_s}\|_{L^\infty(0,T;L^2)}$ can be estimated as in (2.58) with nonlinearities N_{11},N_{22} as in (2.60). A convenient estimate for N_{11} was obtained in (2.85), indeed

$$\begin{aligned}\|N_{11}\|_{L^2_x}&\le|\alpha D|\big(|t|\|\psi_{c_s0}\|^2_{H^2}+\int_0^t\|\psi_{c_s}\|^2_{H^2}dt'\big)\\&=\frac{1}{\rho_{00}\sqrt{c_s}}\big(|t|\|\psi_{c_s0}\|^2_{H^2}+\int_0^t\|\psi_{c_s}\|^2_{H^2}dt'\big).\end{aligned} \tag{2.125}$$

And for N_{22} we can use the inequality (2.87). Here the terms A_{11} and A_{33} are

$$\begin{aligned} A_{11} &= \frac{1}{M^2}\| \int_0^t U(t-t')\Delta(\rho_{c_s} - \tilde{\rho}_{c_s})dt' \|_{L_x^2} \\ &\le \frac{1}{M^2}\int_0^t |t-t'|\, \|\Delta(\rho_{c_s} - \tilde{\rho}_{c_s})\|_{L_x^2} dt' \end{aligned} \tag{2.126}$$

and

$$\begin{aligned} A_{33} &= \| \int_0^t U(t-t')\partial_t^2(\rho_{c_s} - \tilde{\rho}_{c_s})dt' \|_{L_x^2} \\ &\le \big(|t|\, \|\rho_{c_s1} - \rho_1\|_{L_x^2} + \|\rho_{c_s0} - \rho_0\|_{L_x^2} \\ &\quad + \frac{1}{M^2}\| \int_0^t U(t-t')\Delta(\rho_{c_s} - \tilde{\rho}_{c_s})dt' \|_{L_x^2}\big) \\ &\le \big(|t|\, \|\rho_{c_s1} - \rho_1\|_{L_x^2} + \|\rho_{c_s0} - \rho_0\|_{L_x^2} \\ &\quad + \frac{1}{M^2}\int_0^t |t-t'| \|\Delta(\rho_{c_s} - \tilde{\rho}_{c_s})\|_{L_x^2} dt'\big). \end{aligned} \tag{2.127}$$

The term $\|\Delta(\rho_{c_s} - \tilde{\rho}_{c_s})\|_{L_x^2}$ is estimated as that in (2.90), thus

$$\begin{aligned} \|\Delta(\rho_{c_s} - \tilde{\rho}_{c_s})\|_{L_x^2} &= \|U'(t)\Delta(\rho_{c_s0} - \rho_0) + U(t)\Delta(\rho_{c_s1} - \rho_1) \\ &\quad + \int_0^t U(t-t')\Delta\big(F_1(\psi_{c_s}) - \Delta|\tilde{\psi}_{c_s}|^2\big)dt' \|_{L_x^2} \\ &\le \|\rho_{c_s0} - \rho_0\|_{H^2} + |t|\, \|\rho_{c_s1} - \rho_1\|_{H^2} \\ &\quad + \| \int_0^t U(t-t')\Delta\big(\Delta|\psi_{c_s}|^2 - \Delta|\tilde{\psi}_{c_s}|^2\big)dt' \|_{L_x^2} \\ &\quad + |\alpha D| \| \int_0^t U(t-t')\Delta\big(\partial_t\partial_z|\tilde{\psi}_{c_s}|^2\big)dt' \|_{L_x^2}. \end{aligned} \tag{2.128}$$

By using the integration by parts formula, we have

$$\begin{aligned} \|\Delta(\rho_{c_s} - \varrho_{c_s})\|_{L_x^2} &\le \|\rho_{c_s0} - \rho_0\|_{H^2} + |t|\, \|\rho_{c_s1} - \rho_1\|_{H^2} \\ &\quad + \int_0^t |t-t'|\, \|(|\psi_{c_s}| - |\tilde{\psi}_{c_s}|)(|\psi_{c_s}| + |\tilde{\psi}_{c_s}|)\|_{H^4} dt' \\ &\quad + |\alpha D||t| \|\psi_0\|_{H^3}^2 + |\alpha D| \int_0^t \|\tilde{\psi}_{c_s}\|_{H^3}^2 dt' \\ &\le \|\rho_{c_s0} - \rho_0\|_{H^2} + |t|\, \|\rho_{c_s1} - \rho_1\|_{H^2} \\ &\quad + \int_0^t |t-t'|\, \|\psi_{c_s} - \tilde{\psi}_{c_s}\|_{H^4}(\|\psi_{c_s}\|_{H^4} + \|\tilde{\psi}_{c_s}\|_{H^4})dt' \\ &\quad + |\alpha D||t| \|\psi_0\|_{H^3}^2 + |\alpha D| \int_0^t \|\tilde{\psi}_{c_s}\|_{H^3}^2 dt'. \end{aligned} \tag{2.129}$$

But

$$\|\psi_{c_s} - \tilde{\psi}_{c_s}\|_{H^4} \leq \|\psi_{c_s} - \tilde{\psi}_{c_s}\|_{L_x^2}^{\theta} \|\psi_{c_s} - \tilde{\psi}_{c_s}\|_{H^{4+}}^{1-\theta} \quad \text{for some} \quad 0 \leq \theta \leq 1 \tag{2.130}$$

because of interpolation, thereupon

$$\begin{aligned}\|\Delta(\rho_{c_s} - \tilde{\rho}_{c_s})\|_{L_x^2} \leq & \|\rho_{c_s 0} - \rho_0\|_{H^2} + |t|\, \|\rho_{c_s 1} - \rho_1\|_{H^2} \\ & + C_1^{\theta} \int_0^t |t - t'|\, \|\psi_{c_s} - \tilde{\psi}_{c_s}\|_{H^{4+}}^{1-\theta} (\|\psi_{c_s}\|_{H^4} + \|\tilde{\psi}_{c_s}\|_{H^4}) dt' \\ & + |\alpha D||t|\|\psi_0\|_{H^3}^2 + |\alpha D| \int_0^t \|\tilde{\psi}_{c_s}\|_{H^3}^2 dt'. \end{aligned} \tag{2.131}$$

Since the norms $\|\psi_{c_s}\|_{H^{4+}}$ and $\|\tilde{\psi}_{c_s}\|_{H^{4+}}$ are uniformly bounded, and $\|\tilde{\psi}_{c_s} - \psi_{c_s}\|_{L^\infty(0,T;L^2)} \to 0$, then $A_{11}, A_{33} \to 0$, so $\|N_{ii}\|_{L_x^2} \to 0$ for $i = 1, 2, 3$, thereupon $\|\tilde{\rho}_{c_s} - \rho_{c_s}\|_{L^\infty(0,T;L^2)} \to 0$. This completes the proof. ■

Chapter 3

A Modified Zakharov-Rubenchik System

According to Zakharov and Kuznetsov [26], depending on the relations between the group velocity v_g and the sound velocity c_s, the system (ZR_0) permits various simplifications. In the case $v_g < c_s$ and $q|\psi|^2 \ll v_g \Delta k$, where Δk is the width around k of the HF packet, they proposed the system

$$\begin{cases} i(\partial_t \psi + v_g \partial_z \psi) + \dfrac{w''}{2}\partial_z^2 \psi + \dfrac{v_g}{2k_0}\Delta_\perp \psi = (q|\psi|^2 + \beta\rho + \alpha\partial_z \varphi)\psi, \\ -v_g \partial_z \rho + \rho_{00}\Delta\varphi + \alpha\partial_z |\psi|^2 = 0, \\ -v_g \partial_z \varphi + \dfrac{c_s^2}{\rho_{00}}\rho + \beta|\psi|^2 = 0. \end{cases} \tag{ZRK_0}$$

Using a reference frame moving at the group velocity of the carrying waves (the change $z = \tilde{z} + v_g t$) and rescaling the variables as was done for (ZR) in [39] to obtain the non-dimensional form, we have

$$\begin{cases} i\partial_t \psi + \epsilon\partial_z^2 \psi + \sigma_1 \Delta_\perp \psi = (q|\psi|^2 + \mathtt{W}(\rho + \alpha D\partial_z \varphi))\psi, \\ -\sigma_2 \partial_z \rho = \Delta\varphi + \alpha D\partial_z |\psi|^2, \\ -\sigma_2 \partial_z \varphi = \dfrac{1}{M^2}\rho + |\psi|^2, \end{cases} \tag{ZRK_1}$$

where

$$\sigma_1 = -sgn(k_0 v_g), \sigma_2 = -sgn(v_g), \mathtt{W} = \beta^2 \rho_{00}/v_g^2,\ D = |v_g|/\beta\rho_{00}, \text{ and } M = |v_g|/c_s. \tag{3.1}$$

We are going to rewrite the system (ZRK_1) in a convenient form by decoupling the last two equations. We apply the operator ∂_z to them. One gets

$$\begin{cases} i\partial_t\psi + \epsilon\partial_z^2\psi + \sigma_1\Delta_\perp\psi = \Big(q + \dfrac{\alpha\beta}{v_g}\Big)|\psi|^2\psi + \Big(\dfrac{\beta^2\rho_{00}}{v_g^2} + \dfrac{\alpha\gamma}{v_g^3}\Big)\rho\psi \\ \Delta(\rho + M^2|\psi|^2) = M^2\partial_z^2\rho + \dfrac{\alpha v_g^3}{\rho_{00}\gamma}\partial_z^2|\psi|^2, \end{cases} \tag{ZRK}$$

where $\gamma = \beta c_s^2 = \beta v_g^2/M^2 > 0$ is constant.

Note that the nonlinearity in the first equation was rewrited by using that

$$\partial_z\varphi = \frac{1}{-\sigma_2}\big(\frac{1}{M^2}\rho + |\psi|^2\big).$$

We are interested in the behavior of the solutions of the modified Zakharov-Rubenchik system (ZRK) when $M, \beta \to 0$.

We expect the solution of initial value problem associated to converge toward the solution of the Davey-Stewartson system

$$\begin{cases} i\partial_t\psi + \epsilon\partial_z^2\psi + \sigma_1\Delta_\perp\psi = q|\psi|^2\psi + \dfrac{\alpha\gamma}{v_g^3}\rho\psi \\ \Delta\rho = \dfrac{\alpha v_g^3}{\rho_{00}\gamma}\partial_z^2|\psi|^2. \end{cases} \tag{DS}$$

3.1 On the Initial Value Problem for the Modified ZR System

We assume $\mathcal{L} = \Delta$. In this case we have

$$\begin{cases} i\partial_t\psi + \Delta\psi = \Big(q + \dfrac{\alpha\beta}{v_g}\Big)|\psi|^2\psi + \Big(\dfrac{\beta^2\rho_{00}}{v_g^2} + \dfrac{\alpha\gamma}{v_g^3}\Big)\rho\psi \\ \Delta(\rho + M^2|\psi|^2) = M^2\partial_z^2\rho + \dfrac{\alpha v_g^3}{\rho_{00}\gamma}\partial_z^2|\psi|^2, \end{cases} \tag{ZRK}$$

where $\beta c_s^2 = \gamma$ is assumed constant and $\alpha \sim k_0$. Let us consider $n = 2$.

The system above can be writed as

$$\begin{cases} i\partial_t\psi + \Delta\psi = c_1|\psi|^2\psi + c_2\rho\psi, \\ \partial_x^2\rho + c_3\partial_z^2\rho = c_4\partial_z^2|\psi|^2 - M^2\partial_x^2|\psi|^2, \end{cases}$$

where $c_1 = q + \dfrac{\alpha\beta}{v_g}$, $c_2 = \dfrac{\beta^2\rho_{00}}{v_g^2} + \dfrac{\alpha\gamma}{v_g^3}$, $c_3 = 1 - M^2$ and $c_4 = \dfrac{\alpha v_g^3}{\rho_{00}\gamma} - M^2$.

Then it can be regarded as a Zakharov-Schulman system

$$\begin{cases} i\partial_t\psi + \mathcal{L}\psi = \psi\phi \\ \mathcal{L}_1\phi = \mathcal{L}_2(|\psi|^2), \end{cases} \tag{ZS}$$

where $\phi = c_1|\psi|^2 + c_2\rho$, $\mathcal{L} = \Delta$, $\mathcal{L}_1 = \partial_x^2 + c_3\partial_z^2$ and $\mathcal{L}_2 = (c_1 - c_2M^2)\partial_x^2 + (c_1c_3 + c_2c_4)\partial_z^2$.

Note that $\mathcal{L}_1$ is elliptic in the subsonic case $M < 1$.

By setting $\phi = \mathcal{L}_1^{-1}\mathcal{L}_2(|\psi|^2) := \mathsf{T}(|\psi|^2)$ the IVP for system (ZS) becomes

$$i\partial_t\psi + \mathcal{L}\psi = \psi\mathsf{T}(|\psi|^2), \quad \psi(x,0) = \psi_0(x) \tag{3.2}$$

which has a structure similar to the cubic nonlinear Schrödinger equation but with a nonlocal nonlinearity.

The presence of the nonlocal term in the nonlinearity prevents the use of some of the techniques applied to the Schrödinger equation. Note, however, that T is a zeroth-order operator and therefore some tools developed by Kato [20] in the context of the semilinear Schrödinger equation can be applied. In fact, following these ideas, Ghidaglia and Saut [17] obtained a local well-posedness result in $H^1(\mathbb{R}^2)$ for the Davey-Stewartson system.

Recently, by using the contraction mapping principle in an appropriate space, Oliveira, Panthee and Silva [35] proved that

Theorem 3.1 *Let $n = 2, 3$ and $s \geq n/4$. Then for all $\psi_0 \in H^s(\mathbb{R}^n)$, there exists a unique solution*

$$\psi \in C([0,T]; H^s(\mathbb{R}^n))$$

to (3.2) where the life-span $T > 0$ depends exclusively on $\|\psi_0\|_{H^s}$, $T = T(\|\psi_0\|_{H^s})$, and is a continuous decreasing function.

In order to write the ZRK system as a single equation for ψ, we begin by expressing ρ in terms of ψ by solving the second Poisson-like equation. Thus

$$\widehat{\rho}(\xi) = \frac{M^2|\xi|^2 - \dfrac{\alpha v_g^3}{\rho_{00}\gamma}\xi_2^2}{M^2\xi_2^2 - |\xi|^2}\widehat{|\psi|^2}, \tag{3.3}$$

where $\Delta = \partial_x^2 + \partial_z^2$ and $\xi = (\xi_1, \xi_2)$.

Then we have

$$\rho = \mathsf{T}_{Q_M}(|\psi|^2) \qquad \text{(Nonlocal term)}, \tag{3.4}$$

where T_{Q_M} is the singular integral operator defined by

$$\widehat{\mathsf{T}_{Q_M} f} = Q_M \widehat{f}, \qquad Q_M(\xi) = \frac{M^2|\xi|^2 - \dfrac{\alpha v_g^3}{\rho_{00}\gamma}\xi_2^2}{M^2\xi_2^2 - |\xi|^2}. \tag{3.5}$$

Note that if $|\xi| = 1$ then $M^2\xi_2^2 - |\xi|^2 = M^2\xi_2^2 - 1 \neq 0$ for M small enough, so Q_M is of class C^2 on the unit sphere for $M \to 0$. Since Q_M is homogeneous of degree 0, the Hörmander-Mikhlin theorem implies that Q_M is a multiplier for $L^p(\mathbb{R}^2)$, that is, T_{Q_M} is a bounded operator in $L^p(\mathbb{R}^2)$, $1 < p < \infty$.

Therefore there exists a constant $C = C_p > 0$ such that

$$\|\rho\|_{L^p(\mathbb{R}^2)} \leq C\|\psi\|^2_{L^{2p}(\mathbb{R}^2)}, \qquad 1 < p < \infty. \tag{3.6}$$

We have therefore reduced (ZRK) to a nonlinear Schrödinger equation and its IVP associated is

$$\begin{cases} i\partial_t\psi + \Delta\psi = H(\psi), \\ \psi(x, z, 0) = \psi_0(x, z), \end{cases} \tag{NS$_\beta$}$$

with

$$H(\psi) = \Big(q + \frac{\alpha\beta}{v_g}\Big)|\psi|^2\psi + \Big(\frac{\beta^2\rho_{00}}{v_g^2} + \frac{\alpha\gamma}{v_g^3}\Big)\psi\mathsf{T}_{Q_M}(|\psi|^2) \tag{3.7}$$

and integral equation version

$$\psi(t) = e^{it\Delta}\psi_0 - i\int_0^t e^{i(t-t')\Delta}H(\psi)(t')dt'. \tag{3.8}$$

Returning to the system (ZRK) with $M = \beta = 0$ we have

$$\begin{cases} i\partial_t\psi + \Delta\psi = q|\psi|^2\psi + \dfrac{\alpha\gamma}{v_g^3}\rho\psi, \\ \Delta\rho = \dfrac{\alpha v_g^3}{\rho_{00}\gamma}\partial_z^2|\psi|^2. \end{cases} \tag{DS}$$

In this case we obtain

$$\rho = \mathsf{T}_{Q_0}(|\psi|^2) \qquad \text{(Nonlocal term)}, \tag{3.9}$$

with

$$\widehat{\mathsf{T}_{Q_0} f} = Q_0 \widehat{f}, \qquad Q_0(\xi) = \frac{\alpha v_g^3 \xi_2^2}{\rho_{00}\gamma|\xi|^2}. \tag{3.10}$$

Also Q_0 is homogeneous of degree 0 and C^2 on the unit sphere, thereupon T_{Q_0} is a bounded operator in $L^p(\mathbb{R}^2)$, $1 < p < \infty$.

Then we transform the system (DS) to IVP

$$\begin{cases} i\partial_t \psi + \Delta\psi = I(\psi), \\ \psi(x,z,0) = \psi_0(x,z), \end{cases} \tag{NS$_{DS}$}$$

with

$$I(\psi) = q|\psi|^2\psi + \frac{\alpha\gamma}{v_g^3}\psi \mathsf{T}_{Q_0}(|\psi|^2) \tag{3.11}$$

and integral equation version

$$\psi(t) = e^{it\Delta}\psi_0 - i\int_0^t e^{i(t-t')\Delta} I(\psi)(t')dt'. \tag{3.12}$$

This integral formulation for the solution of DS system was the one presented by Ghidaglia-Saut in [17].

Proposition 3.1 *Let us consider the multiplier Q_M. Then*

1. *Q_M is uniformly bounded for any $0 < M^2 \leq 1/2$ and $\xi \neq 0$, that is,*

$$\exists\, C > 0 : \|Q_M\|_{L^\infty} \leq C, \quad C \quad \textit{is independent of} \quad M,$$

2. $\lim_{M\to 0} \|\mathsf{T}_{(Q_M - Q_0)}(f)\|_{L^2} = 0 \quad$ *if* $\quad f \in L^2$.

Proof. Note that

$$|Q_M(\xi) - Q_0(\xi)| = \frac{|M^2(C\xi_2^4 - |\xi|^4)|}{|\xi|^2(|\xi|^2 - M^2\xi_2^2)}, \quad \text{with} \quad C = \frac{\alpha v_g^3}{\rho_{00}\gamma}$$

for M small enough.
Then

$$|Q_M(\xi) - Q_0(\xi)| \leq \frac{|M^2(C\xi_2^4 - |\xi|^4)|}{(1-M^2)|\xi|^2|\xi|^2}$$

and therefore

$$|Q_M(\xi) - Q_0(\xi)| \lesssim \frac{M^2}{1-M^2} \leq 1 \quad \text{if} \quad M^2 \leq 1/2.$$

This implies (1). By using (1) and the dominated convergence theorem we obtain (2). ∎

3.2 Conserved Quantities

We are going to consider the ZRK system

$$\begin{cases} i\partial_t\psi + \Delta\psi = c_1|\psi|^2\psi + c_2\rho\psi, \\ \partial_x^2\rho + c_3\partial_z^2\rho = c_4\partial_z^2|\psi|^2 - M^2\partial_x^2|\psi|^2, \end{cases} \tag{ZRK}$$

where $c_1 = q + \dfrac{\alpha\beta}{v_g}$, $c_2 = \dfrac{\beta^2\rho_{00}}{v_g^2} + \dfrac{\alpha\gamma}{v_g^3}$, $c_3 = 1 - M^2$ and $c_4 = \dfrac{\alpha v_g^3}{\rho_{00}\gamma} - M^2$.

We assume that the solution (ψ, ρ) is such that these functions are smooth and behave properly at infinity.

We multiply the first equation in (ZRK) by $2\bar{\psi}$:

$$2i\bar{\psi}\partial_t\psi + 2\bar{\psi}\Delta\psi = 2c_1|\psi|^4 + 2c_2|\psi|^2\rho. \tag{3.13}$$

Taking the complex conjugate in the equation above we have

$$-2i\psi\partial_t\bar{\psi} + 2\psi\Delta\bar{\psi} = 2c_1|\psi|^4 + 2c_2|\psi|^2\rho. \tag{3.14}$$

Recall that ρ is a real function.

Then the imaginary part gives us

$$(\psi_t\bar{\psi} + \bar{\psi}_t\psi) - i(\psi_{xx}\bar{\psi} - \bar{\psi}_{xx}\psi) - i(\psi_{zz}\bar{\psi} - \bar{\psi}_{zz}\psi) = 0, \tag{3.15}$$

that is

$$(|\psi|^2)_t + 2\text{Im}(\psi_x\bar{\psi})_x + 2\text{Im}(\psi_z\bar{\psi})_z = 0. \tag{3.16}$$

Next, multiplying the first equation in (ZRK) by $2\bar{\psi}_t$, we obtain

$$2i|\psi_t|^2 + 2\bar{\psi}_t\Delta\psi = 2c_1|\psi|^2\psi\bar{\psi}_t + 2c_2\psi\bar{\psi}_t\rho \tag{3.17}$$

whose real part is

$$2(\psi_{xx}\bar{\psi}_t + \bar{\psi}_{xx}\psi_t) + 2(\psi_{zz}\bar{\psi}_t + \bar{\psi}_{zz}\psi_t) = 2c_1|\psi|^2(|\psi|^2)_t + 2c_2\rho(|\psi|^2)_t. \tag{3.18}$$

Adding and subtracting the terms $\psi_x\bar{\psi}_{tx}, \bar{\psi}_x\psi_{tx}, \psi_z\bar{\psi}_{tz}$, and $\bar{\psi}_z\psi_{tz}$ we have

$$\begin{aligned} c_1|\psi|^2(|\psi|^2)_t + c_2\rho(|\psi|^2)_t = {} & [(\psi_x\bar{\psi}_t)_x + (\bar{\psi}_x\psi_t)_x] + [(\psi_z\bar{\psi}_t)_z + (\bar{\psi}_z\psi_t)_z] \\ & - (\psi_x\bar{\psi}_{tx} + \bar{\psi}_x\psi_{tx}) - (\psi_z\bar{\psi}_{tz} + \bar{\psi}_z\psi_{tz}), \end{aligned} \tag{3.19}$$

that is

$$(|\psi_x|^2)_t + (|\psi_z|^2)_t + \frac{c_1}{2}(|\psi|^4)_t + c_2\rho(|\psi|^2)_t = 2\mathrm{Re}[(\psi_x\bar{\psi}_t)_x + (\psi_z\bar{\psi}_t)_z]. \tag{3.20}$$

Let

$$S(\psi) = \int_0^t [\rho(|\psi|^2)_{t'}](t')dt'. \tag{3.21}$$

Thereupon, the quantities

$$\mathcal{M}(\psi) = \int_{\mathbb{R}^2} |\psi|^2 dxdz, \tag{3.22}$$

and

$$\mathcal{E}(\psi) = \int_{\mathbb{R}^2} \big(|\psi_x|^2 + |\psi_z|^2 + \frac{c_1}{2}|\psi|^4 + c_2S(\psi)\big)dxdz, \tag{3.23}$$

are constants of motion because of (3.16) and (3.20).

Chapter 4

Subsonic Regime Results

4.1 Strong Convergence Results

We will deal with the integral form of systems

$$\begin{cases} i\partial_t\psi + \Delta\psi = \Big(q + \dfrac{\alpha\beta}{v_g}\Big)|\psi|^2\psi + \Big(\dfrac{\beta^2\rho_{00}}{v_g^2} + \dfrac{\alpha\gamma}{v_g^3}\Big)\rho\psi \\ \Delta(\rho + M^2|\psi|^2) = M^2\partial_z^2\rho + \dfrac{\alpha v_g^3}{\rho_{00}\gamma}\partial_z^2|\psi|^2, \end{cases} \tag{ZRK}$$

and

$$\begin{cases} i\partial_t\psi + \Delta\psi = q|\psi|^2\psi + \dfrac{\alpha\gamma}{v_g^3}\rho\psi \\ \Delta\rho = \dfrac{\alpha v_g^3}{\rho_{00}\gamma}\partial_z^2|\psi|^2. \end{cases} \tag{DS}$$

We denote by (ψ_β, ρ_β) the solution of the IVP associated to (ZRK) with initial data $\psi_{\beta 0}$. Here we have $\alpha = k_0$ and $M = |v_g|/c_s \ll 1$, so we are going to consider $c_s \to +\infty$ and $\beta = \gamma c_s^{-2}$ with $\gamma > 0$ constant, thereupon $M = |v_g|\sqrt{\gamma^{-1}\beta}$.

Let us denote $T^* = T^*_{\alpha,q,\beta,M} = T^*_{\alpha,\beta}$ the life-time of the solution of (ZRK) such that it does not vanish when $c_s \to +\infty$. Recall that the solution of (DS) is global for small data in L^2.

So we have the following result

Theorem 4.1 *Assume that* $n = 2$,

$$\psi_{\beta 0}, \psi_0 \in H^2, \tag{4.1}$$

$$\sup_{\beta>0} \|\psi_{\beta 0}\|_{H^2} \ll 1 \tag{4.2}$$

$$\|\psi_0\|_{H^1} \leq \inf_{\beta>0} \|\psi_{\beta 0}\|_{H^1} \tag{4.3}$$

and

$$\lim_{\beta\to 0} \|\psi_{\beta 0} - \psi_0\|_{H^1} = 0. \tag{4.4}$$

Then we have

$$i) \lim_{\beta\to 0} T^*_{\alpha,\beta}(\|\psi_{\beta 0}\|_{H^1}) = T^*_{\alpha,0}(\|\psi_0\|_{H^1}), \tag{4.5}$$

$$ii) \lim_{\beta\to 0} \|\psi_\beta - \psi\|_{L^\infty(0,T;L^2)} = \lim_{\beta\to 0} \|\rho_\beta - \rho\|_{L^\infty(0,T;L^2)} = 0 \quad \text{if} \quad T \leq T^*_{\alpha,0}, \tag{4.6}$$

where (ψ, ρ) *is the solution of* (DS) *with initial data* ψ_0.

Proof.

The equality (4.5) is consequences of the hypotheses (4.4). The hypotheses (4.3) ensures that $T^*_{\alpha,\beta}$ is increasing when $\beta \to 0$.

Hypothesis (4.2) and (4.4) ensure that the solution of the system (DS) is global.

Following the ideas of Najman [33] and Machihara [29], we will use the integral versions of (ZR) and (DS) given in Section 3.1 to prove (4.6).

We have

$$\psi_\beta(t) = e^{it\Delta}\psi_{\beta 0} - i\int_0^t e^{i(t-t')\Delta} H(\psi_\beta)(t')dt', \tag{4.7}$$

$$\rho_\beta = \mathsf{T}_{Q_M}(|\psi_\beta|^2), \tag{4.8}$$

with

$$H(\psi_\beta) = \Big(q + \frac{\alpha\beta}{2v_g}\Big)|\psi_\beta|^2\psi_\beta + \Big(\frac{\beta^2\rho_{00}}{v_g^2} + \frac{\alpha\gamma}{2v_g^3}\Big)\psi_\beta \mathsf{T}_{Q_M}(|\psi_\beta|^2), \tag{4.9}$$

and

$$\psi(t) = e^{it\Delta}\psi_0 - i\int_0^t e^{i(t-t')\Delta} I(\psi)(t')dt', \tag{4.10}$$

$$\rho = \mathsf{T}_{Q_0}(|\psi|^2), \tag{4.11}$$

with

$$I(\psi) = q|\psi|^2\psi + \frac{\alpha\gamma}{2v_g^3}\psi \mathsf{T}_{Q_0}(|\psi|^2), \tag{4.12}$$

for $0 < t < T \leq T^*_{\alpha 0}$, $\gamma = 1$. T_{Q_M}, and T_{Q_0} are the multipliers given by (3.5) and (3.10).

Then

$$(\psi_\beta - \psi)(t) = e^{it\Delta}(\psi_{\beta 0} - \psi_0) - i\int_0^t e^{i(t-t')\Delta}(H(\psi_\beta) - I(\psi))dt' \tag{4.13}$$

and

$$\rho_\beta - \rho = \mathsf{T}_{Q_M}(|\psi_\beta|^2) - \mathsf{T}_{Q_0}(|\psi|^2). \tag{4.14}$$

Let $0 < T < T^*_{\alpha 0}$.

Theorem 1.6 and Corollary 1.1 imply that

$$\|\psi_\beta - \psi\|_{L^{q_1}(0,T;L^{p_1})} \lesssim \|\psi_{\beta 0} - \psi_0\|_{L^2_x} + \|H(\psi_\beta) - I(\psi)\|_{L^{q_0'}(0,T;L^{p_0'})} \tag{4.15}$$

where

$$\begin{cases} 2 \leq p_0 < \infty, & \dfrac{2}{q_0} = 1 - \dfrac{2}{p_0} \\ 2 \leq p_1 < \infty, & \dfrac{2}{q_1} = 1 - \dfrac{2}{p_1}. \end{cases} \tag{A2D}$$

We choose $p_1 = p_0 = 2$, then $q_1 = q_0 = \infty$, $p_0' = 2$, $q_0' = 1$.

We have

$$\begin{aligned} H(\psi_\beta) - I(\psi) &= q(|\psi_\beta|^2\psi_\beta - |\psi|^2\psi) + \frac{\alpha}{2v_g^3}(\rho_\beta\psi_\beta - \rho\psi) \\ &\quad + \frac{\alpha\beta}{2v_g}|\psi_\beta|^2\psi_\beta + \frac{\beta^2\rho_{00}}{v_g^2}\rho_\beta\psi_\beta \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{4.16}$$

The terms I_3 and I_4 can be limited as it was done with the term N_1 and N_2 in (2.70), (2.71). Indeed,

$$\|I_3\|_{L^2_x} \leq \frac{|\alpha|\beta}{2|v_g|}\|\psi_\beta\|^3_{L^6_x} \lesssim \beta\|\psi_\beta\|^3_{H^{1+}} \lesssim \beta\|\psi_{\beta 0}\|^3_{H^{1+}} \tag{4.17}$$

and

$$\begin{aligned} \|I_4\|_{L^2_x} &\leq \frac{\beta^2|\rho_{00}|}{v_g^2}\|\rho_\beta\|_{L^2_x}\|\psi_\beta\|_{L^\infty_x} \\ &\lesssim \beta^2\|\psi_\beta\|_{L^4_x}\|\psi_\beta\|_{L^\infty_x} \\ &\lesssim \beta^2\|\psi_\beta\|_{H^1}\|\psi_\beta\|_{H^{1+}} \\ &\lesssim \beta^2\|\psi_{\beta 0}\|^2_{H^{1+}}. \end{aligned} \tag{4.18}$$

Therefore

$$\|I_3\|_{L^1(0,T;L^2)} \lesssim \beta T \quad \text{and} \quad \|I_4\|_{L^1(0,T;L^2)} \lesssim \beta^2 T \tag{4.19}$$

because of (2.81) and the hypothesis (4.40).

Let us estimate I_1 and I_2.

$$\begin{aligned} \|I_2\|_{L^2_x} &= \frac{|\alpha|}{2|v_g^3|}\|\rho_\beta(\psi_\beta - \psi) + \psi(\rho_\beta - \rho)\|_{L^2_x} \\ &\lesssim \|\psi_\beta - \psi\|_{L^2_x}\|\rho_\beta\|_{L^\infty_x} + \|\psi\|_{L^\infty_x}\|\rho_\beta - \rho\|_{L^2_x} \\ &\lesssim \|\psi_\beta - \psi\|_{L^2_x}\|\rho_\beta\|_{H^{1+}} + \|\psi\|_{H^{1+}}\|\rho_\beta - \rho\|_{L^2_x} \\ &=: B_1 + B_2. \end{aligned} \tag{4.20}$$

But,

$$\begin{aligned} \widehat{\rho_\beta - \rho} &= Q_M\widehat{|\psi_\beta|^2} - Q_0\widehat{|\psi|^2} \\ &= Q_M(\widehat{|\psi_\beta|^2} - \widehat{|\psi|^2}) + (Q_M - Q_0)\widehat{|\psi|^2}, \end{aligned} \tag{4.21}$$

hence

$$\begin{aligned} \|\rho_\beta - \rho\|_{L^2_x} &\le \|\mathsf{T}_{Q_M}(|\psi_\beta|^2 - |\psi|^2)\|_{L^2_x} + \|\mathsf{T}_{(Q_M - Q_0)}(|\psi|^2)\|_{L^2_x} \\ &\le C\||\psi_\beta|^2 - |\psi|^2\|_{L^2_x} + \|\mathsf{T}_{(Q_M - Q_0)}(|\psi|^2)\|_{L^2_x} \\ &\lesssim \|(|\psi_\beta| - |\psi|)(|\psi_\beta| + |\psi|)\|_{L^2_x} + \|\mathsf{T}_{(Q_M - Q_0)}(|\psi|^2)\|_{L^2_x} \\ &\lesssim \|\psi_\beta - \psi\|_{L^2_x}(\|\psi_\beta\|_{L^\infty_x} + \|\psi\|_{L^\infty_x}) + \|\mathsf{T}_{(Q_M - Q_0)}(|\psi|^2)\|_{L^2_x} \\ &\lesssim \|\psi_\beta - \psi\|_{L^2_x}(\|\psi_\beta\|_{H^{1+}} + \|\psi\|_{H^{1+}}) + \|\mathsf{T}_{(Q_M - Q_0)}(|\psi|^2)\|_{L^2_x}. \end{aligned} \tag{4.22}$$

Then

$$\begin{aligned} \|B_2\|_{L^1(0,T)} &\le \|\psi\|_{L^\infty(0,T;H^{1+})}\|\psi_\beta - \psi\|_{L^\infty(0,T;L^2)}\|(\|\psi_\beta\|_{H^{1+}} + \|\psi\|_{H^{1+}})\|_{L^1(0,T)} \\ &\quad + \|\psi\|_{L^\infty(0,T;H^{1+})}\|\mathsf{T}_{(Q_M - Q_0)}(|\psi|^2)\|_{L^1(0,T;L^2)} \\ &\lesssim T\|\psi_\beta - \psi\|_{L^\infty(0,T;L^2)} + \|\mathsf{T}_{(Q_M - Q_0)}(|\psi|^2)\|_{L^1(0,T;L^2)}. \end{aligned} \tag{4.23}$$

In the other hand, since Q_M is a multiplier then

$$\begin{aligned} \|\rho_\beta\|_{H^{1+}} &= \Big[\int_{\mathbb{R}^2}(1 + |\xi|^2)^{1+}|Q_M(\xi)\widehat{|\psi_\beta|^2}(\xi)|^2 d\xi\Big]^{1/2} \\ &\lesssim \||\psi_\beta|^2\|_{H^{1+}} \\ &\lesssim \|\psi_{\beta 0}\|^2_{H^{1+}} \le C. \end{aligned} \tag{4.24}$$

bounded by the hypotheses. Thereupon

$$\|B_1\|_{L^1(0,T)} \le \|\psi_\beta - \psi\|_{L^\infty(0,T;L^2)}\|\rho_\beta\|_{L^1(0,T;H^{1+})} \lesssim T\|\psi_\beta - \psi\|_{L^\infty(0,T;L^2)}, \tag{4.25}$$

hence

$$\begin{aligned}\|I_2\|_{L^1(0,T:L^2)} &\leq \|B_1\|_{L^1(0,T)} + \|B_2\|_{L^1(0,T)} \\ &\lesssim T\|\psi_\beta - \psi\|_{L^\infty(0,T;L^2)} + \|\mathsf{T}_{(Q_M-Q_0)}(|\psi|^2)\|_{L^1(0,T;L^2)}. \end{aligned} \tag{4.26}$$

Next, we estimate I_1. We have

$$\begin{aligned}\|I_1\|_{L^2_x} &= |q|\, \|\, |\psi_\beta|^2(\psi_\beta - \psi) + \psi(|\psi_\beta|^2 - |\psi|^2)\|_{L^2_x} \\ &\lesssim \|\, |\psi_\beta|^2(\psi_\beta - \psi)\|_{L^2_x} + \|\psi(|\psi_\beta| - |\psi|)(|\psi_\beta| + |\psi|)\|_{L^2_x} \\ &\lesssim \|\, |\psi_\beta|^2\|_{L^\infty_x}\|\psi_\beta - \psi\|_{L^2_x} + \|\psi_\beta - \psi\|_{L^2_x}\|\, \psi(|\psi_\beta| + |\psi|)\|_{L^\infty} \\ &\lesssim \|\psi_\beta\|^2_{L^\infty_x}\|\psi_\beta - \psi\|_{L^2_x} + \|\psi_\beta - \psi\|_{L^2_x}\|\, |\psi|^2 + |\psi_\beta|^2\|_{L^\infty} \\ &\lesssim (\|\psi_\beta\|^2_{L^\infty_x} + \|\psi\|^2_{L^\infty_x})\|\psi_\beta - \psi\|_{L^2_x} \\ &\lesssim (\|\psi_\beta\|^2_{H^{1+}} + \|\psi\|^2_{H^{1+}})\|\psi_\beta - \psi\|_{L^2_x}, \end{aligned} \tag{4.27}$$

then

$$\begin{aligned}\|I_1\|_{L^1(0,T:L^2)} &\lesssim T\big(\|\psi_\beta\|^2_{L^\infty(0,T;H^{1+})} + \|\psi\|^2_{L^\infty(0,T;H^{1+})}\big)\|\psi_\beta - \psi\|_{L^\infty(0,T;L^2)} \\ &\lesssim T\|\psi_\beta - \psi\|_{L^\infty(0,T;L^2)}. \end{aligned} \tag{4.28}$$

Collecting the information (4.19), (4.26) and (4.28) we obtain that

$$\begin{aligned}\|H(\psi_\beta) - I(\psi)\|_{L^1(0,T;L^2)} \lesssim & T\|\psi_\beta - \psi\|_{L^\infty(0,T;L^2)} + \beta T + \beta^2 T \\ & + \|\mathsf{T}_{(Q_M-Q_0)}(|\psi|^2)\|_{L^1(0,T;L^2)}. \end{aligned} \tag{4.29}$$

Returning to (4.15) we have

$$\begin{aligned}\|\psi_\beta - \psi\|_{L^\infty(0,T;L^2)} &\lesssim \|\psi_{\beta 0} - \psi_0\|_{L^2_x} + \|H(\psi_\beta) - I(\psi)\|_{L^1(0,T;L^2)} \\ &\lesssim \|\psi_{\beta 0} - \psi_0\|_{L^2_x} + T\|\psi_\beta - \psi\|_{L^\infty(0,T;L^2)} \\ &\quad + \|\mathsf{T}_{(Q_M-Q_0)}(|\psi|^2)\|_{L^1(0,T;L^2)} + (\beta + \beta^2)T. \end{aligned} \tag{4.30}$$

If we take $T = T_0$ sufficiently small, then

$$\begin{aligned}\|\psi_\beta - \psi\|_{L^\infty(0,T_0;L^2)} \lesssim & \|\psi_{\beta 0} - \psi_0\|_{L^2_x} + \|\mathsf{T}_{(Q_M-Q_0)}(|\psi|^2)\|_{L^1(0,T_0;L^2)} \\ & + (\beta + \beta^2)T_0. \end{aligned} \tag{4.31}$$

As $\psi_0 \in H^2$ then $\psi \in L^4$, thereupon $\lim\limits_{\beta\to 0}\|\mathsf{T}_{(Q_M-Q_0)}(|\psi|^2)\|_{L^1(0,T_0;L^2)} = 0$ due to Proposition 3.1.

Therefore

$$\|\psi_\beta - \psi\|_{L^\infty(0,T_0;L^2)} \to 0 \quad \text{as} \quad \beta \to 0. \tag{4.32}$$

Previous estimates can be repeated on the time interval $[T_0, 2T_0]$. So we have

$$\|\psi_\beta - \psi\|_{L^\infty(T_0,2T_0;L^2)} \to 0 \quad \text{as} \quad \beta \to 0. \tag{4.33}$$

Repeating this procedure, we obtain eventually

$$\|\psi_\beta - \psi\|_{L^\infty(0,T;L^2)} \to 0 \quad \text{as} \quad \beta \to 0 \quad \text{for any} \quad 0 < T \leq T^*_{\alpha 0}. \tag{4.34}$$

In (4.22) we proved that

$$\begin{aligned}\|\rho_\beta - \rho\|_{L^2_x} &\lesssim \|\psi_\beta - \psi\|_{L^2_x}(\|\psi_\beta\|_{H^{1+}} + \|\psi\|_{H^{1+}}) + \|\mathsf{T}_{(Q_M - Q_0)}(|\psi|^2)\|_{L^2_x} \\ &\lesssim \|\psi_\beta - \psi\|_{L^2_x} + \|\mathsf{T}_{(Q_M - Q_0)}(|\psi|^2)\|_{L^2_x},\end{aligned} \tag{4.35}$$

then by using (4.34) and again the Proposition 3.1, we have

$$\|\rho_\beta - \rho\|_{L^\infty(0,T;L^2)} \to 0 \quad \text{as} \quad \beta \to 0 \quad \text{for any} \quad 0 < T \leq T^*_{\alpha 0}. \tag{4.36}$$

This completes the proof. ■

We have consider the model (ZRK) when $M < 1$ (subsonic regime) where $M := \frac{|v_g|}{c_s}$ is the Mach number defined in the Section 1.1 and we have established the limit behavior of the solutions of (ZRK) when $c_s \to \infty$ while v_g remains constant ($M \to 0$). It is also interesting to think on the case $|v_g| = Mc_s$ considering simultaneously the limits $v_g, c_s \to \infty$.

Let us recall the initial system and the systems limits:

$$\begin{cases} i\partial_t\psi + \epsilon\partial_z^2\psi + \sigma_1\Delta_\perp\psi = \Big(q + \dfrac{\alpha\beta}{v_g}\Big)|\psi|^2\psi + \Big(\dfrac{\beta^2\rho_{00}}{v_g^2} + \dfrac{\alpha\gamma}{v_g^3}\Big)\rho\psi \\ \Delta(\rho + M^2|\psi|^2) = M^2\partial_z^2\rho + \dfrac{\alpha v_g^3}{\rho_{00}\gamma}\partial_z^2|\psi|^2, \end{cases} \tag{ZRK}$$

and

$$\begin{cases} i\partial_t\psi + \epsilon\partial_z^2\psi + \sigma_1\Delta_\perp\psi = q|\psi|^2\psi + \dfrac{\alpha\gamma}{v_g^3}\rho\psi \\ \Delta\rho = \dfrac{\alpha v_g^3}{\rho_{00}\gamma}\partial_z^2|\psi|^2, \end{cases} \tag{DS}$$

where $\mathtt{W} = \beta^2\rho_{00}/v_g^2$, $D = |v_g|/\beta\rho_{00}$, $\gamma = \beta c_s^2$ is a constant and $\alpha = k_0$.

Note that the solutions of these systems depend on v_g in consideration.

We have the operators

$$\widehat{\mathsf{T}_{Q_M} f} = Q_M \widehat{f}, \quad Q_M(\xi) = \frac{M^2|\xi|^2 - \dfrac{\alpha v_g^3}{\rho_{00}\gamma}\xi_2^2}{M^2\xi_2^2 - |\xi|^2} \tag{4.37}$$

and

$$Q_0(\xi) = \frac{\alpha v_g^3 \xi_2^2}{\rho_{00}\gamma|\xi|^2}. \tag{4.38}$$

The main difficulty with Q_M is because of the singularity on two straight lines go through the origin if M is not too small, and not having the possibility to use the Hormander-Mikhlin Theorem. But we are going to assume that $c_s = v_g^2$, so $M = 1/v_g$ and therefore we have good multipliers.

$T^* = T^*_{c_s}$ will be the life-time of the solution of (ZRK).

We have the following result

Theorem 4.2 *Assume that* $n = 2$, $c_s = v_g^2$,

$$\psi_{c_s 0}, \tilde{\psi}_0 \in H^2, \tag{4.39}$$

$$\sup_{c_s > 0} \|\psi_{c_s 0}\|_{H^2} \ll 1 \tag{4.40}$$

$$\|\tilde{\psi}_0\|_{H^1} \leq \inf_{c_s > 0} \|\psi_{c_s 0}\|_{H^1} \tag{4.41}$$

and

$$\lim_{c_s \to \infty} \|\psi_{c_s 0} - \tilde{\psi}_0\|_{H^1} = 0. \tag{4.42}$$

Then we have

$$i) \lim_{c_s \to \infty} T^*_{c_s}(\|\psi_{c_s 0}\|_{H^1}) = T^*_{\infty}(\|\psi_0\|_{H^1}), \tag{4.43}$$

$$ii) \lim_{c_s \to \infty} \|\psi_{c_s} - \tilde{\psi}_{c_s}\|_{L^\infty(0,T;L^2)} = \lim_{c_s \to \infty} \|\rho_{c_s} - \tilde{\rho}_{c_s}\|_{L^\infty(0,T;L^2)} = 0 \quad \text{if} \quad T \leq T^*_{\infty}, \tag{4.44}$$

where (ψ_{c_s}, ρ_{c_s}) *is the solution of* (ZRK) *with initial data* $\psi_{c_s 0}$ *and* $(\tilde{\psi}_{c_s}, \tilde{\rho}_{c_s})$ *is the solution of* (DS) *with initial data* $\tilde{\psi}_0$.

Proof. The proof is similar of that in the Theorem 4.1. The only difference is that the terms I_3, I_4 tend to zero more quickly because of

$$\|I_3\|_{L^2_x} \leq \frac{|\alpha|\beta}{2|v_g|}\|\psi_\beta\|^3_{L^6_x} \lesssim \frac{\beta}{v_g}\|\psi_\beta\|^3_{H^{1+}} \lesssim \frac{\beta}{v_g}\|\psi_{\beta 0}\|^3_{H^{1+}} \tag{4.45}$$

and

$$\begin{aligned}
\|I_4\|_{L_x^2} &\leq \frac{\beta^2|\rho_{00}|}{v_g^2}\|\rho_\beta\|_{L_x^2}\|\psi_\beta\|_{L_x^\infty} \\
&\lesssim \frac{\beta^2}{v_g^2}\|\psi_\beta\|_{L_x^4}\|\psi_\beta\|_{L_x^\infty} \\
&\lesssim \frac{\beta^2}{v_g^2}\|\psi_\beta\|_{H^1}\|\psi_\beta\|_{H^{1+}} \\
&\lesssim \frac{\beta^2}{v_g^2}\|\psi_{\beta 0}\|_{H^{1+}}^2, \qquad (4.46)
\end{aligned}$$

with $\beta = \gamma/c_s^2 \to 0$ and $v_g \to \infty$. ■

Remarks

1. We only know the existence of weak global solutions for ZR in the energy space $H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$ and local well posedness results in the space $H^s(\mathbb{R}^n)\times H^{s-1/2}(\mathbb{R}^n)\times H^{s+1}(\mathbb{R}^n)$ for $s > n/2\,(n = 2, 3)$. Uniqueness of weak solutions and well posedness in the energy space is still an open problem.

2. We have obtained several results regarding the weak and strong convergence of solutions of ZR system to solutions of the Z and DS system. In the case of the ZR system, we provide weak convergence (global in time) of solutions in the space of energy $H^1 \times L^2$, and strong convergence (local in time) of solutions in $L^2 \times L^2$ and $H^{5/2} \times L^2$. We expect to be able to prove strong convergence of solutions in $H^1 \times L^2$.

3. Since solutions of the IVP associated to Z and DS may blow-up in finite time (see [17], [16], [31]) one expects that solutions of the ZR and ZRK have the same behavior. At the present, we are working to prove this conjecture.

4. Existence of solitary wave solution of ZR of the form $(e^{iwt}\psi(x), \rho(x), \varphi(x))$ turns out to be an interesting problem since $(\psi(x), \rho(x), \varphi(x))$ satisfies the system (See [39])

$$\begin{cases} -w\psi + \epsilon\partial_z^2\psi + \sigma_1\Delta_\perp\psi - (q - \mathtt{W}M^2)|\psi|^2\psi - \mathtt{W}(D - M^2\sigma_3)\psi\partial_z\varphi = 0, \\ -M^2\partial_z^2\varphi + \Delta\varphi + (D - \sigma_3 M^2)\partial_z|\psi|^2 = 0. \end{cases}$$

   which is similar to the 3-D system for solitary waves of the Davey-Stewartson system (note that this system ceases to be elliptic in the supersonic case, $|M| > 1$) for which existence [7] and non existence [18] results are known.

Appendix A

Basic Theory

A.1 Function Spaces

Consider an open subset Ω of $\mathbb{R}^n$. Let $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$ the space of C^∞ functions with compact support in Ω and $\mathcal{D}'(\Omega)$ the space of distributions on Ω. A distribution $T \in \mathcal{D}'(\Omega)$ is said to belong to $L^p(\Omega)$ $(1 \leqslant p \leqslant \infty)$ if there exists a function $f \in L^p(\Omega)$ such that

$$\langle T, \varphi \rangle = \int_\Omega f(x)\varphi(x)dx \qquad \forall \varphi \in \mathcal{D}(\Omega).$$

In that case, is well known that f is unique. Let $k \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$. Define

$$W^{k,p}(\Omega) = \Big\{ f \in L^p(\Omega), D^\alpha f \in L^p(\Omega) \quad \forall \alpha \in \mathbb{N}^k \quad \text{such that} \quad |\alpha| \leqslant k \Big\}.$$

$W^{k,p}(\Omega)$ is a Banach space when equipped with the norm defined by

$$\|f\|_{W^{k,p}} = \sum_{|\alpha| \leqslant k} \|D^\alpha f\|_{L^p},$$

for all $f \in W^{k,p}(\Omega)$.

For all k, p as above, we denote by $W_0^{k,p}(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $W^{k,p}(\Omega)$. If $p = 2$, one sets $W^{k,2}(\Omega) = H^k(\Omega)$, $W_0^{k,2}(\Omega) = H_0^k(\Omega)$ and one equips $H^k(\Omega)$ with the following equivalent norm

$$\|f\|_{H^k} = \Big(\sum_{|\alpha| \leqslant k} \|D^\alpha f\|_{L^2}^2 \Big)^{1/2}.$$

Then $H^k(\Omega)$ is a Hilbert space with the scalar product

$$\langle u, v \rangle = \sum_{|\alpha| \leqslant k} \int_\Omega D^\alpha u D^\alpha v dx.$$

The following theorem is a characterization of the spaces $H^k(\mathbb{R}^n)$ by Fourier transform.

Theorem A.1 *Let k be a nonnegative integer.*

1. *A function $u \in L^2(\mathbb{R}^n)$ belongs to $H^k(\mathbb{R}^n)$ if and only if*

$$(1 + |\xi|^k)\widehat{u} \in L^2(\mathbb{R}^n).$$

2. *In addition, there exist a positive constant C such that*

$$\frac{1}{C}\|u\|_{H^k} \leq \|(1 + |\xi|^k)\widehat{u}\|_{L^2} \leq C\|u\|_{H^k}$$

for each $u \in H^k(\mathbb{R}^n)$.

The *fractional* Sobolev spaces are defined by

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \|u\|_{H^s(\mathbb{R}^n)} := \|(1 + |\xi|^2)^{s/2}\widehat{u}(\xi)\|_{L^2(\mathbb{R}^n)} < \infty\},$$

for any $s \in \mathbb{R}$, where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of C^∞-functions decaying at infinity.

It is not difficult to show that if $s_1 \leq s \leq s_2$ then

$$\|u\|_{H^s} \leq \|u\|_{H^{s_1}}^{\theta}\|u\|_{H^{s_2}}^{1-\theta}, \quad \text{with} \quad s = \theta s_1 + (1-\theta)s_2,\ 0 \leq \theta \leq 1.$$

The following results are essential in the theory of partial differential equations.

Theorem A.2 (Sobolev, Gagliardo, Nirenberg, Morrey) *If Ω is open and has a Lipschitz continuous boundary, then*

1. *if $1 \leqslant p < n$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, for every $q \in [p, np/(n-p)]$;*
2. *if $p = n$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, for every $q \in [p, \infty)$;*
3. *if $p > n$ then $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega) \cap C^{0,\alpha}(\Omega)$, where $\alpha = (p-n)/p$.*

Theorem A.3 (Rellich-Kondrachov) *In addition, if Ω is bounded, the embeddings (2) and (3) in Theorem above are compact. The embedding (1) is compact for $q \in [p, np/(n-p))$.*

Theorem A.4 (Sobolev) *If $s > n/2 + k$, then $H^s(\mathbb{R}^n) \hookrightarrow C^k_\infty(\mathbb{R}^n)$, where $C^k_\infty(\mathbb{R}^n)$ is the space of functions with k continuous derivatives vanishing at infinity.*

Theorem A.5 (Sobolev embedding) *Let $p \in [2, +\infty]$.*

1. *If $s > (n/2 - n/p)$ then $H^s(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$.*
2. *If $s = n/2 - n/p$ for $p \neq +\infty$, then $H^s(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$.*

Theorem A.6 *If $s > n/2$, then $H^s(\mathbb{R}^n)$ is an algebra with respect to the product of functions. That is, if $f, g \in H^s(\mathbb{R}^n)$, then*

$$\|fg\|_{H^2} \leq C(s)\|f\|_{H^s}\|g\|_{H^s}.$$

The proof of the results above can be found for example in [2], [14] and [28].

Let X denote a real Banach space, with norm $\|\cdot\|$. The space $L^p(0,T;X)$ consists of all measurable functions $u : [0,T] \to X$ with

$$\|u\|_{L^p(0,T;X)} := \Big(\int_0^T \|u(t)\|dt\Big)^{1/p} < \infty$$

for $1 \leqslant p < \infty$, and

$$\|u\|_{L^\infty(0,T;X)} := \operatorname*{ess\,sup}_{0\leqslant t\leqslant T}\|u(t)\| < \infty.$$

Remark A.1 *Let $I \subset \mathbb{R}$ be an open interval, $f \in L^p(I,X)$ and $g \in L^{p'}(I,X')$. Then*

$$t \mapsto \langle g(t), f(t)\rangle_{X',X}$$

is integrable and

$$\int_I |\langle g(t), f(t)\rangle_{X',X}| \leq \|f\|_{L^p}\|g\|_{L^{p'}}.$$

The following result is related to the preceding remark.

Theorem A.7 *If $1 \leq p < \infty$ and if X is reflexive or if X' is separable, then $(L^p(I,X))' = L^{p'}(I,X')$. In addition, if $1 < p < \infty$ and X is reflexive, then $L^p(I,X)$ is reflexive.*

The following two results are very easy to prove.

Proposition A.1 *Let* $f_m \underset{*}{\rightharpoonup} f \quad in \quad L^\infty((0,\infty); H^s(\mathbb{R}^n))$. *Then*

1. $\partial_x f_m \underset{*}{\rightharpoonup} \partial_x f \quad in \quad L^\infty((0,\infty); H^{s-1}(\mathbb{R}^n))$,
2. $f_m \underset{*}{\rightharpoonup} f \quad in \quad L^\infty((0,\infty); H^{-s}(\mathbb{R}^n))$,
3. *If* $\partial_t f_m \underset{*}{\rightharpoonup} g \quad in \quad L^\infty((0,\infty); H^{-s}(\mathbb{R}^n))$ *then* $g = \partial_t f$.

Proposition A.2 *If* $f_m \underset{*}{\rightharpoonup} f \quad in \quad L^p(I; X')$ *with* $X' = L^q_x$ *or* H^s_x, *then* $f_m \to f \quad in$ $\mathcal{D}'(I \times \mathbb{R}^n)$.

We finish this section by recalling three useful results in the compacity theory.

Lemma A.1 (J. Lions) *Let* $\mathcal{O}$ *be an open and bounded set of* $\mathbb{R}^n_x \times \mathbb{R}_t$ *and* $g_m, g \in L^q(\mathcal{O})$, $1 < q < \infty$, *such that* $\|g_m\|_{L^q(\mathcal{O})} \leqslant C$ *and* $g_m \to g$ *a.e* $\mathcal{O}$. *Then* $g_m \to g$ *weakly in* $L^q(\mathcal{O})$.

Theorem A.8 (Lions-Aubin) *Let* B_0, B, B_1 *be Banach spaces such that* $B_0 \subset B \subset B_1$ *with* B_0, B_1 *reflexives and* $B_0 \underset{c}{\hookrightarrow} B$ *(compact embedding). Let*

$$W = \Big\{ V | V \in L^{p_0}(0,T,B_0), \partial_t V \in L^{p_1}(0,T,B_1) \Big\}, \quad T < \infty,\ 1 < p_0, p_1 < \infty.$$

Then W *is a Banach space with norm*

$$\|V\|_W = \|V\|_{L^{p_0}(0,T,B_0)} + \|\partial_t V\|_{L^{p_1}(0,T,B_1)}$$

and we have $W \underset{c}{\hookrightarrow} L^{p_0}(0,T,B)$.

Definition A.1 *A family* $\mathcal{F}$ *of functions from a topological space* X *to a metric space* (Y,d) *is called* ***equicontinuous*** *at the point* $x \in X$ *if given* $\epsilon > 0$ *there is an open set* O *containing* x *such that* $d(f(x), f(y)) < \epsilon$ *for all* $y \in O$ *and for all* $f \in \mathcal{F}$. *The family is said to be equicontinuous on* X *if it is equicontinuous at each point* x *in* X.

Theorem A.9 (Ascoli) *Let* $\mathcal{F}$ *be an equicontinuous family of functions from a separable metric space* X *to a metric space* Y. *Let* $\{f_m\}$ *be a sequence in* $\mathcal{F}$ *such that for each* x *in* X *the closure of the set* $\{f_m(x) : 0 \leqq m \leqq \infty\}$ *is compact. Then there is a subsequence* $\{f_{m_k}\}$ *which converges pointwise to a continuous function* f, *and the convergence is uniform on each compact subset of* X.

A.2 Multipliers

Let m be a bounded measurable function on $\mathbb{R}^n$. One can then define a linear transformation T_m, whose domain is $L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, by the following relation between Fourier transforms

$$\widehat{\mathsf{T}_m f}(\xi) = m(\xi)\widehat{f}(\xi), \qquad f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n).$$

Definition A.2 *We shall say that m is a multiplier for L^p $(1 \leq p \leq \infty)$ if whenever $f \in L^2 \cap L^p$ then $\mathsf{T}_m f$ is also in L^p, and T_m is bounded, that is,*

$$\|\mathsf{T}_m(f)\|_{L^p} \leq A\|f\|_{L^p}, \qquad f \in L^2 \cap L^p, \quad \text{(with } A \text{ independent of } f\text{)}.$$

Note that if m is a multiplier for L^p and $p < \infty$, then T_m has a unique bounded extension to L^p, which again satisfies the same inequality above. We also write T_m for this extension.

We shall present below an important sufficient condition which contains a large class of multipliers.

Theorem A.10 (Hörmander-Mikhlin) *Suppose that $m(\xi)$ is of class C^k in the complement of the origin of $\mathbb{R}^n$, where $k \in \mathbb{N}$ and $k > n/2$. Assume also that*

$$|\partial^\alpha m(\xi)| \leq B|\xi|^{-|\alpha|} \qquad \text{if} \qquad |\alpha| = \alpha_1 + ... + \alpha_n \leq k.$$

Then $m(\xi)$ is a multiplier for L^p with $1 < p < \infty$.

Proof. See [43]. ■

For example, if $m(\xi)$ is homogeneous of degree 0, i.e. $m(\lambda\xi) = m(\xi)$, $\lambda > 0$, and is of class C^k on the unit sphere, then m is a multiplier for L^p, $1 < p < \infty$.

We will use this result to deal with the system (ZRK) as was done by Ghidaglia and Saut [17] in the study of the Davey-Stewartson system.

A.3 Functionals and the Variational Derivative

Let $J[y]$ be a functional defined on some normed linear space, and let

$$\Delta J[h] = J[y+h] - J[y]$$

be its *increment*, corresponding to the increment $h = h(x)$ of the "independent variable" $y = y(x)$. If y is fixed, $\Delta J[h]$ is a functional of h, in general a nonlinear functional.

Definition A.3 *Suppose that*

$$\Delta J[h] = \delta J[h] + \varepsilon \|h\|,$$

where $\delta J[h]$ is a linear functional and $\varepsilon \to 0$ as $\|h\| \to 0$. Then the functional $J[y]$ is said to be ***differentiable****, and the* ***principal linear part*** *of the increment $\Delta J[h]$, i.e., the linear functional $\delta J[h]$ which differs from $\Delta J[h]$ by an infinitesimal of order higher than 1 relative to $\|h\|$, is called the* ***variation*** *(or* ***differential****) of $J[y]$.*

Remark A.2 *The increment and the variation of $J[y]$, are functionals of two arguments y and h, and to emphasize this fact, we can write $\Delta J[y;h] = \delta J[y;h] + \varepsilon\|h\|$.*

In particular, we will consider a functional

$$J[y] = \int_a^b F(x, y(x), y'(x))dx,$$

where F is continuous and $y(x)$ is in the set of all continuously differentiable functions defined on the interval $[a, b]$.

If we give $y(x)$ an increment $h(x)$, with $h(a) = h(b) = 0$, and by using Taylor's theorem, we obtain

$$\delta J = \int_a^b (F_y h + F_{y'} h')dx$$

and integration by parts give

$$\delta J = \int_a^b (F_y - \frac{d}{dx} F_{y'})h dx.$$

The expression

$$\frac{\delta J}{\delta y} := F_y(x, y, y') - \frac{d}{dx} F_{y'}(x, y, y')$$

is called the **variational** (or **functional**) **derivative** of the functional $J[y]$.

In a more general case, if

$$J[u] = \int_R F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}) dx_1 \ldots dx_n,$$

depending on n independent variables $x_1, ..., x_n$ in the region R, an unknown function u of these variables, and the partial derivatives $u_{x_1}, ..., u_{x_n}$ of u, then

$$\delta J = \int_R (F_u - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_{u_{x_i}}) \Psi(\mathbf{x}) d\mathbf{x}$$

for all admissible $\Psi(\mathbf{x})$ which vanish on the boundary of R, and we have

$$\frac{\delta J}{\delta u} = F_u - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_{u_{x_i}}.$$

We refer the reader to [15] for more details about this issue.

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