Instituto de Matemática Pura e Aplicada

Doctoral Thesis

ON TORUS HOMEOMORPHISMS WHOSE ROTATION SET IS AN INTERVAL

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Abstract

In this work we prove that for a homeomorphism $\tilde{f} : \mathbf{T}^2 \to \mathbf{T}^2$ with a lift $f : \mathbf{R}^2 \to \mathbf{R}^2$ whose rotation set $\rho(f)$ is an interval, either every rational point in $\rho(f)$ is realized by a periodic orbit, or there exists a periodic essential annular set for $\tilde{f}$, satisfying a dissipative-type property. We also give a qualitative description of the dynamics for the case that $\rho(f)$ is a vertical interval containing the origin in its interior.

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0 Notations.

By $\text{pr}_1, \text{pr}_2 : \mathbf{R}^2 \to \mathbf{R}$, we will denote the projections to the first and second coordinate, respectively. Also, if $x \in \mathbf{R}^2$, x_1 and x_2 will denote $\text{pr}_1(x)$ and $\text{pr}_2(x)$, respectively.

For a set $A \subset \mathbf{R}$, the diameter of A is $\text{diam}(A) = \sup_{x,y\in A} |x - y|$. For $A \subset \mathbf{R}^2$, the **horizontal diameter** of A is $\text{diam}_1(A) = \text{diam}(\text{pr}_1(A))$, and the **vertical diameter** of A is $\text{diam}_2(A) = \text{diam}(\text{pr}_2(A))$.

For a set $A \subset \mathbf{R}^2$ and $x \in \mathbf{R}^2$, denote $d(x, A) = \inf_{y\in A} |y - x|$. For $x \in \mathbf{R}^2$ and $r > 0$, denote $B_r(x) = \{y \in \mathbf{R}^2 \,:\, |y - x| < r\}$, and for $A \subset \mathbf{R}^2$, denote $B_r(A) = \{x \in \mathbf{R}^2 \,:\, d(x, A) < r\}$.

For the circle $S^1 = \mathbf{R}/\mathbf{Z}$, and the two-torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$, denote by π, π' and π'' the canonical projections

$$\mathbf{R}^2 \stackrel{\pi}{\to} \mathbf{R} \times S^1 \stackrel{\pi''}{\to} \mathbf{T}^2, \quad \text{and} \quad \pi' = \pi'' \circ \pi.$$

We will denote also by $d(\cdot, \cdot)$ the metric in $\mathbf{T}^2$ or in $\mathbf{R} \times S^1$ induced by the euclidean metric in $\mathbf{R}^2$.

Define $T_1, T_2 : \mathbf{R}^2 \to \mathbf{R}^2$ to be the translations $T_1 : (x_1, x_2) \mapsto (x_1 + 1, x_2)$, $T_2 : (x_1, x_2) \mapsto (x_1, x_2 + 1)$. Also, T_1 and T_2 will denote the translations in $\mathbf{R} \times S^1$, $T_1 : (x_1, x_2) \mapsto (x_1 + 1, x_2)$, and $T_2 : (x_1, x_2) \mapsto (x_1, x_2 + 1 \mod 1)$.

By a curve $\gamma : I \to \mathbf{R}^2$, depending on the context, we mean either γ or $\text{Im}(\gamma) \subset \mathbf{R}^2$. By an **arc**, we mean a compact curve, and if α is an arc, $\dot{\alpha}$ denotes the curve α without its endpoints.

A **line** ℓ is a proper embedding of $\ell : \mathbf{R} \to \mathbf{R}^2$. By Shoenflies' Theorem ([Cai51]), given a line ℓ there exists an orientation preserving homeomorphism h of $\mathbf{R}^2$ such that $h \circ \ell(t) = (0, t)$, for all $t \in \mathbf{R}$. Then, the open half-plane $h^{-1}((0, \infty) \times \mathbf{R})$ is independent of h, and we call it the **right** of ℓ, and denote it by $R(\ell)$. Analogously, we define $L(\ell) = h^{-1}((-\infty, 0) \times \mathbf{R})$ the open half-plane to the **left** of ℓ. The sets $\overline{R}(\ell)$ and $\overline{L}(\ell)$ denote the closures of $R(\ell)$ and $L(\ell)$, resp.

By $\ell \prec \ell'$ we will mean $\ell \subset L(\ell')$.

A closed curve γ in $\mathbf{T}^2$ or in $\mathbf{R} \times S^1$ is **essential** if it is not homotopic to a point, and we say that γ is **vertical** if γ is freely homotopic to a curve of the form $c\beta$, where $c \in \{1, -1\}$ and $\beta(t) = (0, t)$.

A curve γ in $\mathbf{T}^2$ (or in $\mathbf{R} \times S^1$) is **free** for $f : \mathbf{T}^2 \to \mathbf{T}^2$ ($f : \mathbf{R} \times S^1 \to \mathbf{R} \times S^1$) if it is simple and closed, and $f(\gamma) \cap \gamma = \emptyset$, and we say it is **free forever** for f if γ is disjoint from all its iterates by f.

If ℓ, ℓ' are two lines in $\mathbf{R}^2$, we define $(\ell, \ell') = R(\ell) \cap L(\ell')$, and $[\ell, \ell'] = \overline{R}(\ell) \cap \overline{L}(\ell)$. Similarly we define $(\ell, \ell'] = R(\ell) \cap \overline{L}(\ell')$ and $[\ell, \ell') = \overline{R}(\ell) \cap L(\ell')$. If γ and γ' are two disjoint, simple, closed and vertical curves in $\mathbf{T}^2$, we define the topological annuli $(\gamma, \gamma') \subset \mathbf{T}^2$ and $[\gamma, \gamma'] \subset \mathbf{T}^2$ in the following way. Let $\tilde{\gamma} \subset \mathbf{R}^2$ be any lift of γ, and let $\tilde{\gamma}'$ be the first lift of γ' to the right of $\tilde{\gamma}$, that is, $\tilde{\gamma}'$ is the lift of γ' with $\tilde{\gamma} \prec \tilde{\gamma}' \prec T_1(\tilde{\gamma})$. Orient $\tilde{\gamma}$ and $\tilde{\gamma}'$ as going upwards. Define then $(\gamma, \gamma') = \pi'((\tilde{\gamma}, \tilde{\gamma}'))$ and $[\gamma, \gamma'] = \pi'([\tilde{\gamma}, \tilde{\gamma}'])$. In a similar way, if γ and γ' are disjoint, simple, closed and vertical curves in $\mathbf{R} \times S^1$, we define $(\gamma, \gamma') \subset \mathbf{R} \times S^1$ and $[\gamma, \gamma'] \subset \mathbf{R} \times S^1$.

We say that a set A contained in $\mathbf{T}^2$ or in $\mathbf{R} \times S^1$ is **annular** if it is a nested intersection of topological compact annuli.

For a map $f : X \to X$, where X is any metric space, we define an ϵ-chain for f as a set $\{x_i\}_{i=i_0}^{i_1} \subset X$ such that $d(x_{i+1}, f(x_i)) < \epsilon$ for all $i_0 \leq i < i_1$. An ϵ-chain $\{x_i\}_{i=i_0}^{i_1}$ is **periodic** if $x_{i_0} = x_{i_1}$. A point $x \in X$ is **chain recurrent** for f if for all $\epsilon > 0$ there exists a periodic ϵ-chain $\{x_i\}_{i=0}^{n}$ for f with $x_0 = x_n = x$. The **chain recurrent set**, denoted by $CR(f)$, is the set of chain recurrent points for f.

1 Introduction.

Given a homeomorphism $\tilde{f} : \mathbf{T}^n \to \mathbf{T}^n$ homotopic to the identity, the *rotation set* of some lift $f : \mathbf{R}^n \to \mathbf{R}^n$ of $\tilde{f}$ was introduced by Misiurewicz and Ziemian in [MZ89], and it is defined as the set of accumulation points of sequences of the form

$$\left\{ \frac{f^{m_i}(x_i) - x_i}{m_i} \right\}_{i \in \mathbf{N}}$$

where $m_i \to \infty$ and $x_i \in \mathbf{R}^2$.

This set carries dynamical information, and moreover, when $n = 2$ this set has nice geometric properties, like convexity. Also, in the case $n = 2$ much more information can be obtained in base of the theory of planar homeomorphisms (for example, Brouwer theory, Thurston's classification theory, etc.). For this reason we restrict ourselves to the case $n = 2$.

One example of the relation between the rotation set of f, denoted $\rho(f)$, and the dynamics of $\tilde{f}$ is the realization of rational points in $\rho(f)$ by periodic orbits. By a rational point we mean a point of rational coordinates $(p_1/q, p_2/q) \in \rho(f)$ (with $\gcd(p_1, p_2, q) = 1$), and we say that $(p_1/q, p_2/q)$ is realized by a periodic orbit of $\tilde{f}$ if there exists $x \in \mathbf{R}^2$ such that

$$f^q(x) = x + (p_1, p_2).$$

The problem of finding sufficient conditions for the realization of rational points by periodic orbits has been extensively studied. For example, in [Fra88] Franks proved that rational extremal points in $\rho(f)$ are always realized by periodic orbits. In [Fra89] he also proved that every rational point in the interior of the rotation set is realized by a periodic orbit.

Recall that a curve is essential if it is homotopically non-trivial, and a free curve for $\tilde{f}$ is a curve that is disjoint from its image by $\tilde{f}$. In [KK08] Kocsard and Koropecki proved that in the case that $\rho(f)$ has empty interior, if $\tilde{f}$ has no free curves then every rational point if $\rho(f)$ is realized by a periodic orbit. Moreover, it is proved that if there is a rational point in $\rho(f)$ not realized by a periodic orbit, then for every $n \in \mathbf{N}$ there is an essential curve disjoint from its first n iterates by $\tilde{f}$.

Here we will improve this result showing that there is an essential curve that is free forever for $\tilde{f}$, that is, an essential curve that is disjoint of *all* its iterates by $\tilde{f}$. For more results on realization of rational points by periodic orbits, see for example [Fra95] and [JZ98].

The simplest example of a homeomorphism with rational points in its rotation set not realized by periodic orbits is the following. Let $\tilde{g} : S^1 \to S^1$ be a homeomorphism such that $\text{Fix}(\tilde{g}) = \{1/4, 3/4\}$, and such that if $g : \mathbf{R} \to \mathbf{R}$ is a lift of $\tilde{g}$, $x < g(x)$ for any $x \notin \text{Fix}(\tilde{g})$. Define $\tilde{f} : \mathbf{T}^2 \to \mathbf{T}^2$ by $\tilde{f}(x, y) = (\tilde{g}(x), \sin(2\pi x)/2)$ mod $\mathbf{Z}^2$ (see Fig. 1).

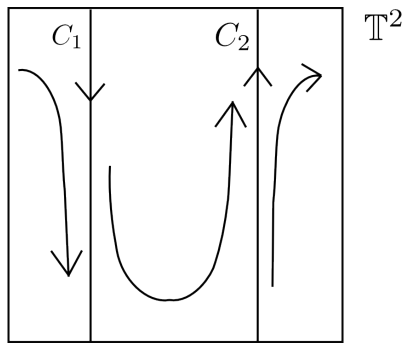

Figure 1:

This homeomorphism $\tilde{f}$ has two invariant circles: $C_1 = \{x \in \mathbf{T}^2 : x_1 = 1/4\}$ and $C_2 = \{x \in \mathbf{T}^2 : x_1 = 3/4\}$. In C_1 the dynamics is a rotation by $\sin(\pi/2)/2 = 1/2$ and in C_2 it is a rotation by $\sin(3\pi/2)/2 = -1/2$. Every point which is not in $C_1 \cup C_2$ has its full-orbit contained in an annulus bounded by these two circles, and has its omega-limit contained either in C_1 or in C_2. By this, one easily sees that $\tilde{f}$ has a lift $f : \mathbf{R}^2 \to \mathbf{R}^2$ such that $\rho(f) = \{0\} \times [-1/2, 1/2]$. Any rational point in $\rho(f)$ which is not extremal is not realized by periodic orbits, as the only periodic points of $\tilde{f}$ are the points $x \in C_1 \cup C_2$, which have period 2 and either $f^2(x) = x + (0, 1)$ or $f^2(x) = x + (0, -1)$. Moreover, $\tilde{f}$ has the following properties:

- The circles C_1 and C_2 are 'semi-attractors'.
- The presence of an invariant circle means that the dynamics of $\tilde{f}$ is 'annular'.

Let us explain what we mean by the terms 'semi-attractor' and 'annular dynamics'. Recall that an essential set in $\mathbf{T}^2$ is a set that is not contained in a topological open disc, and an annular set in $\mathbf{T}^2$ is a set which is a nested intersection of compact topological annuli.

Definition 1.1. Let $h : \mathbf{T}^2 \to \mathbf{T}^2$ be a homeomorphism. An invariant essential annular set $A \subset \mathbf{T}^2$ is a *semi-attractor* for h if A is contained in a topological closed annulus whose boundary consists of two essential curves γ_1 and γ_2 that are free for h and such that:

- either $\omega(x, h) \subset A$ for all $x \in \gamma_1$, or $\alpha(x, h) \subset A$ for all $x \in \gamma_1$, and
- either $\omega(y, h) \subset A$ for all $y \in \gamma_2$, or $\alpha(y, h) \subset A$ for all $y \in \gamma_2$.

If A is a periodic essential annular set with period $q \in \mathbf{N}$, we say that A is a semi-attractor for h if A is a semi-attractor for f^q.

When we say that the dynamics of a torus homeomorphism $\tilde{f}$ is *annular*, we mean that there is a periodic essential annular set A for $\tilde{f}$, and then $\mathbf{T}^2$ is decomposed in the two periodic sets A and A^c. The fact that A is annular implies that A^c is a topological open annulus, and therefore, to understand the dynamics of $\tilde{f}$ it suffices to understand the dynamics of homeomorphisms of the annulus.

The purpose of this work will be to show that in some sense, the dynamics of the example above is a 'model' for the general case of a homeomorphism with rational points in its rotation set which are not realized by periodic orbits. We will show that if a rational point in the rotation set is not realized by a periodic orbit, *then the dynamics is annular*, and moreover, there is a periodic essential annular set which is a semi-attractor.

Before stating our results we mention a theorem for annulus homeomorphisms by Le Calvez, in which this work has been inspired. For a homeomorphism $F : S^1 \times [0,1] \to S^1 \times [0,1]$ isotopic to the identity, the rotation set of some lift $f : \mathbf{R} \times [0,1] \to \mathbf{R} \times [0,1]$ is defined as the set of all accumulation points of sequences of the form

$$\left\{ \frac{f^{m_i}(x_i)_1 - (x_i)_1}{m_i} \right\}_{i \in \mathbf{N}}$$

where $m_i \to \infty$ and $x_i \in \mathbf{R} \times [0,1]$. In this case the rotation set $\rho(f)$ is a compact interval $I \subset \mathbf{R}$ (possibly degenerate). Also, if $\Lambda \subset S^1 \times [0,1]$ is a compact invariant set, we can define the rotation set of Λ, denoted $\rho(\Lambda, f)$, as the set of all accumulation points of sequences of the form

$$\left\{ \frac{f^{m_i}(x_i)_1 - (x_i)_1}{m_i} \right\}_{i \in \mathbf{N}}$$

with $m_i \to \infty$ and $x \in \Pi^{-1}(\Lambda)$, where $\Pi : \mathbf{R} \times [0,1] \to S^1 \times [0,1]$ is the canonical projection. The following theorem by Le Calvez was proven for C^1 diffeomorphisms in [Cal91], but by the results of [Cal05] it is also valid for homeomorphisms (see Theorem 9.1 in that article).

Theorem 1.2. *Let $F : S^1 \times [0,1] \to S^1 \times [0,1]$ be a homeomorphism isotopic to the identity with a lift $f : \mathbf{R} \times [0,1] \to \mathbf{R} \times [0,1]$ that has no fixed points and whose rotation set is an interval containing 0 in its interior.*

Then, there exists a finite non-empty family of free curves for F such that, the maximal invariant set contained between two consecutive curves has rotation set contained either strictly to the right or strictly to the left of 0 (see Fig. 2).

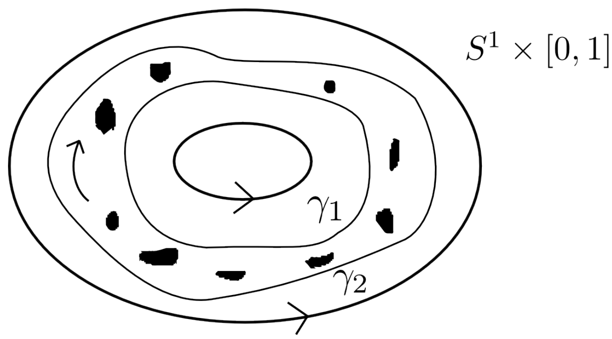

Figure 2: Illustration for Theorem 1.2. The free curves γ_1 and γ_2 are free for F.

One can easily deduce the following corollary. A semi-attractor for a homeomorphism $F : S^1 \times [0,1] \to S^1 \times [0,1]$ is defined in an analogous way as in the definition the torus case; the only modification in Definition 1.1 is to allow one of the curves γ_1 and γ_2 to be a boundary component of $S^1 \times [0,1]$ (and therefore not free for F).

Corollary 1.3. *For a homeomorphism F within the hypotheses of Theorem 1.2, there exists an essential annular F-invariant set A which is a semi-attractor.*

For a homeomorphism $F : S^1 \times [0,1] \to S^1 \times [0,1]$ with a rational point in its rotation set that is not realized by a periodic orbit, one also obtains a dynamical filtration as in Theorem 1.2, and the dissipative-like property of having an essential semi-attractor.

Here we see that the dynamical description given in Theorem 1.2 and its corollary for annulus homeomorphisms is, to some extent, still valid for torus homeomorphisms whose rotation set is an interval with rational points not realized by periodic orbits. Our first result is the following.

Theorem A. *Let $\tilde{f} : \mathbf{T}^2 \to \mathbf{T}^2$ be a homeomorphism homotopic to the identity with a lift $f : \mathbf{R}^2 \to \mathbf{R}^2$ whose rotation set is an interval.*

Then, either every rational point in the rotation set is realized by a periodic orbit, or there exists a periodic essential annular set for $\tilde{f}$ which is also a semi-attractor.

A motivational example for Theorem A is given at the end of this section (see Example 2). A key step in the proof of Theorem A will be to prove that there is an essential curve that is free forever for $\tilde{f}$, which, as we mentioned before, improves a result in [KK08].

To prove that there is an essential curve that is free forever for $\tilde{f}$ we will use the following theorem. It tells us that for torus homeomorphisms having a lift without fixed points and whose rotation set is an interval containing 0, the dynamics is annular, and moreover, *qualitatively the same* as it is shown to be for annulus homeomorphisms in Theorem 1.2. If $\Lambda \subset \mathbf{T}^2$ is a compact $\tilde{f}$-invariant set, we define $\rho(\Lambda, f)$ in a similar way as defined above for the case of $S^1 \times [0, 1]$.

Theorem B. *Let $\tilde{f}$ be a homeomorphism of $\mathbf{T}^2$ homotopic to the identity with a lift $f : \mathbf{R}^2 \to \mathbf{R}^2$ such that:*

- *$\rho(f) = \{0\} \times I$, where I is a non-degenerate interval containing 0 in its interior,*
- *f has no fixed points.*

There exists a non-empty finite family $\{\tilde{l}_i\}_{i=0}^{r-1}$ of curves in $\mathbf{T}^2$ which are simple, closed, vertical, pairwise dijoint and essential, and with the following properties. If

$$\Theta_i := \bigcap_{n \in \mathbf{Z}} \tilde{f}^n \left([\tilde{l}_i, \tilde{l}_{i+1}] \right) \quad \text{for } i \in \mathbf{Z}/r\mathbf{Z},$$

then, for all i,

1. *$\emptyset \neq \Omega(\tilde{f}) \cap [\tilde{l}_i, \tilde{l}_{i+1}] \subset \Theta_i$,*
2. *there is $\epsilon > 0$ such that $\rho(\Theta_i, f)$ is contained either in $\{0\} \times (\epsilon, \infty)$, or in $\{0\} \times (-\infty, -\epsilon)$,*

3. *the curves $\tilde{l}_0, \tilde{l}_1, \ldots, \tilde{l}_{r-1}$ are free forever for $\tilde{f}$, and*

4. *at least one of the sets Θ_i is an annular, essential, $\tilde{f}$-invariant set which is a semi-attractor (see Fig. 3).*

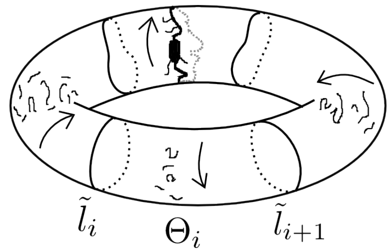

Figure 3: The sets Θ_i and the curves $\tilde{l}_i$. At least one of the Θ_i must be annular and essential.

We do not know if the property of the dynamics being annular is also present in the case that all the rational points in the rotation set are realized by periodic orbits:

Question 1.4. *If $\tilde{f} : \mathbf{T}^2 \to \mathbf{T}^2$ is a homeomorphism with a lift $f : \mathbf{R}^2 \to \mathbf{R}^2$ such that $\rho(f)$ is an interval of the form $\{0\} \times I$, and such that every rational point in $\rho(f)$ is realized by a periodic orbit, does there exist an invariant, essential, annular set for $\tilde{f}$?*

It has been recently announced by Bortollato and Tal that the answer to this question is affirmative in the case that Lebesgue measure λ is ergodic with respect to $\tilde{f}$ and the rotation vector of λ (see section 2.1.2) is of the form $(0, a)$, with $a \in \mathbf{R}$ irrational.

More examples.

We now describe more examples of homeomorphisms $\tilde{f} : \mathbf{T}^2 \to \mathbf{T}^2$ with rational points in its rotation set not realized by periodic orbits.

Example 1. Consider an isotopy $(H_t)_{t\in[0,1]}$ on S^1 such that $H_0 = R_{1/2}$, where $R_{1/2}$ is the rotation in S^1 by $1/2$ and H_1 is a Denjoy homeomorphism $\tilde{g}$. Suppose also that $\tilde{g}$ is such that the isotopy (H_t) lifts to an isotopy $(\tilde{H}_t)_{t\in[0,1]}$ on $\mathbf{R}$ between the translation by $1/2$, and a lift g of $\tilde{g}$ with rotation number $\rho(g) < 0$. Define then $\tilde{f}_0 : \mathbf{T}^2 \to \mathbf{T}^2$ by

$$\tilde{f}_0(x,y) = \begin{cases} (x, H_{2x}(y)), & \text{if } 0 \le x \le 1/2 \\ (x, H_{-2x+2}(y)), & \text{if } 1/2 \le x \le 1 \end{cases}$$

So, the restriction of $\tilde{f}_0$ to the circle $\{x \in \mathbf{T}^2 : x_1 = 1/2\}$ is the Denjoy map $\tilde{g}$, which has a minimal cantor set $K \subset \{x \in \mathbf{T}^2 : x_1 = 1/2\}$, and the restriction of $\tilde{f}_0$ to the circle $C_1 = \{x \in \mathbf{T}^2 : x_1 = 0\}$ is a rotation by $(0, 1/2)$. Let $\varphi : \mathbf{T}^2 \to \mathbf{R}$ be a continuous function such that $\varphi \ge 0$ and $\varphi(x,y) = 0$ if and only if $(x,y) \in K \cup C_1$. Let $\tilde{f}_1 : \mathbf{T}^2 \to \mathbf{T}^2$ be given by $\tilde{f}_1(x,y) = (x + \varphi(x,y), y)$, and then define $\tilde{f} : \mathbf{T}^2 \to \mathbf{T}^2$ by $\tilde{f} = \tilde{f}_0 \circ \tilde{f}_1$ (see Fig. 4).

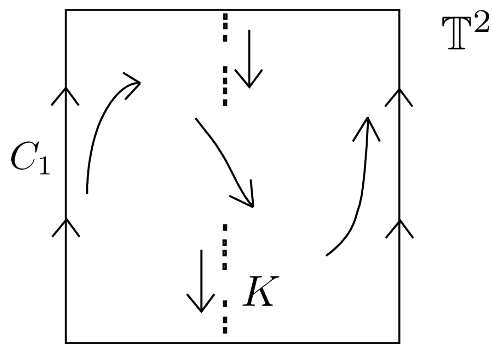

Figure 4: Example 1.

If φ is chosen carefully, $\tilde{f}$ can be made a homeomorphism (e. g., if φ is C^1 and sufficiently close to the constant function $\varphi_0 \equiv 0$). Note that $\tilde{f}$ has an invariant circle $C_1 = \{(x,y) : x = 0\}$, and every point $x \in \mathbf{T}^2 \setminus (C_1 \cup K)$ has its omega limit contained either in C_1 or in K. Therefore $\tilde{f}$ has a lift $f : \mathbf{R}^2 \to \mathbf{R}^2$ with $\rho(f) = [\rho(g), 1/2]$. As the only periodic points of $\tilde{f}$ are the points in C_1, we have that *any non-extremal rational point in $\rho(f)$ is not realized by a periodic orbit.* The fact that $\tilde{f}$ has an invariant circle implies that the dynamics of $\tilde{f}$ is annular, and also $\tilde{f}$ presents the dissipative property of having a semi-attractor, which is the circle C_1.

Example 2. A natural idea is to try to modify last example and replace the rotation in the circle $\{(x,y) : x = 0\}$ by another Denjoy example, in a way that the points not contained in the minimal sets of the Denjoy maps can 'pass through', in hopes of obtaining an example with unbounded orbits in the horizontal direction (see Fig. 5).

More explicitly, one can construct a homeomorphism $\tilde{f} : \mathbf{T}^2 \to \mathbf{T}^2$ with a lift $f : \mathbf{R}^2 \to \mathbf{R}^2$ with the following properties:

- There are two minimal sets $K_1 \subset \{x \in \mathbf{T}^2 \,:\, x_1 = 1/4\}$ and $K_2 \subset \{x \in \mathbf{T}^2 \,:\, x_1 = 3/4\}$, such that the dynamics of $\tilde{f}$ restricted to K_1 and K_2 is the dynamics of a homeomorphism of S^1 restricted to a minimal set.
- For any $x \in \pi'^{-1}(K_1)$ and $y \in \pi'^{-1}(K_2)$ we have $a := \lim_{n\to\infty}(f^n(x)_2 - x_2)/n > 0$ and $b := \lim_{n\to\infty}(f^n(y)_2 - y_2)/n < 0$. Therefore $\rho(f)$ contains the inteval $\{0\} \times [b, a]$.
- $f(x)_1 > x_1$ for all $x \in \mathbf{R}^2 \setminus \pi'^{-1}(K_1 \cup K_2)$.

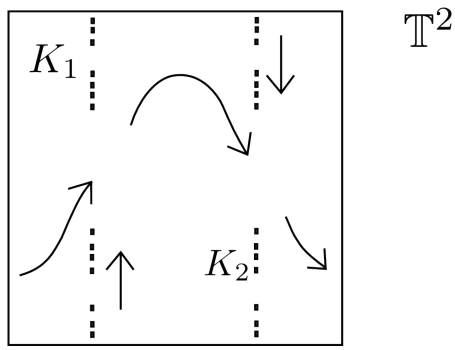

Figure 5: Example 2.

The point $(0, 0) \in \rho(f)$ is not realized by a fixed point, as by definition of $\tilde{f}$ the lift f has no fixed points. One could wonder if this example can be constructed in a way that $\rho(f)$ is an interval of the form $\{0\} \times I$, and such that

$$\limsup \left\{|f^n(x)_1 - x_1| \,:\, x \in \mathbf{R}^2, n \in \mathbf{N}\right\} = \infty. \tag{1}$$

However Theorem B tells us that this is not possible: if in this example $\rho(f)$ is of the form $\{0\} \times I$, then there is an essential annular set A which is $\tilde{f}$-invariant and vertical (that is, A it is contained in a topological annulus of the form $(l, l') \subset \mathbf{T}^2$, where l and l are vertical curves). It is easy to see that the presence of such a set A implies that there is $M > 0$ such that, for any $x \in \mathbf{R}^2$, $|f^n(x)_1 - x_1| < M$ for all n.

So, if this example is constructed in a way that it holds (1), then by Theorem B we have that $\max \mathrm{pr}_1(\rho(f)) > 0$. As $\tilde{f}$ was defined in a way that $\rho(f)$ contains the inteval $\{0\} \times [b, a]$, we then have that $\rho(f)$ has non-empty interior. The fact that $\rho(f)$ has non-empty interior implies that $\tilde{f}$ has infinitely many periodic points, positive entropy, etc. (see [Fra89] and [LM91]).

This work is organized as follows. In section 2 we introduce the preliminary theory used in the proof of Theorem B, which is mainly the following: the rotation set for homeomorphisms of $\mathbf{T}^2$ and some results related to it, the Brouwer theory for planar homeomorphisms developed by Patrice Le Calvez, and Atkinson's Lemma about ergodic cocycles. In section 3 we prove Theorem A assuming Theorem B. Finally, in section 4, which is the longest section of this work, we prove Theorem B.

2 Preliminaries.

2.1 The rotation set.

Denote by $\text{Homeo}(\mathbf{T}^2)$ the set of homeomorphisms of $\mathbf{T}^2$, and by $\text{Homeo}_*(\mathbf{T}^2)$ the elements of $\text{Homeo}(\mathbf{T}^2)$ which are homotopic to the identity. Let $\tilde{f} \in \text{Homeo}_*(\mathbf{T}^2)$ and let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be a lift of $\tilde{f}$.

Definition 2.1 ([MZ89])**.** The *rotation set of* f is defined as

$$\rho(f) = \bigcap_{m=1}^{\infty} \text{cl}\left(\bigcup_{n=m}^{\infty} \left\{ \frac{f^n(x) - x}{n} \ : \ x \in \mathbf{R}^2 \right\} \right) \subset \mathbf{R}^2.$$

The *rotation set of a point* $x \in \mathbf{R}^2$ is defined by

$$\rho(x, f) = \bigcap_{m=1}^{\infty} \text{cl} \left\{ \frac{f^n(x) - x}{n} \ : \ n > m \right\}.$$

If the above set consists of a single point $v \in \mathbf{R}^2$, we call v the *rotation vector of* x. If $\Lambda \subset \mathbf{T}^2$ is a compact $\tilde{f}$-invariant set, we define the *rotation set of* Λ as

$$\rho(\Lambda, f) = \bigcap_{m=1}^{\infty} \text{cl}\left(\bigcup_{n=m}^{\infty} \left\{ \frac{f^n(x) - x}{n} \ : \ x \in \pi'^{-1}(\Lambda) \right\} \right) \subset \mathbf{R}^2.$$

Remark 2.2. It is easy to see that for integers n, m_1, m_2,

$$\rho(T_1^{m_1} T_2^{m_2} f^n) = n\rho(f) + (m_1, m_2).$$

Then, the rotation set of any other lift of $\tilde{f}$ is an integer translate of $\rho(f)$, and we can talk of the 'rotation set of $\tilde{f}$' if we keep in mind that it is defined modulo $\mathbf{Z}^2$.

Theorem 2.3 ([MZ89])**.** *Let $\tilde{f} : \mathbf{T}^2 \to \mathbf{T}^2$ be a homeomorphism, let $\Lambda \subset \mathbf{T}^2$ be a compact $\tilde{f}$-invariant set, and let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be a lift of $\tilde{f}$. Then:*

- *The set $\rho(\Lambda, f)$ is compact.*
- *The set $\rho(f)$ is compact and convex, and every extremal point of $\rho(f)$ is the rotation vector of some point.*

Given $A \in \text{GL}(2, \mathbf{Z})$, we denote by $\tilde{A}$ the homeomorphism of $\mathbf{T}^2$ lifted by A. If $\tilde{h} \in \text{Homeo}(\mathbf{T}^2)$, there is a unique $A \in \text{GL}(2, \mathbf{Z})$ such that for every lift h of $\tilde{h}$, the map $h - A$ is bounded (in fact, $\mathbf{Z}^2$-periodic). Then $\tilde{h}$ is isotopic to $\tilde{A}$.

Lemma 2.4. *Let $\tilde{f} \in Homeo_*(\mathbf{T}^2)$, $A \in \mathrm{GL}(2, \mathbf{Z})$ and $\tilde{h} \in \mathrm{Homeo}(\mathbf{T}^2)$ isotopic to A. Let f and h be lifts of $\tilde{f}$ and $\tilde{h}$ to $\mathbf{R}^2$. Then*

$$\rho(hfh^{-1}) = A\rho(f).$$

In particular, $\rho(AfA^{-1}) = A\rho(f)$.

For a proof of this lemma, see for example [KK08].

Remark 2.5. If $\rho(f)$ is segment of rational slope, there exists $A \in \mathrm{GL}(2, \mathbf{Z})$ such that $A\rho(f)$ is a vertical segment. Indeed, if $\rho(f)$ is a segment of slope p/q (with p and q coprime integers), we can find $x, y \in \mathbf{Z}$ such that $px + qy = 1$, and letting

$$A = \begin{pmatrix} p & -q \\ y & x \end{pmatrix}$$

we have that $\det(A) = 1$, and since $A(q, p) = (0, 1)$, $A\rho(f)$ is vertical.

2.1.1 The rotation set and periodic orbits.

Recall that we say that a rational point $(p_1/q, p_2/q) \in \rho(f)$ is realized by a periodic orbit if there exists $x \in \mathbf{R}^2$ such that

$$f^q(x) = x + (p_1, p_2).$$

We mention the following realization results.

Theorem 2.6 ([Fra88])**.** *If a rational point of $\rho(f)$ is extremal, then it is realized by a periodic orbit.*

Theorem 2.7 ([Fra89])**.** *Any rational point in the interior of $\rho(f)$ is realized by a periodic orbit.*

The following theorem is stated for diffemorphisms in [Cal91], p. 106, but its proof remains valid for homeomorphisms using the results in [Cal05] (see p. 9 of that article).

Theorem 2.8. *If a rational point belongs to a line of irrational slope which bounds a closed half-plane that contains $\rho(f)$, then this point is realized by a periodic orbit.*

2.1.2 The rotation set and invariant measures.

For a compact $\tilde{f}$-invariant set $\Lambda \subset \mathbf{T}^2$, we denote by $\mathcal{M}_{\tilde{f}}(\Lambda)$ the family of $\tilde{f}$-invariant probability measures with support in Λ, and $\mathcal{M}_{\tilde{f}} = \mathcal{M}_{\tilde{f}}(\mathbf{T}^2)$. Define the *displacement function* $\phi : \mathbf{T}^2 \to \mathbf{R}^2$ by

$$\phi(\tilde{x}) = f(x) - x, \quad \text{for } x \in \pi'^{-1}(\tilde{x}).$$

This is well defined, as any two preimages of $\tilde{x}$ by the projection $\pi' : \mathbf{T}^2 \to \mathbf{R}^2$ differ by an element of $\mathbf{Z}^2$, and f is $\mathbf{Z}^2$-periodic. Now, for $\mu \in \mathcal{M}_{\tilde{f}}$, we define the *rotation vector* of μ as

$$\rho(\mu, f) = \int \phi \, d\mu.$$

Then, we define the sets

$$\rho_{mes}(\Lambda, f) = \left\{\rho(\mu, f) \, : \, \mu \in \mathcal{M}_{\tilde{f}}(\Lambda)\right\},$$

and

$$\rho_{erg}(\Lambda, f) = \left\{\rho(\mu) \, : \, \mu \text{ is ergodic for } \tilde{f} \text{ and } \operatorname{supp}(\mu) \subset \Lambda\right\}.$$

When $\Lambda = \mathbf{T}^2$ we simply write $\rho_{mes}(f)$ and $\rho_{erg}(f)$.

Proposition 2.9 ([MZ89])**.** *It holds the following:*

$$\rho(f) = \rho_{mes}(f) = \operatorname{conv}(\rho_{erg}(f)).$$

When Λ is a proper (compact, invariant) subset of $\mathbf{T}^2$, the set $\rho(\Lambda, f)$ is not necessarily convex. However, we have the following.

Proposition 2.10. *It holds*

$$\operatorname{conv}\rho(\Lambda, f) = \rho_{mes}(\Lambda, f),$$

and therefore, if $v \in \mathbf{R}^2$ is an extremal point of $\operatorname{conv}\rho(\Lambda, f)$, there exists an ergodic measure μ for $\tilde{f}$ with $\rho(\mu, f) = v$.

Proof. We first observe that $\rho_{mes}(\Lambda, f)$ is convex. To see this, let $r_1, r_2 \in \rho_{mes}(\Lambda, f)$, and let $\mu_1, \mu_2 \in \mathcal{M}_{\tilde{f}}(\Lambda)$ be such that $\rho(\mu_1, f) = r_1$ and $\rho(\mu_2, f) = r_2$. The set $\mathcal{M}_{\tilde{f}}(\Lambda)$ is convex, and then for all $t \in [0, 1]$, $t\mu_1 + (1-t)\mu_2$ belongs to $\mathcal{M}_{\tilde{f}}(\Lambda)$. Also, for $t \in [0, 1]$,

$$t \cdot r_1 + (1-t)r_2 = t \int \phi \, d\mu_1 + (1-t) \int \phi \, d\mu_2 =$$

$$= \int \phi \, d(t\mu_1 + (1-t)\mu_2) = \rho(t\mu_1 + (1-t)\mu_2, f),$$

and then $tr_1 + (1-t)r_2 \in \rho_{mes}(\Lambda, f)$. Therefore $\rho_{mes}(\Lambda, f)$ is convex.

Thus, to prove the inclusion $\mathrm{conv}(\rho(\Lambda, f)) \subset \rho_{mes}(\Lambda, f)$ it suffices to prove that $\rho(\Lambda, f) \subset \rho_{mes}(\Lambda, f)$. Let $v \in \rho(\Lambda, f)$. There exists then a sequence $\{x_n\}_n$ in $\pi'^{-1}(\Lambda)$ and a sequence of natural numbers $\{m_n\}_n$ such that

$$\lim_n \frac{(f^{m_n}(x_n) - x_n)}{m_n} = v.$$

Define a sequence of probability measures $\{\delta_n\}_n$ by

$$\delta_n = \frac{\delta_{x_n} + \delta_{f(x_n)} + \cdots + \delta_{f^{m_n-1}(x_n)}}{m_n},$$

and let μ be an accumulation point of $\{\delta_n\}_n$ in the space $\mathcal{M}_{\tilde{f}}(\Lambda)$ of Borel probability measures in Λ, equipped with the weak-* topology. Then, μ is $\tilde{f}$-invariant. Choosing a subsequence, we can assume that $\delta_n \to \mu$. Then

$$\rho(\mu, f) = \lim_n \int \phi \, d(\delta_n) = \lim_n \frac{f^{m_n}(x_n) - x_n}{m_n} = v,$$

and therefore $\rho(\Lambda, f) \subset \rho_{mes}(\Lambda, f)$.

Now we prove the inclusion $\rho_{mes}(\Lambda, f) \subset \mathrm{conv}(\rho(\Lambda, f))$. As $\rho_{mes}(\Lambda, f)$ is convex, it suffices to show that the extremal points of $\rho_{mes}(\Lambda, f)$ are contained in $\mathrm{conv}(\rho(\Lambda, f))$. Actually, we will show that the extremal points of $\rho_{mes}(\Lambda, f)$ are contained in $\rho(\Lambda, f)$.

Consider the vector space $C(\mathbf{T}^2)$ of continuous maps from $\mathbf{T}^2$ to $\mathbf{R}$, and consider the dual vector space $C'(\mathbf{T}^2)$ of $C(\mathbf{T}^2)$, that is, the space of linear functionals from $C(\mathbf{T}^2)$ to $\mathbf{R}$. We know that $C'(\mathbf{T}^2)$ is isomorphic to the vector space $\mathcal{M}_s(\mathbf{T}^2)$ of signed measures in $\mathbf{T}^2$ (see for example [Fol84]). Consider the linear map $L_{\tilde{f}} : \mathcal{M}_s(\mathbf{T}^2) \to \mathbf{R}^2$ given by

$$L_{\tilde{f}}(\mu) = \int \phi \, d\mu.$$

The map $L_{\tilde{f}}$ is linear, and

$$L_{\tilde{f}}(\mathcal{M}_{\tilde{f}}(\Lambda)) = \rho_{mes}(\Lambda, f).$$

Let w be an extremal point of $\rho_{mes}(\Lambda, f)$. We show now that there is $x \in \Lambda$ such that $\rho(x, f) = w$. Recall the following fact from convex analysis: if $T : E_1 \to E_2$ is a linear map between vector spaces, and $C \subset E_1$ is convex, then $T(C)$ is a convex subset of E_2 and for any extremal point $v \in T(C)$, the set $T^{-1}(v) \subset C$ contains an extremal point of C (see for ex. [Roc97]). As $L_{\tilde{f}}$ is linear and the sets $\mathcal{M}_{\tilde{f}}(\Lambda)$ and $\rho_{mes}(\Lambda, f)$ are convex, we have that the preimage by $L_{\tilde{f}}$ of the extremal point w contains an extremal point of $\mathcal{M}_{\tilde{f}}(\Lambda)$, that is, an ergodic measure μ for $\tilde{f}$ with support in Λ. By Birkhoff's Theorem, there exists $x \in \mathrm{supp}(\mu) \subset \Lambda$ such that

$$\int \phi \, d\mu = \lim_n \frac{1}{n} \sum_{i=0}^{n-1} \phi(\tilde{f}^n(x)) = \lim_n \frac{f^n(x) - x}{n} = \rho(x, f).$$

As $w = L_{\tilde{f}}(\mu) = \int \phi d\mu$, we have that $\rho(x, f) = w$, as desired. With this we have that the set of extremal points of $\rho_{mes}(\Lambda, f)$ is contained in $\rho(\Lambda, f)$, and as mentioned above, this gives us the inclusion

$$\rho_{mes}(\Lambda, f) \subset \mathrm{conv}(\rho(\Lambda, f)).$$

Then $\rho_{mes}(\Lambda, f) = \mathrm{conv}(\rho(\Lambda, f))$, and this finishes the proof of the first claim of the proposition.

For the second claim, let w be an extremal point of $\mathrm{conv}\, \rho(\Lambda, f) = \rho_{mes}(\Lambda, f)$. Then, just notice that we proved that the set $L_{\tilde{f}}^{-1}(w)$ contains an ergodic measure μ, such that $\rho(\mu, f) = L_{\tilde{f}}(\mu) = w$, as desired. This finishes the proof of the lemma. ■

2.2 Brouwer Theory.

In [Bro12], Brouwer proved the following theorem for homeomorphisms of the plane, known as the Brouwer Translation Theorem:

Theorem 2.11. *Let h be a homeomorphism of $\mathbf{R}^2$ without fixed points. Then:*

1. *For all point $x \in \mathbf{R}^2$ there exists a line ℓ passing through x such that*

$$\ell \prec h(\ell) \quad \text{and} \quad h^{-1}(\ell) \prec \ell.$$

2. *There exists a cover of $\mathbf{R}^2$ by open invariant disks where h is conjugate to a translation.*

A line satisfying item (1) is called a *Brouwer line* for h. By item (2) we have that h has no periodic points, and moreover, every point is wandering for h. The proofs of this theorem use the Brouwer Translation Lemma, which states that if a homeomorphism of the plane has no fixed points, then it has no periodic points. In [Fra88] Franks proved the following stronger property of non-recurrence:

Theorem 2.12 (Franks' Lemma). *If h is a plane homeomorphism without fixed points, then there does not exist a sequence $(U_i)_{i\in\mathbf{Z}/n\mathbf{Z}}$ of pairwise disjoint open disks and a sequence of integers $(m_i)_{i\in\mathbf{Z}/n\mathbf{Z}}$ such that*

$$f^{m_i}(U_i)\cap U_{i+1}\neq\emptyset \quad \text{for all } i\in\mathbf{Z}/n\mathbf{Z}.$$

As a corollary one obtains the following.

Theorem 2.13 ([Fra89]). *Let h be an orientation preserving homeomorphism of the plane, without fixed points and which is the lift of a homeomorphism of $\mathbf{T}^2$. Then, there exists $\epsilon>0$ such that there are no periodic ϵ-chains for h.*

In [Cal04], Le Calvez showed the following remarkable and much stronger version of the Brouwer Translation Theorem.

Theorem 2.14. *Let h be a plane homeomorphism without fixed points. There exists a topological oriented foliation $\mathcal{F}$ of the plane such that each leaf of $\mathcal{F}$ is a Brouwer line for h.*

In [Cal05] it is proved the following equivariant version of Theorem 2.14.

Theorem 2.15. *Let G be a discrete group of orientation preserving homeomorphisms of the plane, whose action is free and proper. Let h be a plane homeomorphism without fixed points that commutes with the elements of G. Then, there exists a topological oriented foliation $\mathcal{F}$ of the plane, invariant under the action of G, and such that each leaf of $\mathcal{F}$ is a Brouwer line for h.*

Then, for example, if h is the lift of a torus homeomorphism, then h commutes with the elements of the group G generated by the horizontal and vertical translations $T_1:(x,y)\mapsto(x+1,y)$ and $T_2:(x,y)\mapsto(x,y+1)$, and we obtain a topological oriented foliation of the plane by Brouwer curves which projects to a topological oriented foliation of $\mathbf{T}^2$.

Passing to the universal cover, one can prove that the following theorem is equivalent to Theorem 2.15 (see [Cal05]).

Theorem 2.16. *Let M be a surface and $(H_t)_{t\in[0,1]}$ an isotopy in M joining the identity to a homeomorphism f. For all $z\in M$ we define the arc $\gamma_z:t\mapsto H_t(z)$. We suppose that f does not have any contractible fixed point z, that is, a fixed point z such that γ_z is a closed curve homotopic to a point. Then there exists an oriented topological foliation $\mathcal{F}$ in M and for all $z\in M$ an arc positively transverse to $\mathcal{F}$ joining z to $f(z)$ that is homotopic with fixed extremes to the arc γ_z.*

In [Cal05], as an application of Theorem 2.16, it is proved the following Theorem. The statement in [Cal05] (Theorem 9.1) is for orbits, instead of ϵ-chains as it is stated here. However, the result is easily adaptable for the case of ϵ-chains (see Proposition 8.2 in that article). We include a sketch of the proof.

Theorem 2.17. *Let $\tilde{f} : \mathbf{T}^2 \to \mathbf{T}^2$ be a homeomorphism isotopic to the identity without contractible fixed points. Fix an isotopy $(\tilde{H}_t)_{t\in[0,1]}$ in $\mathbf{T}^2$ between $\tilde{f}$ and the identity, and let $\mathcal{F}$ be the foliation of $\mathbf{T}^2$ transverse to $(\tilde{H}_t)_{t\in[0,1]}$ given by Theorem 2.16. Let $(H_t)_{t\in[0,1]}$ be the isotopy in $\mathbf{R}^2$ which is the lift of $(\tilde{H}_t)$ and satisfies $H_0 = \mathrm{Id}$, and let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be the lift of $\tilde{f}$ given by $f = H_1$. Let $\hat{\mathcal{F}}$ be the lift of $\mathcal{F}$ to $\mathbf{R} \times S^1$.*

There exists $\epsilon > 0$ such that, if $\hat{x}, \hat{y} \in \mathbf{R} \times S^1$, are points with lifts $x, y \in \mathbf{R}^2$ and:

- *there is an ϵ-chain for f from x to $x + (0, m)$ for some $m \in \mathbf{N}$, and*
- *there is an ϵ-chain for f from y to $y + (0, -n)$ for some $n \in \mathbf{N}$,*

then there exists a compact leaf $l \in \hat{\mathcal{F}}$ which is an essential curve that separates $\hat{x}$ from $\hat{y}$ (that is, $\hat{x}$ and $\hat{y}$ belong to different connected components of $\mathbf{R} \times S^1 \setminus l$). In particular $\hat{x} \neq \hat{y}$.

Sketch of the Proof. Let $F : \mathbf{R}\times S^1 \to \mathbf{R}\times S^1$ be the lift of $\tilde{f}$ such that $F\circ\pi = \pi\circ f$. Let $(\hat{H}_t)_{t\in[0,1]}$ be the isotopy in $\mathbf{R} \times S^1$ between F and the identity which is the lift of the isotopy $(\tilde{H}_t)_{t\in[0,1]}$. By Theorem 2.16, for every $\hat{x} \in \mathbf{R} \times S^1$ there exists an arc which is positively transverse to $\hat{\mathcal{F}}$, joins $\hat{x}$ to $F(\hat{x})$ and is homotopic with fixed extremes to the arc $\gamma_{\hat{x}} :\mapsto \hat{H}_t(\hat{x})$. By this, one can easily see that, for any $\hat{x} \in \mathbf{R} \times S^1$ there exists $\epsilon > 0$ such that any point $\hat{z}$ in $B_\epsilon(\hat{x})$ can be joined to any point $\hat{z}'$ in $B_\epsilon(F(\hat{x}))$ by an arc which is positively transverse to $\hat{\mathcal{F}}$ and homotopic to an arc of the form $\gamma_{\hat{z}\hat{x}}\gamma_{\hat{x}}\gamma_{F(\hat{x})\hat{z}'}$, where $\gamma_{\hat{z}\hat{x}}$ joins $\hat{z}$ to $\hat{x}$ in $B_\epsilon(\hat{x})$ and $\gamma_{F(\hat{x})\hat{z}'}$ joins $F(\hat{x})$ to $\hat{z}'$ in $B_\epsilon(F(\hat{x}))$, and where the product of two arcs stands for their concatenation.

As F is the lift of the homeomorphism $f : \mathbf{T}^2 \to \mathbf{T}^2$, and as $\mathbf{T}^2$ is compact, there exists $\eta > 0$ such that for any point $\hat{x} \in \mathbf{R} \times S^1$, any point in $B_\eta(\hat{x})$ can be joined to any point in $B_\eta(F(\hat{x}))$ by an arc positively transverse to $\hat{\mathcal{F}}$ as above. Also, by the continuity of F, there is $0 < \epsilon < \eta$ such that for any $\hat{x} \in \mathbf{R} \times S^1$, if $\{\hat{x}_i\}_{i=0}^n$ is a periodic ϵ-chain for F with $\hat{x}_0 = \hat{x}_n = \hat{x}$, then $\hat{x}_{n-1} \in B_\eta(F^{-1}(\hat{x}))$.

Suppose then that there are $\hat{x}, \hat{y} \in \mathbf{R} \times S^1$ with lifts $x, y \in \mathbf{R}^2$ such that there is an ϵ-chain $\{x_i\}_{i=0}^{n_1}$ for f with $x_0 = x$ and $x_{n_1} = x + (0, m)$ for some $m \in \mathbf{N}$, and an ϵ-chain $\{y_i\}_{i=0}^{n_2}$ for f with $y_0 = y$ and $y_{n_2} = y + (0, -n)$ for some $n \in \mathbf{N}$. Then, we can construct a sequence of arcs $(\gamma_n)_{n=1}^{n_1}$ positively transverse to $\mathcal{F}$, and such that:

- γ_1 joins x_0 to $f(x_0)$,
- γ_i joins $f(x_{i-2})$ to $f(x_{i-1})$ for $2 \leq i \leq n_1 - 2$,
- γ_{n_1-1} joins $f(x_{n_1-3})$ to $f^{-1}(x + (0, m))$, and
- γ_{n_1} joins $f^{-1}(x + (0, m))$ to $x + (0, m)$.

Then, letting $\gamma = \Pi_{i=1}^{n_1} \gamma_i$, we have that γ is an arc positively transverse to $\mathcal{F}$ joining x to $x + (0, m)$.

Analogously, we construct an arc β positively transverse to $\mathcal{F}$ and joining y to $y + (0, -n)$. In [Cal05] it is proved that γ and β project to disjoint (not necessarily simple) loops $\tilde{\gamma}$, and $\tilde{\beta}$ in $\mathbf{R} \times S^1$, and there is a connected component U of $\mathbf{R} \times S^1 \setminus (\tilde{\gamma} \cup \tilde{\beta})$ which is a topological essential annulus. As $\tilde{\gamma}$ and $\tilde{\beta}$ are positively transverse to $\hat{\mathcal{F}}$, then $\hat{\mathcal{F}}$ is transverse to the border of U, either inwards or outwards. By the Poincaré Bendixon theorem, there exists a closed essential leaf l contained in U. As the points $\hat{x}$ and $\hat{y}$ belong to the border of U, l separates x from y. ■

2.3 Atkinson's Lemma.

Let $T : X \to X$ be a measurable map of the metric space X, and suppose that T is ergodic with respect to a probability measure μ. Let $\varphi \in L^1(\mu)$.

Then, Birkhoff's theorem tells us that for μ-almost every point $x \in X$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \varphi(T^i(x)) = \int \varphi d\mu,$$

that is, the temporal averages for x and φ converge to the spacial average of φ. In particular, if $\int \varphi d\mu = 0$, for μ-a. e. x

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \varphi(T^i(x)) = 0.$$

In this case, however, one may have that the Birkhoff sums diverge: there can be a measurable set $A \subset X$ of positive μ-measure such that for any $x \in A$,

$$\limsup_{n\to\infty} \left| \sum_{i=0}^{n} \varphi(T^i(x)) \right| = \infty.$$

On the other hand, by the Poincaré Recurrence Theorem, μ-almost every point $x \in X$ is recurrent: for any open set U containing x, there is $n \in \mathbf{N}$ such that $f^n(x) \in U$.

The following Theorem, known as Atkinson's Lemma, tells us that there is a total measure subset of X such that for any point x in this set we can find infinitely many iterates $T^n(x)$ with both recurrence and small Birkhoff sums.

Theorem 2.18 ([Atk76])**.** *Let (X, μ) be a probability space, and suppose that μ is ergodic with respect to a measurable transformation $T : X \to X$. Let $\phi : X \to \mathbf{R}$ be a measurable function with $\int \phi d\mu = 0$. Then, there exists a full measure set $\widetilde{X} \subset X$ such that for any $x \in \widetilde{X}$, any $\epsilon > 0$, and any set of positive measure $A \subset X$ containing x, it holds that*

$$T^n(x) \in A \quad \text{and} \quad \left| \sum_{i=0}^{n-1} \phi(T^i(x)) \right| < \epsilon$$

for infinitely many values of $n \in \mathbf{N}$.

3 Proof of Theorem A from Theorem B.

If $\rho(f)$ has irrational slope and contains a rational point, then by Theorem 2.8 this point is realized by a periodic orbit. Then we are left then with the case that $\rho(f)$ has rational slope and contains rational points.

We suppose there is a rational point $v = (p_1/q, p_2/q)$ contained in $\rho(f)$ that is not realized by a periodic orbit, an we will prove there is a periodic essential annular set for f which is a semi-attractor. By Remark 2.5, there is $A \in \mathrm{GL}(2, \mathbf{Z})$ such that $\rho(AfA^{-1}) = A\rho(f)$ is a vertical segment, containing the rational point $v' = (p'_1/q, p'_2/q)$ given by $v' = Av$. By Remark 2.2, if $g_0 = (AfA^{-1})^{q'}$, then $\rho(g_0) = q'\rho(AfA^{-1})$, and then $\rho(g_0)$ is a vertical interval containing the point $w = q'v' = (p'_1, p'_2) \in \mathbf{Z}^2$. We know that $\rho(T_1^{-p'_1}T_2^{-p'_2}g_0) = T_1^{-p'_1}T_2^{-p'_2}\rho(g_0)$, and therefore, if $g = T_1^{-p'_1}T_2^{-p'_2}g_0$, $\rho(g)$ is a vertical interval containing the point $T_1^{p'_1}T_2^{p'_2}(w) = (0,0)$. Let $\tilde{A}$, $\tilde{g}$ and $\tilde{g}_0$ be the homeomorphisms of $\mathbf{T}^2$ lifted by A, g and g_0, respectively. Then $\tilde{g} = \tilde{g}_0$, and as $g_0 = (AfA^{-1})^{q'} = Af^{q'}A^{-1}$ we have that $\tilde{g}$ and $\tilde{f}^{q'}$ are conjugate by $\tilde{A}$.

We now show that g has no fixed points. As $v = (p_1/q, p_2/q) \in \rho(f)$ is not realized by a periodic point for $\tilde{f}$, then $v' = (p'_1/q', p'_2/q') \in \rho(AfA^{-1})$ is not realized by a periodic point for $\tilde{A}\tilde{f}\tilde{A}^{-1}$, and then $w = q'v' = (p'_1, p'_2) \in \rho(g_0)$ is not realized by a fixed point for $\tilde{g}_0$. Therefore $(0,0) = T_1^{p'_1}T_2^{p'_2}(w) \in \rho(g)$ is not realized by a fixed point for $\tilde{g}$, and therefore g has no fixed points.

Then, by Theorem B, there exists an annular, essential, $\tilde{g}$-invariant set $B \subset \mathbf{T}^2$, which is a semi-attractor. As $\tilde{f}^{q'}$ and $\tilde{g}$ are conjugate by $\tilde{A}$, $\tilde{A}(B) \subset \mathbf{T}^2$ is an annular set such that $\tilde{f}^{q'}(\tilde{A}(B)) = \tilde{A}(B)$. The property of having a periodic annular set which is a semi-attractor is clearly invariant by topological conjugacy, and this finishes the proof of Theorem A.

4 Proof of Theorem B.

4.1 Outline of the proof.

In section 4.2 we will define a simple but useful property satisfied by compact connected sets in $\mathbf{R}^2$. Informally, that property is defined as follows. A compact connected set $C \subset \mathbf{R}^2$ is said to satisfy property **PL** with respect to $p \in \mathbf{R}^2$ if $\mathrm{pr}_2(C)$ is 'large' and $p \in \mathbf{R}^2$ is 'to the left' of C. Then, it is proved that if $h : \mathbf{R}^2 \to \mathbf{R}^2$ is a homeomorphism and C satisfies property **PL** with respect to p, then $h(C)$ intersects $R(v(h(p)))$, where $v(h(p))$ is the straight vertical line passing through $h(p)$ oriented upwards (see Fig. 6). That is, if $\mathrm{pr}_2(C)$ is 'large enough', and p is 'to the left' of C, then C is *pushed* to the right by p under iteration by h. The term 'large enough' will depend on the homeomorphism h. This property will be an important tool in the proof of the theorem.

In section 4.3 we prove a preliminar version of Theorem B. Namely, it is proved the existence of a finite family of simple, closed, vertical, pairwise disjoint curves $\tilde{l}_i \subset \mathbf{T}^2$, such that for each i, the maximal invariant set Θ_i of $[\tilde{l}_i, \tilde{l}_{i+1}]$ is non-empty, and $\rho(\tilde{\Theta}_i, f)$ is contained either in $\{0\} \times (\epsilon, \infty)$ or in $\rho(\tilde{\Lambda}_i, f) \subset \{0\} \times (-\infty, \epsilon)$, for some $\epsilon > 0$. This will give us that the family $\{\tilde{l}_i\}$ is such that it holds item (2) from Theorem B (see Fig. 3).

The essence of the proof of this preliminary version comes from the results of [Cal05] (see Theorem 1.2). However, this still does not allow us to conclude Theorem B; *a priori* the curves $\tilde{l}_i$ could be not free forever for $\tilde{f}$, or they could even be not free for $\tilde{f}$. As explained in the introduction, there could be points in $\mathbf{T}^2$ spinning sidewards at a sublinear speed; that is, $\tilde{x} \in \mathbf{T}^2$ such that $\lim_{n\to\infty} |\mathrm{pr}_1(f^n(x))| = \infty$, and $\limsup_{n\to\infty} |\mathrm{pr}_1(f^n(x) - x)|/n = 0$, for $x \in \pi'^{-1}(\tilde{x})$. In section 4.4 we will prove that the curves $\tilde{l}_i$ are free forever for $\tilde{f}$. Finally, in section 4.5 we will prove it holds items (1) and (4) from the theorem; that is, for each i the set $\Omega(\tilde{f}) \cap [\tilde{l}_i, \tilde{l}_{i+1}]$ is non-empty and contained in Θ_i, and at least one of the sets Θ_i is an annular and essential semi-attractor.

4.2 Properties PR and PL.

Definition 4.1. Let $h : \mathbf{R}^2 \to \mathbf{R}^2$ be a lift of a homeomorphism $\tilde{h} : \mathbf{T}^2 \to \mathbf{T}^2$, with $\tilde{h}$ isotopic to the identity. Let $C \subset \mathbf{R}^2$ compact and connected, $k \in \mathbf{R}^+$ and $p \in \mathbf{R}^2$. We say that C satisfies the property $\mathbf{PL}(k, p)$ if the following hold (see Fig. 6):

1. There exist horizontal (disjoint) straight lines $r_1 \prec r_2$, (oriented as going to the right) such that $r_1 \cap C \neq \emptyset$, $r_2 \cap C \neq \emptyset$, and such that the strip $(r_1, r_2) \subset \mathbf{R}^2$ contains a ball of radius k centered in p.
2. The point p belongs to the (unique) connected component of $(r_1, r_2) \backslash C$ which is unbounded to the left.

Analogously, we say that C satisfies the property $\mathbf{PR}(k, p)$ if it holds item (1) from property $\mathbf{PL}(k, p)$ and p belongs to the (unique) connected component of $(r_1, r_2) \setminus C$ which is unbounded to the right.

The following lemma will be a crucial tool in the proof of Theorem B.

Lemma 4.2. *Let $h : \mathbf{R}^2 \to \mathbf{R}^2$ be a lift of a homeomorphism $\tilde{h} : \mathbf{T}^2 \to \mathbf{T}^2$ isotopic to the identity, and for $x \in \mathbf{R}^2$, denote by $v(x) \subset \mathbf{R}^2$ the vertical straight line that passes through x. There exists $k > 0$ such that if a compact connected set $C \subset \mathbf{R}^2$ satisfies $\mathbf{PL}(k, p)$ (resp. $\mathbf{PR}(k, p)$) for some $p \in \mathbf{R}^2$, then $h(C) \cap R(v(h(p))) \neq \emptyset$ (resp. $h(C) \cap L(v(h(p))) \neq \emptyset$, see Fig. 6).*

Proof. First observe that as h is the lift of a homeomorphism of $\mathbf{T}^2$, $\|h - \mathrm{Id}\|_0 < \infty$. Define $k = 2\|h - \mathrm{Id}\|_0 + 1$. Suppose that C satisfies the property $\mathbf{PL}(k, p)$ for some $p \in \mathbf{R}^2$ (the case of $\mathbf{PR}(k, p)$ is similar). Then there are two horizontal straight lines $r_1 \prec r_2$ intersecting C and such that (r_1, r_2) contains a ball of radius k centered in p. Observe that by the definition of k, $\min \mathrm{pr}_2(h(r_1)) > h(p)_2 > \max \mathrm{pr}_2(h(r_2))$, and then if w is the horizontal straight line passing through $h(p)$, we have

$$w \subset (h(r_1), h(r_2)). \tag{2}$$

Define U_L (resp. U_R) to be the connected component of $(r_1, r_2) \setminus C$ unbounded to the left (resp. right). As $\|h - \mathrm{Id}\|_0 < \infty$, $h(U_L)$ is unbounded to the left and bounded to the right, and also $h(U_R)$ is unbounded to the right and bounded to the left.

We claim that for this choice of k, we have $h(C) \cap R(v(h(p))) \neq \emptyset$. If this was not the case, then we would have that $C \cap w_+ = \emptyset$, where $w_+ = w \cap R(v(h(p)))$. By (2), $w_+ \subset (h(r_1), h(r_2))$, and therefore w_+ is contained in $h(U_R)$. Then $h(p)$ belongs to $h(U_R)$, which is unbounded to the right, which contradicts the fact that p belongs to a connected component of $(r_1, r_2) \setminus C$ bounded to the right. We must have then that $h(C) \cap R(v(h(p))) \neq \emptyset$, and this proves the lemma. ∎

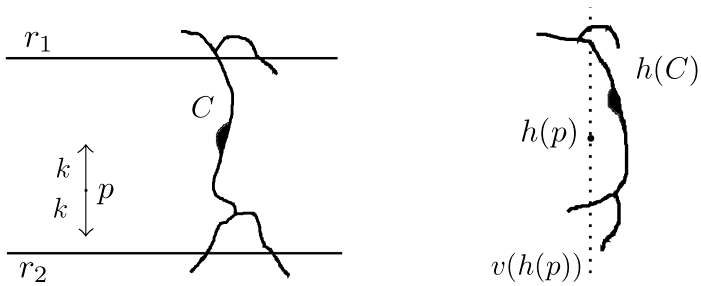

Figure 6: Left: a set C satisfying property $\mathbf{PL}(k,p)$. Right: $h(C) \cap R(v(h(p))) \neq \emptyset$.

4.3 Construction of the curves $\tilde{l}_i$ and proof of item (2).

The following proposition gives us a finite family of curves for which it holds item (2) of Theorem B. It is an analogous of Proposition 2.35 in [Cal91] for the torus case.

Proposition 4.3. *Within the hypotheses of Theorem B, there exists a finite family of pairwise disjoint, simple, closed, essential curves $\tilde{l}_i \subset \mathbf{T}^2$ such that if Θ_i is the maximal invariant set of $[\tilde{l}_i, \tilde{l}_{i+1}]$, then Θ_i is non-empty, and $\rho(\Theta_i, f)$ is contained either in $\{0\} \times (0, \infty)$ or in $\{0\} \times (-\infty, 0)$.*

Also, the family $\{\tilde{l}_i\}$ can be chosen such that, for any $i \in \mathbf{Z}/r\mathbf{Z}$, if $\rho(\Theta_i, f) \subset \{0\} \times (0, \infty)$ then $\rho(\Theta_{i+1}, f) \subset \{0\} \times (-\infty, 0)$, and if $\rho(\Theta_i, f) \subset \{0\} \times (-\infty, 0)$ then $\rho(\Theta_{i+1}, f) \subset \{0\} \times (0, \infty)$.

Remark 4.4. As the sets Θ_i are compact, the sets $\rho(\Theta_i, f)$ are also compact (see Theorem 2.3). Therefore, the fact that $\rho(\Theta_i, f)$ is contained either in $\{0\} \times (0, \infty)$ or in $\{0\} \times (-\infty, 0)$ means actually that there is $\epsilon > 0$ such that $\rho(\Theta_i, f)$ is contained either in $\{0\} \times (\epsilon, \infty)$ or in $\{0\} \times (-\infty, -\epsilon)$, and therefore for any i, every point in Θ_i rotates with linear speed either 'upwards' or 'downwards'; that is, for a fixed i, either $\liminf_{n\to\infty}(f^n(x) - x)_2/n > 0$ for every $x \in \Theta_i$, or $\limsup_{n\to\infty}(f^n(x) - x)_2/n < 0$ for every $x \in \Theta_i$.

To prove Proposition 4.3 it will be convenient to work on the lift $\mathbf{R} \times S^1$ of $\mathbf{T}^2$. We will first prove the following.

Lemma 4.5. *For $\tilde{f}$ and f as in Theorem B, let $F : \mathbf{R} \times S^1 \to \mathbf{R} \times S^1$ be the lift of $\tilde{f}$ such that $F \circ \pi = \pi \circ f$. Then:*

1. *$CR(F) \neq \emptyset$, and $CR(F) = \Lambda^+ \cup \Lambda^-$, where Λ^+ and Λ^- are closed disjoint*

F-invariant sets such that, denoting $\widetilde{\Lambda}^{\pm} = \pi''(\Lambda^{\pm}) \subset \mathbf{T}^2$, we have $\rho(\widetilde{\Lambda}^{+}, f) \subset \{0\} \times (\epsilon, \infty)$ and $\rho(\widetilde{\Lambda}^{-}, f) \subset \{0\} \times (-\infty, -\epsilon)$, for some $\epsilon > 0$.

2. *There exist simple, closed, essential curves $l_0 \prec l_1 \prec \cdots l_r = T_1(l_0)$ on $\mathbf{R} \times S^1$ which are free for F, and such that they 'separate' Λ^+ from Λ^-, that is:*

   (a) $CR(F) \bigcap \cup_{i=0}^{r} l_i = \emptyset$,

   (b) for $0 \leq i < r$, the set $\Lambda_i := CR(F) \cap (l_i, l_{i+1})$ is compact, non-empty and F-invariant,

   (c) for $0 \leq i < r$, either $\Lambda_i \subset \Lambda^+$ or $\Lambda_i \subset \Lambda^-$, and

   (d) if $\Lambda_i \subset \Lambda^+$, then $\Lambda_{i+1} \subset \Lambda^-$, and if $\Lambda_i \subset \Lambda^-$ then $\Lambda_{i+1} \subset \Lambda^+$, for any $0 \leq i < r - 1$ (see Fig. 7).

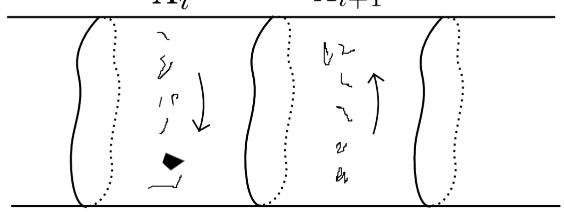

Figure 7: The sets Λ_i and the curves l_i.

Proof. First we observe the following elementary fact. There exists an isotopy $(\tilde{H}_t)_{t\in[0,1]}$ between the identity and $\tilde{f}$ with the property that if (H_t) is the lift of $(\tilde{H}_t)$ with $H_0 = \mathrm{Id}$, then $H_1 = f$. To see this just observe that if $(\tilde{H}'_t)_{t\in[0,1]}$ is any isotopy between the identity and $\tilde{f}$, and if (H'_t) is the lift of $(\tilde{H}'_t)$ with $H'_0 = \mathrm{Id}$, then $H'_1 = f + (a, b)$, for some $a, b \in \mathbf{Z}$. Defining $H_t = H'_t + t(-a, -b)$, for $t \in [0, 1]$, we have that (H_t) is an isotopy between the identity and f wich projects to an isotopy $(\tilde{H}_t)$ on $\mathbf{T}^2$ between the identity and $\tilde{f}$ with the desired properties.

Now, let $\mathcal{F}$ be the Brouwer foliation of $\mathbf{T}^2$ transversal to $\tilde{H}$ given by Theorem 2.16. Let $\hat{\mathcal{F}}$ be the lift of $\mathcal{F}$ to $\mathbf{R} \times S^1$.

Part 1.
$CR(F)$ **is non-empty**. Let $\phi : \mathbf{T}^2 \to \mathbf{R}$ be given by

$$\phi(\tilde{x}) = f(x)_1 - x_1,$$

where $x \in \mathbf{R}^2$ is any lift of $\tilde{x}$, and let $\phi_1 = \mathrm{pr}_1 \circ \phi$. Then

$$\sum_{i=0}^{n-1} \phi_1(\tilde{f}^i(\tilde{x})) = f^n(x)_1 - x_1.$$

Let μ be any ergodic measure for f. By hypothesis, $\int \phi_1 d\mu = \rho(\mu, f)_1 = 0$, and by Atkinson's Theorem 2.18 there exists a full μ-measure set $X \subset \mathbf{T}^2$ such that for any $\tilde{x} \in X$, $x \in \pi'^{-1}(\tilde{x})$ and $\epsilon > 0$ we have that

$$d(\tilde{f}^n(\tilde{x}), \tilde{x}) < \epsilon, \quad \text{and} \quad \left| \sum_{i=0}^{n-1} \phi_1(\tilde{f}^i(\tilde{x})) \right| = |f^n(x)_1 - x_1| < \epsilon$$

for infinitely many values of $n \in \mathbf{N}$. This means that $\pi(x)$ is recurrent for F, and in particular $CR(F) \neq \emptyset$.

Definition of Λ^+ and Λ^-. As f has no fixed points by hypothesis, by Franks' Lemma 2.13 there is $\epsilon_1 > 0$ such that f has no periodic ϵ_1-chains, and by Theorem 2.17 there is $\epsilon_2 > 0$ such that, if there are $\hat{x}, \hat{y} \in \mathbf{R} \times S^1$ with lifts $x, y \in \mathbf{R}^2$ such that:

- there is an ϵ_2-chain for f from x to $x + (0, m)$ for some $m \in \mathbf{N}$, and
- there is an ϵ_2-chain for f from y to $y + (0, -n)$ for some $n \in \mathbf{N}$,

then there exists a compact leaf of $\hat{\mathcal{F}}$ that separates $\hat{x}$ from $\hat{y}$. Let $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$.

We define $\Lambda^+ \subset \mathbf{R} \times S^1$ as the set of points $\hat{x} \in CR(F)$ such that, if $x \in \pi^{-1}(\hat{x})$, there exists an ϵ_0-chain $\{x_i\}_{i=0}^n$ for f, with $x_0 = x$ and $x_n = x + (0, m)$ for some $m \in \mathbf{N}$. Analogously, we define $\Lambda^- \subset \mathbf{R} \times S^1$ as the set of points $\hat{x} \in CR(F)$ such that, if $x \in \pi^{-1}(\hat{x})$, there exists an ϵ_0-chain $\{x_i\}_{i=0}^n$ for f, with $x_0 = x$ and $x_n = x + (0, -m)$, for some $m \in \mathbf{N}$.

Λ^+ and Λ^- are non-empty. We prove that Λ^+ is non-empty; the case of Λ^- is similar. By our hypotheses and by Proposition 2.10, there exists an ergodic measure μ with respect to $\tilde{f}$ with $\rho(\mu)_2 > 0$. By Birkhoff's Theorem, there exists a set $X \subset \mathbf{T}^2$ of full μ-measure such that for $\tilde{x} \in X$ and $x \in \pi'^{-1}(\tilde{x})$, we have

$$\rho(\tilde{x}, f) = \lim_n \sum_{i=0}^{n} \phi(\tilde{f}^i(\tilde{x})) = \int \phi d\mu = \rho(\mu, f).$$

By Atkinson's Lemma 2.18, there exists a full measure set $X' \subset \mathbf{T}^2$ such that if $\tilde{x} \in X'$ and $x \in \pi'^{-1}(\tilde{x})$, then for all $\epsilon > 0$ there are infinitely many values of

$n > 0$ such that

$$d(\tilde{f}^n(\tilde{x}), \tilde{x}) < \epsilon \quad \text{and} \quad \left| \sum_{i=0}^{n-1} \phi_1(\tilde{f}^i(\tilde{x})) \right| = |f^n(x)_1 - x_1| < \epsilon$$

for and for infinitely many values of $n \in \mathbf{N}$.

Let $\tilde{y} \in X \cap X'$ and $y \in \pi'^{-1}(\tilde{y})$. Then, given $\epsilon > 0$ there are increasing sequences $\{r_n\}_n$ and $\{s_n\}_n$ of integers such that

$$|f^{r_n}(y) - y - (0, s_n)| < \epsilon,$$

and also $\rho(\tilde{y}, f)_2 = \rho(\mu, f)_2 > 0$. Therefore $\lim_n s_n = \infty$, and in particular $s_n > 0$ for n sufficiently large. As $d(\hat{y}, F^{r_n}(\hat{y})) < \epsilon$, and as the choice of $\epsilon > 0$ was arbitrary, we have that $\hat{y}$ is recurrent for F, and in particular $\hat{y} \in CR(F)$. Therefore $\hat{y} \in \Lambda^+$, and Λ^+ is non-empty.

It holds $CR(F) = \Lambda^+ \cup \Lambda^-$. Observe that by the definition of the sets Λ^+ and Λ^-, we have that $\Lambda^+ \cup \Lambda^- \subset CR(F)$, and then we only need to prove that $CR(F) \subset \Lambda^+ \cup \Lambda^-$. Suppose by contradiction that there is $\hat{x} \in CR(F) \setminus (\Lambda^+ \cup \Lambda^-)$. By definition of Λ^+ and Λ^-, there exists an ϵ_0-chain for f starting and ending in x, that is, a periodic ϵ_0-chain for f. By definition of ϵ_0, we have $\epsilon_0 \leq \epsilon_1$, where ϵ_1 is the constant given by Franks' Lemma 2.13, and therefore that lemma implies that there is a fixed point for f, a contradiction. Therefore we must have $CR(F) \subset \Lambda^+ \cup \Lambda^-$ as we wanted.

The sets Λ^+ and Λ^- are disjoint and closed. We will prove that $\overline{\Lambda^+} \cap \overline{\Lambda^-} = \emptyset$. As $CR(F) = \Lambda^+ \cup \Lambda^-$ and $CR(F)$ is closed, this will imply that Λ^+ and Λ^- are closed and disjoint. Suppose by contradiction that there is $\hat{x} \in \overline{\Lambda^+} \cap \overline{\Lambda^-}$. Let $\hat{y} \in \Lambda^+$ and $\hat{z} \in \Lambda^-$, be such that $d(\hat{y}, \hat{x}) < \epsilon_0/3$ and $d(\hat{z}, \hat{x}) < \epsilon_0/3$, and let $\{y_i\}_{i=0}^{n_1}$, and $\{z_i\}_{i=0}^{n_2}$ be $\epsilon_0/3$-chains for f such that $y_0 \in \pi^{-1}(\hat{y})$, $y_{n_1} = y_0 + (0, m_1)$, $z_0 \in \pi^{-1}(\hat{z})$, $|z_0 - y_0| < 2\epsilon_0/3$ and $z_{n_2} = y_0 + (0, -m_2)$, for some $m_1, m_2 \in \mathbf{N}$. We now show that we can concatenate integer translates of these chains $\{y_i\}$ and $\{z_i\}$ to get a periodic chain for f. For each $0 \leq i < m_2$ define the $\epsilon_0/3$-chain $\{y_l^i\}_{l=0}^{n_1}$ for f as the translate of $\{y_l\}_{l=0}^{n_1}$ by $T_2^{im_1}$, that is,

$$y_l^i = T_2^{im_1} y_l, \quad \text{for } 0 \leq l < n_1,$$

and for each $0 \leq j < m_1$, define the $\epsilon_0/3$-chain $\{z_k^j\}_{k=0}^{n_2}$ for f as the translate of $\{z_k\}_{k=0}^{n_2}$ by $T_2^{m_1 m_2 - j m_2}$, that is,

$$z_k^j = T_2^{m_1 m_2 - j m_2} z_k, \quad \text{for } 0 \leq k < n_2.$$

Define then the ϵ_0-chain $\{w_i\}_{i=0}^{n_1m_2+n_2m_1}$ for f as the concatenation of the chains defined above, given by

$$w_{in_1+l} = y_l^i, \quad \text{for } 0 \leq i < m_2 \text{ and } 0 \leq l < n_1,$$

$$w_{m_2n_1+jn_2+k} = z_k^j \quad \text{for } 0 \leq j < m_1 \text{ and } 0 \leq k < n_2, \text{ and}$$

$$w_{n_1m_2+n_2m_1} = w_0.$$

Then, $\{w_i\}_{i=0}^{n_1m_2+n_2m_1}$ is a periodic ϵ_0-chain for f. By Franks' Lemma 2.13 this is a contradiction, and therefore there cannot be $\hat{x} \in \overline{\Lambda^+} \cap \overline{\Lambda^-}$. As we mentioned, this implies that Λ^+ and Λ^- are closed and disjoint.

Before proving that the sets Λ^+ and Λ^- are F-invariant and the last claim of Part 1, we will prove Part 2.

Part 2.
Construction of the family $\{l_i\}_{i=0}^r$. By Theorem 2.17 and by the definition of the sets Λ^+ and Λ^-, for each $x \in \Lambda^+$, $y \in \Lambda^-$, there exists a compact leaf $l \in \hat{\mathcal{F}}$ that separates x from y. So, the set $\mathcal{F}_c$ of compact leaves of $\hat{\mathcal{F}}$ is not empty. The union of the compact leaves of a foliation of $\mathbf{T}^2$ is compact (see for ex. [Hae62]), and as $\hat{\mathcal{F}}$ is a lift of a foliation of $\mathbf{T}^2$, the set $\cup\mathcal{F}_c$ is closed as a subset of $\mathbf{R} \times S^1$ ($\cup\mathcal{F}_c$ denotes the union of the elements of $\mathcal{F}_c$). Observe that, as the leaves of $\mathcal{F}$ are Brouwer lines for f, the elements of $\mathcal{F}_c$ are free curves for F.

Claim: $CR(F) \cap \mathcal{F}_c = \emptyset$.
Let $l \in \mathcal{F}_c$, and without loss of generality, assume that $l \prec F(l)$. Let $\delta := d(l, F(l)) > 0$, and let $x \in l$. Observe that $F(\overline{R}(l)) = \overline{R}(F(l))$, and then if $\{x_i\}_{i=0}^r$ is any $\delta/2$-chain with $x_0 = x$, it holds that $d(x_i, l) > \delta/2$ for all $0 < i \leq r$, and therefore x is not chain recurrent for F. As the choice of $l \in \mathcal{F}_c$ and $x \in l$ was arbitrary, we have that $CR(F) \cap \mathcal{F}_c = \emptyset$, which proves our claim.

This claim gives us that $CR(F)$ has an open cover $\mathcal{U}'$ whose elements are the connected components of $\mathbf{R} \times S^1 \setminus \cup\mathcal{F}_c$, which are sets of the form (l, l'), with $l, l' \in \mathcal{F}_c$. By definition of the sets Λ^+ and Λ^-, and by Theorem 2.17 we have that for any element (l, l') of $\mathcal{U}'$,

$$\text{either } CR(F) \cap (l, l') \subset \Lambda^+, \text{ or } CR(F) \cap (l, l') \subset \Lambda^-.$$

Now, fix $l_* \in \mathcal{F}_c$. The compact set $CR(F) \cap [l_*, T_1(l_*)]$ has a finite subcover $\mathcal{U}'' \subset \mathcal{U}'$, of the form $\mathcal{U}'' = \{(l'_{2i}, l'_{2i+1})\}_{i=0}^{r'-1}$. We reindex the curves l'_i in a way that $l'_i \prec l'_{i+1}$ for $0 \le i < 2r' - 1$, and we extract from the family of compact leaves $\{l'_i\}_{i=0}^{2r'-1}$ a subfamily $\{l_i\}_{i=0}^{r-1}$ which is minimal with respect to the following property: if $l_r = T_1(l_0)$, then for each $0 \le i < r$

$$\text{either } \emptyset \neq CR(F) \cap (l_i, l_{i+1}) \subset \Lambda^+, \text{ or } \emptyset \neq CR(F) \cap (l_i, l_{i+1}) \subset \Lambda^-.$$

As a consequence we have that, if for $0 \le i < r$ we define

$$\Lambda_i = CR(F) \cap (l_i, l_{i+1}),$$

then,

- $\Lambda_i \neq \emptyset$ for all $0 \le i < r$, and
- if $\Lambda_i \subset \Lambda^+$ then $\Lambda_{i+1} \subset \Lambda^-$, and if $\Lambda_i \subset \Lambda^-$ then $\Lambda_{i+1} \subset \Lambda^+$, for any $i \in \mathbf{Z}/r\mathbf{Z}$.

This concludes the construction of the family $\{l_i\}_{i=0}^{r-1}$ satisfying items (a), (c) and (d) from Part 2 of the lemma. We also have that $\Lambda_i \neq \emptyset$ for all $0 \le i < r$, so to prove that $\{l_i\}_{i=0}^{r-1}$ satisfies item (b) it remains to prove that Λ_i is F-invariant, for each $0 \le i < r$.

For any $0 \le i < r$, Λ_i is F-invariant. Fix $i \in \{0, \dots, r-1\}$. First we prove that $F(\Lambda_i) \subset \Lambda_i$. As $CR(F)$ is F-invariant and $\Lambda_i = CR(F) \cap (l_i, l_{i+1})$, to show that $F(\Lambda_i) \subset \Lambda_i$ it suffices to show that if $x \in \Lambda_i$ then $F(x) \in (l_i, l_{i+1})$.

Suppose this is not true. Then there exists $x_0 \in \Lambda_i$ such that $F(x_0) \notin (l_i, l_{i+1})$. Without loss of generality suppose that $F(x_0) \in \overline{R}(l_{i+1})$. Then, as l_{i+1} is free for F we must have that $l_{i+1} \prec F(l_{i+1})$. Let $\delta_1 := d(l_i, F(l_i)) > 0$. By the continuity of F there is $\delta_2 > 0$ such that if $d(x, l_i) < \delta_2$, then $F(x) \in R(l_i)$ and $d(F(x), l_i) > \delta_1/2$. Let $\delta = \min\{\delta_2, \delta_1/2\}$, and let $\{y_i\}_{i=0}^s$ be any δ-chain for F with $y_0 = x_0$. Then $d(y_1, l_i) < \delta_2$ and then $F(y_1) \in R(l_i)$ and $d(F(y_1), l_i) > \delta_1/2$. Therefore, $y_2 \in R(l_i)$, and $F(y_2) \in R(F(l_i))$. Then $y_3 \in R(l_i)$. By induction, we get that $y_n \in R(l_i)$ for all $n \ge 2$. As $\{y_i\}_{i=0}^s$ was an arbitrary δ-chain with $y_0 = x_0$, we then have that x_0 is not δ-chain recurrent, which contradicts that $x_0 \in \Lambda_i \subset CR(F)$. This contradiction gives us that $F(x_0)$ must be contained in (l_i, l_{i+1}), and therefore $F(\Lambda_i) \subset \Lambda_i$.

Now we prove that $F^{-1}(\Lambda_i) \subset \Lambda_i$. Applying the arguments in last paragraph to F^{-1} we get that $F^{-1}(CR(F^{-1}) \cap (l_i, l_{i+1})) \subset CR(F^{-1}) \cap (l_i, l_{i+1})$, and as $CR(F) = CR(F^{-1})$ we get that $F^{-1}(\Lambda_i) \subset \Lambda_i$.

As the choice of i was arbitrary, we conclude that for any i, Λ_i is F-invariant, as we wanted. This finishes the proof of Part 2.

Now we proceed to the remaining part of the proof of Part 1.

The sets Λ^+ and Λ^- are F-invariant. We proved that for each i the set Λ_i is F-invariant, and contained either in Λ^+ or in Λ^-. As both Λ^+ and Λ^- are contained in $\cup\Lambda_i$, we conclude that Λ^+ and Λ^- are F-invariant.

There is $\epsilon > 0$ such that $\rho(\widetilde{\Lambda}^+, f) \subset \{0\} \times (\epsilon, \infty)$, and $\rho(\widetilde{\Lambda}^-, f) \subset \{0\} \times (-\infty, -\epsilon)$. We will deal only with the case of $\rho(\widetilde{\Lambda}^+, f)$; the case of $\rho(\widetilde{\Lambda}^-, f)$ is analogous. As Λ^+ is closed and F-invariant, $\widetilde{\Lambda}^{\pm}$ is a compact $\tilde{f}$-invariant set. Let v^- the lower endpoint of $\operatorname{conv}\rho(\widetilde{\Lambda}^+, f)$. It suffices to prove that $(v^-)_2 > 0$. Suppose by contradiction that $(v^-)_2 \leq 0$.

By Proposition 2.10, as v^- is an extremal point of $\operatorname{conv}\rho(\widetilde{\Lambda}, f)$, there exists an ergodic measure μ for $\tilde{f}$ with $\rho(\mu, f) = v^-$ and therefore $\operatorname{supp}(\mu) \subset \pi''(\Lambda^+)$. As $(v^-)_2 \leq 0$, then, as in the proof that Λ^+ and Λ^- are non-empty, with the aid of Birkhoff's Theorem and Atkinson's Lemma 2.18, we find a point $\tilde{x} \in \operatorname{supp}(\mu)$ such that

$$f^n(x) - x - (0, -m) < \epsilon_0$$

for $x \in \pi'^{-1}(\tilde{x})$ and for some $n \in \mathbf{N}$ and $m \in \mathbf{N}_0$. If $m > 0$ this means that $\pi(x) \in \Lambda^-$, and as $x \in \pi'^{-1}(\operatorname{supp}(\mu))$ and $\operatorname{supp}(\mu) \subset \pi''(\Lambda^+)$, we have that $\pi(x) \in \pi(\pi'^{-1}(\pi''(\Lambda^+))) = \pi''^{-1}(\pi''(\Lambda^+)) = \Lambda^+$. Therefore $\Lambda^+ \cap \Lambda^- \neq \emptyset$, which is a contradiction, and then we cannot have that $m > 0$. If $m = 0$ we have that x is ϵ_0-chain recurrent for f, which by the definition of ϵ_0 and by Franks' Lemma 2.13 is a contradiction. Therefore we cannot have that $(v^-)_2 \leq 0$, as we wanted. This finishes the proof of Part 1, and of the lemma. ■

Remark 4.6. The fact that the sets $\Lambda_i \subset (l_i, l_{i+1})$ are non-empty and F-invariant implies the following. If $\ell \subset \mathbf{R}^2$ is a lift of l_i and if $\ell' \subset \mathbf{R}^2$ is the lift of l_{i+1} such that $\ell \prec \ell' \prec T_1(\ell)$, then

$$\bigcap_{n\in\mathbf{Z}} f^n\left((\ell, \ell')\right) = \pi^{-1}\left(\bigcap_{n\in\mathbf{Z}} F^n\left((l_i, l_{i+1})\right)\right) \supset \pi^{-1}(\Lambda_i) \neq \emptyset.$$

Proof of Proposition 4.3. By construction, the curves l_i are lifts of leaves from a foliation of $\mathbf{T}^2$; that is, the curves $\tilde{l}_i = \pi''(l_i) \subset \mathbf{T}^2$ are also compact leaves from a foliation of $\mathbf{T}^2$ (and therefore pairwise disjoint). As $\pi'' : \mathbf{R} \times S^1 \to \mathbf{T}^2$ is a covering map, the curves $\tilde{l}_i$ are also essential, and by the definition of π'' it is easy to see that they are vertical. For any $0 \le i < r$, if $\Theta_i \subset \mathbf{T}^2$ is as in Theorem B and $\Lambda_i \subset \mathbf{R} \times S^1$ is as in Lemma 4.5, we observe that Θ_i is non-empty: by item 2-(b) of Lemma 4.5, $\emptyset \neq \pi''(\Lambda_i) \subset (\tilde{l}_i, \tilde{l}_{i+1})$ is $\tilde{f}$-invariant, and then $\pi''(\Lambda_i) \subset \Theta_i$.

We now prove that it holds (2) from Theorem B; that is, $\rho(\Theta_i, f)$ is contained either in $\{0\} \times (0, \infty)$ or in $\{0\} \times (-\infty, 0)$ for each i.

Let $i \in \{0, 1, \ldots, r-1\}$, and suppose first that $\Lambda_i \subset \Lambda^+$, where Λ^+ is as in Lemma 4.5. Then we will prove that $\rho(\Theta_i, f) \subset \{0\} \times (0, \infty)$. Let v^- be the lower endpoint of the interval $\mathrm{conv}(\rho(\Theta_i, f))$. To prove that $\rho(\Theta_i, f) \subset \{0\} \times (0, \infty)$, it suffices then to prove that $(v^-)_2 > 0$. By contradiction, suppose that $(v^-)_2 \le 0$.

By Proposition 2.10, we can find an ergodic measure μ with support contained in Θ_i and with $\rho(\mu, f) = v^-$. As in the proof in Lemma 4.5 that the sets Λ^+ and Λ^- are non-empty, with the use of Atkinson's Lemma we can find a point $\tilde{x} \in \mathrm{supp}(\mu)$ such that, for any $\epsilon > 0$ and $x \in \pi'^{-1}(\tilde{x})$, it holds

$$|f^n(x) - x - (0, m)| < \epsilon \text{ for some } m \le 0 \text{ and } n > 0. \tag{3}$$

Therefore if $\hat{x} \in \pi''^{-1}(\tilde{x}) \subset \mathbf{R} \times S^1$, $\hat{x}$ is recurrent for F, and in particular $\hat{x} \in CR(F)$. As $\tilde{x} \in \mathrm{supp}(\mu) \subset [\tilde{l}_i, \tilde{l}_{i+1}]$, $\hat{x} \in [l_i, l_{i+1}]$. Therefore $\hat{x} \in CR(F) \cap [l_i, l_{i+1}] = \Lambda_i$. As in (3) $\epsilon > 0$ is arbitrary and $m \le 0$, we have that $\hat{x} \in \Lambda^-$, and then $\Lambda_i \cap \Lambda^- \neq \emptyset$, which contradicts our assumption that $\Lambda_i \subset \Lambda^+$. Therefore we must have $(v^-)_2 > 0$, and this proves that $\rho(\Theta_i, f) \subset \{0\} \times (0, \infty)$, as we wanted.

Similarily, if $\Lambda_i \subset \Lambda^-$ we prove that $\rho(\Theta_i, f)$ is contained in $\{0\} \times (-\infty, 0)$. The choice of $i \in \{0, \ldots, r-1\}$ was arbitrary, and then we have that for the family $\{\tilde{l}_i\}_{i=0}^{r-1}$ it holds item (2) from Theorem B ■

4.4 Proof of item (3).

4.4.1 Main ideas.

In this section we prove item (3) from Theorem B; that is, we prove that the curves $\tilde{l}_i \subset \mathbf{T}^2$ are free forever for $\tilde{f}$. We start by specifying the following.

Notation. From now on, r will denote the cardinality of the family $\{\tilde{l}_i\}_{i=0}^{r-1}$ obtained in Proposition 4.3.

We claim that we may assume that each of the closed curves $\tilde{l}_i$ is a straight vertical circle. To see this, note that it is easy to define a homeomorphism $\tilde{h} : \mathbf{T}^2 \to \mathbf{T}^2$ in the isopoty class of the identity such that $\tilde{h}(\tilde{l}_i)$ is a straight vertical circle, for all $0 \le i < r$. For a simple closed curve, the property of being free forever for $\tilde{f}$ is invariant by topological conjugacy, so to prove that the curves $\tilde{l}_i$ are free forever for $\tilde{f}$, it suffices to prove that the curves $\tilde{h}(\tilde{l}_i)$ are free forever for $\tilde{h}\tilde{f}\tilde{h}^{-1}$. As $\tilde{h}$ is isotopic to the identity, there is a lift h of $\tilde{h}$ such that $\rho(hfh^{-1}) = \rho(f)$, and then in the proof of Theorem A we can work with $\tilde{h}\tilde{f}\tilde{h}^{-1}$ instead of $\tilde{f}$. This proves our claim.

So from now on we make the following assumption:

Assumption 4.7. *For each $0 \le i < r$, the curve $\tilde{l}_i$ is a straight vertical circle.*

By construction of the family $\{\tilde{l}_i\}$, if $\ell \subset \mathbf{R}^2$ is a lift of some $\tilde{l}_i$ then ℓ is a Brouwer curve for f, so either $\ell \prec f(\ell)$ or $f(\ell) \prec \ell$. However, under the hypothesis that $\rho(f)$ is an interval of the form $\{0\} \times I$, *we cannot have $T_1(\ell) \prec f^n(\ell)$, nor $f^n(\ell) \prec T_1^{-1}(\ell)$ for any $n \in \mathbf{Z}$.* Let us show this. Suppose for example that

$$T_1(\ell) \prec f^{n_0}(\ell) \quad \text{for some } n_0 \in \mathbf{N}. \tag{4}$$

As f commutes with T_1, $T_1^2(\ell) \prec f^{n_0}(T_1(\ell))$, and then $T_2(\ell) \prec f^{2n_0}(\ell)$. By induction, we get that

$$T_1^k(\ell) \prec f^{kn_0}(\ell) \quad \text{for all } k \in \mathbf{N}. \tag{5}$$

As we are under Assumption 4.7, the curves ℓ_i are straight and vertical, and then (5) gives us that for any point $x \in \ell$, it holds $f^{kn_0}(x)_1 - (x)_1 > k$ for each $k \in \mathbf{N}$. This means that $\max \mathrm{pr}_1(\rho(f)) > 1/n_0 > 0$, a contradiction, and therefore (4) is not possible. One discards analogously the case $n_0 < 0$ and the case that $f^n(\ell) \prec T_1^{-1}(\ell)$ for some $n \in \mathbf{Z}$.

By this, we conclude the following:

Remark 4.8. In order to prove that the curves $\tilde{l}_i$ are free forever for $\tilde{f}$, it suffices to show that for any i, if $\ell \subset \mathbf{R}^2$ is a lift of $\tilde{l}_i$, then

$$f^n(\ell) \cap T_1^{-1}(\ell) = \emptyset = f^n(\ell) \cap T_1(\ell) \quad \text{for all } n \in \mathbf{Z}.$$

As we will be working basically in $\mathbf{R}^2$ with lifts of the curves $\tilde{l}_i$, we start by fixing a family of such lifts.

Definition 4.9 (The curves ℓ_i)**.** For $i \in \mathbf{Z}$ we define a lift $\ell_i \subset \mathbf{R}^2$ of the curve $\tilde{l}_{i \bmod r}$ in the following way. First define $\ell_0 \subset \mathbf{R}^2$ as any lift of $\tilde{l}_0$. Then, for each $1 \le i < r$ define ℓ_i as the lift of $\tilde{l}_i$ such that $\ell_0 \prec \ell_i \prec T_1(\ell_0)$ (reindexing the curves $\tilde{l}_i$ we may assume that $\ell_i \prec \ell_{i+1}$ for all $0 \le i \le r-2$). Then, for every $0 \le i < r$ and $j \in \mathbf{N}$ we define $\ell_{jr+i} = T_1^j \ell_i$ (see Fig. 8).

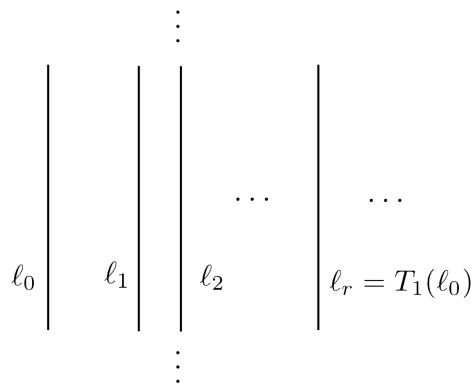

Figure 8: The curves ℓ_i. By Assumption 4.7 they are straight lines.

Strategy of the proof. To prove that the curves $\tilde{l}_i$ are free forever for $\tilde{f}$ it will be enough to prove the following:

Claim 4.10. There exists $i_0 \in \mathbf{N}_0$ such that $f^n(\ell_{i_0}) \subset L(\ell_{i_0+1})$ for all $n \in \mathbf{Z}$.

Let us see how Claim 4.10 implies that the curves $\tilde{l}_i$ are free forever for $\tilde{f}$. Suppose on the contrary that it holds (4.10) and there is $i_1 \in \{0, \ldots, r-1\}$ such that the curve $\tilde{l}_{i_1}$ is not free forever for $\tilde{f}$. By Remark 4.8, there is $n_0 \in \mathbf{Z}$ such that, either $f^{n_0}(\ell_{i_1}) \cap T_1(\ell_{i_1}) \neq \emptyset$, or $f^{n_0}(\ell_{i_1}) \cap T_1^{-1}(\ell_{i_1}) \neq \emptyset$. Suppose that it holds the former (the latter case is similar). Let $\ell_* \subset \mathbf{R}^2$ be the integer translate of ℓ_{i_1} such that $\ell_* \subset (T_1^{-1}(\ell_{i_0}), \ell_{i_0}]$, and $\ell_{i_0} \prec T_1(\ell_*)$. Then, by the periodicity of f we have that $f^{n_0}(\ell_*) \cap T_1(\ell_*) \neq \emptyset$, and this implies that $f^{n_0}(\ell_{i_0}) \cap R(\ell_{i_0+1}) \neq \emptyset$, which contradicts (4.10). Therefore (4.10) implies that the curves $\tilde{l}_i$ are free forever for $\tilde{f}$.

To prove Claim 4.10 we will proceed by contradiction. First note the following.

Claim 4.11. If Claim 4.10 does not hold, there is $n_0 > 0$ such that, either $f^{n_0}(\ell_i) \cap \ell_{i+1} \neq \emptyset$ for all i, or $f^{n_0}(\ell_i) \cap \ell_{i-1} \neq \emptyset$ for all i.

Proof. Observe that for any i and n, we cannot have $\ell_{i+1} \prec f^n(\ell_i)$ nor $f^n(\ell_i) \prec \ell_{i-1}$. This is because f preserves orientation and, for any i, the maximal invariant set of (ℓ_i, ℓ_{i+1}) is non-empty (see Remark 4.6). From this we obtain that if Claim 4.10 does not hold, then for each i there is $N_i > 0$ such that either $f^{N_i}(\ell_i) \cap \ell_{i+1} \neq \emptyset$, or $f^{N_i}(\ell_i) \cap \ell_{i-1} \neq \emptyset$. By the fact that f is a orientation preserving homeomorphism it is easy to see that we must actually have either $f^{N_i}(\ell_i) \cap \ell_{i+1} \neq \emptyset$ for all i, or $f^{N_i}(\ell_i) \cap \ell_{i-1} \neq \emptyset$ for all i. By the periodicity of f, defining $n_0 = \min_{0 \leq i < r} N_i$ the claim follows. ■

By Claim 4.11 and by our hypothesis that $\rho(f)$ is of the form $\{0\} \times I$, to prove Claim 4.10 it suffices to prove the following.

Claim 4.12. If Claim 4.10 does not hold, we have the following possibilities:

1. There is $n_0 > 0$ such that $f^{n_0}(\ell_i) \cap \ell_{i+1} \neq \emptyset$ for all i, and $\max \mathrm{pr}_1(\rho(f)) > 0$.
2. There is $n_0 > 0$ such that $f^{n_0}(\ell_i) \cap \ell_{i-1} \neq \emptyset$ for all i, and $\min \mathrm{pr}_1(\rho(f)) < 0$.

Therefore, to prove that the curves $\tilde{l}_i$ are free forever for $\tilde{f}$, the rest of Section 4.4 will be dedicated to the proof of Claim 4.12. For future reference we make the following explicit statement:

Remark 4.13. The fact that $\rho(f)$ is of the form $\{0\} \times I$, together with Claim 4.12 imply that the curves $\tilde{l}_i$ are free forever for $\tilde{f}$.

To end section 4.4.1 we explain the ideas of how we will prove Claim 4.12. By Claim 4.11, there is $n_0 > 0$ such that either

$$f^{n_0}(\ell_i) \cap \ell_{i+1} \neq \emptyset \text{ for all } i, \tag{6}$$

or

$$f^{n_0}(\ell_i) \cap \ell_{i-1} \neq \emptyset \text{ for all } i. \tag{7}$$

We suppose it holds (6) and we will outline the proof that $\max \mathrm{pr}_1(\rho(f)) > 0$ (the proof that $\min \mathrm{pr}_1(\rho(f)) < 0$ for the case it holds (7) is similar). We want to show that there is a sequence of points $x_k \in \mathbf{R}^2$ and a natural number N such that $\mathrm{pr}_1(f^{Nk}(x_k) - x_k) \geq k$ for all $k \in \mathbf{N}$. Observe that (6) implies in particular that $\ell_i \prec f(\ell_i)$ for all i. By Proposition 4.3, the sets $\Theta_0, \Theta_1 \subset \mathbf{T}^2$ which are the maximal invariant sets of $[\tilde{l}_0, \tilde{l}_1]$ and $[\tilde{l}_1, \tilde{l}_2]$, resp., are non-empty, and we have either $\rho(\Theta_0, f) \subset \{0\} \times (0, \infty)$ and $\rho(\Theta_1, f) \subset \{0\} \times (-\infty, 0)$, or viceversa. To fix ideas, suppose that $\rho(\Theta_0, f) \subset \{0\} \times (0, \infty)$ and $\rho(\Theta_1, f) \subset \{0\} \times (-\infty, 0)$.

Using the fact that the points of $\pi'^{-1}(\Theta_0) \cap (\ell_0, \ell_1)$ go upwards under iteration by f, and the points of $\pi'^{-1}(\Theta_1) \cap (\ell_1, \ell_2)$ go downwards, we will prove the following:

$$f^{n_0}(\ell_0) \cap \ell_1 \neq \emptyset \text{ implies that } f^{n_0+m}(\ell_0) \cap R(\ell_1) \text{ gets}$$

$$\text{vertically stretched as } m \to \infty,$$

where by 'vertically stretched' we mean that the vertical diameter of some connected components of $f^{n_0+m}(\ell_0) \cap R(\ell_1)$ grows to infinity as $m \to \infty$. Using this we will get that, for some m large enough, there is a connected component C_1 of

$f^{n_0+m}(\ell_0) \cap R(\ell_2)$ that satisfies property **PL** (cf. sec. 4.2) with respect to a point $p_1 \in \ell_1$ such that $f^{n_0}(p_1) \in \ell_2$ (such a point exists; just take $p_1 \in \ell_1 \cap f^{-n_0}(\ell_2)$). Then, Lemma 4.2 will give us that $f^{n_0}(C_1) \cap R(\ell_2) \neq \emptyset$ (see Fig. 9).

Proceeding inductively, we will prove that for any n, there is an iterate of ℓ_0 that intersects ℓ_n. Moreover, and this is a crucial fact, we will prove that this is done at a uniform speed: there exists an integer $N > 0$ such that for any $n \in \mathbf{N}$, $f^{Nn}(\ell_0)$ intersects ℓ_n. In particular there exists a sequence of points x_k in ℓ_0 such that $\mathrm{pr}_1(f^{Nk}(x_k) - x_k) \geq k$ for all $k \in \mathbf{N}$, and therefore $\max \mathrm{pr}_1(\rho(f)) > 0$. This will conclude the proof that in the case it holds (6) then $\max \mathrm{pr}_1(\rho(f)) > 0$. The arguments for proving that $\min \mathrm{pr}_1(\rho(f)) < 0$ in the case it holds (7) are analogous. Therefore, this concludes the outline of the proof of Claim 4.12.

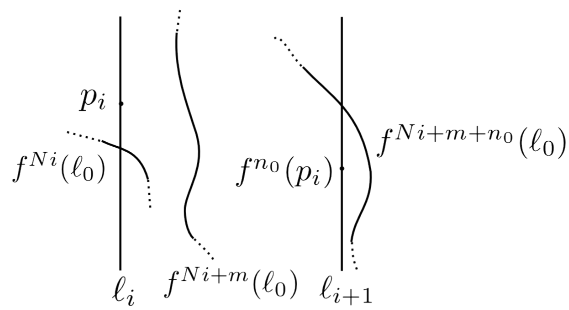

Figure 9: $f^{Ni}(\ell_0)$ intersects ℓ_i. Then, $f^{Ni+m}(\ell_0)$ gets vertically stretched in $R(\ell_{ir})$ for large m, and after that, p_i 'pushes rightwards' $f^{Ni+m}(\ell_0)$ under iteration by f^{n_0}.

This section is organized as follows. In section 4.4.2 we prove a lemma relating properties **PL** and **PR** to the iterates of the curves ℓ_i, which formalizes some ideas explained above. In section 4.4.3 we define and study the sets R^i_∞ and L^i_∞, which will be our references for the horizontal displacement of the iterates $f^n(\ell_0)$. In section 4.4.4 we prove that the curves $\tilde{l}_i$ are free forever for $\tilde{f}$ assuming some lemmas that will be proved in section 4.4.5.

4.4.2 The properties PL and PR, and the curves ℓ_i.

The following lemma is a convenient application of Lemma 4.2.

Lemma 4.14. *Let $i, j \in \mathbf{N}$ and suppose that $n \in \mathbf{Z}$ is such that $f^n(\ell_i) \cap \ell_j \neq \emptyset$. Then, there exists a constant $K > 0$ such that, if $C \subset \mathbf{R}^2$ is a continuum contained in the open strip bounded by ℓ_i and ℓ_j and such that* $\mathrm{diam}_2(C) \geq K$*, then:*

- *If $i < j$, then $f^n(C) \cap R(\ell_j) \neq \emptyset$.*
- *If $j < i$, then $f^n(C) \cap L(\ell_j) \neq \emptyset$.*

Proof. Without loss of generality suppose that $n > 0$. By Lemma 4.2 applied to f^n there is a constant $k > 0$ such that, if C is a continuum that satisfies the property $\mathbf{PL}(k, p)$ (or $\mathbf{PR}(k, p)$) for some $p \in \mathbf{R}^2$, then $f^n(C) \cap R(v(f^n(p))) \neq \emptyset$ (resp., $f^n(C) \cap L(v(f^n(p))) \neq \emptyset$).

We treat the case $i < j$, the case $i > j$ being similar. By hypothesis $f^n(\ell_i) \cap \ell_j \neq \emptyset$. Take $x \in f^{-n}(\ell_j) \cap \ell_i$ and define $K = k + 1$. Suppose that C is a continuum contained in (ℓ_i, ℓ_j) and such that $\operatorname{diam}_2(C) \geq K$. Then there is $s \in \mathbf{Z}$ such that

$$((T_2^s(x))_2 - k, (T_2^s(x))_2 + k) \subset \operatorname{pr}_2(C).$$

As we are under Assumption 4.7, the line ℓ_i is straight, and as $C \subset R(\ell_i)$, it is easy to see that C satisfies property $\mathbf{PL}(k, x)$. Therefore as $f^n(T_2^s(x)) \in \ell_j$, Lemma 4.2 gives us that $f^n(C) \cap R(\ell_j) = f^n(C) \cap R(v(f^n(T_2^s(x)))) \neq \emptyset$, as we wanted. ■

4.4.3 The sets L^i_∞, R^i_∞, and X_i.

In this section we study an important tool in this work. For each $i \in \mathbf{N}$, define the sets

$$R^i_\infty = \bigcap_{n \in \mathbf{Z}} R\left(f^n(\ell_i)\right), \quad L^i_\infty = \bigcap_{n \in \mathbf{Z}} L(f^{-n}(\ell_{i+1})), \quad \text{and} \quad X_i = L^i_\infty \cup R^i_\infty$$

(see Fig. 10.)

As we are under Assumption 4.19, $\ell_i \prec f(\ell_i)$ for all i, and therefore we have that

$$R^i_\infty = \{x \in \mathbf{R}^2 \ : \ f^{-n}(x) \in R(\ell_i) \ \forall n \geq 0\},$$

and

$$L^i_\infty = \{x \in \mathbf{R}^2 \ : \ f^n(x) \in L(\ell_{i+1}) \ \forall n \geq 0\}.$$

The following lemmas study some properties of these sets.

Lemma 4.15. *For every $i \geq 0$:*

1. *if C is a connected component of R^i_∞, then $\sup \operatorname{pr}_1(C) = +\infty$,*
2. *if C' is a connected component of L^i_∞, then $\inf \operatorname{pr}_1(C') = -\infty$, and*

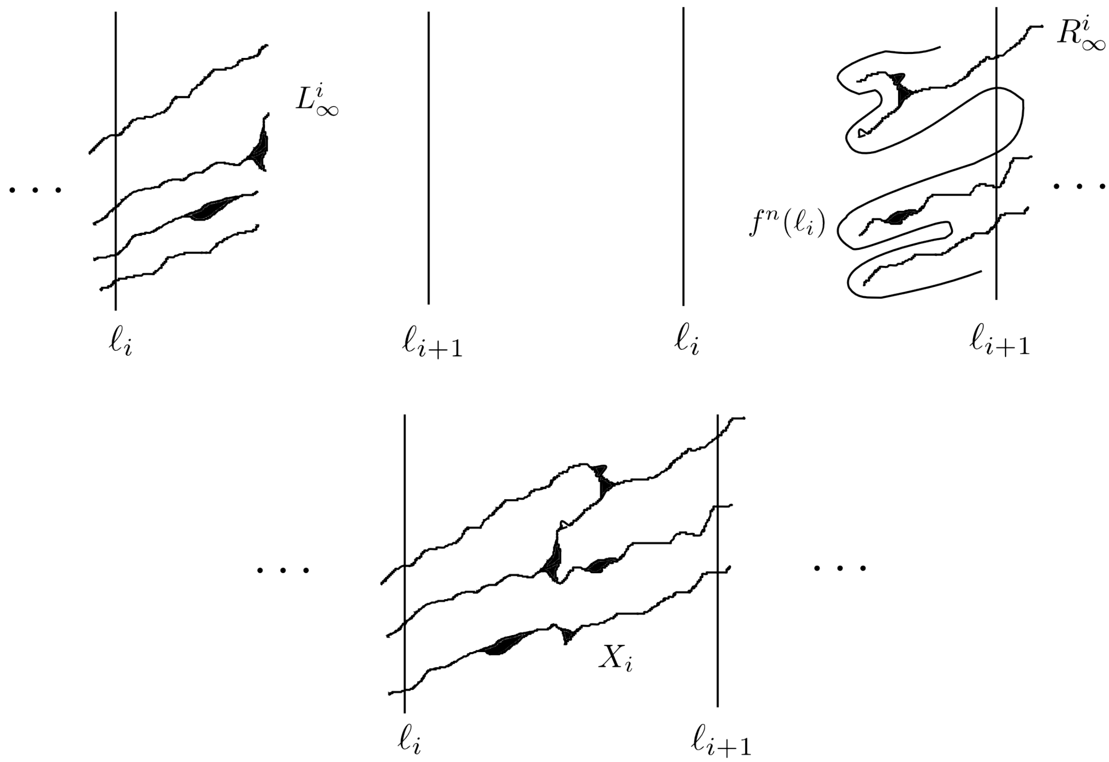

Figure 10: Some examples of the sets L_∞^i, R_∞^i and X_i.

3. the connected components of $\mathbf{R}^2 \setminus X_i$ *are simply connected.*

Proof. Let $S = \mathbf{R} \times S^1 \cup \{\infty\} \cup \{-\infty\}$ be the two-point compactification of $\mathbf{R} \times S^1$, which is homeomorphic to S^2, and let $j : \mathbf{R} \times S^1 \hookrightarrow S$ be the inclusion. By assumption 4.7, the curves $\pi(\ell_i) \subset \mathbf{R} \times S^1$ are vertical circles, and then the sets $D_n := j(\pi(\overline{R}(f^n(\ell_i)))) \cup \{\infty\}$, and $D'_n := j(\pi(\overline{L}(f^{-n}(\ell_i)))) \cup \{-\infty\}$ are topological closed discs in S, for any n and i. Observe that

$$\widehat{L}_i := j(\pi(L_\infty^i)) \cup \{-\infty\} = \bigcap_{n \in \mathbf{N}} D'_n,$$

and

$$\widehat{R}_i := j(\pi(R_\infty^i)) \cup \{\infty\} = \bigcap_{n \in \mathbf{N}} D_n.$$

As we are under Assumption 4.19, $\ell_i \prec f(\ell_i)$ for any i, and then $D_{n+1} \subset D_n$ for all n. Therefore the sets $\widehat{L}_i$ and $\widehat{R}_i$ are nested intersections of topological closed discs, and thus they are compact and connected.

Observe that, for every i, $L^i_\infty \cap R^i_\infty = \cap_{n\in\mathbf{Z}} f^n((\ell_i, \ell_{i+1}))$. By Remark 4.6, $L^i_\infty \cap R^i_\infty \neq \emptyset$, and then $\widehat{L}_i \cap \widehat{R}_i \neq \emptyset$. Therefore, $\widehat{X}_i := j(X_i) \cup \{\infty\} \cup \{-\infty\}$ is compact and connected, as it is the union of $\widehat{L}_i$ and $\widehat{R}_i$, which are connected sets with nonempty intersection.

(1). It suffices to show that, for $x \in \widehat{R}_i \setminus \{\infty\}$, if C_x is the connected component of $\widehat{R}_i \setminus \{\infty\}$ containing x, then $\infty \in \overline{C_x}$. For each n, let α_n be an arc contained in D_n, such that $\alpha_n(0) = \infty$ and $\alpha_n(1) = x$. Let $B \subset S$ be a ball that contains ∞, and does not contain x. For each n, let $\beta_n \subset \alpha_n$ be an arc contained in B^c with extremes $\beta_n(0) \in \partial B$ and $\beta_n(1) = x$. Then, take an accumulation point $C \subset S$ of the sequence (β_n) in the Hausdorff topology. As $\beta_n \subset \alpha_n \subset D_n$, and as $D_n \subset D_{n-1}$ for all n, we have that $\beta_n \subset D_k$ for all $k \leq n$. Therefore C is contained in D_n for all n; that is, $C \subset \widehat{R}_i = \cap_{n\geq 0} D_n$. Also, C is compact, connected, contains x, and intersects ∂B. Therefore, the connected component C_x must contain C, and then C_x intersects ∂B. As B was an arbitrarily small ball containing ∞, this means that $\infty \in \overline{C_x}$, as we wanted.

(2). The proof is analogous to (1).

(3). First we will prove that the connected components of $S \setminus \widehat{X}_i$ are simply connected. To see this just note that if there was a connected component U_0 of $S \setminus \widehat{X}_i$ not simply connected, then there would exist a simple closed curve $\gamma \subset U_0$ separating two connected components of ∂U_0, but as $\partial U_0 \subset \widehat{X}_i$, we would have that $\widehat{X}_i$ is not connected, a contradiction. Then each connected component of $S \setminus \widehat{X}_i$ must be simply connected.

Now, let V be any connected component of X_i^c. Then $j(V) \subset S$ is a connected component of $S \setminus \widehat{X}_i$, and therefore simply connected. As $j : \mathbf{R}^2 \to S \setminus \{\infty\}$ is a homeomorphism, V must be also simply connected. ∎

Corollary 4.16. *For each $i \geq 0$, the connected components of $X_i^c \cap (\ell_i, \ell_{i+1})$ are simply connected.*

Proof. If $A, B \subset \mathbf{R}^2$ are simply connected sets, it is easy to see that each connected component of $A \cap B$ is simply connected. Then, if U is any connected component of X_i^c, each connected component of $U \cap R(\ell_i)$ is simply connected, and then each connected component of $U \cap (\ell_i, \ell_{i+1}) = U \cap R(\ell_i) \cap L(\ell_{i+1})$ is simply connected. As any connected component of $X_i^c \cap (\ell_i, \ell_{i+1})$ is of the form $U_0 \cap (\ell_i, \ell_{i+1})$, for some connected component U_0 of X_i^c, we then have that any connected component of $X_i^c \cap (\ell_i, \ell_{i+1})$ is simply connected. ∎

The following lemma is an application of Lemma 4.14.

Lemma 4.17. *There exists a constant M_0 such that, for any $i \geq 0$, any connected component of $R^i_\infty \cap L(\ell_{i+1})$ has vertical diameter less than M_0, and also any connected component of $L^i_\infty \cap R(\ell_i)$ has vertical diameter less than M_0.*

Proof. First we treat the case of $R^i_\infty \cap L(\ell_{i+1})$. As we are under Assumption 4.19, we have that $f^{-n_0}(\ell_{i+1}) \cap \ell_i \neq \emptyset$ for all $i \geq 0$. By Lemma 4.14, there exists a constant $K_0 > 0$ such that if $C \subset \mathbf{R}^2$ is a continuum contained in (ℓ_i, ℓ_{i+1}) with $\mathrm{diam}_2(C) > K_0$, then $f^{-n_0}(C) \cap L(\ell_i) \neq \emptyset$. Therefore, for any $i \geq 0$, any connected component C_0 of $R^i_\infty \cap L(\ell_{i+1})$ must have vertical diameter less than K_0, because otherwise $f^{-n_0}(C_0)$ would intersect $L(\ell_i)$, which contradicts the definition of R^i_∞.

Analogously, by Assumption 4.19, we have that $f^{n_0}(\ell_i) \cap \ell_{i+1} \neq \emptyset$ for all $i \geq 0$, and by Lemma 4.14, if $C \subset \mathbf{R}^2$ is a continuum contained in (ℓ_i, ℓ_{i+1}), with $\mathrm{diam}_2(C) > K_0$ then $f^{n_0}(C) \cap R(\ell_{i+1}) \neq \emptyset$. Therefore, for any $i \geq 0$, any component C_0 of $L^i_\infty \cap R(\ell_i)$ must have vertical diameter less than K_0, because otherwise $f^{n_0}(C_0)$ would intersect $R(\ell_{i+1})$, which contradicts the definition of L^i_∞.

Setting $M_0 = K_0$, the lemma follows. ∎

Lemma 4.18. *There exists $M_1 > 0$ such that for any $i \geq 0$, any connected component of $X_i^c \cap (\ell_i, \ell_{i+1})$ has vertical diameter less than M_1.*

Proof. Let $i \geq 0$, and let $x \in L^i_\infty \cap R^i_\infty$. Let C_1 and C_2 be the connected components of $R^i_\infty \cap L(\ell_{i+1})$ and $L^i_\infty \cap R(\ell_i)$, respectively, that contain x. By Lemma 4.15 C_1 is unbounded to the right and C_2 is unbounded to the left, so C_1 intersects ℓ_{i+1} and C_2 intersects ℓ_i. The set $C = C_1 \cup C_2$ is connected and as it intersects both ℓ_i and ℓ_{i+1}, it separates (ℓ_i, ℓ_{i+1}), that is, $(\ell_i, \ell_{i+1}) \setminus C$ is not connected. Also, by Lemma 4.17, there is a constant M_0 such that $\mathrm{diam}_2(C_i) \leq M_0$ for $i = 1, 2$, and then $\mathrm{diam}_2(C) \leq 2M_0$. Thus, $C \cap T_2^{3M_0}(C) = C \cap T_2^{-3M_0}(C) = \emptyset$.

Now, consider the set

$$A = \bigcup_{n \in \mathbf{Z}} T_2^{3M_0 n}(C).$$

The connected components of $(\ell_i, \ell_{i+1}) \setminus A$ have then vertical diameter less than $\mathrm{diam}_2(C) + 3M_0 \leq 4M_0$. As $A \subset X_i$, any connected component of $X_i^c \cap (\ell_i, \ell_{i+1})$ is contained in a connected component of $(\ell_i, \ell_{i+1}) \setminus A$, and therefore is bounded by $4M_0$. Therefore, making $M_1 := 4M_0$, the lemma follows. ∎

4.4.4 Proof of item (3) from Theorem B.

We start by recalling the statements of Claims 4.10 and 4.12:

Claim (4.10)**.** *There exists* $i_0 \in \mathbf{N}_0$ *such that* $f^n(\ell_{i_0}) \subset L(\ell_{i_0+1})$ *for all* $n \in \mathbf{Z}$*.*

Claim (4.12)**.** *If Claim 4.10 does not hold, we have the following possibilities:*

1. *There is* $n_0 > 0$ *such that* $f^{n_0}(\ell_i) \cap \ell_{i+1} \neq \emptyset$ *for all* i*, and* $\max \mathrm{pr}_1(\rho(f)) > 0$*.*
2. *There is* $n_0 > 0$ *such that* $f^{n_0}(\ell_i) \cap \ell_{i-1} \neq \emptyset$ *for all* i*, and* $\min \mathrm{pr}_1(\rho(f)) < 0$*.*

Also, we recall that Claim 4.12 implies that the curves $\tilde{l}_i$ are free forever for f (see Remark 4.13 in section 4.4.1), and therefore, to prove item (3) from Theorem B it suffices to prove Claim 4.12.

For the sake of simplicity in subsequent statements, we will subdivide Claim 4.12 in the following two claims.

Claim (4.12a)**.** *If there is* $n_0 > 0$ *such that* $f^{n_0}(\ell_i) \cap \ell_{i+1} \neq \emptyset$ *for all* i*, then* $\max \mathrm{pr}_1(\rho(f)) > 0$*.*

Claim (4.12b)**.** *If there is* $n_0 > 0$ *such that* $f^{n_0}(\ell_i) \cap \ell_{i-1} \neq \emptyset$ *for all* i*, then* $\min \mathrm{pr}_1(\rho(f)) < 0$*.*

The proof of both Claim 4.12a and Claim 4.12b implies Claim 4.12. To see this just recall that if Claim 4.10 does not hold, by Claim 4.11 there is $n_0 > 0$ such that either $f^{n_0}(\ell_i) \cap \ell_{i+1} \neq \emptyset$ for all i, or $f^{n_0}(\ell_i) \cap \ell_{i-1} \neq \emptyset$ for all i.

Now we will state two central lemmas (Lemmas 4.20 and 4.22), and using these lemmas, we will prove Claim 4.12a. The proof of Claim 4.12b will be totally analogous. In section 4.4.5 we will proceed to the proof of Lemmas 4.20 and 4.22.

As we want to prove Claim 4.12a, we will work under the assumption that there is $n_0 > 0$ such that $f^{n_0}(\ell_i) \cap \ell_{i+1} \neq \emptyset$ for all i. Unless stated the contrary, this assumption will be implicit from now on:

Assumption 4.19. *There is* $n_0 > 0$ *such that* $f^{n_0}(\ell_i) \cap \ell_{i+1} \neq \emptyset$ *for all* i*.*

Recall that if α is an arc, $\dot{\alpha}$ denotes the open arc which is α without its endpoints. The first of our two central lemmas gives us, for each $i \geq 0$, a connected component U_i of $X_i^c \cap (\ell_i, \ell_{i+1})$ and an arc α_i such that $\dot{\alpha}_i$ is contained in U_i, α_i intersects both ℓ_i and $\partial U_i \setminus \ell_i$, and α_i exits the strip (ℓ_i, ℓ_{i+1}) under an amount of iterates by f which is uniform on i.

Definition and Lemma 4.20 (The sets U_i and the curves α_i)**.** *For each $i \geq 0$ there exists a connected component U_i of $X_i^c \cap (\ell_i, \ell_{i+1})$ and an arc α_i such that:*

- $\alpha_i(0) \in \ell_i \setminus L^i_\infty$,
- $\alpha_i(1) \in R^i_\infty \setminus L^i_\infty$, *and*
- $\alpha_i(t) \in U_i$ *for* $0 < t < 1$ *(see Fig. 11).*

Also, there is $N_1 \in \mathbf{N}$ such that $f^{N_1}(\alpha_i) \subset R(\ell_{i+1})$ for any $i \geq 0$.

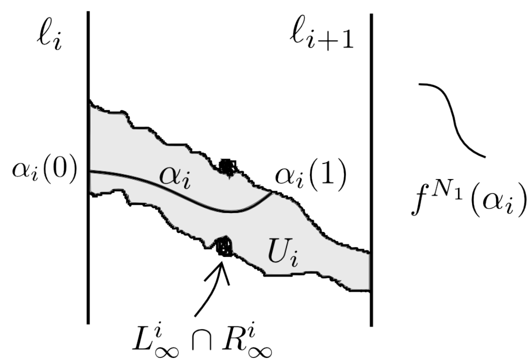

Figure 11: The sets U_i and the curves α_i. In this example $\overline{U}_i$ intersects ℓ_{i+1}. In general we may have $\overline{U}_i \cap \ell_{i+1} = \emptyset$.

Before stating the second lemma, we make a definition which we now loosely explain. For $i \geq 0$ let U_i and α_i be as in Definition 4.20. We will say that a curve $\tilde{\gamma}$ has *good intersection* with U_i if $\tilde{\gamma}$ contains an arc γ such that $\dot{\gamma}$ is contained in U_i, γ is 'to the right' of ℓ_i, one endpoint of γ has its forward orbit contained in (ℓ_i, ℓ_{i+1}), and the other endpoint of γ is in α_i.

Definition 4.21 (good intersection)**.** Let $\{U_i\}_{i\geq 0}$ and $\{\alpha_i\}_{i\geq 0}$ be as in Lemma 4.20. Let $j \in \mathbf{N}_0$ and $s \in \mathbf{Z}$. We say that a curve $\tilde{\gamma}$ has *good intersection* with $T_2^s(U_j)$ if $\tilde{\gamma}$ contains an arc γ such that:

- one endpoint of γ lies in $T_2^s(\partial U_j) \cap L^j_\infty$,
- the other endpoint of γ lies in $T_2^s(\alpha_j)$, and
- $\dot{\gamma} \subset T_2^s(\overline{U}_j) \setminus X_j$ (see Fig. 12).

The second of our two central lemmas gives us a constant $N_2 \in \mathbf{N}$ such that, for any i and any curve β_i contained in $[\ell_i, \ell_{i+1}]$ satisfying some specific conditions,

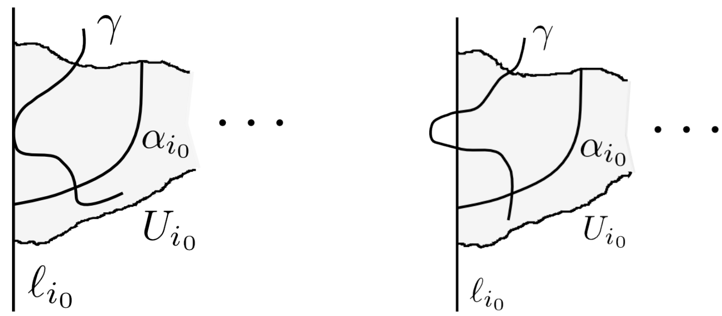

Figure 12: Left: γ has good intersection with U_{i_0}. Right: γ does not have good intersection with U_{i_0}.

we have that $f^{N_2}(\beta_i)$ has good intersection with some vertical integer translate of U_{i+1}. These conditions are the following: $\dot{\beta}_i$ is contained in $X_i^c \cap (\ell_i, \ell_{i+1})$, one extreme of β_i is in ℓ_{i+1}, and the other extreme remains in (ℓ_i, ℓ_{i+1}) under forward iteration by f.

Lemma 4.22. *There exists $N_2 > 0$ such that for any $i \geq 0$, and any arc β_i such that:*

- $\beta_i(0) \in L^i_\infty$,
- $\beta_i(1) \in \ell_{i+1}$, *and*
- $\beta_i(t) \in X_i^c \cap (\ell_i, \ell_{i+1})$ *for* $0 < t < 1$,

then $f^{N_2}(\beta_i)$ has good intersection with $T_2^s(U_{i+1})$, for some $s \in \mathbf{Z}$ (see Fig. 13).

We emphasize that the constant N_2 is independent of i and of β_i. Our two main lemmas 4.20 and 4.22 give us the following.

Lemma 4.23. *There exists $N_3 > 0$ such that, for each $n \geq 0$, $f^{nN_3}(\ell_0)$ has good intersection with $T_2^{s_n}(U_n)$, for some $s_n \in \mathbf{Z}$.*

Proof. Let N_1 and N_2 be the constants given by Lemmas 4.20, and 4.22, respectively, and set $N_3 := N_1 + N_2$. We proceed by induction.

Step $n = 0$. It follows by the definitions that ℓ_0 has good intersection with U_0.

Step n. We suppose that $f^{N_3(n-1)}(\ell_0)$ has good intersection with $T_2^{s_{n-1}}(U_{n-1})$

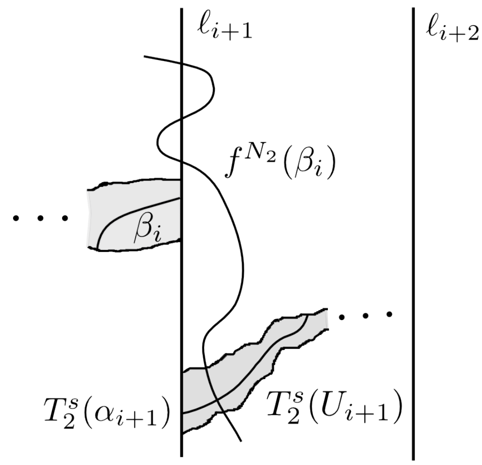

Figure 13: Illustration for Lemma 4.22.

for some $s_{n-1} \in \mathbf{Z}$ and we will prove that $f^{N_3 n}(\ell_0)$ has good intersection with $T_2^{s_n}(U_n)$ for some $s_n \in \mathbf{Z}$.

By the definition of good intersection, there exists an arc γ_{n-1} contained in $f^{N_3(n-1)}(\ell_0)$ such that:

- $\gamma_{n-1}(0) \in L_\infty^{n-1}$ (and therefore $f^j(\gamma_{n-1}(0)) \in (\ell_{n-1}, \ell_n)$ for all $j \geq 0$),
- $\dot{\gamma}_{n-1} \subset T_2^{s_{n-1}}(\overline{U}_{n-1}) \setminus X_{i-1}$, and
- $\gamma_{n-1}(1) \in T_2^{s_{n-1}}(\alpha_{n-1})$ (see Fig. 14).

By Lemma 4.20, we have that $f^{N_1}(\gamma_{n-1}(1)) \in R(\ell_n)$, and as $f^{N_1}(\gamma_{n-1}(0)) \in (\ell_{n-1}, \ell_n)$, we have that

$$f^{N_1}(\gamma_{n-1}) \cap \ell_n \neq \emptyset.$$

Let η be the arc contained in $f^{N_1}(\gamma_{n-1})$ with endpoints

- $\eta(0) = f^{N_1}(\gamma_{n-1}(0)) \in L_\infty^{n-1}$, and
- $\eta(1) = f^{N_1}(\gamma_{n-1}(t^*)) \in \ell_n$, where $t^* = \min\{t \in [0,1] \,:\, f^{N_1} \circ \gamma_{n-1}(t) \in \ell_n\}$.

Then $\dot{\eta} \subset X_{n-1}^c \cap (\ell_{n-1}, \ell_n)$. Therefore η is an arc satisfying the hypotheses of Lemma 4.22, and then by that lemma $f^{N_2}(\eta)$ has good intersection with $T_2^{s_n}(U_n)$, for some $s_n \in \mathbf{Z}$. As $\eta \subset f^{N_3(n-1)+N_1}(\ell_0)$, we have that $f^{N_3(n-1)+N_1+N_2}(\ell_0) = f^{N_3 n}(\ell_0)$ has good intersection with $T_2^{s_1}(U_n)$, which finishes the n-th induction step, and therefore the proof of the lemma. ■

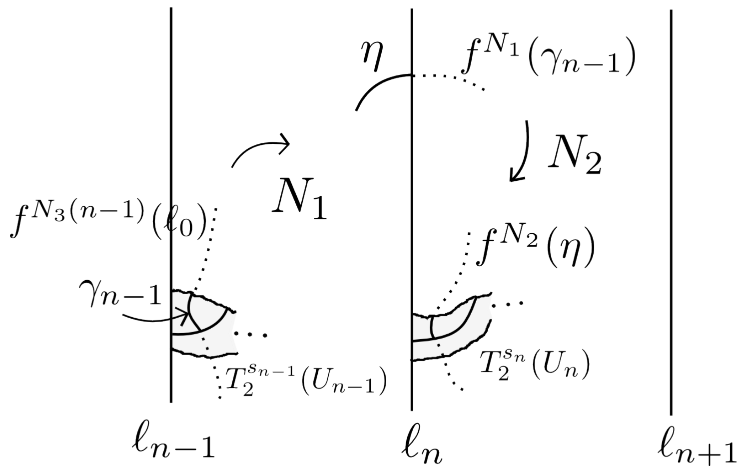

Figure 14: Illustration for Lemma 4.23. The curve $f^{N_1+N_2}(\gamma_{n-1})$ has good intersection with $T_2^{s_n}(U_n)$

.

From this lemma, the proof of Claim 4.12a is immediate:

Proof of Claim 4.12a Note that setting $N' = rN_3$, by Lemma 4.23 we have that $f^{N'n}(\ell_0)$ has good intersection with $T_2^{s_{rn}}(U_{rn})$, and in particular, there exists a sequence $\{x_n\}_n \subset \ell_0$ such that

$$f^{N'n}(x_n)_1 - (x_n)_1 > n \quad \text{for all } n \geq 0.$$

Then $\max \mathrm{pr}_1(\rho(f^{N'})) > 1$, and therefore $\max \mathrm{pr}_1(\rho(f)) > 1/N' > 0$. ■

The proof of Claim 4.12b is totally analogous to that of Claim 4.12a. Therefore, to conclude the proof of Claim 4.12 it suffices to prove Lemmas 4.20 and 4.22. This will also conclude the proof of item (3) from Theorem B (recall that Claim 4.12 implies item (3) from Theorem B; see Remark 4.13 in section 4.4.1).

4.4.5 Proof of Lemmas 4.20 and 4.22.

We recall that in Lemmas 4.20 and 4.22 Assumption 4.19 is implicit; that is, we assume that there is $n_0 > 0$ such that $f^{n_0}(\ell_i) \cap \ell_{i+1} \neq \emptyset$ for all i. We proceed now to the proof of Lemma 4.20. We recall the statement:

Lemma (4.20)**.** *For each $0 \leq i < r$ there exists a connected component U_i of $X_i^c \cap (\ell_i, \ell_{i+1})$ and an arc α_i such that:*

1. $\alpha_i(0) \in \ell_i \setminus L^i_\infty$,

2. $\alpha_i(1) \in R^i_\infty \setminus L^i_\infty$, *and*

3. $\alpha_i(t) \in U_i$ *for* $0 < t < 1$ *(see Fig. 11).*

Also, there is $N_1 \in \mathbf{N}$ *such that* $f^{N_1}(\alpha_i) \subset R(\ell_{i+1})$ *for any* $0 \leq i < r$.

Proof. By Assumption 4.19, for each $0 \leq i < r$ we have that $f^{n_0}(\ell_i) \cap R(\ell_{i+1}) \neq \emptyset$. Then there exists an arc $I \subset \ell_{i+1}$ such that $I \subset L(f^{n_0}(\ell_i))$, and then $I \cap R^i_\infty = \emptyset$. Also, as $f^{-1}(\ell_{i+1}) \subset L(\ell_{i+1})$, then $I \cap L^i_\infty = \emptyset$, and therefore $I \subset X^c_i$.

Let $\widetilde{U}_i$ be the connected component of X^c_i that contains I. Let $\hat{J}$ be the maximal open arc contained in $\ell_{i+1} \cap \widetilde{U}_i$ that contains I, and let J be the closure of $\hat{J}$. By Lemma 4.18 J is compact, and we can give J a parametrization $J : [0, 1] \to \ell_{i+1}$, with orientation coinciding with the upwards orientation of ℓ_{i+1}. We observe that $J(1) \notin L^i_\infty$, because $L^i_\infty \cap \ell_{i+1} = \emptyset$, and then

$$J(1) \in R^i_\infty \setminus L^i_\infty. \tag{8}$$

Now, as $I \subset L(f^{n_0}(\ell_i))$ and $I \subset J$, then $f^{-n_0}(J) \cap L(\ell_i) \neq \emptyset$, and as $J(1) \in R^i_\infty$,

$$f^{-n_0}(J(1)) \in R(\ell_i),$$

and we can define $t^*_i = \max\{t \in [0, 1] : f^{-n_0} \circ J(t) \in \ell_i\}$. Now, for $0 \leq i < r$, define $\alpha_i : [0, 1] \to \mathbf{R}^2$ as (any reparametrization of) $f^{-n_0} \circ J|_{[t^*_i,1]}$, and define U_i as the connected component of $X^c_i \cap (\ell_i, \ell_{i+1})$ whose closure contains α_i. By the invariance of the sets L^i_∞ and R^i_∞ and by (8) we have that

$$\alpha_i(1) = f^{-n_0} \circ J(1) \in \partial U_i \cap (R^i_\infty \setminus L^i_\infty),$$

so item (2) of the lemma holds for α_i. Also, as $\hat{J} \subset X^c_i$ and by the invariance of X^c_i, $f^{-n_0}\hat{J} \subset X^c_i$, and in particular $f^{-n_0} \circ J(t^*_i) \in X^c_i$. Then

$$\alpha_i(0) = f^{-n_0} \circ J(t^*_i) \in \ell_i \setminus L^i_\infty,$$

and by definition of U_i and α_i, $\alpha(t) \in U_i$ for $0 < t < 1$, so items (1) and (3) hold for U_i and α_i, and we have found, for $0 \leq i < r$, α_i and U_i as required. Then, for $0 \leq i < r$ and $j \in \mathbf{N}$, define $U_{i+jr} = T^j_1 U_i$ and $\alpha_i = T^j_1 \alpha_i$. By the periodicity of f, items (1) to (3) hold for α_i and U_i, for any $i \geq 0$.

Finally, we define N_1. By the definition of the curves α_i, we have that, for any i,

$$\alpha_i \subset X^c_i \cup R^i_\infty.$$

For any point $x \in X_i^c \cup R_\infty^i$ there exists $n \in \mathbf{N}$ such that $f^n(x) \in R(\ell_{i+1})$ (by the definition of the sets X_i and R_∞^i). Then by the compacity of each curve $\alpha_i \subset \mathbf{R}^2$, for each $0 \leq i < r$ there exists $n_i \in \mathbf{N}$ such that $f^{n_i}(\alpha_i) \subset R(\ell_{i+1})$. By definition of α_i for $i \geq r$ and by the periodicity of f, $f^{n_i}(\alpha_{i+rn}) \subset R(\ell_{i+rn})$ for any $n \geq 0$ and $0 \leq i < r$. So taking $N_1 = \max_{0 \leq i < r}\{n_i\}$, we have that $f^{N_1}(\alpha_i) \subset R(\ell_{i+1})$ for any $i \geq 0$, as we wanted. ■

The proof of Lemma 4.22 is quite long and technical. We will first prove some previous results (lemmas 4.24, 4.26, and 4.27), and then we will proceed to the proof of Lemma 4.22.

We start with the next lemma, which tells us that the points that remain under iteration by f in a strip (ℓ_i, ℓ_{i+1}), must go either upwards or downwards uniformly.

Lemma 4.24. *Given $m > 0$ there exists $N \in \mathbf{N}$ such that, if:*

- $i \geq 0$,
- $n \in \mathbf{Z}$, $|n| \geq N$,
- $x \in (\ell_i, \ell_{i+1})$ *and* $f^n(x) \in (\ell_i, \ell_{i+1})$,

then:

- *If* $\rho(\Theta_i, f) \subset \{0\} \times \mathbf{R}^+$, *then* $f^n(x)_2 - x_2 > m$ *if* $n > 0$ *and* $x_2 - f^n(x)_2 > m$ *if* $n < 0$.
- *If* $\rho(\Theta_i, f) \subset \{0\} \times \mathbf{R}^-$, *then* $x_2 - f^n(x)_2 > m$ *if* $n > 0$ *and* $f^n(x)_2 - x_2 > m$ *if* $n < 0$.

Proof. We deal with the case that $n > 0$, the case $n < 0$ being similar. By contradiction, suppose the lemma does not hold. That means that there exist $m_0 > 0$, $i_0 \geq 0$, and sequences $\{x_n\}_n \subset (\ell_{i_0}, \ell_{i_0+1})$, $\{s_n\} \subset \mathbf{Z}$, such that:

- $s_n \to \infty$ as $n \to \infty$,
- $f^{s_n}(x_n) \in (\ell_{i_0}, \ell_{i_0+1})$ for all $n \in \mathbf{N}$,
- $f^{s_n}(x_n)_2 - (x_n)_2 < m_0$ for all $n \in \mathbf{N}$, if $\rho(\Theta_{i_0}, f) \subset \{0\} \times \mathbf{R}^+$, and
- $(x_n)_2 - f^{s_n}(x_n)_2 < m_0$ for all $n \in \mathbf{N}$, if $\rho(\Theta_{i_0}, f) \subset \{0\} \times \mathbf{R}^-$.

Therefore

$$\lim_n |f^{s_n}(x_n)_2 - (x_n)_2| / s_n = 0.$$

Define the sequence of probability measures $\{\delta_n\}_n$ in $\mathbf{T}^2$ by

$$\delta_n = \frac{\delta_{\pi'(x_n)} + \delta_{\pi'(f(x_n))} + \cdots + \delta_{\pi'(f^{s_n-1}(x_n))}}{s_n},$$

and let δ be an accumulation point of $\{\delta_n\}_n$ in $\mathcal{M}_{\tilde{f}}(\mathbf{T}^2)$. Then δ is $\tilde{f}$-invariant, and

$$\rho(\delta, f) = \int \phi \, d\delta = \lim_n \int \phi \, d(\delta_n) = \lim_n \frac{1}{s_n}(f^{s_n}(x_n) - x_n) = 0,$$

where $\phi : \mathbf{T}^2 \to \mathbf{R}^2$ is the displacement function defined in section 2.1.2. Also, as $\mathrm{supp}(\delta)$ is $\tilde{f}$-invariant and is contained in $[\tilde{l}_{i_0}, \tilde{l}_{i_0+1}]$, $\mathrm{supp}(\delta)$ must be contained in Θ_{i_0}, where Θ_{i_0} is the maximal invariant set of $[\tilde{l}_{i_0}, \tilde{l}_{i_0+1}]$. This means that $(0,0) \in \rho(\Theta_{i_0}, f)$, and this is a contradiction by Proposition 4.3. This concludes the proof of the lemma. ■

As a corollary, we get that there is a maximum amount of displacement downwards, if $\rho(\Theta_i, f) \subset \{0\} \times \mathbf{R}^+$, for points that remain in (ℓ_i, ℓ_{i+1}) under iteration by f. An analogous statement is obtained for the case $\rho(\Theta_i, f) \subset \{0\} \times \mathbf{R}^-$.

Corollary 4.25. *There exists $c > 0$ such that for any $i \geq 0$, and any connected component V of $X_i^c \cap (\ell_i, \ell_{i+1})$, we have that:*

- *If $\rho(\Theta_i, f) \subset \{0\} \times \mathbf{R}^+$, $f^n(V) \cap L(\ell_{i+1}) \cap A_c^- = \emptyset$ for all $n \geq 0$, where A_c^- is the half-plane $A_c^- = \{x \in \mathbf{R}^2 \ : \ y_2 - x_2 > c \text{ for all } y \in V\}$.*
- *If $\rho(\Theta_i, f) \subset \{0\} \times \mathbf{R}^-$, $f^n(V) \cap L(\ell_{i+1}) \cap A_c^+ = \emptyset$ for all $n \geq 0$, where A_c^+ is the half-plane $A_c^+ = \{x \in \mathbf{R}^2 \ : \ x_2 - y_2 > c \text{ for all } y \in V\}$ (see Fig. 15).*

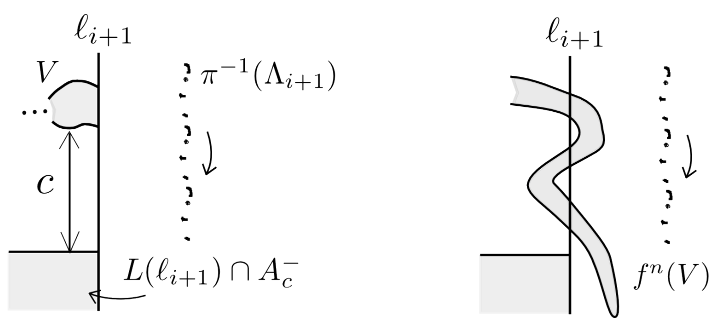

Figure 15: Illustration of Corollary 4.25 for the case $\rho(\Theta_i, f) \subset \{0\} \times \mathbf{R}^+$ and $\rho(\Theta_{i+1}, f) \subset \{0\} \times \mathbf{R}^-$. Left: The sets V and A_c^-. Right: $f^n(V) \cap L(\ell_{i+1}) \cap A_c^- = \emptyset$ for all $n \geq 0$.

Proof. By Lemma 4.24 there exists $N_0 > 0$ such that if:

- $i \geq 0$,
- $n > N_0$, and
- $x \in (\ell_i, \ell_{i+1})$ and $f^n(x) \in (\ell_i, \ell_{i+1})$,

then $f^n(x)_2 - x_2 > 0$ if $\rho(\Theta_i, f) \subset \{0\} \times \mathbf{R}^+$, and $x_2 - f^n(x)_2 > 0$ if $\rho(\Theta_i, f) \subset \{0\} \times \mathbf{R}^-$. Let $i \geq 0$ be such that $\rho(\Theta_i, f) \subset \{0\} \times \mathbf{R}^+$. Let V be any connected component of $X_i^c \cap (\ell_i, \ell_{i+1})$. Then we have that for any $x \in V$, either $f^{N_0}(x) \in R(\ell_{i+1})$ or $f^{N_0}(x)_2 > x_2$. Similarly, if $j \geq 0$ is such that $\rho(\Theta_j, f) \subset \{0\} \times \mathbf{R}^-$ and V' is any connected component of $X_j^c \cap (\ell_j, \ell_{j+1})$, we have that for any $x' \in V'$ either $f^{N_0}(x') \in R(\ell_{j+1})$ or $f^{N_0}(x')_2 < x_2$. Making $c = N_0 \|f - \mathrm{Id}\|_0$, the lemma follows. ∎

Lemma 4.26. *There exists $N_4 \in \mathbf{N}$ such that for all $i \geq 0$, $f^{N_4}(\ell_i) \cap \ell_{i+2} \neq \emptyset$.*

Proof. Fix $i \in \{0, \dots, r-1\}$. By Assumption 4.19, $f^{n_0}(\ell_i) \cap \ell_{i+1} \neq \emptyset$.
Claim. There exists an arc β contained in $f^{n_0}(\ell_i)$ such that $\beta(0) \in L_\infty^i$, $\beta(1) \in \ell_{i+1}$ and $\dot{\beta} \subset X_i^c \cap (\ell_i, \ell_{i+1})$ (see Fig. 16).
To see this, first note that as $\ell_i \cap L_\infty^i \neq \emptyset$, and as L_∞^i is f-invariant,

$$f^{n_0}(\ell_i) \cap L_\infty^i \neq \emptyset.$$

On the other hand, observe that as the points in R_∞^i have backward orbits contained in $R(\ell_i)$, then $f^{n_0}(\ell_i) \cap R_\infty^i = \emptyset$. By this and by the fact that $L_\infty^i \cap \ell_{i+1} = \emptyset$ (by definition of L_∞^i), we have that

$$f^{n_0}(\ell_i) \cap \ell_{i+1} \subset X_i^c.$$

Therefore there exist $a \in f^{n_0}(\ell_i) \cap L_\infty^i$, $b \in f^{n_0}(\ell_i) \cap (\ell_{i+1} \setminus X_i)$, and we define $\tilde{\beta}$ to be the arc contained in $f^{n_0}(\ell_i)$ with endpoints a and b. As $\tilde{\beta}$ is compact, the sets $\tilde{\beta} \cap L_\infty^i$ and $\tilde{\beta} \cap \ell_{i+1}$ are compact and disjoint. Also, as $f^{n_0}(\ell_i) \cap R_\infty^i = \emptyset$, $\tilde{\beta} \subset L_\infty^i \cup X_i^c$. Therefore there exists an arc $\beta \subset \tilde{\beta}$ with one endpoint in $\tilde{\beta} \cap L_\infty^i$, with the other endpoint in $\tilde{\beta} \cap \ell_{i+1}$, and with $\dot{\beta} \subset X_i^c \cap (\ell_i, \ell_{i+1})$. As $\beta \subset f^{n_0}(\ell_i)$, this proves our claim.

Let V be the connected component of X_i^c such that $\dot{\beta} \subset V$. Without loss of generality suppose that $\rho(\Theta_{i+1}, f) \subset \{0\} \times \mathbf{R}^-$. By Lemma 4.18 there is a

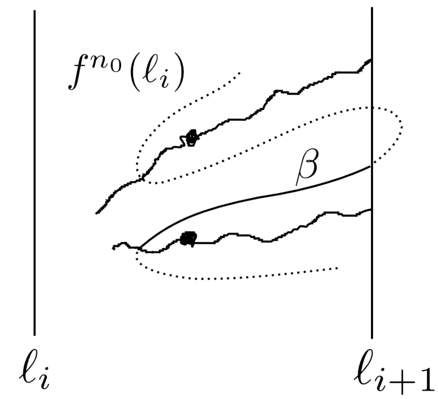

Figure 16:

constant M_1 such that $\mathrm{diam}_2(V) \leq M_1$. By Lemma 4.25 there is a constant $c > 0$ such that for all $n > 0$,

$$f^n(V) \cap \{x \in \mathbf{R}^2 \,:\, y_2 - x_2 > c \text{ for all } y \in V\} \subset R(\ell_{i+1}). \tag{9}$$

As $f^{n_0}(\ell_i) \cap \ell_{i+1} \neq \emptyset$, by Lemma 4.14 there is a constant $K_0 > 0$ such that if C is a continuum contained in (ℓ_i, ℓ_{i+1}) with $\mathrm{diam}_2(C) \geq K_0$ then $f^{n_0}(C) \cap R(\ell_{i+1}) \neq \emptyset$. By Lemma 4.24 there is a constant $N_0 > 0$ such that:

- if $x \in [\ell_{i+1}, \ell_{i+2}]$, then for any iterate $f^n(x)$ with $n \geq N_0$ such that $f^n(x) \in [\ell_{i+1}, \ell_{i+2}]$ we have that $x_2 - f^n(x)_2 > M_1 + c + K_0$, and
- if $y \in [\ell_i, \ell_{i+1}]$, then for any iterate $f^n(y)$ with $n \geq N_0$ such that $f^n(y) \in [\ell_i, \ell_{i+1}]$ we have that $f^n(y)_2 - y_2 > 0$

(recall that $\rho(\Theta_i, f) \subset \{0\} \times \mathbf{R}^+$ and $\rho(\Theta_{i+1}, f) \subset \{0\} \times \mathbf{R}^-$).

We have two possibilities: either $f^{N_0}(\beta(1)) \in \overline{R}(\ell_{i+2})$, or $f^{N_0}(\beta(1)) \in (\ell_{i+1}, \ell_{i+2})$. If $f^{N_0}(\beta(1)) \in \overline{R}(\ell_{i+2})$ we conclude that, as $f^{N_0}(\beta) \subset f^{n_0+N_0}(\ell_0)$,

$$f^{n_0+N_0}(\ell_0) \cap \ell_{i+2} \neq \emptyset.$$

Otherwise, if $f^{N_0}(\beta(1)) \in (\ell_{i+1}, \ell_{i+2})$, by the choice of N_0 and as $\mathrm{diam}_2(V) \leq M_1$ we have

$$x_2 - f^{N_0}(\beta(1))_2 > c + K_0$$

for all $x \in V$, and in particular

$$\beta(0)_2 - f^{N_0}(\beta(1))_2 > c + K_0. \tag{10}$$

As $\beta(0) \in L^i_\infty$, all the forward iterates of $\beta(0)$ stay in (ℓ_i, ℓ_{i+1}) and therefore, by the choice of N_0,

$$f^{N_0}(\beta(0))_2 > \beta(0)_2, \tag{11}$$

and by (9), as $\beta \subset V$,

$$f^{N_0}(\beta) \cap \{x \in \mathbf{R}^2 \,:\, y_2 - x_2 > c \text{ for all } y \in V\} \subset R(\ell_{i+1}). \tag{12}$$

Thus if $f^{N_0}(\beta(1)) \in (\ell_{i+1}, \ell_{i+2})$, by (10), (11) and (12) we conclude that $f^{N_0}(\beta)$ contains an arc γ such that $\gamma \subset R(\ell_{i+1})$ and $\mathrm{diam}_2(\gamma) > K_0$. Then, by the choice of the constant K_0 and by the fact that $f^{n_0}(\ell_{i+1}) \cap \ell_{i+2} \neq \emptyset$, we have

$$f^{n_0}(\gamma) \cap \ell_{i+2} \neq \emptyset.$$

As $\gamma \subset f^{n_0+N_0}(\beta) \subset f^{2n_0+N_0}(\ell_i)$, setting $N_4' = 2n_0 + N_0$ we obtain that $f^{N_4'}(\ell_i) \cap \ell_{i+2} \neq \emptyset$.

Therefore, in both cases $f^{N_0}(\beta(1)) \in \overline{R}(\ell_{i+2})$, and $f^{N_0}(\beta(1)) \in (\ell_{i+1}, \ell_{i+2})$ we have that $f^{N_4'}(\ell_i) \cap \ell_{i+2} \neq \emptyset$. This constant N_4' was obtained for a fixed i, so in this way for each $0 \leq i < r$ we obtain constants $N_4^i \in \mathbf{N}$ such that $f^{N_4^i}(\ell_i) \cap \ell_{i+2} \neq \emptyset$. Letting $N_4 = \max_{0 \leq i < r} N_4^i$, by the periodicity of f we obtain that $f^{N_4}(\ell_i) \cap \ell_{i+2} \neq \emptyset$ for all $i \geq 0$, and this proves the lemma. ■

Now we give our last lemma before the proof of Lemma 4.22. It tells us that, for a curve β contained in $[\ell_i, \ell_{i+1}]$ satisfying the hypotheses of Lemma 4.22, there is an iterate of β intersecting ℓ_{i+2}, and moreover, this iterate is independent of β and i.

Lemma 4.27. *There exists $N_5 > 0$ such that for any $i \geq 0$, and any arc β_i such that:*

- $\beta_i(0) \in L^i_\infty$,
- $\beta_i(1) \in \ell_{i+1}$, *and*
- $\beta_i(t) \in X_i^c \cap (\ell_i, \ell_{i+1})$ *for* $0 < t < 1$,

then $f^{N_5}(\beta_i) \cap \ell_{i+2} \neq \emptyset$ (see Fig. 17).

Proof. Let $i \in \{0, \ldots, r-1\}$, and suppose first that $\rho(\Theta_{i+1}, f) \subset \{0\} \times \mathbf{R}^-$. By Lemma 4.26, there is $N_4 \in \mathbf{N}$ such that

$$f^{N_4}(\ell_i) \cap \ell_{i+2} \neq \emptyset. \tag{13}$$

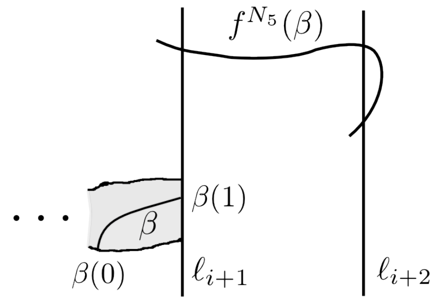

Figure 17: Illustration for Lemma 4.27.

Then, by Lemma 4.14 there is a constant $K_0 > 0$ such that if C is a continuum contained in (ℓ_i, ℓ_{i+2}) with $\text{diam}_2 \geq K_0$ then $f^{N_4}(C) \cap R(\ell_{i+2}) \neq \emptyset$.

By Lemma 4.24, there exists $N_0 > 0$ such that:

- if $x \in [\ell_{i+1}, \ell_{i+2}]$ then for all $n \geq N_0$ such that $f^n(x) \in [\ell_{i+1}, \ell_{i+2}]$ we have $x_2 - f^n(x)_2 > 2K_0 + 1/2$.
- if $y \in [\ell_i, \ell_{i+1}]$ then for all $n \geq N_0$ such that $f^n(y) \in [\ell_i, \ell_{i+1}]$ we have $f^n(y)_2 - y_2 > 0$

(recall that $\rho(\Theta_i, f) \subset \{0\} \times \mathbf{R}^+$ and $\rho(\Theta_{i+1}, f) \subset \{0\} \times \mathbf{R}^-$).
Let $\delta > 0$ be such that

$$|f^{N_4}(x) - f^{N_4}(y)| < 1/2 \quad \text{for all } x, y \in \mathbf{R}^2 \text{ such that } |y - x| < \delta.$$

Let $n_1 > 0$ be such that

$$f^{n_1}(\ell_{i+1}) \subset B_\delta(R_\infty^{i+1}) \cup \overline{R}(\ell_{i+2}). \tag{14}$$

Let $n_2 > 0$ be such that

$$f^{n_2}(x) \in \overline{R}(\ell_{i+2}) \quad \text{for all } x \in B_\delta(R_\infty^{i+1}) \setminus B_\delta(L_\infty^{i+1}). \tag{15}$$

Let β_i be as in the statement of the lemma. As $\beta_i(1) \in \ell_{i+1}$, by (14) we have

$$f^{n_1}(\beta_i(1)) \in B_\delta(R_\infty^{i+1}) \cup \overline{R}(\ell_{i+2}).$$

We have four possibilities:
1.

$$f^{n_1}(\beta_i(1)) \in \overline{R}(\ell_{i+2}).$$

2. $f^{n_1}(\beta_i(1)) \in B_\delta(R_\infty^{i+1}) \setminus B_\delta(L_\infty^{i+1})$, and therefore by (15),

$$f^{n_1+n_2}(\beta_i) \cap R(\ell_{i+2}) \neq \emptyset.$$

3. $f^{n_1}(\beta_i(1)) \in B_\delta(L_\infty^{i+1})$ and $\mathrm{diam}_2(f^{n_1}(\beta_i)) \geq K_0$. In this case, as $f^{n_1}(\beta_i) \subset R(\ell_i)$, by Lemma 4.14 and by (13) we have that

$$f^{n_1+N_4}(\beta_i) \cap \ell_{i+2} \neq \emptyset.$$

4. $f^{n_1}(\beta_i(1)) \in B_\delta(L_\infty^{i+1})$ and $\mathrm{diam}_2(f^{n_1}(\beta_i)) < K_0$. To treat this case, first note that as $\beta_i(0) \in L_\infty^i$, we have $f^{n_1+N_0}(\beta_i(0)) \in (\ell_i, \ell_{i+1})$, and by the choice of N_0,

$$f^{n_1+N_0}(\beta_i(0))_2 > f^{n_1}(\beta_i(0))_2. \tag{16}$$

Now, either

$$f^{n_1+N_0}(\beta_i(1)) \in \overline{R}(\ell_{i+2}),$$

or $f^{n_1+N_0}(\beta_i(1)) \in (\ell_{i+1}, \ell_{i+2})$. In this case, by the choice of N_0, δ, and by the fact that $f^{n_1}(\beta_i(1)) \in B_\delta(L_\infty^{i+1}) \cap (\ell_i, \ell_{i+1})$,

$$f^{n_1}(\beta_i(1))_2 - f^{n_1+N_0}(\beta_i(1))_2 > 2K_0. \tag{17}$$

Also, by (16), (17), and as $\mathrm{diam}_2(f^{n_1}(\beta_i)) < K_0$, we have

$$f^{n_1+N_0}(\beta_i(0))_2 - f^{n_1+N_0}(\beta_i(1))_2 > K_0.$$

and therefore, $\mathrm{diam}_2(f^{n_1+N_0}(\beta_i)) > K_0$. As $f^{n_1+N_0}(\beta_i) \subset R(\ell_i)$, by (13) and by the choice of the constant K_0 we conclude that

$$f^{n_1+N_0+N_4}(\beta_i) \cap \ell_{i+2} \neq \emptyset.$$

Therefore, letting $N_5^i = \max\{n_1 + n_2, n_1 + N_0 + N_4\}$ we have that, in any of these four cases $f^{N_5^i}(\beta_i) \cap \ell_{i+1} \neq \emptyset$. In a similar way we prove that if $\rho(\Theta_{i+1}, f) \subset \{0\} \times \mathbf{R}^+$, it also holds that $f^{N_5^i}(\beta_i) \cap \ell_{i+1} \neq \emptyset$. As the choice of $i \in \{0, \ldots, r-1\}$ was arbitrary, if we let $N_5 = \max_{0 \leq i < r} N_5^i$, by the periodicity of f we have that $f^{N_5}(\beta_i) \cap \ell_{i+1} \neq \emptyset$ for all $i \geq 0$, and the lemma follows. ■

Now we are ready to prove Lemma 4.22. We recall the statement of the lemma, and the definition of good intersection of an arc with a translate of some U_i.

Definition (good intersection)**.** *Let $\{U_i\}_{i\geq 0}$ and $\{\alpha_i\}_{i\geq 0}$ be as in Lemma 4.20. Let $j \in \mathbf{N}_0$ and $s \in \mathbf{Z}$. We say that a curve $\tilde{\gamma}$ has good intersection with $T_2^s(U_j)$ if $\tilde{\gamma}$ contains an arc γ such that:*

- *one endpoint of* γ *lies in* $T_2^s(\partial U_j) \cap L_\infty^j$,
- *the other endpoint of* γ *lies in* $T_2^s(\alpha_j)$, *and*
- $\dot{\gamma} \subset T_2^s(\overline{U}_j) \setminus X_j$ *(see Fig. 12).*

Lemma (4.22)**.** *There exists* $N_2 > 0$ *such that, if* $i \geq 0$*, and if* β_i *is an arc such that:*

- $\beta_i(0) \in L_\infty^i$,
- $\beta_i(1) \in \ell_{i+1}$, *and*
- $\beta_i(t) \in X_i^c \cap (\ell_i, \ell_{i+1})$ *for* $0 < t < 1$,

then $f^{N_2}(\beta_i)$ *has good intersection with* $T_2^s(U_{i+1})$*, for some* $s \in \mathbf{Z}$*.*

Proof. Fix $i \in \{0, \ldots, r-1\}$. First we treat the case $\rho(\Theta_{i+1}, f) \subset \{0\} \times \mathbf{R}^-$.

By Lemma 4.18 there exists a constant M_1 such that every connected component of $X_i^c \cap (\ell_i, \ell_{i+1})$ has diameter less or equal than M_1. Let V be the connected component of $X_i^c \cap (\ell_i, \ell_{i+1})$ that contains $\dot{\beta}_i$. By Lemma 4.25 there is a constant c such that, if $S \subset \mathbf{R}^2$ be the half-plane given by

$$S = \{x \in \mathbf{R}^2 \, : \, y_2 - x_2 > c \text{ for any } y \in V\},$$

then $f^n(\beta_i) \cap S \subset R(\ell_{i+1})$ for all $n \geq 0$.

Let $s \in \mathbf{Z}$ be such that

$$T_2^s(U_{i+1}) \subset S \quad \text{and} \quad T_2^{s+1}(U_{i+1}) \cap S^c \neq \emptyset.$$

By Lemma 4.27 there is $N_5 > 0$ such that $f^{N_5}(\beta_i) \cap \ell_{i+2} \neq \emptyset$. Let

$$c_1 = 2M_1 + c + N_5 \|f - \mathrm{Id}\|_0 + 1.$$

By Lemma 4.24, there exists $N_0 > 0$ such that if x and $f^{-N_0}(x)$ are contained in (ℓ_{i+1}, ℓ_{i+2}) then $f^{-N_0}(x)_2 - x_2 > c_1$ (recall that $\rho(\Theta_{i+1}, f) \subset \{0\} \times \mathbf{R}^-$). In particular,

$$f^{-N_0}(z)_2 - z_2 > c_1 \quad \text{for any } z \in R_\infty^{i+1} \cap L(\ell_{i+2}). \tag{18}$$

As $\mathrm{diam}_2(U_{i+1}) < M_1$ and by the definition of s, if $z \in \partial T_2^s(U_{i+1})$,

$$z + (0, M_1 + 1) \in S^c. \tag{19}$$

If $y \in V$ and $z \in f^{N_5}(\beta_i)$, we have

$$z_2 - y_2 \leq M_1 + N_5 \|f - \mathrm{Id}\|_0 . \tag{20}$$

Then, by the definition of c_1, by (18), (19) and (20), we have that for any point z in $\partial T_2^s(U_{i+1}) \cap R_\infty^{i+1}$,

$$f^{-N_0}(z)_2 > y_2 \quad \text{for any } y \in f^{N_5}(\beta_i) \cap (\ell_{i+1}, \ell_{i+1}) \tag{21}$$

(see Fig. 18).

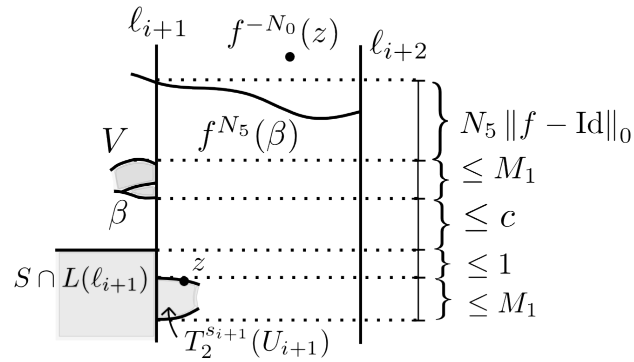

Figure 18: If $z \in \partial T_2^s(U_{i+1}) \cap R_\infty^{i+1}$, then $f^{-N_0}(z)$ is above $f^{N_5}(\beta_i) \cap (\ell_{i+1}, \ell_{i+2})$

Now, let $\beta_i^1 : [0, \infty) \to \mathbf{R}^2$ be a proper embedding such that

- $\beta_i^1(0) = f^{N_5}(\beta_i(0))$,
- $\beta_i^1(t) \in L(f^{-N_0}(\ell_{i+1}))$ for all $t > 0$, and
- $-\infty < \inf\{\beta_i^1(t)_2 \,:\, t \geq 0\}$.

Let β_i^2 be the arc contained in $f^{N_5}(\beta_i)$ with endpoints $f^{N_5}(\beta_i(0))$ and $f^{N_5}(\beta_i)(t_*)$, where $t_* = \min\{t \,:\, f^{N_5}(\beta_i(t)) \in \ell_{i+2}\}$. Let $\beta_i^3 : [0, \infty) \to \mathbf{R}^2$ be a curve contained in ℓ_{i+2}, starting in $f^{N_5}(\beta(t_*))$ and going upwards to infinity. Consider then the open unbounded disc $D \subset \mathbf{R}^2$ whose boundary is $\beta_i^1 \cup \beta_i^2 \cup \beta_i^3$ (see Fig. 19).

Observe that D is bounded from below (that is, $\inf \mathrm{pr}_2(D) > -\infty$). By (21), $f^{-N_0}(z) \in D$ for any $z \in \partial T_2^s(U_{i+1}) \cap R_\infty^{i+1}$. In particular, if α_{i+1} is as in Definition 19, then $\alpha_{i+1}(1) \in R_\infty^{i+1}$ and $f^{-N_0}(T_2^s(\alpha_{i+1}(1))) \in D$, or equivalently

$$T_2^s(\alpha_{i+1}(1)) \in f^{N_0}(D). \tag{22}$$

Note that by the definition of D,

$$f^{N_0}(\partial D) \cap (\ell_{i+1}, \ell_{i+2}) = f^{N_0}(\beta_i^2) \cap (\ell_{i+1}, \ell_{i+2}), \tag{23}$$

and then, by the definition of $S \subset \mathbf{R}^2$ and by the choice of the constant c,

$$f^{N_0}(\partial D) \cap S = f^{N_0}(\beta_i^2) \cap S \subset R(\ell_{i+1}). \tag{24}$$

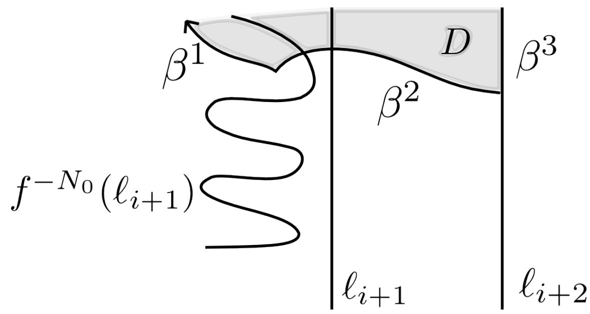

Figure 19: Illustration of the disk D.

So, we see that $f^{N_0}(\beta_i^2)$ must intersect $T_2^s(\alpha_{i+1})$: otherwise, by (22), (23) and (24), $f^{N_0}(D)$ would contain the connected set $S \cap L(\ell_{i+1}) \cup T_2^s(\alpha_{i+1})$, but this is not possible, because both D and $f^{N_0}(D)$, are bounded from below.

Observe that, as $\beta_i \subset \overline{L}(\ell_{i+1})$, $f^n(\beta_i) \cap R_\infty^{i+1} = \emptyset$, for all $n \geq 0$. Then,

$$f^{N_0}(\beta_i^2) \cap R_\infty^{i+1} \subset f^{N_5+N_0}(\beta_i) \cap R_\infty^{i+1} = \emptyset, \tag{25}$$

and therefore $\alpha_{i+1}(1) \notin f^{N_0}(\beta_i^2)$, which implies

$$f^{N_0}(\beta_i^2) \cap T_2^s(\alpha_{i+1}) \subset \text{int}(E), \tag{26}$$

where $E \subset \mathbf{R}^2$ is the set $E = (S \cap L(\ell_{i+1})) \cup T_2^s(U_{i+1})$ (see Fig. 20).

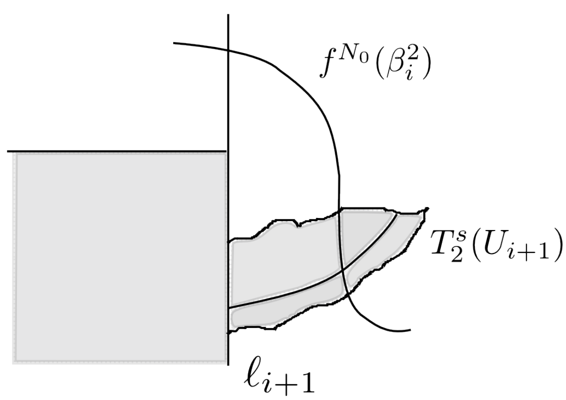

Figure 20: The set E (colored in gray).

Observe that by the definition of the integer s, $E \subset S$. Also, as $\beta_i^2(0) \in L_\infty^i$, $f^{N_0}(\beta_i^2(0)) \in L(\ell_{i+1})$, and then by (24)

$$f^{N_0}(\beta_i^2(0)) \in S^c \subset E^c. \tag{27}$$

So, by (26) and (27), $f^{N_0}(\beta_i^2) \cap \partial E \neq \emptyset$. By (24) and (25), we have

$$f^{N_0}(\beta_i^2) \cap \partial E \subset T_2^s(\partial U_{i+1}) \setminus R_\infty^{i+1}.$$

By this, by (24), and as $f^{N_0}(\beta_i^2) \cap T_2^s(\alpha_{i+1}) \neq \emptyset$, there exist $0 \leq t_1 < t_2 \leq 1$ such that:

- $f^{N_0}(\beta_i^2(t_1)) \in T_2^s(\partial U_{i+1} \setminus R_\infty^{i+1})$,
- $f^{N_0}(\beta_i^2(t_2)) \in T_2^s(\alpha_{i+1})$, and
- $f^{N_0}(\beta_i^2(t)) \in T_2^s(U_{i+1})$ for all $t_1 < t < t_2$.

This means that $f^{N_0}(\beta_i^2) \subset f^{N_5+N_0}(\beta_i)$ has good intersection with $T_2^s(U_{i+1})$.

Now suppose that $\rho(\Theta_{i+1}, f) \subset \{0\} \times \mathbf{R}^+$. In this case we have that

$$z_2 - f^{-N_0}(z)_2 > c_1 \quad \text{for any } z \in R_\infty^{i+1}.$$

If we define

$$S' = \{x \in \mathbf{R}^2 \,:\, x_2 - y_2 > c \text{ for any } y \in V\},$$

and let $s' \in \mathbf{Z}$ be such that

$$T_2^{s'}(U_{i+1}) \subset S \quad \text{and} \quad T_2^{s'-1}(U_{i+1}) \cap S^c \neq \emptyset,$$

then all the arguments above work to show that $f^{N_5+N_0}(\beta_i)$ has good intersection with $T_2^{s'}(U_{i+1})$.

The choice of this integer N_0 was made for a fixed i. So, for any $0 \leq i < r$ we obtain in this way an integer N_0^i, such that $f^{N_5+N_0^i}(\beta_i)$ has good intersection with $T_2^{s_i}(U_{i+1})$ for some $s_i \in \mathbf{Z}$. Setting $N_2 = N_5 + \max_{0 \leq i < r}\{N_0^i\}$, by the periodicity of f we have that, for all $n \in \mathbf{N}$, $f^{N_2}(\beta_{i+nr})$ has good intersection with $T_2^{s_i}(U_{i+nr+1})$. This concludes the proof of the lemma. ∎

4.5 Proof of items (1) and (4).

First we prove item (4) from Theorem B; that is, at least one of the sets Θ_i is an annular, essential, $\tilde{f}$-invariant set, which is also a semi-attractor. Observe that for each i, Θ_i is non-empty: by item 2-(b) of Lemma 4.5, $\emptyset \neq \pi''(\Lambda_i) \subset (\tilde{l}_i, \tilde{l}_{i+1})$ is $\tilde{f}$-invariant, and then $\pi''(\Lambda_i) \subset \Theta_i$. By definition Θ_i is $\tilde{f}$-invariant.

Let $\{\ell_i\}$ be the family of lifts of the curves $\tilde{l}_i$ from Definition 4.9. By Claim 4.10:

$$\text{there is } i_0 \text{ such that } f^n(\ell_{i_0}) \subset L(\ell_{i_0+1}) \text{ for all } n \in \mathbf{Z}. \tag{28}$$

With this, we now prove that Θ_{i_0} is an essential annular set. Recall that by definition,

$$\Theta_{i_0} = \bigcap_{n\in\mathbf{Z}} f^n\left([\tilde{l}_{i_0}, \tilde{l}_{i_0+1}]\right) = \bigcap_{n\in\mathbf{Z}} [\tilde{f}^n(\tilde{l}_{i_0}), f^n(\tilde{l}_{i_0+1})].$$

Recall also that the curves ℓ_{i_0} and ℓ_{i_0+1} are Brouwer curves for f. Suppose first that $\ell_{i_0} \prec f(\ell_{i_0})$ and $\ell_{i_0+1} \prec f(\ell_{i_0+1})$. For $n \in \mathbf{N}$ let $A_n = [\tilde{f}^n(\tilde{l}_{i_0}), f^{-n}(\tilde{l}_{i_0+1})]$. Then $A_{n+1} \subset A_n$, and $\Theta_{i_0} = \cap_{n\in\mathbf{N}} A_n$. By (28), for each n, $\tilde{f}^n(\tilde{l}_{i_0}) \cap f^{-n}(\tilde{l}_{i_0+1}) = \emptyset$. Then, for each n the set A_n is a topological closed essential annulus, and therefore Θ_{i_0} is an essential annular set. One deals analogously with the cases $f(\ell_{i_0}) \prec \ell_{i_0}$ and $f(\ell_{i_0+1}) \prec \ell_{i_0+1}$.

Now, either Θ_{i_0} is a semi-attractor, or it is a *repellor*, that is, $f^{-1}([\tilde{l}_{i_0}, \tilde{l}_{i_0+1}]) \subset [\tilde{l}_{i_0}, \tilde{l}_{i_0+1}]$ and $\alpha(x, \tilde{f}) \subset \Theta_{i_0}$ for any $x \in [\tilde{l}_{i_0}, \tilde{l}_{i_0+1}]$. In the case that Θ_{i_0} it is a repellor, using the fact that the curves $\tilde{l}_i$ are free for $\tilde{f}$ it is easy to verify that there is $i_1 \neq i_0$ such that Θ_{i_1} is an annular set that is an *attractor*, that is, $f([\tilde{l}_{i_1}, \tilde{l}_{i_1+1}]) \subset [\tilde{l}_{i_1}, \tilde{l}_{i_1+1}]$ and $\omega(x, \tilde{f}) \subset \Theta_{i_1}$ for any $x \in [\tilde{l}_{i_1}, \tilde{l}_{i_1+1}]$. In particular, Θ_{i_1} is a semi-attractor. Therefore at least one of the sets Θ_i is a semi-attractor, and this concludes the proof of item (4) from Theorem B.

Now we prove item (1) from Theorem B; that is, $\emptyset \neq \Omega(\tilde{f}) \cap [\tilde{l}_i, \tilde{l}_{i+1}] \subset \Theta_i$, for all i. By definition of the sets Θ_i and as the curves $\tilde{l}_i$ are free for $\tilde{f}$, each Θ_i is disjoint from the curves $\tilde{l}_i$. Also, the curves $\tilde{l}_i$ are free forever for $\tilde{f}$, and we proved that some of the sets Θ_i is an annular $\tilde{f}$-invariant set. With this and using that $\tilde{f}$ preserves orientation we have that for any i, if $S_i \subset \mathbf{T}^2$ is the topological open annulus bounded by $\tilde{l}_i$ and $\tilde{f}(\tilde{l}_i)$, then

$$\tilde{f}^n(S_i) \cap S_i = \emptyset \quad \text{for all } n \in \mathbf{Z}.$$

Also, using that the curves $\tilde{l}_i$ are free for $\tilde{f}$ and the existence of an invariant annular set, it is not difficult to see that the curves $\tilde{l}_i$ are contained in the wandering set for $\tilde{f}$. Therefore, the closed topological annulus $\overline{S}_i$ is contained in the wandering set for $\tilde{f}$.

Now, let $i \in \{0, \ldots, r-1\}$ and let $\tilde{x} \in [\tilde{l}_i, \tilde{l}_{i+1}] \setminus \Theta_i$. We want to show that $\tilde{x}$ is wandering for $\tilde{f}$. By definition of Θ_i there exists $n \in \mathbf{Z}$ such that $\tilde{f}^n(\tilde{x}) \in \mathbf{T}^2 \setminus [\tilde{l}_i, \tilde{l}_{i+1}]$. Let $x \in \pi'^{-1}(\tilde{x}) \cap (\ell_i, \ell_{i+1})$ and without loss of generality, suppose that $f^n(x) \in R(\ell_{i+1})$ and $n \in \mathbf{N}$. This means that $x \in [f^{-n}(\ell_{i+1}), f^{-n+1}(\ell_{i+1})]$, that is, $\tilde{x} \in [\tilde{f}^{-n}(\tilde{l}_{i+1}), \tilde{f}^{-n+1}(\tilde{l}_{i+1})]$. Then, by last paragraph $\tilde{x}$ is wandering for $\tilde{f}$, as we wanted. This concludes the proof of item (1) of Theorem B.

4.6 Conclusion of the proof of Theorem B.

In section 4.3 we have constructed the non-empty family $\{\tilde{l}_i\}_{i=0}^{r-1}$ of curves in $\mathbf{T}^2$ which are simple, closed, pairwise disjoint and essential, and such that the maximal invariant set Θ_i of $[\tilde{l}_i, \tilde{l}_{i+1}]$ is non-empty for all $i \in \mathbf{Z}/r\mathbf{Z}$. It was also proved in that section that the curves $\tilde{l}_i$ satisfy item (2) from Theorem B (cf. Proposition 4.3 and Remark 4.4). In section 4.4 we proved that the curves $\tilde{l}_i$ satisfy item (3) from Theorem B; that is, they are free forever for $\tilde{f}$. Finally, in section 4.5 we showed that these curves satisfy items (1) and (4) from Theorem B: at least one of the sets Θ_i is an annular, essential, $\tilde{f}$-invariant set, which also has the dissipative-type property of being a semi-attractor, the non-wandering set of $\tilde{f}$ intersects Θ_i for each i, and moreover, the non-wandering set of $\tilde{f}$ is contained in $\cup\Theta_i$. This concludes the proof of Theorem B.

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