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Hydrodynamical Limit and Large Deviations Principle for the Exclusion Process with Slow Bonds

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Doctoral Thesis
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Rio de Janeiro, 2011

If people do not believe that mathematics is simple, it is only because they not realize how complicated life is.

John Von Neumann

Agradecimentos

Esta parte de agradecimentos é uma tarefa difícil. Já que é impossível registrar num papel o agradecimento a todas pessoas que contribuíram de alguma forma para a conclusão desta tese. Se durante estes quatro anos muitas pessoas deram a sua contribuição, também existem muitos, que participaram da minha caminhada antes destes quatro anos, e que contribuíram para a realização desta tese. Sendo assim, a lista de pessoas a agradecer é imensa. Tentando não cometer injustiças, mas certa de que não agradecerei a todos, começo a expressar a minha gratidão pelas pessoas que ajudaram-me nesta caminhada.

Vou começar agradecendo ao Claudio Landim, pela orientação durante estes 4 anos e principalmente pela sua maneira singular de incentivar-me a enfrentar as dificuldades. Gostaria também de agradecer aos participantes da banca: Antonio Galves, Pablo Ferrari, Milton Jara e Robert Morris pelas contribuições para esta tese. E, falando em contribuições para esta tese, gostaria de registrar meu profundo agradecimento aos colaboradores nos trabalhos que a compõe: Tertuliano Franco, Patrícia Gonçalves e Glauco Valle.

Sou grata ao professor Artur Lopes, meu orientador no mestrado na UFRGS, por ter encorajado-me a fazer doutorado no IMPA. Também agradeço aos meus orientadores de iniciação científica da UFPel: Andrei Bourchtein e Lioudmila Bourchtein, por terem apresentado-me para a pesquisa em Matemática. Deixo aqui meus sinceros agradecimentos para todos meus ex-professores, que contribuiram na minha formação acadêmica. Destaco alguns deles: Sérgio Luiz Cardoso de Oliveira, Alexandre Tavares Baraviera e Roberto Imbuzeiro Oliveira. Os outros não mencionados não são menos importantes.

Agradeço às agências CNPq e FAPERJ pelo suporte financeiro. Gostaria de agradecer ao IMPA, por ter me acolhido durante estes 4 anos. Acredito que parte da excelência desta intituição seja devido à dedicação de seus funcionários: Fátima Russo, Dion Visgueiro, Ana Paula Rodrigues, Samantha Nunes, Juliana Bressan, Antonio Carlos, Kenia Cardoso, Andrea Nascimento, Isabel Cherques, Josenildo Salustino, Pará, Rogerio Silva, Luis Carlos Moura e todos os outros não mencionados, mas não menos importantes. Agradeço à eles todo o apoio e atenção durante a minha estadia nesta instituição.

Devo contar que esta não foi uma caminhada fácil, principalmente pelas intercorrências pessoais que me atropelaram. Esses percalços, longe de obscurecerem o trajeto, aumentaram-lhe o brilho. E, ao invés de me deterem, impulsionaram-me com mais força. Quero agradecer à espontânea generosidade de meus amigos: Tertuliano Franco, Marcelo Hilário, Patricia Cirilo, Cristina Lizana, Vanessa Barros, Anna Macedo, Maria Andrade, Julio Daniel, Pietro da Silva, Clarissa Codá, Wanderson Costa, Mirtes Santos, Renato Soares, Fernando Marek, Alan Prata, Augusto Teixeira, Ives Macedo, Dey Salvador, Emilio Vital Brazil, Cristina Vital Brazil, José Manuel Urrea. Os quais fizeram possível a transformação de momentos de angústia e sofrimento em momentos de alegria e superação. Temo aqui estar sendo injusta, mas se eu fosse enumerar todos, eu não terminaria de escrever estes agradecimentos tão cedo.

As vezes pequenos gestos fazem muita diferença, por isso quero agradecer especialmente à Jacob Palis, Vladas Sidoravicius, Luis Florit, Felipe Linares, Maria Eulália Vares, André Nachbin, Enrique Pujals, Marcelo Vianna, Paulo Sad, Mariluce Pessoa, Barbara Melo e

Clarissa Gazzaneo. Estas pessoas nem podem imaginar o quanto incentivaram-me à superar os obstáculos que a vida apresentou-me e concluir esta tese.

Quando eu parecia estar desamparada, foi quando fui muito bem amparada por excelentes profissionais: o ortopedista Vincenzo Giordano Neto e a fisioterapeuta Anna Macedo. A este dois seres humanos toda a minha gratidão ainda é pouco. Eles fizeram-me entender o sentido da frase: "Se toda a medicina não está na bondade menos vale dela separada." (Miguel Couto).

Gostaria de agradecer a *toda a minha família*, que é a base de tudo! Destaco aqui o agradecimento especial a minha mãe, pela sua calma e pelo seu amor incondicional, e aos meus avós (Celda e Lindolpho Neumann), por sempre terem incentivado-me e apoiado-me na vida e na carreira acadêmica. Mas, todos os outros integrantes da minha família são igualmente importantes para mim. E como falar de incentivo, apoio, amor e dedicação, sem não falar do meu parceiro na vida: *Alexandre Augusto Pereira Feijó.* A conclusão desta etapa também é mérito dele. Nesta caminhada ele esteve sempre ao meu lado, ajudando-me a enfrentar os obstáculos e fazendo-me feliz. Eu não tenho palavras para agradecê-lo!!!

Resumo

Esta tese de doutorado é composta por três partes, todas elas relacionadas com o processo de exclusão simétrico com elos lentos. Tais elos são os que têm a menor taxa de passagem de partículas, chamada de condutância. O objetivo desta tese é entender o comportamento coletivo de sistemas microscópicos, através de procedimentos limites, obtendo leis macroscópicas determiníticas. A primeira parte é um princípio grandes desvios para o limite de escala da medida empírica, contendo também o limite hidrodinâmico do processo de exclusão simétrico com elo lento e o limite hidrodinâmico do processo de exclusão fracamente assimétrico com elo lento. A segunda parte trata do limite hidrodinâmico da medida empírica, na presença de elos lentos com condutância $N^{-\beta}$, onde N é o parâmetro de escala. Três comportamentos diferentes são exibidos, correspondendo aos casos $\beta \in [0,1)$, $\beta = 1$ e $\beta > 1$. A terceira parte é um problema d-dimensional. Neste trabalho os elos lentos têm sua posição espacial associada a uma superfície suave e fechada, modelando uma membrana que diminui a passagem de partículas. É apresentado o limite hidrodinâmico para esse modelo.

Palavras-Chaves: Elo lento; Limite Hidrodinâmico; Grandes Desvios

Abstract

This PhD thesis consists of three parts, all of them related to the symmetric exclusion process in the presence of slow bonds, which are particular bonds with smaller rate of passage of particles, called conductance. The first part is a large deviation principle for the scaling limit of empirical measure, containing also the hydrodynamical limit of the symmetric exclusion process with slow bonds and the hydrodynamical limit of weakly asymmetric exclusion process with slows bonds. The second part deals with the hydrodynamical limit of the empirical measure in the presence of slow bonds with conductance $N^{-\beta}$, where N is the scaling parameter. Three different behaviors are exhibited, corresponding to the cases $\beta \in [0,1)$, $\beta = 1$ or $\beta > 1$. The third part is a d-dimensional problem. There, the slow bonds have a spatial position associated to a smooth closed surface, modeling a membrane slowing down the passage of particles. It is presented the hydrodynamical limit of such model.

Key words: Slow bond; Hydrodynamical limit; Large deviations

Contents

List of Figures

Introduction

The exclusion process is a continuous time interacting particle system that has been the subject of intense studies during the last decades due to the fact that, in one hand, it provides insights on the dynamical aspects of some models from statistical physics, and, in the other hand it is, up to some extent, mathematically tractable.

The exclusion process on the discrete d-dimensional torus with N sites, $\mathbb{T}_N^d = (\mathbb{Z}/N\mathbb{Z})^d$, is described by particles that move as independent random walks on a graph except for the exclusion rule that prevents two particles from occupying the same site, or vertex. The state space is therefore the set of configurations with at most one particle per site in the discrete torus, namely $\{0,1\}^{\mathbb{T}_N^d}$. In the symmetric case, the process evolves as follows: at each bond we associate a exponential clock, which is independent of the exponential clock for any other bond. When this clock rings, the occupancies of the sites connected by the bond are exchanged. The exchange rate for a bond is simply the parameter of the exponential clock associated to it. Sometimes it will be called the conductance of this bond. We allow the conductance to vary from bond to bond, and if $x, y \in \mathbb{T}_N^d$ are the nearest-neighbors, then we denote the conductance of the bond (x,y) by $\xi_{x,y}^N = \xi_{y,x}^N > 0$. The specification of the exchange rates determines the environment for the exclusion process.

We would like understand the collective behavior of the microscopic system above (the exclusion process). For this, we will need derive, through a limit procedure, deterministic macroscopic laws. Such laws will characterize the collective behavior of our system. The limit procedure mentioned earlier provides a bridge between the macroscopic and microscopic systems, which is a central problem in the clasical statistical mechanics. More specifically, when we say limit procedure, one wishes to prove at least the convergence of the time-evolution of the spatial density of particles to the solution of a macroscopic equation. The density of particles is also called by the empirical measure associated to the process. For symmetric exclusion process, it has been shown that the time evolution of the density of particles satisfies a parabolic evolution equation. This is the so called hydrodynamic limit, and it corresponds to a law of large numbers for the empirical measure.

The hydrodynamic limit says little or nothing about the rate of convergence. As a very natural consequence, it is almost unavoidable to ask oneself about deviations from the behavior hydrodynamic. The large deviations rate function, associated to the dynamical, measures the exponential decay of the asymptotic probability of deviations from the hydrodynamical evolution, when the scaling parameter diverges. Thus, we are naturally led to the investigation and identification of the large deviations rate function in the set of the empirical measures of the interacting particle systems.

This thesis studies the hydrodynamical behavior and large deviations principle of symmetric exclusion processes in non-homogeneous environments, where the non-homogeneity is due to the presence of slow bond. While an usual bond has exchange rate one, a slow bond has a lower exchange rate. With respect to the scaling parameter, we assume that a slow bond has exchange rate of order N^{-1} in the first and third part this work and of order $N^{-\beta}$, for all $\beta \geq 0$, in the second part.

When the environment is homogeneous, the exclusion process has a well-known hydrodynamical behavior under diffusive scaling. Recently, attention has been raised by the hydrodynamic behavior of interacting particle systems with random or inhomogeneous media. One relevant and puzzling problem is to consider particle systems with slow bonds and to analyze the macroscopic effect on the hydrodynamic profiles, depending on the *strength* at these bonds.

We present a brief review about some results on hydrodynamic behavior of the exclusion process in random or inhomogeneous media. In [6] the author considered the one-dimensional exclusion process with suitable random conductances $\{c_k : k \geq 1\}$. Assuming that $\{c_k^{-1} : k \geq 1\}$ satisfy a Law of Large Numbers, he proved that the randomness of the media is not present in the macroscopic time evolution of the density of particles. In [7], the authors assumed that the conductance over the bond $[\frac{x}{N}, \frac{x+1}{N}]$ is equal to $[N(W(x+1/N) - W(x/N))]^{-1}$, where W is an α-stable subordinator of a Lévy Process. In this case, the randomness survives in the continuum, by replacing in the hydrodynamical equation the usual Laplacian by a generalized operator $\frac{d}{dx}\frac{d}{dW}$, which results in the weak heat equation. In the same line of such quenched result, [9] shows the analogous behavior for a general, but non-random, strictly increasing function W. All the cited so far works are restricted to the one-dimensional setting, and strongly based on convergence results for diffusions or random walks in one-dimensional inhomogeneous media. Even the d-dimensional case treated in [21] has considered a class of non-homogeneous environments that could be decomposed, in a proper sense, into d one-dimensional cases. General sufficient conditions for the hydrodynamical limit of exclusion process in inhomogeneous media were established in [14]. All the works above have in common the association of exponential clocks to the *bonds*, the Bernoulli product measure as invariant measure, and, in some sense, the similarity with to the symmetric simple exclusion process.

In [19], it is studied the totally asymmetric simple exclusion process with a single bond having its clock parameter smaller than the other bonds. Such “slow bond", not only slows down the passage of particles across it, but also has a macroscopical impact since it disturbs the hydrodynamic profile. Somewhat intermediate between the symmetric and asymmetric case, in [2] it is considered a single asymmetric bond in the exclusion process, when the model is considered on the torus. This unique asymmetric bond gives rise to a flux in the torus and also influences the macroscopic evolution of the density of particles.

In the asymmetric case, e.g. [19] and [2], the slow bond parameter does not need to be rescaled, in order to have a macroscopic influence. Nevertheless, in the symmetric case, from [7] and [9], we see that the parameter at the slow bond must be of order N^{-1} in order to have macroscopical impact. As a consequence, one can observe a distinct behavior of slow bonds in symmetric and asymmetric settings.

About large deviations for the symmetric exclusion process in non-homogeneous environment, there is no previous references. However, the model with a slow bond has a strong similarity with models involving boundaries, as it was shown in the part II of this thesis. As we can see there, when the slow bond has conductance $N^{-\beta}$, the hydrodynamical behavior is driven by three different PDE's with distinct boundary conditions, corresponding to the respective values of β. So, this similarity with models involving boundaries allows us to apply techniques of two previous works, namely [1] and [8]. The main difficulty for establishing large deviations of symmetric exclusion process with a slow bond of parameter N^{-1} was due to the behavior near the slow bond. In the previous works [1] and [8] the authors have considered exclusion process with fixed rate of incoming and outcoming particles at the boundaries, leading to Dirichlet boundary conditions, therefore with time-independent values at the boundaries. On the other hand, the hydrodynamical behavior for the model with a slow bond, when seen as a boundary, is driven by Neumann boundary conditions, whose solutions are time-dependent at the boundaries. There is still plenty to do on this subject, with a large number of natural open questions. For instance, the large deviations for the model in the Part III of this thesis; the large deviations for cases $\beta \in (0,1)$ and $\beta > 1$ in Part II. In what follows we describe the content of each part of this thesis.

In Part I, **Hydrodynamics and large deviations of exclusion processes with slow bond**, we analyze the one-dimensional symmetric exclusion with slow bond on the torus $\mathbb{T}$. This is a joint work with Tertuliano Franco. Let us introduce more precisely this model. For a point $a \in \mathbb{T}$ fixed, the bond associated to the point a is taken as the bond that contains the point a in the natural embedding of the discrete torus in the continuous torus, $\frac{1}{N}\mathbb{T}_N \subset \mathbb{T}$, see the Figure 1. If a is a common vertex of two bonds, we consider the bond lying in the left side of point a. Note it is assumed an orientation in the continuous torus.

All the bonds have conductance equal to one, except the bond that is associate to the point $a \in \mathbb{T}$, which is called the slow bond. The conductance of this slow bond is chosen as N^{-1}.

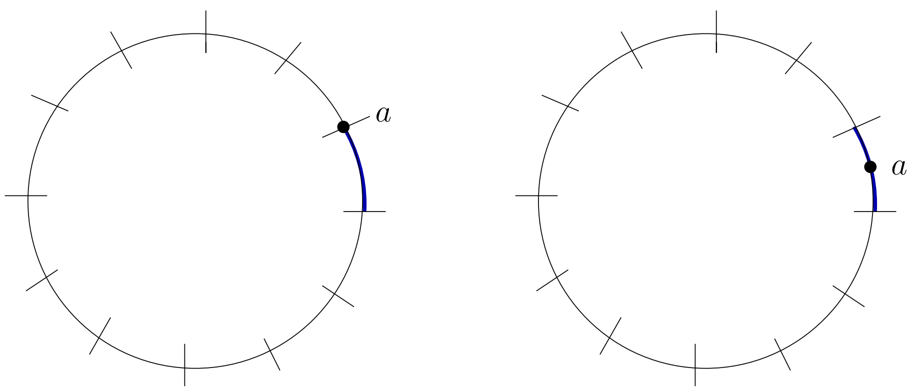

Figure 1: The slow bond is the bond associated with a.

This model is a particular case of the model considered in [9]. We propose a simpler proof of a hydrodynamic limit of this process. Besides, the proof and statement have a suitable form to be applied in the proof of large deviations. The time evolution of the density of particles $\rho(t,\cdot)$, in the diffusive scaling limit, it is described by the partial differential equation with Neumann's boundary conditions

$$\begin{cases} \partial_t \rho \;=\; \Delta\rho \\ \rho(0,\cdot) \;=\; \gamma(\cdot) \\ \partial_u\rho_t(a^+) \;=\; \partial_u\rho_t(a^-) \;=\; \rho_t(a^+)-\rho_t(a^-)\,,\ \forall t\in[0,T]\,, \end{cases} \tag{1}$$

where a^+ and a^- stand for the right and left side of a macroscopic point a related to a slow bond. More precisely, $\rho(t,\cdot)$ is a solution of the corresponding integral equation

$$\begin{aligned} &\langle \rho_t, H_t\rangle \;-\; \langle \gamma, H_0\rangle \;-\int_0^t \langle \rho_s, (\partial_s+\Delta)H_s\rangle\, ds \\ &-\int_0^t \big\{\rho_s(a^+)\partial_u H_s(a^+)-\rho_s(a^-)\partial_u H_s(a^-)\big\}\, ds \\ &+\int_0^t \big\{\rho_s(a^+)-\rho_s(a^-)\big\}\big\{H_s(a^+)-H_s(a^-)\big\}\, ds \;=\; 0\,, \end{aligned} \tag{2}$$

for all test functions $H\in C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$ and for all $t\in[0,T]$. This space $C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$ of test functions is defined in the following way: a function $H:[0,T]\times\mathbb{T}\to\mathbb{R}$ is said to belong to $C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$, if H restricted to $[0,T]\times\mathbb{T}\backslash\{a\}$ belongs to $C^{1,2}([0,T]\times\mathbb{T}\backslash\{a\})$ and H has a $C^{1,2}$ extension to $[0,T]\times[a,1+a]$, where we are identifying $(a,1+a)$ with $\mathbb{T}\backslash\{a\}$. This space of test functions should not be misunderstood with $C^{1,2}([0,T]\times\mathbb{T})$, since a typical function of $C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$ can have a discontinuity at the point $a\in\mathbb{T}$.

The proof for the large deviations upper bound can be outlined in the following scheme. For each measure μ_N on $\{0,1\}^{\mathbb{T}_N}$, denote by $\mathbb{P}_{\mu_N}$ the probability measure on the space of the trajectories $\mathcal{D}\big([0,T],\{0,1\}^{\mathbb{T}_N}\big)$ induced by the initial state μ_N and the Markov process described above. Besides, let $\big\{\exp\{NJ_\theta(\pi^N)\}\big\}_{\theta\in\mathcal{A}}$ be a family of mean-one positive martingales that can be expressed as function of the empirical measure π^N. Let $\mathcal{K}$ be a compact set in the space of trajectories. Then,

$$\begin{aligned} \mathbb{P}_{\mu_N}\big[\pi^N\in\mathcal{K}\big] &= \mathbb{E}_{\mu_N}\Big[\exp\big\{-NJ_\theta(\pi^N)\big\}\exp\big\{NJ_\theta(\pi^N)\big\}\mathbf{1}_{\{\pi^N\in\mathcal{K}\}}\Big] \\ &\le \exp\Big\{-N\inf_{\pi\in\mathcal{K}} J_\theta(\pi)\Big\}\mathbb{E}_{\mu_N}\Big[\exp\big\{NJ_\theta(\pi^N)\big\}\mathbf{1}_{\{\pi^N\in\mathcal{K}\}}\Big] \\ &\le \exp\Big\{-N\inf_{\pi\in\mathcal{K}} J_\theta(\pi)\Big\}\,. \end{aligned} \tag{3}$$

Therefore, minimizing over θ in $\mathcal{A}$, we get

$$\varlimsup_{N\to\infty}\tfrac{1}{N}\log\mathbb{P}_{\mu_N}\big[\pi^N\in\mathcal{K}\big]\le -\sup_{\theta\in\mathcal{A}}\inf_{\pi\in\mathcal{K}} J_\theta(\pi)\,,$$

and it just remains to justify the exchange between the supremum and the infimum. This is done through the Minimax Lemma, see [16, Lemma A2.3.3]. The extension to closed sets (not only compact sets) is made in Section 4.3, following the standard way of proving that the family of probabilities $\{\mathbb{P}_{\mu_N}\}_N$ is exponentially tight. The natural way to find a family of mean-one positive martingales is to consider the Radon-Nikodym derivative of a (sufficiently large) family of small perturbation of the original process with respect to the original process itself. In our case, the small perturbation is given by the weakly asymmetric exclusion process with a slow bond, indexed on the class of functions $H \in C^{1,2}\big([0,T] \times \overline{\mathbb{T}\backslash\{a\}}\big)$.

Unfortunately, this Radon-Nikodym derivative is not a function of the empirical measure, so the argument in (3) can not be applied. This is the main difficulty in the proof of a upper bound: we have to show that the Radon-Nikodym derivative is superexponentially close to a function of the empirical measure. Here, superexponentially means that the difference between the Radon-Nikodym derivative and a function of the empirical measure has expectation of order smaller than $\exp\{-CN\}$, for any chosen $C > 0$. This is done using the results of Chapter 3.

Closely related to this question, it is the energy of trajectories. Following steps of [1] and [8], we define an energy (see Definition (1.4.1)) and in Section 3.1 we prove that, in the limit, the trajectories with infinity energy are not relevant. The idea is that, in order to obtain the large deviations, one looks to the probability of observing events far from the expected limit trajectory. However, it not necessary to consider all kind of trajectories. In the limit, some kinds of trajectories are hardly seen, in the sense that they are of probability of order even smaller than exponential. The advantages of such restriction to the class of trajectories of finite energy are clear.

Next, we describe the strategy of the proof of the lower bound. We start proving a law of large numbers for a empirical measure evolving according to the perturbations such as those considered in the proof of the upper bound. This was made in Chapter 5 for the class of pertubations $H \in C^{1,2}([0,T] \times \mathbb{T})$. The second step consists in proving that the entropy of the perturbation of the process with respect to process itself, when divided by N, converges, as N tends to infinity, to the rate function I^*. This rate function is evaluated in the solution of the hydrodynamical equation associated to the perturbation of the original process. At this point is possible to conclude the proof of the lower bound for the set of all paths $\pi(t,du) = \rho(t,u)du$ such that ρ is a unique weak solution of the hydrodynamical equation associated to the perturbation of the original process.

The part dedicated to this work is divided as follows. In Chapter 1, we introduce notation and state the main results, namely Theorem 1.2.2, Theorem 1.3.2 and Theorem 1.4.1. In Chapter 2 we prove the Theorem 1.2.2. In Chapter 3, we establish the energy estimates. In Chapter 4, we prove the upper bound that is the part (i) of the Theorem 1.4.1. In Chapter 5 we prove the Theorem 1.3.2. Finally, the lower bound that is the part (ii) of the Theorem 1.4.1 is proved in the Chapter 6.

The Part II of this thesis, **Hydrodynamical behavior of symmetric exclusion with slow bonds of parameter** $N^{-\beta}$, gives a complete characterization of the hydrodynamic limit scenario for the one-dimensional exclusion process with slow bonds above. It is a joint work with Tertuliano Franco and Patrícia Gonçalves, see [10].

Here, all the bonds have an exponential clock with parameter equal to 1, except for k of bonds. As in the first work, in order to select those k bonds, we start with k macroscopic points $b_1, \ldots, b_k \in \mathbb{T}$, and consider the respective associated bonds in $\frac{1}{N}\mathbb{T}_N \subset \mathbb{T}$, see Figure 1. Those bonds will be also called slow bonds and their conductances are equal to $N^{-\beta}$, with $\beta \in [0, +\infty)$. The scale here will be taken to be diffusive in all bonds.

We prove that the time evolution of the empirical density of particles, in the diffusive scaling, has a distinct behavior according to the range of the parameter β. Note that the hydrodynamical limit for the $\beta = 1$ was already treated in the first work. However, in this work we present a simple proof for this hydrodynamical behavior. The reason why the proof here is simpler is that we consider the test functions as being the subset of functions $H \in C^{1,2}\big([0,T] \times \overline{\mathbb{T}\backslash\{b_1, \ldots, b_k\}}\big)$ satisfying

$$\partial_u H_s(b_i^+) = \partial_u H_s(b_{i+1}^-) = H_s(b_i^+) - H_s(b_{i+1}^-)\,, \quad \forall s \in [0,T] \quad \text{and} \quad \forall i = 1, ..., k\,. \tag{4}$$

This simplify the proof of characterization of the limit points, because we avoid the use of the *Replacement Lemma*. And the condition (4) prevents us to work with the integrals over the boundary of $\mathbb{T}\backslash\{b_1, ..., b_k\}$, which appear in the integral equation (2). In the first work we have considered the set $C^{1,2}\big([0,T] \times \overline{\mathbb{T}\backslash\{a\}}\big)$, because the choice of $C^{1,2}\big([0,T] \times \overline{\mathbb{T}\backslash\{a\}}\big)$ seems to be the best adapted to large deviations.

If $\beta \in [0, 1)$, the conductances in these slow bonds do not converge to zero fast enough in order to appear in the hydrodynamical limit. As a consequence, there is no macroscopical influence of the slow bonds in the continuum and we obtain the hydrodynamical equation as the usual heat equation. The proof of the last result is based on the *Replacement Lemma*, and the range parameter of β for which it holds in the sense that, it only works for $\beta \in [0, 1)$.

As β increases, the conductance at the slow bonds decreases and the passage of particles through these bonds becomes more difficult. In fact, for $\beta \in (1, +\infty)$, the clock parameters tends to zero faster than at the critical value $\beta = 1$ and each slow bond gives rise to a barrier in the continuum limit. Macroscopically this phenomena gives leads to the usual heat equation with Neumann's boundary conditions at each macroscopic point $\{b_i : i = 1, ..., k\}$. This means that the spatial derivative of ρ at each $\{b_i : i = 1, ..., k\}$ equals to zero and, physically, this represents an *isolated boundary*. Moreover, the uniqueness of weak solutions of such equation says explicitly that the macroscopic evolution of the density of particles is independent for each interval $[b_i, b_{i+1}]$, however the passage of particles in the discrete torus through the slow bonds is still possible. The proof of this result is also based on the *Replacement Lemma* and requires sharp energy estimates.

Since the regime $\beta = 1$ was already known from previous works, the main contribution of this article is the complete characterization of the three distinct behaviors for the time evolution of the empirical density of particles, exhibiting a phase transition depending on the parameter of the conductance at the slow bonds. As far as we know, no similar phenomena were exploited before for the hydrodynamic limit of interacting particle systems. Moreover, for the regime $\beta \in (1, \infty)$ the density evolves according to the heat equation with Neumann's boundary conditions, which has a meaningful physical interpretation. This the other great novelty developed in this paper. So far, partial differential equations with Dirichlet's boundary conditions could be approached by e.g. studying interacting particle systems in contact with reservoirs. Here, by considering partial differential equations with Neumann's boundary conditions, we give a step towards extending the set of treatable partial differential equations by the hydrodynamic limit theory.

In order to achieve our goal, the main difficulties appear in the characterization of limit points for each regime of β. We overcome this difficulty by developing a suitable *Replacement Lemma*, which allows us to replace the product of site occupancies by functions of the empirical measure in the continuum limit. Furthermore, that lemma is also crucial for characterizing the behavior near the slow bonds.

Our result can also be extended to non-degenerate exclusion type models as introduced in [13]. In such models, particles interact with hard core exclusion and the rate of exchange between two consecutive sites is influenced by the number of particles in the vicinity of the exchanging sites. The jump rate is strictly positive, so that all the configurations are erdogic, in the sense that a move to an unoccupied site can always occur. It was shown in [13] that the hydrodynamical equation for such models is given by a non-linear partial equation. Having established the *Replacement Lemma*, the extension of our results to these models is almost standard [9]. We also believe that our method is robust enough fitting other models such as independent random walks, the zero-range process, the generalized exclusion process, when a finite number of slow bonds is present.

The chapter dedicated to this work is divided as follows. In Section 7.1, we introduce notation and state the main result, namely Theorem 7.1.1. In Section 7.2 we make precise the scaling limit and sketch the proof of Theorem 7.1.1. In Section 7.3, we prove tightness for any range of the parameter β. In Section 7.4, we prove the *Replacement Lemma* and we establish the energy estimates, which are fundamental for characterizing the limit points and the uniqueness of weak solutions of the partial differential equations considered here. In Section 7.5 we characterize the limit points as weak solutions of the corresponding partial differential equations. Finally, uniqueness of weak solutions is refereed to Section 7.6.

The Part III of this thesis is the work **Hydrodynamic limit for a type of exclusion processes with slow bonds in dimension** ≥ 2, [12]. It is a joint work with Tertuliano Franco and Glauco Valle, accepted for publication in Journal of Applied Probability (June 2011).

We now describe the exclusion processes which we are concerned. Let $\{e_j : j = 1, ..., d\}$ be the canonical basis of $\mathbb{R}^d$ and $\Lambda \subset \mathbb{T}^d$ be a simple connected region with smooth boundary $\partial\Lambda$. If the bond $[\frac{x}{N}, \frac{x+e_j}{N}] \in N^{-1}\mathbb{T}^d_N$ has one vertex in each of the regions Λ and $\Lambda^{\complement}$, its exchange rate is defined as N^{-1} times the absolute value of the inner product between e_j and the normal exterior vector to $\partial\Lambda$. For others edges, the exchange rate is defined as one. This means that the slow bonds are among those crossing the boundary of Λ. We call this process the exclusion process with slow bonds over $\partial\Lambda$.

We can interpret $\partial\Lambda$ as a permeable membrane, which slows down the passage of particles between the regions Λ and $\Lambda^{\complement}$. For this type of exclusion process, the membrane does not completely prevent the passage of particles, and still survives in the continuum limit, appearing explicitly in the hydrodynamic equation. The exchange rate of particles for a bond crossing $\partial\Lambda$ is smaller if the bond is close to a tangent line of $\partial\Lambda$. Note that this assumption has physical meaning, take for example cases of reflections in several physical models: Partial reflection of light crossing a medium with different refraction indexes, mechanical systems where particles try to cross some interface, etc. However the direction of the speed of particles is not changed as usually occur in physical reflection. Our definition of the exchange rates also allows a strong convergence result for the empirical measures associated to the exclusion process making simpler the proof of the hydrodynamic limit.

The hydrodynamical equation of the exclusion process with slow bonds over $\partial\Lambda$ is a parabolic partial differential equation $\partial_t\rho = \mathcal{L}_\Lambda\rho$, where the operator $\mathcal{L}_\Lambda$ is a sort of d-dimensional Krein-Feller operator. Without the presence of slow bonds, the operator $\mathcal{L}_\Lambda$ would be replaced by the laplacian operator acting on C^2 functions and the hydrodynamical equation would therefore be the heat equation. Here, the existence of the membrane modifies the domain, and thus the operator itself. In fact, we observe that the proper domain for $\mathcal{L}_\Lambda$ contains functions that are discontinuous over $\partial\Lambda$. Geometrically, $\mathcal{L}_\Lambda$ glues the discontinuity of a function around $\partial\Lambda$ and then behaves like the laplacian.

One possible approach to prove the hydrodynamic limit for the exclusion process with slow bonds over $\partial\Lambda$ is through Gamma convergence. In [14], this approach and the conditions for it to hold are discussed, see also [6]. There, the coersiveness condition would require some kind of Rellich-Kondrachov's Theorem (namely, the compact embedding in L^2 of some sort of Sobolev space supporting an extension of $\mathcal{L}_\Lambda$, see [4]). In the method presented here, we go in this direction, but instead of reach the hypotheses in [14], we have used similar analytical tools in order to obtain a short and simple proof of uniqueness of the hydrodynamic equation. We also show that the extension of $\mathcal{L}_\Lambda$ satisfies the Hille-Yoshida Theorem. On the other hand, the convergence from discrete to continuous that we present here is made in a very direct way, and it was inspired by the convergence of the discrete laplacian to the continuous laplacian.

The chapter dedicated to this work is divided as follows: In Section 8.1, we define the model and state all results contained in the paper; Section 8.2 is devoted to prove the results concerning the continuous operator $\mathcal{L}_\Lambda$; In Section 8.3, the hydrodynamic limit is proved.

Part I

Hydrodynamics and large deviations for the exclusion processes with slow bond

Chapter 1

Notation and Results

1.1 The model

Let $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$ be the one-dimensional discrete torus with N points. The points of $\mathbb{T}_N$, called sites, are represented by the last characters of the alphabet (x, y and z). Denote by η the configurations of the state space $\{0,1\}^{\mathbb{T}_N}$ so that $\eta(x) = 0$, if site x is vacant and $\eta(x) = 1$, if site x is occupied.

We define now the exclusion process with state space $\{0,1\}^{\mathbb{T}_N}$ and with conductance $\{\xi^N_{x,x+1}; x \in \mathbb{T}_N\}$ at the bond of vertices $x, x+1$. The dynamics of this Markov process can be described as follows. At each bond of vertices $x, x+1$, we associate an exponential clock of parameter $\xi^N_{x,x+1}$, which are independent of the exponential clock for any other bonds. When this clock rings, the values of η at the vertices of this bond are exchanged. This process can also be characterized in terms of its infinitesimal generator L_N, which acts on local functions $f : \{0,1\}^{\mathbb{T}_N} \to \mathbb{R}$ as

$$(L_N f)(\eta) = \sum_{x \in \mathbb{T}_N} \xi^N_{x,x+1} \left[f(\eta^{x,x+1}) - f(\eta) \right], \tag{1.1}$$

where $\eta^{x,x+1}$ is the configuration obtained from η by exchanging the variables $\eta(x)$, $\eta(x+1)$:

$$(\eta^{x,x+1})(y) = \begin{cases} \eta(x+1), & \text{if } y = x \\ \eta(x), & \text{if } y = x+1 \\ \eta(y), & \text{otherwise}. \end{cases} \tag{1.2}$$

Denote by $\mathbb{T} = [0,1)$ the one-dimensional continuous torus, where we are identifying the values 0 and 1. Fix a point $a \in \mathbb{T}$. In the model, it is assumed that the jump rates are given by

$$\xi^N_{x,x+1} = \xi^N_{x+1,x} = \begin{cases} \frac{1}{N}, & \text{if } a \in \left(\frac{x}{N}, \frac{x+1}{N}\right] \\ 1, & \text{otherwise}. \end{cases} \tag{1.3}$$

To simplify notation, sometimes we denote $\xi^N_{x,x+1}$ by ξ^N_x. In some parts of this work, we will consider $a = 0$, but in other parts we will write the results for a general $a \in \mathbb{T}$. Such double choice was taken aiming to simplify notation in some parts and, when necessary, to make clear that all results apply to a finite number of slow bonds associated to points $a^1, \dots, a^k \in \mathbb{T}$. We write a_N for the site of the left side of the slow bond in the discrete torus $\mathbb{T}_N$, $a \in \big(\frac{a_N}{N}, \frac{a_N+1}{N}\big]$. For instance, when $a = 0$, then $a_N = -1$.

A simple computation shows that the Bernoulli product measures $\{\nu_\alpha^N; 0 \le \alpha \le 1\}$ are invariant, in fact reversible, for the dynamics. The measure ν_α^N is obtained by placing a particle at each site, independently from other sites, with probability α. Thus, ν_α^N is a product measure over $\{0,1\}^{\mathbb{T}_N}$ with marginals given by

$$\nu_\alpha^N\{\eta; \eta(x) = 1\} \;=\; \alpha\,, \quad \text{for}\, x \in \mathbb{T}_N\,.$$

Denote by $\{\eta_t^N; t \in [0,T]\}$ the Markov process on $\{0,1\}^{\mathbb{T}_N}$ associated to the generator L_N, defined in (1.1), *speeded up* by N^2. When the dependency of N is evident, sometimes, we omit the index N of η_t^N. Let $\mathcal{D}\big([0,T], \{0,1\}^{\mathbb{T}_N}\big)$ be the path space of càdlàg trajectories with values in $\{0,1\}^{\mathbb{T}_N}$. For a measure μ_N on $\{0,1\}^{\mathbb{T}_N}$, denote by $\mathbb{P}_{\mu_N}$ the probability measure on $\mathcal{D}\big([0,T], \{0,1\}^{\mathbb{T}_N}\big)$ induced by the initial state μ_N and the Markov process $\{\eta_t^N; t \in [0,T]\}$. Expectation with respect to $\mathbb{P}_{\mu_N}$ is denoted by $\mathbb{E}_{\mu_N}$.

A sequence of probability measures $\{\mu_N; N \ge 1\}$ is said to be associated to a profile $\rho_0 : \mathbb{T} \to [0,1]$ if

$$\lim_{N\to\infty} \mu_N \Big[\Big| \tfrac{1}{N} \sum_{x \in \mathbb{T}_N} H(\tfrac{x}{N}) \eta(x) - \int H(u) \rho_0(u) du \Big| > \delta \Big] \;=\; 0\,, \tag{1.4}$$

for every $\delta > 0$ and every continuous functions $H : \mathbb{T} \to \mathbb{R}$.

The quantity just introduced in the definition above can be reformulated in terms of empirical measures. Let $\mathcal{M}$ be the space of positive measures on $\mathbb{T}$ with total mass bounded by one endowed with the weak topology. Consider the measure $\pi^N \in \mathcal{M}$, which is obtained by rescaling space by N and by assigning mass N^{-1} to each particle:

$$\pi^N(\eta, du) \;=\; \tfrac{1}{N} \sum_{x \in \mathbb{T}_N} \eta(x)\, \delta_{\frac{x}{N}}(du)\,,$$

where δ_u is the Dirac measure concentrated on u. The measure $\pi^N(\eta, du)$ is called the empirical measure associated to the configuration η. The dependence in η will frequently be omitted to keep notation as simple as possible. With this notation $\frac{1}{N} \sum_{x \in \mathbb{T}_N} H(\frac{x}{N}) \eta(x)$ is the integral of H with respect to the empirical measure π^N, denoted by $\langle \pi^N, H \rangle$.

We consider the time evolution of the empirical measure π_t^N associated to the Markov process $\{\eta_t^N; t \ge 0\}$ by:

$$\pi_t^N(du) \;=\; \pi^N(\eta_t, du) \;=\; \tfrac{1}{N} \sum_{x \in \mathbb{T}_N} \eta_t(x)\, \delta_{\frac{x}{N}}(du)\,. \tag{1.5}$$

Fix $T > 0$. Let $\mathcal{D}\big([0,T],\mathcal{M}\big)$ be the space of $\mathcal{M}$-valued càdlàg trajectories $\pi : [0,T] \to \mathcal{M}$ endowed with the *Skorohod* topology. Notice that $\{\pi_t^N; 0 \le t \le T\}$ belongs to $\mathcal{D}\big([0,T],\mathcal{M}\big)$ and inherits the Markov property from $\{\eta_t^N; t \ge 0\} \in \mathcal{D}\big([0,T],\{0,1\}^{\mathbb{T}_N}\big)$.

For each probability measure μ_N on $\{0,1\}^{\mathbb{T}_N}$, denote by $\mathbb{Q}_{\mu_N}$ the measure on the path space $\mathcal{D}\big([0,T],\mathcal{M}\big)$ induced by the measure μ_N and the empirical process π_t^N introduced in (1.5).

The exclusion process with slow bond has a related random walk on $\frac{1}{N}\mathbb{T}_N$ that describes the evolution of the system with a single particle. Thus particles in the exclusion process evolve independently as such random walk except for the hard core interaction. To simplify notation later, we introduce here the generator of this random walk with conductances $\xi^N_{x,x+1}$, which is given by

$$(\mathbb{L}_N H)(\tfrac{x}{N}) = \xi^N_{x,x+1}\big\{H(\tfrac{x+1}{N}) - H(\tfrac{x}{N})\big\} + \xi^N_{x,x-1}\big\{H(\tfrac{x-1}{N}) - H(\tfrac{x}{N})\big\}\,, \tag{1.6}$$

for x in $\mathbb{T}_N$ and a function $H : \frac{1}{N}\mathbb{T}_N \to \mathbb{R}$. We will not distinguish the notation for functions H defined on $\mathbb{T}$ and on $\mathbb{T}_N$.

The indicator function of a set A will be written by $\mathbf{1}_A(u)$, which is one when $u \in A$ and zero otherwise. Given a function $f : \mathbb{T} \to \mathbb{R}$, we will denote $f(a^-)$ and $f(a^+)$, respectively, for the left and right side limits of f at the point $a \in \mathbb{T}$. In the case that $a = 0$, we can also use the notation $f(1)$ and $f(0)$ for denote, respectively, the left and right side limits of f at the point $0 \in \mathbb{T}$. We are going to use the notation $g_t(u)$ to denote $g(t,u)$, for a function $g : [0,T] \times \mathbb{T}$. It must cause no confusion with the notation for time derivative, namely $\partial_t g(t,u)$.

1.2 Hydrodynamic limit of exclusion process with slow bond

1.2.1 The hydrodynamic equation

For a non-negative integer k denote by $C^k(\mathbb{T})$ the set of continuous functions from $\mathbb{T}$ to $\mathbb{R}$ with continuous derivatives of order up to k. The set $C^0(\mathbb{T})$ will be written just as $C(\mathbb{T})$. For non-negative integers j and k denote by $C^{j,k}([0,T] \times \mathbb{T})$ the set of continuous functions from $[0,T] \times \mathbb{T}$ to $\mathbb{R}$ with continuous derivatives of order up to j in the temporal coordinate, $t \in [0,T]$, and k in the spatial coordinate, $u \in \mathbb{T}$.

The study of exclusion process dynamics in presence of a slow bond requires the use of functions defined in the continuous torus, which must be smooth except possibly at the point $a \in \mathbb{T}$. In such a way, consider the following

Definition 1.2.1. *Denote by $C^{1,2}\big([0,T] \times \overline{\mathbb{T}\backslash\{a\}}\,\big)$ the space of functions $H : [0,T] \times \mathbb{T} \to \mathbb{R}$ such that*

1. *H restricted to $[0,T]\times\mathbb{T}\backslash\{a\}$ belongs to $C^{1,2}([0,T]\times\mathbb{T}\backslash\{a\})$;*

2. *H has a $C^{1,2}$ extension to $[0,T]\times[a,1+a]$, where we are identifying $(a,1+a)$ with $\mathbb{T}\backslash\{a\}$.*

It is of worth pointing out the meaning of such definition. Note this space of test functions should not be misunderstood with $C^{1,2}([0,T]\times\mathbb{T})$, since a typical function of $C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$ can have a discontinuity at the point $a\in\mathbb{T}$.

Denote by $\langle\cdot,\cdot\rangle$ the inner product in $L^2(\mathbb{T})$ and by ρ_t the function $\rho(t,\cdot)$.

Definition 1.2.2. *Consider a bounded density profile $\gamma:\mathbb{T}\to\mathbb{R}$. A bounded function $\rho:[0,T]\times\mathbb{T}\to\mathbb{R}$ is said to be a weak solution of the parabolic differential equation*

$$\begin{cases} \partial_t\rho \,=\, \Delta\rho \\ \rho(0,\cdot) \,=\, \gamma(\cdot) \\ \partial_u\rho_t(a^+)=\partial_u\rho_t(a^-)=\rho_t(a^+)-\rho_t(a^-)\,,\ \forall t\in[0,T]\,, \end{cases} \tag{1.7}$$

if the following two conditions are fulfilled:

(1) $\rho\in L^2\big(0,T;\mathcal{H}^1(\mathbb{T}\backslash\{a\})\big)$;

(2) For all functions $H\in C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$ and for all $t\in[0,T]$, ρ satisfies the integral equation

$$\begin{aligned} \langle\rho_t,H_t\rangle \,-\, \langle\gamma,H_0\rangle \,=\, & \int_0^t \langle\rho_s,(\partial_s+\Delta)H_s\rangle\,ds \\ +\int_0^t & \big\{\rho_s(a^+)\partial_uH_s(a^+)-\rho_s(a^-)\partial_uH_s(a^-)\big\}\,ds \\ -\int_0^t & \big\{\rho_s(a^+)-\rho_s(a^-)\big\}\big\{H_s(a^+)-H_s(a^-)\big\}\,ds\,. \end{aligned} \tag{1.8}$$

Remark 1.2.1. *For a heuristics about why we denote the integral equation* (1.8) *in the way* (1.7)*, one should multiply both sides of* (1.7) *by a test function $H\in C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$, integrate in space and time and then perform twice a formal integration by parts, obtaining the equation*

$$\begin{aligned} \langle\rho_t,H_t\rangle \,-\, \langle\gamma,H_0\rangle \,=\, & \int_0^t \langle\rho_s,(\partial_s+\Delta)H_s\rangle\,ds \\ +\int_0^t & \big\{\rho_s(a^+)\partial_uH_s(a^+)-\rho_s(a^-)\partial_uH_s(a^-)\big\}\,ds \\ -\int_0^t & \big\{\partial_u\rho_s(a^+)H_s(a^+)-\partial_u\rho_s(a^-)H_s(a^-)\big\}\,ds\,, \end{aligned}$$

Applying the formal boundary conditions about ρ, one gets (1.8)*. Besides, it shows that any solution in the strong sense of* (1.7) *is a weak solution of* (1.7)*.*

In Section 2.6, we show uniqueness of integral solutions. Existence follows from the tightness of the sequence of probability measures $\mathbb{Q}_{\mu_N}$ introduced in Section 2.2 and the characterization of limit points given in Section 2.5.

We are now in position to state the main result of this section:

Theorem 1.2.2. *Fix a continuous initial profile $\gamma : \mathbb{T} \to [0,1]$ and consider a sequence of probability measures μ_N on $\{0,1\}^{\mathbb{T}_N}$ associated to γ in the sense* (1.4)*. Then, for any $t \in [0,T]$,*

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\left[\left|\frac{1}{N}\sum_{x\in\mathbb{T}_N} H(\tfrac{x}{N})\eta_t(x) - \int H(u)\rho(t,u)du\right| > \delta\right] = 0\,,$$

for every $\delta > 0$ and every function $H \in C(\mathbb{T})$. Here, ρ is the unique weak solution of the linear equation (1.7) *with $\rho_0 = \gamma$.*

1.3 Hydrodynamic limit of weakly asymmetric exclusion process with a slow bond

For each function $H \in C^{1,2}([0,T]\times\mathbb{T})$, consider the time inhomogeneous Markov process whose generator at time t is given by

$$\begin{aligned}(L_{N,t}^H f)(\eta) = &\sum_{x\in\mathbb{T}_N} \xi_x^N\, e^{H_t(\frac{x+1}{N})-H_t(\frac{x}{N})}\,\eta(x)\big(1-\eta(x{+}1)\big)\big[f(\eta^{x,x+1})-f(\eta)\big]\\ &+\sum_{x\in\mathbb{T}_N} \xi_x^N\, e^{-H_t(\frac{x+1}{N})+H_t(\frac{x}{N})}\,\eta(x{+}1)\big(1-\eta(x)\big)\big[f(\eta^{x,x+1})-f(\eta)\big]\,,\end{aligned} \tag{1.9}$$

for all $f : \{0,1\}^{\mathbb{T}_N} \to \mathbb{R}$, where $\eta^{x,x+1}$ is defined in (1.2) and the conductance ξ_x^N was defined in (1.3). Notice that, if the function $H \in C^{1,2}([0,T]\times\mathbb{T})$ is constant, the infinitesimal generator $L_{N,t}^H$ is equal to the infinitesimal generator L_N, which is defined in (1.1).

For each probability measure μ_N on $\{0,1\}^{\mathbb{T}_N}$, denote by $\mathbb{P}_{\mu_N}^H$ ($\mathbb{Q}_{\mu_N}^H$, respectively) the probability measure on the space of trajectories $\mathcal{D}\big([0,T],\{0,1\}^{\mathbb{T}_N}\big)$ ($\mathcal{D}\big([0,T],\mathcal{M}\big)$, respectively) corresponding to the inhomogeneous Markov process η_t (π_t^N, respectively) with generator L_N^H defined in (1.9) accelerated by N^2 and starting from μ_N.

1.3.1 The hydrodynamic equation

Definition 1.3.1. *Consider a bounded density profile $\gamma : \mathbb{T} \to \mathbb{R}$ and $H \in C^{1,2}([0,T]\times\mathbb{T})$. A function $\rho : [0,T]\times\mathbb{T} \to [0,1]$ is said to be a weak solution of the partial differential equation*

$$\left\{\begin{array}{l}\partial_t\rho \,=\, \Delta\rho - 2\,\partial_u\big(\chi(\rho)\partial_u H\big)\\ \rho(0,\cdot) \,=\, \gamma(\cdot)\\ \partial_u\rho_t(a^+) \,=\, 2\,\chi\big(\rho_t(a^+)\big)\,\partial_u H_t(a^+) + \rho_t(a^+) - \rho_t(a^-),\ \forall\, t\in[0,T]\\ \partial_u\rho_t(a^-) \,=\, 2\,\chi\big(\rho_t(a^-)\big)\,\partial_u H_t(a^-) + \rho_t(a^+) - \rho_t(a^-),\ \forall\, t\in[0,T]\end{array}\right. \tag{1.10}$$

if the following two conditions are fulfilled:

(1) $\rho \in L^2\big(0,T;\mathcal{H}^1(\mathbb{T}\backslash\{a\})\big)$ *;*

(2) For all functions G *in* $C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\,\big)$*, and all* $t\in[0,T]$*,* ρ *satisfies the integral equation*

$$\begin{aligned}\langle\rho_t,G_t\rangle\;-\;\langle\gamma,G_0\rangle = &\int_0^t\big\langle\rho_s,(\partial_s+\Delta)G_s\big\rangle\,ds+2\int_0^t\big\langle\chi(\rho_s)\partial_uH_s\,,\partial_uG_s\big\rangle\,ds\\ &+\int_0^t\big\{\rho_s(a^+)\partial_uG_s(a^+)-\rho_s(a^-)\partial_uG_s(a^-)\big\}\,ds\\ &-\int_0^t\big\{\rho_s(a^+)-\rho_s(a^-)\big\}\big\{G_s(a^+)-G_s(a^-)\big\}\,ds\,,\end{aligned}\tag{1.11}$$

where ρ_s *is the notation for* $\rho(s,\cdot)$*,* $\chi(\alpha)=\alpha(1-\alpha)$*.*

Remark 1.3.1. *Notice that the boundary integral is well defined by assumption (1). Choosing* H *as the constant function identically equals one, we get the hydrodynamical equation* (1.7)*, as expected. Besides, the expressions* $\partial_u\rho_t(a^+)$ *and* $\partial_u\rho_t(a^-)$ *appearing in* (1.10) *differ only on its first parcel.*

At least, the reason we call the integral equation (1.11) *in the way* (1.10) *is the same reason that we denote the integral equation* (1.8) *in the way* (1.7)*. Suppose* ρ *is a smooth solution of* (1.11)*. Multiplying both sides by a test function* G *smooth in* $\mathbb{T}\backslash\{a\}$*, integrating in space and time, and then performing the respective integrations by parts, some boundary integrals will appear. These boundary conditions imposed in* (1.10) *are the exact conditions needed to obtain the integral equation* (1.11)*. Or else, any solution in the strong sense of* (1.10) *is a weak solution of* (1.10)*.*

In Section 5.4, we shall prove uniqueness of such weak solutions. Existence of solutions follows from the tightness of the sequence of probability measures $\mathbb{Q}^H_{\mu_N}$ introduced in Section 5.1 and the characterization of limit points given in Section 5.3.

We are now in position to state the main result of this section:

Theorem 1.3.2. *Let* $H\in C^{1,2}([0,T]\times\mathbb{T})$*. Fix a continuous initial profile* $\gamma:\mathbb{T}\to[0,1]$ *and consider a sequence of probability measures* μ_N *on* $\{0,1\}^{\mathbb{T}_N}$ *associated to* γ *in the sense* (1.4)*. Then, for any* $t\in[0,T]$*,*

$$\lim_{N\to\infty}\mathbb{P}^H_{\mu_N}\Bigg[\Big|\tfrac{1}{N}\sum_{x\in\mathbb{T}_N}G(\tfrac{x}{N})\eta_t(x)-\int G(u)\rho(t,u)du\Big|>\delta\Bigg]\;=\;0\,,$$

for every $\delta>0$ *and every function* $G\in C(\mathbb{T})$ *. Here,* ρ *is the unique weak solution of the diferential equation* (1.10) *with* $\rho_0=\gamma$*.*

1.4 Large deviations

Denote by $\mathcal{M}_0$ the subset of $\mathcal{M}$ of all absolutely continuous measures with density bounded by 1:

$$\mathcal{M}_0 = \Big\{ \omega \in \mathcal{M};\, \omega(du) = \rho(u)du \quad \text{and} \quad 0 \le \rho \le 1 \quad \text{almost surely} \Big\}.$$

The set $\mathcal{M}_0$ is a closed subset of $\mathcal{M}$ endowed with the weak topology (see A.2.1). This property is inherited by $\mathcal{D}\big([0,T],\mathcal{M}_0\big)$, which is a closed subset of $\mathcal{D}\big([0,T],\mathcal{M}\big)$ for the Skorohod topology.

Denote by ∂_u the partial derivative of a function with respect to the space variable. Let $L^2([0,T]\times\mathbb{T})$ be the Hilbert space of measurable functions $H:[0,T]\times\mathbb{T}\to\mathbb{R}$ such that

$$\int_0^T \int_{\mathbb{T}} \big(H(s,u)\big)^2\, du\, ds \;<\; \infty\,,$$

endowed with the scalar product $\langle\!\langle H, G\rangle\!\rangle$ defined by

$$\langle\!\langle H, G\rangle\!\rangle \;=\; \int_0^T \int_{\mathbb{T}} H(s,u)\, G(s,u)\, du\, ds\,.$$

Definition 1.4.1. *For $H \in C^{0,1}([0,T]\times\mathbb{T})$ with compact support contained in $[0,T]\times(\mathbb{T}\backslash\{a\})$, define $\mathcal{E}_H : D([0,T],\mathcal{M}) \to \overline{\mathbb{R}}$ by*

$$\mathcal{E}_H(\pi) \;=\; \begin{cases} \langle\!\langle \partial_u H, \rho\rangle\!\rangle - 2\langle\!\langle H, H\rangle\!\rangle\,, & \textit{if } \; \pi \in D([0,T],\mathcal{M}_0)\,, \\ +\infty\,, & \textit{otherwise}\,. \end{cases}$$

Futhermore, define the energy functional $\mathcal{E} : D([0,T],\mathcal{M}) \to \overline{\mathbb{R}}$ by

$$\mathcal{E}(\pi) = \sup_H \mathcal{E}_H(\pi)\,, \tag{1.12}$$

where the supremum is taken over all $H \in C^{0,1}([0,T]\times\mathbb{T})$ with compact support contained in $[0,T]\times(\mathbb{T}\backslash\{a\})$.

In Section 2.4, we prove that $\pi \in \mathcal{D}([0,T],\mathcal{M})$ with $\mathcal{E}(\pi) < \infty$, there exists $\rho \in L^2(0,T;\mathcal{H}^1(\mathbb{T}\backslash\{a\}))$, such that $\pi(t,du) = \rho_t(u)du$.

Definition 1.4.2. *Let $H \in C^{1,2}([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\,)$, define for $\pi \in \mathcal{D}([0,T],\mathcal{M})$ with $\mathcal{E}(\pi) < \infty$, $\pi(t,du) = \rho_t(u)du$,*

$$\begin{aligned} \ell_H(\pi) = \langle \rho_T, H_T\rangle - \langle \rho_0, H_0\rangle - \int_0^T \langle \rho_t, (\partial_t + \Delta) H_t\rangle\, dt \\ - \int_0^T \{\rho_t(a^+)\partial_u H_t(a^+) - \rho_t(a^-)\partial_u H_t(a^-)\}\, dt \end{aligned} \tag{1.13}$$

and

$$
\begin{aligned}
\hat{J}_H(\pi) = \ell_H(\pi) &- \int_0^T \langle \chi(\rho_t), (\partial_u H_t)^2 \rangle \, dt \\
&- \int_0^T \rho_t(a^-)(1-\rho_t(a^+))\big(e^{H_t(a^+)-H_t(a^-)} - 1\big) \, dt \\
&- \int_0^T \rho_t(a^+)(1-\rho_t(a^-))\big(e^{-H_t(a^+)+H_t(a^-)} - 1\big) \, dt \,.
\end{aligned} \tag{1.14}
$$

Definition 1.4.3. *For all function $H \in C^{1,2}([0,T] \times \overline{\mathbb{T}\backslash\{a\}})$, we define the functional $J_H : \mathcal{D}([0,T], \mathcal{M}) \to \overline{\mathbb{R}}$, by*

$$
J_H(\pi) \;=\; \begin{cases} \hat{J}_H(\pi), & \text{if } \mathcal{E}(\pi) < \infty \,, \\ +\infty, & \text{otherwise.} \end{cases}
$$

Define $I : \mathcal{D}([0,T], \mathcal{M}) \to [0,\infty]$ by

$$
I(\pi) \;=\; \sup_{H \in C^{1,2}([0,T] \times \overline{\mathbb{T}\backslash\{a\}})} J_H(\pi)
$$

and $I^* : \mathcal{D}([0,T], \mathcal{M}) \to [0,\infty]$ by

$$
I^*(\pi) \;=\; \sup_{H \in C^{1,2}([0,T] \times \mathbb{T})} J_H(\pi) \,.
$$

Denote by $\mathcal{D}^0\big([0,T], \mathcal{M}_0\big)$ the subset of $\mathcal{D}\big([0,T], \mathcal{M}_0\big)$ consisting of all paths $\pi(t,du) = \rho(t,u)du$ such that there exists $H \in C^{1,2}([0,T] \times \mathbb{T})$ that $\rho = \rho^H$ is a unique weak solution of (1.10).

We are now in position to state the main result of this section:

Theorem 1.4.1. *The sequence of measures $\{\mathbb{Q}_{\nu_\alpha^N}; N \geq 1\}$ satisfies:*

(i) **Upper bound:** *Let $\mathcal{C}$ be a closed subset of $\mathcal{D}\big([0,T], \mathcal{M}\big)$. Then*

$$
\varlimsup_{N \to \infty} \tfrac{1}{N} \log \mathbb{Q}_{\nu_\alpha^N}[\mathcal{C}] \leq - \inf_{\pi \in \mathcal{C}} I(\pi) \,.
$$

(ii) **Lower bound:** *Let $\mathcal{O}$ be an open subset of $\mathcal{D}\big([0,T], \mathcal{M}\big)$. Then*

$$
\varliminf_{N \to \infty} \tfrac{1}{N} \log \mathbb{Q}_{\nu_\alpha^N}[\mathcal{O}] \geq - \inf_{\pi \in \mathcal{O} \cap \mathcal{D}^0([0,T], \mathcal{M}_0)} I^*(\pi) \,.
$$

The item (i) of the theorem above is proved in Chapter 4. The other item of the last theorem is proved in Chapter 6.

Chapter 2

Hydrodynamic limit for the exclusion process with slow bond

2.1 Scaling Limit

We begin by recalling that for a function $H : \mathbb{T} \to \mathbb{R}$, $\langle \pi_t^N, H\rangle$ stands for the integral of H with respect to π_t^N:

$$\langle \pi_t^N, H\rangle \;=\; \tfrac{1}{N}\sum_{x\in\mathbb{T}_N} H(\tfrac{x}{N})\,\eta_t(x)\,.$$

This notation should not be mistaken with the inner product in $L^2(\mathbb{T})$. Also, when π_t has a density ρ, $\pi(t,du) = \rho(t,u)du$, we sometimes write $\langle \rho_t, H\rangle$ for $\langle \pi_t, H\rangle$.

We also recall that $\mathbb{Q}_{\mu_N}$ is the measure on the path space $\mathcal{D}([0,T],\mathcal{M})$ induced by the probability measure μ_N on $\{0,1\}^{\mathbb{T}_N}$ and the empirical process π_t^N introduced in the Chapter 1.

Proposition 2.1.1. *Fix a continuous profile $\rho_0 : \mathbb{T} \to [0,1]$ and consider a sequence $\{\mu_N : N \geq 1\}$ of measures on $\{0,1\}^{\mathbb{T}_N}$ associated to ρ_0 in the sense* (1.4). *Let $\mathbb{Q}$ be the probability measure on $\mathcal{D}([0,T],\mathcal{M})$ concentrated on the deterministic path $\pi(t,du) = \rho(t,u)du$, where ρ is the unique weak solution of* (1.7). *Then, the sequence of probability measures $\mathbb{Q}_{\mu_N}$ converges weakly to $\mathbb{Q}$, as $N \to \infty$.*

It is straightforward to obtain the Theorem 1.2.2 as a corollary of the previous proposition.

The proof of the Proposition 2.1.1 is divided in three parts. In Section 2.2, we show that the sequence $\{\mathbb{Q}_{\mu_N} : N \geq 1\}$ is tight and in Section 2.5 we characterize the limit points of this sequence. For that we have proved that all limit points of this sequence are concentrated on weak solutions of the hydrodynamic equation (1.7). As a consequence, we have the existence of weak solutions of (1.7) with initial condition γ. The uniqueness of weak solutions of (1.7) is presented in Section (2.6) and this implies the uniqueness of limit points of the sequence $\{\mathbb{Q}_{\mu_N} : N \geq 1\}$.

2.2 Tightness

Proposition 2.2.1. *The sequence of measures $\{\mathbb{Q}_{\mu_N}, N \geq 1\}$ is tight in the Skorohod space $\mathcal{D}\big([0,T],\mathcal{M}\big)$.*

Proof. It is well known (see Proposition 4.1.7 in [16]) that, in order to prove such tightness, it is enough to show tightness of the real-valued processes $\{\langle \pi_t^N, H\rangle; 0 \leq t \leq T\}$ for a set of smooth functions $H : \mathbb{T} \to \mathbb{R}$ dense in $C(\mathbb{T})$ for the uniform topology. Furthermore, if a sequence of distributions in $\mathcal{D}\big([0,T],\mathbb{R}\big)$ endowed with the uniform topology is tight, then it is also tight in $\mathcal{D}\big([0,T],\mathbb{R}\big)$ endowed with the Skorohod topology.

Here, we prove that the sequence $\{\langle \pi_t^N, H_t\rangle; 0 \leq t \leq T\}$ is tight in $\mathcal{D}\big([0,T],\mathbb{R}\big)$, endowed with the uniform topology, for $H \in C^{1,2}\big([0,T] \times \overline{\mathbb{T}\backslash\{a\}}\big)$. Notice that $C^2(\mathbb{T})$ is a subset of the $C^{1,2}\big([0,T] \times \overline{\mathbb{T}\backslash\{a\}}\big)$, then the set $C^{1,2}\big([0,T] \times \overline{\mathbb{T}\backslash\{a\}}\big)$ is dense in in $C(\mathbb{T})$ for the uniform topology.

Fix $H \in C^{1,2}\big([0,T] \times \overline{\mathbb{T}\backslash\{a\}}\big)$. By definition $\{\langle \pi_t^N, H_t\rangle; 0 \leq t \leq T\}$ is tight in $D([0,T],\mathbb{R})$ endowed with the uniform topology if, for the boundedness,

$$\lim_{m\to\infty} \sup_N \mathbb{P}_{\mu_N}\Big[\sup_{0\leq t\leq T} |\langle \pi_t^N, H_t\rangle| > m\Big] = 0\,, \tag{2.1}$$

and, for the equicontinuity,

$$\lim_{\delta\to 0} \varlimsup_{N\to\infty} \mathbb{P}_{\mu_N}\Big[\sup_{|t-s|\leq\delta} |\langle \pi_t^N, H_t\rangle - \langle \pi_s^N, H_s\rangle| > \varepsilon\Big] = 0\,, \quad \text{for all } \varepsilon > 0\,. \tag{2.2}$$

The limit in (2.1) is trivial since $|\langle \pi_t^N, H\rangle| \leq \|H\|_\infty$. So we only need to prove (2.2).

By Dynkyn's formula (see appendix in [16]), for any function $H \in C^{1,2}\big([0,T] \times \overline{\mathbb{T}\backslash\{a\}}\big)$,

$$M_t^N(H) = \langle \pi_t^N, H_t\rangle - \langle \pi_0^N, H_0\rangle - \int_0^t \big\{\langle \pi_s^N, \partial_s H_s\rangle + N^2 L_N \langle \pi_s^N, H_s\rangle\big\}\, ds \tag{2.3}$$

is a martingale. By the previous expression, (2.2) follows from

$$\lim_{\delta\to 0} \varlimsup_{N\to\infty} \mathbb{P}_{\mu_N}\Big[\sup_{|t-s|\leq\delta} |M_t^N - M_s^N| > \varepsilon\Big] = 0\,, \quad \text{for all } \varepsilon > 0\,, \tag{2.4}$$

and

$$\lim_{\delta\to 0} \varlimsup_{N\to\infty} \mathbb{P}_{\mu_N}\Big[\sup_{0\leq t-s\leq\delta} \Big|\int_s^t N^2 L_N \langle \pi_r^N, H\rangle dr\Big| > \varepsilon\Big] = 0\,, \quad \text{for all } \varepsilon > 0\,. \tag{2.5}$$

To prove (2.4) it is enough to prove that the quadratic variation of the martingale $M_t^N(H)$, $\langle M^N(H)\rangle_t$, converges uniformly to zero in $L^1(\mathbb{P}_{\mu_N})$, as $N \to \infty$. The quadratic variation $\langle M^N(H)\rangle_t$ is equal to

$$\int_0^t N^2 \Big[L_N \langle \pi_s^N, H_s\rangle^2 - 2\langle \pi_s^N, H\rangle L_N \langle \pi_s^N, H_s\rangle\Big]\, ds\,.$$

Just applying the definition of the generator L_N, the quadratic variation, $\langle M^N(H)\rangle_t$, can be rewritten as

$$\int_0^t \sum_{x\in\mathbb{T}_N} \xi^N_{x,x+1}\Big[\big(\eta_s(x)-\eta_s(x+1)\big)\big(H_s(\tfrac{x+1}{N})-H_s(\tfrac{x}{N})\big)\Big]^2\,ds\,. \tag{2.6}$$

Since any function $H\in C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$ has bounded first derivative in $\mathbb{T}\backslash\{a\}$, there is a constant $C>0$ such that $\big|H_s(\frac{x+1}{N})-H_s(\frac{x}{N})\big|\leq CN^{-1}$ for any $x\neq a_N$. Applying this inequality in (2.6) and recalling that $\xi^N_{a_N,a_N+1}=\xi^N_{a_N}=N^{-1}$, we get the bound

$$\langle M^N(H)\rangle_t\;\leq\;\frac{T(C^2+2\|H\|^2_\infty)}{N}\,.$$

The next step is to analyze the limit (2.5). Recall the definition of the operator $\mathbb{L}_N$ given in (1.6). By a simple changing of variables, it is easy to see that $N^2L_N\langle\pi^N_s,H_s\rangle=\langle\pi^N_s,N^2\mathbb{L}_NH_s\rangle$, which on his hand is

$$\begin{aligned}\langle\pi^N_s,N^2\mathbb{L}_NH_s\rangle=\tfrac{1}{N}\sum_{\substack{x\neq a_N\\ x\neq a_N+1}}&\eta_s(x)\Big[N^2\big(H_s(\tfrac{x+1}{N})+H_s(\tfrac{x-1}{N})-2H_s(\tfrac{x}{N})\big)\Big]\\ &+\eta_s(a_N)\Big[\big(H_s(\tfrac{a_N+1}{N})-H_s(\tfrac{a_N}{N})\big)+N\big(H_s(\tfrac{a_N-1}{N})-H_s(\tfrac{a_N}{N})\big)\Big]\\ &+\eta_s(a_N+1)\Big[N\big(H_s(\tfrac{a_N+2}{N})-H_s(\tfrac{a_N+1}{N})\big)+\big(H_s(\tfrac{a_N}{N})-H_s(\tfrac{a_N+1}{N})\big)\Big]\,.\end{aligned} \tag{2.7}$$

Since $H\in C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$, $H\big|_{[0,T]\times\mathbb{T}\backslash\{a\}}\in C^{1,2}([0,T]\times\mathbb{T}\backslash\{a\})$ with bounded derivatives of first and second order. By Taylor expansion, we obtain that

$$|N^2L_N\langle\pi^N_s,H\rangle|\;\leq\;2\Big[\|\Delta H\|_\infty+\|\partial_uH\|_\infty+\|H\|_\infty\Big]\,.$$

We have concluded that there is a constant $C>0$ depending only H such that

$$\Big|\int_r^t N^2L_N\langle\pi^N_s,H\rangle ds\Big|\leq C|t-r|\,,$$

which implies the limit (2.5).

□

2.3 Replacement Lemma

We obtain fundamental results that allow us to replace the mean occupation of a site by the mean density of particles in a small macroscopic box around this site. This result implies that the limit trajectories must belong to some Sobolev space, what will be clarified later.

Denote by $H_N(\mu_N|\nu_\alpha^N)$ the entropy of a probability measure μ_N with respect to the invariant state ν_α^N. For a precise definition and properties of the entropy, we refer the reader to [16]. In Proposition A.1.8 in the Appendix we review a classical result saying that there exists a finite constant $K_0 := K_0(\alpha)$, such that

$$H_N(\mu_N|\nu_\alpha^N) \;\leq\; K_0 N\,, \tag{2.8}$$

for any probability measure $\mu_N \in \{0,1\}^{\mathbb{T}_N}$.

Denote by $\langle\cdot,\cdot\rangle_{\nu_\alpha^N}$ the scalar product of $L^2(\nu_\alpha^N)$ and denote by $\mathfrak{D}_N$ the Dirichlet form of f, which is the convex and lower semicontinuous functional (see Corollary A1.10.3 of [16]) defined by

$$\mathfrak{D}_N(f) \;=\; \langle -L_N\sqrt{f}\,,\,\sqrt{f}\rangle_{\nu_\alpha^N}\,,$$

where f is a probability density with respect to ν_α^N (i.e. $f \geq 0$ and $\int f d\nu_\alpha^N = 1$). An elementary computation shows that

$$\mathfrak{D}_N(f) \;=\; \sum_{x\in\mathbb{T}_N} \frac{\xi_{x,x+1}^N}{2} \int \Big(\sqrt{f(\eta^{x,x+1})} - \sqrt{f(\eta)}\Big)^2\, d\nu_\alpha^N(\eta)\,,$$

By Theorem A1.9.2 of [16], if $\{S_t^N : t \geq 0\}$ stands for the semi-group associated to the generator $N^2\mathcal{L}_N$, then

$$H_N(\mu_N S_t^N|\nu_\alpha^N) \,+\, N^2\int_0^t \mathfrak{D}_N(f_s^N)\,ds \;\leq\; H_N(\mu_N|\nu_\alpha^N)\,,$$

provided f_s^N stands for the Radon-Nikodym derivative of $\mu_N S_s^N$ (the distribution of η_s starting from μ_N) with respect to ν_α^N.

From this point, we denote the integer part of εN, namely $\lfloor \varepsilon N\rfloor$, simply by εN. Next, we define the local density of particles, which corresponds to the mean occupation in a box around a given site. We represent this empirical density in the box of size ℓ around a given site x by $\eta^\ell(x)$. The idea is to define a box around the site x in such way it avoids the slow bond.

Definition 2.3.1. *If $x \in \mathbb{T}_N$ is such that $\frac{x}{N} \in (a-\varepsilon, a)$, then the empirical density is defined by*

$$\eta^{\varepsilon N}(x) \;=\; \tfrac{1}{\varepsilon N} \sum_{y=a_N-\varepsilon N+1}^{a_N} \eta(y)\,.$$

Otherwise, if, let us say, $\frac{x}{N} \notin (a-\varepsilon, a)$, then the empirical density is defined by

$$\eta^{\varepsilon N}(x) \;=\; \tfrac{1}{\varepsilon N} \sum_{y=x+1}^{x+\varepsilon N} \eta(y)\,.$$

According to the previous definition of local density of particles, we define an approximation of identity in the continuous torus given by

$$\iota_\varepsilon^a(u,v) = \begin{cases} \frac{1}{\varepsilon}\,\mathbf{1}_{(a-\varepsilon,a)}(u)\,, & \text{if } \; v\in(a-\varepsilon,a) \\ \\ \frac{1}{\varepsilon}\,\mathbf{1}_{(v,v+\varepsilon)}(u)\,, & \text{otherwise.} \end{cases} \tag{2.9}$$

We also define the convolution

$$(\psi * \iota_\varepsilon^a)(v) = \langle \psi, \iota_\varepsilon^a(\cdot,v)\rangle\,,$$

for all function $\psi:\mathbb{T}\to\mathbb{R}$ or measure ψ in $\mathbb{T}$.

To simplify notation, we define the functions

$$g_1:\{0,1\}^{\mathbb{Z}}\to\mathbb{R} \;\text{ by }\; g_1(\eta)=\eta(0)(1-\eta(1)) \tag{2.10}$$

and

$$\tilde g_1:[0,1]\times[0,1]\to\mathbb{R} \;\text{ by }\; \tilde g_1(\alpha,\beta) \;=\; \alpha(1-\beta)\,.$$

Also,

$$g_2:\{0,1\}^{\mathbb{Z}}\to\mathbb{R} \;\text{ by }\; g_2(\eta)=\eta(1)(1-\eta(0)) \tag{2.11}$$

and

$$\tilde g_2:[0,1]\times[0,1]\to\mathbb{R} \;\text{ by }\; \tilde g_2(\alpha,\beta) \;=\; \beta(1-\alpha)\,.$$

Lemma 2.3.1. *Fix a function $F:\mathbb{T}\to\mathbb{R}$. Let be f density with respect to ν_α^N. Then, for any $A>0$,*

$$\begin{aligned} &\tfrac{1}{N}\sum_{x\neq a_N}\int F(\tfrac{x}{N})\Big\{\tau_x g_i(\eta)-\tilde g_i(\eta^{\varepsilon N}(x),\eta^{\varepsilon N}(x+1))\Big\}f(\eta)\,d\nu_\alpha^N(\eta) \\ &\qquad\le\; 12A\varepsilon\sum_{x\neq a_N}\big(F(\tfrac{x}{N})\big)^2 \;+\; \tfrac{3}{A}\,\mathfrak{D}_N(f)\,,\quad \forall i=1,2\,, \end{aligned} \tag{2.12}$$

$$\begin{aligned} &F(\tfrac{a_N}{N})\int\Big\{\tau_{a_N} g_i(\eta)-\tilde g_i(\eta^{\varepsilon N}(a_N),\eta^{\varepsilon N}(a_N+1))\Big\}f(\eta)\,d\nu_\alpha^N(\eta) \\ &\qquad\le\; 6A\varepsilon N\big(F(\tfrac{a_N}{N})\big)^2 \;+\; \tfrac{3}{A}\,\mathfrak{D}_N(f)\,,\quad \forall i=1,2\,, \end{aligned} \tag{2.13}$$

$$\tfrac{1}{N}\sum_{x\in\mathbb{T}_N}\int F(\tfrac{x}{N})\{\eta(x)-\eta^{\varepsilon N}(x)\}f(\eta)\,d\nu_\alpha^N(\eta)\le\; 4A\varepsilon\sum_{x\in\mathbb{T}_N}\big(F(\tfrac{x}{N})\big)^2 \;+\; \tfrac{1}{A}\,\mathfrak{D}_N(f) \tag{2.14}$$

and

$$\int\{\eta(x)-\eta^{\varepsilon N}(x)\}f(\eta)\,d\nu_\alpha^N(\eta)\le\; 4NA\varepsilon \;+\; \tfrac{1}{A}\,\mathfrak{D}_N(f)\,,\qquad \forall x\in\mathbb{T}_N\,. \tag{2.15}$$

Proof. We handle with the inequalities (2.12) and (2.13) for $i=1$. The proof for the others inequalities are analogous. First of all, rewrite $\eta(x)(1-\eta(x+1))-\eta^{\varepsilon N}(x)(1-\eta^{\varepsilon N}(x+1))$ as

$$\eta(x)-\eta^{\varepsilon N}(x)-\eta(x)(\eta(x+1)-\eta^{\varepsilon N}(x+1))-\eta^{\varepsilon N}(x+1)(\eta(x)-\eta^{\varepsilon N}(x))\,.$$

We will consider only the function $\eta(x)(\eta(x+1)-\eta^{\varepsilon N}(x+1))$, for the others functions the proof is analogous. Then we claim that for f density with respect to ν_α^N and for any $A>0$, it is true that

$$\begin{aligned}&\tfrac{1}{N}\sum_{x\neq a_N}\int F(\tfrac{x}{N})\eta(x)\Big\{\eta(x+1)-\eta^{\varepsilon N}(x+1)\Big\}f(\eta)\,d\nu_\alpha^N(\eta)\\&\leq 4A\varepsilon\sum_{x\neq a_N}\big(F(\tfrac{x}{N})\big)^2+\tfrac{1}{A}\mathfrak{D}_N(f)\end{aligned}\tag{2.16}$$

and

$$\begin{aligned}&\int F(\tfrac{a_N}{N})\eta(a_N)(\eta(a_N+1)-\eta^{\varepsilon N}(a_N+1))f(\eta)\,d\nu_\alpha^N(\eta)\\&\leq 2A\varepsilon N\big(F(\tfrac{a_N}{N})\big)^2+\tfrac{1}{A}\mathfrak{D}_N(f)\,.\end{aligned}\tag{2.17}$$

Recall Definition 2.3.1. First we will analyse a site x such that $\frac{x+1}{N}\notin(a-\varepsilon,a]$. In this case,

$$\begin{aligned}&\int F(\tfrac{x}{N})\eta(x)(\eta(x+1)-\eta^{\varepsilon N}(x+1))f(\eta)\,d\nu_\alpha^N(\eta)\\&=\int F(\tfrac{x}{N})\eta(x)\Big\{\tfrac{1}{\varepsilon N}\sum_{y=x+2}^{x+1+\varepsilon N}(\eta(x+1)-\eta(y))\Big\}f(\eta)\,d\nu_\alpha^N(\eta)\,.\end{aligned}$$

In the expression above, writing $\eta(x+1)-\eta(y)$ as a telescopic sum, we get

$$\int F(\tfrac{x}{N})\eta(x)\Big\{\tfrac{1}{\varepsilon N}\sum_{y=x+2}^{x+1+\varepsilon N}\sum_{z=x+1}^{y-1}(\eta(z)-\eta(z+1))\Big\}f(\eta)\,d\nu_\alpha^N(\eta)\,.$$

Now, rewriting the last expression as twice the half and making the change of variables $\eta\mapsto\eta^{z,z+1}$ (using that the probability ν_α^N is invariant) it becomes as

$$\tfrac{1}{2\varepsilon N}\sum_{y=x+2}^{x+1+\varepsilon N}\sum_{z=x+1}^{y-1}F(\tfrac{x}{N})\int\eta(x)(\eta(z)-\eta(z+1))(f(\eta)-f(\eta^{z,z+1}))\,d\nu_\alpha^N(\eta)\,.$$

Since $a-b=(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$, applying the Cauchy-Schwarz inequality, for any $A>0$, we bound the previous expression from above by

$$\begin{aligned}&\tfrac{1}{2\varepsilon N}\sum_{y=x+2}^{x+1+\varepsilon N}\sum_{z=x+1}^{y-1}\tfrac{A}{\xi_{z,z+1}^N}\big(F(\tfrac{x}{N})\big)^2\int\Big(\sqrt{f(\eta)}+\sqrt{f(\eta^{z,z+1})}\Big)^2\,d\nu_\alpha^N(\eta)\\&+\tfrac{1}{2\varepsilon N}\sum_{y=x+2}^{x+1+\varepsilon N}\sum_{z=x+1}^{y-1}\tfrac{\xi_{z,z+1}^N}{A}\int\Big(\sqrt{f(\eta)}-\sqrt{f(\eta^{z,z+1})}\Big)^2\,d\nu_\alpha^N(\eta)\,.\end{aligned}$$

The second sum above is bounded by

$$\tfrac{1}{A\varepsilon N}\sum_{y=x+2}^{x+1+\varepsilon N}\sum_{z\in\mathbb{T}_N}\tfrac{\xi^N_{z,z+1}}{2}\int\Big(\sqrt{f(\eta)}-\sqrt{f(\eta^{z,z+1})}\Big)^2\,d\nu^N_\alpha(\eta)\leq\tfrac{1}{A}\mathfrak{D}_N(f)\,.$$

Since $\xi^N_{z,z+1}=1$ for all $z\in\{x+1,\dots,x+\varepsilon N\}$ and f is density with respect to ν^N_α, it yields the boundedness of the first sum by

$$\tfrac{1}{\varepsilon N}\sum_{y=x+2}^{x+1+\varepsilon N}\sum_{z=x+1}^{y-1}2A\big(F(\tfrac{x}{N})\big)^2\leq 2A\varepsilon N\big(F(\tfrac{x}{N})\big)^2\,.$$

Thus, for any site x such that $\frac{x+1}{N}\notin(a-\varepsilon,a]$, we have that

$$\int F(\tfrac{x}{N})\eta(x)(\eta(x+1)-\eta^{\varepsilon N}(x+1))f(\eta)\,d\nu^N_\alpha(\eta)\leq 2A\varepsilon N\big(F(\tfrac{x}{N})\big)^2+\tfrac{1}{A}\mathfrak{D}_N(f)\,.$$

Taking $x=a_N$ in the last inequality, we obtain the inequality (2.17).

In order to achieve (2.16), we need to analyse the other sites. Let x be a site such that $\frac{x+1}{N}\in(a-\varepsilon,a]$. In this case,

$$\begin{aligned}&\int F(\tfrac{x}{N})\eta(x)(\eta(x+1)-\eta^{\varepsilon N}(x+1))f(\eta)\,d\nu^N_\alpha(\eta)\\&=\int F(\tfrac{x}{N})\eta(x)\Big\{\tfrac{1}{\varepsilon N}\sum_{y=a_N-\varepsilon N+1}^{a_N}(\eta(x+1)-\eta(y))\Big\}f(\eta)\,d\nu^N_\alpha(\eta)\,.\end{aligned}$$

Now, we split the last summation into two blocs: $\{a_N-\varepsilon N+1,\cdots,x\}$ and $\{x+1,\cdots,a_N\}$. Then we proceed by writing $\eta(x+1)-\eta(y)$ as a telescopic sum, getting

$$\begin{aligned}&F(\tfrac{x}{N})\int\eta(x)\Big\{\tfrac{1}{\varepsilon N}\sum_{y=a_N-\varepsilon N+1}^{x}\sum_{z=y}^{x}(\eta(z+1)-\eta(z))\Big\}f(\eta)\,d\nu^N_\alpha(\eta)\\&+F(\tfrac{x}{N})\int\eta(x)\Big\{\tfrac{1}{\varepsilon N}\sum_{y=x+2}^{a_N}\sum_{z=x+1}^{y-1}(\eta(z)-\eta(z+1))\Big\}f(\eta)\,d\nu^N_\alpha(\eta)\,.\end{aligned}$$

Then, by the same arguments used above and since $\xi^N_{z,z+1}=1$ for all z in the range $\{a_N-\varepsilon N+1,\dots,a_N-1\}$, we bound the previous expression by

$$4A\varepsilon N\big(F(\tfrac{x}{N})\big)^2+\tfrac{1}{A}\mathfrak{D}_N(f)\,.$$

This conclude the claim (2.16).

$\square$

Proposition 2.3.2 (Replacement Lemma)**.** *Given a bounded function* $F:\mathbb{T}\to\mathbb{R}$, *then*

$$\varlimsup_{\varepsilon\to 0}\varlimsup_{N\to\infty}\mathbb{E}_{\mu_N}\Big[\Big|\int_0^t \tfrac{1}{N}\sum_{x\in\mathbb{T}_N} F(\tfrac{x}{N})\{\eta_s(x)-\eta_s^{\varepsilon N}(x)\}\,ds\Big|\Big] \;=\; 0\,,$$

$$\varlimsup_{\varepsilon\to 0}\varlimsup_{N\to\infty}\mathbb{E}_{\mu_N}\Big[\Big|\int_0^t \tfrac{1}{N}\sum_{x\neq a_N} F(\tfrac{x}{N})\Big\{\tau_x g_i(\eta)-\tilde{g}_i(\eta^{\varepsilon N}(x),\eta^{\varepsilon N}(x{+}1))\Big\}\,ds\Big|\Big] \;=\; 0\,,\quad \forall i=1,2$$

and

$$\varlimsup_{\varepsilon\to 0}\varlimsup_{N\to\infty}\mathbb{E}_{\mu_N}\Big[\Big|\int_0^t F(\tfrac{a_N}{N})\Big\{\tau_{a_N} g_i(\eta)-\tilde{g}_i(\eta^{\varepsilon N}(a_N),\eta^{\varepsilon N}(a_N{+}1))\Big\}\,ds\Big|\Big] \;=\; 0\,,\quad \forall i=1,2\,,$$

where g_i *and* $\tilde{g}_i$, $i=1,2$ *were defined in* (2.10) *and* (2.11)*.*

Proof. We will prove the first limit, the other ones are similar.

Using the definition of the entropy and Jensen's Inequality, the expectation is bounded from above by

$$\tfrac{H_N(\mu_N|\nu_\alpha^N)}{\gamma N} \;+\; \tfrac{1}{\gamma N}\log\mathbb{E}_{\nu_\alpha^N}\Big[\exp\Big\{\gamma\Big|\int_0^t \tfrac{1}{N}\sum_{x\in\mathbb{T}_N} F(\tfrac{x}{N})\{\eta_s(x)-\eta_s^{\varepsilon N}(x)\}\,ds\Big|\Big\}\Big]\,,$$

for all $\gamma>0$. In view of (2.8), to prove this proposition, it is enough to show that the second term vanishes as $N\to\infty$ and then $\varepsilon\downarrow 0$ for every $\gamma>0$. Since $e^{|x|}\le e^{x}+e^{-x}$ and by Proposition A.2.7, we may remove the absolute value inside the exponential. Thus, to complete the prove of this proposition, we need to show that

$$\tfrac{1}{\gamma N}\log\mathbb{E}_{\nu_\alpha^N}\Big[\exp\Big\{\gamma\int_0^t \tfrac{1}{N}\sum_{x\in\mathbb{T}_N} F(\tfrac{x}{N})\{\eta_s(x)-\eta_s^{\varepsilon N}(x)\}\,ds\Big\}\Big] \;=\; 0\,,$$

for every $\gamma>0$.

By Feynman-Kac formula (c.f. Proposition A.1.7 and [16], Lemma 7.2, p. 336), for each fixed N the previous expression is bounded from above by

$$t\,\sup_f\Big\{\int \tfrac{1}{N}\sum_{x\in\mathbb{T}_N} F(\tfrac{x}{N})\Big\{\eta(x)-\eta^{\varepsilon N}(x)\Big\}f(\eta)\,d\nu_\alpha^N(\eta)-\tfrac{N}{\gamma}\mathfrak{D}_N(f)\Big\}\,,$$

where the supremum is carried over all density functions f with respect to ν_α^N. From inequality (2.14) of the Lemma 2.3.1, and assumption over the function F, the previous expression is less than or equal to

$$t\,\sup_f\Big\{4A\varepsilon\sum_{x\in\mathbb{T}_N}\big(F(\tfrac{x}{N})\big)^2+\tfrac{1}{A}\mathfrak{D}_N(f)-\tfrac{N}{\gamma}\mathfrak{D}_N(f)\Big\}\,.$$

Here, if we were proving the others limits of the statement of this proposition, we would have that use the others inequalities of the Lemma 2.3.1.

Letting $A = \frac{\gamma}{N}$, the last expression becomes

$$\tfrac{4\gamma\varepsilon t}{N} \sum_{x\in\mathbb{T}_N} \big(F(\tfrac{x}{N})\big)^2 .$$

For all $\gamma > 0$, this expression vanishes as $N \to \infty$ and then $\varepsilon \downarrow 0$, which concludes the proof of first limit in statement of the lemma. For the second limit, one needs just multiply and divide the expectation there by N and proceed as before. $\square$

Proposition 2.3.3 (Replacement Lemma for a Single Site)**.** *For each bounded function $F : [0,T] \times \mathbb{T}$, site $x \in \mathbb{T}_N$ and $\varepsilon > 0$, let*

$$V_{N,\varepsilon}^{F,x}(t,\eta) = F(t, \tfrac{x}{N})\{\eta(x) - \eta^{\varepsilon N}(x)\}$$

Then,

$$\varlimsup_{\varepsilon\downarrow 0}\ \varlimsup_{N\to\infty}\ \mathbb{E}_{\nu_\alpha^N}\Big[\Big|\int_0^T V_{N,\varepsilon}^{F,a_N}(t,\eta_t)\,dt\Big|\Big] = 0 \,.$$

Proof. This proof follows by same method as in Proposition 2.3.2. The unique difference is that to apply the inequality (2.15) of the Lemma 2.3.1.

$\square$

2.4 Sobolev spaces

We prove in this section that any limit point $\mathbb{Q}^*$ of the sequence $\mathbb{Q}_{\mu_N}^N$ is concentrated on trajectories $\rho(t,u)du$ which belongs to the Sobolev space, which will be defined ahead. Let $\mathbb{Q}^*$ be a limit point of the sequence $\mathbb{Q}_{\mu_N}^N$ and assume without loss of generality that the sequence $\mathbb{Q}_{\mu_N}^N$ converges to $\mathbb{Q}^*$.

We repeat here the definition of the Sobolev Space from [4].

Definition 2.4.1 (Sobolev space)**.** *The Sobolev space $\mathcal{H}^1(\mathbb{T}\backslash\{a\})$ consists of all locally summable functions $\zeta : \mathbb{T}\backslash\{a\} \to \mathbb{R}$ such that there exists $\partial\zeta \in L^2(\mathbb{T}\backslash\{a\})$ satisfying*

$$\langle \partial_u G, \zeta\rangle \;=\; -\langle G, \partial\zeta\rangle \,,$$

for all $G \in C^\infty(\mathbb{T}\backslash\{a\})$ with compact support. For $\zeta \in \mathcal{H}^1(\mathbb{T}\backslash\{a\})$, we define the norm

$$\|\zeta\|_{\mathcal{H}^1(\mathbb{T}\backslash\{a\})} \;=\; \|\partial\zeta\|_{L^2} \,.$$

Definition 2.4.2. *The space $L^2(0,T;\mathcal{H}^1(\mathbb{T}\backslash\{a\}))$ consists of all measurable functions $\xi : [0,T] \to \mathcal{H}^1(\mathbb{T}\backslash\{a\})$ with*

$$\|\xi\|_{L^2(0,T;\mathcal{H}^1(\mathbb{T}\backslash\{a\}))} \;:=\; \Big(\int_0^T \|\xi_t\|^2_{\mathcal{H}^1(\mathbb{T}\backslash\{a\})}\,dt\Big)^{1/2} \;<\; \infty \,.$$

We refer to [4] for more informations about Sobolev spaces.

Proposition 2.4.1. *The measure $\mathbb{Q}^*$ is concentrated on paths $\rho(t,u)du$ such that ρ belongs to $L^2(0,T;\mathcal{H}^1(\mathbb{T}\backslash\{a\}))$.*

The proof is based on the Riez Representation Theorem and follows from the next lemmata.

Lemma 2.4.2. *Fix any function $H:\mathbb{T}\to\mathbb{R}$ and let f be a density with respect to ν_α^N. Then,*

$$\begin{aligned}&\int \tfrac{1}{\varepsilon N}\sum_{x\in\mathbb{T}_N} H(\tfrac{x}{N})\Big\{\eta(x-\varepsilon N)-\eta(x)\Big\}f(\eta)\,d\nu_\alpha^N(\eta)\\ &\leq N\,\mathfrak{D}_N(f)+\tfrac{2}{N}\sum_{x\in\mathbb{T}_N}\big(H(\tfrac{x}{N})\big)^2\{1+\tfrac{1}{\varepsilon}\boldsymbol{1}_{(a-\varepsilon,a]}(\tfrac{x}{N})\}\,.\end{aligned}$$

Moreover, this inequality is remains valid replacing $\{\eta(x-\varepsilon N)-\eta(x)\}$ by $\{\eta(x)-\eta(x+\varepsilon N)\}$.

Proof. This proof follows the same steps as in the Lemma 2.3.1. One begins by writing as a telescopic sum,

$$\begin{aligned}&\int \tfrac{1}{\varepsilon N}\sum_{x\in\mathbb{T}_N} H(\tfrac{x}{N})\Big\{\eta(x_0)-\eta(x_1)\Big\}f(\eta)\,d\nu_\alpha^N(\eta)\\ &=\tfrac{1}{\varepsilon N}\sum_{x\in\mathbb{T}_N} H(\tfrac{x}{N})\sum_{y=x_0}^{x_1-1}\int\Big\{\eta(y)-\eta(y+1)\Big\}f(\eta)\,d\nu_\alpha^N(\eta)\,,\end{aligned}$$

where $x_0=x-\varepsilon N$ and $x_1=x$ or $x_0=x$ and $x_1=x+\varepsilon N$.

Now, following the same arguments as in Lemma 2.3.1, we bound the previous expression by

$$\begin{aligned}&\tfrac{1}{\varepsilon N}\sum_{x\in\mathbb{T}_N}\big(H(\tfrac{x}{N})\big)^2\sum_{y=x_0}^{x_1-1}\tfrac{A}{2\xi^N_{y,y+1}}\int\{\sqrt{f(\eta)}+\sqrt{f(\eta^{y,y+1})}\}^2\,d\nu_\alpha^N(\eta)\\ &+\tfrac{1}{\varepsilon N}\sum_{x\in\mathbb{T}_N}\sum_{y=x_0}^{x_1-1}\tfrac{\xi^N_{y,y+1}}{2A}\int\{\sqrt{f(\eta)}-\sqrt{f(\eta^{y,y+1})}\}^2\,d\nu_\alpha^N(\eta)\,.\end{aligned}\tag{2.18}$$

The second sum above is less than or equal to $\frac{1}{A}\mathfrak{D}_N(f)$. Since f is density with respect to ν_α^N and also the definition of $\xi^N_{y,y+1}$, the first sum in (2.18) is less than or equal to

$$\tfrac{1}{\varepsilon N}\sum_{x\in\mathbb{T}_N}\big(H(\tfrac{x}{N})\big)^2\sum_{y=x_0}^{x_1-1}\tfrac{2A}{\xi^N_{y,y+1}}\leq\tfrac{2A}{\varepsilon N}\sum_{x\in\mathbb{T}_N}\big(H(\tfrac{x}{N})\big)^2\{\varepsilon N+N\mathbf{1}_{(a-\varepsilon,a]}(\tfrac{x}{N})\}\,.$$

This inequality is true for $x_0=x-\varepsilon N$ and $x_1=x$ or $x_0=x$ and $x_1=x+\varepsilon N$. Choosing $A=\frac{1}{N}$, it yields the inequality in the statement of the lemma. $\square$

For a function $H : \mathbb{T} \to \mathbb{R}$, $\varepsilon > 0$ and a positive integer N, define $U_N(\varepsilon, H, \eta)$ by

$$
\begin{aligned}
U_N(\varepsilon, H, \eta) = & \tfrac{1}{\varepsilon N} \sum_{x \in \mathbb{T}_N} H(\tfrac{x}{N}) \Big\{ \eta(x - \varepsilon N) - \eta(x) \Big\} \\
& - \tfrac{2}{N} \sum_{x \in \mathbb{T}_N} \big(H(\tfrac{x}{N}) \big)^2 \{ 1 + \tfrac{1}{\varepsilon} \mathbf{1}_{(a-\varepsilon, a]}(\tfrac{x}{N}) \} \,.
\end{aligned}
\tag{2.19}
$$

Recall the definition of the constant K_0 given in (2.8).

Lemma 2.4.3. *For every $k \geq 1$, consider the functions $H_1, \ldots, H_k$ from $[0, T] \times \mathbb{T}$ to $\mathbb{R}$. Then, for every $\varepsilon > 0$,*

$$
\varlimsup_{\delta \downarrow 0} \varlimsup_{N \to \infty} \mathbb{E}_{\mu_N} \Big[\max_{1 \leq i \leq k} \Big\{ \int_0^T U_N(\varepsilon, H_i(s, \cdot), \eta_s^{\delta N}) \, ds \Big\} \Big] \;\leq\; K_0 \,.
$$

Proof. It follows from the Replacement Lemma that in order to prove the lemma we just need to show that

$$
\varlimsup_{N \to \infty} \mathbb{E}_{\mu_N} \Big[\max_{1 \leq i \leq k} \Big\{ \int_0^T U_N(\varepsilon, H_i(s, \cdot), \eta_s) \, ds \Big\} \Big] \;\leq\; K_0 \,.
$$

By the entropy and Jensen's Inequalities, for each fixed N, the previous expectation is bounded from above by

$$
\frac{H(\mu_N | \nu_\alpha^N)}{N} \;+\; \tfrac{1}{N} \log \mathbb{E}_{\nu_\alpha^N} \Big[\exp \Big\{ \max_{1 \leq i \leq k} \Big\{ N \int_0^T U_N(\varepsilon, H_i(s, \cdot), \eta_s) \, ds \Big\} \Big\} \Big] \,.
$$

By (2.8), the first term is bounded by K_0. Since $\exp\{\max_{1 \leq j \leq k} a_j\}$ is bounded from above by $\sum_{1 \leq j \leq k} \exp\{a_j\}$ and since $\varlimsup_N N^{-1} \log\{a_N + b_N\}$ is less than or equal to the maximum of $\varlimsup_N N^{-1} \log a_N$ and $\varlimsup_N N^{-1} \log b_N$, the limit, as $N \to \infty$, of the previous expression is less than or equal to

$$
K_0 \;+\; \max_{1 \leq i \leq k} \varlimsup_{N \to \infty} \tfrac{1}{N} \log \mathbb{E}_{\nu_\alpha^N} \Big[\exp \Big\{ N \int_0^T U_N(\varepsilon, H_i(s, \cdot), \eta_s) \, ds \Big\} \Big] \,.
$$

We now prove that, for each fixed i the limit above is nonpositive.

Fix $1 \leq i \leq k$. By Feynman-Kac's formula and the variational formula for the largest eigenvalue of a symmetric operator, for each fixed N, the second term in the previous expression is bounded from above by

$$
\int_0^T \sup_f \Big\{ \int U_N(\varepsilon, H_i(s, \cdot), \eta_s) f(\eta) d\nu_\alpha^N(\eta) - N \mathfrak{D}_N(f) \Big\} \, ds \,.
$$

In last formula the supremum is taken over all probability densities f with respect to ν_α^N. Applying the Lemma 2.4.2, the result is straightforward. $\square$

Lemma 2.4.4.

$$E_{\mathbb{Q}^*}\Big[\sup_H \Big\{ \int_0^T \int_{\mathbb{T}} \partial_u H(s,u)\, \rho_s(u)\, du\, ds \;-\; 2\int_0^T \int_{\mathbb{T}} H(s,u)^2\, du\, ds \Big\}\Big] \;\le\; K_0\,,$$

where the supremum is carried over all functions H in $C^{0,1}([0,T]\times\mathbb{T})$ with compact support in $[0,T]\times(\mathbb{T}\backslash\{a\})$.

Proof. Consider a sequence $\{H_\ell,\ \ell\ge 1\}$ dense (with respect to the norm $\|H\|_\infty + \|\partial_u H\|_\infty$) in the subset $C^{0,1}([0,T]\times\mathbb{T})$ of the functions with support contained in $[0,T]\times(\mathbb{T}\backslash\{a\})$. Recall that we suppose that $\mathbb{Q}^N_{\mu_N}$ converges to $\mathbb{Q}^*$. By Lemma 2.4.3, for every $k\ge 1$

$$\begin{aligned}\overline{\lim_{\delta\downarrow 0}}\, E_{\mathbb{Q}^*}\Big[\max_{1\le i\le k}\Big\{ \tfrac{1}{\varepsilon}\int_0^T \int_{\mathbb{T}} H_i(s,u)\,[\rho^\delta_s(u-\varepsilon) - \rho^\delta_s(u)]\, du\, ds \\ -\; 2\int_0^T \int_{\mathbb{T}} \big(H_i(s,u)\big)^2\,\{1+\tfrac{1}{\varepsilon}\mathbf{1}_{(a-\varepsilon,a]}(u)\}\, du\, ds\Big\}\Big] \;\le\; K_0\,,\end{aligned}$$

where $\rho^\delta(u) := (\rho * \iota^a_\delta)(u)$. Letting $\delta\downarrow 0$, we obtain

$$\begin{aligned}E_{\mathbb{Q}^*}\Big[\max_{1\le i\le k}\Big\{ \tfrac{1}{\varepsilon}\int_0^T \int_{\mathbb{T}} H_i(s,u)\,[\rho_s(u-\varepsilon) - \rho_s(u)]\, du\, ds \\ -\; 2\int_0^T \int_{\mathbb{T}} \big(H_i(s,u)\big)^2\,\{1+\tfrac{1}{\varepsilon}\mathbf{1}_{(a-\varepsilon,a]}(u)\}\, du\, ds\Big\}\Big] \;\le\; K_0\,.\end{aligned}$$

Changing variables in the first integral,

$$\begin{aligned}E_{\mathbb{Q}^*}\Big[\max_{1\le i\le k}\Big\{ \int_0^T \int_{\mathbb{T}} \tfrac{1}{\varepsilon}[H_i(s,u+\varepsilon) - H_i(s,u)]\rho_s(u)\; du\, ds \\ -\; 2\int_0^T \int_{\mathbb{T}} \big(H_i(s,u)\big)^2\,\{1+\tfrac{1}{\varepsilon}\mathbf{1}_{(a-\varepsilon,a]}(u)\}\, du\, ds\Big\}\Big] \;\le\; K_0\,.\end{aligned}$$

Since H_i in $C^{0,1}([0,T]\times\mathbb{T})$ with compact support in $[0,T]\times(\mathbb{T}\backslash\{a\})$. Making $\varepsilon\downarrow 0$ in the last inequality, we obtain

$$\begin{aligned}E_{\mathbb{Q}^*}\Big[\max_{1\le i\le k}\Big\{ \int_0^T \int_{\mathbb{T}} \partial_u H_i(s,u)\rho(s,u)\; du\, ds \\ -\; 2\int_0^T \int_{\mathbb{T}} \big(H_i(s,u)\big)^2\, du\, ds\Big\}\Big] \;\le\; K_0\,.\end{aligned}$$

To conclude the proof it remains to apply the Monotone Convergence Theorem and recall that $\{H_\ell,\ \ell\ge 1\}$ is a dense sequence (with respect to the norm $\|H\|_\infty + \|\partial_u H\|_\infty$) in the subset of functions of $C^{0,1}([0,T]\times\mathbb{T})$ with support contained in $[0,T]\times(\mathbb{T}\backslash\{a\})$.

□

Proof of Proposition 2.4.1. Denote by $\ell : C^{0,1}([0,T]\times\mathbb{T}) \to \mathbb{R}$ the linear functional defined by

$$\ell(H) \;=\; \int_0^T \int_{\mathbb{T}} \partial_u H(s,u)\, \rho_s(u)\, du\, ds\,.$$

Since $C^{0,1}([0,T]\times\mathbb{T})$ with support contained in $[0,T]\times(\mathbb{T}\backslash\{a\})$ is dense in $L^2([0,T]\times\mathbb{T})$, by Lemma 2.4.4 and Proposition A.1.1, ℓ is $\mathbb{Q}^*$-almost surely bounded functional in $C^{0,1}([0,T]\times\mathbb{T})$, we can extend it to a $\mathbb{Q}^*$-almost surely bounded functional in $L^2([0,T]\times\mathbb{T})$. In particular, by Riesz representation theorem, there exists a function G in $L^2([0,T]\times\mathbb{T})$ such that

$$\ell(H) \;=\; -\int_0^T \int_{\mathbb{T}} H(s,u)\, G(s,u)\, du\, ds\,.$$

One can use the Lemma A.1.9 to conclude the proof of the proposition. □

2.5 Characterization of limit points

We prove in this section that all limit points $\mathbb{Q}^*$ of the sequence $\mathbb{Q}_{\mu_N}$ are concentrated on absolutely continuous trajectories $\pi(t,du) = \rho(t,u)du$, whose density $\rho(t,u)$ is a weak solution of the hydrodynamic equation (1.7) with $\gamma = \rho_0$.

Let $\mathbb{Q}^*$ be a limit point of the sequence $\mathbb{Q}_{\mu_N}$ and assume, without loss of generality, that $\mathbb{Q}_{\mu_N}$ converges to $\mathbb{Q}^*$.

Since there is at most one particle per site, $\mathbb{Q}^*$ is concentrated on trajectories $\pi_t(du)$ which are absolutely continuous with respect to the Lebesgue measure, $\pi_t(du) = \rho(t,u)du$, and whose density ρ is non-negative and bounded by 1. For more explications we refer the reader to [16].

In Proposition 2.4.1, we proved that $\rho(t,\cdot)$ belongs to $L^2\big(0,T;\mathcal{H}^1(\mathbb{T}\backslash\{a\})\big)$. It is well known that the Sobolev space $\mathcal{H}^1(\mathbb{T}\backslash\{a\})$ has special properties: all its elements are absolutely continuous functions with bounded variation, c.f. [4], therefore with well defined lateral limits. Such property is inherited by $L^2\big(0,T;\mathcal{H}^1(\mathbb{T}\backslash\{a\})\big)$ in the sense that we can integrate in time the lateral limits.

Let $H \in C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\,\big)$. We begin by claiming that

$$\begin{aligned}\mathbb{Q}^*\Bigg[\pi_\cdot : \langle \rho_t, H_t\rangle \;-\; \langle \gamma, H_0\rangle \;-\; \int_0^t \langle \rho_s, (\partial_s+\Delta)H_s\rangle\, ds \\ -\int_0^t \big\{\rho_s(a^+)\partial_u H_s(a^+) - \rho_s(a^-)\partial_u H_s(a^-)\big\}\, ds \\ +\int_0^t \big\{\rho_s(a^+) - \rho_s(a^-)\big\}\big\{H_s(a^+) - H_s(a^-)\big\}\, ds \;=\; 0, \quad \forall t\in[0,T]\Bigg] \;=\; 1\,.\end{aligned} \tag{2.20}$$

In order to prove the last equality, its enough to show that, for every $\delta > 0$

$$\mathbb{Q}^*\Bigg[\pi.: \sup_{0\le t\le T}\Bigg|\langle \rho_t, H_t\rangle - \langle \gamma, H_0\rangle - \int_0^t \langle \rho_s, (\partial_s+\Delta)H_s\rangle\, ds - \int_0^t \big\{\rho_s(a^+)\partial_u H_s(a^+) - \rho_s(a^-)\partial_u H_s(a^-)\big\}\, ds + \int_0^t \big\{\rho_s(a^+) - \rho_s(a^-)\big\}\big\{H_s(a^+) - H_s(a^-)\big\}\, ds\Bigg| > \delta\Bigg] = 0\,.$$

Since the boundary integrals are not well-defined in the whole Skorohod space $\mathcal{D}\big([0,T],\mathcal{M}_0\big)$, we cannot use directly Portmanteau's Theorem. To avoid this technical obstacle, fix $\varepsilon>0$, which will be taken small later. Adding and subtracting the convolution of $\rho_t(u)$ with $\iota_\varepsilon := \iota_\varepsilon^a$, recall definition (2.9). Then, we can see the probability above is less than or equal to the sum of

$$\mathbb{Q}^*\Bigg[\pi.: \sup_{0\le t\le T}\Bigg|\langle \rho_t, H_t\rangle - \langle \gamma, H_0\rangle - \int_0^t \langle \rho_s, (\partial_s+\Delta)H_s\rangle\, ds - \int_0^t \big\{(\rho_s*\iota_\varepsilon)(a^+)\partial_u H_s(a^+) - (\rho_s*\iota_\varepsilon)(a^-)\partial_u H_s(a^-)\big\}\, ds + \int_0^t \big\{(\rho_s*\iota_\varepsilon)(a^+) - (\rho_s*\iota_\varepsilon)(a^-)\big\}\big\{H_s(a^+) - H_s(a^-)\big\}\, ds\Bigg| > \delta/3\Bigg], \tag{2.21}$$

$$\mathbb{Q}^*\Bigg[\pi.: \sup_{0\le t\le T}\Bigg|\int_0^t \big\{(\rho_s*\iota_\varepsilon)(a^+)\partial_u H_s(a^+) - (\rho_s*\iota_\varepsilon)(a^-)\partial_u H_s(a^-)\big\}\, ds - \int_0^t \big\{\rho_s(a^+)\partial_u H_s(a^+) - \rho_s(a^-)\partial_u H_s(a^-)\big\}\, ds\Bigg| > \delta/3\Bigg]$$

and

$$\mathbb{Q}^*\Bigg[\pi.: \sup_{0\le t\le T}\Bigg|\int_0^t \big\{(\rho_s*\iota_\varepsilon)(a^+) - (\rho_s*\iota_\varepsilon)(a^-)\big\}\big\{H_s(a^+) - H_s(a^-)\big\}\, ds - \int_0^t \big\{\rho_s(a^+) - \rho_s(a^-)\big\}\big\{H_s(a^+) - H_s(a^-)\big\}\, ds\Bigg| > \delta/3\Bigg].$$

The convolutions above are suitable averages of ρ around the boundary point 0. Therefore, as $\varepsilon\downarrow 0$, the set inside the three previous probabilities decreases to a set of null probability. It remains to deal with (2.21). We want use the Portmanteau's Theorem and Proposition

A.2.7, and conclude that the probability (2.21) is bounded from above by

$$\varlimsup_{N\to\infty} \mathbb{Q}_{\mu_N}\Bigg[\pi_\cdot : \sup_{0\le t\le T}\Bigg|\langle\rho_t, H_t\rangle \;-\; \langle\gamma, H_0\rangle \;-\; \int_0^t \langle\rho_s, (\partial_s+\Delta)H_s\rangle\, ds$$
$$- \int_0^t \big\{(\rho_s * \iota_\varepsilon)(a^+)\partial_u H_s(a^+) - (\rho_s * \iota_\varepsilon)(a^-)\partial_u H_s(a^-)\big\}\, ds$$
$$+ \int_0^t \big\{(\rho_s * \iota_\varepsilon)(a^+) - (\rho_s * \iota_\varepsilon)(a^-)\big\}\big\{H_s(a^+) - H_s(a^-)\big\}\, ds\Bigg| > \delta/3\Bigg]. \tag{2.22}$$

Although the functions H_t, H_0, $(\partial_s+\Delta)H_s$, $\iota_\varepsilon(\cdot, a^-)$ and $\iota_\varepsilon(\cdot, a^+)$ may not belong to $C(\mathbb{T})$, we can proceed as in Subsections 7.5.2 and 8.3.2 (see (7.23) and (8.32)) in order to justify why (2.21) is bounded from above by (2.22). Next we outline the main arguments involved in that procedure. Before applying the Portmanteau's Theorem, we replace these functions by continuous functions such that the new functions coincide with the original functions in the torus except on small neighborhood of the points of the discontinuity the functions H_t, H_0, $(\partial_s+\Delta)H_s$, $\iota_\varepsilon(\cdot, a^-)$ and $\iota_\varepsilon(\cdot, a^+)$ and their L^∞-norm are bounded from above by L^∞-norm of the respective original function. Using the rule that has only one particle per site, the set where we compare this change has small probability. Thus, we have continuous functions and we are able to apply the Portmanteau's Theorem and Proposition A.2.7. After using the Portmanteau's Theorem, let us return to the original functions, by the same arguments as above. Then, we obtain the expression (2.22).

If we consider the discrete torus as embedded in the continuous torus, a_N is the closest site to the left of a and a_N+1 is the closest site to the right of a. Since $(\pi_s^N * \iota_\varepsilon)(\frac{x}{N}) = \eta_s^{\varepsilon N}(x)$, for all $x \in \mathbb{T}_N$. Using the definition of $\mathbb{Q}_{\mu_N}$, we can rewrite the previous expression as

$$\varlimsup_{N\to\infty} \mathbb{P}_{\mu_N}\Bigg[\sup_{0\le t\le T}\Bigg|\langle\pi_t^N, H_t\rangle \;-\; \langle\pi_0^N, H_0\rangle \;-\; \int_0^t \langle\pi_s^N, (\partial_s+\Delta)H_s\rangle\, ds$$
$$- \int_0^t \Big\{\eta_s^{\varepsilon N}(a_N+1)\partial_u H_s(a^+) - \eta_s^{\varepsilon N}(a_N)\partial_u H_s(a^-)\Big\}\, ds$$
$$+ \int_0^t \big\{\eta_s^{\varepsilon N}(a_N+1) - \eta_s^{\varepsilon N}(a_N)\big\}\big\{H_s(a^+) - H_s(a^-)\big\}\, ds\Bigg| > \delta/3\Bigg].$$

The next step is to add and subtract $N^2 L_N\langle\pi_s^N,\, H_s\rangle$ and the previous probability becomes now bounded from above by the sum of

$$\varlimsup_{N\to\infty} \mathbb{P}_{\mu_N}\Bigg[\sup_{0\le t\le T}\Bigg|\langle\pi_t^N, H_t\rangle \;-\; \langle\pi_0^N, H_0\rangle \;-\; \int_0^t \langle\pi_s^N, \partial_s H_s\rangle + N^2 L_N\langle\pi_s^N,\, H_s\rangle\, ds\Bigg| > \delta/6\Bigg]$$

and

$$
\begin{aligned}
\varlimsup_{N\to\infty}\ \mathbb{P}_{\mu_N}\Bigg[\sup_{0\le t\le T}\Bigg|\int_0^t\ N^2L_N\langle\pi_s^N,\ H_s\rangle\,ds - \int_0^t\langle\pi_s^N,\Delta H_s\rangle\,ds\\
-\int_0^t\Big\{\eta_s^{\varepsilon N}(a_N+1)\partial_u H_s(a^+) - \eta_s^{\varepsilon N}(a_N)\partial_u H_s(a^-)\Big\}\,ds \qquad\qquad (2.23)\\
+\int_0^t\big\{\eta_s^{\varepsilon N}(a_N+1)-\eta_s^{\varepsilon N}(a_N)\big\}\big\{H_s(a^+)-H_s(a^-)\big\}\,ds\Bigg|\ >\ \delta/6\Bigg].
\end{aligned}
$$

The expression inside the first probability is the martigale $M_t^N(H)$ defined in (2.3). Using the fact that the martingale $M_t^N(H)$ converges to zero in $L^2(\mathbb{P}_{\mu_N})$, which is proved in Proposition 2.2.1, and Doob's inequality, the first probability is equal to

$$
\lim_{N\to\infty}\mathbb{P}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|M_t^N(H)\Big| > \delta/6\Big]\ =\ 0\,,
$$

for every $\delta>0$,

We going to show now that the second probability above is null. By expression (2.7) for $N^2L_N\langle\pi_s^N,\ H_s\rangle$ the probability (2.23) is less than or equal to the sum of

$$
\begin{aligned}
\varlimsup_{N\to\infty}\ \mathbb{P}_{\mu_N}\Bigg[\sup_{0\le t\le T}\Bigg|\ \int_0^t\langle\pi_s^N,\Delta H_s\rangle\,ds\\
-\int_0^t\tfrac{1}{N}\sum_{\substack{x\neq a_N\\ x\neq a_N+1}}\ \eta_s(x)N^2[H_s(\tfrac{x+1}{N})+H_s(\tfrac{x-1}{N})-2H_s(\tfrac{x}{N})]\,ds\Bigg|\ >\ \delta/18\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\varlimsup_{N\to\infty}\ \mathbb{P}_{\mu_N}\Bigg[\sup_{0\le t\le T}\Bigg|\int_0^t\Big\{\eta_s^{\varepsilon N}(a_N+1)\partial_u H_s(a^+)-\eta_s^{\varepsilon N}(a_N)\partial_u H_s(a^-)\Big\}\,ds\\
-\int_0^t\Big\{\eta_s(a_N{+}1)N\nabla_N H_{a_N+1}\ -\ \eta_s(a_N)N\nabla_N H_{a_N-1}\Big\}\,ds\Bigg|\ >\ \delta/18\Bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
\varlimsup_{N\to\infty}\ \mathbb{P}_{\mu_N}\Bigg[\sup_{0\le t\le T}\Bigg|\int_0^t\big\{\eta_s^{\varepsilon N}(a_N+1)-\eta_s^{\varepsilon N}(a_N)\big\}\big\{H_s(a^+)-H_s(a^-)\big\}\,ds\\
-\int_0^t\big\{\eta_s(a_N+1)-\eta_s(a_N)\big\}\nabla_N H_{a_N}\,ds\Bigg|\ >\ \delta/18\Bigg].
\end{aligned}
$$

Since $H\in C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$, the discrete Laplacian, which applied to H_s converges uniformly to the continuous Laplacian of H_s. Then the first probability is null. To prove that the others probabilities are null, we observe that $N\nabla_N F_x$ converges uniformly to $\partial_u F_s$, as $N \to \infty$ and $\nabla_N F_{a_N}$ converges uniformly to $F_s(a^+) - F_s(a^-)$, as $N \to \infty$, since $F \in C^{1,2}\big([0,T] \times \overline{\mathbb{T}\backslash\{a\}}\big)$. By the rule of maximum of one particle per site and approximation of integral by Riemann sums, for the previous probabilities be null, we need prove that the probabilities

$$\varlimsup_{N\to\infty} \mathbb{P}_{\mu_N}\Bigg[\sup_{0\le t\le T}\Bigg|\int_0^t \big\{\eta_s^{\varepsilon N}(a_N+1) - \eta_s(a_N+1)\big\}\partial_u H_s(a^+) \\ - \big\{\eta_s^{\varepsilon N}(a_N) - \eta_s(a_N)\big\}\partial_u H_s(a^-)\, ds\Bigg| > \delta\Bigg]$$

and

$$\varlimsup_{N\to\infty} \mathbb{P}_{\mu_N}\Bigg[\sup_{0\le t\le T}\Bigg|\int_0^t \Big\{\big\{\eta_s^{\varepsilon N}(a_N+1) - \eta_s^{\varepsilon N}(a_N)\big\} \\ - \big\{\eta_s(a_N+1) - \eta_s(a_N)\big\}\Big\}\big\{H_s(a^+) - H_s(a^-)\big\}\, ds\Bigg| > \delta\Bigg].$$

converge to zero, as $\varepsilon \downarrow 0$, $\forall \delta > 0$. It follows by Replacement Lemma 2.3.3.

Proposition 2.5.1. *Fix a Borel measurable profile $\gamma : \mathbb{T} \to [0,1]$ and consider a sequence $\{\mu_N : N \ge 1\}$ of probability measures on $\{0,1\}^{\mathbb{T}_N}$ associated to γ in the sense of* (1.4)*. Then any limit point of $\mathbb{Q}_{\mu_N}$ is concentrated on absolutely continuous paths $\pi_t(du) = \rho(t,u)du$, with positive density ρ_t bounded by* 1*, such that ρ is a weak solutions of* (1.7) *with initial condition γ.*

Proof. Let $\{H_i : i \ge 1\}$ be a countable dense set of functions on $C^{1,2}\big([0,T] \times \overline{\mathbb{T}\backslash\{a\}}\big)$, with respect to the norm $\|H\|_\infty + \|\partial_u H\|_\infty + \|\partial_u^2 H\|_\infty$. Provided by (2.20) and intercepting a countable number of sets of probability one, is straightforward to extend (2.20) for all functions $H \in C^{1,2}\big([0,T] \times \overline{\mathbb{T}\backslash\{a\}}\big)$ simultaneously. □

2.6 Uniqueness of weak solutions

This section is devoted to the uniqueness of weak solutions of (1.7). To simplify notation, along this section, we will consider $a = 0$ and sometimes we will denote $0^+ = 0$ and $0^- = 1$.

Denote $L^2(\mathbb{T})^{\perp 1}$ the subspace of functions $g \in L^2(\mathbb{T})$ with zero mean, or else, satisfying

$$\int_{\mathbb{T}} g(u)\, du = 0\,.$$

Definition 2.6.1. *Denote by $\mathcal{H}^2_{bc}(\mathbb{T})$ the set of functions $H : \mathbb{T} \to \mathbb{R}$ such that H is twice differentiable with $H \in C(\mathbb{T})$, $\partial_u H$ is absolutely continuous and $\Delta H \in L^2(\mathbb{T})^{\perp 1}$. Moreover, H satisfies the boundary conditions:*

$$\partial_u H(0^+) = \partial_u H(0^-) = H(0^+) - H(0^-)\,.$$

Proposition 2.6.1. *Let $\rho : [0,T] \times \mathbb{T} \to \mathbb{R}$ be a weak solution of the parabolic differential equation* (1.7) *with initial condition $\gamma : \mathbb{T} \to \mathbb{R}$. Then, for all $t \in [0,T]$ and for all $H \in \mathcal{H}^2_{bc}(\mathbb{T})$, holds*

$$\langle \rho_t, H\rangle \;-\; \langle \gamma, H\rangle \;=\; \int_0^t \langle \rho_s, \Delta H\rangle \, ds\,, \tag{2.24}$$

for all $t \in [0,T]$.

Proof. Let $H \in \mathcal{H}^2_{\mathrm{bc}}(\mathbb{T})$. Here, we will denote $0^+ = 0$ and $0^- = 1$. Hence H satisfies

$$\partial_u H(0) = \partial_u H(1) = H(0) - H(1)\,.$$

Consider $h_n \in C(\mathbb{T})$ such that $\int h_n(x)\,dx = 0$ and h_n converges to ΔH and β_n converging to $\partial_u H(0)$, and define

$$H_n(x) = H(0) + \beta_n x + \int_0^x \int_0^y h_n(z)\,dz\,dy\,.$$

Notice that $H_n \in C^{1,2}\big([0,T] \times \overline{\mathbb{T}\backslash\{0\}}\,\big)$, $\partial_u H_n(0) = \partial_u H_n(1)$ and $\Delta H_n = h_n$. Then,

$$\begin{aligned} \langle \rho_t, H_n\rangle \;-\; \langle \gamma, H_n\rangle \;=\; & \int_0^t \langle \rho_s, h_n\rangle \, ds \\ + \int_0^t \big\{\rho_s(0) - \rho_s(1)\big\}\partial_u H_n(0)\, ds & \\ - \int_0^t \big\{\rho_s(0) - \rho_s(1)\big\}\big\{H_n(0) - H_n(1)\big\}\, ds\,. & \end{aligned} \tag{2.25}$$

Since H_n converges to H and h_n converges to ΔH, we just need to analyse the boundary terms. By definition of H_n, $\partial_u H_n(0) = \partial_u H_n(1) = \beta_n$ converges to $\partial_u H(0)$. Using that h_n converges to ΔH in L^2 and β_n converges to $\partial_u H(0)$, we get $H_n(1)$ converges to

$$H(0) + \partial_u H(0) + \int_0^1 \int_0^y \Delta H(z)\,dz\,dy\,. \tag{2.26}$$

By definition of the set $\mathcal{H}^2_{\mathrm{bc}}(\mathbb{T})$, we have that

$$\int_0^y \Delta H(z) = \partial_u H(y) - \partial_u H(0)\,.$$

The expression (2.26) is equal to $H(1)$. Thus, $H_n(0) - H_n(1)$ converges to $H(0) - H(1) = \partial_u H(0)$. One can obtain the equation (2.24).

□

The next step is to construct the inverse of the operator $\Delta : \mathcal{H}^2_{\mathrm{bc}}(\mathbb{T}) \to L^2(\mathbb{T})$. For $g \in L^2(\mathbb{T})^{\perp 1}$, define

$$[(-\Delta)^{-1} g](x) \;=\; \int_0^1 G(x,z) g(z)\,dz, \tag{2.27}$$

where the function $G : [0,1] \times [0,1] \to \mathbb{R}$ is given by

$$G(x,z) \;=\; \frac{x(1-z)}{2} - (x-z)\mathbf{1}_{\{0\le z\le x\le 1\}}\,.$$

Proposition 2.6.2. *The operator $(-\Delta)^{-1}$ enjoys the following properties:*

(a) $\forall\, g \in L^2(\mathbb{T})^{\perp 1}$, $(-\Delta)^{-1}g \in C^1(\mathbb{T}\backslash\{0\})$ *and* $\partial_u(-\Delta)^{-1}g$ *is absolutely continuous in* $\mathbb{T}\backslash\{0\}$*, both having finite side limits around the point* 0*;*

(b) $\forall\, g \in L^2(\mathbb{T})^{\perp 1}$, $[(-\Delta)^{-1}g](0^+)-[(-\Delta)^{-1}g](0^-) = \partial_u[(-\Delta)^{-1}g](0^+) = \partial_u[(-\Delta)^{-1}g](0^-)$*;*

(c) $\forall\, g \in L^2(\mathbb{T})^{\perp 1}$, $(-\Delta)^{-1}g \in \mathcal{H}^2_{bc}(\mathbb{T})$*;*

(d) $\forall\, g \in L^2(\mathbb{T})^{\perp 1}$, $-\Delta(-\Delta)^{-1}g = g$*;*

(e) The operators $-\Delta : \mathcal{H}^2_{bc}(\mathbb{T}) \to L^2(\mathbb{T})^{\perp 1}$ *and* $(-\Delta)^{-1} : L^2(\mathbb{T})^{\perp 1} \to \mathcal{H}^2_{bc}(\mathbb{T})$ *are symmetric and nonnegative;*

(f) $\forall\, g \in \mathcal{H}^2_{bc}(\mathbb{T})$, $\int_{\mathbb{T}} \Delta g(u)\, du = 0$.

Proof. Let g be a function in $L^2(\mathbb{T})^{\perp 1}$. By the definition of $(-\Delta)^{-1}$,

$$[(-\Delta)^{-1}g](x) \;=\; x\int_{\mathbb{T}} \frac{(1-z)}{2} g(z)\, dz - x\int_0^x g(z)\, dz + \int_0^x z g(z)\, dz\,, \tag{2.28}$$

which easily implies (a). Item (b) follows by the assumption g has zero mean. Items (a) and (b) imply (c). Deriving (2.28) twice and recalling item (c), we obtain (d).

Let $g, h \in \mathcal{H}^2_{\mathrm{bc}}(\mathbb{T})$. Integrating by parts,

$$\langle -\Delta g, h\rangle \;=\; \langle \partial_u g, \partial_u h\rangle + \partial_u g(0^+)h(0^+) - \partial_u g(0^-)h(0^-)\,.$$

Since $g, h \in \mathcal{H}^2_{\mathrm{bc}}(\mathbb{T})$, these functions satisfy the boundary conditions $g(0^+) - g(0^-) = \partial_u g(0^+) = \partial_u g(0^-)$ and $h(0^+) - h(0^-) = \partial_u h(0^+) = \partial_u h(0^-)$. Putting them together we observe that

$$\langle -\Delta g, h\rangle \;=\; \langle \partial_u g, \partial_u h\rangle + \partial_u g(0^+)\partial_u h(0^+), \tag{2.29}$$

which implies symmetry and non-negativity. The same holds for $(-\Delta)^{-1}$, using the item (d). Item (f) follows from expression (2.29) with $h = -1$.

□

Proposition 2.6.3. *Let ρ be a weak solution of the hydrodynamic equation* (1.7) *with zero initial condition. For all $t \in [0,T]$, holds the equality*

$$\langle \rho_t, (-\Delta)^{-1}\rho_t\rangle = -2\int_0^t \langle \rho_s, \rho_s\rangle\, ds\,. \tag{2.30}$$

In particular, since the hydrodynamical equation (1.7) *is linear, there is at most one weak solution with initial condition ρ_0.*

Proof. Notice that the mean of a weak solution of (1.7) is constant in time, therefore $\rho_t \in L^2(\mathbb{T})^{\perp 1}$ for any time $t \in [0,T]$.

Take a partition $0 = t_0 < t_1 < \cdots < t_n = T$ of the interval $[0,T]$, so that

$$\begin{aligned}
&\langle \rho_t, (-\Delta)^{-1}\rho_t\rangle - \langle \rho_0, (-\Delta)^{-1}\rho_0\rangle \\
&= \sum_{k=0}^{n-1} \langle \rho_{t_{k+1}}, (-\Delta)^{-1}\rho_{t_{k+1}}\rangle - \langle \rho_{t_{k+1}}, (-\Delta)^{-1}\rho_{t_k}\rangle \\
&+ \sum_{k=0}^{n-1} \langle \rho_{t_{k+1}}, (-\Delta)^{-1}\rho_{t_k}\rangle - \langle \rho_{t_k}, (-\Delta)^{-1}\rho_{t_k}\rangle\,.
\end{aligned}$$

We handle the second term, the first one being similar because $(-\Delta)^{-1}$ is a symmetric operator. Since ρ is a weak solution of (1.7), ρ_{t_k} belongs to $L^2(\mathbb{T})^{\perp 1}$ and recalling Proposition 2.6.1 and Proposition 2.6.2 item (c),

$$\begin{aligned}
&\langle \rho_{t_{k+1}}, (-\Delta)^{-1}\rho_{t_k}\rangle - \langle \rho_{t_k}, (-\Delta)^{-1}\rho_{t_k}\rangle \\
&= -\int_{t_k}^{t_{k+1}} \langle \rho_s, \rho_s\rangle\, ds + \int_{t_k}^{t_{k+1}} \langle \rho_s, \rho_s - \rho_{t_k}\rangle\, ds\,.
\end{aligned} \tag{2.31}$$

The sum over k of the first term in the right side of (2.31) is exactly the expression that we announced in (2.30). We shall treat the remainder. Let $\iota_\delta : \mathbb{T} \to \mathbb{R}$ be an smooth approximation of identity and $\Phi_\delta : \mathbb{T} \to \mathbb{R}$ a smooth function bounded by one, equals to zero in the interval $(-\delta, \delta)$, and equals to one in $\mathbb{T}\backslash(-2\delta, 2\delta)$. Define

$$\rho_s^\delta(u) \;=\; (\rho_s * \iota_\delta)(u)\Phi_\delta(u)\,.$$

It is of easy verification that $\rho_s^\delta \in \mathcal{H}^2_{\mathrm{bc}}(\mathbb{T})$, for any $s \in [0,T]$, and also that $\rho_s^\delta(\cdot)$ converges to $\rho_s(\cdot)$ in $L^2(\mathbb{T})$, when $\delta \downarrow 0$. Adding and subtracting ρ^δ, the second term on the r.h.s of (2.31) can be written as

$$\int_{t_k}^{t_{k+1}} \langle \rho_s - \rho_s^\delta, \rho_s - \rho_{t_k}\rangle\, ds + \int_{t_k}^{t_{k+1}} \langle \rho_s^\delta, \rho_s - \rho_{t_k}\rangle\, ds\,. \tag{2.32}$$

Fix $\varepsilon > 0$. Since $\rho_s^\delta(\cdot)$ converges to $\rho_s(\cdot)$ in $L^2(\mathbb{T})$, applying the Dominated Convergence Theorem, the sum in k of the first term in (2.32) is bounded in modulus by ε for some $\delta(\varepsilon)$ small.

Fix now such $\delta = \delta(\varepsilon)$. Since $\rho_s^\delta \in \mathcal{H}^2_{\mathrm{bc}}(\mathbb{T})$ and since ρ is a weak solution of (1.7), the second term in (2.32) is equal to

$$\int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \langle \rho_r, \Delta\rho_s^\delta\rangle\, dr\, ds\,,$$

whose modulus is bounded by $C(\rho,\delta)(t_{k+1} - t_k)^2$, concluding the proof of (2.30).

To see that this implies the uniqueness of solutions with a initial solution ρ_0, note that the hydrodynamical equation (1.7) is linear, being enough to prove there is a unique weak solution with zero initial condition. Besides, if ρ is a weak solution of (1.7) with zero initial condition, then $\rho_t \in L^2(\mathbb{T})^{\perp}$ for any time $t \in [0,T]$. Then, by the item (e) of the Proposition 2.6.2, $\langle \rho_t, (-\Delta)^{-1}\rho_t \rangle \geq 0$, for all $t \in [0,T]$. Using (2.30), we have $\langle \rho_t, (-\Delta)^{-1}\rho_t \rangle = 0$, for all $t \in [0,T]$. From item (d), fixed $t \in [0,T]$, there exists $f_t \in \mathcal{H}^2_{\mathrm{bc}}(\mathbb{T})$ such that $\rho_t = (-\Delta) f_t$, and thus

$$\langle \rho_t, (-\Delta)^{-1}\rho_t \rangle = \langle -\Delta f_t, f_t \rangle \; = \; \langle \partial_u f_t, \partial_u f_t \rangle + (\partial_u f_t(0^+))^2 \,.$$

Then, $\partial_u f_t(u) = 0$, u - almost surely and for all $t \in [0,T]$. Since $\rho_t = (-\Delta) f_t$, we have $\rho_t(u) = 0$, u - almost surely and for all $t \in [0,T]$. This concludes the proof.

$\square$

Chapter 3

Superexponential Estimates

In this chapter, we present some results needed in order to show a Large Deviations Principle for our model. Given $a \in \mathbb{T}$, recall that a_N denotes the site of the left side of the slow bond in the discrete torus $\mathbb{T}_N$. In the others chapters, we can choose $a = 0$, but here a is any. Such choice has been taken since, with this notation, becomes more clear that all results are immediately generalized for finite slow bonds associated to finite points $a^1, \dots, a^k \in \mathbb{T}$. Before proceeding we introduce some tools that we use in the sequel.

For the Large Deviations, the Replacement Lemma presented in Section 2.3.2 is not enough, because we need to prove that the difference between cylinder functions and functions of the density field are superexponentially small, that is, of order smaller that $\exp\{-CN\}$, for all $C > 0$.

Proposition 3.0.4 (Superexponential Estimate)**.** *Let $F_i \colon [0,T] \times \mathbb{T} \to \mathbb{R}$, $i = 1, 2$, such that*

$$\varlimsup_{N\to\infty} \int_0^T \Big\{ (F_2(t, \tfrac{a_N}{N}))^2 + \tfrac{1}{N} \sum_{x \neq a_N} \big(F_1(t, \tfrac{x}{N})\big)^2 \Big\}\, dt \,<\, \infty\,.$$

For each $\varepsilon > 0$, consider

$$\begin{aligned} V_{N,\varepsilon}^{F_1,F_2}(t,\eta) = \tfrac{1}{N} \sum_{x \neq a_N} F_1(t, \tfrac{x}{N}) \Big\{ \tau_x g_1(\eta) - \tilde{g}_1(\eta^{\varepsilon N}(x), \eta^{\varepsilon N}(x{+}1)) \Big\} \\ + F_2(t, \tfrac{a_N}{N}) \Big\{ \tau_{a_N} g_1(\eta) - \tilde{g}_1(\eta^{\varepsilon N}(a_N), \eta^{\varepsilon N}(a_N{+}1)) \Big\}\,, \end{aligned}$$

where g_1 and $\tilde{g}_1$ were defined in (2.10)*. Then, for any $\delta > 0$,*

$$\varlimsup_{\varepsilon \downarrow 0} \varlimsup_{N\to\infty} \tfrac{1}{N} \log \mathbb{P}_{\nu_\alpha^N} \Big[\Big| \int_0^T V_{N,\varepsilon}^{F_1,F_2}(t, \eta_t)\, dt \Big| > \delta \Big] = -\infty\,. \tag{3.1}$$

Finally, it is true the same result with g_2 and $\tilde{g}_2$ replacing by g_1 and $\tilde{g}_1$.

Proof. Using the Proposition A.2.7, it is enough to prove (3.1) without the absolute value for $V_{N,\varepsilon}^{F_1,F_2}$ and $V_{N,\varepsilon}^{-F_1,-F_2}$.

Let $C>0$, by Chebyshev exponential inequality, we get

$$\begin{aligned}&\mathbb{P}_{\nu_\alpha^N}\Big[\int_0^t V_{N,\varepsilon}^{F_1,F_2}(s,\eta_s)\,ds>\delta\Big]\\&\leq \exp\{-C\delta N\}\mathbb{E}_{\nu_\alpha^N}\Big[\exp\Big\{CN\int_0^t V_{N,\varepsilon}^{F_1,F_2}(s,\eta_s)\,ds\Big\}\Big]\,.\end{aligned}$$

To conclude the proof of the theorem it is therefore enough to show that

$$\overline{\lim_{\varepsilon\downarrow 0}}\;\overline{\lim_{N\to\infty}}\;\tfrac{1}{N}\log\mathbb{E}_{\nu_\alpha^N}\left[\exp\Big\{\int_0^T CN\;V_{N,\varepsilon}^{F_1,F_2}(t,\eta_t)\,dt\Big\}\right]\leq 0\,, \tag{3.2}$$

for every $C>0$, because in this case we would have proved that left hand side of (3.1) is bounded from above by $-C\delta$ for every $C>0$ and it would remain to let C increase to ∞.

From Feynman-Kac formula (see [16], Lemma 7.2, p. 336 and Proposition A.1.7), for each fixed N the previous expectation is bounded from above by

$$\exp\Big\{\int_0^T\sup_f\Big[\int CN\,V_{N,\varepsilon}^{F_1,F_2}(t,\eta)f(\eta)d\nu_\alpha^N(\eta)-N^2\mathfrak{D}_N(f)\Big]\,dt\Big\}\,,$$

where the supremum is carried over all density functions f with respect to ν_α^N. Replacing the expression of $V_{N,\varepsilon}^{F_1,F_2}(t,\eta)$ and using the Lemma 2.3.1, the expression in (3.2) becomes bounded from above by

$$\int_0^T\sup_f\Big[6CA\varepsilon\Big(2\sum_{x\neq a_N}\big(F_1(t,\tfrac{x}{N})\big)^2+N(F_2(t,\tfrac{a_N}{N}))^2\Big)+\tfrac{6C}{A}\mathfrak{D}_N(f)-N\mathfrak{D}_N(f)\Big]\,dt\,.$$

Choosing $A=\frac{6C}{N}$, the expression above becomes

$$36C^2\varepsilon\int_0^T\Big(\tfrac{2}{N}\sum_{x\neq a_N}\big(F_1(t,\tfrac{x}{N})\big)^2+(F_2(t,\tfrac{a_N}{N}))^2\Big)\,dt\,.$$

For all $C>0$, this expression vanishes as $N\to\infty$ and then $\varepsilon\downarrow 0$, which concludes this proof. $\square$

Proposition 3.0.5 (Superexponential Estimate for a single site)**.** *For each bounded function $F:[0,T]\times\mathbb{T}$, site $x\in\mathbb{T}_N$ and each $\varepsilon>0$, let*

$$V_{N,\varepsilon}^{F,x}(t,\eta)=F(t,\tfrac{x}{N})\{\eta(x)-\eta^{\varepsilon N}(x)\}$$

Then, for any $\delta>0$,

$$\overline{\lim_{\varepsilon\downarrow 0}}\;\overline{\lim_{N\to\infty}}\;\frac{1}{N}\log\mathbb{P}_{\nu_\alpha^N}\Big[\Big|\int_0^T V_{N,\varepsilon}^{F,x}(t,\eta_t)\,dt\Big|>\delta\Big]=-\infty.$$

Proof. This proof follows by same method as in Proposition 3.0.4. The unique difference is that to apply the inequality (2.15) of the Lemma 2.3.1, we need to see that

$$\mathbb{P}_{\nu_\alpha^N}\Big[\Big|\int_0^T V_{N,\varepsilon}^{F,x}(t,\eta_t)\,dt\Big| > \delta\Big] \le \mathbb{P}_{\nu_\alpha^N}\Big[\Big|\int_0^T \{\eta(x)-\eta^{\varepsilon N}(x)\}\,dt\Big| > \tfrac{\delta}{\|F\|_\infty}\Big]\,.$$

□

3.1 Energy estimates

We prove in this section an energy estimate. It permits to exclude paths with infinite energy in the large deviations regime. The energy is presented in Section 1.4, more specifically, in Definifion 1.4.1. By Lemma 2.4.4 and Proposition A.1.1, if π has finite energy, its density ρ belongs to $L^2\big(0,T;\mathcal{H}^1(\mathbb{T}\backslash\{a\})\big)$. The next proposition is the fundamental result needed to obtain the energy estimates.

Proposition 3.1.1.

$$\overline{\lim_{\varepsilon\downarrow 0}}\;\overline{\lim_{N\to\infty}}\;\tfrac{1}{N}\log\mathbb{P}_{\mu_N}\Big[\mathcal{E}_H(\pi^N * \iota_\varepsilon^a)\ge l\Big] \le -l+K_0\,.$$

Proof. We begin by claiming that, for enough small $\varepsilon>0$, holds the equality

$$\int_0^T\!\!\int_{\mathbb{T}} \partial_v H(t,v)(\pi_t^N * \iota_\varepsilon^a)(v)dvdt = \int_0^T \tfrac{1}{\varepsilon N}\sum_{x\in\mathbb{T}_N} H(t,\tfrac{x}{N})\,[\eta_t(x)-\eta_t(x+\varepsilon N)]dt\,. \tag{3.3}$$

Since H has support contained in $[0,T]\times(\mathbb{T}\backslash\{a\})$, there exists some $\varepsilon_0>0$ such that $H(t,v)$ vanishes if $v\in(a-\varepsilon_o,a+\varepsilon_0)$, for all $t\in[0,T]$. Applying Fubini's Theorem,

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}} \partial_u H(t,v)(\pi_t^N * \iota_\varepsilon^a)(v)\,dv\,dt\\ &=\int_0^T \tfrac{1}{N}\sum_{x\in\mathbb{T}_N}\eta_t(x)\Big(\int_{\mathbb{T}}\partial_u H(t,v)\iota_\varepsilon^a(\tfrac{x}{N},v)\,dv\Big)\,dt\,.\end{aligned}$$

From the definition of ι_ε^a given in (2.9) and taking $0<\varepsilon<\varepsilon_0$, the last expression is equal to

$$\begin{aligned}&\int_0^T \tfrac{1}{N}\sum_{x\in\mathbb{T}_N}\eta_t(x)\Big(\int_{\mathbb{T}\backslash(a-\varepsilon,a+\varepsilon)}\partial_u H(t,v)\tfrac{1}{\varepsilon}\mathbf{1}_{(v,v+\varepsilon)}(\tfrac{x}{N})\,dv\Big)\,dt\\ &=\int_0^T \tfrac{1}{N}\sum_{x\in\mathbb{T}_N}\eta_t(x)\Big(\tfrac{1}{\varepsilon}\mathbf{1}_{\mathbb{T}\backslash(a-\varepsilon,a+\varepsilon)}(\tfrac{x}{N})[H_t(\tfrac{x}{N})-H_t(\tfrac{x}{N}-\varepsilon)]\Big)\,dt\,.\end{aligned}$$

Using again that $H(t,v)$ vanishes if $v\in(a-\varepsilon,a+\varepsilon)$, for all $t\in[0,T]$, the expression above is equal to

$$\int_0^T \tfrac{1}{\varepsilon N}\sum_{x\in\mathbb{T}_N}\eta_t(x)[H_t(\tfrac{x}{N})-H_t(\tfrac{x}{N}-\varepsilon)]\,dt\,,$$

concluding the claim.

Applying the definition of energy and (3.3), for enough small $\varepsilon > 0$, we have that

$$\begin{aligned}\mathcal{E}_H(\pi^N * \iota^a_\varepsilon) = &\int_0^T \tfrac{1}{\varepsilon N} \sum_{x\in\mathbb{T}_N} H(t, \tfrac{x}{N})\,[\eta_t(x) - \eta_t(x+\varepsilon N)]dt \\ &- 2\int_0^T \int_{\mathbb{T}} \big(H(t,u)\big)^2\, du\, dt\,.\end{aligned}$$

Let us introduce the notation

$$V_N(\varepsilon, H, \eta) = \tfrac{1}{\varepsilon N} \sum_{x\in\mathbb{T}_N} H(\tfrac{x}{N})\{\eta(x) - \eta(x+\varepsilon N)\} - \tfrac{2}{N}\sum_{x\in\mathbb{T}_N}\big(H(\tfrac{x}{N})\big)^2\,.$$

To prove the statement of proposition, just left to show that

$$\varlimsup_{\varepsilon\downarrow 0}\ \varlimsup_{N\to\infty} \tfrac{1}{N}\log \mathbb{P}_{\mu_N}\Big[\int_0^T V_N(\varepsilon, H_t, \eta_t)\, dt \geq l\Big] \leq -l + K_0\,.$$

By Tchebychev exponential inequality,

$$\begin{aligned}&\tfrac{1}{N}\log \mathbb{P}_{\mu_N}\Big[\int_0^T V_N(\varepsilon, H_t, \eta_t) dt \geq l\Big] \\ &\leq \tfrac{1}{N}\log \mathbb{E}_{\mu_N}\Big[\exp\Big\{N\int_0^T V_N(\varepsilon, H_t, \eta_t)dt\Big\}\Big] - l\,,\end{aligned}$$

and the proof reduces to the statement

$$\varlimsup_{\varepsilon\downarrow 0}\ \varlimsup_{N\to\infty} \tfrac{1}{N}\log \mathbb{E}_{\mu_N}\Big[\exp\Big\{N\int_0^T V_N(\varepsilon, H_t, \eta_t)dt\Big\}\Big] \leq K_0\,.$$

From definition of the entropy and Jensen's Inequality, the expectation is bounded from above by

$$K_0 + \tfrac{1}{N}\log \mathbb{E}_{\nu^N_\alpha}\Big[\exp\Big\{N\int_0^T V_N(\varepsilon, H_t, \eta_t)dt\Big\}\Big]\,.$$

By the Feynman-Kac formula and the variational formula for the largest eigenvalue of a symmetric operator, for each fixed N,

$$\begin{aligned}&\tfrac{1}{N}\log \mathbb{E}_{\nu^N_\alpha}\Big[\exp\Big\{N\int_0^T V_N(\varepsilon, H_t, \eta_t)dt\Big\}\Big] \\ &\leq \int_0^T \sup_f \Big\{\int V_N(\varepsilon, H_t, \eta) f(\eta) d\nu^N_\alpha(\eta) - N\mathfrak{D}_N(f)\Big\}\, dt\,.\end{aligned}$$

The supremum is taken over all probability densities f with respect to ν^N_α. Using the Lemma 2.4.2, the last expression is bounded from above by

$$\int_0^T \tfrac{2}{\varepsilon N} \sum_{\frac{x}{N}\in(a-\varepsilon, a]} \big(H_t(\tfrac{x}{N})\big)^2\, dt\,.$$

Since H has compact support, for $\varepsilon > 0$ enough small the expression above vanishes. $\square$

Corollary 3.1.2. *Let $\{H_j\}$ dense in the subset $C^{0,1}([0,T]\times\mathbb{T})$ of the functions with support contained in $[0,T]\times(\mathbb{T}\backslash\{a\})$. Then,*

$$\varlimsup_{\varepsilon\downarrow 0}\varlimsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}_{\mu_N}\Big[\max_{1\le j\le k}\mathcal{E}_{H_j}(\pi^N*\iota_\varepsilon^a)\ge l\Big]\le -l+K_0T\,. \tag{3.4}$$

Proof. Follows from the fact $\exp\{\max_{1\le j\le k}a_j\}$ is bounded from above by $\sum_{j=1}^k\exp\{a_j\}$ and by Proposition A.2.7

□

Chapter 4

Large Deviations Upper Bound

Recall that $\mathbb{P}_{\nu_\alpha^N}$ and $\mathbb{P}^H_{\nu_\alpha^N}$ are probabilities measures on the space $\mathcal{D}([0,T],\{0,1\}^{\mathbb{T}_N})$. The probability $\mathbb{P}_{\nu_\alpha^N}$ corresponds to the homogeneous Markov process η_t with generator L_N defined in (1.1) accelerated by N^2 and starting from ν_α^N. For $H \in C^{1,2}([0,T]\times\overline{\mathbb{T}\backslash\{a\}})$, the probability $\mathbb{P}^H_{\nu_\alpha^N}$ corresponds to the inhomogeneous Markov process η_t with generator L_N^H defined in (1.9) accelerated by N^2 and starting from the invariant measure ν_α^N.

Handwaving, the prove of the large deviations upper bound is constructed by an optimization over a class of mean-one positive martingales, which must be functions of the process, or, as in our case, close to functions of the process. In Section 4.1, we will obtain one good candidate to mean-one positive martingale, the Radon-Nikodym derivative of the measure $\mathbb{P}^H_{\nu_\alpha^N}$ with respect to $\mathbb{P}_{\nu_\alpha^N}$. Unfortunately, $\mathbf{d}\mathbb{P}^H_{\nu_\alpha^N}/\mathbf{d}\mathbb{P}_{\nu_\alpha^N}$ is not a function of the empirical measure, see your expression in (4.3). The first step in the proof of a large deviations principle is therefore to show that $\mathbf{d}\mathbb{P}^H_{\nu_\alpha^N}/\mathbf{d}\mathbb{P}_{\nu_\alpha^N}$ is superexponentially close to a function of the empirical measure. Here superexponentially means that the L^1-norm of the difference between the Radon-Nikodym derivative and a function of the empirical measure has expectation of order smaller than $\exp\{-CN\}$ for all $C>0$. In Chapter 3, we prove the superexponential estimates.

4.1 Radon-Nikodym derivative

By $(\mathbf{d}\mathbb{P}^H_{\nu_\alpha^N}/\mathbf{d}\mathbb{P}_{\nu_\alpha^N})(t)$ let us denote the Radon-Nikodym derivative of $\mathbb{P}^H_{\nu_\alpha^N}$ with respect to $\mathbb{P}_{\nu_\alpha^N}$ restricted to the σ-algebra generated by $\{\eta_s,\, 0\le s\le t\}$. It is a general fact of stochastic process that $(\mathbf{d}\mathbb{P}^H_{\nu_\alpha^N}/\mathbf{d}\mathbb{P}_{\nu_\alpha^N})(t)$ is a mean-one positive martingale. The explicit formula of the Radon-Nikodym derivative of a Markov process with respect to another one (see Appendix of [16]) shows that $(\mathbf{d}\mathbb{P}^H_{\nu_\alpha^N}/\mathbf{d}\mathbb{P}_{\nu_\alpha^N})(T)$ is equal to

$$\exp\left\{N\Big[\langle\pi_T^N,H_T\rangle-\langle\pi_0^N,H_0\rangle-\tfrac{1}{N}\int_0^T e^{-N\langle\pi_t^N,H_t\rangle}(\partial_t+N^2L_N)e^{N\langle\pi_t^N,H_t\rangle}dt\Big]\right\}. \tag{4.1}$$

In what follows of this section, we present some simple and long calculations in order to arrive at a suitable form of $(\mathbf{d}\mathbb{P}^H_{\nu^N_\alpha}/\mathbf{d}\mathbb{P}_{\nu^N_\alpha})(T)$, which we will denote by $\mathbf{d}\mathbb{P}^H_{\nu^N_\alpha}/\mathbf{d}\mathbb{P}_{\nu^N_\alpha}$. In a first reading, the reader can assume (4.3), and go ahead with no future difficulties.

Denote $\nabla_N H_x = H_t(\frac{x+1}{N}) - H_t(\frac{x}{N})$. By simple calculations, $\mathrm{d}\mathbb{P}^H_{\nu^N_\alpha}/\mathrm{d}\mathbb{P}_{\nu^N_\alpha}$ can be rewritten as

$$\begin{aligned}
\exp\Bigg\{ & N\langle \pi^N_T, H_T\rangle - N\langle \pi^N_0, H_0\rangle - N\int_0^T \langle \pi^N_t, \partial_t H_t\rangle\, dt \\
& - N^2\int_0^T \sum_{x\in\mathbb{T}_N} \xi^N_{x,x+1}\eta_t(x)\big(1-\eta_t(x+1)\big)\big(e^{\nabla_N H_x}-1\big)\, dt \\
& - N^2\int_0^T \sum_{x\in\mathbb{T}_N} \xi^N_{x,x+1}\eta_t(x+1)\big(1-\eta_t(x)\big)\big(e^{-\nabla_N H_x}-1\big)\, dt \Bigg\}.
\end{aligned}$$

Using the definition of ξ^N and performing some more calculations, the last expression becomes

$$\begin{aligned}
\exp\Bigg\{ & N\langle \pi^N_T, H_T\rangle - N\langle \pi^N_0, H_0\rangle - N\int_0^T \langle \pi^N_t, \partial_t H_t\rangle\, dt \\
& - N^2\int_0^T \sum_{\substack{x\neq a_N\\ x\neq a_N+1}} \eta_t(x)(\nabla_N H_x - \nabla_N H_{x-1})\, dt \\
& - N^2\int_0^T \sum_{x\neq a_N} \eta_t(x)\big(1-\eta_t(x+1)\big)\big(e^{\nabla_N H_x} - \nabla_N H_x - 1\big)\, dt \\
& - N^2\int_0^T \sum_{x\neq a_N} \eta_t(x+1)\big(1-\eta_t(x)\big)\big(e^{-\nabla_N H_x} + \nabla_N H_x - 1\big)\, dt \\
& - N^2\int_0^T \big\{\eta_t(a_N+1)\nabla_N H_{a_N+1} - \eta_t(a_N)\nabla_N H_{a_N-1}\big\}\, dt \\
& - N\int_0^T \eta_t(a_N)\big(1-\eta_t(a_N+1)\big)\big(e^{\nabla_N H_{a_N}}-1\big)\, dt \\
& - N\int_0^T \eta_t(a_N+1)\big(1-\eta_t(a_N)\big)\big(e^{-\nabla_N H_{a_N}}-1\big)\, dt \Bigg\}.
\end{aligned}$$

Since $H\in C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$ and by Taylor's expansion up to the second order, we get

$$\Big|\tfrac{1}{N}\sum_{\substack{x\neq a_N\\ x\neq a_N+1}} \eta_t(x)N^2(\nabla_N H_x - \nabla_N H_{x-1}) - \tfrac{1}{N}\sum_{x\in\mathbb{T}_N}\eta_t(x)\Delta H_t(\tfrac{x}{N})\Big| = O_H(\tfrac{1}{N}).$$

Using again the Taylor's expansion up to the second order and the elementary inequality $|e^u - 1 - u - (1/2)u^2| \le (1/6)|u|^3 e^{|u|}$, the expression

$$\big|N^2\big(e^{\nabla_N H_x} - \nabla_N H_x - 1\big) - \tfrac{1}{2}(\partial_u H_t)^2(\tfrac{x}{N})\big|$$

is $O_H(\frac{1}{N})$, for each $x \neq a_N$. By the same reason, the expression

$$\big|N^2\big(e^{-\nabla_N H_x} + \nabla_N H_x - 1\big) - \tfrac{1}{2}(\partial_u H_t)^2(\tfrac{x}{N})\big|$$

is also $O_H(\frac{1}{N})$, for each $x \neq a_N$. It is also easy to see that $|N\nabla_N H_{a_N+1} - \partial_u H_t(\frac{a_N+1}{N})| = O_H(\frac{1}{N})$ and $|N\nabla_N H_{a_N-1} - \partial_u H_t(\frac{a_N}{N})| = O_H(\frac{1}{N})$.

Putting together the facts above, we can rewrite the Radon-Nikodym derivative, $\frac{\mathbf{d}\mathbb{P}^H_{\nu^N_\alpha}}{\mathbf{d}\mathbb{P}_{\nu^N_\alpha}}$, as

$$\begin{aligned}
\exp\Bigg\{ & N\langle \pi^N_T, H_T\rangle - N\langle \pi^N_0, H_0\rangle - N\int_0^T \langle \pi^N_t, (\partial_t + \Delta)H_t\rangle\, dt \\
& - N\int_0^T \tfrac{1}{N}\sum_{x\neq a_N} \eta_t(x)\big(1-\eta_t(x{+}1)\big)\tfrac{1}{2}(\partial_u H_t)^2(\tfrac{x}{N})\, dt \\
& - N\int_0^T \tfrac{1}{N}\sum_{x\neq a_N} \eta_t(x{+}1)\big(1-\eta_t(x)\big)\tfrac{1}{2}(\partial_u H_t)^2(\tfrac{x}{N})\, dt \\
& - N\int_0^T \big\{\eta_t(a_N+1)\partial_u H_t(\tfrac{a_N+1}{N}) - \eta_t(a_N)\partial_u H_t(\tfrac{a_N}{N})\big\}\, dt \\
& - N\int_0^T \eta_t(a_N)\big(1-\eta_t(a_N{+}1)\big)\big(e^{\nabla_N H_{a_N}} - 1\big)\, dt \\
& - N\int_0^T \eta_t(a_N+1)\big(1-\eta_t(a_N)\big)\big(e^{-\nabla_N H_{a_N}} - 1\big)\, dt - NO_{H,T}(\tfrac{1}{N})\Big]\Bigg\}.
\end{aligned}$$

To write this in a simple form, we introduce some notation. For each function $H \in C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$, we consider the linear functional $\ell^{int}_H : \mathcal{D}\big([0,T],\mathcal{M}\big) \to \mathbb{R}$ given by

$$\ell^{int}_H(\pi) = \langle \pi_T, H_T\rangle - \langle \pi_0, H_0\rangle - \int_0^T \langle \pi_t, (\partial_t + \Delta)H_t\rangle\, dt\,. \tag{4.2}$$

Recall the notation g_1, $\tilde{g}_1$, g_2 and $\tilde{g}_2$ defined in (2.10) and (2.11). With this notation we may write the Radon-Nikodym derivative $\mathbf{d}\mathbb{P}^H_{\nu^N_\alpha}/\mathbf{d}\mathbb{P}_{\nu^N_\alpha}$ as

$$\begin{aligned}
\exp\Bigg\{ & N\Big[\ell^{int}_H(\pi^N) - \int_0^T \tfrac{1}{2N}\sum_{x\neq a_N}\big\{\tau_x g_1(\eta_t) + \tau_x g_2(\eta_t)\big\}(\partial_u H_t)^2(\tfrac{x}{N})\, dt\Big] \\
& - N\int_0^T \big\{\eta_t(a_N+1)\partial_u H_t(\tfrac{a_N+1}{N}) - \eta_t(a_N)\partial_u H_t(\tfrac{a_N}{N})\big\}\, dt
\end{aligned} \tag{4.3}$$

$$
\begin{aligned}
&- N \int_0^T \big\{ \tau_{a_N} g_1(\eta_t) \big(e^{\nabla_N H_{a_N}} - 1 \big) + \tau_{a_N} g_2(\eta_t) \big(e^{-\nabla_N H_{a_N}} - 1 \big) \big\} \, dt \\
&- N \, O_{H,T}(\tfrac{1}{N}) \Bigg\} .
\end{aligned}
$$

We begin by defining a set where the Radon-Nikodym derivative $\mathbf{d}\mathbb{P}^H_{\nu^N_\alpha} / \mathbf{d}\mathbb{P}_{\nu^N_\alpha}$ is close to a function of the empirical measure. Consider

$$
V^1_{N,\varepsilon}(t,\eta) = V^{F_1,F_2}_{N,\varepsilon}(t,\eta) \, , \quad V^2_{N,\varepsilon}(t,\eta) = V^{G_1,G_2}_{N,\varepsilon}(t,\eta) \, ,
$$

$$
V^3_{N,\varepsilon}(t,\eta) = V^{\partial H, a_N}_{N,\varepsilon}(t,\eta) \, , \quad V^4_{N,\varepsilon}(t,\eta) = V^{\partial H, a_N+1}_{N,\varepsilon}(t,\eta) \, ,
$$

where $V^{F_1,F_2}_{N,\varepsilon}$, $V^{G_1,G_2}_{N,\varepsilon}$, $V^{\partial H, a_N}_{N,\varepsilon}$ and $V^{\partial H, a_N+1}_{N,\varepsilon}$ have been defined in Propositions 3.0.4 and 3.0.5 with $F_1(t,u) = \frac{1}{2}(\partial_u H_t)^2(u)$, $F_2(t, \frac{a_N}{N}) = e^{\nabla_N H_{a_N}} - 1$, $G_1(t,u) = \frac{1}{2}(\partial_u H_t)^2(u)$ and $G_2(t, \frac{a_N}{N}) = e^{-\nabla_N H_{a_N}} - 1$.

Define $B^H_{\delta,\varepsilon}$ as the set of trajectories $\{\eta_t\}_{0 \le t \le T}$ such that

$$
B^H_{\delta,\varepsilon} = \left\{ \eta \in D([0,T], \{0,1\}^{\mathbb{T}_N}) \, ; \; \Big| \int_0^T V^i_{N,\varepsilon}(t,\eta_t) \, dt \Big| \le \delta \, , i = 1,2,3,4 \right\} . \tag{4.4}
$$

From Propositions 3.0.4 and 3.0.5, the set $B^H_{\delta,\varepsilon}$ has probability superexponentially close to 1: for each $\delta > 0$ and $H \in C^{1,2}([0,T] \times \overline{(0,1)}\,)$,

$$
\varlimsup_{\varepsilon \downarrow 0} \varlimsup_{N \to \infty} \tfrac{1}{N} \log \mathbb{P}_{\nu^N_\alpha} \Big[(B^H_{\delta,\varepsilon})^{\complement} \Big] = -\infty \, . \tag{4.5}
$$

Since $\eta^{\varepsilon N}(x) = (\pi^N * \iota^a_\varepsilon)(\frac{x}{N})$. In view of this identity and the expression (4.3) for $\mathbf{d}\mathbb{P}^H_{\nu^N_\alpha} / \mathbf{d}\mathbb{P}_{\nu^N_\alpha}$, on $B^H_{\delta,\varepsilon}$ the Radon-Nikodym derivative can be written as a function of the empirical measure modulo some small errors, i.e., $\mathbf{d}\mathbb{P}^H_{\nu^N_\alpha} / \mathbf{d}\mathbb{P}_{\nu^N_\alpha}$ restricted to $B^H_{\delta,\varepsilon}$ is equal to

$$
\begin{aligned}
\exp \Bigg\{ N \Bigg[& \ell^{int}_H(\pi^N) - \int_0^T \tfrac{1}{2N} \sum_{x \ne a_N} \tilde{g}_1 \Big((\pi^N_t * \iota^a_\varepsilon)(\tfrac{x}{N}), (\pi^N_t * \iota^a_\varepsilon)(\tfrac{x+1}{N}) \Big) (\partial_u H_t)^2(\tfrac{x}{N}) \, dt \\
& - \int_0^T \tfrac{1}{2N} \sum_{x \ne a_N} \tilde{g}_2 \Big((\pi^N_t * \iota^a_\varepsilon)(\tfrac{x}{N}), (\pi^N_t * \iota^a_\varepsilon)(\tfrac{x+1}{N}) \Big) (\partial_u H_t)^2(\tfrac{x}{N}) \, dt \\
& - \int_0^T \Big[(\pi^N_t * \iota^a_\varepsilon)(\tfrac{a_N+1}{N}) \partial_u H_t(\tfrac{a_N+1}{N}) - (\pi^N_t * \iota^a_\varepsilon)(\tfrac{a_N}{N}) \partial_u H_t(\tfrac{a_N}{N}) \Big] \, dt \\
& - \int_0^T \tilde{g}_1 \Big((\pi^N_t * \iota^a_\varepsilon)(\tfrac{a_N}{N}), (\pi^N_t * \iota^a_\varepsilon)(\tfrac{a_N+1}{N}) \Big) (e^{\nabla_N H_{a_N}} - 1) \, dt \\
& - \int_0^T \tilde{g}_2 \Big((\pi^N_t * \iota^a_\varepsilon)(\tfrac{a_N}{N}), (\pi^N_t * \iota^a_\varepsilon)(\tfrac{a_N+1}{N}) \Big) (e^{-\nabla_N H_{a_N}} - 1) \, dt \\
& + O_{H,T}(\tfrac{1}{N}) \; + \; O(\delta) \Bigg] \Bigg\} .
\end{aligned} \tag{4.6}
$$

The Radon-Nikodym derivative already is one function of the empirical measure more small errors, but to conclude the upper bound large deviations we will need take some limits. For these operations will be true to ensure that the boundary terms in (4.6) are well defined. For this reason we will make $\pi^N * \iota_\varepsilon^a$ more smooth. In this way, we will replace π^N by $\pi^N * \iota_\gamma$, where ι_γ is a continuous approximation of identity. Thus, we will work with $(\pi^N * \iota_\gamma) * \iota_\varepsilon^a$. Since $\pi^N * \iota_\gamma$ belongs to $\mathcal{D}([0,T],\mathcal{M}_0)$, by Lemma A.3.8, $\mathcal{E}((\pi^N * \iota_\gamma) * \iota_\varepsilon^a)$ is finite. This will ensure that the boundary terms are well defined.

For this fact, we will need of the next technical lemmata, which proofs are in the end of section.

Let $f : \mathbb{T} \to \mathbb{R}_+$, any continuous function such that the support of f is contained in $[-\frac{1}{4}, \frac{1}{4}]$, $\|f\|_\infty \leq 4$, $f(0) > 0$, $\|f\|_{L^1} = 1$ and $f(u) = f(1-u)$ for all $u \in \mathbb{T}$. Define now the continuous approximation of identity ι_γ by $\iota_\gamma(u) = \frac{1}{\gamma} f(\frac{u}{\gamma})$.

Lemma 4.1.1.
$$|(\pi_t^N * \iota_\varepsilon^a)(v) - ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(v)| \leq \tfrac{\gamma}{\varepsilon}\,,$$
uniformly in $v \in \mathbb{T}$, N and in $t \in [0,T]$.

Lemma 4.1.2. *Recalling that ℓ_H^{int} is the linear functional defined in* (4.2).

$$\ell_H^{int}(\pi^N) = \ell_H^{int}\Big((\pi^N * \iota_\gamma) * \iota_\varepsilon^a\Big) + O_H(\varepsilon) + O_H(\tfrac{\gamma}{\varepsilon})\,,$$

uniformly in N.

Lemma 4.1.3. *For $i = 1, 2$, the function*

$$\Big|\tilde{g}_i\Big((\pi_t^N * \iota_\varepsilon^a)(\tfrac{x}{N}), (\pi_t^N * \iota_\varepsilon^a)(\tfrac{x+1}{N})\Big) - \tilde{g}_i\Big(((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(\tfrac{x}{N}), ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(\tfrac{x+1}{N})\Big)\Big|$$

is equal to $O(\frac{\gamma}{\varepsilon})$.

The Lemmata 4.1.1, 4.1.2 and 4.1.3 allow to replace π^N by $\pi_t^N * \iota_\gamma$ in the expression of Radon-Nikodym derivative (4.6). Then, restricted to the set $B_{\delta,\varepsilon}^H$, the Radon-Nikodym derivative $\mathbf{d}\mathbb{P}_{\nu_\alpha^N}^H / \mathbf{d}\mathbb{P}_{\nu_\alpha^N}$ becomes

$$\begin{aligned} \exp\Bigg\{ N \Bigg[& \ell_H^{int}\Big((\pi^N * \iota_\gamma) * \iota_\varepsilon^a\Big) \\ & - \int_0^T \tfrac{1}{2N} \sum_{x \neq a_N} \tilde{g}_1\Big(((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(\tfrac{x}{N}), ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(\tfrac{x+1}{N})\Big)(\partial_u H_t)^2(\tfrac{x}{N})\, dt \\ & - \int_0^T \tfrac{1}{2N} \sum_{x \neq a_N} \tilde{g}_2\Big(((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(\tfrac{x}{N}), ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(\tfrac{x+1}{N})\Big)(\partial_u H_t)^2(\tfrac{x}{N})\, dt \end{aligned} \tag{4.7}$$

$$- \int_0^T \Big[((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(\tfrac{a_N+1}{N}) \partial_u H_t(\tfrac{a_N+1}{N}) - ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(\tfrac{a_N}{N}) \partial_u H_t(\tfrac{a_N}{N}) \Big] \, dt$$

$$- \int_0^T \tilde{g}_1 \Big(((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(\tfrac{a_N}{N}), ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(\tfrac{a_N+1}{N}) \Big) (e^{\nabla_N H_{a_N}} - 1) \, dt$$

$$- \int_0^T \tilde{g}_2 \Big(((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(\tfrac{a_N}{N}), ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(\tfrac{a_N+1}{N}) \Big) (e^{-\nabla_N H_{a_N}} - 1) \, dt$$

$$+ \, O_{H,T}(\tfrac{1}{N}) \, + \, O(\delta) \, + \, O_H(\varepsilon) \, + \, O_H(\tfrac{\gamma}{\varepsilon}) \Bigg] \Bigg\} .$$

Lemma 4.1.4. *Let χ be the function defined by $\chi(\alpha) = \alpha(1-\alpha)$, for all $\alpha \in [0,1]$. Then, for $i = 1, 2$,*

$$\Big| \tfrac{1}{N} \sum_{x \neq a_N} \tilde{g}_i \Big(((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(\tfrac{x}{N}), ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(\tfrac{x+1}{N}) \Big) (\partial_u H_t)^2 (\tfrac{x}{N})$$
$$- \int_{\mathbb{T}} \chi \Big(((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(v) \Big) (\partial_u H_t)^2 (v) \, dv \Big| = O_{H,\varepsilon}(\tfrac{1}{N}) \, ,$$

uniformly in $t \in [0, T]$.

Lemma 4.1.5. *The function*

$$\Big| ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(\tfrac{a_N+1}{N}) \partial_u H_t(\tfrac{a_N+1}{N}) - ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(\tfrac{a_N}{N}) \partial_u H_t(\tfrac{a_N}{N})$$
$$- ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(a^+) \partial_u H_t(a^+) - ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(a^-) \partial_u H_t(a^-) \Big|$$

is $O_{H,T,\varepsilon,\gamma}(\frac{1}{N})$, uniformly in $t \in [0, T]$.

Lemma 4.1.6.

$$\Big| \tilde{g}_1 \Big(((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(a^-), ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(a^+) \Big) \Big(e^{H_t(a^+) - H_t(a^-)} - 1 \Big)$$
$$- \tilde{g}_1 \Big(((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(\tfrac{a_N}{N}), ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(\tfrac{a_N+1}{N}) \Big) (e^{\nabla_N H_{a_N}} - 1) \Big|$$

is $O_{H,T,\varepsilon,\gamma}(\frac{1}{N})$, uniformly in $t \in [0, T]$. The result analogous is valid for $\tilde{g}_2$.

Using the Lemmata 4.1.4, 4.1.5 and 4.1.6, we can rewrite the expression (4.7) of the Radon-Nikodyn derivative $\mathbf{d}\mathbb{P}^H_{\nu^N_\alpha} / \mathbf{d}\mathbb{P}_{\nu^N_\alpha}$, on the set $B^H_{\delta,\varepsilon}$, as

$$\exp \Bigg\{ N \Bigg[\ell_H^{int} \Big((\pi^N * \iota_\gamma) * \iota_\varepsilon^a \Big) - \int_0^T \int_{\mathbb{T}} \chi \Big(((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(v) \Big) (\partial_u H_t)^2 (v) \, dv \, dt$$

$$
\begin{aligned}
&- \int_0^T \Big[((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(a^+) \partial_u H_t(a^+) - ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(a^-) \partial_u H_t(a^-) \Big]\, dt \\
&- \int_0^T \tilde{g}_1 \Big(((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(a^-), ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(a^+) \Big) (e^{H_t(a^+) - H_t(a^-)} - 1)\, dt \\
&- \int_0^T \tilde{g}_2 \Big(((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(a^-), ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(a^+) \Big) (e^{-H_t(a^+) + H_t(a^-)} - 1)\, dt \\
&+ O_{H,T,\varepsilon,\gamma}(\tfrac{1}{N}) \,+\, O(\delta) \,+\, O_H(\varepsilon) \,+\, O_H(\tfrac{\gamma}{\varepsilon}) \Big] \Big] \Big\} .
\end{aligned}
\tag{4.8}
$$

Now, we need write the Radon-Nikodym derivative in a short expression. This will be useful for future manipulations in the upper bound of large deviations. One can see a similarity between the expression above and the expression of the funcional $\hat{J}_H$, defined in (1.14). Before we continue with this replacement, we must clarify some details.

We begin observing that the functional ℓ_H, defined in (1.13), can be written in another form. Indeed, recalling the Definition (4.2) of the functional $\ell_H^{int} : \mathcal{D}\big([0,T], \mathcal{M}\big) \to \mathbb{R}$

$$
\ell_H^{int}(\pi) = \langle \pi_T, H_T \rangle - \langle \pi_0, H_0 \rangle - \int_0^T \langle \pi_t, (\partial_t + \Delta) H_t \rangle \, dt \,,
$$

we obtain the follows expression for

$$
\ell_H(\pi) = \ell_H^{int}(\pi) - \int_0^T \{ \rho_t(a^+) \partial_u H_t(a^+) - \rho_t(a^-) \partial_u H_t(a^-) \} \, dt \,.
$$

An important observation is that $(\pi^N * \iota_\gamma) * \iota_\varepsilon^a$ has energy finite, it follows by Lemma A.3.8. Now, we just need to remember the expression of $\hat{J}_H$ and the Definition 1.4.3 of the functional J_H, to be able to rewrite the expression (4.8). Thus, the Radon-Nikodym derivative $\mathbf{d}\mathbb{P}^H_{\nu_\alpha^N} / \mathbf{d}\mathbb{P}_{\nu_\alpha^N}$ restricted to the set $B^H_{\delta,\varepsilon}$ is equal to

$$
\exp \Big\{ N \Big[J_H \Big((\pi^N * \iota_\gamma) * \iota_\varepsilon^a \Big) + O_{H,T,\varepsilon,\gamma}(\tfrac{1}{N}) + O(\delta) + O_H(\varepsilon) + O_H(\tfrac{\gamma}{\varepsilon}) \Big] \Big\} . \tag{4.9}
$$

Unfortunately, the set $\{\pi; \mathcal{E}(\pi) < \infty\}$ not is closed. In sense of Proposition A.1.2, it must to be finite the functional in a closed set. In this way, we introduce the next definitions.

Definition 4.1.1. *Let $A_{k,l}$, $A_{k,l}^{\varepsilon,\gamma}$ and $A_{k,l}^{\varepsilon}$ be the subsets of trajectories given by*

$$
\begin{aligned}
A_{k,l} &= \{ \pi \in \mathcal{D}\big([0,T], \mathcal{M}\big) ; \ \max_{1 \le j \le k} \mathcal{E}_{H_j}(\pi) \le l \} \,, \\
A_{k,l}^{\varepsilon} &= \big\{ \pi \in \mathcal{D}\big([0,T], \mathcal{M}\big) ; \ \pi * \iota_\varepsilon^a \in A_{k,l} \big\} \,, \\
A_{k,l}^{\varepsilon,\gamma} &= \big\{ \pi \in \mathcal{D}\big([0,T], \mathcal{M}\big) ; \ (\pi * \iota_\gamma) * \iota_\varepsilon^a \in A_{k,l} \big\} \,.
\end{aligned}
$$

Proposition 4.1.7. *For fixed $\varepsilon, \gamma, k, l$, the set $A^{\varepsilon,\gamma}_{k,l}$ is closed.*

Proof. It is sufficient to show that the function $\psi : \mathcal{D}\big([0,T],\mathcal{M}\big) \to \overline{\mathbb{R}}$ given by $\psi(\pi) = \mathcal{E}_{H_j}((\pi^N * \iota_\gamma) * \iota^a_\varepsilon)$ is continuous. Let $\{\pi^n_t; t \in [0,T]\}_n$ converging to $\{\pi_t; t \in [0,T]\}$ on $\mathcal{D}([0,T],\mathcal{M})$. Therefore, $\pi^n_t \stackrel{\omega^*}{\to} \pi_t$, almost surely in time. For such t, $\pi_t * \iota_\gamma = \lim_{n\to\infty} \pi^n_t * \iota_\gamma$, since ι_γ is a continuous function. By Dominated Convergence Theorem,

$$((\pi_t * \iota_\gamma) * \iota^a_\varepsilon)(v) = \int_{\mathbb{T}} \lim_{n\to\infty} (\pi^n_t * \iota_\gamma)(u)\, \iota^a_\varepsilon(u,v)\, du = \lim_{n\to\infty} ((\pi^n_t * \iota_\gamma) * \iota^a_\varepsilon)(v)\,. \tag{4.10}$$

Again by the Dominated Convergence Theorem,

$$\begin{aligned}
&\langle\!\langle \partial_u H_j, (\pi_t * \iota_\gamma) * \iota^a_\varepsilon \rangle\!\rangle = \int_0^T \int_{\mathbb{T}} \partial_u H_j(t,v)((\pi_t * \iota_\gamma) * \iota^a_\varepsilon)(v)\, dv\, dt \\
&= \lim_{n\to\infty} \int_0^T \int_{\mathbb{T}} \partial_u H_j(t,v)((\pi^n_t * \iota_\gamma) * \iota^a_\varepsilon)(v)\, dv\, dt = \lim_{n\to\infty} \langle\!\langle \partial_u H_j, (\pi^n * \iota_\gamma) * \iota^a_\varepsilon \rangle\!\rangle\,.
\end{aligned}$$

□

Proposition 4.1.8. *For k, l fixed,*

$$\varlimsup_{\varepsilon\downarrow 0} \varlimsup_{\gamma\downarrow 0} \varlimsup_{N\to\infty} \tfrac{1}{N} \log \mathbb{P}_{\nu^N_\alpha}\Big[\{\pi^N \in (A^{\varepsilon,\gamma}_{k,l})^{\complement}\}\Big] \leq -l + K_0 T\,.$$

Proof. For all $r > 0$,

$$\begin{aligned}
&\mathbb{P}_{\nu^N_\alpha}\Big[\max_{1\leq j\leq k} \mathcal{E}_{H_j}((\pi^N * \iota_\gamma) * \iota^a_\varepsilon) \geq l\Big] \leq \mathbb{P}_{\nu^N_\alpha}\Big[\max_{1\leq j\leq k} \mathcal{E}_{H_j}(\pi^N * \iota^a_\varepsilon) \geq l - r\Big] \\
&+ \mathbb{P}_{\nu^N_\alpha}\Big[\max_{1\leq j\leq k} \mathcal{E}_{H_j}\Big((\pi^N * \iota_\gamma) * \iota^a_\varepsilon - \pi^N * \iota^a_\varepsilon\Big) \geq r\Big]\,.
\end{aligned}$$

By Proposition 4.1.1,

$$\max_{1\leq j\leq k} \mathcal{E}_{H_j}\Big((\pi^N * \iota_\gamma) * \iota^a_\varepsilon - \pi^N * \iota^a_\varepsilon\Big) = \max_{1\leq j\leq k} \Big\langle\!\Big\langle \partial_u H_j,\, (\pi^N * \iota_\gamma) * \iota^a_\varepsilon - \pi^N * \iota^a_\varepsilon \Big\rangle\!\Big\rangle \leq \tfrac{C\gamma}{\varepsilon}\,,$$

where $C = C(\{H\}_{1\leq j\leq k})$. Therefore,

$$\mathbb{P}_{\nu^N_\alpha}\Big[\max_{1\leq j\leq k} \mathcal{E}_{H_j}((\pi^N * \iota_\gamma - \pi^N) * \iota^a_\varepsilon) \geq r\Big] \leq \mathbb{P}_{\nu^N_\alpha}\Big[\tfrac{C\gamma}{\varepsilon} \geq r\Big]\,,$$

which can be only one or zero, independently of N, assuming the value zero if γ is enoughly small. Then,

$$\begin{aligned}
&\varlimsup_{\gamma\downarrow 0} \varlimsup_{N\to\infty} \tfrac{1}{N} \log \mathbb{P}_{\nu^N_\alpha}\Big[\max_{1\leq j\leq k} \mathcal{E}_{H_j}((\pi^N * \iota_\gamma) * \iota^a_\varepsilon) \geq l\Big] \\
&\leq \varlimsup_{N\to\infty} \tfrac{1}{N} \log \mathbb{P}_{\nu^N_\alpha}\Big[\max_{1\leq j\leq k} \mathcal{E}_{H_j}(\pi^N * \iota^a_\varepsilon) \geq l - r\Big]\,.
\end{aligned}$$

By corollary 3.1.2, we get

$$\varlimsup_{\varepsilon\downarrow 0} \varlimsup_{\gamma\downarrow 0} \varlimsup_{N\to\infty} \tfrac{1}{N} \log \mathbb{P}_{\nu^N_\alpha}\Big[\max_{1\leq j\leq k} \mathcal{E}_{H_j}((\pi^N * \iota_\gamma) * \iota^a_\varepsilon) \geq l\Big] \leq -l + K_0 T + r\,.$$

Because r is arbitrary, it finishes the proof. □

Fix a sequence $\{F_i\}_{i\geq 1}$ of smooth non-negative functions dense in the subset of non-negative functions $C(\mathbb{T})$ for the uniform topology. For $i \geq 1$ and $j \geq 1$, define the set

$$D_i^j = \Big\{\pi \in \mathcal{D}([0,T],\mathcal{M});\, 0 \leq \langle \pi_t, F_i\rangle \leq \int_{\mathbb{T}} F_i(u)\,du + \tfrac{1}{j}\|F_i'\|_\infty,\, 0\leq t\leq T\Big\}, \tag{4.11}$$

and for $m \geq 1$ and $j \geq 1$, let

$$E_m^j = \bigcap_{i=1}^m D_i^j\,.$$

Proposition 4.1.9. *It hold:*

(i) *For $i \geq 1$ and $j \geq 1$, the set D_i^j is a closed subset of $\mathcal{D}\big([0,T],\mathcal{M}\big)$;*

(ii) *$D([0,T],\mathcal{M}_0) = \cap_{j\geq 1}\cap_{m\geq 1} E_m^j$;*

(iii) *For $m \geq 1$ and $j \geq 1$, $\overline{\lim}_{N\to\infty} \frac{1}{N}\log \mathbb{P}_{\nu_\alpha^N}[\{\pi^N \in (E_m^j)^\complement\}] = -\infty$.*

Proof. **(i)** This statement follows from the fact for F_i continuous, then the function that associate for each $\pi \in \mathcal{D}\big([0,T],\mathcal{M}\big)$ the number $\sup_{0\leq t\leq T}\langle \pi_t, F_i\rangle$ is continuous.

(ii) The inclusion $D([0,T],\mathcal{M}_0) \subset \cap_{j\geq 1}\cap_{m\geq 1} E_m^j$ is trivial. The other hand follows approximating indicators functions of open intervals by an suitable sequence in $\{F_i\}_{i\geq 1}$ and in j.

(iii) The probability $\mathbb{P}_{\nu_\alpha^N}[\{\pi^N \in (E_m^j)^\complement\}]$ is

$$\mathbb{P}_{\nu_\alpha^N}\Big[\bigcup_{i=1}^m\Big\{\tfrac{1}{N}\sum_{x\in\mathbb{T}_N} F_i(\tfrac{x}{N})\eta_t(x) > \int_{\mathbb{T}} F_i(u)\,du + \tfrac{1}{j}\|F_i'\|_\infty,\ \text{for some } t\in[0,T]\Big\}\Big]\,.$$

From the inequality

$$\Big|\tfrac{1}{N}\sum_{x\in T_N} F_i(\tfrac{x}{N}) - \int_{\mathbb{T}} F_i(u)\,du\Big| \leq \sum_{x\in T_N}\int_{[\frac{x}{N},\frac{x+1}{N})} |F_i(\tfrac{x}{N}) - F_i(u)|\,du \leq \tfrac{\|F_i'\|_\infty}{N}\,,$$

the probability $\mathbb{P}_{\nu_\alpha^N}[\{\pi^N \in (E_m^j)^\complement\}]$ becomes zero for N sufficiently large. □

From Lemma A.3.8, $\mathcal{E}((\pi * \iota_\gamma) * \iota_\varepsilon^a) < \infty$, for all $\pi \in \mathcal{D}([0,T],\mathcal{M})$. Then, we define

$$J_{H,\gamma,\varepsilon,\zeta}^{k,l,m,j}(\pi) \;=\; \begin{cases} \hat{J}_H\Big((\pi*\iota_\gamma)*\iota_\varepsilon^a\Big), & \text{if } \ \pi \in A_{k,l}^{\zeta,\gamma}\cap E_m^j\,,\\ +\infty, & \text{otherwise.}\end{cases}$$

Finally, the Radon-Nikodym derivative $\mathbf{d}\mathbb{P}_{\nu_\alpha^N}^H/\mathbf{d}\mathbb{P}_{\nu_\alpha^N}$ restricted to the set $\{\pi^N \in A_{k,l}^{\zeta,\gamma}\cap E_m^j\}\cap B_{\delta,\varepsilon}^H$ is

$$\exp\Big\{N\Big[J_{H,\gamma,\varepsilon,\zeta}^{k,l,m,j}(\pi^N) + O_{H,T,\varepsilon,\gamma}(\tfrac{1}{N}) + O(\delta) + O_H(\varepsilon) + O_H(\tfrac{\gamma}{\varepsilon})\Big]\Big\}\,. \tag{4.12}$$

Here toward the end of the section we present the proofs of lemmata above.

Proof of Lemma 4.1.1. Writing the expression $|(\pi_t^N * \iota_\varepsilon^a)(v) - ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(v)|$ as

$$\Big|\tfrac{1}{N}\sum_{x\in\mathbb{T}_N}\eta_t(x)\iota_\varepsilon^a(\tfrac{x}{N},v) - \int_{\mathbb{T}}\tfrac{1}{N}\sum_{x\in\mathbb{T}_N}\eta_t(x)\iota_\gamma(u-\tfrac{x}{N})\iota_\varepsilon^a(u,v)\,du\Big|\,.$$

Using the rule of maximum of one particle per site, the last expression is bounded by

$$\tfrac{1}{N}\sum_{x\in\mathbb{T}_N}\Big|\iota_\varepsilon^a(\tfrac{x}{N},v) - \int_{\mathbb{T}}\iota_\gamma(u-\tfrac{x}{N})\iota_\varepsilon^a(u,v)\,du\Big|\,.$$

Fix N, v and ε, then $\iota_\varepsilon^a(\cdot,v)$ is the indicator function of an open interval $(z,z+\varepsilon)$, for $z=v$ or $z=a-\varepsilon$. The summand above is possibly not zero only if $\frac{x}{N}$ belongs to the open intervals $(z-\frac{\gamma}{4},z+\frac{\gamma}{4})$ or $(z+\varepsilon-\frac{\gamma}{4},z+\varepsilon+\frac{\gamma}{4})$. The summands are bounded by $\frac{1}{\varepsilon}$, and the number of non zero summands is of order γN, which concludes the proof. □

Proof of Lemma 4.1.2. First we compare $\ell_H^{int}(((\pi^N * \iota_\gamma) * \iota_\varepsilon^a))$ with $\ell_H^{int}((\pi^N * \iota_\varepsilon^a))$. Using the Lemma 4.1.1, we obtain the difference between this functions is

$$\Big|\Big\langle((\pi_T^N * \iota_\gamma) * \iota_\varepsilon^a) - (\pi_T^N * \iota_\varepsilon^a), H_T\Big\rangle - \Big\langle((\pi_0^N * \iota_\gamma) * \iota_\varepsilon^a) - (\pi_0^N * \iota_\varepsilon^a), H_0\Big\rangle$$
$$- \int_0^T \Big\langle((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a) - (\pi_t^N * \iota_\varepsilon^a), (\partial_t+\Delta)H_t\Big\rangle\,dt\Big| \le C(H)\tfrac{\gamma}{\varepsilon}\,.$$

Then, we need only analyze the expression below

$$\Big|\ell_H^{int}((\pi^N * \iota_\varepsilon^a)) - \ell_H^{int}(\pi^N)\Big| = \Big|\Big\langle(\pi_T^N * \iota_\varepsilon^a) - \pi_T^N, H_T\Big\rangle - \Big\langle(\pi_0^N * \iota_\varepsilon^a) - \pi_0^N, H_0\Big\rangle$$
$$- \int_0^T \Big\langle\,(\pi_t^N * \iota_\varepsilon^a) - \pi_t^N, (\partial_t+\Delta)H_t\Big\rangle\,dt\Big|\,.$$

We handle only the first term, because the others terms are similar. Thus,

$$\Big\langle(\pi_t^N * \iota_\varepsilon^a), H_t\Big\rangle = \int_{\mathbb{T}}(\pi_t^N * \iota_\varepsilon^a)(v)H_t(v)dv = \int_{\mathbb{T}}\tfrac{1}{N}\sum_{y\in\mathbb{T}_N}\eta_t(y)\iota_\varepsilon^a(\tfrac{y}{N},v)H_t(v)dv$$
$$= \tfrac{1}{N}\sum_{y\in\mathbb{T}_N}\eta_t(y)\int_{\mathbb{T}}H_t(v)\,\iota_\varepsilon^a(\tfrac{y}{N},v)\,dv = \langle\pi_t^N, H_t\rangle + O_H(\varepsilon)\,.$$

This approximation holds uniformly in time and N, since $H\in C^{1,2}([0,T]\times\overline{(0,1)}\,)$ and there is at most one particle per site. Therefore,

$$|\ell_H^{int}(\pi^N * \iota_\varepsilon^a) - \ell_H^{int}(\pi^N)| = O_H(\varepsilon)\,.$$

□

Proof of Lemma 4.1.3. This proof follows by the definition of $\tilde{g}_1$ and $\tilde{g}_2$ (see (2.10) and (2.11)), the triangular inequality and the Lemma 4.1.1. $\square$

Proof of Lemma 4.1.4. Consider $i = 1$. To simplify notation, denote

$$f^N(\tfrac{x}{N}) := \tilde{g}_1\Big(((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(\tfrac{x}{N}), ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(\tfrac{x+1}{N})\Big)$$

and

$$g^N(v) := \tilde{g}_1\Big(((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(v), ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(v)\Big) = \chi\Big(((\pi_t^N * \iota_\gamma) * \iota_\varepsilon^a)(v)\Big)\,.$$

From the definition of ι_ε^a, if $x \neq a_N$,

$$|(\varrho * \iota_\varepsilon^a)(\tfrac{x}{N}) - (\varrho * \iota_\varepsilon^a)(v)| \leq \frac{\|\varrho\|_\infty}{\varepsilon N}\,, \qquad \forall v \in [\tfrac{x}{N}, \tfrac{x+1}{N}]\,,$$

where ϱ is any bounded function defined on the torus. The same inequality is still valid with $x+1$ replacing x in left side of inequality. Since $\|\pi_t^N * \iota_\gamma\|_\infty \leq 4$, if $x \neq a_N$,

$$|f^N(\tfrac{x}{N}) - g^N(v)| = O(\tfrac{1}{\varepsilon N}), \qquad \forall v \in [\tfrac{x}{N}, \tfrac{x+1}{N}]\,.$$

Then,

$$\begin{aligned}
&\Big|\tfrac{1}{N}\sum_{x\neq a_N} f^N(\tfrac{x}{N})(\partial_u H_t)^2(\tfrac{x}{N}) - \int_{\mathbb{T}} g^N(v)(\partial_u H_t)^2(v)\,dv\Big|\\
&\leq \Big|\tfrac{1}{N}\sum_{x\neq a_N} f^N(\tfrac{x}{N})\Big[(\partial_u H_t)^2(\tfrac{x}{N}) - N\int_{\mathbb{T}} \mathbf{1}_{[\frac{x}{N},\frac{x+1}{N})}(v)(\partial_u H_t)^2(v)\,dv\Big]\Big|\\
&+ \Big|\sum_{x\neq a_N}\int_{\mathbb{T}} \mathbf{1}_{[\frac{x}{N},\frac{x+1}{N})}(v)\Big[f^N(\tfrac{x}{N}) - g^N(v)\Big](\partial_u H_t)^2(v)\,dv\Big|\\
&+ \Big|\int_{\mathbb{T}} \mathbf{1}_{[\frac{a_N}{N},\frac{a_N+1}{N})}(v) g^N(v)(\partial_u H_t)^2(v)\,dv\Big|\\
&\leq \tfrac{1}{N}\sum_{x\neq a_N}\Big|(\partial_u H_t)^2(\tfrac{x}{N}) - N\int_{\mathbb{T}} \mathbf{1}_{[\frac{x}{N},\frac{x+1}{N})}(v)(\partial_u H_t)^2(v)\,dv\Big|\\
&+ O(\tfrac{1}{\varepsilon N})\int_{\mathbb{T}} |(\partial_u H_t)^2(v)|\,dv + \tfrac{1}{N}\|(\partial_u H_t)^2\|_\infty\,.
\end{aligned}$$

The first sum is $O_H(\frac{1}{N})$, since H belongs to $C^{1,2}([0,T]\times\overline{(0,1)}\,)$, which finishes the proof. $\square$

Proof of Lemma 4.1.5. This proof follows by fact that $\iota_\varepsilon^a(\cdot, \frac{a_N}{N})$, $\iota_\varepsilon^a(\cdot, \frac{a_N+1}{N})$, $\partial_u H_t(\frac{a_N}{N})$ and $\partial_u H_t(\frac{a_N+1}{N})$ converges to $\iota_\varepsilon^a(\cdot, a^-)$, $\iota_\varepsilon^a(\cdot, a^+)$, $\partial_u H_t(a^-)$ and $\partial_u H_t(a^+)$, respectively, as N increases to infinity. $\square$

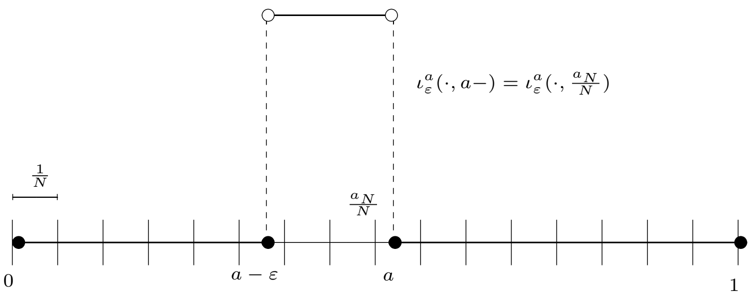

Figure 4.1: $\iota^a_\varepsilon(\cdot, a^-)$ and $\iota^a_\varepsilon(\cdot, \frac{a_N}{N})$

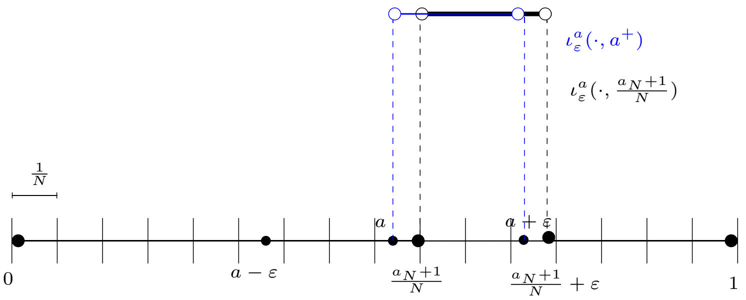

Figure 4.2: $\iota^a_\varepsilon(\cdot, \frac{a_N+1}{N})$ and $\iota^a_\varepsilon(\cdot, a^+)$

Proof of Lemma 4.1.6. We only analyze the first statement, the second one is just the same argument. By definition of $\tilde{g}_1$, the expression in the left side of the first equality is bounded above by

$$\begin{aligned}
&\Big|((\pi^N_t * \iota_\gamma) * \iota^a_\varepsilon)(\tfrac{a_N}{N})(e^{\nabla_N H_{a_N}} - 1) - ((\pi^N_t * \iota_\gamma) * \iota^a_\varepsilon)(a^-)(e^{H_t(a^+)-H_t(a^-)} - 1)\Big| \\
&+ \Big|((\pi^N_t * \iota_\gamma) * \iota^a_\varepsilon)(\tfrac{a_N}{N})((\pi^N_t * \iota_\gamma) * \iota^a_\varepsilon)(\tfrac{a_N+1}{N})(e^{\nabla_N H_{a_N}} - 1) \\
&- ((\pi^N_t * \iota_\gamma) * \iota^a_\varepsilon)(a^-)((\pi^N_t * \iota_\gamma) * \iota^a_\varepsilon)(a^+)(e^{H_t(a^+)-H_t(a^-)} - 1)\Big| .
\end{aligned}$$

The conclusion follows by fact that $\iota^a_\varepsilon(\cdot, \frac{a_N}{N})$, $\iota^a_\varepsilon(\cdot, \frac{a_N+1}{N})$ and $e^{\nabla_N H_{a_N}} - 1$ converges to $\iota^a_\varepsilon(\cdot, a^-)$, $\iota^a_\varepsilon(\cdot, a^+)$ and $e^{H_t(a^+)-H_t(a^-)} - 1$, respectively, as N increases to infinity.

□

4.2 Upper bound for compact sets

Proposition 4.2.1 (Upper bound for compact sets)**.** *For every $\mathcal{K}$ compact subset of the space $\mathcal{D}([0,T], \mathcal{M})$.*

$$\varlimsup_{N\to\infty} \frac{1}{N} \log \mathbb{Q}_{\nu^N_\alpha}[\mathcal{K}] \leq - \inf_{\pi\in\mathcal{K}} I(\pi) \,.$$

For proof this proposition we need analyze that happens with open sets.

Let $\mathcal{O}$ be an open set of $\mathcal{D}([0,T],\mathcal{M})$ and fix $H \in C^{1,2}([0,T]\times\overline{(0,1)}\,)$. Then,

$$
\begin{aligned}
&\varlimsup_{N\to\infty} \tfrac{1}{N}\log \mathbb{Q}_{\nu^N_\alpha}[\mathcal{O}] \;=\; \varlimsup_{N\to\infty} \tfrac{1}{N}\log \mathbb{P}_{\nu^N_\alpha}[\pi^N\in\mathcal{O}]\\
&= \max\Big\{ \varlimsup_{N\to\infty} \tfrac{1}{N}\log \mathbb{P}_{\nu^N_\alpha}[\{\pi^N \in \mathcal{O}\cap A^{\zeta,\gamma}_{k,l}\cap E^j_m\}\cap B^H_{\delta,\varepsilon}]\,,\\
&R^l_k(\zeta,\gamma),\, R^j_m,\, R^\delta_H(\varepsilon)\Big\},
\end{aligned}
$$

where we have denoted

$$
\begin{aligned}
&R^l_k(\zeta,\gamma) = \varlimsup_{N\to\infty} \tfrac{1}{N}\log \mathbb{P}_{\nu^N_\alpha}[\{\pi^N \in (A^{\zeta,\gamma}_{k,l})^\complement\}]\,,\\
&R^j_m = \varlimsup_{N\to\infty} \tfrac{1}{N}\log \mathbb{P}_{\nu^N_\alpha}[\{\pi^N \in (E^j_m)^\complement\}]\,,\\
&R^\delta_H(\varepsilon) = \varlimsup_{N\to\infty} \;\tfrac{1}{N}\log \mathbb{P}_{\nu^N_\alpha}[(B^H_{\delta,\varepsilon})^\complement]\,.
\end{aligned}
$$

Using Propositions 4.1.8 and 4.1.9 and the limit (4.5), the expressions above satisfy

$$
\varlimsup_{\zeta\downarrow 0}\,\varlimsup_{\gamma\downarrow 0}\; R^l_k(\zeta,\gamma) \le -l + K_0T\,,\quad R^j_m = -\infty\,,\quad \text{and}\quad \varlimsup_{\varepsilon\downarrow 0}\; R^\delta_H(\varepsilon) = -\infty\,.
$$

Just applying the Radon-Nikodym derivative, and using the expression (4.12),

$$
\begin{aligned}
&\mathbb{P}_{\nu^N_\alpha}\Big[\{\pi^N \in \mathcal{O}\cap A^{\zeta,\gamma}_{k,l}\cap E^j_m\}\cap B^H_{\delta,\varepsilon}\Big]\\
&= \mathbb{E}^H_{\nu^N_\alpha}\left[\Big(\frac{\mathrm{d}\mathbb{P}^H_{\nu^N_\alpha}}{\mathrm{d}\mathbb{P}_{\nu^N_\alpha}}\Big)^{-1}\mathbf{1}\Big\{\{\pi^N \in \mathcal{O}\cap A^{\zeta,\gamma}_{k,l}\cap E^j_m\}\cap B^H_{\delta,\varepsilon}\Big\}\right]\,.\\
&= \mathbb{E}^H_{\nu^N_\alpha}\Bigg[\exp\Big\{N\Big[-J^{k,l,m,j}_{H,\gamma,\varepsilon,\zeta}(\pi^N) + O_{H,T,\varepsilon,\gamma}(\tfrac{1}{N}) + O(\delta) + O_H(\varepsilon) + O_H(\tfrac{\gamma}{\varepsilon})\Big]\Big\}\\
&\qquad \mathbf{1}\Big\{\{\pi^N \in \mathcal{O}\cap A^{\zeta,\gamma}_{k,l}\cap E^j_m\}\cap B^H_{\delta,\varepsilon}\Big\}\Bigg]\,.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\tfrac{1}{N}\log \mathbb{P}_{\nu^N_\alpha}[\{\pi^N \in \mathcal{O}\cap A^{\zeta,\gamma}_{k,l}\cap E^j_m\}\cap B^H_{\delta,\varepsilon}]\\
&\le \sup_{\pi\in\mathcal{O}}\{-J^{k,l,m,j}_{H,\gamma,\varepsilon,\zeta}(\pi)\} + O_{H,T,\varepsilon,\gamma}(\tfrac{1}{N}) + O(\delta) + O_H(\varepsilon) + O_H(\tfrac{\gamma}{\varepsilon})\,.
\end{aligned}
$$

For all $\gamma,\varepsilon,\zeta,\delta>0$, $k,l,m,j\in\mathbb{N}$ and $H\in C^{1,2}([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\,)$,

$$
\begin{aligned}
&\varlimsup_{N\to\infty} \tfrac{1}{N}\log \mathbb{Q}_{\nu^N_\alpha}[\mathcal{O}]\\
&\le \max\Big\{\sup_{\pi\in\mathcal{O}}\{-J^{k,l,m,j}_{H,\gamma,\varepsilon,\zeta}(\pi)\} + O(\delta) + O_H(\varepsilon) + O_H(\tfrac{\gamma}{\varepsilon}), R^l_k(\zeta,\gamma),\, R^j_m,\, R^\delta_H(\varepsilon)\Big\}\\
&= \max\Big\{\sup_{\pi\in\mathcal{O}}\{-J^{k,l,m,j}_{H,\gamma,\varepsilon,\zeta}(\pi)\} + O(\delta) + O_H(\varepsilon) + O_H(\tfrac{\gamma}{\varepsilon}), R^l_k(\zeta,\gamma),\, R^\delta_H(\varepsilon)\Big\}\,.
\end{aligned}
$$

Optimizing over $\gamma, \varepsilon, \zeta, \delta, k, l, m, j, H$, the right side of the above inequality is bounded by

$$
\begin{aligned}
&\inf_{\substack{\gamma,\varepsilon,\zeta,\delta,\\ k,l,m,j,H}} \max\Big\{ \sup_{\pi\in\mathcal{O}}\{-J^{k,l,m,j}_{H,\gamma,\varepsilon,\zeta}(\pi)\} + O(\delta) + O_H(\varepsilon) + O_H(\tfrac{\gamma}{\varepsilon}),\ R^l_k(\zeta,\gamma),\ R^\delta_H(\varepsilon)\Big\} \\
&= \inf_{\substack{\gamma,\varepsilon,\zeta,\delta,\\ k,l,m,j,H}} \sup_{\pi\in\mathcal{O}} \max\Big\{-J^{k,l,m,j}_{H,\gamma,\varepsilon,\zeta}(\pi) + O(\delta) + O_H(\varepsilon) + O_H(\tfrac{\gamma}{\varepsilon}), R^l_k(\zeta,\gamma), R^\delta_H(\varepsilon)\Big\}\,.
\end{aligned}
\tag{4.13}
$$

Proposition 4.2.2. *For fixed $\gamma, \varepsilon, \zeta, \delta, k, l, m, j, H$, the functional*

$$
\max\Big\{ - J^{k,l,m,j}_{H,\gamma,\varepsilon,\zeta}(\pi) + O(\delta) + O_H(\varepsilon) + O_H(\tfrac{\gamma}{\varepsilon}),\ R^l_k(\zeta,\gamma),\ R^\delta_H(\varepsilon)\Big\}.
$$

is upper semi-continuous in $\mathcal{D}([0,T],\mathcal{M})$.

Proof. The unique term that depends on π in the functional of the statement of this Proposition is $J^{k,l,m,j}_{H,\gamma,\varepsilon,\zeta}(\pi)$. By the definition of the functional and the Propositions A.1.2, 4.1.7, 4.1.9, we only need to prove the continuity of $\hat{J}((\pi * \iota_\gamma) * \iota^a_\varepsilon)$ in $\mathcal{D}([0,T],\mathcal{M})$.

Let π^n be a sequence in $\mathcal{D}([0,T],\mathcal{M})$ converging to some π. In particular, π^n_t converges to π_t in $\mathcal{M}$, for almost all $t \in [0,T]$. Recall that $\mathcal{M}$ is endowed with the weak topology. According to (4.10) and iterated aplications of Dominated Convergence Theorem yields the continuity of $\hat{J}((\pi * \iota_\gamma) * \iota^a_\varepsilon)$. □

Provided by the proposition above, we may apply the Minimax Lemma [16, Lemma A2.3.3], interchanging supremum with infimum in (4.13), and passing to compacts sets. Then, for all $\mathcal{K} \subset \mathcal{D}([0,T],\mathcal{M})$ compact,

$$
\begin{aligned}
&\overline{\lim_{N\to\infty}} \frac{1}{N} \log \mathbb{Q}_{\nu^N_\alpha}[\mathcal{K}] \\
&\leq \sup_{\pi\in\mathcal{K}} \inf_{\substack{\gamma,\varepsilon,\zeta,\delta,\\ k,l,m,j,H}} \max\Big\{ - J^{k,l,m,j}_{H,\gamma,\varepsilon,\zeta}(\pi) + O(\delta) + O_H(\varepsilon) + O_H(\tfrac{\gamma}{\varepsilon}), R^l_k(\zeta,\gamma),\ R^\delta_H(\varepsilon)\Big\}\,.
\end{aligned}
\tag{4.14}
$$

Proposition 4.2.3. *For all $\pi \in \mathcal{D}([0,T],\mathcal{M})$,*

$$
\overline{\lim_{\varepsilon\downarrow 0}}\ \overline{\lim_{l\to\infty}}\ \overline{\lim_{k\to\infty}}\ \overline{\lim_{\zeta\downarrow 0}}\ \overline{\lim_{\gamma\downarrow 0}}\ \overline{\lim_{j\to\infty}}\ \overline{\lim_{m\to\infty}}\ J^{k,l,m,j}_{H,\gamma,\varepsilon,\zeta}(\pi) \geq J_H(\pi)\,.
$$

Proof. Let $\pi \in \mathcal{D}([0,T],\mathcal{M})$. Be begin by taking limits in m and j:

$$
\overline{\lim_{j\to\infty}}\ \overline{\lim_{m\to\infty}}\ J^{k,l,m,j}_{H,\gamma,\varepsilon,\zeta}(\pi) = \begin{cases} \hat{J}_H((\pi * \iota_\gamma) * \iota^a_\varepsilon), & \text{if } \ \pi \in A^{\zeta,\gamma}_{k,l} \cap \mathcal{D}([0,T],\mathcal{M}_0)\,. \\ +\infty, & \text{otherwise,} \end{cases}
$$

The last equality follows from the fact: if π does not belong to $\mathcal{D}([0,T],\mathcal{M}_0)$, there exist some m and j such that $\pi \notin E^j_m$. To check this, one needs only to apply the definition of a measure be absolutely continuous with respect to Lebesgue, and the density of the functions $\{F_i\}_{i\geq 1}$, whose appear in (4.11). We step to the next limit. We claim that

$$
\overline{\lim_{\gamma\downarrow 0}} \begin{cases} \hat{J}_H((\pi * \iota_\gamma) * \iota^a_\varepsilon), & \text{if } \ \pi \in A^{\zeta,\gamma}_{k,l} \cap \mathcal{D}([0,T],\mathcal{M}_0) \\ +\infty, & \text{otherwise} \end{cases}
$$

$$\geq \begin{cases} \hat{J}_H(\pi * \iota_\varepsilon^a), & \text{if } \ \pi \in A_{k,l+1}^{\zeta} \cap \mathcal{D}([0,T],\mathcal{M}_0)\,, \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.15}$$

If $\pi \in A_{k,l}^{\zeta,\gamma} \cap \mathcal{D}([0,T],\mathcal{M}_0)$,

$$\max_{1\leq j\leq k} \mathcal{E}_{H_j}(\pi * \iota_\zeta^a) \leq l + \max_{1\leq j\leq k} \big\langle\!\big\langle \partial_u H_j,\ [\pi * \iota_\zeta^a - (\pi * \iota_\gamma) * \iota_\zeta^a] \big\rangle\!\big\rangle\,.$$

Since

$$(\pi_t * \iota_\zeta^a)(v) - ((\pi_t * \iota_\gamma) * \iota_\zeta^a)(v) = \int_{\mathbb{T}} \rho_t(z) \Big[\iota_\zeta^a(z,v) - \int_{\mathbb{T}} \iota_\gamma(-(z-u)) \iota_\zeta^a(u,v)\, du \Big]\, dz\,, \tag{4.16}$$

for ζ fixed, we can choose enough small γ such that π belongs to $A_{k,l+1}^{\zeta} \cap \mathcal{D}([0,T],\mathcal{M}_0)$. At this point, we must to analyse the semicontinuity of the terms which sum compounds $\hat{J}_H$, see definition 1.14. By the Proposition A.2.6

$$-\int_0^T \int_{\mathbb{T}} (\partial_u H)^2(t,u)\, \chi(\rho_t(u))\, du\, dt$$

is lower semicontinuous and by A.2.5, ℓ_H^{int} is continuous. It remains to check the terms associated to the point a, namely

$$\begin{aligned} &-\int_0^T \{\rho_t(a^+)\partial_u H_t(a^+) - \rho_t(a^-)\partial_u H_t(a^-)\}\, dt\,, \\ &-\int_0^T \rho_t(a^-)(1-\rho_t(a^+))(e^{H_t(a^+)-H_t(a^-)} - 1)\, dt \quad \text{and} \\ &-\int_0^T \rho_t(a^+)(1-\rho_t(a^-))(e^{-H_t(a^+)+H_t(a^-)} - 1)\, dt\,. \end{aligned}$$

From simple calculations, one can verify that, for fixed $\varepsilon > 0$, $(\pi * \iota_\gamma) * \iota_\varepsilon^a$ converges uniformly to $\pi * \iota_\varepsilon^a$ in a left (and right) neighborhood of a, see (4.16). Notice that, from the definition of ι_ε^a, the left and right limits around a of $\pi * \iota_\varepsilon^a$ are well defined.

The ensuing step is to take the limit in $\zeta \downarrow 0$. We claim that

$$\varlimsup_{\zeta\downarrow 0} \begin{cases} \hat{J}_H(\pi * \iota_\varepsilon^a), & \text{if } \ \pi \in A_{k,l+1}^{\zeta} \cap \mathcal{D}([0,T],\mathcal{M}_0)\,, \\ +\infty, & \text{otherwise.} \end{cases}$$

$$\geq \begin{cases} \hat{J}_H(\pi * \iota_\varepsilon^a), & \text{if } \ \pi \in A_{k,l+2} \cap \mathcal{D}([0,T],\mathcal{M}_0)\,, \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.17}$$

Indeed, if $\pi \in A_{k,l+1}^{\zeta} \cap \mathcal{D}([0,T],\mathcal{M}_0)$, then

$$\begin{aligned} \max_{1\leq j\leq k} \mathcal{E}_{H_j}(\pi) &= \max_{1\leq j\leq k} \mathcal{E}_{H_j}(\pi * \iota_\zeta^a) + \max_{1\leq j\leq k} \langle\!\langle \partial_u H_j, \pi - \pi * \iota_\zeta^a \rangle\!\rangle \\ &\leq l{+}1 \ + \max_{1\leq j\leq k} \int_0^T\!\!\int_{\mathbb{T}} \partial_u H_j(t,u)(\rho_t(u) - (\pi_t * \iota_\zeta^a)(u))\, du\, dt\,. \end{aligned}$$

It is possible choose the ζ such that the integral is less than or equal to 1, because the Lebesgue Differentiation Theorem.

Passing the limit in $k \to \infty$, we will have

$$\overline{\lim_{k\to\infty}} \begin{cases} \hat{J}_H(\pi * \iota_\varepsilon^a), & \text{if } \ \pi \in A_{k,l+2} \cap \mathcal{D}([0,T],\mathcal{M}_0)\,, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$= \begin{cases} \hat{J}_H(\pi * \iota_\varepsilon^a), & \text{if } \ \mathcal{E}(\pi) \le l+2 \text{ and } \pi \in \mathcal{D}([0,T],\mathcal{M}_0)\,, \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.18}$$

Now, we take limit in $l \to \infty$,

$$\overline{\lim_{l\to\infty}} \begin{cases} \hat{J}_H(\pi * \iota_\varepsilon^a), & \text{if } \ \mathcal{E}(\pi) \le l+2 \text{ and } \pi \in \mathcal{D}([0,T],\mathcal{M}_0)\,, \\ +\infty, & \text{otherwise.} \end{cases}$$

$$\ge \begin{cases} \hat{J}_H(\pi * \iota_\varepsilon^a), & \text{if } \ \mathcal{E}(\pi) < \infty\,, \\ +\infty, & \text{otherwise.} \end{cases}$$

Finally we can make $\varepsilon \downarrow 0$

$$\overline{\lim_{\varepsilon\downarrow 0}} \begin{cases} \hat{J}_H(\pi * \iota_\varepsilon^a), & \text{if } \ \mathcal{E}(\pi) < \infty\,, \\ +\infty, & \text{otherwise.} \end{cases} = J_H(\pi)\,.$$

The last equality is true, because for π in the set $\{\pi;\, \mathcal{E}(\pi) < \infty\}$ we have $\pi_t(du) = \rho_t(u)du$ and it is well defined the left and right limits around a of ρ_t, $\forall t \in [0,T]$.

□

Applying the proposition above in (4.14), we have that

$$\overline{\lim_{N\to\infty}} \tfrac{1}{N} \log \mathbb{Q}_{\nu_\alpha^N}[\mathcal{K}] \le \sup_{\pi\in\mathcal{K}} \inf_H \{-J_H(\pi)\} = -\inf_{\pi\in\mathcal{K}} \sup_H \ J_H(\pi) = -\inf_{\pi\in\mathcal{K}} I(\pi)\,.$$

This concludes the proof of the upper bound for compact sets.

4.3 Upper bound for closed sets

Proposition 4.3.1 (Upper bound for closed sets)**.** *For every $\mathcal{C}$ closed subset of $\mathcal{D}([0,T],\mathcal{M})$.*

$$\overline{\lim_{N\to\infty}} \tfrac{1}{N} \log \mathbb{Q}_{\nu_\alpha^N}[\mathcal{C}] \le -\inf_{\pi\in\mathcal{C}} I(\pi)\,.$$

As we shall see in the next proposition, the upper bound for closed sets is an immediate consequence of upper bound for compact sets and exponential tightness. By exponential tightness, we mean that for all $n \in \mathbb{N}$ there exist compact sets $K_n \subset \mathcal{D}([0,T],\mathcal{M})$ such that

$$\overline{\lim_{N\to\infty}} \tfrac{1}{N} \log Q_N[K_n^{\complement}] \le -n\,.$$

Proposition 4.3.2. *If the sequence of probability $\{Q_N\}_N$ is exponentially tight and holds the inequality*
$$\varlimsup_{N\to\infty} \tfrac{1}{N}\log Q_N[K] \le -\inf_{\pi\in K} I(\pi)\,,$$
for any compact set K, then $\{Q_N\}_N$ satisfies
$$\varlimsup_{N\to\infty} \tfrac{1}{N}\log Q_N[C] \le -\inf_{\pi\in C} I(\pi)\,,$$
for any closed set C.

Proof. Let C be a closed set. Since $Q_N[C] \le Q_N[C\cap K_n] + Q_N[K_n^{\complement}]$ and $C\cap K_n$ is compact,
$$\begin{aligned}
&\varlimsup_{N\to\infty} \tfrac{1}{N}\log Q_N[C] \le \max\Big\{\varlimsup_{N\to\infty} \tfrac{1}{N}\log Q_N[C\cap K_n],\ \varlimsup_{N\to\infty} \tfrac{1}{N}\log Q_N[K_n^{\complement}]\Big\}\\
&\le \max\Big\{-\inf_{\pi\in C\cap K_n} I(\pi),\ -n\Big\} \le \max\Big\{-\inf_{\pi\in C} I(\pi),\ -n\Big\}\,.
\end{aligned}$$
Since n is arbitrary, the inequality follows. $\square$

The rest of this section is concerned about exponential tightness, whose proof is essentially the same as found in [16]. For sake of completeness, we include here all the steps involved, emphasizing the slight differences. First of all, we claim that the exponential tightness is just a consequence of

Lemma 4.3.3. *For every $\varepsilon>0$ and every continuous function $H:\mathbb{T}\to\mathbb{R}$,*
$$\lim_{\delta\downarrow 0}\varlimsup_{N\to\infty} \tfrac{1}{N}\log \mathbb{Q}_{\nu_\alpha^N}\Bigg[\sup_{|t-s|\le\delta} |\langle \pi_t, H\rangle - \langle\pi_s, H\rangle| > \varepsilon\Bigg] = \infty\,.$$

Indeed, suppose the lemma above. Let $H_l\in C^2(\mathbb{T})$ be a dense set of functions in $C(\mathbb{T})$ for the uniform topology. For each $\delta>0$ and $\varepsilon>0$, denote by $C_{l,\delta,\varepsilon}$ the following set of paths:
$$C_{l,\delta,\varepsilon} = \Big\{\pi\in\mathcal{D}([0,T],\mathcal{M})\,;\ \sup_{|t-s|\le\delta} |\langle\pi_t, H_l\rangle - \langle\pi_s, H_l\rangle| \le \varepsilon\Big\}\,.$$
Therefore, from the Lemma 4.3.3,
$$\lim_{\delta\downarrow 0}\varlimsup_{N\to\infty} \tfrac{1}{N}\log \mathbb{Q}_{\nu_\alpha^N}\Big[\pi\notin C_{l,\delta,\varepsilon}\Big] = \infty\,,$$
for each $l\ge 1$ and $\varepsilon>0$. Thus, for each positive integers l,m and n, there exists $\delta=\delta(l,m,n)$ such that
$$\mathbb{Q}_{\nu_\alpha^N}\Big[\pi\notin C_{l,\delta,\frac{1}{m}}\Big] \le e^{-Nnml}$$
for all N large enough. Modifying δ, if necessary, we may extend this inequality for all positive integers N. Consider K_n^o defined by
$$K_n^o = \bigcap_{l\ge 1, m\ge 1} C_{l,\delta(l,m,n),\frac{1}{m}}\,.$$

From Arzelá-Ascoli, $K_n = K_n^o \cap \mathcal{D}([0,T],\mathcal{M}_0)$ is a compact set for each $n \geq 1$. On the other hand, since there is at most one particle per site, $\mathbb{Q}_{\nu_\alpha^N}[K_n] = \mathbb{Q}_{\nu_\alpha^N}[K_n^o]$. Furthermore, by construction,

$$\mathbb{Q}_{\nu_\alpha^N}\Big[\pi \notin K_n^o\Big] \leq \sum_{l\geq 1, m\geq 1} e^{-Nnml} \leq C\, e^{-Nn}\,,$$

where C is a constant not depending in the parameters. In particular,

$$\varlimsup_{N\to\infty} \tfrac{1}{N} \log \mathbb{Q}_{\nu_\alpha^N}\Big[\pi \notin K_n^o\Big] \;\leq\; -n\,,$$

which is the exponential tightness. Therefore, it just remains to prove Lemma 4.3.3.

Proof of Lemma 4.3.3. Fix $\varepsilon > 0$ and $H : \mathbb{T} \to \mathbb{R}$ continuous. Firstly, notice that

$$\begin{aligned}
&\Big\{\sup_{|t-s|\leq\delta} |\langle \pi_t, H\rangle - \langle \pi_s, H\rangle| > \varepsilon\Big\}\\
&\subset \bigcup_{k=0}^{\lfloor T\delta^{-1}\rfloor} \Big\{\sup_{k\delta\leq t<(k+1)\delta} |\langle \pi_t, H\rangle - \langle \pi_{k\delta}, H\rangle| > \tfrac{\varepsilon}{4}\Big\}\,.
\end{aligned}$$

We have here $\frac{\varepsilon}{4}$ instead of $\frac{\varepsilon}{3}$ due to the presence of jumps. Since we are concerned only about dynamical large deviations, the initial measure can be taken as the equilibrium measure. Using the useful fact

$$\varlimsup_{N\to\infty} \tfrac{1}{N} \log(a_N + b_N) = \max\Big\{\varlimsup_{N\to\infty} \tfrac{1}{N} \log(a_N),\, \varlimsup_{N\to\infty} \tfrac{1}{N} \log(b_N)\Big\}\,, \tag{4.19}$$

and the invariance of the measure, in order to prove the Lemma 4.3.3, it is enough to show that

$$\lim_{\delta\downarrow 0} \varlimsup_{N\to\infty} \tfrac{1}{N} \log \mathbb{Q}_{\nu_\alpha^N}\Big[\sup_{0\leq t\leq\delta} |\langle \pi_t, H\rangle - \langle \pi_0, H\rangle| > \varepsilon\Big] \;=\; \infty\,, \tag{4.20}$$

for every $\varepsilon > 0$ and every $H \in C^2(\mathbb{T})$. Recall that

$$\begin{aligned}
M_t^{c,H} \;=\; &\exp\Big\{N\Big[\langle \pi_t^N, cH\rangle - \langle \pi_0^N, cH\rangle\\
&-\tfrac{1}{N}\int_0^t e^{-N\langle \pi_s^N, cH\rangle}(\partial_s + N^2 L_N) e^{N\langle \pi_s^N, cH\rangle}\, ds\Big]\Big\}
\end{aligned}$$

is a positive martingale equal to 1 at time 0. The constant c above will be chosen *a posteriori* as enoughly large. Rewriting the expression above, using the fact that H not depends of time, we get

$$\begin{aligned}
M_t^{c,H} \;=\; &\exp\Big\{Nc\langle \pi_t^N, H\rangle - Nc\langle \pi_0^N, H\rangle\\
&-\int_0^t \sum_{|x-y|=1} N^2\, \xi_{x,y}^N\, [e^{c[H(\frac{y}{N}) - H(\frac{x}{N})]} - 1]\, \eta_s(x)(1-\eta_s(y))\, ds\Big\}\,.
\end{aligned}$$

Now, to obtain (4.20), are sufficient the limits

$$\lim_{\delta\downarrow 0}\varlimsup_{N\to\infty}\tfrac{1}{N}\log\mathbb{Q}_{\nu_\alpha^N}\Big[\sup_{0\le t\le\delta}\Big|\tfrac{1}{N}\log M_t^{c,H}\Big|>c\,\varepsilon\Big]\;=\;-\infty \tag{4.21}$$

and

$$\lim_{\delta\downarrow 0}\varlimsup_{N\to\infty}\tfrac{1}{N}\log\mathbb{Q}_{\nu_\alpha^N}\Big[\sup_{0\le t\le\delta}\Big|\int_0^t U_N(H,s,\eta_s)\,ds\Big|>c\varepsilon\Big]=-\infty\,, \tag{4.22}$$

where

$$U_N(H,s,\eta_s)=\sum_{|x-y|=1}N\xi^N_{x,y}\,[e^{c[H(\frac{y}{N})-H(\frac{x}{N})]}-1]\eta_s(x)(1-\eta_s(y))\,.$$

We claim now that the expression $|\int_0^t U_N(H,s,\eta_s)\,ds|$ is bounded by $C(c,H)t$. For sites x such that $a\notin[\frac{x-1}{N},\frac{x+1}{N}]$, one just needs to expand the exponential with the Taylor's formula and use $H\in C^2(\mathbb{T})$. The other sites are in number of two, and $H\in C^1(\mathbb{T})$ guarantees the limitation.

Provided by the previous boundedness, we conclude that for δ enoughly small the probability in (4.22) vanishes.

On the other hand, to prove (4.21), observe we can neglect the absolute value, since

$$\begin{aligned}&\mathbb{Q}_{\nu_\alpha^N}\Big[\sup_{0\le t\le\delta}\Big|\tfrac{1}{N}\log M_t^{c,H}\Big|>c\,\varepsilon\Big]\\ &\le\mathbb{Q}_{\nu_\alpha^N}\Big[\sup_{0\le t\le\delta}\tfrac{1}{N}\log M_t^{c,H}>c\,\varepsilon\Big]+\mathbb{Q}_{\nu_\alpha^N}\Big[\sup_{0\le t\le\delta}\tfrac{1}{N}\log M_t^{c,H}<-c\,\varepsilon\Big]\end{aligned}$$

and again (4.19). Because $M_t^{c,H}$ is a mean one positive martingale, we can apply Doob's Inequality, which yields

$$\mathbb{Q}_{\nu_\alpha^N}\Big[\sup_{0\le t\le\delta}\tfrac{1}{N}\log M_t^{c,H}>c\,\varepsilon\Big]\;=\;\mathbb{Q}_{\nu_\alpha^N}\Big[\sup_{0\le t\le\delta}M_t^{c,H}>e^{c\,\varepsilon N}\Big]\;\le\;\frac{1}{e^{c\varepsilon N}}\,.$$

Passing the log function and dividing by N, we get

$$\lim_{\delta\downarrow 0}\varlimsup_{N\to\infty}\tfrac{1}{N}\log\mathbb{Q}_{\nu_\alpha^N}\Big[\sup_{0\le t\le\delta}\tfrac{1}{N}\log M_t^{c,H}>c\varepsilon\Big]\le -c\,\varepsilon\,.$$

Since c is arbitrary large, it finishes the proof. □

Chapter 5

Hydrodynamic limit for the weakly asymmetric exclusion process with a slow bond

Recall that $\mathbb{P}_{\nu_\alpha^N}$ and $\mathbb{P}^H_{\nu_\alpha^N}$ are probabilities measures on the space $\mathcal{D}([0,T],\{0,1\}^{\mathbb{T}_N})$. The probability $\mathbb{P}_{\nu_\alpha^N}$ corresponds to the homogeneous Markov process η_t with generator L_N defined in (1.1) accelerated by N^2 and starting from ν_α^N. For $H \in C^{1,2}([0,T]\times\mathbb{T})$, the probability $\mathbb{P}^H_{\nu_\alpha^N}$ corresponds to the inhomogeneous Markov process η_t with generator L_N^H defined in (1.9) accelerated by N^2 and starting from the invariant measure ν_α^N.

We call $\mathbb{Q}^H_{\nu_\alpha^N}$ the probability measure on the space of trajectories $\mathcal{D}\big([0,T],\mathcal{M}\big)$ corresponding to the inhomogeneous Markov process π_t^N with generator L_N^H defined in (1.9) accelerated by N^2 and starting from ν_α^N.

Proposition 5.0.4. *Consider a bounded density profile $\rho_0 : \mathbb{T}\to\mathbb{R}$ and $H \in C^{1,2}([0,T]\times\mathbb{T})$. The sequence of probabilities $\{\mathbb{Q}^H_{\mu_N};\, N\geq 1\}$ converges in distribution to the probability measure concentrated on the absolutely continuous path $\pi_t(du) = \rho(t,u)du$ whose density $\rho(t,u)$ is the unique weak solution of the partial differential equation* (1.10).

It is straightforward to obtain Theorem 1.3.2 as a corollary of the previous proposition. The proof of this result is divided in two parts. In Section 5.1, we show that the sequence $\{\mathbb{Q}^H_{\mu_N};\, N\geq 1\}$ is tight, in Section 5.3 we characterize the limit points of this sequence. We prove the uniqueness of a weak solutions of the partial differential equation (1.10) in Section 5.4.

5.1 Tightness

Proposition 5.1.1. *For $H \in C^{1,2}([0,T]\times\mathbb{T})$ fixed, the sequence of measures $\{\mathbb{Q}^H_{\mu_N};\, N\geq 1\}$ is tight in the Skorohod topology of $\mathcal{D}\big([0,T],\mathcal{M}\big)$.*

Proof. In order to prove tightness of the sequence of measures $\{\mathbb{Q}^H_{\mu_N} : N \geq 1\}$ induced in the Skorohod space $\mathcal{D}\big([0,T],\mathcal{M}\big)$ by the random elements $\{\pi^N_t : 0 \leq t \leq T\}$. We will use same arguments presented in Section 2.2. We begin by considering the martingale

$$M^H_{N,t}(G) = \langle \pi^N_t, G_t\rangle - \langle \pi^N_0, G_0\rangle - \int_0^t \langle \pi^N_s, \partial_s G_s\rangle + N^2 L^H_{N,s}\langle \pi^N_s, G_s\rangle \, ds\,, \tag{5.1}$$

with $H \in C^{1,2}([0,T]\times\mathbb{T})$ and $G \in C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$. The generator $L^H_{N,s}$ was defined in (1.9). Firstly, we show that the $L^2(\mathbb{P}^H_{\mu_N})$-norm of this martingale vanishes as $N\to\infty$.

The quadratic variation of $M^H_{N,t}(G)$ is given by

$$\langle M^H_N(G)\rangle_t = \int_0^t N^2\Big[L^H_{N,s}\langle \pi^N_s, G_s\rangle^2 - 2\langle \pi^N_s, G_s\rangle L^H_{N,s}\langle \pi^N_s, G_s\rangle\Big] ds$$

Applying the definition of the generator $L^H_{N,s}$, the quadratic variation can be rewritten as

$$\begin{aligned}\langle M^H_N(G)\rangle_t = &\int_0^t \sum_{x\in\mathbb{T}_N} \xi^N_{x,x+1} e^{\nabla_N H_x}\eta_s(x)(1-\eta_s(x+1))(\nabla_N G_x)^2\, ds\\ &+\int_0^t \sum_{x\in\mathbb{T}_N} \xi^N_{x,x+1} e^{-\nabla_N H_x}\eta_s(x+1)(1-\eta_s(x))(\nabla_N G_x))^2\, ds\,,\end{aligned}$$

where $\nabla_N F_x$ denotes the diference $F_s(\frac{x+1}{N}) - F_s(\frac{x}{N})$. Using the definition of ξ^N and that $H \in C^{1,2}([0,T]\times\mathbb{T})$ and $G \in C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$, the expression above is bounded from above by $TC_{H,G}N^{-1}$.

To conclude the proof of tightness, we need to proof that the term inside the integral in (5.1) is bounded uniformly in $s\in[0,T]$. For this, it is enough to prove that $N^2 L^H_{N,s}\langle \pi^N_s, G_s\rangle$ is equal to

$$\begin{aligned}&\tfrac{1}{N}\sum_{\substack{x\neq a_N\\ x\neq a_N+1}} \eta_s(x) N^2[G_s(\tfrac{x+1}{N}) + G_s(\tfrac{x-1}{N}) - 2G_s(\tfrac{x}{N})]\\ &+\tfrac{1}{N}\sum_{x\neq a_N}\Big[\eta_s(x)(1-\eta_s(x+1)) + \eta_s(x+1)(1-\eta_s(x))\Big] N\nabla_N H_x\, N\nabla_N G_x\\ &+\eta_s(a_N+1)N\nabla_N G_{a_N+1} \,-\, \eta_s(a_N)N\nabla_N G_{a_N-1}\\ &+\eta_s(a_N)(1-\eta_s(a_N+1))\, e^{\nabla_N H_{a_N}}\, \nabla_N G_{a_N}\\ &-\eta_s(a_N+1)(1-\eta_s(a_N))\, e^{-\nabla_N H_{a_N}}\, \nabla_N G_{a_N} \,+\, O_{H,G}(\tfrac{1}{N})\,.\end{aligned}\tag{5.2}$$

Indeed, for obtain the last equality, we use the expression (1.9) of the generator $L^H_{N,s}$ and write $N^2 L^H_{N,s}\langle \pi^N_s, G_s\rangle$ as

$$\begin{aligned}&N\sum_{x\neq a_N}\Big[e^{\nabla_N H_x}\eta_s(x)(1-\eta_s(x+1)) - e^{-\nabla_N H_x}\eta_s(x+1)(1-\eta_s(x))\Big]\nabla_N G_x\\ &+\eta_s(a_N)(1-\eta_s(a_N+1))\, e^{\nabla_N H_{a_N}}\, \nabla_N G_{a_N}\\ &-\eta_s(a_N+1)(1-\eta_s(a_N))\, e^{-\nabla_N H_{a_N}}\, \nabla_N G_{a_N}\,.\end{aligned}$$

Using Taylor's expansion in the functions $e^{\nabla_N H_x}$ and $e^{-\nabla_N H_x}$, the first term above becomes

$$
\begin{aligned}
& N \sum_{x\neq a_N} (\eta_s(x) - \eta_s(x{+}1))\nabla_N G_x \;+\; O_{H,G}(\tfrac{1}{N}) \\
& + \tfrac{1}{N} \sum_{x\neq a_N} \Big[\eta_s(x)(1-\eta_s(x{+}1)) + \eta_s(x{+}1)(1-\eta_s(x))\Big] N\nabla_N H_x \, N\nabla_N G_x \, .
\end{aligned}
$$

Making change of variables in the first term above, we get

$$
\begin{aligned}
& \tfrac{1}{N} \sum_{\substack{x\neq a_N \\ x\neq a_N+1}} \eta_s(x) N^2 [G_s(\tfrac{x+1}{N}) + G_s(\tfrac{x-1}{N}) - 2G_s(\tfrac{x}{N})] \\
& \;+\; \eta_s(a_N{+}1) N\nabla_N G_{a_N+1} - \eta_s(a_N) N\nabla_N G_{a_N-1} \, .
\end{aligned}
$$

This establishes the formula (5.2). □

5.2 Sobolev space

In this section, we prove that any limit point $\mathbb{Q}^H_*$ of the sequence $\mathbb{Q}^H_{\mu_N}$ is concentrated on trajectories $\rho_t(u)du$ such that $\rho_t(u)$ belongs to the sobolev space $L^2\big(0,T;\mathcal{H}^1(\mathbb{T}\backslash\{a\})\big)$ defined in 2.4.2.

Proposition 5.2.1. *The measure $\mathbb{Q}^H_*$ is concentrated on paths $\rho_t(u)du$ such that ρ belongs to $L^2\big(0,T;\mathcal{H}^1(\mathbb{T}\backslash\{a\})\big)$.*

This proof follows the same steps in the proof of Proposition 2.4.1. The main difference here is we are going to use the estimate about the Radon-Nikodym $\mathbf{d}\mathbb{P}^H_{\nu^N_\alpha}/\mathbf{d}\mathbb{P}_{\nu^N_\alpha}$ derivative presented in (4.3) and then profit the results already reached for the probability $\mathbb{P}_{\nu^N_\alpha}$ (symmetric case). After that, we proceed in the same way as in Proposition 2.4.1.

The Radon-Nikodym derivative of the probabilities $\mathbb{P}_{\nu^N_\alpha}$ and $\mathbb{P}^H_{\nu^N_\alpha}$ has the expression (4.3). Using the rule of maximum of one particle per site, we get

$$
\left\| \frac{\mathbf{d}\mathbb{P}^H_{\nu^N_\alpha}}{\mathbf{d}\mathbb{P}_{\nu^N_\alpha}} \right\|_\infty \leq e^{C(H,T)\,N} \, . \tag{5.3}
$$

Lemma 5.2.2 (Replacement Lemma)**.** *Given a bounded function $G:\mathbb{T}\to\mathbb{R}$, then*

$$
\varlimsup_{\varepsilon\to 0} \varlimsup_{N\to\infty} \mathbb{E}^H_{\mu_N}\Big[\Big|\int_0^t \tfrac{1}{N}\sum_{x\in\mathbb{T}_N} G(\tfrac{x}{N})\{\eta_s(x) - \eta^{\varepsilon N}_s(x)\}\, ds\Big|\Big] \;=\; 0\,,
$$

$$
\varlimsup_{\varepsilon\to 0} \varlimsup_{N\to\infty} \mathbb{E}^H_{\mu_N}\Big[\Big|\int_0^t \tfrac{1}{N}\sum_{x\neq a_N} G(\tfrac{x}{N})\Big\{\tau_x g_i(\eta) - \tilde{g}_i(\eta^{\varepsilon N}(x),\eta^{\varepsilon N}(x{+}1))\Big\}\, ds\Big|\Big] \;=\; 0\,, \quad \forall i=1,2
$$

and

$$\overline{\lim_{\varepsilon\to 0}}\,\overline{\lim_{N\to\infty}}\,\mathbb{E}^H_{\mu_N}\Big[\Big|\int_0^t G(\tfrac{a_N}{N})\Big\{\tau_{a_N}g_i(\eta)-\tilde{g}_i(\eta^{\varepsilon N}(a_N),\eta^{\varepsilon N}(a_N{+}1))\Big\}\,ds\Big|\Big] \;=\; 0\,, \quad \forall i=1,2\,,$$

where g_i and $\tilde{g}_i$, $i=1,2$ were defined in (2.10) *and* (2.11).

Proof. The proof follows by the same arguments of the Lemma 2.3.2. But, it has one difference: we need use the Radon-Nikodym derivative.

We only prove the first limit, as in Lemma 2.3.2. From definition of the entropy and Jensen's Inequality, the expectation is bounded from above by

$$\frac{H_N(\mu_N|\nu^N_\alpha)}{\gamma N} \;+\; \frac{1}{\gamma N}\log\mathbb{E}^H_{\nu^N_\alpha}\left[\exp\left\{\gamma\Big|\int_0^t\sum_{x\in\mathbb{T}_N}G(\tfrac{x}{N})\Big\{\eta_s(x)-\eta^{\varepsilon N}_s(x)\Big\}ds\Big|\right\}\right],$$

for all $\gamma>0$. In view of (2.8), to prove the lemma, it is enough to show that the second term vanishes as $N\to\infty$ and then $\varepsilon\downarrow 0$ for every $\gamma>0$. Here is the difference: we use (5.3) in the second term above, getting the boundedness of it by

$$\frac{C(H,T)}{\gamma}+\frac{1}{\gamma N}\log\mathbb{E}_{\nu^N_\alpha}\left[\exp\left\{\gamma\Big|\int_0^t\sum_{x\in\mathbb{T}_N}G(\tfrac{x}{N})\Big\{\eta_s(x)-\eta^{\varepsilon N}_s(x)\Big\}ds\Big|\right\}\right].$$

The same way that in Proposition 2.3.2 gives the result.

□

Recall the definition of U_N in (2.19),

$$\begin{aligned}U_N(\varepsilon,H,\eta) =&\tfrac{1}{\varepsilon N}\sum_{x\in\mathbb{T}_N}H(\tfrac{x}{N})\,\Big\{\eta(x-\varepsilon N)-\eta(x)\Big\}\\ &-\tfrac{2}{N}\sum_{x\in\mathbb{T}_N}(H(\tfrac{x}{N}))^2\{1+\tfrac{1}{\varepsilon}\mathbf{1}_{(a-\varepsilon,a]}(\tfrac{x}{N})\}\,.\end{aligned}$$

Lemma 5.2.3. *For every $k\geq 1$, consider the functions $G_1,\dots,G_k$ defined on $[0,T]\times(\mathbb{T}\backslash\{a\})$ taking values in $\mathbb{R}$. Then, for every $\varepsilon>0$,*

$$\overline{\lim_{\delta\downarrow 0}}\,\overline{\lim_{N\to\infty}}\,\mathbb{E}^H_{\mu^N}\Big[\max_{1\leq i\leq k}\Big\{\int_0^T U_N(\varepsilon,G_i(s,\cdot),\eta^{\delta N}_s)\,ds\Big\}\Big]\;\leq\; K_0+C(H,T)\,.$$

Proof. It follows from the Replacement Lemma that in order to prove the lemma we just need to show that

$$\overline{\lim_{N\to\infty}}\,\mathbb{E}^H_{\mu^N}\Big[\max_{1\leq i\leq k}\Big\{\int_0^T U_N(\varepsilon,G_i(s,\cdot),\eta_s)\,ds\Big\}\Big]\;\leq\; K_0+C(H,T)\,.$$

By the entropy and Jensen's Inequalities, for each fixed N, the previous expectation is bounded from above by

$$\frac{H(\mu^N|\nu_\alpha^N)}{N} + \tfrac{1}{N}\log \mathbb{E}^H_{\nu_\alpha^N}\Big[\exp\Big\{\max_{1\le i\le k}\Big\{N\int_0^T U_N(\varepsilon, G_i(s,\cdot),\eta_s)\,ds\Big\}\Big\}\Big].$$

By (2.8), the first term is bounded by K_0. Using the Radon-Nikodym derivative and (5.3), the second term is bounded from above by

$$C(H,T) + \tfrac{1}{N}\log \mathbb{E}_{\nu_\alpha^N}\Big[\exp\Big\{\max_{1\le i\le k}\Big\{N\int_0^T U_N(\varepsilon, G_i(s,\cdot),\eta_s)\,ds\Big\}\Big\}\Big].$$

Now, this proof follows for the same arguments of the Lemma 2.4.3.

□

Lemma 5.2.4.

$$\mathbb{E}_{\mathbb{Q}^H_*}\Big[\sup_G\Big\{\int_0^T\int_{\mathbb{T}}\partial_u G(s,u)\,\rho_s(u)\,du\,ds\; - \;2\int_0^T\int_{\mathbb{T}} G(s,u)^2\,du\,ds\Big\}\Big] \le K_0 + C(H,T),$$

where the supremum is carried over all functions G in $C^{0,1}([0,T]\times\mathbb{T})$ with compact support in $[0,T]\times(\mathbb{T}\backslash\{a\})$.

The proof of this lemma is the same of 2.4.4, replacing Lemma 2.4.3 by Lemma 5.2.3, and will be omitted.

Proof of Proposition 5.2.1. Analogously as in the Proposition 2.4.1, denote by $\ell^{int}: C^{0,1}([0,T]\times\mathbb{T})\to\mathbb{R}$ the linear functional defined by

$$\ell^{int}(G) = \int_0^T\int_{\mathbb{T}}\partial_u G_s(u)\,\rho_s(u)\,du\,ds.$$

Using the Lemma 5.2.4 and Proposition A.1.1 and proceeding as in proof of 2.4.1, we obtain that ℓ^{int} is $\mathbb{Q}^H_*$-almost surely bounded functional in $L^2([0,T]\times\mathbb{T})$. To conclude we apply the Riesz representation theorem. □

5.3 Characterization of limit points

This section is devoted to prove that all limit points of the sequence $\{\mathbb{Q}^H_{\mu_N}: N\ge 1\}$ are concentrated on trajectories of measures absolutely continuous with respect to the Lebesgue measure: $\pi(t,du) = \rho_t(u)du$, whose density $\rho_t(u)$ is a weak solution of the hydrodynamic equation (1.10).

Let $\mathbb{Q}^H_*$ be a limit point of the sequence $\{\mathbb{Q}^H_{\mu_N}: N\ge 1\}$ and assume, without lost of generality, that $\{\mathbb{Q}^H_{\mu_N}: N\ge 1\}$ converges to $\mathbb{Q}^H_*$. The existence of $\mathbb{Q}^H_*$ is guaranteed by Proposition 5.1.1.

Since there is at most one particle per site, it is easy to show that Q_*^H is concentrated on trajectories $\pi_t(du)$ which are absolutely continuous with respect to the Lebesgue measure, $\pi_t(du) = \rho_t(u)du$ and whose density $\rho_t(\cdot)$ is non-negative and bounded by 1 (for more details see [16]).

In Proposition 5.2.1, we proved that $\rho(t,\cdot)$ belongs to $L^2\big(0,T;\mathcal{H}^1(\mathbb{T}\backslash\{a\})\big)$. It is well known that the Sobolev space $\mathcal{H}^1(\mathbb{T}\backslash\{a\})$ has special properties: all its elements are absolutely continuous functions with bounded variation, c.f. [4], therefore with lateral limits well-defined. Such property is inherited by $L^2\big(0,T;\mathcal{H}^1(\mathbb{T}\backslash\{a\})\big)$ in the sense that we can integrate in time the lateral limits.

Let $G \in C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$. We begin by claiming that

$$\begin{aligned}\mathbb{Q}_*^H\Bigg[\pi_\cdot :\ &\langle \rho_t, G_t\rangle \,-\, \langle \rho_0, G_0\rangle \,-\, \int_0^t \langle \rho_s, (\partial_s+\Delta)G_s\rangle\, ds\\ &-\,2\int_0^t \langle \chi(\rho_s),\, \partial_u H_s \partial_u G_s\rangle\, ds\\ &-\int_0^t \big\{\rho_s(a^+)\partial_u G_s(a^+) - \rho_s(a^-)\partial_u G_s(a^-)\big\}\, ds\\ &+\int_0^t \big\{\rho_s(a^+)-\rho_s(a^-)\big\}\big\{G_s(a^+)-G_s(a^-)\big\}\, ds \,=\, 0, \quad \forall t\in[0,T]\Bigg] \,=\, 1\,.\end{aligned} \tag{5.4}$$

In order to prove the equality above, its enough to show that, for every $\delta > 0$

$$\begin{aligned}\mathbb{Q}_*^H\Bigg[\pi_\cdot :\ \sup_{0\le t\le T}\Bigg|&\langle \rho_t, G_t\rangle \,-\, \langle \rho_0, G_0\rangle \,-\, \int_0^t \langle \rho_s, (\partial_s+\Delta)G_s\rangle\, ds\\ &-\,2\int_0^t \langle \chi(\rho_s),\, \partial_u H_s \partial_u G_s\rangle\, ds\\ &-\int_0^t \big\{\rho_s(a^+)\partial_u G_s(a^+) - \rho_s(a^-)\partial_u G_s(a^-)\big\}\, ds\\ &+\int_0^t \big\{\rho_s(a^+)-\rho_s(a^-)\big\}\big\{G_s(a^+)-G_s(a^-)\big\}\, ds\Bigg| \,>\, \delta\Bigg] = 0\,.\end{aligned}$$

Since the boundary integrals and the integral with χ are not well-defined in the whole Skorohod space $\mathcal{D}\big([0,T],\mathcal{M}_0\big)$, we cannot use directly Portmanteau's Theorem. To avoid this technical obstacle, fix $\varepsilon > 0$, which will be taken small later. Adding and subtracting the convolution of $\rho_t(u)$ with $\iota_\varepsilon := \iota_\varepsilon^a$, it is defined in (2.9). Then probability above is less than or equal to the sum of

$$\begin{aligned}\mathbb{Q}_*^H\Bigg[\pi_\cdot :\ \sup_{0\le t\le T}\Bigg|&\langle \rho_t, G_t\rangle \,-\, \langle \rho_0, G_0\rangle \,-\, \int_0^t \langle \rho_s, (\partial_s+\Delta)G_s\rangle\, ds\\ &-\,2\int_0^t \langle \chi(\rho_s*\iota_\varepsilon),\, \partial_u H_s \partial_u G_s\rangle\, ds\end{aligned}$$

$$
\begin{aligned}
&- \int_0^t \big\{(\rho_s * \iota_\varepsilon)(a^+)\partial_u G_s(a^+) - (\rho_s * \iota_\varepsilon)(a^-)\partial_u G_s(a^-)\big\}\, ds \\
&+ \int_0^t \big\{(\rho_s * \iota_\varepsilon)(a^+) - (\rho_s * \iota_\varepsilon)(a^-)\big\}\big\{G_s(a^+) - G_s(a^-)\big\}\, ds \,\Bigg| > \delta/4 \Bigg],
\end{aligned} \tag{5.5}
$$

$$
\mathbb{Q}_*^H \Bigg[\pi_\cdot : \sup_{0\le t\le T} \Bigg| 2\int_0^t \langle \chi(\rho_s * \iota_\varepsilon),\, \partial_u H_s \partial_u G_s\rangle\, ds - 2\int_0^t \langle \chi(\rho_s),\, \partial_u H_s \partial_u G_s\rangle\, ds \Bigg| > \delta/4 \Bigg],
$$

$$
\begin{aligned}
\mathbb{Q}_*^H \Bigg[\pi_\cdot : \sup_{0\le t\le T} \Bigg| &\int_0^t \big\{(\rho_s * \iota_\varepsilon)(a^+)\partial_u G_s(a^+) - (\rho_s * \iota_\varepsilon)(a^-)\partial_u G_s(a^-)\big\}\, ds \\
&- \int_0^t \big\{\rho_s(a^+)\partial_u G_s(a^+) - \rho_s(a^-)\partial_u G_s(a^-)\big\}\, ds \Bigg| > \delta/4 \Bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{Q}_*^H \Bigg[\pi_\cdot : \sup_{0\le t\le T} \Bigg| &\int_0^t \big\{(\rho_s * \iota_\varepsilon)(a^+) - (\rho_s * \iota_\varepsilon)(a^-)\big\}\big\{G_s(a^+) - G_s(a^-)\big\}\, ds \\
&- \int_0^t \big\{\rho_s(a^+) - \rho_s(a^-)\big\}\big\{G_s(a^+) - G_s(a^-)\big\}\, ds \Bigg| > \delta/4 \Bigg].
\end{aligned}
$$

The set inside the three previous probabilities decreases to a set of null probability, as $\varepsilon \downarrow 0$. It remains to deal with (5.5). By Portmanteau's Theorem, Proposition A.2.7 and the exclusion rule, (5.5) is bounded from above by

$$
\begin{aligned}
\varliminf_{N\to\infty} \mathbb{Q}_{\mu_N}^H \Bigg[\pi_\cdot : \sup_{0\le t\le T} \Bigg| &\langle \pi_t, G_t\rangle - \langle \pi_0, G_0\rangle - \int_0^t \langle \pi_s, (\partial_s + \Delta) G_s\rangle\, ds \\
&- 2\int_0^t \langle \chi(\pi_s * \iota_\varepsilon),\, \partial_u H_s \partial_u G_s\rangle\, ds \\
&- \int_0^t \big\{(\pi_s * \iota_\varepsilon)(a^+)\partial_u G_s(a^+) - (\pi_s * \iota_\varepsilon)(a^-)\partial_u G_s(a^-)\big\}\, ds \\
&+ \int_0^t \big\{(\rho_s * \iota_\varepsilon)(a^+) - (\rho_s * \iota_\varepsilon)(a^-)\big\}\big\{G_s(a^+) - G_s(a^-)\big\}\, ds \Bigg| > \delta/4 \Bigg].
\end{aligned}
$$

In the step above, we need to be carefully. Because the functions inside the probability above may not continuous. For mores details, we recommend to see the Section 2.5 or Subsections 7.5.2 and 8.3.2.

If we consider the discrete torus as embedded in the continuous torus, $a_N = -1$ is the closest site to the left of 0 and $a_N + 1 = 0$ is the closest site to the right of $a = 0$. Since $(\pi_s^N * \iota_\varepsilon)(\frac{x}{N}) = \eta_s^{\varepsilon N}(x)$, for all $x \in \mathbb{T}_N$. Using the definition of $\mathbb{Q}_{\mu_N}^H$, we can rewrite the previous expression as

$$
\begin{aligned}
\varlimsup_{N\to\infty} \; \mathbb{P}^H_{\mu_N}\Bigg[\sup_{0\le t\le T}\Bigg| &\langle \pi^N_t, G_t\rangle \;-\; \langle \pi^N_0, G_0\rangle \;-\; \int_0^t \langle \pi^N_s, (\partial_s+\Delta)G_s\rangle\, ds \\
&- 2\int_0^t \langle \chi(\pi^N_s * \iota_\varepsilon),\, \partial_u H_s \partial_u G_s\rangle\, ds \\
&- \int_0^t \Big\{\eta^{\varepsilon N}_s(a_N+1)\partial_u G_s(a^+) - \eta^{\varepsilon N}_s(a_N)\partial_u G_s(a^-)\Big\}\, ds \\
&+ \int_0^t \big\{\eta^{\varepsilon N}_s(a_N+1) - \eta^{\varepsilon N}_s(a_N)\big\}\big\{G_s(a^+) - G_s(a^-)\big\}\, ds\Bigg| \;>\; \delta/4\Bigg]\,.
\end{aligned}
$$

The next step is to add and subtract $N^2 L^H_{N,s}\langle \pi^N_s,\, G_s\rangle$ and the previous probability becomes now bounded from above by the sum of

$$
\varlimsup_{N\to\infty} \; \mathbb{P}^H_{\mu_N}\Bigg[\sup_{0\le t\le T}\Bigg|\langle \pi^N_t, G_t\rangle \;-\; \langle \pi^N_0, G_0\rangle \;-\; \int_0^t \langle \pi^N_s, \partial_s G_s\rangle + N^2 L^H_{N,s}\langle \pi^N_s,\, G_s\rangle\, ds\Bigg| \;>\; \delta/8\Bigg]
$$

and

$$
\begin{aligned}
\varlimsup_{N\to\infty} \; \mathbb{P}^H_{\mu_N}\Bigg[\sup_{0\le t\le T}\Bigg| &\int_0^t N^2 L^H_{N,s}\langle \pi^N_s,\, G_s\rangle\, ds - \int_0^t \langle \pi^N_s, \Delta G_s\rangle\, ds \\
&- 2\int_0^t \langle \chi(\pi^N_s * \iota_\varepsilon),\, \partial_u H_s \partial_u G_s\rangle\, ds \\
&- \int_0^t \Big\{\eta^{\varepsilon N}_s(a_N+1)\partial_u G_s(a^+) - \eta^{\varepsilon N}_s(a_N)\partial_u G_s(a^-)\Big\}\, ds \\
&+ \int_0^t \big\{\eta^{\varepsilon N}_s(a_N+1) - \eta^{\varepsilon N}_s(a_N)\big\}\big\{G_s(a^+) - G_s(a^-)\big\}\, ds\Bigg| \;>\; \delta/8\Bigg]\,.
\end{aligned} \tag{5.6}
$$

The expression inside the first probability is the martigale $M^H_{N,t}(G)$ defined in (5.1). Using the fact that the martingale $M^H_{N,t}(G)$ converges to zero in $L^2(\mathbb{P}^H_{\mu_N})$, which is proved in Proposition 5.1.1, and Doob's inequality, the first probability above is equal to

$$
\lim_{N\to\infty} \mathbb{P}^H_{\mu_N}\Big[\sup_{0\le t\le T}\Big|M^H_{N,t}(G)\Big| > \delta/8\Big] \;=\; 0\,,
$$

for every $\delta>0$,

We will show that the probability (5.6) is null. By expression (5.2) for $N^2 L^H_{N,s}\langle \pi^N_s,\, G_s\rangle$ the expression (5.6) is less than or equal to

$$
\begin{aligned}
\varlimsup_{N\to\infty} \; \mathbb{P}^H_{\mu_N}\Bigg[\sup_{0\le t\le T}\Bigg| &\int_0^t \langle \pi^N_s, \Delta G_s\rangle\, ds \\
- \int_0^t \tfrac{1}{N} \sum_{\substack{x\neq a_N\\ x\neq a_N+1}} &\eta_s(x) N^2 [G_s(\tfrac{x+1}{N}) + G_s(\tfrac{x-1}{N}) - 2G_s(\tfrac{x}{N})]\, ds\Bigg| \;>\; \delta/32\Bigg]\,,
\end{aligned} \tag{5.7}
$$

$$
\begin{aligned}
&\varlimsup_{N\to\infty}\ \mathbb{P}^H_{\mu_N}\Bigg[\sup_{0\le t\le T}\Bigg|2\int_0^t\langle\chi(\pi^N_s*\iota_\varepsilon),\,\partial_uH_s\partial_uG_s\rangle\,ds\\
&-\int_0^t\tfrac{1}{N}\sum_{x\neq a_N}\Big[\eta_s(x)(1-\eta_s(x{+}1))+\eta_s(x{+}1)(1-\eta_s(x))\Big]N\nabla_NH_x\,N\nabla_NG_x\,ds\Bigg|\;>\;\delta/32\Bigg],
\end{aligned}
\tag{5.8}
$$

$$
\begin{aligned}
&\varlimsup_{N\to\infty}\ \mathbb{P}^H_{\mu_N}\Bigg[\sup_{0\le t\le T}\Bigg|\int_0^t\Big\{\eta^{\varepsilon N}_s(a_N+1)\partial_uG_s(a^+)-\eta^{\varepsilon N}_s(a_N)\partial_uG_s(a^-)\Big\}\,ds\\
&-\int_0^t\Big\{\eta_s(a_N{+}1)N\nabla_NG_{a_N+1}\;-\;\eta_s(a_N)N\nabla_NG_{a_N-1}\Big\}\,ds\Bigg|\;>\;\delta/32\Bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
&\varlimsup_{N\to\infty}\ \mathbb{P}^H_{\mu_N}\Bigg[\sup_{0\le t\le T}\Bigg|\int_0^t\big\{\eta^{\varepsilon N}_s(a_N+1)-\eta^{\varepsilon N}_s(a_N)\big\}\big\{G_s(a^+)-G_s(a^-)\big\}\,ds\\
&+\int_0^t\Big\{\eta_s(a_N)(1-\eta_s(a_N{+}1))e^{\nabla_NH_{a_N}}\\
&-\eta_s(a_N{+}1)(1-\eta_s(a_N))e^{-\nabla_NH_{a_N}}\Big\}\nabla_NG_{a_N}\,ds\Bigg|\;>\;\delta/32\Bigg].
\end{aligned}
$$

Since $G\in C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$, the discrete laplacian of G converges uniformly to the continuous laplacian of G, and therefore the expression (5.7) is null. To prove that the others probabilities are null, we observe that $N\nabla_NH_x$ and $N\nabla_NG_x$ converge uniformly to ∂_uH_s and ∂_uG_s, as $N\to\infty$, respectively. Since $H\in C^{1,2}([0,T]\times\mathbb{T})$ and $G\in C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$, $\nabla_NH_{a_N}$ and $\nabla_NG_{a_N}$ converge uniformly to 0 and $G_s(a^+)-G_s(a^-)$, as $N\to\infty$, respectively. By the rule of maximum of one particle per site and approximation of integral by Riemann sums, in order to be null (5.7) and (5.8), it is suficient to show that go to zero the expressions

$$
\begin{aligned}
&\varlimsup_{N\to\infty}\ \mathbb{P}^H_{\mu_N}\Bigg[\sup_{0\le t\le T}\Bigg|\int_0^t\tfrac{1}{N}\sum_{x\neq a_N}\Big[\eta^{\varepsilon N}_s(x)(1-\eta^{\varepsilon N}_s(x{+}1))\\
&\qquad-\eta_s(x)(1-\eta_s(x{+}1))\Big]\,\partial_uH_s(\tfrac{x}{N})\partial_uG_s(\tfrac{x}{N})\,ds\Bigg|\;>\;\delta\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
&\varlimsup_{N\to\infty}\ \mathbb{P}^H_{\mu_N}\Bigg[\sup_{0\le t\le T}\Bigg|\int_0^t\tfrac{1}{N}\sum_{x\neq a_N}\Big[\eta^{\varepsilon N}_s(x{+}1)(1-\eta^{\varepsilon N}_s(x))\\
&\qquad-\eta_s(x{+}1)(1-\eta_s(x))\Big]\,\partial_uH_s(\tfrac{x}{N})\partial_uG_s(\tfrac{x}{N})\,ds\Bigg|\;>\;\delta\Bigg],
\end{aligned}
$$

$$\overline{\lim_{N\to\infty}}\ \mathbb{P}^{H}_{\mu_N}\Bigg[\sup_{0\le t\le T}\Bigg|\int_0^t \big\{\eta_s^{\varepsilon N}(a_N+1)-\eta_s(a_N{+}1)\big\}\partial_u G_s(a^+)$$

$$-\big\{\eta_s^{\varepsilon N}(a_N)-\eta_s(a_N)\big\}\partial_u G_s(a^-)\,ds\Bigg| > \delta\Bigg],$$

and

$$\overline{\lim_{N\to\infty}}\ \mathbb{P}^{H}_{\mu_N}\Bigg[\sup_{0\le t\le T}\Bigg|\int_0^t \big\{\eta_s^{\varepsilon N}(a_N+1)-\eta_s(a_N+1)\big\}\big\{G_s(a^+)-G_s(a^-)\big\}$$

$$-\big\{\eta_s^{\varepsilon N}(a_N)+\eta_s(a_N)\big\}\big\{G_s(a^+)-G_s(a^-)\big\}\,ds\Bigg| > \delta\Bigg],$$

converge to zero, as $\varepsilon\downarrow 0$, $\forall\delta>0$. It follows by Replacement Lemma 5.2.2.

Proposition 5.3.1. *Fix a Borel measurable profile $\rho_0:\mathbb{T}\to[0,1]$ and consider a sequence $\{\mu_N : N\ge 1\}$ of probability measures on $\{0,1\}^{\mathbb{T}_N}$ associated to ρ_0 in the sense of* (1.4)*. Then any limit point of $\mathbb{Q}^H_{\mu_N}$ is concentrated on absolutely continuous paths $\pi_t(du)=\rho(t,u)du$, with positive density ρ_t bounded by 1, such that ρ is a weak solutions of* (1.10) *with initial condition ρ_0.*

Proof. Let $\{G_i : i\ge 1\}$ be a countable dense set of functions on $C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$, with respect to the norm $\|G\|_\infty+\|\partial_u G\|_\infty+\|\partial_u^2 G\|_\infty$. Provided by (5.4) and intercepting a countable number of sets of probability one, is straightforward to extend (5.4) for all functions $G\in C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$ simultaneously. □

5.4 Uniqueness of weak solutions

This section is devoted to the uniqueness of weak solutions of (1.10). To simplify notation, along this section, we will consider $a=0$ and sometimes we will denote $0^+=0$ and $0^-=1$.

Proposition 5.4.1. *Let $\rho:[0,T]\times\mathbb{T}\to\mathbb{R}$ be a weak solution of the parabolic differential equation* (1.10) *with initial condition $\gamma:\mathbb{T}\to\mathbb{R}$. Then, for all $t\in[0,T]$ and for all $G\in\mathcal{H}^2_{bc}(\mathbb{T})$, holds*

$$\langle\rho_t,G\rangle\,-\,\langle\gamma,G\rangle=\int_0^t\big\langle\rho_s,\Delta G\big\rangle\,ds+2\int_0^t\big\langle\chi(\rho_s)\partial_u H_s\,,\partial_u G\big\rangle\,ds\,, \tag{5.9}$$

for all $t\in[0,T]$.

Proof. This proof is like to proof 2.6.1, we will denote $0^+=0$ and $0^-=1$. Let $G\in\mathcal{H}^2_{\rm bc}(\mathbb{T})$. Consider $g_n\in C(\mathbb{T})$ such that $\int g_n(x)\,dx=0$ and g_n converges to ΔG and β_n converging to $\partial_u G(0)$, and define

$$G_n(x)=G(0)+\beta_n x+\int_0^x\int_0^y g_n(z)\,dz\,dy\,.$$

Notice that $G_n \in C^{1,2}\big([0,T] \times \overline{\mathbb{T}\backslash\{0\}}\big)$, $\partial_u G_n(0) = \partial_u G_n(1)$ and $\Delta G_n = g_n$. Then,

$$
\begin{aligned}
\langle \rho_t, G_n\rangle \,-\, \langle \gamma, G_n\rangle \,=\, & \int_0^t \langle \rho_s, g_n\rangle\, ds + 2\int_0^t \big\langle \chi(\rho_s)\partial_u H_s\,, \partial_u G_n\big\rangle\, ds \\
+ & \int_0^t \{\rho_s(0) - \rho_s(1)\}\partial_u G_n(0)\, ds \\
- & \int_0^t \big\{\rho_s(0) - \rho_s(1)\big\}\big\{G_n(0) - G_n(1)\big\}\, ds\,.
\end{aligned}
\tag{5.10}
$$

Since G_n converges to G, g_n converges to ΔG and $\partial_u G_n = \beta_n + \int_0^x g_n(z)\, dz$ converges to $\partial_u G(0) + \int_0^x \Delta G(z)\, dz = \partial_u G$, one can conclude this proof like as in Proposition 2.6.1. □

Recall the definition of the inverse operator $(-\Delta)^{-1} : L^2(\mathbb{T}) \to \mathcal{H}^2_{\text{bc}}(\mathbb{T})$ in (2.27).

Proposition 5.4.2. *Let ρ and λ be two weak solutions of the asymmetric equation with pertubation $H \in C^{1,2}([0,T] \times \mathbb{T})$ given in Definition 1.3.1, with respective initial conditions ρ_0 and λ_0. For all $t \in [0,T]$, holds the equality*

$$
\begin{aligned}
\big\langle \rho_t,\, (-\Delta)^{-1}\lambda_t\big\rangle - \big\langle \rho_0,\, (-\Delta)^{-1}\lambda_0\big\rangle = -2\int_0^t \big\langle \rho_s, \lambda_s\big\rangle\, ds \\
+2\int_0^t \big\langle \chi(\rho_s)\partial_u H_s,\, \partial_u(-\Delta)^{-1}\lambda_s\big\rangle\, ds + 2\int_0^t \big\langle \chi(\lambda_s)\partial_u H_s,\, \partial_u(-\Delta)^{-1}\rho_s\big\rangle\, ds\,.
\end{aligned}
\tag{5.11}
$$

Proof. This proof is very similar to proof of the Proposition 2.6.3. The mean of a weak solution of (1.10) is also constant in time, thus $\rho_t, \lambda_t \in L^2(\mathbb{T})^{\perp 1}$ for any time $t \in [0,T]$.

Take a partition $0 = t_0 < t_1 < \cdots < t_n = T$ of the interval $[0,T]$ and, as like in Proposition 2.6.3, we write

$$
\begin{aligned}
& \big\langle \rho_t,\, (-\Delta)^{-1}\lambda_t\big\rangle - \big\langle \rho_0,\, (-\Delta)^{-1}\lambda_0\big\rangle \\
& = \sum_{k=0}^{n-1} \big\langle \rho_{t_{k+1}},\, (-\Delta)^{-1}\lambda_{t_{k+1}}\big\rangle - \big\langle \rho_{t_{k+1}},\, (-\Delta)^{-1}\lambda_{t_k}\big\rangle \\
& + \sum_{k=0}^{n-1} \big\langle \rho_{t_{k+1}},\, (-\Delta)^{-1}\lambda_{t_k}\big\rangle - \big\langle \rho_{t_k},\, (-\Delta)^{-1}\lambda_{t_k}\big\rangle\,.
\end{aligned}
\tag{5.12}
$$

Since ρ is a weak solution of (1.7), λ_{t_k} belongs to $L^2(\mathbb{T})^{\perp 1}$ and recalling the Proposition 5.4.1 and the Proposition 2.6.2 item (c), the second term above can be written as

$$
\begin{aligned}
& \big\langle \rho_{t_{k+1}},\, (-\Delta)^{-1}\lambda_{t_k}\big\rangle - \big\langle \rho_{t_k},\, (-\Delta)^{-1}\lambda_{t_k}\big\rangle \\
& = -\int_{t_k}^{t_{k+1}} \langle \rho_s, \lambda_{t_k}\rangle\, ds + 2\int_{t_k}^{t_{k+1}} \big\langle \chi(\rho_s)\partial_u H_s,\, \partial_u(-\Delta)^{-1}\lambda_{t_k}\big\rangle\, ds \\
& = -\int_{t_k}^{t_{k+1}} \langle \rho_s, \lambda_s\rangle\, ds + 2\int_{t_k}^{t_{k+1}} \big\langle \chi(\rho_s)\partial_u H_s,\, \partial_u(-\Delta)^{-1}\lambda_s\big\rangle\, ds + R_n^{k,1}(\rho,\lambda) + R_n^{k,2}(\rho,\lambda)\,,
\end{aligned}
\tag{5.13}
$$

where

$$R_n^{k,1}(\rho,\lambda) \;=\; \int_{t_k}^{t_{k+1}} \langle \rho_s, \lambda_s - \lambda_{t_k}\rangle \, ds$$

and

$$R_n^{k,2}(\rho,\lambda) \;=\; 2\int_{t_k}^{t_{k+1}} \big\langle \chi(\rho_s)\partial_u H_s,\; \partial_u(-\Delta)^{-1}(\lambda_{t_k}-\lambda_s)\big\rangle \, ds\,.$$

The first term in (5.12) is similar to the second one, because $(-\Delta)^{-1}$ is a symmetric operator,

$$\begin{aligned}
&\big\langle \rho_{t_{k+1}},\, (-\Delta)^{-1}\lambda_{t_{k+1}}\big\rangle - \big\langle \rho_{t_{k+1}},\, (-\Delta)^{-1}\lambda_{t_k}\big\rangle \\
&= -\int_{t_k}^{t_{k+1}} \langle \rho_s, \lambda_s\rangle \, ds + 2\int_{t_k}^{t_{k+1}} \big\langle \chi(\lambda_s)\partial_u H_s,\; \partial_u(-\Delta)^{-1}\rho s\big\rangle \, ds + R_n^{k,3}(\lambda,\rho) + R_n^{k,4}(\lambda,\rho)\,,
\end{aligned} \tag{5.14}$$

where

$$R_n^{k,3}(\lambda,\rho) \;=\; \int_{t_k}^{t_{k+1}} \langle \lambda_s, \rho_s - \rho_{t_{k+1}}\rangle \, ds$$

and

$$R_n^{k,4}(\lambda,\rho) \;=\; 2\int_{t_k}^{t_{k+1}} \big\langle \chi(\lambda_s)\partial_u H_s,\; \partial_u(-\Delta)^{-1}(\rho_{t_{k+1}}-\rho_s)\big\rangle \, ds\,.$$

The sum over k of the firsts two terms in the right side of (5.13) and of (5.14) is exactly the expression that we announced in (5.11). We shall treat the remainder.

We claim that

$$\sum_{k=0}^{n-1} \big\{ R_n^{k,1}(\rho,\lambda) + R_n^{k,2}(\rho,\lambda) + R_n^{k,3}(\lambda,\rho) + R_n^{k,4}(\lambda,\rho)\big\}$$

converges to zero, as $n \to \infty$.

If we prove this claim, the proof of this proposition is completed. To prove this claim we will proceed as in Proposition 2.6.3. Let $\iota_\delta : \mathbb{T} \to \mathbb{R}$ be an smooth approximation of identity and $\Phi_\delta : \mathbb{T} \to \mathbb{R}$ a smooth function bounded by one, equals to zero in the interval $(-\delta, \delta)$, and equals to one in $\mathbb{T}\backslash(-2\delta, 2\delta)$. Define

$$\rho_s^\delta(u) \;=\; (\rho_s * \iota_\delta)(u)\Phi_\delta(u)\,.$$

It is of easy verification that $\rho_s^\delta, \lambda_s^\delta \in \mathcal{H}^2_{\mathrm{bc}}(\mathbb{T})$, for any $s \in [0,T]$, and also that $\rho_s^\delta(\cdot)$ converges to $\rho_s(\cdot)$ in $L^2(\mathbb{T})$, as $\delta \downarrow 0$.

Adding and subtracting ρ^δ, $R_n^{k,1}(\rho,\lambda)$ can be written as

$$\int_{t_k}^{t_{k+1}} \langle \rho_s - \rho_s^\delta, \lambda_s - \lambda_{t_k}\rangle \, ds + \int_{t_k}^{t_{k+1}} \langle \rho_s^\delta, \lambda_s - \lambda_{t_k}\rangle \, ds\,. \tag{5.15}$$

Fix $\varepsilon > 0$. Since $\rho_s^\delta(\cdot)$ converges to $\rho_s(\cdot)$ in $L^2(\mathbb{T})$, applying the Dominated Convergence Theorem, the sum in k of the first term in (5.15) is bounded in modulus by εt for some $\delta(\varepsilon)$ small.

Fix now such $\delta = \delta(\varepsilon)$. Since $\rho_s^\delta \in \mathcal{H}^2_{\text{bc}}(\mathbb{T})$ and since λ is a weak solution of (1.10), the second term in (5.15) is equal to

$$\int_{t_k}^{t_{k+1}} \Big\{ \int_{t_k}^{s} \langle \lambda_r, \Delta\rho_s^\delta\rangle \, dr + 2\int_{t_k}^{s} \big\langle \chi(\lambda_r)\partial_u H_s,\, \partial_u \rho_r^\delta \big\rangle \, dr \Big\} \, ds \,,$$

whose modulus is bounded by $C(\rho, H, \delta(\varepsilon))(t_{k+1}-t_k)^2$. Thus,

$$\sum_{k=0}^{n-1} R_n^{k,1}(\rho,\lambda) \;\le\; \varepsilon\, t \;+\; C(\rho,H,\delta(\varepsilon))\; t \max_{k\in\{0,\cdots,n-1\}} |t_{k+1}-t_k| \,.$$

Taking the limit on expression above, as $n \to \infty$, and recalling that $\varepsilon > 0$ is any, we get

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} R_n^{k,1}(\rho,\lambda) = 0 \,.$$

Now, we use the Young's inequality, then $R_n^{k,2}(\rho,\lambda)$ is bounded from above by

$$\varepsilon \int_{t_k}^{t_{k+1}} \big\langle \chi(\rho_s)\partial_u H_s,\, \chi(\rho_s)\partial_u H_s \big\rangle \, ds + \tfrac{1}{\varepsilon} \int_{t_k}^{t_{k+1}} \big\langle \partial_u(-\Delta)^{-1}(\lambda_{t_k}-\lambda_s),\, \partial_u(-\Delta)^{-1}(\lambda_{t_k}-\lambda_s) \big\rangle \, ds \,,$$

for all $\varepsilon > 0$. Integration by parts, the second term above is equal to

$$\begin{aligned} &\tfrac{1}{\varepsilon} \int_{t_k}^{t_{k+1}} \big\langle \lambda_{t_k}-\lambda_s,\, (-\Delta)^{-1}(\lambda_{t_k}-\lambda_s) \big\rangle \, ds \\ &+\tfrac{1}{\varepsilon} \int_{t_k}^{t_{k+1}} \Big\{ \big[(-\Delta)^{-1}(\lambda_{t_k}-\lambda_s)\big](1) \big[\partial_u(-\Delta)^{-1}(\lambda_{t_k}-\lambda_s)\big](1) \Big\} \, ds \\ &-\tfrac{1}{\varepsilon} \int_{t_k}^{t_{k+1}} \Big\{ \big[(-\Delta)^{-1}(\lambda_{t_k}-\lambda_s)\big](0) \big[\partial_u(-\Delta)^{-1}(\lambda_{t_k}-\lambda_s)\big](0) \Big\} \, ds \,. \end{aligned} \tag{5.16}$$

Using the Proposition 2.6.2 item (b), the two last terms in the expression above can be written as

$$-\tfrac{1}{\varepsilon} \int_{t_k}^{t_{k+1}} \Big\{ \big[(-\Delta)^{-1}(\lambda_{t_k}-\lambda_s)\big](0) - \big[(-\Delta)^{-1}(\lambda_{t_k}-\lambda_s)\big](1) \Big\}^2 \, ds \,.$$

We will use that λ is a weak solution of (1.10), $(-\Delta)^{-1}(\lambda_{t_k}-\lambda_s)$ belongs to $\mathcal{H}^2_{\text{bc}}(\mathbb{T})$ and the Proposition 5.4.1, then the first term in (5.16) is equal to

$$\tfrac{1}{\varepsilon} \int_{t_k}^{t_{k+1}} \Big\{ -\int_{t_k}^{s} \langle \lambda_r, (\lambda_{t_k}-\lambda_s\rangle \, dr + 2\int_{t_k}^{s} \big\langle \chi(\lambda_r)\partial_u H_s,\, \partial_u(-\Delta)^{-1}(\lambda_{t_k}-\lambda_s) \big\rangle \, dr \Big\} \, ds \,.$$

Thus, there exists the constants $C(H) > 0$ and $C(\lambda, H) > 0$ such that

$$\sum_{k=0}^{n-1} R_n^{k,2}(\rho,\lambda) \;\le\; \varepsilon\, C(H)\, t \;+\; \tfrac{1}{\varepsilon}\, C(\lambda,H)\; t \max_{k\in\{0,\cdots,n-1\}} |t_{k+1}-t_k| \,.$$

One can conclude that the limit of $\sum_{k=0}^{n-1} R_n^{k,2}(\rho,\lambda)$ is zero, as $n\to\infty$. For finish the proof of claim, we proceed with $R_n^{k,3}(\lambda,\rho)$ and with $R_n^{k,4}(\lambda,\rho)$ in the same way that with $R_n^{k,1}(\rho,\lambda)$ and $R_n^{k,2}(\rho,\lambda)$.

□

Corollary 5.4.3. *Let ρ and λ be two weak solutions of the asymmetric equation with perturbation $H\in C^{1,2}([0,T]\times\mathbb{T})$ given in Definition 1.3.1, with respective initial conditions ρ_0 and λ_0. Then,*

$$\langle \rho_t-\lambda_t,\,(-\Delta)^{-1}(\rho_t-\lambda_t)\rangle \le \langle \rho_0-\lambda_0,\,(-\Delta)^{-1}(\rho_0-\lambda_0)\rangle\, e^{ct}\,, \tag{5.17}$$

for all $t\in[0,T]$ and some constant $c\in\mathbb{R}$. In particular, there exists at most one weak solution of the asymmetric equation with perturbation H.

Proof. From Proposition 5.4.2, the expression

$$\langle \rho_t-\lambda_t,\,(-\Delta)^{-1}(\rho_t-\lambda_t)\rangle \tag{5.18}$$

is equal to

$$-2\int_0^t \langle \rho_s-\lambda_s,\,\rho_s-\lambda_s\rangle\,ds + 4\int_0^t \big\langle [\chi(\rho_s)-\chi(\lambda_s)]\partial_u H_s\,,\;\partial_u(-\Delta)^{-1}(\rho_s-\lambda_s)\big\rangle\,ds\,.$$

Let us estimate the second integral in the expression above. By Young's inequality, for any $\varepsilon>0$,

$$\begin{aligned}
&4\int_0^t \langle [\chi(\rho_s)-\chi(\lambda_s)]\partial_u H_s\,,\;\partial_u\rho_s-\partial_u\lambda_s\rangle\,ds\\
&\le 2\varepsilon\, C(H)\int_0^t \big\langle \chi(\rho_s)-\chi(\lambda_s),\,\chi(\rho_s)-\chi(\lambda_s)\big\rangle\,ds\\
&+\frac{2}{\varepsilon}\int_0^t \big\langle \partial_u(-\Delta)^{-1}(\rho_s-\lambda_s),\,\partial_u(-\Delta)^{-1}(\rho_s-\lambda_s)\big\rangle\,ds\,.
\end{aligned}$$

By hypothesis, ρ and λ take values in the interval $[0,1]$. Therefore, because $\chi(\cdot)$ is a Lipschitz function,

$$2\varepsilon\, C(H)\int_0^t \big\langle \chi(\rho_s)-\chi(\lambda_s),\,\chi(\rho_s)-\chi(\lambda_s)\big\rangle\,ds \le \varepsilon\, C_H\int_0^t \langle \rho_s-\lambda_s,\,\rho_s-\lambda_s\rangle\,ds\,.$$

By integration by parts and Proposition 2.6.2 item (b), the term

$$\frac{2}{\varepsilon}\int_0^t \big\langle \partial_u(-\Delta)^{-1}(\rho_s-\lambda_s),\,\partial_u(-\Delta)^{-1}(\rho_s-\lambda_s)\big\rangle\,ds$$

is equal to

$$\begin{aligned}
&\frac{2}{\varepsilon}\int_0^t \langle \rho_s-\lambda_s,\,(-\Delta)^{-1}(\rho_s-\lambda_s)\rangle\,ds\\
&-\frac{2}{\varepsilon}\int_0^t \big\{(-\Delta)^{-1}(\rho_s-\lambda_s)(0)-(-\Delta)^{-1}(\rho_s-\lambda_s)(1)\big\}^2\,ds\,.
\end{aligned}$$

We conclude that, for enough small ε, the expression (5.18) is bounded from above by

$$\frac{2}{\varepsilon}\int_0^t \langle \rho_s-\lambda_s,\,(-\Delta)^{-1}(\rho_s-\lambda_s)\rangle\,ds\,.$$

Thus, it just remains to apply Gronwall's inequality to obtain (5.17).

To see that this implies the uniqueness of solutions with the same initial condition γ. Use item (d) of the Proposition 2.6.2, for fixed $t\in[0,T]$, to obtain $f_t\in\mathcal{H}^2_{\rm bc}(\mathbb{T})$ such that $\rho_t-\lambda_t=(-\Delta)f_t$, and thus

$$0=\langle \rho_t-\lambda_t,\,(-\Delta)^{-1}(\rho_t-\lambda_t)\rangle=\big\langle -\Delta f_t,f_t\big\rangle \;=\; \langle \partial_u f_t,\partial_u f_t\rangle+(\partial_u f_t(0^+))^2\,.$$

Then, $\partial_u f_t(u)=0$, u - almost surely and for all $t\in[0,T]$. Since $\rho_t-\lambda_t=(-\Delta)f_t$, we have $\rho_t(u)=\lambda_t(u)$, u - almost surely and for all $t\in[0,T]$. This concludes the proof. □

Chapter 6

Large Deviations Lower Bound

In this chapter we will present in Proposition 6.0.6 the lower bound of the Large Deviation Principle. For this, we will need the next lemmata.

Lemma 6.0.4. *For each function $H \in C^{1,2}([0,T] \times \mathbb{T})$. Let ρ^H be a unique weak solution of* (1.10)*. Then,*

$$I^*(\rho^H) = \hat{J}_H(\rho^H) = \int_0^T \left\langle \chi(\rho_t^H), (\partial_u H_t)^2 \right\rangle dt\,. \tag{6.1}$$

Proof. Using that ρ^H is weak solution of (1.10), for all $G \in C^{1,2}([0,T] \times \mathbb{T})$, we get

$$\hat{J}_G(\rho^H) = \int_0^T \left\langle \chi(\rho_t^H), (\partial_u H_t)^2 \right\rangle dt - \int_0^T \left\langle \chi(\rho_t^H), (\partial_u H_t - \partial_u G_t)^2 \right\rangle dt\,.$$

Then,

$$I^*(\rho^H) \;=\; \sup_{G \in C^{1,2}([0,T]\times\mathbb{T})} \hat{J}_G(\rho^H) \;=\; \hat{J}_H(\rho^H) \;=\; \int_0^T \left\langle \chi(\rho_t^H), (\partial_u H_t)^2 \right\rangle dt\,.$$

□

Lemma 6.0.5. *For each $H \in C^{1,2}([0,T] \times \mathbb{T})$. Denote by $\boldsymbol{H}\big(\mathbb{P}^H_{\nu^N_\alpha} | \mathbb{P}_{\nu^N_\alpha}\big)$ the entropy of a probability measure $\mathbb{P}^H_{\nu^N_\alpha}$ with respect to a probability measure $\mathbb{P}_{\nu^N_\alpha}$. We refer to [16, Section A1.8] for a precise definition. Then,*

$$\lim_{N\to\infty} \tfrac{1}{N} \boldsymbol{H}\big(\mathbb{P}^H_{\nu^N_\alpha} | \mathbb{P}_{\nu^N_\alpha}\big) = I(\rho^H)\,,$$

where ρ^H is a unique weak solution of (1.10).

Proof. By the explicit formula for entropy

$$\tfrac{1}{N} \boldsymbol{H}\big(\mathbb{P}^H_{\nu^N_\alpha} | \mathbb{P}_{\nu^N_\alpha}\big) = \tfrac{1}{N} \mathbb{E}^H_{\nu^N_\alpha}\left[\log \frac{\mathbf{d}\mathbb{P}^H_{\nu^N_\alpha}}{\mathbf{d}\mathbb{P}_{\nu^N_\alpha}}\right]. \tag{6.2}$$

Recall the definition of the set $B^H_{\delta,\varepsilon}$ given in (4.4). We claim the probability $(B^H_{\delta,\varepsilon})^{\complement}$ with respect to $\mathbb{P}^H_{\nu^N_\alpha}$ is superexponentially small. Indeed, using (5.3)

$$\mathbb{P}^H_{\nu^N_\alpha}\Big[(B^H_{\delta,\varepsilon})^{\complement}\Big] = \mathbb{E}_{\nu^N_\alpha}\left[\frac{\mathbf{d}\mathbb{P}^H_{\nu^N_\alpha}}{\mathbf{d}\mathbb{P}_{\nu^N_\alpha}}\,\mathbf{1}_{(B^H_{\delta,\varepsilon})^{\complement}}\right] \le e^{C(H,T)N}\mathbb{P}_{\nu^N_\alpha}\Big[(B^H_{\delta,\varepsilon})^{\complement}\Big]\,.$$

By (4.5), we get

$$\overline{\lim_{\varepsilon\downarrow 0}}\ \overline{\lim_{N\to\infty}}\ \tfrac{1}{N}\log\mathbb{P}^H_{\nu^N_\alpha}\Big[(B^H_{\delta,\varepsilon})^{\complement}\Big] = -\infty\,. \tag{6.3}$$

From (6.3) and the fact that $\frac{1}{N}\log\frac{\mathbf{d}\mathbb{P}^H_{\nu^N_\alpha}}{\mathbf{d}\mathbb{P}_{\nu^N_\alpha}}$ is bounded by $C(H,T)$, the right hand side of (6.2) is equal to

$$\tfrac{1}{N}\mathbb{E}^H_{\nu^N_\alpha}\left[\log\frac{\mathbf{d}\mathbb{P}^H_{\nu^N_\alpha}}{\mathbf{d}\mathbb{P}_{\nu^N_\alpha}}\,\mathbf{1}_{B^H_{\delta,\varepsilon}}\right] + o_N(1)\,, \tag{6.4}$$

for all $\delta > 0$ and each ε enoughly small ($\varepsilon < \varepsilon(\delta)$). Applying the expression (4.9) for the Radon-Nikodym derivative, $\frac{1}{N}\log\frac{\mathbf{d}\mathbb{P}^H_{\nu^N_\alpha}}{\mathbf{d}\mathbb{P}_{\nu^N_\alpha}}$ on the set $B^H_{\delta,\varepsilon}$ is equal to

$$\hat{J}_H\big((\pi^N * \iota_\gamma) * \iota^a_\varepsilon\big) + O_{H,T,\varepsilon,\gamma}(\tfrac{1}{N}) + O(\delta) + O_H(\varepsilon) + O_H(\tfrac{\gamma}{\varepsilon})\,,$$

for all $\delta > 0$ and all ε and γ small enough. Since this expression is bounded and the probability of $(B^H_{\delta,\varepsilon})^{\complement}$ with respect to $\mathbb{P}^H_{\nu^N_\alpha}$ vanishes as N increases to infinity, the expression (6.4) becomes

$$\mathbb{E}^H_{\nu^N_\alpha}\Big[\hat{J}_H\big((\pi^N * \iota_\gamma) * \iota^a_\varepsilon\big)\Big] + O_{H,T,\varepsilon,\gamma}(\tfrac{1}{N}) + O(\delta) + O_H(\varepsilon) + O_H(\tfrac{\gamma}{\varepsilon}) + o_N(1)\,,$$

for all $\delta > 0$ and all ε and γ small enough. The functional $\rho \mapsto \hat{J}_H\big((\rho * \iota_\gamma) * \iota^a_\varepsilon\big)$ is continuous with respect to the Skorohod topology with ε and γ fixed, see the Proposition A.2.4. By Proposition 5.0.4 the sequence $\mathbb{Q}^H_{\mu_N}$ converges weakly to the probability concentrated on the weak solution of (1.10). In particular, as N increases to infinity the previous expectation converges to

$$\hat{J}_H\big((\rho^H * \iota_\gamma) * \iota^a_\varepsilon\big) + O(\delta) + O_H(\varepsilon) + O_H(\tfrac{\gamma}{\varepsilon})\,.$$

It remains to let $\gamma \downarrow 0$ and $\varepsilon \downarrow 0$, then $\delta \downarrow 0$ and recall identity (6.1).

□

Recall that $\mathcal{D}^0\big([0,T],\mathcal{M}_0\big)$ is the subset of $\mathcal{D}\big([0,T],\mathcal{M}_0\big)$ that consists of all paths $\pi(t,du) = \rho(t,u)du$ such that there exists $H \in C^{1,2}([0,T]\times\mathbb{T})$ that $\rho = \rho^H$ is a unique weak solution of (1.10).

Proposition 6.0.6. *Let $\mathcal{O}$ be an open set of $\mathcal{D}\big([0,T],\mathcal{M}\big)$. Then*

$$\underline{\lim}_{N\to\infty}\ \tfrac{1}{N}\log\mathbb{Q}_{\nu^N_\alpha}[\mathcal{O}] \ge -\inf_{\pi\in\mathcal{O}\cap\mathcal{D}^0([0,T],\mathcal{M}_0)} I^*(\pi)\,.$$

Proof. This proof is essentially the same as found in [16]. Let $\pi \in \mathcal{O} \cap \mathcal{D}^0\big([0,T],\mathcal{M}_0\big)$, there exists $H \in C^{1,2}([0,T]\times\mathbb{T})$ such that $\pi(t,du) = \rho_t^H(u)du$, where ρ^H is a weak solution of (1.10). Denote by $\mathbb{P}^H_{\nu_\alpha^N,\mathcal{O}}$ the probability on space $\mathcal{D}\big([0,T],\{0,1\}^{\mathbb{T}_N}\big)$ given by

$$\mathbb{P}^H_{\nu_\alpha^N,\mathcal{O}}[A] = \frac{\mathbb{P}^H_{\nu_\alpha^N}[A, \pi^N \in \mathcal{O}]}{\mathbb{P}^H_{\nu_\alpha^N}[\pi^N \in \mathcal{O}]}\,,$$

for all measurable set A of $\mathcal{D}\big([0,T],\{0,1\}^{\mathbb{T}_N}\big)$. Using this probability, we may rewrite $\frac{1}{N}\log \mathbb{Q}_{\nu_\alpha^N}[\mathcal{O}]$ as

$$\tfrac{1}{N}\log \mathbb{E}^H_{\nu_\alpha^N,\mathcal{O}}\Big[\frac{\mathbf{d}\mathbb{P}_{\nu_\alpha^N}}{\mathbf{d}\mathbb{P}^H_{\nu_\alpha^N}}\Big] + \tfrac{1}{N}\log \mathbb{Q}^H_{\nu_\alpha^N}[\mathcal{O}]\,.$$

By Proposition 5.0.4, since $\mathcal{O}$ is a neighborhood that contains ρ^H, the second expression above converges to 0 as N increases to infinity. Applying Jensen's inequality, the first one is bounded below by

$$\mathbb{E}^H_{\nu_\alpha^N,\mathcal{O}}\Big[\tfrac{1}{N}\log\frac{\mathbf{d}\mathbb{P}_{\nu_\alpha^N}}{\mathbf{d}\mathbb{P}^H_{\nu_\alpha^N}}\Big]\,.$$

Using the definition of probability $\mathbb{P}^H_{\nu_\alpha^N,\mathcal{O}}$ and expression (6.2), the last expression becomes

$$\frac{1}{\mathbb{Q}^H_{\nu_\alpha^N}[\mathcal{O}]}\Bigg\{ -\tfrac{1}{N}\boldsymbol{H}\big(\mathbb{P}^H_{\nu_\alpha^N}|\mathbb{P}_{\nu_\alpha^N}\big) - \mathbb{E}^H_{\nu_\alpha^N}\Big[\tfrac{1}{N}\log\frac{\mathbf{d}\mathbb{P}_{\nu_\alpha^N}}{\mathbf{d}\mathbb{P}^H_{\nu_\alpha^N}}\,\mathbf{1}_{\{\pi^N\in\mathcal{O}^\complement\}}\Big]\Bigg\}\,.$$

Once again, by Proposition 5.0.4, $\mathbb{Q}^H_{\nu_\alpha^N}[\mathcal{O}]$ converges to 1 as $N\to\infty$. Since by (5.3) the expression $\frac{1}{N}\log\frac{\mathbf{d}\mathbb{P}_{\nu_\alpha^N}}{\mathbf{d}\mathbb{P}^H_{\nu_\alpha^N}}$ is bounded, the second term inside braces vanishes as $N\to\infty$. Therefore,

$$\varliminf_{N\to\infty}\tfrac{1}{N}\log\mathbb{Q}_{\nu_\alpha^N}[\mathcal{O}] \geq \lim_{N\to\infty} -\tfrac{1}{N}\boldsymbol{H}\big(\mathbb{P}^H_{\nu_\alpha^N}|\mathbb{P}_{\nu_\alpha^N}\big) = -I^*(\rho^H)\,.$$

□

Part II

Hydrodynamical behavior of symmetric exclusion with slow bonds of parameter $N^{-\beta}$

Chapter 7

Hydrodynamical behavior of symmetric exclusion with slow bonds of parameter $N^{-\beta}$

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To be appear in the Annales de l'Institut Henri Poincaré: Probability and Statistics (B).

7.1 Notation and Results

Let $\mathbb{T}_N = \{1, \ldots, N\}$ be the one-dimensional discrete torus with N points. At each site, we allow at most one particle. Therefore, we will be concerned about the state space $\{0,1\}^{\mathbb{T}_N}$. Configurations will be denoted by the Greek letter η, so that $\eta(x) = 1$, if the site x is occupied, otherwise $\eta(x) = 0$.

We define now the exclusion process with state space $\{0,1\}^{\mathbb{T}_N}$ and with conductance $\{\xi^N_{x,x+1}\}_x$ at the bond of vertices $x, x+1$. The dynamics of this Markov process can be described as follows. At each bond of vertices $x, x+1$, we associate an exponential clock of parameter $\xi^N_{x,x+1}$. When this clock rings, the value of η at the vertices of this bond are exchanged. This process can also be characterized in terms of its infinitesimal generator $\mathcal{L}_N$, which acts on local functions $f : \{0,1\}^{\mathbb{T}_N} \to \mathbb{R}$ as

$$\mathcal{L}_N f(\eta) = \sum_{x \in \mathbb{T}_N} \xi^N_{x,x+1} \Big[f(\eta^{x,x+1}) - f(\eta) \Big] ,$$

where $\eta^{x,x+1}$ is the configuration obtained from η by exchanging the variables $\eta(x)$ and $\eta(x+1)$:

$$(\eta^{x,x+1})(y) = \begin{cases} \eta(x+1), & \text{if } y = x\,, \\ \eta(x), & \text{if } y = x+1\,, \\ \eta(y), & \text{otherwise.} \end{cases}$$

The Bernoulli product measures $\{\nu_\alpha^N : 0 \le \alpha \le 1\}$ are invariant and in fact, reversible, for the dynamics introduced above. Namely, ν_α^N is a product measure on $\{0,1\}^{\mathbb{T}_N}$ with marginal at site x in $\mathbb{T}_N$ given by

$$\nu_\alpha^N\{\eta : \eta(x) = 1\} \;=\; \alpha.$$

Denote by $\mathbb{T}$ the one-dimensional continuous torus $[0,1)$. The exclusion process with a slow bond at each point $b_1 \dots, b_k \in \mathbb{T}$ is defined with the following conductances:

$$\xi_{x,x+1}^N \;=\; \begin{cases} N^{-\beta}, & \text{if } \;\{b_1,\dots,b_k\} \cap (\frac{x}{N}, \frac{x+1}{N}] \neq \varnothing\,, \\[1em] \;1, & \text{otherwise}\,. \end{cases}$$

The conductances are chosen in such a way that particles cross bonds at rate one, except k particular bonds in which the dynamics is slowed down by a factor $N^{-\beta}$, with $\beta \in [0,\infty)$. Each one of these particular bonds contains the macroscopic point $b_i \in \mathbb{T}$; or b_i coincides with some vertex $\frac{x}{N}$ and the slow bond is chosen as the bond to the left of $\frac{x}{N}$. To simplify notation, we denote by Nb_i the left vertex of the slow bond containing b_i.

Denote by $\{\eta_t := \eta_{tN^2} : t \ge 0\}$ the Markov process on $\{0,1\}^{\mathbb{T}_N}$ associated to the generator $\mathcal{L}_N$ *speeded up* by N^2. Although η_t depends on N and β, we are not indexing it on that in order not to overload notation. Let $D(\mathbb{R}_+, \{0,1\}^{\mathbb{T}_N})$ be the path space of càdlàg trajectories with values in $\{0,1\}^{\mathbb{T}_N}$. For a measure μ_N on $\{0,1\}^{\mathbb{T}_N}$, denote by $\mathbb{P}_{\mu_N}^\beta$ the probability measure on $D(\mathbb{R}_+, \{0,1\}^{\mathbb{T}_N})$ induced by the initial state μ_N and the Markov process $\{\eta_t : t \ge 0\}$ and denote by $\mathbb{E}_{\mu_N}^\beta$ the expectation with respect to $\mathbb{P}_{\mu_N}^\beta$.

Definition 7.1.1. *A sequence of probability measures $\{\mu_N : N \ge 1\}$ on $\{0,1\}^{\mathbb{T}_N}$ is said to be associated to a profile $\rho_0 : \mathbb{T} \to [0,1]$ if for every $\delta > 0$ and every continuous functions $H : \mathbb{T} \to \mathbb{R}$*

$$\lim_{N\to\infty} \mu_N\Big\{\eta : \Big|\tfrac{1}{N} \sum_{x\in\mathbb{T}_N} H(\tfrac{x}{N})\, \eta(x) - \int_{\mathbb{T}} H(u)\, \rho_0(u) du\Big| > \delta\Big\} \;=\; 0. \tag{7.1}$$

Now we introduce an operator which corresponds to the generator of the random walk in $\mathbb{T}_N$ with conductance $\xi_{x,x+1}^N$ at the bond of vertices $x, x+1$. This operator acts on $H : \mathbb{T} \to \mathbb{R}$ as

$$\mathbb{L}_N H(\tfrac{x}{N}) = \xi_{x,x+1}^N \Big[H\Big(\tfrac{x+1}{N}\Big) - H\Big(\tfrac{x}{N}\Big)\Big] + \xi_{x-1,x}^N \Big[H\Big(\tfrac{x-1}{N}\Big) - H\Big(\tfrac{x}{N}\Big)\Big]\,. \tag{7.2}$$

We will not differentiate the notation for functions H defined on $\mathbb{T}$ and on $\mathbb{T}_N$. The indicator function of a set A will be written by $\mathbf{1}_A(u)$, which is one when $u \in A$ and zero otherwise.

7.1.1 The Operator $\frac{d}{dx}\frac{d}{dW}$

Given the points $b_1, \ldots, b_k \in \mathbb{T}$, define the measure $W(du)$ in the torus $\mathbb{T}$ by

$$W(du) \;=\; du + \delta_{b_1}(du) + \cdots + \delta_{b_k}(du)\,,$$

so that W is the Lebesgue measure on the torus $\mathbb{T}$ plus the sum of the Dirac measure in each of the $\{b_i : i = 1, ..., k\}$.

Let $\mathcal{H}^1_W$ be the set of functions F in $L^2(\mathbb{T})$ such that for $x \in \mathbb{T}$

$$F(x) \;=\; a \;+\; \int_{(0,x]} \Big(b + \int_0^y f(z)\,dz\Big) W(dy),$$

for some function f in $L^2(\mathbb{T})$ and $a, b \in \mathbb{R}$ such that

$$\int_0^1 f(x)\,dx \;=\; 0\,, \quad \int_{(0,1]} \Big(b + \int_0^y f(z)\,dz\Big) W(dy) \;=\; 0\,. \tag{7.3}$$

Define the operator

$$\begin{aligned}&\frac{d}{dx}\frac{d}{dW} : \mathcal{H}^1_W \to L^2(\mathbb{T})\\ &\frac{d}{dx}\frac{d}{dW} F = f.\end{aligned}$$

For more details we refer the reader to [9].

7.1.2 The hydrodynamical equations

Consider a continuous density profile $\gamma : \mathbb{T} \to [0, 1]$. Denote by $\langle \cdot, \cdot \rangle$ the inner product in $L^2(\mathbb{T})$, by ρ_t a function $\rho(t, \cdot)$ and for an integer n denote by $C^n(\mathbb{T})$ the set of continuous functions from $\mathbb{T}$ to $\mathbb{R}$ and with continuous derivatives of order up to n. For $\mathcal{I}$ an interval of $\mathbb{T}$, here and in the sequel, for n and m integers, we use the notation $C^{n,m}([0, T] \times \mathcal{I})$ to denote the set of functions defined on the domain $[0, T] \times \mathcal{I}$, that are of class C^n in time and C^m in space.

Definition 7.1.2. *A bounded function $\rho : [0, T] \times \mathbb{T} \to \mathbb{R}$ is said to be a weak solution of the parabolic differential equation with initial condition $\gamma(\cdot)$:*

$$\begin{cases} \partial_t \rho \;=\; \partial_u^2 \rho \\ \rho(0, \cdot) \;=\; \gamma(\cdot) \end{cases} \tag{7.4}$$

if, for $t \in [0, T]$ and $H \in C^2(\mathbb{T})$, $\rho(t, \cdot)$ satisfies the integral equation

$$\langle \rho_t, H \rangle \;-\; \langle \gamma, H \rangle \;-\; \int_0^t \langle \rho_s, \partial_u^2 H \rangle\, ds \;=0.$$

Definition 7.1.3. *A bounded function $\rho : [0,T] \times \mathbb{T} \to \mathbb{R}$ is said to be a weak solution of the parabolic differential equation with initial condition $\gamma(\cdot)$:*

$$\begin{cases} \partial_t \rho & = \dfrac{d}{dx}\dfrac{d}{dW}\rho \\ \rho(0,\cdot) & = \gamma(\cdot) \end{cases} \tag{7.5}$$

if, for $t \in [0,T]$ and $H \in \mathcal{H}^1_W$, $\rho(t,\cdot)$ satisfies the integral equation

$$\langle \rho_t, H \rangle \; - \; \langle \gamma, H \rangle - \int_0^t \Big\langle \rho_s, \frac{d}{dx}\frac{d}{dW} H \Big\rangle \, ds \; = \; 0 \, .$$

Following the notation of [4], denote by $L^2(0,T;\mathcal{H}^1(a,b))$ the space of functions $\varrho \in L^2([0,T] \times [a,b])$ for which there exists a function in $L^2([0,T] \times [a,b])$, denoted by $\partial_u \varrho$, satisfying

$$\int_0^T \int_a^b (\partial_u H)(s,u) \, \varrho(s,u) \, du \, ds \; = \; - \int_0^T \int_a^b H(s,u) \, (\partial_u \varrho)(s,u) \, du \, ds \, ,$$

for any $H \in C^{0,1}([0,T] \times [a,b])$ with compact support in $[0,T] \times (a,b)$.

Definition 7.1.4. *Let $[b_i, b_{i+1}] \subset \mathbb{T}$. A bounded function $\rho : [0,T] \times [b_i, b_{i+1}] \to \mathbb{R}$ is said to be a weak solution of the parabolic differential equation with Neumann's boundary conditions in the cylinder $[0,T] \times [b_i, b_{i+1}]$ and with initial condition $\gamma(\cdot)$:*

$$\begin{cases} \partial_t \rho \; = \; \partial_u^2 \rho \\ \rho(0,\cdot) \; = \; \gamma(\cdot) \\ \partial_u \rho(t, b_i) = \partial_u \rho(t, b_{i+1}) = 0, \; \forall t \in [0,T] \end{cases} \tag{7.6}$$

if, for $t \in [0,T]$ and $H \in C^{1,2}([0,T] \times [b_i, b_{i+1}])$, $\rho(t,\cdot)$ satisfies the integral equation

$$\begin{aligned} & \int_{b_i}^{b_{i+1}} \rho(t,u) \, H(t,u) \, du - \int_{b_i}^{b_{i+1}} \gamma(u) \, H(0,u) \, du \\ & - \int_0^t \int_{b_i}^{b_{i+1}} \rho(s,u) \, \{ \partial_u^2 H(s,u) + \partial_s H(s,u) \} \, du \, ds \\ & + \int_0^t \partial_u H(s, b_{i+1}) \, \rho(s, b_{i+i}^-) \, ds - \int_0^t \partial_u H(s, b_i) \, \rho(s, b_i^+) \, ds \; = \; 0 \end{aligned} \tag{7.7}$$

and $\rho(t,\cdot)$ belongs to $L^2(0,T;\mathcal{H}^1(b_i, b_{i+1}))$.

Since in Definition 7.1.4 we impose $\rho \in L^2(0,T;\mathcal{H}^1(b_i, b_{i+1}))$, the integrals are well-defined at the boundary. This is a consequence of the following two facts. On one hand, it follows from the assumption that $\rho(t,\cdot) \in \mathcal{H}^1(b_i, b_{i+1})$, almost surely in $t \in [0,T]$. On the other hand, it is well-known that functions belonging to $\mathcal{H}^1(b_i, b_{i+1})$ and with sided limits at b_i and b_{i+1} are absolutely continuous with respect to the Lebesgue measure, see [18] for instance. We refer the reader to [4] for classical results about Sobolev spaces.

Heuristically, in order to establish an integral equation for the weak solution of the heat equation with Neumann's boundary conditions as above, one should multiply (7.6) by a test function H and perform twice a formal integration by parts to arrive at (7.7).

We are now in position to state the main result of this paper:

Theorem 7.1.1. *Fix $\beta \in [0,\infty)$. Consider the exclusion process with k slow bonds corresponding to macroscopic points $b_1,\dots,b_k \in \mathbb{T}$ and with conductance $N^{-\beta}$ at each one of these slow bonds.*

Fix a continuous initial profile $\gamma : \mathbb{T} \to [0,1]$. Let $\{\mu_N : N \geq 1\}$ be a sequence of probability measures on $\{0,1\}^{\mathbb{T}_N}$ associated to γ. Then, for any $t \in [0,T]$, for every $\delta > 0$ and every $H \in C(\mathbb{T})$, it holds that

$$\lim_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big\{\eta_\cdot : \Big|\frac{1}{N}\sum_{x\in\mathbb{T}_N} H(\tfrac{x}{N})\,\eta_t(x) - \int_{\mathbb{T}} H(u)\,\rho(t,u)du\Big| > \delta\Big\} \;=\; 0\,,$$

where :

- *if $\beta \in [0,1)$, $\rho(t,\cdot)$ is the unique weak solution of (7.4);*
- *if $\beta = 1$, $\rho(t,\cdot)$ is the unique weak solution of (7.5);*
- *if $\beta \in (1,\infty)$, in each cylinder $[0,T]\times[b_i,b_{i+1}]$, $\rho(t,\cdot)$ is the unique weak solution of (7.6).*

Remark 7.1.2. *The assumption that all slow bonds have exactly the same conductance is not necessary at all. In fact, last result is true when considering each slow bond containing the macroscopic point b_i with conductance $N^{-\beta_i}$. In that case, we would obtain a parabolic differential equation with the behavior at each $[b_i,b_{i+1}]$ given by the regime of the corresponding β_i as above. Another straightforward generalization is to consider conductances not exactly equal to $N^{-\beta}$, but of order $N^{-\beta}$, in the sense that the quotient with $N^{-\beta}$ converges to one. For sake of clarity, we present the proof under the conditions of Theorem 7.1.1.*

7.2 Scaling Limit

Let $\mathcal{M}$ be the space of positive measures on $\mathbb{T}$ with total mass bounded by one, endowed with the weak topology. Let $\pi^N_t \in \mathcal{M}$ be the empirical measure at time t associated to η_t, namely, it is the measure on $\mathbb{T}$ obtained by rescaling space by N and by assigning mass N^{-1} to each particle:

$$\pi^N_t \;=\; \frac{1}{N}\sum_{x\in\mathbb{T}_N} \eta_t(x)\,\delta_{x/N}\,, \tag{7.8}$$

where δ_u is the Dirac measure concentrated on u. For an integrable function $H : \mathbb{T} \to \mathbb{R}$, $\langle \pi^N_t, H\rangle$ stands for the integral of H with respect to π^N_t:

$$\langle \pi^N_t, H\rangle \;=\; \frac{1}{N}\sum_{x\in\mathbb{T}_N} H(\tfrac{x}{N})\,\eta_t(x)\,.$$

This notation is not to be mistaken with the inner product in $L^2(\mathbb{R})$. Also, when π_t has a density ρ, namely when $\pi(t,du) = \rho(t,u)du$, we sometimes write $\langle \rho_t, H\rangle$ for $\langle \pi_t, H\rangle$.

Fix $T > 0$. Let $D([0,T],\mathcal{M})$ be the space of $\mathcal{M}$-valued càdlàg trajectories $\pi : [0,T] \to \mathcal{M}$ endowed with the *Skorohod* topology. For each probability measure μ_N on $\{0,1\}^{\mathbb{T}_N}$, denote by $\mathbb{Q}^{\beta,N}_{\mu_N}$ the measure on the path space $D([0,T],\mathcal{M})$ induced by the measure μ_N and the empirical process π^N_t introduced in (7.8).

Fix a continuous profile $\gamma : \mathbb{T} \to [0,1]$ and consider a sequence $\{\mu_N : N \geq 1\}$ of measures on $\{0,1\}^{\mathbb{T}_N}$ associated to γ. Let $\mathbb{Q}^\beta$ be the probability measure on $D([0,T],\mathcal{M})$ concentrated on the deterministic path $\pi(t,du) = \rho(t,u)du$, where:

- if $\beta \in [0,1)$, $\rho(t,\cdot)$ is the unique weak solution of (7.4);
- if $\beta = 1$, $\rho(t,\cdot)$ is the unique weak solution of (7.5);
- if $\beta \in (1,\infty)$, in each cylinder $[0,T] \times [b_i, b_{i+1}]$, $\rho(t,\cdot)$ is the unique weak solution of (7.6).

Proposition 7.2.1. *As $N \uparrow \infty$, the sequence of probability measures $\{\mathbb{Q}^{\beta,N}_{\mu_N} : N \geq 1\}$ converges weakly to $\mathbb{Q}^\beta$.*

The proof of this result is divided into three parts. In the next section, we show that the sequence $\{\mathbb{Q}^{\beta,N}_{\mu_N} : N \geq 1\}$ is tight, for any $\beta \in [0,\infty)$. In Section 7.5 we characterize the limit points of this sequence for each regime of the parameter β. Uniqueness of weak solutions is presented in Section 7.6 and this implies the uniqueness of limit points of the sequence $\{\mathbb{Q}^{\beta,N}_{\mu_N} : N \geq 1\}$. In the fifth section, we prove a suitable *Replacement Lemma* for each regime of β, which is crucial in the task of characterizing limit points and uniqueness.

7.3 Tightness

Proposition 7.3.1. *For any fixed $\beta \in [0,\infty)$, the sequence of measures $\{\mathbb{Q}^{\beta,N}_{\mu_N} : N \geq 1\}$ is tight in the Skorohod topology of $D([0,T],\mathcal{M})$.*

Proof. In order to prove tightness of $\{\pi^N_t : 0 \leq t \leq T\}$ it is enough to show tightness of the real-valued processes $\{\langle \pi^N_t, H\rangle : 0 \leq t \leq T\}$ for $H \in C(\mathbb{T})$. In fact, c.f. [16] it is enough to show tightness of $\{\langle \pi^N_t, H\rangle : 0 \leq t \leq T\}$ for a dense set of functions in $C(\mathbb{T})$ with respect to the uniform topology. For that purpose, fix $H \in C^2(\mathbb{T})$. By Dynkin's formula,

$$M^N_t(H) = \langle \pi^N_t, H\rangle - \langle \pi^N_0, H\rangle - \int_0^t N^2 \mathcal{L}_N \langle \pi^N_s, H\rangle \, ds\,, \tag{7.9}$$

is a martingale with respect to the natural filtration $\mathcal{F}_t := \sigma(\eta_s : s \leq t)$. In order to prove tightness of $\{\langle \pi^N_t, H\rangle : N \geq 1\}$, we prove tightness of the sequence of the martingales and the integral terms in the decomposition above. We start by the former.

We begin by showing that the $L^2(\mathbb{P}^{\beta}_{\mu_N})$-norm of the martingale above vanishes as $N \to +\infty$. The quadratic variation of $M_t^N(H)$ is given by

$$\langle M^N(H)\rangle_t \;=\; \int_0^t \sum_{x\in\mathbb{T}_N} \xi^N_{x,x+1}\Big[(\eta_s(x)-\eta_s(x+1))(H(\tfrac{x+1}{N})-H(\tfrac{x}{N}))\Big]^2 ds. \tag{7.10}$$

It is easy to show that $\langle M^N(H)\rangle_t \;\leq\; \frac{T}{N}\|\partial_u H\|^2_\infty$. Here and in the sequel we use the notation $\|H\|_\infty := \sup_{u\in\mathbb{T}}|H(u)|$.

Thus, $M_t^N(H)$ converges to zero as $N\to+\infty$ in $L^2(\mathbb{P}^{\beta}_{\mu_N})$. Notice that above we used the trivial bound $\xi^N_{x,x+1}\leq 1$. By Doob's inequality, for every $\delta>0$,

$$\lim_{N\to\infty}\mathbb{P}^{\beta}_{\mu_N}\Bigg[\sup_{0\leq t\leq T}|M_t^N(H)|>\delta\Bigg]=0\,, \tag{7.11}$$

which implies tightness of the sequence of martingales $\{M_t^N(H); N\geq 1\}$. Now, we need to examine tightness of the integral term in (7.9).

Denote by Γ_N the subset of sites $x\in\mathbb{T}_N$ such that x has some adjacent slow bond, namely, $\xi^N_{x,x+1}=N^{-\beta}$ or $\xi^N_{x-1,x}=N^{-\beta}$. The term $N^2\mathcal{L}_N\langle\pi^N_s,H\rangle$ appearing inside the time integral in (7.9) is explicitly given by

$$\begin{aligned}
&N\sum_{x\notin\Gamma_N}\eta_s(x)\Big[H(\tfrac{x+1}{N})+H(\tfrac{x-1}{N})-2H(\tfrac{x}{N})\Big]\\
&+N\sum_{x\in\Gamma_N}\eta_s(x)\Big[\xi^N_{x,x+1}\{H(\tfrac{x+1}{N})-H(\tfrac{x}{N})\}+\xi^N_{x-1,x}\{H(\tfrac{x-1}{N})-H(\tfrac{x}{N})\}\Big]\,.
\end{aligned}$$

By Taylor expansion on H, the absolute value of the first sum above is bounded by $\|\partial^2_u H\|_\infty$. Since there are at most $2k$ elements in Γ_N, $\xi_{x,x+1}\leq 1$ and since there is only one particle per site, the absolute value of the second sum above is bounded by $2\,k\|\partial_u H\|_\infty$. Therefore, there exists a constant $C:=C(H,k)>0$, such that $|N^2\mathcal{L}_N\langle\pi^N_s,H\rangle|\leq C$, which yields

$$\Bigg|\int_r^t N^2\mathcal{L}_N\langle\pi^N_s,H\rangle ds\Bigg|\leq C|t-r|\,.$$

By Proposition 4.1.6 of [16], last inequality implies tightness of the integral term. This concludes the proof. □

7.4 Replacement Lemma and Sobolev Spaces

In this section, we obtain fundamental results that allow us to replace the mean occupation of a site by the mean density of particles in a small macroscopic box around this site. This result implies that the limit trajectories must belong to some Sobolev space, this will be clear later. Before proceeding we introduce some tools that we use in the sequel.

Denote by $H_N(\mu_N|\nu_\alpha)$ the entropy of a probability measure μ_N with respect to the invariant state ν_α. For a precise definition and properties of the entropy, we refer the reader to [16]. In Proposition A.1.8 in the Appendix we review a classical result saying that there exists a finite constant $K_0 := K_0(\alpha)$, such that

$$H_N(\mu_N|\nu_\alpha) \;\leq\; K_0 N, \tag{7.12}$$

for any probability measure $\mu_N \in \{0,1\}^{\mathbb{T}_N}$.

Denote by $\langle \cdot, \cdot \rangle_{\nu_\alpha}$ the scalar product of $L^2(\nu_\alpha)$ and denote by $\mathfrak{D}_N$ the Dirichlet form, which is the convex and lower semicontinuous functional (see Corollary A1.10.3 of [16]) defined as:

$$\mathfrak{D}_N(f) \;=\; \langle -L_N\sqrt{f}\,,\,\sqrt{f}\rangle_{\nu_\alpha},$$

where f is a probability density with respect to ν_α (i.e. $f \geq 0$ and $\int f d\nu_\alpha = 1$). An elementary computation shows that

$$\mathfrak{D}_N(f) \;=\; \sum_{x\in\mathbb{T}_N} \frac{\xi^N_{x,x+1}}{2} \int \Big(\sqrt{f(\eta^{x,x+1})} - \sqrt{f(\eta)}\Big)^2 d\nu_\alpha\,.$$

By Theorem A1.9.2 of [16], if $\{S_t^N : t \geq 0\}$ stands for the semi-group associated to the generator $N^2\mathcal{L}_N$, then

$$H_N(\mu_N S_t^N|\nu_\alpha) \;+\; N^2\int_0^t \mathfrak{D}_N(f_s^N)\,ds \;\leq\; H_N(\mu_N|\nu_\alpha)\,,$$

provided f_s^N stands for the Radon-Nikodym derivative of $\mu_N S_s^N$ (the distribution of η_s starting from μ_N) with respect to ν_α.

7.4.1 Replacement Lemma

Now, we define the local density of particles, which corresponds to the mean occupation in a box around a given site. We represent this empirical density in the box of size ℓ around a given site x by $\eta^\ell(x)$. For $\beta \in [0,1)$, this box can be chosen in the usual way, but for $\beta \in [1,\infty)$, this box must avoid the slow bond. From this point on, we denote the integer part of εN, namely $\lfloor \varepsilon N \rfloor$, simply by εN.

Definition 7.4.1. *For $\beta \in [0,1)$, define the empirical density by*

$$\eta^{\varepsilon N}(x) \;=\; \tfrac{1}{\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N} \eta(y)\,.$$

Definition 7.4.2. *For $\beta \in [1,\infty)$, if x is such that $\{Nb_1,\dots,Nb_k\} \cap \{x,\dots,x+\varepsilon N\} = \varnothing$, then the empirical density is defined by*

$$\eta^{\varepsilon N}(x) \;=\; \tfrac{1}{\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N} \eta(y)\,.$$

Otherwise, if, let us say, $Nb_i \in \{x,\dots,x+\varepsilon N\}$ for some $i = 1,..,k$, then the empirical density is defined by

$$\eta^{\varepsilon N}(x) \;=\; \tfrac{1}{\varepsilon N}\sum_{y=Nb_i-\varepsilon N+1}^{Nb_i} \eta(y)\,.$$

Since we are considering a finite number of slow bonds, the distance between two consecutive macroscopic points related to two consecutive slow bonds is at least ε, for ε sufficiently small. As a consequence, we can suppose, without lost of generality that in the previous definition, b_i is unique.

Lemma 7.4.1. *Fix $\beta \in [0,1)$. Let f be a density with respect to the invariant measure ν_α. Then,*

$$\int \{\eta(x) - \eta^{\varepsilon N}(x)\} f(\eta)\nu_\alpha(d\eta) \;\leq\; 2(kN^{\beta-1}+\varepsilon) + N\,\mathfrak{D}_N(f)\,, \forall x \in \mathbb{T}_N.$$

Proof. From Definition 7.4.1 we have that

$$\int \{\eta(x) - \eta^{\varepsilon N}(x)\} f(\eta)\nu_\alpha(d\eta) = \int \Big\{\tfrac{1}{\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}(\eta(x)-\eta(y))\Big\} f(\eta)\,\nu_\alpha(d\eta)\,.$$

Writing $\eta(x)-\eta(y)$ as a telescopic sum, the last expression becomes equal to

$$\int \Big\{\tfrac{1}{\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}\sum_{z=x}^{y-1}(\eta(z)-\eta(z+1))\Big\} f(\eta)\,\nu_\alpha(d\eta)\,.$$

Rewriting the expression above as twice the half and making the transformation $\eta \mapsto \eta^{z,z+1}$ (for which the probability ν_α is invariant) it becomes as:

$$\tfrac{1}{2\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}\sum_{z=x}^{y-1}\int \{\eta(z)-\eta(z+1)\}(f(\eta)-f(\eta^{z,z+1}))\,\nu_\alpha(d\eta)\,.$$

Since $(a-b) = (\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$ and by the Cauchy-Schwarz's inequality, for any $A>0$, we bound the previous expression from above by

$$\tfrac{1}{2\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}\sum_{z=x}^{y-1}\frac{A}{\xi^N_{z,z+1}}\int \{\eta(z)-\eta(z+1)\}^2\Big(\sqrt{f(\eta)}+\sqrt{f(\eta^{z,z+1})}\Big)^2\,\nu_\alpha(d\eta)$$
$$+\tfrac{1}{2\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}\sum_{z=x}^{y-1}\frac{\xi^N_{z,z+1}}{A}\int \Big(\sqrt{f(\eta)}-\sqrt{f(\eta^{z,z+1})}\Big)^2\,\nu_\alpha(d\eta)\,.$$

The second sum above is bounded by

$$\tfrac{1}{2\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}\sum_{z\in\mathbb{T}_N}\frac{\xi^N_{z,z+1}}{A}\int\Big(\sqrt{f(\eta)}-\sqrt{f(\eta^{z,z+1})}\Big)^2\nu_\alpha(d\eta)=\tfrac{1}{A}\mathfrak{D}_N(f)\,.$$

On the other hand, since f is a density, the first sum is bounded from above by

$$\tfrac{1}{2\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}\sum_{z=x}^{y-1}\frac{4A}{\xi^N_{z,z+1}}\leq\tfrac{1}{\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}2A(kN^\beta+\varepsilon N)=2A(kN^\beta+\varepsilon N)\,.$$

Notice that the term kN^β comes from the existence of k slow bonds. Choosing $A=\frac{1}{N}$, the proof ends. □

Lemma 7.4.2 (Replacement Lemma)**.** *Fix $\beta\in[0,1)$. Let $b\in\mathbb{T}$ and let x be the right (or left) vertex of the bond containing the macroscopic point b. Then,*

$$\varlimsup_{\varepsilon\to 0}\varlimsup_{N\to\infty}\mathbb{E}^\beta_{\mu_N}\Big[\Big|\int_0^t\{\eta_s(x)-\eta_s^{\varepsilon N}(x)\}\,ds\Big|\Big]\;=\;0\,.$$

Proof. From Jensen's inequality together with the entropy inequality (see for example Appendix 1 of [16]), for any $\gamma\in\mathbb{R}$ (which will be chosen large), the expectation appearing on the statement of the Lemma is bounded from above by

$$\frac{H_N(\mu_N|\nu_\alpha)}{\gamma N}+\frac{1}{\gamma N}\log\mathbb{E}_{\nu_\alpha}\Big[\exp\Big\{\gamma N\Big|\int_0^t\{\eta_s(x)-\eta_s^{\varepsilon N}(x)\}\,ds\Big|\Big\}\Big]\,. \tag{7.13}$$

By Proposition A.1.8, $H_N(\mu_N|\nu_\alpha)\leq K_0\,N$, so that it remains to focus on the second summand above. Since $e^{|x|}\leq e^x+e^{-x}$ and

$$\varlimsup_N\tfrac{1}{N}\log(a_N+b_N)=\max\Big\{\varlimsup_N\tfrac{1}{N}\log a_N,\varlimsup_N\tfrac{1}{N}\log b_N\Big\}\,, \tag{7.14}$$

we can remove the modulus inside the exponential. By Feynman-Kac's formula, see Lemma A1.7.2 of [16] and Proposition A.1.7, the second term on the right hand side of (7.13) is less than or equal to

$$t\sup_{f\text{ density}}\Big\{\int\{\eta(x)-\eta^{\varepsilon N}(x)\}f(\eta)\nu_\alpha(d\eta)-N\,\mathfrak{D}_N(f)\Big\}\,.$$

Applying Lemma 7.4.1 and recalling that γ is arbitrarily large, the proof finishes. □

The next two results are concerned with both cases $\beta=1$ and $\beta\in(1,\infty)$.

Lemma 7.4.3. *Fix $\beta\in[1,\infty)$. Let f be a density with respect to the invariant measure ν_α. Then,*

$$\int\{\eta(x)-\eta^{\varepsilon N}(x)\}f(\eta)\nu_\alpha(d\eta)\leq N\mathfrak{D}_N(f)+4\varepsilon\,,\forall x\in\mathbb{T}_N.$$

Moreover, given a function $H:\mathbb{T}\to\mathbb{R}$:

$$\tfrac{1}{N}\sum_{x\in\mathbb{T}_N}\int H(\tfrac{x}{N})\{\eta(x)-\eta^{\varepsilon N}(x)\}f(\eta)\nu_\alpha(d\eta)\leq N\mathfrak{D}_N(f)+\tfrac{4\varepsilon}{N}\sum_{x\in\mathbb{T}_N}\Big(H(\tfrac{x}{N})\Big)^2\,.$$

Proof. Recall the Definition 7.4.2. Let first x be a site such that there is no slow bond connecting two sites in $\{x,\dots,x+\varepsilon N\}$. In this case,

$$\begin{aligned}
&\int H(\tfrac{x}{N})\{\eta(x)-\eta^{\varepsilon N}(x)\}f(\eta)\nu_\alpha(d\eta)\\
&=\int H(\tfrac{x}{N})\Big\{\tfrac{1}{\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}(\eta(x)-\eta(y))\Big\}f(\eta)\nu_\alpha(d\eta)\,,
\end{aligned}$$

and following the same arguments as in Lemma 7.4.1, we bound the previous expression from above by

$$\begin{aligned}
&\tfrac{(H(\frac{x}{N}))^2}{2\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}\sum_{z=x}^{y-1}\int\frac{A}{\xi^N_{z,z+1}}\{\eta(z)-\eta(z+1)\}^2\Big(\sqrt{f(\eta)}+\sqrt{f(\eta^{z,z+1})}\Big)^2\nu_\alpha(d\eta)\\
&+\frac{1}{2\varepsilon N}\sum_{y=x+1}^{x+\varepsilon N}\sum_{z=x}^{y-1}\int\frac{\xi^N_{z,z+1}}{A}\{\eta(z)-\eta(z+1)\}^2\Big(\sqrt{f(\eta)}-\sqrt{f(\eta^{z,z+1})}\Big)^2\nu_\alpha(d\eta)\,.
\end{aligned}$$

Since $\xi^N_{z,z+1}=1$ for all $z\in\{x,\dots,x+\varepsilon N-1\}$, it yields the boundedness of the previous expression by

$$2\varepsilon NA\left(H(\tfrac{x}{N})\right)^2+\frac{\mathfrak{D}_N(f)}{A}\,.$$

Let now x be a site such that $Nb_i\in\{x,\dots,x+\varepsilon N\}$ for some $i=1,\dots,k$. In this case,

$$\begin{aligned}
&\int H(\tfrac{x}{N})\{\eta(x)-\eta^{\varepsilon N}(x)\}f(\eta)\nu_\alpha(d\eta)\\
&=\int H(\tfrac{x}{N})\frac{1}{\varepsilon N}\sum_{y=Nb_i-\varepsilon N+1}^{Nb_i}\Big\{\eta(x)-\eta(y)\Big\}f(\eta)\nu_\alpha(d\eta)
\end{aligned}\tag{7.15}$$

Now we split the last summation into two cases, $y>x$ and $y<x$ and then we proceed by writing $\eta(x)-\eta(y)$ as a telescopic sum as in Lemma 7.4.1. Then, by the same arguments of Lemma 7.4.1 and since $\xi^N_{z,z+1}=1$ for all z in the range $\{Nb_i-\varepsilon N+1,\dots,Nb_i-1\}$, we bound the previous expression by

$$4\varepsilon NA\Big(H(\tfrac{x}{N})\Big)^2+\frac{\mathfrak{D}_N(f)}{A}\,.$$

Now the first claim of the lemma follows by taking the particular case $H(\frac{x}{N})=1$ and choosing $A=\frac{1}{N}$.

Finally, if in (7.15) we sum over $x\in\mathbb{T}_N$ and then divide by N, one concludes the second claim of the lemma. □

Lemma 7.4.4 (Replacement Lemma)**.** *Fix $\beta\in[1,\infty)$. Then, for every $x\in\mathbb{T}_N$*

$$\varlimsup_{\varepsilon\to0}\varlimsup_{N\to\infty}\mathbb{E}^\beta_{\mu_N}\Big[\Big|\int_0^t\{\eta_s(x)-\eta_s^{\varepsilon N}(x)\}\,ds\Big|\Big]\;=\;0\,.$$

Moreover, given a function $H : \mathbb{T} \to \mathbb{R}$ satisfying

$$\varlimsup_{N\to\infty} \tfrac{1}{N} \sum_{x\in\mathbb{T}_N} \Big(H(\tfrac{x}{N})\Big)^2 \;<\; \infty\,,$$

also holds

$$\varlimsup_{\varepsilon\to 0}\varlimsup_{N\to\infty} \mathbb{E}^{\beta}_{\mu_N}\Big[\Big|\int_0^t \tfrac{1}{N}\sum_{x\in\mathbb{T}_N} H(\tfrac{x}{N})\{\eta_s(x)-\eta_s^{\varepsilon N}(x)\}\,ds\Big|\Big] \;=\; 0\,.$$

Proof. The proof follows exactly the same arguments in Lemma 7.4.2. Therefore, is sufficient to show that the expressions

$$t\sup_{f \text{ density}}\Big\{\int\{\eta(x)-\eta^{\varepsilon N}(x)\}f(\eta)d\nu_\alpha - N\mathfrak{D}_N(f)\Big\}$$

and

$$t\sup_{f \text{ density}}\Big\{\int \tfrac{1}{N}\sum_{x} H(\tfrac{x}{N})\{\eta(x)-\eta^{\varepsilon N}(x)\}f(\eta)d\nu_\alpha - N\mathfrak{D}_N(f)\Big\}\,,$$

vanish as $N\to+\infty$, which is an immediate consequence of Lemma 7.4.3. □

In the next subsection, we will need the following variation of Lemma 7.4.3:

Lemma 7.4.5. *Let $H : \mathbb{T}\to\mathbb{R}$ and let f be a density with respect to ν_α. Then, for every $x\in\mathbb{T}_N$*

$$\begin{aligned}&\int \frac{1}{\varepsilon N}\sum_{x\in\mathbb{T}_N} H(\tfrac{x}{N})\Big\{\eta(x)-\eta(x+\varepsilon N)\Big\}f(\eta)\,\nu_\alpha(d\eta)\\ &\le N\mathfrak{D}_N(f) + \frac{2}{\varepsilon N}\sum_{x\in\mathbb{T}_N}\Big(H(\tfrac{x}{N})\Big)^2\Big\{\varepsilon + N^{\beta-1}\sum_{i=1}^{k}\mathbf{1}_{[b_i,b_i+\varepsilon)}(\tfrac{x}{N})\Big\}\,.\end{aligned}$$

The proof of the last lemma follows the same steps as above and for that reason will be omitted. Nevertheless, we sketch the idea of the proof. One begins by writing $\eta(x)-\eta(x+\varepsilon N)$ as a telescopic sum and proceeding as in Lemma 7.4.3. The only relevant difference in this case is that is not possible to avoid the slow bonds inside the telescopic sum, and therefore the upper bound depends on β.

7.4.2 Sobolev Spaces

We prove in this subsection that any limit point $\mathbb{Q}^{\beta}_{*}$ of the sequence $\{\mathbb{Q}^{\beta,N}_{\mu_N} : N\ge 1\}$ is concentrated on trajectories $\rho(t,u)du$ with finite energy, meaning that $\rho(t,u)$ belongs to some Sobolev space. For $\beta\in[0,1)$, this result is an immediate consequence of the uniqueness of weak solutions of the heat equation. The case $\beta=1$ is a particular case of the one considered in [9]. Therefore, we will treat here the remaining case $\beta\in(1,\infty)$. Such result will play an important role in the uniqueness of weak solutions of (7.6).

Let $\mathbb{Q}^{\beta}_{*}$ be a limit point of $\{\mathbb{Q}^{\beta,N}_{\mu_N} : N \geq 1\}$ and assume without lost of generality that the whole sequence converges weakly to $\mathbb{Q}^{\beta}_{*}$.

Proposition 7.4.6. *The measure $\mathbb{Q}^{\beta}_{*}$ is concentrated on paths $\pi(t,u) = \rho(t,u)du$. Moreover, there exists a function in $L^2([0,T]\times\mathbb{T})$, denoted by $\partial_u\rho$, such that*

$$\int_0^T \int_{\mathbb{T}} (\partial_u H)(s,u)\,\rho(s,u)\,du\,ds \;=\; -\int_0^T \int_{\mathbb{T}} H(s,u)\,(\partial_u\rho)(s,u)\,du\,ds\,,$$

for all H in $C^{0,1}([0,T]\times\mathbb{T})$ whose support is contained in $[0,T]\times(\mathbb{T}\backslash\{b_1,\ldots,b_k\})$.

The previous result follows from the next lemma. Recall the definition of the constant K_0 given in (7.12).

Lemma 7.4.7.

$$E_{\mathbb{Q}^{\beta}_{*}}\Big[\sup_H \Big\{ \int_0^T \int_{\mathbb{T}} (\partial_u H)(s,u)\,\rho(s,u)\,du\,ds \\ -\;2\int_0^T \int_{\mathbb{T}} \Big(H(s,u)\Big)^2\,du\,ds\Big\}\Big] \;\leq\; K_0\,,$$

where the supremum is carried over all functions H in $C^{0,1}([0,T]\times\mathbb{T})$ with support contained in $[0,T]\times(\mathbb{T}\backslash\{b_1,\ldots,b_k\})$.

We start by showing Proposition 7.4.6 assuming the last result. Later and independently we will prove the previous lemma.

Proof of Proposition 7.4.6. Denote by $\ell : C^{0,1}([0,T]\times\mathbb{T}) \to \mathbb{R}$ the linear functional defined by

$$\ell(H) \;=\; \int_0^T \int_{\mathbb{T}} (\partial_u H)(s,u)\,\rho(s,u)\,du\,ds\,.$$

Since the set of functions $H \in C^{0,1}([0,T]\times\mathbb{T})$ with support contained in $[0,T]\times(\mathbb{T}\backslash\{b_1,\ldots,b_k\})$ is dense in $L^2([0,T]\times\mathbb{T})$ and since by Lemma 7.4.7, ℓ is a $\mathbb{Q}^{\beta}_{*}$-a.s. bounded functional in $C^{0,1}([0,T]\times\mathbb{T})$, we can extend it to a $\mathbb{Q}^{\beta}_{*}$-a.s. bounded functional in $L^2([0,T]\times\mathbb{T})$. In particular, by the Riesz Representation Theorem, there exists a function G in $L^2([0,T]\times\mathbb{T})$ such that

$$\ell(H) \;=\; -\int_0^T \int_{\mathbb{T}} H(s,u)\,G(s,u)\,du\,ds\,.$$

This finishes the proof. □

For a smooth function $H\colon \mathbb{T}\to\mathbb{R}$, $\varepsilon > 0$ and a positive integer N, define $V_N(\varepsilon, H, \eta)$ by

$$V_N(\varepsilon,H,\eta) \;=\; \tfrac{1}{\varepsilon N}\sum_{x\in\mathbb{T}_N} H(\tfrac{x}{N})\{\eta(x) - \eta(x+\varepsilon N)\} - \tfrac{2}{N}\sum_{x\in\mathbb{T}_N}\Big(H(\tfrac{x}{N})\Big)^2\,.$$

In order to prove the Lemma 7.4.7, we need the following technical result:

Lemma 7.4.8. *Consider $H_1, \dots, H_k$ functions in $C^{0,1}([0,T]\times\mathbb{T})$ with support contained in $[0,T]\times(\mathbb{T}\backslash\{b_1,\dots,b_k\})$. Hence, for every $\varepsilon>0$:*

$$\varlimsup_{\delta\to 0}\varlimsup_{N\to\infty}\mathbb{E}^{\beta}_{\mu^N}\Big[\max_{1\le i\le k}\Big\{\int_0^T V_N(\varepsilon,H_i(s,\cdot),\eta_s^{\delta N})\,ds\Big\}\Big]\;\le\;K_0\,. \tag{7.16}$$

Proof. It follows from Lemma 7.4.4 that in order to prove (7.16), we just need to show that

$$\varlimsup_{N\to\infty}\mathbb{E}^{\beta}_{\mu^N}\Big[\max_{1\le i\le k}\Big\{\int_0^T V_N(\varepsilon,H_i(s,\cdot),\eta_s)\,ds\Big\}\Big]\;\le\;K_0\,.$$

By the entropy and the Jensen's inequality, for each fixed N, the previous expectation is less than or equal to

$$\frac{H(\mu^N|\nu_\alpha)}{N}\;+\;\frac{1}{N}\log\mathbb{E}_{\nu_\alpha}\Big[\exp\Big\{\max_{1\le i\le k}N\int_0^T\;V_N(\varepsilon,H_i(s,\cdot),\eta_s)ds\Big\}\Big]\,.$$

By (7.12), the first term above is bounded by K_0. Since $\exp\{\max_{1\le j\le k}a_j\}$ is bounded from above by $\sum_{1\le j\le k}\exp\{a_j\}$ and by (7.14), the limit as $N\uparrow\infty$, of the second term of the previous expression is less than or equal to

$$\max_{1\le i\le k}\varlimsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}_{\nu_\alpha}\Big[\exp\Big\{N\int_0^T\;V_N(\varepsilon,H_i(s,\cdot),\eta_s)ds\Big\}\Big]\,.$$

We now prove that, for each fixed i the limit above is nonpositive.

Fix $1\le i\le k$. By the Feynman-Kac's formula and the variational formula for the largest eigenvalue of a symmetric operator, for each fixed N, the previous expectation is bounded from above by

$$\int_0^T\sup_f\Big\{\int V_N(\varepsilon,H_i(s,\cdot),\eta)f(\eta)\nu_\alpha(d\eta)-N\mathfrak{D}_N(f)\Big\}\,ds.$$

In last formula the supremum is taken over all probability densities f with respect to ν_α. By assumption, each of the functions $\{H_i: i=1,\dots,k\}$ vanishes in a neighborhood of each $b_i\in\mathbb{T}$. This together with Lemma 7.4.5, imply that the previous expression has nonpositive limsup. This is enough to conclude. $\square$

We define now an approximation of the identity in the continuous torus given by

$$\iota_\varepsilon(u,v)=\begin{cases}\frac{1}{\varepsilon}\,\mathbf{1}_{(v,v+\varepsilon)}(u)\,, & \text{if }\; v\in\mathbb{T}\backslash\cup_{i=1}^k(b_i-\varepsilon,b_i)\,,\\ \frac{1}{\varepsilon}\,\mathbf{1}_{(b_1-\varepsilon,b_1)}(u)\,, & \text{if }\; v\in(b_1-\varepsilon,b_1)\,,\\ \qquad\vdots & \qquad\qquad\vdots\\ \frac{1}{\varepsilon}\,\mathbf{1}_{(b_k-\varepsilon,b_k)}(u)\,, & \text{if }\; v\in(b_k-\varepsilon,b_k)\,.\end{cases} \tag{7.17}$$

The convolution of a measure π with ι_ε is defined by

$$(\pi * \iota_\varepsilon)(v) \;=\; \int \iota_\varepsilon(u,v)\,\pi(du)\,.$$

For a function ρ, the convolution $\rho * \iota_\varepsilon$ is understood as the convolution of the measure $\rho(u)\,du$ with ι_ε. Recall Definition 7.4.2. At this point, an important remark is the equality

$$\eta_t^{\varepsilon N}(x) = (\pi_t^N * \iota_\varepsilon)(\tfrac{x}{N})\,, \tag{7.18}$$

which is of straightforward verification.

Proof of Lemma 7.4.7. Consider a sequence $\{H_i : i \geq 1\}$ dense (with respect to the norm $\|H\|_\infty + \|\partial_u H\|_\infty$) in the subset of $C^{0,1}([0,T]\times\mathbb{T})$ of functions with support contained in $[0,T]\times(\mathbb{T}\backslash\{b_1,\dots,b_k\})$.

Recall that we suppose that $\{\mathbb{Q}_{\mu_N}^{\beta,N} : N\geq 1\}$ converges to $\mathbb{Q}_*^\beta$. By (7.16) and (7.18), for every $k\geq 1$,

$$\varlimsup_{\delta\to 0} E_{\mathbb{Q}_*^\beta}\Big[\max_{1\leq i\leq k}\Big\{\frac{1}{\varepsilon}\int_0^T\int_{\mathbb{T}} H_i(s,u)\,\Big\{\rho_s^\delta(u)-\rho_s^\delta(u+\varepsilon)\Big\}\,du\,ds \\ -\;2\int_0^T\int_{\mathbb{T}}(H_i(s,u))^2\,du\,ds\Big\}\Big] \;\leq\; K_0\,,$$

where $\rho_s^\delta(u) = (\rho_s * \iota_\delta)(u)$ as defined above. Letting $\delta\downarrow 0$, performing a change of variables and then letting $\varepsilon\downarrow 0$, we obtain that

$$E_{\mathbb{Q}_*^\beta}\Big[\max_{1\leq i\leq k}\Big\{\int_0^T\int_{\mathbb{T}}(\partial_u H_i)(s,u)\rho(s,u)\;du\,ds \\ -\;2\int_0^T\int_{\mathbb{T}}(H_i(s,u))^2\;du\,ds\Big\}\Big] \;\leq\; K_0\,.$$

To conclude the proof it remains to apply the Monotone Convergence Theorem and recall that $\{H_i : i\geq 1\}$ is a dense sequence (with respect to the norm $\|H\|_\infty + \|\partial_u H\|_\infty$) in the subset of functions of $C^{0,1}([0,T]\times\mathbb{T})$ with support contained in $[0,T]\times(\mathbb{T}\backslash\{b_1\dots,b_k\})$. □

Remark 7.4.9. *In terms of Sobolev spaces, we have just proved that, for $\beta\in(1,\infty)$, Q_*^β-almost surely, the limit trajectory $\rho(t,u)du$ is such that $\rho(t,u)$ belongs to $L^2(0,T;\mathcal{H}^1(b_i,b_{i+1}))$, in each cylinder $[0,T]\times(b_i,b_{i+1})$. Notice that in view of the presence of slow bonds and of Lemma 7.4.5 is it not possible to obtain the same result considering the whole space $L^2(0,T;\mathcal{H}^1(\mathbb{T}))$.*

7.5 Characterization of Limit Points

We prove in this section that all limit points $\mathbb{Q}_*^\beta$ of the sequence $\{\mathbb{Q}_{\mu_N}^{\beta,N} : N \geq 1\}$ are concentrated on trajectories of measures absolutely continuous with respect to the Lebesgue measure: $\pi(t,du) = \rho(t,u)du$, whose density $\rho(t,u)$ is a weak solution of the hydrodynamic equation (7.4), (7.5) or (7.6), for each corresponding value of β.

Let $\mathbb{Q}_*^\beta$ be a limit point of the sequence $\{\mathbb{Q}_{\mu_N}^{\beta,N} : N \geq 1\}$ and assume, without lost of generality, that $\{\mathbb{Q}_{\mu_N}^{\beta,N} : N \geq 1\}$ converges to $\mathbb{Q}_*^\beta$. The existence of $\mathbb{Q}_*^\beta$ is guaranteed by Proposition 7.3.1.

Since there is at most one particle per site, it is easy to show that $\mathbb{Q}_*^\beta$ is concentrated on trajectories $\pi_t(du)$ which are absolutely continuous with respect to the Lebesgue measure, $\pi_t(du) = \rho(t,u)du$ and whose density $\rho(\cdot)t,\cdot$ is non-negative and bounded by 1 (for more details see [16]). We distinguish the regime of β in different subsections below. In all the cases, we will make use of the martingale $M_t^N(H)$ defined in (7.9). By a simple change of variables, the integral term in (7.9) can be rewritten as a function of the empirical measure, such that:

$$M_t^N(H) \;=\; \langle \pi_t^N, H\rangle \;-\; \langle \pi_0^N, H\rangle \;-\; \int_0^t \langle \pi_s^N, N^2\,\mathbb{L}_N H\rangle\, ds\,, \tag{7.19}$$

where $\mathbb{L}_N$ was defined in (7.2).

We notice here that, for any choice of H, $M_t^N(H)$ is a martingale. In due course we impose extra conditions on H in order to identify the density $\rho(t,\cdot)$ as a weak solution of the corresponding weak equation depending on the regime of the parameter β.

7.5.1 Characterization of Limit Points for $\beta \in [0,1)$

Here, we want to show that $\rho(t,\cdot)$ is a weak solution of (7.4). Let $H \in C^2(\mathbb{T})$. We begin by claiming that

$$\mathbb{Q}_*^\beta\Big[\pi_\cdot : \langle \pi_t, H\rangle \;-\; \langle \pi_0, H\rangle \;-\; \int_0^t \langle \pi_s, \partial_u^2 H\rangle\, ds \;=\; 0, \forall t\in[0,T]\Big] \;=\; 1. \tag{7.20}$$

In order to prove the last claim, it is enough to show that, for every $\delta > 0$:

$$\mathbb{Q}_*^\beta\Big[\pi_\cdot : \sup_{0\leq t\leq T}\Big|\langle \pi_t, H\rangle \;-\; \langle \pi_0, H\rangle \;-\; \int_0^t \langle \pi_s, \partial_u^2 H\rangle\, ds\Big| \;>\; \delta\Big] \;=0.$$

By Portmanteau's Theorem and Proposition A.2.7, last probability is bounded from above by

$$\varliminf_{N\to\infty} \mathbb{Q}_{\mu_N}^{\beta,N}\Big[\pi_\cdot : \sup_{0\leq t\leq T}\Big|\langle \pi_t, H\rangle \;-\; \langle \pi_0, H\rangle \;-\; \int_0^t \langle \pi_s, \partial_u^2 H\rangle\, ds\Big| \;>\; \delta\Big]$$

since the supremum above is a continuous function in the Skorohod metric. Adding and subtracting $\langle \pi_s^N, N^2\,\mathbb{L}_N H\rangle$ in the integral term above and recalling the definition of $\mathbb{Q}_{\mu_N}^{\beta,N}$, the previous expression is bounded from above by

$$\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\langle \pi^N_t, H\rangle - \langle \pi^N_0, H\rangle - \int_0^t \langle \pi^N_s, N^2\,\mathbb{L}_N H\rangle\,ds\Big| > \delta/2\Big]$$
$$+\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\int_0^t \langle \pi^N_s, \partial_u^2 H - N^2\,\mathbb{L}_N H\rangle\,ds\Big| > \delta/2\Big].$$

By (7.19) and (7.11), the first term in last expression is null. By the definition of Γ_N given in Section 7.3 and since there is only one particle per site, the second term in last expression becomes bounded by

$$\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\tfrac{T}{N}\sum_{x\notin\Gamma_N}\Big|\partial_u^2 H\Big(\frac{x}{N}\Big) - N^2\,\mathbb{L}_N H\Big(\frac{x}{N}\Big)\Big| > \delta/4\Big]$$
$$+\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\int_0^t \tfrac{1}{N}\sum_{x\in\Gamma_N}\Big\{\partial_u^2 H(\tfrac{x}{N}) - N^2\,\mathbb{L}_N H(\tfrac{x}{N})\Big\}\,\eta_s(x)\,ds\Big| > \delta/4\Big].$$

Outside Γ_N, the operator $N^2\,\mathbb{L}_N$ coincides with the discrete Laplacian and since $H \in C^2(\mathbb{T})$, the first term in last expression is zero. Recall that there are $2k$ elements in Γ_N. Applying the triangular inequality, the second expression in the previous sum becomes bounded by

$$\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\tfrac{2kT}{N}\|\partial_u^2 H\|_\infty > \delta/8\Big]$$
$$+\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\sum_{x\in\Gamma_N}\int_0^t N\,\mathbb{L}_N H(\tfrac{x}{N})\,\eta_s(x)\,ds\Big| > \delta/8\Big].$$

For large N, the first probability vanishes. Now we deal with the second term. We associate to each slow bond containing a point b_i, a unique pair of sites in Γ_N, namely Nb_i and Nb_i+1. By the triangular inequality, in order to show that the second expression above is zero, it is sufficient to verify that

$$\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\int_0^t \{N\,\mathbb{L}_N H(\tfrac{Nb_i}{N})\,\eta_s(Nb_i)$$
$$+N\,\mathbb{L}_N H(\tfrac{Nb_i+1}{N})\,\eta_s(Nb_i+1)\}\,ds\Big| > \delta/8k\Big] = 0,$$

for each $i = 1,\dots,k$. The expression inside the integral above can be explicitly written as

$$\Big\{N\,[H(\tfrac{Nb_i-1}{N}) - H(\tfrac{Nb_i}{N})] + N^{1-\beta}\,[H(\tfrac{Nb_i+1}{N}) - H(\tfrac{Nb_i}{N})]\Big\}\,\eta_s(Nb_i)$$
$$+\Big\{N^{1-\beta}\,[H(\tfrac{Nb_i}{N}) - H(\tfrac{Nb_i+1}{N})] + N\,[H(\tfrac{Nb_i+2}{N}) - H(\tfrac{Nb_i+1}{N})]\Big\}\,\eta_s(Nb_i+1).$$

Since H is smooth and $\beta \in [0,1)$, the terms inside the parenthesis involving $N^{1-\beta}$ converge to zero and the terms involving N converge to plus or minus the space derivative of H at b_i. Therefore, again by the triangular inequality, it remains to show that, for any $\delta > 0$,

$$\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\int_0^t \partial_u H(b_i)\Big\{\eta_s(Nb_i) - \eta_s(Nb_i+1)\Big\}\,ds\Big| > \delta\Big] \tag{7.21}$$

equals to zero. The integral inside the probability above is continuous as a function of the time t. Moreover, it has a Lipschitz constant bounded by $|\partial_u H(b_i)|$. If $\partial_u H(b_i) = 0$, then there is nothing to do. Otherwise, let $t_0 = 0 < t_1 < \cdots < t_n = T$ be a partition of $[0,T]$ with mesh bounded by $\delta(|2\partial_u H(b_i)|)^{-1}$. Notice the partition is fixed, depending only on the function H. By the triangular inequality, (7.21) is bounded by

$$\sum_{j=0}^{n} \varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\Big|\int_0^{t_j} \partial_u H(b_i)\Big\{\eta_s(Nb_i) \;-\; \eta_s(Nb_i+1)\Big\}\,ds\Big| \;>\; \delta/2\Big].$$

Therefore, we just need to prove that, for any $\delta > 0$ and any $t \in [0,T]$

$$\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\Big|\int_0^{t} \Big\{\eta_s(Nb_i) \;-\; \eta_s(Nb_i+1)\Big\}\,ds\Big| \;>\; \delta\Big] \;=0.$$

Applying Markov's inequality, we bound the previous probability by

$$\delta^{-1}\,\mathbb{E}^{\beta}_{\mu_N}\Big[\Big|\int_0^{t} \Big\{\eta_s(Nb_i) \;-\; \eta_s(Nb_i+1)\Big\}\,ds\Big|\Big].$$

Now, in order to conclude it is enough to do the following. First add and subtract the empirical mean in the box of size εN around Nb_i and Nb_i+1. Then, by the triangular inequality and since $|\eta_s^{\varepsilon N}(x) - \eta_s^{\varepsilon N}(x+1)| \le \frac{2}{\varepsilon N}$, the term involving the two empirical means vanish. For the other two terms, we invoke Lemma 7.4.2. This finishes the claim.

Proposition 7.5.1. *For $\beta \in [0,1)$, any limit point of $\mathbb{Q}^{\beta,N}_{\mu_N}$ is concentrated in absolutely continuous paths $\pi_t(du) = \rho(t,u)\,du$, with positive density $\rho(t,\cdot)$ bounded by 1, such that $\rho(t,\cdot)$ is a weak solution of* (7.4).

Proof. Let $\{H_i : i \ge 1\}$ be a countable dense set of functions on $C^2(\mathbb{T})$, with respect to the norm $\|H\|_\infty + \|\partial_u^2 H\|_\infty$. Provided by (7.20) and intercepting a countable number of sets of probability one, is straightforward to extend (7.20) for all functions $H \in C^2(\mathbb{T})$ simultaneously. □

7.5.2 Characterization of Limit Points for $\beta = 1$

The idea in this case is to show that $\rho(t,\cdot)$ is an integral solution of (7.5) for a small domain of functions and then extend this set to $\mathcal{H}^1_W$.

Let $\mathcal{C}_W \subset \mathcal{H}^1_W$ be the set of functions H in $L^2(\mathbb{T})$ such that for $x \in \mathbb{T}$

$$H(x) \;=\; a \;+\; \int_{(0,x]} \Big(b + \int_0^y h(z)dz\Big) W(dy),$$

for some function h in $C(\mathbb{T})$ and $a, b \in \mathbb{R}$ satisfying

$$\int_0^1 h(x)\,dx \;=\; 0\,, \quad \int_{(0,1]} \Big(b + \int_0^y h(z)\,dz\Big) W(dy) \;=\; 0\,.$$

Note that a function in $\mathcal{C}_W$ is continuous in $\mathbb{T}\backslash\{b_1,...,b_k\}$ and well defined everywhere. Now, fix a function $H \in \mathcal{C}_W$ and define the martingale $M_t^N(H)$ as in (7.9). We aim that, for every $\delta > 0$, the result in (7.11) holds for $H \in \mathcal{C}_W$. In fact, this was already shown, for $H \in C^2(\mathbb{T})$, in the proof of Proposition 7.3.1. By (7.10), for $t \in [0,T]$

$$\langle M^N(H)\rangle_t \leq T \sum_{x\in\mathbb{T}_N} \xi^N_{x,x+1}\Big[H(\tfrac{x+1}{N}) - H(\tfrac{x}{N})\Big]^2.$$

Since $H \in \mathcal{C}_W$, H is differentiable with bounded derivative, except at the points $b_1,\dots,b_k$. Therefore, for any pair $x, x+1$ such that there is no b_i between $\frac{x}{N}$ and $\frac{x+1}{N}$, the following inequality holds

$$\xi^N_{x,x+1}\Big[\,H(\tfrac{x+1}{N}) - H(\tfrac{x}{N})\,\Big]^2 \leq \frac{1}{N^2}\|\partial_u^2 H\|^2_\infty.$$

On the other hand, if there is some $\{b_i : i = 1,..,k\}$ in the interval $[\frac{x}{N}, \frac{x+1}{N})$, then $\xi^N_{x,x+1} = N^{-\beta}$ and in this case we get to:

$$\xi^N_{x,x+1}\Big[\,H(\tfrac{x+1}{N}) - H(\tfrac{x}{N})\,\Big]^2 \leq \frac{4}{N^{2\beta}}\|H\|^2_\infty\,.$$

Since there are only finite k slow bonds, we conclude that the quadratic variation of $M_t^N(H)$ vanishes as $N \to \infty$. Now, Doob's inequality is enough to conclude. As above, by a simple change of variables, we may rewrite the martingale $M_t^N(H)$ in terms of the empirical measure as in (7.19). Now we want to analyze the integral term in the martingale decomposition (7.19).

Lemma 7.5.2. *For any $H \in \mathcal{C}_W$,*

$$\lim_{N\to\infty} \tfrac{1}{N} \sum_{x\in\mathbb{T}_N} \Big|\, N^2\mathbb{L}_N H(\tfrac{x}{N}) - \tfrac{d}{dx}\tfrac{d}{dW} H(\tfrac{x}{N})\,\Big| \;=\; 0\,.$$

Proof. Recall the definition of the set Γ_N given in Section 7.3 and rewrite the previous sum as

$$\tfrac{1}{N} \sum_{x\notin\Gamma_N} \Big|\, N^2\mathbb{L}_N H(\tfrac{x}{N}) - \tfrac{d}{dx}\tfrac{d}{dW} H(\tfrac{x}{N})\Big| + \tfrac{1}{N} \sum_{x\in\Gamma_N} \Big|\, N^2\mathbb{L}_N H(\tfrac{x}{N}) - \tfrac{d}{dx}\tfrac{d}{dW} H(\tfrac{x}{N})\Big|. \tag{7.22}$$

Outside $b_1,\dots,b_k$, the operator $\frac{d}{dx}\frac{d}{dW}$ coincides with the Laplacian, and outside Γ_N, the discrete operator $N^2\,\mathbb{L}_N$ coincides with the discrete Laplacian. Hence, the first term above is equal to

$$\tfrac{1}{N} \sum_{x\notin\Gamma_N} \Big|\, N^2\Big(H(\tfrac{x+1}{N}) + H(\tfrac{x-1}{N}) - 2H(\tfrac{x}{N})\Big) - \partial_u^2 H(\tfrac{x}{N})\,\Big|\,.$$

It is easy to verify that $H \in C^2(\mathbb{T}\backslash\{b_1,\dots,b_k\})$ and has bounded derivatives. Thus, by a Taylor expansion on H, it follows that the previous sum converges to zero as $N \to +\infty$. On the other hand, the second sum in (7.22) is bounded by the sum of

$$\tfrac{1}{N} \sum_{x\in\Gamma_N} \Big|\, \tfrac{d}{dx}\tfrac{d}{dW} H(\tfrac{x}{N})\,\Big|$$

and

$$\sum_{x\in\Gamma_N}\Big|\,N\xi^N_{x,x+1}\Big[H(\tfrac{x+1}{N})-H(\tfrac{x}{N})\Big]+N\xi^N_{x-1,x}\Big[H(\tfrac{x-1}{N})-H(\tfrac{x}{N})\Big]\,\Big|\,.$$

Since $H\in C_W$, $\frac{d}{dx}\frac{d}{W}H$ is a continuous function, therefore bounded. Since Γ_N has k elements, the first sum above converges to zero as $N\to+\infty$. It remains to analyze the second sum above, where now the definition of the domain C_W is crucial. For each $x\in\Gamma_N$, one of the conductances above is equal to N^{-1}. Let us suppose that $\xi^N_{x,x+1}=N^{-1}$ and $\xi^N_{x-1,x}=1$, the other case being completely analogous. In this case, there exists some $b_i\in(\frac{x}{N},\frac{x+1}{N}]$. From the definition of C_W and the measure W, the function H has a discontinuity at b_i of size

$$\int_0^{b_i}h(dz)\,dz\,.$$

Besides that, the function H has also sided-derivatives at b_i of the same value. With this in mind, is easy to see that

$$[H(\tfrac{x+1}{N})-H(\tfrac{x}{N})]+N[H(\tfrac{x-1}{N})-H(\tfrac{x}{N})]$$

converges to zero as $N\to\infty$. Recalling there are finite $2k$ elements in Γ_N, we finish the proof of the lemma. □

Now, fix $H\in C_W$ and take a continuous function H^ε which coincides with H in $\mathbb{T}\setminus\cup_{i=1}^k(b_i-\varepsilon,b_i+\varepsilon)$ and that $\|H^\varepsilon\|_\infty\le\|H\|_\infty$. The choice of ε will be determined later. Notice that

$$\sup_{0\le t\le T}|\langle\pi_t,H^\varepsilon-H\rangle|\le\sup_{0\le t\le T}\sum_{i=1}^k\int_{(b_i-\varepsilon,b_i+\varepsilon)}\rho(t,u)\,|H^\varepsilon(u)-H(u)|\,du\le 4\,k\,\varepsilon\,\|H\|_\infty\,.$$

For every $\delta>0$,

$$\mathbb{Q}^\beta_*\Big[\pi.:\sup_{0\le t\le T}\Big|\langle\pi_t,H\rangle-\langle\pi_0,H\rangle-\int_0^t\langle\pi_s,\tfrac{d}{dx}\tfrac{d}{dW}H\rangle\,ds\Big|>\delta\Big]\tag{7.23}$$

$$\le\mathbb{Q}^\beta_*\Big[\pi.:\sup_{0\le t\le T}\Big|\langle\pi_t,H^\varepsilon\rangle-\langle\pi_0,H^\varepsilon\rangle-\int_0^t\langle\pi_s,\tfrac{d}{dx}\tfrac{d}{dW}H\rangle\,ds\Big|>\delta/3\Big]$$
$$+2\,\mathbb{Q}^\beta_*\Big[\pi.:\sup_{0\le t\le T}\Big|\langle\pi_t,H^\varepsilon-H\rangle\Big|>\delta/3\Big]\,.$$

By a suitable choice of ε, the second probability in the sum above is null. Since H^ε and $\frac{d}{dx}\frac{d}{dW}H$ are continuous, by the Portmanteau's Theorem and Proposition A.2.7, it holds that

$$\mathbb{Q}^\beta_*\Big[\pi:\sup_{0\le t\le T}\Big|\langle\pi_t,H^\varepsilon\rangle-\langle\pi_0,H^\varepsilon\rangle-\int_0^t\langle\pi_s,\tfrac{d}{dx}\tfrac{d}{dW}H\rangle\,ds\Big|>\delta/3\Big]$$
$$\le\varliminf_{N\to\infty}\mathbb{Q}^{\beta,N}_{\mu_N}\Big[\pi:\sup_{0\le t\le T}\Big|\langle\pi_t,H^\varepsilon\rangle-\langle\pi_0,H^\varepsilon\rangle-\int_0^t\langle\pi_s,\tfrac{d}{dx}\tfrac{d}{dW}H\rangle\,ds\Big|>\delta/3\Big]$$
$$=\varliminf_{N\to\infty}\mathbb{P}^\beta_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\langle\pi^N_t,H^\varepsilon\rangle-\langle\pi^N_0,H^\varepsilon\rangle-\int_0^t\langle\pi^N_s,\tfrac{d}{dx}\tfrac{d}{dW}H\rangle\,ds\Big|>\delta/3\Big]\,.$$

Notice that the last equality is just the definition of the measure $\mathbb{Q}^{\beta,N}_{\mu_N}$. Since there is only one particle per site, it holds that $\sup_{0\le t\le T}\big|\langle \pi^N_t, H^\varepsilon - H\rangle\big| \le 4\,k\,\varepsilon\|H\|_\infty$, since H^ε coincides with H in $\mathbb{T}\backslash \cup_{i=1}^k (b_i-\varepsilon, b_i+\varepsilon)$. Adding and subtracting $\langle \pi^N_s, N^2\,\mathbb{L}_N H\rangle$, $\langle \pi^N_t, H\rangle$ and $\langle \pi^N_0, H\rangle$, we obtain that

$$
\begin{aligned}
&\varlimsup_{N\to\infty} \mathbb{P}^\beta_{\mu_N}\Big[\sup_{0\le t\le T} |\langle \pi^N_t, H^\varepsilon\rangle \;-\; \langle \pi^N_0, H^\varepsilon\rangle \;-\; \int_0^t \langle \pi^N_s\,,\, \tfrac{d}{dx}\tfrac{d}{dW} H\rangle\, ds| \;>\; \delta/3\Big]\\
&\le \varlimsup_{N\to\infty} \mathbb{P}^\beta_{\mu_N}\Big[\sup_{0\le t\le T} \Big|\langle \pi^N_t, H\rangle \;-\; \langle \pi^N_0, H\rangle \;-\; \int_0^t \langle \pi^N_s, N^2\,\mathbb{L}_N H\rangle\, ds\Big| \;>\; \delta/12\Big]\\
&+ \varlimsup_{N\to\infty} \mathbb{P}^\beta_{\mu_N}\Big[\tfrac{1}{N}\sum_{x\in\mathbb{T}_N} \Big|N^2 \mathbb{L}_N H(\tfrac{x}{N}) - \tfrac{d}{dx}\tfrac{d}{dW} H(\tfrac{x}{N})\Big| \;>\; \delta/12\Big]\\
&+ 2\varlimsup_{N\to\infty} \mathbb{P}^\beta_{\mu_N}\Big[\sup_{0\le t\le T} \Big|\langle \pi^N_t, H^\varepsilon - H\rangle\Big| \;>\; \delta/12\Big]\,.
\end{aligned}
$$

With another suitable choice of ε, the third probability in the sum above is null. Lemma 7.5.2 implies that the second probability above is zero for N sufficiently large. Recall we proved that (7.11) holds for $H\in\mathcal{C}_W$, so that the first term in the sum above is zero. Finally, from the previous computations we conclude that (7.23) is zero for any $\delta>0$. Therefore, $\mathbb{Q}^\beta_*$ is concentrated on absolutely continuous paths $\pi_t(du)=\rho(t,u)\,du$ with positive density bounded by 1 and for any fixed $H\in\mathcal{C}_W$, $\mathbb{Q}^\beta_*$ a.s.

$$
\langle \rho_t, H\rangle - \langle \rho_0, H\rangle \;=\; \int_0^t \Big\langle \rho_s\,,\, \tfrac{d}{dx}\tfrac{d}{dW} H\Big\rangle\, ds\,, \qquad \text{for all } t\in[0,T]\,. \tag{7.24}
$$

Proposition 7.5.3. *For $\beta=1$, any limit point of $\mathbb{Q}^{\beta,N}_{\mu_N}$ is concentrated in absolutely continuous paths $\pi_t(du)=\rho(t,u)\,du$, with positive density $\rho(t,\cdot)$ bounded by 1, such that $\rho(t,\cdot)$ is a weak solution of (7.5).*

Proof. By a density argument, (7.24) also holds, Q^β_* a.s., for all $H\in C_W$ simultaneously. It remains to extend (7.24) for $H\in\mathcal{H}^1_W$. For that purpose fix $H\in\mathcal{H}^1_W$. Thus, for $x\in\mathbb{T}$

$$
H(x) = \alpha + \int_{(0,x]} \Big(\beta + \int_0^y h(z)\,dz\Big)\, W(dy)\,,
$$

with $\alpha,\beta\in\mathbb{R}$, $h\in L^2(\mathbb{T})$ satisfying (7.3). Let $h_n\in C(\mathbb{T})$ converging to $h\in L^2(\mathbb{T})$. Define

$$
H_n(x) = \alpha_n + \int_{(0,x]} \Big(\beta_n + \int_0^y h_n(z)\,dz\Big)\, W(dy)\,,
$$

where $\alpha_n\to\alpha$ and $\beta_n\to\beta$. By the Dominated Convergence Theorem, it follows that H_n converges uniformly to H. Therefore (7.24) is true for all $H\in\mathcal{H}^1_W$. □

7.5.3 Characterization of Limit Points for $\beta \in (1,\infty)$

In this regime of the parameter β, Proposition 7.4.6 says that Q_*^β is concentrated on trajectories absolutely continuous with respect to the Lebesgue measure $\pi_t(du) = \rho(t,u)\,du$ such that, for each interval (b_i,b_{i+1}), $\rho(t,\cdot)$ belongs to $L^2(0,T;\mathcal{H}^1(b_i,b_{i+1}))$. It is well known that the Sobolev space $\mathcal{H}^1(a,b)$ has the following properties: all its elements are absolutely continuous functions with bounded variation, c.f. [4] and [18], therefore with lateral limits well-defined. Such property is inherited by $L^2(0,T;\mathcal{H}^1(b_i,b_{i+1}))$ in the sense that we can integrate in time the lateral limits. Therefore, $Q_*^\beta a.s.$, for each $i=1,\dots,k$ and for any $t\in[0,T]$:

$$\int_0^t \rho(s,b_i^+)\,ds < \infty \quad \text{and} \quad \int_0^t \rho(s,b_{i+1}^-)\,ds < \infty.$$

To simplify notation, in this subsection we denote $a=b_i$ and $b=b_{i+1}$. Fix $h\in C^2(\mathbb{T})$ and define $H:[0,T]\times\mathbb{T}\to\mathbb{R}$ by $H(t,u) = h(t,u)\,\mathbf{1}_{[a,b]}(u)$.

Recall that $\pi_t(du)=\rho(t,u)du$. We begin by claiming that

$$\begin{aligned}\mathbb{Q}_*^\beta\Big[\pi_\cdot:\ &\langle\rho_t,H_t\rangle - \langle\rho_0,H_0\rangle - \int_0^t \langle\rho_s,\partial_u^2H_s+\partial_sH_s\rangle\,ds\\ &-\int_0^t \partial_uH(s,a^+)\,\rho(s,a^+)\,ds + \int_0^t \partial_uH(s,b^-)\,\rho(s,b^-)\,ds = 0, \forall t\in[0,T]\Big] = 1\,.\end{aligned} \tag{7.25}$$

In order to prove (7.25), it is enough to show that, for every $\delta>0$

$$\begin{aligned}\mathbb{Q}_*^\beta\Big[\pi:\ \sup_{0\le t\le T}\Big|&\langle\rho_t,H_t\rangle - \langle\rho_0,H_0\rangle - \int_0^t \langle\rho_s,\partial_u^2H_s+\partial_sH_s\rangle\,ds\\ &-\int_0^t \partial_uH(s,a^+)\,\rho(s,a^+)\,ds + \int_0^t \partial_uH(s,b^-)\,\rho(s,b^-)\,ds\Big| > \delta\Big] = 0\,.\end{aligned}$$

Since the boundary integrals are not well-defined in the whole Skorohod space $D([0,T],\mathcal{M})$, we cannot use directly Portmanteau's Theorem. To avoid this technical obstacle, fix $\varepsilon>0$, which will be taken small later. Adding and subtracting the convolution of $\rho(t,u)$ with ι_ε, the probability above is less than or equal to the sum of

$$\begin{aligned}\mathbb{Q}_*^\beta\Big[\pi_\cdot:\ \sup_{0\le t\le T}\Big|&\langle\rho_t,H_t\rangle - \langle\rho_0,H_0\rangle - \int_0^t \langle\rho_s,\partial_u^2H_s+\partial_sH_s\rangle\,ds\\ &-\int_0^t \partial_uH(s,a^+)\,(\rho_s*\iota_\varepsilon)(a)\,ds + \int_0^t \partial_uH(s,b^-)\,(\rho_s*\iota_\varepsilon)(b-\varepsilon)\,ds\Big| > \delta/2\Big]\end{aligned} \tag{7.26}$$

and

$$\begin{aligned}\mathbb{Q}_*^\beta\Big[\pi:\ \sup_{0\le t\le T}\Big|&\int_0^t \partial_uH(s,a^+)\,(\rho_s*\iota_\varepsilon)(a)\,ds - \int_0^t \partial_uH(s,b^-)\,(\rho_s*\iota_\varepsilon)(b-\varepsilon)\,ds\\ &-\int_0^t \partial_uH(s,a^+)\,\rho(s,a^+)\,ds + \int_0^t \partial_uH(s,b^-)\,\rho(s,b^-)\,ds\Big| > \delta/2\Big].\end{aligned}$$

where ι_ε and the convolution $\rho*\iota_\varepsilon$ were defined in (7.17). The convolutions above are suitable averages of ρ around the boundary points a and b. Therefore, as $\varepsilon \downarrow 0$, the set inside the previous probability decreases to a set of null probability. It remains to deal with (7.26).

By Portmanteau's Theorem, Proposition A.2.7 and since there is only one particle per site, (7.26) is bounded from above by

$$\begin{aligned}\varliminf_{N\to\infty} \; \mathbb{Q}^{\beta,N}_{\mu_N}\Big[\pi_\cdot : \sup_{0\le t\le T}\Big|\langle \pi_t, H\rangle \;-\; \langle \pi_0, H_0\rangle \;-\; \int_0^t \langle \pi_s, \partial_u^2 H_s + \partial_s H_s\rangle\, ds \\ - \int_0^t \partial_u H(s,a^+)\,(\pi_s * \iota_\varepsilon)(a)\, ds + \int_0^t \partial_u H(s,b^-)\,(\pi_s * \iota_\varepsilon)(b-\varepsilon)\, ds\Big| \;>\; \delta/2\Big]\,.\end{aligned}$$

Now, by the definition of $\mathbb{Q}^{\beta,N}_{\mu_N}$, we can rewrite the previous expression as

$$\begin{aligned}\varliminf_{N\to\infty} \; \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\langle \pi^N_t, H_t\rangle \;-\; \langle \pi^N_0, H_0\rangle \;-\; \int_0^t \langle \pi^N_s, \partial_u^2 H_s + \partial_s H_s\rangle\, ds \\ - \int_0^t \partial_u H(s,a^+)\,\eta_s^{\varepsilon N}(Na+1)\, ds + \int_0^t \partial_u H(s,b^-)\,\eta_s^{\varepsilon N}(Nb)\, ds\Big| \;>\; \delta/2\Big]\,.\end{aligned}$$

If we consider the discrete torus as embedded in the continuous torus, $Na+1$ is the closest site to the right of a and Nb is the closest site to the left of b. The next step is to add and subtract $\langle \pi^N_s, N^2\,\mathbb{L}_N H\rangle$ and the previous probability becomes now bounded from above by the sum of

$$\varlimsup_{N\to\infty} \; \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\langle \pi^N_t, H_t\rangle \;-\; \langle \pi^N_0, H_0\rangle - \int_0^t \langle \pi^N_s, N^2\,\mathbb{L}_N H_s + \partial_s H_s\rangle\, ds\Big| \;>\; \delta/4\Big]$$

and

$$\begin{aligned}\varlimsup_{N\to\infty} \; \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\int_0^t \langle \pi^N_s, N^2\,\mathbb{L}_N H_s\rangle\, ds \;-\; \int_0^t \langle \pi^N_s, \partial_u^2 H_s\rangle\, ds \\ - \int_0^t \partial_u H(s,a^+)\,\eta_s^{\varepsilon N}(Na+1)\, ds + \int_0^t \partial_u H(s,b^-)\,\eta_s^{\varepsilon N}(Nb)\, ds\Big| \;>\; \delta/4\Big]\,.\end{aligned}$$

Repeating similar computations to the ones performed in Section 7.3 we can show (7.11) for a test function H that depends also on time. Therefore the first probability above is null. Now we focus on showing that the second probability above is null. Recalling the definition of $H(s,\cdot)$ above, we have that $H(s,\cdot)$ is zero outside the interval $[a,b]$. Besides that, for the set of vertices $\{Na+2,\dots,Nb-1\}$, the discrete operator $N^2\,\mathbb{L}_N$ coincides with the discrete Laplacian, which applied to $H(s,\cdot)$ converges uniformly to the continuous Laplacian of $H(s,\cdot)$. Hence, by the triangular inequality, it is enough to show that, for any $\delta>0$:

$$\begin{aligned}\overline{\lim_{N\to\infty}}\ \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|&\tfrac{1}{N}\int_0^t\{N^2\,\mathbb{L}_N H_s(\tfrac{Na}{N})-\partial_u^2 H_s(\tfrac{Na}{N})\}\,\eta_s(Na)\,ds\\
&+\tfrac{1}{N}\int_0^t\{N^2\,\mathbb{L}_N H_s(\tfrac{Na+1}{N})-\partial_u^2 H_s(\tfrac{Na+1}{N})\}\,\eta_s(Na+1)\,ds\\
&+\tfrac{1}{N}\int_0^t\{N^2\,\mathbb{L}_N H_s(\tfrac{Nb}{N})-\partial_u^2 H_s(\tfrac{Nb}{N})\}\,\eta_s(Nb)\,ds\\
&+\tfrac{1}{N}\int_0^t\{N^2\,\mathbb{L}_N H_s(\tfrac{Nb+1}{N})-\partial_u^2 H_s(\tfrac{Nb+1}{N})\}\,\eta_s(Nb+1)\,ds\\
&-\int_0^t\partial_u H(s,a^+)\,\eta_s^{\varepsilon N}(Na+1)\,ds+\int_0^t\partial_u H(s,b^-)\,\eta_s^{\varepsilon N}(Nb)\,ds\Big|\;>\delta\Big]=0.\end{aligned}$$

Since $h\in C^2(\mathbb{T})$, the term involving the Laplacian above is bounded. Now, by the triangular inequality, it is sufficient to show that, for any $\delta>0$:

$$\begin{aligned}\overline{\lim_{N\to\infty}}\ \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|&\int_0^t N\mathbb{L}_N H_s(\tfrac{Na}{N})\,\eta_s(Na)ds+\int_0^t N\mathbb{L}_N H_s(\tfrac{Na+1}{N})\,\eta_s(Na+1)ds\\
&+\int_0^t N\,\mathbb{L}_N H_s(\tfrac{Nb}{N})\,\eta_s(Nb)\,ds+\int_0^t N\,\mathbb{L}_N H_s(\tfrac{Nb+1}{N})\,\eta_s(Nb+1)\,ds\\
&-\int_0^t\partial_u H(s,a^+)\,\eta_s^{\varepsilon N}(Na+1)\,ds+\int_0^t\partial_u H(s,b^-)\,\eta_s^{\varepsilon N}(Nb)\,ds\Big|\;>\;\delta\Big]\;=0.\end{aligned}$$

For each one of the four vertices appearing inside the previous probability, the operator $\mathbb{L}_N$ has two conductances, one equals to $N^{-\beta}$ and the other equals to 1. Since $\beta>1$, the terms involving $N^{-\beta}$ converge to zero. The terms involving the conductances equal to 1, converge to plus or minus the lateral space derivatives of H. Recall from definition of H that $\partial_u H(s,a^-)=\partial_u H(s,b^+)=0$ for all $0\le s\le t$. From this, it remains to show that for any $\delta>0$

$$\begin{aligned}\overline{\lim_{N\to\infty}}\ \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|&\int_0^t\partial_u H(s,a^+)\,\eta_s(Na+1)\,ds-\int_0^t\partial_u H(s,b^-)\,\eta_s(Nb)\,ds\\
&-\int_0^t\partial_u H(s,a^+)\,\eta_s^{\varepsilon N}(Na+1)\,ds+\int_0^t\partial_u H(s,b^-)\,\eta_s^{\varepsilon N}(Nb)\,ds\Big|\;>\;\delta\Big],\end{aligned}$$

is null. Last expression is bounded from above by

$$\begin{aligned}&\overline{\lim_{N\to\infty}}\ \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\int_0^t\partial_u H(s,a^+)\Big\{\eta_s(Na+1)-\eta_s^{\varepsilon N}(Na+1)\Big\}\,ds\Big|\;>\;\delta/2\Big]\\
&+\overline{\lim_{N\to\infty}}\ \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\int_0^t\partial_u H(s,b^-)\Big\{\eta_s(Nb)-\eta_s^{\varepsilon N}(Nb)\Big\}\,ds\Big|\;>\;\delta/2\Big].\end{aligned}$$

The integral inside the probability above is a continuous function of the time t. Moreover, it has a bounded Lipschitz constant. The same argument as the one used in (7.21) together with Lemma 7.4.4 imply that the previous expression converges to zero when $\varepsilon \downarrow 0$, which proves (7.25).

Proposition 7.5.4. *For $\beta \in (1,\infty)$, any limit point of $\{\mathbb{Q}^{\beta,N}_{\mu_N} : N \geq 1\}$ is concentrated in absolutely continuous paths $\pi_t(du) = \rho(t,u)\,du$, with positive density $\rho(t,\cdot)$ bounded by 1, such that $\rho(t,\cdot)$ is a weak solution of (7.6) in each cylinder $[0,T] \times [b_i, b_{i+1}]$.*

Proof. Given (7.25), it remains to extend the result for all functions H and all cylinders $[0,T] \times [b_i, b_{i+1}]$ simultaneously. Intercepting a countable number of sets of probability one and applying a density argument as in Proposition 7.5.1, the statement follows. □

7.6 Uniqueness of Weak Solutions

The uniqueness of weak solutions of (7.4) is standard and we refer to [16] for a proof. It remains to prove uniqueness of weak solutions of the parabolic differential equations (7.5) and (7.6). In both cases, by linearity it suffices to check the uniqueness for $\gamma(\cdot) \equiv 0$. Notice that existence of weak solutions of (7.4), (7.5) and (7.6) is guaranteed by tightness of the process as proved in Section 7.3, together with the characterization of limit points as proved in Section 7.5.

7.6.1 Uniqueness of weak solutions of (7.5)

Let $\rho : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}$ be a weak solution of (7.5) with $\gamma \equiv 0$. By Definition 7.1.3, for all $H \in \mathcal{H}^1_W$ and all $t > 0$

$$\langle \rho_t, H \rangle \;=\; \int_0^t \Big\langle \rho_s, \frac{d}{dx}\frac{d}{dW} H \Big\rangle \, ds \,. \tag{7.27}$$

From Theorem 1 of [9], the operator $-\frac{d}{dx}\frac{d}{dW}$ has a countable number of eigenvalues $\{\lambda_n : n \geq 0\}$ and eigenvectors $\{F_n : n \geq 0\}$. All eigenvalues have finite multiplicity, $0 = \lambda_0 \leq \lambda_1 \leq \cdots$ and $\lim_{n\to\infty} \lambda_n = \infty$. Moreover, the eigenvectors $\{F_n : n \geq 0\}$ form a complete orthonormal system in $L^2(\mathbb{T})$. For $t > 0$, define

$$R(t) = \sum_{n\in\mathbb{N}} \frac{1}{n^2(1+\lambda_n)} \langle \rho_t, F_n \rangle^2 .$$

Notice that $R(0) = 0$ and since ρ_t belongs to $L^2(\mathbb{T})$, $R(t)$ is well defined for all $t \geq 0$. By (7.27), it follows that $\frac{d}{dt}\langle \rho_t, F_n\rangle^2 = -2\lambda_n \langle \rho_t, F_n \rangle^2$. Thus

$$(\tfrac{d}{dt}R)(t) = -\sum_{n\in\mathbb{N}} \frac{2\lambda_n}{n^2(1+\lambda_n)} \langle \rho_t, F_n \rangle^2 \,,$$

because $\sum_{n\leq N} \frac{-2\lambda_n}{n^2(1+\lambda_n)} \langle \rho_t, F_n \rangle^2$ converges uniformly to $\sum_{n\in\mathbb{N}} \frac{-2\lambda_n}{n^2(1+\lambda_n)} \langle \rho_t, F_n \rangle^2$, as N increases to infinity. Therefore $R(t) \geq 0$ and $(\frac{d}{dt}R)(t) \leq 0$, for all $t > 0$ and since $R(0) = 0$, it follows that $R(t) = 0$ for all $t > 0$. As a consequence of $\{F_n : n \geq 0\}$ being a complete orthonormal system, it follows that $\langle \rho_t, \rho_t \rangle = 0$, which is enough to conclude.

7.6.2 Uniqueness of weak solutions of (7.6)

At first, we begin with an auxiliary lemma on integration by parts.

Lemma 7.6.1. *Let $\rho(t,\cdot)$ be a function in the Sobolev space $L^2(0,T;\mathcal{H}^1(a,b))$. Then, for any $H \in C^{0,1}([0,T]\times[a,b])$:*

$$\begin{aligned}
&\int_0^T \int_a^b \rho(s,u)\,\partial_u H(s,u)\,du\,ds \\
&= -\int_0^T \int_a^b \partial_u \rho(s,u)\,H(u,s)\,du\,ds + \int_0^T \Big\{\rho(s,b)\,H(s,b) - \rho(s,a)\,H(s,a)\Big\}\,ds\,.
\end{aligned}$$

Notice the partial derivative in ρ is the weak derivative, while the partial derivative in H is the usual one. Besides that, the function H is smooth, but possibly not null at the boundary $[0,T]\times\{a,b\}$, and therefore is not valid the integration by parts in the sense of $L^2(0,T;\mathcal{H}^1(a,b))$, which has no boundary integrals.

Proof. Fix $\varepsilon>0$ and write $H = H^\varepsilon + (H - H^\varepsilon)$, where H^ε coincides with H in the region $[0,T]\times(a+\varepsilon, b-\varepsilon)$, has compact support contained in $[0,T]\times(a,b)$ and belongs to $C^{0,1}([0,T]\times(a,b))$. By the assumptions on H^ε, we have that

$$\begin{aligned}
&\int_0^T \int_a^b \rho(s,u)\,\partial_u H(s,u)\,du\,ds \\
&= -\int_0^T \int_a^b \partial_u \rho(s,u)\,H^\varepsilon(s,u)\,du\,ds + \int_0^T \int_a^b \rho(s,u)\partial_u(H-H^\varepsilon)(s,u)\,du\,ds\,.
\end{aligned}$$

Last result is a consequence of H^ε having compact support strictly contained in the open set (a,b). Let $f_\varepsilon:[a,b]\to\mathbb{R}$ be the function such that $f(u)=1$ if $u\in(a+\varepsilon,b-\varepsilon)$, $f(a)=f(b)=0$, and interpolated linearly otherwise. The decomposition $H = H\,f^\varepsilon + H(1-f^\varepsilon)$ can be done, but now the function $H\,f^\varepsilon$ does not have the properties as required above for H^ε. Nevertheless, taking a suitable approximating sequence of functions H^ε, it follows that

$$\begin{aligned}
&\int_0^T \int_a^b \rho(s,u)\,\partial_u H(s,u)\,du\,ds \\
&= -\int_0^T \int_a^b \Big\{\partial_u\rho(s,u)H(s,u)f^\varepsilon(u) + \rho(s,u)\partial_u\Big(H(s,u)(1-f^\varepsilon(u))\Big)\Big\}\,du\,ds.
\end{aligned}$$

Taking the limit as $\varepsilon\downarrow 0$ yields the statement of the lemma. $\square$

Let $\rho(t,\cdot)$ be a weak solution of (7.6) with $\gamma\equiv 0$. Provided by Lemma 7.6.1, for any function $H\in C^{1,2}([0,T]\times(b_i,b_{i+1}))$,

$$\int_{b_i}^{b_{i+1}} \rho_t(u)H(t,u)\,du + \int_0^t \int_{b_i}^{b_{i+1}} \Big\{\partial_u\rho_s(u)\partial_u H(s,u) - \rho_s(u)\partial_s H(s,u)\Big\}du\,ds = 0.$$

From this point, uniqueness is a particular case of a general result in [17], namely Theorem III.4.1. In sake of completeness, we sketch an adaptation of it to our particular case. Denote by $W^1_{2,T} = W^1_{2,T}([0,T]\times(a,b))$ the space of functions with one weak derivative in space and time, both belonging to $L^2([0,T]\times(a,b))$ and vanishing at time T. By extending the previous equality to $H\in W^1_{2,T}$ it follows that

$$\int_0^T\int_{b_i}^{b_{i+1}}\Big\{\partial_u\rho_s(u)\,\partial_u H(s,u)-\rho_s(u)\,\partial_s H(s,u)\Big\}\,du\,ds\;=\;0\,. \tag{7.28}$$

It is not difficult to show that the function

$$H(s,u)\;=\;-\int_s^T\rho(r,u)\,dr$$

belongs to $W^1_{2,T}$. Replacing last function in (7.28), then we can rewrite (7.28) as

$$\int_0^T\int_{b_i}^{b_{i+1}}\Big\{\frac{1}{2}\partial_s(\partial_u H(s,u))^2-(\partial_s H(s,u))^2\Big\}\,du\,ds\;=\;0\,.$$

By Fubini's Theorem we get to

$$\frac{1}{2}\int_{b_i}^{b_{i+1}}\Big\{(\partial_u H(T,u))^2\;-(\partial_u H(0,u))^2\Big\}\,du-\int_0^T\int_{b_i}^{b_{i+1}}(\partial_s H(s,u))^2\,du\,ds\;=\;0\,.$$

By the definition of H, its weak space derivative vanishes at time T, so that the first integral above is null. Therefore, $\partial_s H$ is identically null, and by the definition of H above, this implies that ρ vanishes, finishing the proof.

Part III

Hydrodynamic limit for a type of exclusion processes with slow bonds in dimension ≥ 2

Chapter 8

Hydrodynamic Limit for a type of Exclusion Processes with slow bonds in dimension ≥ 2

Joint work with Tertuliano Franco and Glauco Valle.
To be appear in the Journal of Applied Probability 48.2 (June 2011).

8.1 Notation and Results

Let $\mathbb{T}^d$ be the d-dimensional torus, which is $[0,1)^d$ with periodic boundary conditions, and $\mathbb{T}_N^d$ be the discrete torus with N^d points, i.e., $\{0, ..., N-1\}^d$ with periodic boundary conditions. We denote by $\eta = (\eta(x))_{x\in\mathbb{T}_N^d}$ a typical configuration in the state space $\Omega_N = \{0,1\}^{\mathbb{T}_N^d}$, for which, $\eta(x) = 0$ means that site x is vacant, and $\eta(x) = 1$ that site x is occupied. If a bond of $N^{-1}\mathbb{T}_N^d$ has vertices $\frac{x}{N}$ and $\frac{y}{N}$, it will be denoted by $[\frac{x}{N}, \frac{y}{N}]$.

Recall that $\{e_j : j = 1, ..., d\}$ is the canonical basis of $\mathbb{R}^d$. The symmetric nearest neighbor exclusion process with exchange rates $\xi_{x,y}^N > 0$, $x, y \in \mathbb{T}_N^d$, $|x-y| = 1$, is a Markov process with configuration space Ω_N, whose generator L_N acts on functions $f : \Omega_N \to \mathbb{R}$ as

$$(L_N f)(\eta) = \sum_{x\in\mathbb{T}_N^d} \sum_{j=1}^{d} \xi_{x,x+e_j}^N \left[f(\eta^{x,x+e_j}) - f(\eta)\right], \tag{8.1}$$

where $\eta^{x,x+e_j}$ is the configuration obtained from η by exchanging the variables $\eta(x)$ and $\eta(x+e_j)$:

$$(\eta^{x,x+e_j})(y) = \begin{cases} \eta(x+e_j), & \text{if } y = x\,, \\ \eta(x), & \text{if } y = x+e_j\,, \\ \eta(y), & \text{otherwise}\,. \end{cases}$$

Let ν_α^N, $\alpha \in [0,1]$, be the Bernoulli product measure Ω_N, i.e., the product measure whose marginals have Bernoulli distribution with parameter α. Then $\{\nu_\alpha^N : 0 \leq \alpha \leq 1\}$ is a family of invariant, in fact reversible, measures for any symmetric exclusion process.

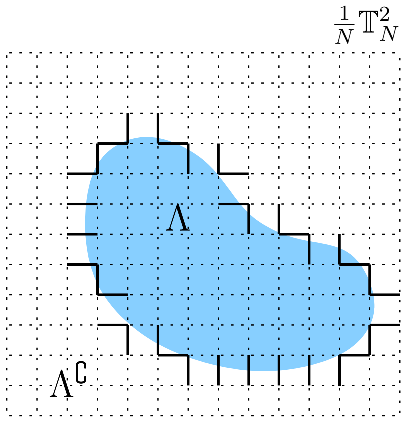

Figure 8.1: The darker region corresponds to Λ. The bolded bonds have exchanges rates $\frac{|\vec{\zeta}_{x,j}\cdot e_j|}{N}$, any other bond has exchange rate 1.

Now, fix a simple connected region $\Lambda \subset \mathbb{T}^d$ with smooth boundary $\partial\Lambda$. Denote by $\vec{\zeta}(u)$ the normal unitary exterior vector to the smooth surface $\partial\Lambda$ in the point $u \in \partial\Lambda$. If $\frac{x}{N} \in \Lambda$ and $\frac{x+e_j}{N} \in \Lambda^{\complement}$, or $\frac{x}{N} \in \Lambda^{\complement}$ and $\frac{x+e_j}{N} \in \Lambda$, we define $\vec{\zeta}_{x,j}$ as a vector $\vec{\zeta}(u)$ evaluated in an arbitrary but fixed point $u \in \partial\Lambda \cap [x, x+e_j]$. The exclusion process with slow bonds over $\partial\Lambda$ is a symmetric nearest neighbor exclusion process with exchange rates $\xi^N_{x,x+e_j} = \xi^N_{x+e_j,x}$ given by

$$\begin{cases} \dfrac{|\vec{\zeta}_{x,j}\cdot e_j|}{N}\,, & \text{if } \frac{x}{N} \in \Lambda \text{ and } \frac{x+e_j}{N} \in \Lambda^{\complement}\,, \text{ or } \frac{x}{N} \in \Lambda^{\complement} \text{ and } \frac{x+e_j}{N} \in \Lambda\,, \\ \\ 1\,, & \text{otherwise}, \end{cases} \tag{8.2}$$

for $j = 1, \dots, d$, and for every $x \in \mathbb{T}^d_N$. In this case, the exchange rate of a bond crossing the boundary $\partial\Lambda$ is also of order N^{-1}, but it depends on the angle of incidence: the crossing of $\partial\Lambda$ by a particle gets harder to happen as the direction of entrance gets closer to the tangent plane to the surface $\partial\Lambda$.

From now on, the rates in the definition of L_N will always be given by (8.2). Denote by $\{\eta^N_t : t \geq 0\}$ a Markov process with state space Ω_N and generator L_N speeded up by N^2. Let $D(\mathbb{R}_+, \Omega_N)$ be the Skorohod space of càdlàg trajectories taking values in Ω_N. For a measure μ on Ω_N, denote by $\mathbb{P}^N_\mu$ the probability measure on $D(\mathbb{R}_+, \Omega_N)$ induced by the initial state μ and the Markov process $\{\eta^N_t : t \geq 0\}$. The expectation with respect to $\mathbb{P}^N_\mu$ is going to be denoted by $\mathbb{E}^N_\mu$.

A sequence of probability measures $\{\mu_N : N \geq 1\}$ is said to be associated to a profile $\gamma : \mathbb{T}^d \to [0,1]$ if μ_N is a probability measure on Ω_N, for every N, and

$$\lim_{N\to\infty} \mu_N \left\{ \left| \frac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} H(\tfrac{x}{N})\eta(x) - \int H(u)\gamma(u)du \right| > \delta \right\} = 0 \tag{8.3}$$

for every $\delta > 0$, and every continuous function $H : \mathbb{T}^d \to \mathbb{R}$.

The exclusion process with slow bonds over $\partial\Lambda$ has a related random walk on $N^{-1}\mathbb{T}_N^d$ that describes the evolution of the system with a single particle. Thus particles in the exclusion process evolve independently as such random walk except for the hard core interaction. To simplify notation later, we introduce here the generator of this random walk, which is given by

$$(\mathbb{L}_N H)(\tfrac{x}{N}) = \sum_{j=1}^{d} \Big\{ \xi^N_{x,x+e_j} \Big[H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N}) \Big] + \xi^N_{x,x-e_j} \Big[H(\tfrac{x-e_j}{N}) - H(\tfrac{x}{N}) \Big] \Big\}, \tag{8.4}$$

for every $H : N^{-1}\mathbb{T}_N^d \to \mathbb{R}$ and every $x \in \mathbb{T}_N^d$. We will not differentiate the notation for functions H defined on $\mathbb{T}^d$ and on $N^{-1}\mathbb{T}_N^d$.

8.1.1 The Operator $\mathcal{L}_\Lambda$

Here we define the operator $\mathcal{L}_\Lambda$ and state its main properties. First, its domain is defined as a set of functions that are two times continuously differentiable inside and outside Λ and satisfy some additional conditions related to their behavior at $\partial\Lambda$. Such conditions are imposed in order to have good properties of $\mathcal{L}_\Lambda$ that allows us to conclude the uniqueness of solutions of the hydrodynamic equation, and obtain a strong convergence result for the empirical measures in the proof of the hydrodynamic limit. The necessity of these conditions are going to be made clear later in the text.

Definition 8.1.1. *Recall that $\vec{\zeta}$ denotes the normal exterior vector to the surface $\partial\Lambda$. The domain $\mathfrak{D}_\Lambda \subset L^2(\mathbb{T}^d)$ will be the set of functions $H \in L^2(\mathbb{T}^d)$, such that $H(u) = h(u) + \lambda \mathbf{1}_\Lambda(u)$, where:*

- ***(i)*** $\lambda \in \mathbb{R}$;
- ***(ii)*** $h \in C^2(\mathbb{T}^d)$;
- ***(iii)*** $\nabla h|_{\partial\Lambda}(u) = -\lambda\,\vec{\zeta}(u)$.

Now, we define the operator $\mathcal{L}_\Lambda : \mathfrak{D}_\Lambda \to L^2(\mathbb{T}^d)$ by

$$\mathcal{L}_\Lambda H \;= \Delta h\,.$$

Geometrically, the operator $\mathcal{L}_\Lambda$ removes the discontinuity around the surface $\partial\Lambda$ and then acts like the laplacian operator.

Remark 8.1.1. *It is not entirely obvious why there exist functions $h \in C^2(\mathbb{T}^d)$ such that $\nabla h|_{\partial\Lambda}(u) = -\lambda\,\vec{\zeta}(u)$, for $\lambda \neq 0$. For an example of such a function, consider firstly $g : \mathbb{T}^d \to \mathbb{R}$ defined by*

$$g(u) \;=\; \begin{cases} \lambda\, dist\,(u, \partial\Lambda)\,, & \text{if } \; u \in \Lambda^{\complement}\,, \\ -\lambda\, dist\,(u, \partial\Lambda)\,, & \text{if } \; u \in \Lambda\,. \end{cases}$$

Since $\partial\Lambda$ has no self intersection and is smooth, it is simple to check that there exists a sufficiently small $\varepsilon > 0$ such that

$$V = \{u \in \mathbb{T}^d : \, dist\,(u, \partial\Lambda) < \varepsilon\}$$

has smooth boundary and without self intersection. Thus, the function g is smooth in the open neighborhood V of $\partial\Lambda$, and satisfies the condition $\nabla g|_{\partial\Lambda}(u) = -\lambda\,\vec{\zeta}(u)$. However, g is not differentiable in the space $\mathbb{T}^d$. To solve this problem, it is enough to multiply g by $\sum_i \Phi_i$, where $\{\Phi_i\}$ is a partition of unity such that the support of any Φ_i is contained in V and $\sum_i \Phi_i(u) = 1$ for all $u \in U \subset V$, U an open set containing $\partial\Lambda$. Finally, the function

$$h(u) \;=\; g(u) \sum_i \Phi_i(u)$$

satisfies the required conditions.

For the next result we need to introduce some notation. We denote by $\mathbb{I}$ the identity operator in $L^2(\mathbb{T}^d)$ and by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ and $\|\cdot\|$ its usual inner product and norm:

$$\langle\!\langle f, g \rangle\!\rangle \;=\; \int_{\mathbb{T}^d} f(u)\, g(u)\, du \quad \text{and} \quad \|f\| = \sqrt{\langle\!\langle f, f \rangle\!\rangle}\;, \quad f,\; g \in L^2(\mathbb{T}^d)\,.$$

Theorem 8.1.2. *There exists a Hilbert space $(\mathcal{H}^1_\Lambda, \langle\!\langle \cdot, \cdot \rangle\!\rangle_{1,\Lambda})$ which is compactly embedded in $L^2(\mathbb{T}^d)$ such that $\mathfrak{D}_\Lambda \subset \mathcal{H}^1_\Lambda$ and $\mathcal{L}_\Lambda$ can be extended to $\mathcal{L}_\Lambda : \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^d)$ in such a way that the extension enjoys the following properties:*

(a) The domain $\mathcal{H}^1_\Lambda$ is dense in $L^2(\mathbb{T}^d)$;

(b) The operator $\mathcal{L}_\Lambda$ is self-adjoint and non-positive: $\langle\!\langle H, -\mathcal{L}_\Lambda H \rangle\!\rangle \geq 0$, for all H in $\mathcal{H}^1_\Lambda$;

(c) The operator $\mathbb{I} - \mathcal{L}_\Lambda : \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^d)$ is bijective and $\mathfrak{D}_\Lambda$ is a core for it;

(d) The operator $\mathcal{L}_\Lambda$ is dissipative, i.e.,

$$\|\mu H - \mathcal{L}_\Lambda H\| \geq \mu \|H\|\,,$$

for all $H \in \mathcal{H}^1_\Lambda$ and $\mu > 0$;

(e) The eigenvalues of $-\mathcal{L}_\Lambda$ form a countable set $0 = \mu_0 \leq \mu_1 \leq \cdots$ with $\lim_{n\to\infty} \mu_n = \infty$, and all these eigenvalues have finite multiplicity;

(f) There exists a complete orthonormal basis of $L^2(\mathbb{T}^d)$ composed of eigenvectors of $-\mathcal{L}_\Lambda$.

In view of (a), (c) and (d), by the Hille-Yoshida Theorem, $\mathcal{L}_\Lambda$ is the generator of a strongly continuous contraction semigroup in $L^2(\mathbb{T}^d)$.

The space $\mathcal{H}^1_\Lambda$ will be defined in Section 8.2. The name has been chosen in analogy to the notation used for Sobolev spaces.

8.1.2 The hydrodynamic equation

Consider a bounded Borel measurable profile $\rho_0 : \mathbb{T}^d \to \mathbb{R}$. A bounded function $\rho : \mathbb{R}_+ \times \mathbb{T}^d \to \mathbb{R}$ is said to be a weak solution of the parabolic differential equation

$$\begin{cases} \partial_t \rho \;=\; \mathcal{L}_\Lambda \rho \\ \rho(0,\cdot) \;=\; \rho_0(\cdot)\,, \end{cases} \tag{8.5}$$

if for all functions H in $\mathcal{H}^1_\Lambda$ and all $t > 0$, ρ satisfies the integral equation

$$\langle\!\langle \rho_t, H \rangle\!\rangle \;-\; \langle\!\langle \rho_0, H \rangle\!\rangle - \int_0^t \langle\!\langle \rho_s, \mathcal{L}_\Lambda H \rangle\!\rangle \, ds \;=\; 0, \tag{8.6}$$

where ρ_t is the notation for $\rho(t,\cdot)$. We prove in Subsection 8.3.3 the uniqueness of weak solutions of (8.5). Existence follows from the convergence result for the empirical measures associated to the diffusively rescaled exclusion processes with slow bonds over Λ, this is discussed in Section 8.3. Here we do not use time dependent test functions as usual in the definition of weak solution, but we have a well posed problem and we do not need a solution in a stronger sense to prove the hydrodynamic limit which is the next stated theorem.

Theorem 8.1.3. *Fix a Borel measurable initial profile $\gamma : \mathbb{T}^d \to [0,1]$ and consider a sequence of probability measures μ_N on Ω_N associated to γ. Then, for any $t \geq 0$,*

$$\lim_{N\to\infty} \mathbb{P}^N_{\mu_N} \Big\{ \Big| \tfrac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} H(x/N)\, \eta_t(x) - \int_{\mathbb{T}^d} H(u)\rho(t,u)du \Big| > \delta \Big\} \;=\; 0\,,$$

for every $\delta > 0$ and every function $H \in C(\mathbb{T}^d)$, where ρ is the unique weak solution of the differential equation (8.5) *with $\rho_0 = \gamma$.*

8.2 The operator $\mathcal{L}_\Lambda$

We begin by studying properties of $\mathcal{L}_\Lambda$ defined on the domain $\mathfrak{D}_\Lambda$ and we consider the extension afterwards.

Lemma 8.2.1. *The domain $\mathfrak{D}_\Lambda$ is dense in $L^2(\mathbb{T}^d)$.*

Proof. It is enough to prove that there exists a subset of $\mathfrak{D}_\Lambda$ which is dense in $L^2(\mathbb{T}^d)$. All smooth functions with support contained in $\mathbb{T}^d \backslash \partial\Lambda$ belong to $\mathfrak{D}_\Lambda$, which is clearly a dense subset of $L^2(\mathbb{T}^d)$, since $\partial\Lambda$ is a smooth zero Lebesgue measure surface that divides $\mathbb{T}^d \backslash \partial\Lambda$ in two disjoint open regions. □ □

From now on, we use ℓ_d to denote the d-dimensional Lebesgue measure on $\mathbb{T}^d$.

Lemma 8.2.2. *The operator $-\mathcal{L}_\Lambda : \mathfrak{D}_\Lambda \to L^2(\mathbb{T}^d)$ is symmetric and non-negative. Furthermore, it satisfies a Poincaré inequality, which means that there exists a finite constant $C > 0$ such that*

$$\|H\|^2 \leq C \,\langle\!\langle -\mathcal{L}_\Lambda H, H \rangle\!\rangle + \Big(\int_{\mathbb{T}^d} H(x)\, dx \Big)^2 \tag{8.7}$$

for all functions $H \in \mathfrak{D}_\Lambda$.

Proof. Let $H, G \in \mathfrak{D}_\Lambda$. Write $H = h + \lambda_h \mathbf{1}_\Lambda$ and $G = g + \lambda_g \mathbf{1}_\Lambda$, as in Definition 8.1.1. By the first Green identity and condition (iii) in Definition 8.1.1, we have that

$$\begin{aligned} \lambda_h \int_\Lambda \Delta g \, du &= \lambda_h \int_{\partial\Lambda} (\nabla g \cdot \vec{\zeta}) \; dS \;=\; -\lambda_h \, \lambda_g \, \mathrm{Vol}_{d-1}(\partial\Lambda) \\ &= \lambda_g \int_{\partial\Lambda} (\nabla h \cdot \vec{\zeta}) \; dS \;=\; \lambda_g \int_\Lambda \Delta h \, du \,, \end{aligned} \tag{8.8}$$

where dS is a infinitesimal element of volume of $\partial\Lambda$ and $\mathrm{Vol}_{d-1}(\partial\Lambda)$ is its $(d-1)$-dimensional volume. Thus,

$$\begin{aligned} \langle\!\langle H, -\mathcal{L}_\Lambda G \rangle\!\rangle &= \langle\!\langle h + \lambda_h \mathbf{1}_\Lambda, -\Delta g \rangle\!\rangle = -\int_{\mathbb{T}^d} h \, \Delta g \, du - \lambda_h \int_\Lambda \Delta g \, du \\ &= -\int_{\mathbb{T}^d} g \, \Delta h \, du - \lambda_g \int_\Lambda \Delta h \, du \;=\; \langle\!\langle -\mathcal{L}_\Lambda H, G \rangle\!\rangle \,. \end{aligned}$$

For the non-negativeness, using (8.8) above,

$$\begin{aligned} \langle\!\langle H, -\mathcal{L}_\Lambda H \rangle\!\rangle &= -\int_{\mathbb{T}^d} h \, \Delta h \, du - \lambda_h \int_\Lambda \Delta h \, du \\ &= \int_{\mathbb{T}^d} |\nabla h|^2 \, du + \lambda_h^2 \, \mathrm{Vol}_{d-1}(\partial\Lambda) \;\geq\; 0 \,. \end{aligned}$$

It remains to prove the Poincaré inequality. Write

$$\|H\|^2 - \Big(\int_{\mathbb{T}^d} H(x)\, dx \Big)^2 = \int_{\mathbb{T}^d} \Big[H(u) - \int_{\mathbb{T}^d} H(v) \; dv \Big]^2 \, du \,,$$

which can be rewritten as

$$\int_{\mathbb{T}^d} \Big[\Big(h(u) - \int_{\mathbb{T}^d} h(v)\, dv \Big) + \lambda_h \Big(\mathbf{1}_\Lambda(u) - \ell_d(\Lambda) \Big) \Big]^2 \, du \,.$$

Now apply the inequality $(a+b)^2 \leq 2\,(a^2 + b^2)$ to the previous expression to obtain that it is bounded by

$$2 \int_{\mathbb{T}^d} \Big(h(u) - \int_{\mathbb{T}^d} h(v)\, dv \Big)^2 du + 2\, \lambda_h^2 \Big(\ell_d(\Lambda) - (\ell_d(\Lambda))^2 \Big) \,.$$

By the usual Poincaré inequality, see [4], the last expression is less than or equal to

$$2\,C_1 \int_{\mathbb{T}^d} |\nabla h(u)|^2\, du + 2\,\lambda_h^2 \Big(\ell_d(\Lambda) - (\ell_d(\Lambda))^2 \Big)\,.$$

Choosing a constant $C_2 > 0$ such that $\ell_d(\Lambda) - (\ell_d(\Lambda))^2 \leq C_2 \mathrm{Vol}_{d-1}(\partial\Lambda)$, the previous expression is bounded above by

$$2\,\max\{C_1, C_2\}\,\langle\!\langle -\mathcal{L}_\Lambda H, H \rangle\!\rangle\,,$$

which finishes the proof with $C = 2\,\max\{C_1, C_2\}$. □ □

Denote by $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{1,\Lambda}$ the inner product on $\mathfrak{D}_\Lambda$ defined by

$$\langle\!\langle F, G \rangle\!\rangle_{1,\Lambda} \;=\; \langle\!\langle F, G \rangle\!\rangle \;+\; \langle\!\langle F, -\mathcal{L}_\Lambda G \rangle\!\rangle\,.$$

Let $\mathcal{H}^1_\Lambda$ be the set of all functions F in $L^2(\mathbb{T}^d)$ for which there exists a sequence $\{F_n : n \geq 1\}$ in $\mathfrak{D}_\Lambda$ such that F_n converges to F in $L^2(\mathbb{T}^d)$ and F_n is Cauchy for the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{1,\Lambda}$. Such sequence $\{F_n\}$ is called admissible for F. For F, G in $\mathcal{H}^1_\Lambda$, define

$$\langle\!\langle F, G \rangle\!\rangle_{1,\Lambda} \;=\; \lim_{n\to\infty} \langle\!\langle F_n, G_n \rangle\!\rangle_{1,\Lambda}\,, \tag{8.9}$$

where $\{F_n\}$, $\{G_n\}$ are admissible sequences for F, G, respectively. By [20, Proposition 5.3.3], the limit exists and does not depend on the admissible sequence chosen. Moreover, $\mathcal{H}^1_\Lambda$ endowed with the scalar product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{1,\Lambda}$ just defined is a real Hilbert space. From now on, we consider $\mathcal{H}^1_\Lambda$ with the norm induced by $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{1,\Lambda}$, unless we mention that we are going to use the L^2-norm.

Lemma 8.2.3. *The embedding $\mathcal{H}^1_\Lambda \subset L^2(\mathbb{T}^d)$ is compact.*

Proof. Let $\{H_n\}$ a bounded sequence in $\mathcal{H}^1_\Lambda$. Fix $\{F_n\}$ as a sequence in $\mathfrak{D}_\Lambda$ such that $\|F_n - H_n\| \to 0$ and $\{F_n\}$ is also bounded in $\mathcal{H}^1_\Lambda$. Thus, to get a convergent subsequence of $\{H_n\}$, it is sufficient to find a convergent subsequence of $\{F_n\}$ in $L^2(\mathbb{T}^d)$. Write $F_n = f_n + \lambda_n \mathbf{1}_\Lambda$, with $f_n \in C^2(\mathbb{T}^d)$. Then,

$$\langle\!\langle F_n, F_n \rangle\!\rangle_{1,\Lambda} = \langle\!\langle f_n + \lambda_n \mathbf{1}_\Lambda, f_n + \lambda_n \mathbf{1}_\Lambda \rangle\!\rangle + \langle\!\langle f_n + \lambda_n \mathbf{1}_\Lambda, -\Delta f_n \rangle\!\rangle\,.$$

Expanding the right hand side and using (8.8), we get that

$$\langle\!\langle F_n, F_n \rangle\!\rangle_{1,\Lambda} = \|f_n\|^2 + \lambda_n^2 \ell_d(\Lambda) + 2\lambda_n \int_\Lambda f_n(u)\, du + \|\nabla f_n\|^2 + \lambda_n^2\, \mathrm{Vol}_{d-1}(\partial\Lambda)\,,$$

which is greater or equal to

$$\|f_n\|^2 + \lambda_n^2 \ell_d(\Lambda) - \lambda_n^2 - \ell_d(\Lambda) \int_\Lambda f_n^2(u)\, du + \|\nabla f_n\|^2 + \lambda_n^2\, \mathrm{Vol}_{d-1}(\partial\Lambda)$$

$$= \Big(\ell_d(\Lambda) - 1 + \mathrm{Vol}_{d-1}(\partial\Lambda)\Big)\,\lambda_n^2 + (1-\ell_d(\Lambda))\int_\Lambda f_n^2(u)\,du + \int_{\Lambda^\complement} f_n^2(u)\,du + \|\nabla f_n\|^2$$
$$\geq \Big(\mathrm{Vol}_{d-1}(\partial\Lambda) - \ell_d(\Lambda^\complement)\Big)\,\lambda_n^2 + (1-\ell_d(\Lambda))\,\|f_n\|^2 + \|\nabla f_n\|^2\,.$$

If we put $\tilde f_n = f_n + \lambda_n$, and write $F_n = \tilde f_n - \lambda_n \mathbf{1}_{\Lambda^\complement}$, an analogous computation shows that $\langle\!\langle F_n, F_n\rangle\!\rangle_{1,\Lambda}$ is greater or equal than

$$\Big(\mathrm{Vol}_{d-1}(\partial\Lambda) - \ell_d(\Lambda)\Big)\,\lambda_n^2 + (1-\ell_d(\Lambda^\complement))\,\|\tilde f_n\|^2 + \|\nabla \tilde f_n\|^2\,.$$

By the classical isoperimetric inequality on the torus (see [3, Lemma 4.6] for the statement and a direct proof), we have that

$$\max\{\,\mathrm{Vol}_{d-1}(\partial\Lambda) - \ell_d(\Lambda^\complement)\,,\,\mathrm{Vol}_{d-1}(\partial\Lambda) - \ell_d(\Lambda)\,\} > 0\,.$$

Since $\{\langle\!\langle F_n, F_n\rangle\!\rangle_{1,\Lambda}\}$ is a bounded sequence, we conclude that $\{\lambda_n\}$ is bounded, as well the sequence $\{\|f_n\|^2 + \|\nabla f_n\|^2\}$. By the Rellich-Kondrachov Compactness Theorem, see [4, Theorem 5.7.1], $\{f_n\}$ has a convergent subsequence in $L^2(\mathbb{T}^d)$. From this subsequence, choosing a convergent subsequence of $\{\lambda_n\}$ finishes the proof. □ □

Lemma 8.2.4. *The image of* $\mathbb{I} - \mathcal{L}_\Lambda : \mathfrak{D}_\Lambda \to L^2(\mathbb{T}^d)$ *is dense in* $L^2(\mathbb{T}^d)$.

Proof. By a similar argument to the one found in Lemma 8.2.1, it is enough to show that any smooth function f with support contained in $\mathbb{T}^d\backslash\partial\Lambda$ belongs to $(\mathbb{I} - \mathcal{L}_\Lambda)(\mathfrak{D}_\Lambda)$. Therefore, we need to find a function h in $C^2(\mathbb{T}^d)$ with support in $\mathbb{T}^d\backslash\partial\Lambda$ such that

$$h - \Delta h \,=\, f\,. \tag{8.10}$$

From the classical theory of second-order elliptic equations, e.g., see [4, Theorem 5.7.1], there exists $h \in C^2$ satisfying (8.10). □

□

Proof of Theorem 8.1.2. **(a)** Since $\mathfrak{D}_\Lambda \subset \mathcal{H}^1_\Lambda$, it follows from Lemma 8.2.1 that $\mathcal{H}^1_\Lambda$ is dense in $L^2(\mathbb{T}^d)$.

(b) Denote $\mathbb{I} - \mathcal{L}_\Lambda = \mathcal{A} : \mathfrak{D}_\Lambda \to \mathbb{L}^2(\mathbb{T}^d)$. From Lemma 8.2.2, $\mathcal{A}$ is linear, symmetric and strongly monotone on the Hilbert space $L^2(\mathbb{T}^d)$. By strongly monotone, we mean that there exists $c > 0$ such that

$$\langle\!\langle \mathcal{A}\,H, H\rangle\!\rangle \geq\, c\,\|H\|^2\,,\quad \forall H \in \mathfrak{D}_\Lambda\,.$$

In this case, $\mathcal{A}$ satisfies the inequality above with $c = 1$. By [20, Theorem 5.5.a], in the conditions above, the Friedrichs extension $\mathcal{A} : \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^2)$ is self-adjoint, bijective and strongly monotone. By an abuse of notation, define now the extension $\mathcal{L}_\Lambda : \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^2)$ as $(\mathbb{I} - \mathcal{A})$. Since $\mathbb{I}$ and $\mathcal{A}$ are self-adjoint in $\mathcal{H}^1_\Lambda$, this property is inherited by $\mathcal{L}_\Lambda : \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^2)$.

For non-positiveness, note that

$$\langle\!\langle -\,\mathcal{L}_\Lambda\,H, H\rangle\!\rangle = \langle\!\langle -(\mathbb{I} - \mathcal{A})H, H\rangle\!\rangle = -\langle\!\langle H, H\rangle\!\rangle + \langle\!\langle \mathcal{A}\,H, H\rangle\!\rangle \geq 0\,.$$

(c) As mentioned in the proof of (b) above, the Friedrichs extension $\mathcal{A} : \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^2)$ is bijective. So it remains to show that $\mathfrak{D}_\Lambda$ is a core of $\mathcal{A} : \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^2)$. For any operator B, denote by $\mathcal{G}(B)$ the graphic of B. Then $\mathfrak{D}_\Lambda$ is a core for $\mathcal{A}$, if the closure of $\mathcal{G}(\mathcal{A}|_{\mathfrak{D}_\Lambda})^{L^2\times L^2}$ in $L^2 \times L^2$ is equal to $\mathcal{G}(\mathcal{A})$. Since $\mathcal{A}$ is self-adjoint, $\mathcal{A}$ is a closed operator, or else, $\mathcal{G}(\mathcal{A})$ is a closed set. Thus the closure of $\mathcal{G}(\mathcal{A}|_{\mathfrak{D}_\Lambda})$ is a subset of $\mathcal{G}(\mathcal{A})$. Let $H \in \mathcal{H}^1_\Lambda$, from Lemma 8.2.4, there exists a sequence $\{H_n\}$ in $\mathfrak{D}_\Lambda$ such that $\mathcal{A}H_n$ converges to $\mathcal{A}H$ in L^2. Hence, as proved in [20, Theorem 5.5.a], $\mathcal{A}^{-1}$ is a bounded linear operator, and H_n converges to H in L^2, which yields that the closure of $\mathcal{G}(\mathcal{A}|_{\mathfrak{D}_\Lambda})$ contains $\mathcal{G}(\mathcal{A})$.

(d) Fix a function H in $\mathcal{H}^1_\Lambda$ and $\mu > 0$. Put $G = (\mu\mathbb{I} - \mathcal{L}_\Lambda)H$. Taking the inner product with respect to H on both sides of this equality, we obtain that

$$\mu \langle\!\langle H, H\rangle\!\rangle \;+\; \langle\!\langle -\mathcal{L}_\Lambda H, H\rangle\!\rangle \;=\; \langle\!\langle H, G\rangle\!\rangle \;\le\; \langle\!\langle H, H\rangle\!\rangle^{1/2} \langle\!\langle G, G\rangle\!\rangle^{1/2} \,.$$

Since H belongs to $\mathcal{H}^1_\Lambda$, by (b), the second term on the left hand side is positive. Therefore, $\mu\|H\| \le \|G\| = \|(\mu\mathbb{I} - \mathcal{L}_\Lambda)H\|$.

(e) and (f) We have seen that the operator $(\mathbb{I} - \mathcal{L}_\Lambda) : \mathfrak{D}_\Lambda \to L^2(\mathbb{T})$ is symmetric and strongly monotone. By Lemma 8.2.3 , the embedding $\mathcal{H}^1_\Lambda \subset L^2(\mathbb{T}^d)$ is compact. Therefore, by [20, Theorem 5.5.c], the Friedrichs extension $\mathcal{A} : \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^d)$, satisfies claims (e) and (f) with $1 \le \lambda_1 \le \lambda_2 \le \cdots$, $\lambda_n \uparrow \infty$. In particular, the operator $-\mathcal{L}_\Lambda = (\mathcal{A} - \mathbb{I})$ has the same property with $0 \le \mu_1 \le \mu_2 \le \cdots$, $\mu_n \uparrow \infty$. Since 0 is an eigenvalue of $-\mathcal{L}_\Lambda$, a constant function is an eigenfunction with eigenvalue 0, then (e) and (f) also hold. □ □

8.3 Scaling Limit

Let $\mathcal{M}$ be the space of positive Radon measures on $\mathbb{T}^d$ with total mass bounded by one endowed with the weak topology. For a measure $\pi \in \mathcal{M}$ and a measurable π-integrable function $H : \mathbb{T}^d \to \mathbb{R}$, we denote by $\langle \pi, H\rangle$ the integral of H with respect to π.

Recall that $\{\eta^N_t : t \ge 0\}$ denote a Markov process with state space Ω_N and generator L_N speeded up by N^2. Let $\pi^N_t \in \mathcal{M}$ be the empirical measure at time t associated to $\{\eta^N_t : t \ge 0\}$, which is the random measure in $\mathcal{M}$ given by

$$\pi^N_t \;=\; \frac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} \eta^N_t(x)\, \delta_{x/N} \,, \tag{8.11}$$

where δ_u is the Dirac measure concentrated on u.

Note that

$$\langle \pi^N_t, H\rangle \;=\; \tfrac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} H(\tfrac{x}{N})\eta^N_t(x) \,,$$

for the empirical measures, and $\langle \pi, H\rangle = \langle\!\langle \rho, H\rangle\!\rangle$, for absolutely continuous measures π with L^2 bounded density ρ, and $H \in L^2(\mathbb{T}^d)$.

Fix $T > 0$. Let $D([0,T], \mathcal{M})$ be the space of $\mathcal{M}$-valued càdlàg trajectories $\pi : [0,T] \to \mathcal{M}$ endowed with the *Skorohod* topology. Then, the $\mathcal{M}$-valued process $\{\pi^N_t : t \ge 0\}$ is a random element of $D([0,T],\mathcal{M})$ whose distribution is determined by the initial distribution of $\{\eta_t^N : t \geq 0\}$. For each probability measure μ on Ω_N, denote by $\mathbb{Q}_{\mu}^{\Lambda,N}$ the distribution of $\{\pi_t^N : t \geq 0\}$ on the path space $D([0,T],\mathcal{M})$, when η_0^N has distribution μ.

Proposition 8.3.1. *Fix a Borel measurable profile $\gamma : \mathbb{T}^d \to [0,1]$ and consider a sequence $\{\mu_N : N \geq 1\}$ of measures on Ω_N associated to γ in the sense of* (8.3). *Then there exists a unique weak solution ρ of* (8.5) *with initial condition γ and the sequence of probability measures $\mathbb{Q}_{\mu_N}^{\Lambda,N}$ converges weakly to $\mathbb{Q}_{\Lambda}^{\gamma}$ as $N \uparrow \infty$, where $\mathbb{Q}_{\Lambda}^{\gamma}$ is the probability measure on $D([0,T],\mathcal{M})$ concentrated on the deterministic path $\pi(t,du) = \rho(t,u)du$.*

It is straightforward to obtain Theorem 8.1.3 as a corollary of the previous proposition. The proof of Proposition 8.3.1 follows directly from the uniqueness of weak solutions of (8.5), proved in Subsection 8.3.3, and the next two results:

Proposition 8.3.2. *For any sequence $\{\mu_N : N \geq 1\}$ of probability measures with μ_N concentrated on Ω_N, the sequence of measures $\{\mathbb{Q}_{\mu_N}^{\Lambda,N} : N \geq 1\}$ is tight.*

Proposition 8.3.3. *Fix a Borel measurable profile $\gamma : \mathbb{T}^d \to [0,1]$ and consider a sequence $\{\mu_N : N \geq 1\}$ of probability measures on Ω_N associated to γ in the sense of* (8.3). *Then any limit point of $\mathbb{Q}_{\mu_N}^{\Lambda,N}$ is concentrated on absolutely continuous trajectories that are weak solutions of* (8.5) *with initial condition γ.*

Proof of Proposition 8.3.1. By Proposition 8.3.2, the set of measures $\{\mathbb{Q}_{\mu_N}^{\Lambda,N} : N \geq 1\}$ is tight. Since the Skorohod space $D([0,T],\mathcal{M})$ is Polish, by Prohorov's Theorem, tightness is equivalent to relative compactness (for the weak convergence). By the relative compactness, in order to prove the convergence of the sequence $(Q_{\mu_N}^{\Lambda,N})_{N\geq 1}$ to the probability measure $\mathbb{Q}_{\Lambda}^{\gamma}$, it is enough to show that any convergent subsequence of $(Q_{\mu_N}^{\Lambda,N})_{N\geq 1}$ has limit equal to $\mathbb{Q}_{\Lambda}^{\gamma}$. Let $\mathbb{Q}^*$ be a limit of a convergent subsequence. By Proposition 8.3.3, $\mathbb{Q}^*$ is concentrated on trajectories $\pi(t,du) = \rho(t,u)\,du$ such that $\rho(t,u)$ is a weak solution of (8.5) with initial condition γ. Uniqueness of weak solutions of (8.5) proved in Section 8.3.3 implies that $\mathbb{Q}^* = \mathbb{Q}_{\Lambda}^{\gamma}$. $\square$ $\square$

In Subsection 8.3.1, we prove Proposition 8.3.2 and in Subsection 8.3.2 we prove Proposition 8.3.3. As a consequence, we have the existence of solutions of (8.5) with initial condition γ. We complete the proof in Subsection (8.3.3) showing the uniqueness of weak solutions of (8.5).

8.3.1 Tightness

Here we prove Proposition 8.3.2. Let $D([0,T],\mathbb{R})$ be the space of $\mathbb{R}$-valued càdlàg trajectories with domain $[0,T]$ endowed with the *Skorohod* topology. To prove tightness of $\{\pi_t^N : 0 \leq t \leq T\}$ in $D([0,T],\mathcal{M})$, it is enough to show tightness in $D([0,T],\mathbb{R})$ of the real-valued processes $\{\langle \pi_t^N, H\rangle : 0 \leq t \leq T\}$ for a set of functions $H : \mathbb{T}^d \to \mathbb{R}$ which is dense in the space of continuous real functions on $\mathbb{T}^d$ endowed with the uniform topology, see [16]. Furthermore, if a sequence of distributions in $D([0,T],\mathbb{R})$ endowed with the uniform topology is tight, then it is also tight in $D([0,T],\mathbb{R})$ endowed with the Skorohod topology. Here we prove tightness of $\{\langle \pi_t^N, H\rangle : 0 \le t \le T\}$ in $D([0,T],\mathbb{R})$, endowed with the uniform topology, for $H \in C^2(\mathbb{T}^d)$.

Fix $H \in C^2(\mathbb{T}^d)$. By definition $\{\langle \pi_t^N, H\rangle : 0 \le t \le T\}$ is tight in $D([0,T],\mathbb{R})$ endowed with the uniform topology if, for the boundedness,

$$\lim_{m\to\infty} \sup_{N} \mathbb{P}^N_{\mu_N}\left[\sup_{0\le t\le T} |\langle \pi_t^N, H\rangle| > m\right] = 0\,, \tag{8.12}$$

and, for the equicontinuity,

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \mathbb{P}^N_{\mu_N}\left[\sup_{|t-s|\le\delta} |\langle \pi_t^N, H\rangle - \langle \pi_s^N, H\rangle| > \varepsilon\right] = 0\,,\ \text{ for all } \varepsilon > 0\,. \tag{8.13}$$

The limit in (8.12) is trivial since

$$|\langle \pi_t^N, H\rangle| \le \sup_{u\in\mathbb{T}^d} |H(u)|\,.$$

So we only need to prove (8.13). By Dynkyn's formula (see appendix in [16]),

$$M_t^N = \langle \pi_t^N, H\rangle - \langle \pi_0^N, H\rangle - \int_0^t N^2 L_N \langle \pi_s^N, H\rangle ds \tag{8.14}$$

is a martingale. By the previous expression, (8.13) follows from

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \mathbb{P}^N_{\mu_N}\left[\sup_{|t-s|\le\delta} |M_t^N - M_s^N| > \varepsilon\right] = 0\,,\ \text{ for all } \varepsilon > 0\,, \tag{8.15}$$

and

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \mathbb{P}^N_{\mu_N}\left[\sup_{0\le t-s\le\delta} \left|\int_s^t N^2 L_N \langle \pi_r^N, H\rangle dr\right| > \varepsilon\right] = 0\,,\ \text{ for all } \varepsilon > 0\,. \tag{8.16}$$

Indeed, we show the stronger results below:

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \mathbb{E}^N_{\mu^N}\left[\sup_{|t-s|\le\delta} |M_t^N - M_s^N|\right] = 0\,, \tag{8.17}$$

and

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \mathbb{E}^N_{\mu^N}\left[\sup_{0\le t-s\le\delta} \left|\int_s^t N^2 L_N \langle \pi_r^N, H\rangle dr\right|\right] = 0\,. \tag{8.18}$$

To verify (8.17), we use the quadratic variation of M_t^N that we denote by $\langle M_t^N \rangle$. By Doob's inequality, we have that

$$\begin{aligned} \mathbb{E}_{\mu^N}^N \Big[\sup_{|t-s|\leq\delta} |M_t^N - M_s^N| \Big] &\leq 2\,\mathbb{E}_{\mu^N}^N \Big[\sup_{0\leq t\leq T} |M_t^N| \Big] \\ &\leq 2\,\mathbb{E}_{\mu^N}^N \Big[\sup_{0\leq t\leq T} |M_t^N|^2 \Big]^{\frac{1}{2}} \leq 4\,\mathbb{E}_{\mu^N}^N \big[\langle M_T^N \rangle \big]^{\frac{1}{2}} . \end{aligned}$$

Since

$$\langle M_t^N \rangle = \int_0^t N^2 [L_N \langle \pi_s^N, H\rangle^2 - 2\langle \pi_s^N, H\rangle L_N \langle \pi_s^N, H\rangle] ds \,,$$

we obtain by a straightforward computation that

$$\langle M_t^N \rangle = \int_0^t N^2 \sum_{j=1}^d \sum_{x\in\mathbb{T}_N^d} \xi_{x,x+e_j}^N \tfrac{1}{N^{2d}} \Big[(\eta_s(x) - \eta_s(x+e_j))(H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N})) \Big]^2 ds \,.$$

Therefore, since $\xi_{x,x+e_j}^N \leq 1$,

$$\begin{aligned} \langle M_t^N \rangle &\leq \frac{T}{N^{2d-2}} \sum_{j=1}^d \sum_{x\in\mathbb{T}_N^d} \xi_{x,x+e_j}^N \Big[H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N}) \Big]^2 \\ &\leq \frac{Td}{N^d} \Big(\sup_{u\in\mathbb{T}^d} |\nabla H(u)\cdot e_j| \Big)^2 . \end{aligned} \tag{8.19}$$

Thus, M_t^N converges to zero in L^2 and (8.17) holds.

We finish the proof by verifying (8.18). Write

$$N^2 L_N \langle \pi_s^N, H\rangle = \tfrac{1}{N^{d-2}} \sum_{j=1}^d \sum_{x\in\mathbb{T}_N^d} \xi_{x,x+e_j}^N ((\eta_s(x) - \eta_s(x+e_j)) \big(H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N}) \big)$$

$$= \tfrac{1}{N^{d-2}} \sum_{j=1}^d \sum_{x\in\mathbb{T}_N^d} \eta_s(x) \Big[\xi_{x,x+e_j}^N \Big(H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N}) \Big) + \xi_{x,x-e_j}^N \Big(H(\tfrac{x-e_j}{N}) - H(\tfrac{x}{N}) \Big) \Big] .$$

Define $\Gamma_N \subset \mathbb{T}_N^d$ as the set of vertices whose have some adjacent edge with exchange rate not equal to one. Then $N^2 L_N \langle \pi_s^N, H\rangle$ is equal to the sum of

$$\tfrac{1}{N^{d-2}} \sum_{j=1}^d \sum_{x\notin\Gamma_N} \eta_s(x) \Big[H(\tfrac{x+e_j}{N}) + H(\tfrac{x-e_j}{N}) - 2H(\tfrac{x}{N}) \Big] \tag{8.20}$$

and

$$+ \tfrac{1}{N^{d-2}} \sum_{j=1}^d \sum_{x\in\Gamma_N} \eta_s(x) \Big[\xi_{x,x+e_j}^N \Big(H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N}) \Big) + \xi_{x,x-e_j}^N \Big(H(\tfrac{x-e_j}{N}) - H(\tfrac{x}{N}) \Big) \Big] . \tag{8.21}$$

By the Taylor expansion (remember $H \in C^2$), the absolute value of the summand in (8.20) is bounded by $N^{-2} \sup_{u\in\mathbb{T}^d} |\Delta H(u)|$. Considering the factor N^{-d+2} in front of the sum, we conclude that the expression (8.20) is bounded in absolute value by $d \sup_{u\in\mathbb{T}^d} |\Delta H(u)|$.

Since there are in order of N^{d-1} vertices in Γ_N, and $\xi_{x,x+e_j} \leq 1$, the absolute value of the expression (8.21) is bounded by

$$\tfrac{1}{N^{d-2}} \sum_{j=1}^{d} \sum_{x\in\Gamma_N} \Big[|H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N})| + |H(\tfrac{x-e_j}{N}) - H(\tfrac{x}{N})| \Big] \leq 2d \sup_{u\in\mathbb{T}^d} |\nabla H(u) \cdot e_j| \,.$$

By the boundedness of expressions (8.20) and (8.21), there exists $C > 0$, depending only on H, such that $|N^2 L_N \langle \pi^N_s, H\rangle| \leq C$, which yields

$$\Big| \int_r^t N^2 L_N \langle \pi^N_s, H\rangle ds \Big| \leq C(t-r)\,,$$

and (8.18) holds.

8.3.2 Characterization of limit points

Let $\gamma : \mathbb{T}^d \to [0,1]$ be a Borel measurable profile and consider a sequence $\{\mu_N : N \geq 1\}$ of measures on Ω_N associated to γ in the sense of (8.3). We prove Proposition 8.3.3 in this subsection, i.e., that all limit points $\mathbb{Q}^*$ of the sequence $\mathbb{Q}^{\Lambda,N}_{\mu_N}$ are concentrated on absolutely continuous trajectories $\pi(t, du) = \rho(t, u)du$, whose density $\rho(t, u)$ is a weak solution of the hydrodynamic equation (8.5) with γ as initial condition.

Let $\mathbb{Q}^*$ be a limit point of the sequence $\mathbb{Q}^{\Lambda,N}_{\mu_N}$ and assume, without loss of generality, that $\mathbb{Q}^{\Lambda,N}_{\mu_N}$ converges to $\mathbb{Q}^*$.

Since there is at most one particle per site, $\mathbb{Q}^*$ is concentrated on trajectories $\pi_t(du)$ which are absolutely continuous with respect to the Lebesgue measure, $\pi_t(du) = \rho(t, u)du$, and whose density ρ is non-negative and bounded by 1, see [16, Chapter 4].

We shall prove the following result:

Lemma 8.3.4. *Any limit point $\mathbb{Q}^*$ of $\mathbb{Q}^{\Lambda,N}_{\mu_N}$ is concentrated on absolutely continuous trajectories $\pi_t(du) = \rho(t, u)du$ such that, for any $H \in \mathfrak{D}_\Lambda$,*

$$\langle\!\langle \rho_t, H\rangle\!\rangle - \langle\!\langle \gamma, H\rangle\!\rangle \;=\; \int_0^t \langle\!\langle \rho_s\,,\, \mathcal{L}_\Lambda H\rangle\!\rangle \, ds\,. \tag{8.22}$$

By the previous lemma we can show Proposition 8.3.3.

Proof of Proposition 8.3.3. It just remains to extend the equality (8.22) to functions $H \in \mathcal{H}^1_\Lambda$. By Theorem 8.1.2, the set $\mathfrak{D}_\Lambda$ is a core for the Friedrichs extension $\mathbb{I} - \mathcal{L}_\Lambda : \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^d)$. Thus, for any $H \in \mathcal{H}^1_\Lambda$, there exists a sequence $H_n \in \mathfrak{D}_\Lambda$ such that $H_n \to H$ and $(\mathbb{I} - \mathcal{L}_\Lambda)\, H_n \to (\mathbb{I} - \mathcal{L}_\Lambda)\, H$, both in $L^2(\mathbb{T}^d)$. This implies that $\mathcal{L}_\Lambda\, H_n \to \mathcal{L}_\Lambda\, H$ in $L^2(\mathbb{T}^d)$. Replacing H_n in equality (8.22), and taking the limit as $n \to \infty$ finishes the proof. □ □

The remain of this section is devoted to the proof of Lemma 8.3.4. Fix a function $H \in \mathfrak{D}_\Lambda$ and define the martingale M_t^N by

$$\langle \pi_t^N, H\rangle \;-\; \langle \pi_0^N, H\rangle \;-\; \int_0^t N^2 L_N \langle \pi_s^N, H\rangle\, ds\,. \tag{8.23}$$

We claim that, for every $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}^N \Big[\sup_{0\le t\le T} \big|M_t^N\big| \;>\; \delta\Big] \;=\; 0\,. \tag{8.24}$$

For $H \in C^2$, this follows from Chebyshev inequality and the estimates done in the proof of tightness, where we have shown that

$$\lim_{N\to\infty} \mathbb{E}_\mu^N \left[\sup_{0\le t\le T} |M_t^N|\right] \;\le\; \lim_{N\to\infty} \mathbb{E}_\mu^N \left[\sup_{0\le t\le T} \langle M_t^N\rangle\right]^{\frac{1}{2}} = 0\,. \tag{8.25}$$

For $H = h + \lambda\,\mathbf{1}_\Lambda$ in $\mathfrak{D}_\Lambda$, the first inequality in (8.19) is still valid and

$$\begin{aligned}
\langle M_t^N\rangle \;&\le\; \tfrac{T}{N^{2d-2}} \sum_{j=1}^{d} \sum_{x\in\mathbb{T}_N^d} \xi_{x,x+e_j}^N \Big[H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N})\Big]^2 \\
&=\; \tfrac{T}{N^{2d-2}} \sum_{j=1}^{d} \sum_{x\notin\Gamma_N} \Big[h(\tfrac{x+e_j}{N}) - h(\tfrac{x}{N})\Big]^2 && (8.26)\\
&\qquad + \tfrac{T}{N^{2d-2}} \sum_{j=1}^{d} \sum_{x\in\Gamma_N} \xi_{x,x+e_j}^N \Big[H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N})\Big]^2, && (8.27)
\end{aligned}$$

where Γ_N is also defined in the proof of tightness. The expression (8.26) goes to zero as N increases, since the function h is Lipschitz. For the expression in (8.27), let $x \in \Gamma_N$. If $\frac{x}{N} \in \Lambda$ and $\frac{x+e_j}{N} \in \Lambda^\complement$, then $\xi_{x,x+e_j}^N \le \frac{1}{N}$. The same occurs if $\frac{x}{N} \in \Lambda^\complement$ and $\frac{x+e_j}{N} \in \Lambda$. If $\frac{x}{N}, \frac{x+e_j}{N}$ both belong to Λ or $\Lambda^\complement$, the exchange rate $\xi_{x,x+e_j}^N$ is one, but $|H(\frac{x+e_j}{N}) - H(\frac{x}{N})| = |h(\frac{x+e_j}{N}) - h(\frac{x}{N})| \le \frac{1}{N}\sup_{u\in\mathbb{T}^d} |\nabla H(u)\cdot e_j|$. In both cases, the expression (8.27) is of order $O(N^{-d})$. Therefore, from (8.25), we obtain (8.24). □

The next step is to show that we can replace $N^2\mathbb{L}_N$ by the continuous operator $\mathcal{L}_\Lambda$ in the martingale formula (8.23) and that the resulting expression still converges to zero in probability. This will follow from the ensuing proposition. Recall the definition of $\mathbb{L}_N$ given in (8.4).

Proposition 8.3.5. *For any $H \in \mathfrak{D}_\Lambda$,*

$$\lim_{N\to\infty} \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \Big| N^2\mathbb{L}_N H(\tfrac{x}{N}) - \mathcal{L}_\Lambda H(\tfrac{x}{N})\Big| = 0\,. \tag{8.28}$$

Proof. As usual, put $H = h + \lambda\,\mathbf{1}_\Lambda$, where $h \in C^2(\mathbb{T}^d)$. Rewrite the sum in (8.28) as

$$\frac{1}{N^d}\sum_{x\notin\Gamma_N}\Big|N^2\mathbb{L}_N H(\tfrac{x}{N}) - \mathcal{L}_\Lambda H(\tfrac{x}{N})\Big| + \frac{1}{N^d}\sum_{x\in\Gamma_N}\Big|N^2\mathbb{L}_N H(\tfrac{x}{N}) - \mathcal{L}_\Lambda H(\tfrac{x}{N})\Big|\,.$$

The first term above is equal to

$$\frac{1}{N^d}\sum_{x\notin\Gamma_N}\Big|N^2\Big(h(\tfrac{x+e_j}{N}) + h(\tfrac{x-e_j}{N}) - 2h(\tfrac{x}{N})\Big) - \Delta h(\tfrac{x}{N})\Big|\,,$$

which converges to zero because $h \in C^2$. The second one is less than or equal to the sum of

$$\frac{1}{N^d}\sum_{x\in\Gamma_N}|\Delta h(\tfrac{x}{N})| \tag{8.29}$$

and

$$\frac{1}{N^{d-1}}\sum_{x\in\Gamma_N}\sum_{j=1}^{d}\Big|N\xi^N_{x,x+e_j}(H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N})) \\ + N\xi^N_{x,x-e_j}(H(\tfrac{x-e_j}{N}) - H(\tfrac{x}{N}))\Big|\,. \tag{8.30}$$

Since there are $O(N^{d-1})$ terms in Γ_N, the expression in (8.29) converges to zero as $N\to\infty$. Since $\partial\Lambda$ is smooth, the quantity of points $x\in\Gamma_N$ for which both $\xi^N_{x,x+e_j}$ and $\xi^N_{x,x-e_j}$ are different of one is negligible. Therefore, we must only worry about points $x\in\Gamma_N$ such that, for some j, only one of $\xi^N_{x,x+e_j}$ and $\xi^N_{x,x-e_j}$ is of order N^{-1}. This occurs in one of the following four cases: $\frac{x}{N}\in\Lambda$, $\frac{x-e_j}{N}\in\Lambda$ and $\frac{x+e_j}{N}\in\Lambda^\complement$; $\frac{x}{N}\in\Lambda$, $\frac{x-e_j}{N}\in\Lambda^\complement$ and $\frac{x+e_j}{N}\in\Lambda$; $\frac{x}{N}\in\Lambda^\complement$, $\frac{x-e_j}{N}\in\Lambda$ and $\frac{x+e_j}{N}\in\Lambda^\complement$; $\frac{x}{N}\in\Lambda^\complement$, $\frac{x-e_j}{N}\in\Lambda^\complement$ and $\frac{x+e_j}{N}\in\Lambda$. The analysis of these cases are analogous, thus we only consider the first one. Suppose $\frac{x}{N}\in\Lambda$, $\frac{x-e_j}{N}\in\Lambda$ and $\frac{x+e_j}{N}\in\Lambda^\complement$. In this case, the summand in (8.30) can be rewritten as

$$\begin{aligned}&N\xi^N_{x,x+e_j}(H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N})) + N\xi^N_{x,x-e_j}(H(\tfrac{x-e_j}{N}) - H(\tfrac{x}{N}))\\ &= |\vec{\zeta}_{x,j}\cdot e_j|\,[H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N})]\;+N\,[H(\tfrac{x-e_j}{N}) - H(\tfrac{x}{N})]\,,\end{aligned}$$

which becomes uniformly (in $x\in\Gamma_N$) close to

$$-\lambda|\vec{\zeta}_{x,j}\cdot e_j|\,\operatorname{sgn}\Big(\vec{\zeta}_{x,j}\cdot e_j\Big) - \tfrac{\partial h}{\partial u_j}(\tfrac{x}{N}) = -\lambda\,\vec{\zeta}_{x,j}\cdot e_j - \tfrac{\partial h}{\partial u_j}(\tfrac{x}{N})\,.$$

The condition $\nabla h|_{\partial\Lambda}(u) = -\lambda\,\vec{\zeta}(u)$, which was imposed in the definition of $\mathfrak{D}_\Lambda$, implies that

$$\lim_{N\to\infty} N\xi^N_{x,x+e_j}(H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N})) + N\xi^N_{x,x-e_j}(H(\tfrac{x-e_j}{N}) - H(\tfrac{x}{N})) = 0\,.$$

Therefore, the terms in (8.30) converge uniformly to zero, and the same holds for the whole sum. □ □

Corollary 8.3.6. *For $H \in \mathfrak{D}_\Lambda$ and for every $\delta > 0$,*

$$\lim_{N\to\infty} \mathbb{Q}^{\Lambda,N}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\langle \pi^N_t, H\rangle \;-\; \langle \pi^N_0, H\rangle \;-\; \int_0^t \langle \pi^N_s, \mathcal{L}_\Lambda H\rangle\, ds\Big| \;>\; \delta\Big] \;=\; 0\,.$$

Proof. By a simple calculation, the martingale defined in (8.23) can be rewritten as

$$M^N_t = \langle \pi^N_t, H\rangle \;-\; \langle \pi^N_0, H\rangle \;-\; \int_0^t \langle \pi^N_s, N^2\mathbb{L}_N\, H\rangle\, ds\,.$$

The result follows from Proposition 8.3.5 and expression (8.24). □ □

At this point we have all the ingredients needed to prove Lemma 8.3.4, which says that, under $\mathbb{Q}^*$, with probability one, (8.22) holds for any $H \in \mathfrak{D}_\Lambda$. In order to prove this, it is enough to show that, for any $\delta > 0$, and any $H \in \mathfrak{D}_\Lambda$,

$$\mathbb{Q}^*\Big[\sup_{0\le t\le T}\Big|\langle \pi_t, H\rangle \;-\; \langle \pi_0, H\rangle \;-\; \int_0^t \langle \pi_s, \mathcal{L}_\Lambda H\rangle\; ds\Big| \;>\; \delta\Big] \;=\; 0\,. \tag{8.31}$$

So let H be a fixed function in $\mathfrak{D}_\Lambda$. The idea to estimate the probability in (8.31) is to apply Portmanteau's Theorem to replace Q^* by $\mathbb{Q}^{\Lambda,N}_{\mu_N}$ and then use Corollary 8.3.6. But to obtain an appropriate inequality we need the set

$$\Big\{\sup_{0\le t\le T}\Big|\langle \pi_t, H\rangle \;-\; \langle \pi_0, H\rangle \;-\; \int_0^t \langle \pi_s, \mathcal{L}_\Lambda H\rangle\; ds\Big| \;>\; \delta\Big\}$$

to be open in $D([0,T],\mathcal{M})$. In order to guarantee this, we need H to be continuous which is not the case. To solve this problem, we use approximations of H by smooth functions.

For $\varepsilon > 0$, define

$$(\partial\Lambda)^\varepsilon = \{u \in \mathbb{T}^d;\, \mathrm{dist}(u, \partial\Lambda) \le \varepsilon\}\,.$$

Let H^ε be a smooth function which coincides with H on $\mathbb{T}^d \backslash (\partial\Lambda)^\varepsilon$ and $\sup_{\mathbb{T}^d}|H^\varepsilon| \le \sup_{\mathbb{T}^d}|H|$. Fix $\delta > 0$. By the triangular inequality,

$$\begin{aligned}
&\mathbb{Q}^*\Big[\sup_{0\le t\le T}\Big|\langle \pi_t, H\rangle \;-\; \langle \pi_0, H\rangle \;-\; \int_0^t \langle \pi_s, \mathcal{L}_\Lambda H\rangle\; ds\Big| \;>\; \delta\Big]\\
&\le \mathbb{Q}^*\Big[\sup_{0\le t\le T}\Big|\langle \pi_t, H^\varepsilon\rangle \;-\; \langle \pi_0, H^\varepsilon\rangle \;-\; \int_0^t \langle \pi_s, \mathcal{L}_\Lambda H\rangle\; ds\Big| \;>\; \delta/3\Big]\\
&+ 2\,\mathbb{Q}^*\Big[\sup_{0\le t\le T}\Big|\langle \pi_t, H^\varepsilon - H\rangle\Big| \;>\; \delta/3\Big]\,.
\end{aligned} \tag{8.32}$$

Recall that $\mathbb{Q}^*$ is concentrated on trajectories $\pi_t(du) = \rho(t,u)du$ whose density ρ is non-negative and bounded above by 1. Then, under $\mathbb{Q}^*$,

$$\begin{aligned}
\sup_{0\le t\le T}|\langle \pi_t, H^\varepsilon - H\rangle| \;&\le\; \sup_{0\le t\le T}\int_{(\partial\Lambda)^\varepsilon} \rho(t,u)\,|H^\varepsilon(u) - H(u)|\,du\\
&\le\; 2\,\ell_d((\partial\Lambda)^\varepsilon)\,\sup_{u\in\mathbb{T}^d}|H(u)|\,.
\end{aligned}$$

Therefore, for small enough ε, the second probability in the right hand side of inequality (8.32) is null. So it remains to show that

$$\mathbb{Q}^*\Big[\sup_{0\le t\le T}\Big|\langle \pi_t, H^\varepsilon\rangle \ - \ \langle \pi_0, H^\varepsilon\rangle \ - \ \int_0^t \langle \pi_s, \mathcal{L}_\Lambda H\rangle \, ds\Big| \ > \ \delta/3\Big] = 0\,.$$

If G_1, G_2, G_3 are continuous functions, the application from $D([0,T],\mathcal{M})$ to $\mathbb{R}$ that associates to a trajectory $\{\pi_t, 0\le t\le T\}$ the number

$$\sup_{0\le t\le T}\Big|\langle \pi_t, G_1\rangle \ - \ \langle \pi_0, G_2\rangle \ - \ \int_0^t \langle \pi_s, G_3\rangle \, ds\Big|$$

is continuous in the Skorohod metric. Notice that H^ε and $\mathcal{L}_\Lambda H$ are continuous functions. By Portmanteau's Theorem,

$$\begin{aligned}
&\mathbb{Q}^*\Big[\sup_{0\le t\le T}\Big|\langle \pi_t, H^\varepsilon\rangle \ - \ \langle \pi_0, H^\varepsilon\rangle \ - \ \int_0^t \langle \pi_s, \mathcal{L}_\Lambda H\rangle \, ds\Big| \ > \ \delta/3\Big] \\
&\le \varliminf_{N\to\infty} \mathbb{Q}^{\Lambda,N}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\langle \pi^N_t, H^\varepsilon\rangle \ - \ \langle \pi^N_0, H^\varepsilon\rangle \ - \ \int_0^t \langle \pi^N_s, \mathcal{L}_\Lambda H\rangle \, ds\Big| \ > \ \delta/3\Big]\,, \qquad (8.33)
\end{aligned}$$

since $\mathbb{Q}^{\Lambda,N}_{\mu_N}$ converges weakly to $\mathbb{Q}^*$ and the above set is open.

Now we replace H^ε by H. This may be confusing to the reader, however the previous introduction of H^ε was a necessary step in the proof. From this point, to deal with the right hand side in (8.33), we need Corollary 8.3.6. Hence H^ε should be replaced by H.

By definition,

$$\begin{aligned}
\sup_{0\le t\le T}\Big|\langle \pi^N_t, H^\varepsilon - H\rangle\Big| &\le \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N}\Big|H^\varepsilon(x/N) - H(x/N)\Big| \\
&\le \Big(\ell_d((\partial\Lambda)^\varepsilon) + O(\tfrac{1}{N})\Big)\, 2\sup_{u\in\mathbb{T}}|H(u)|\,,
\end{aligned}$$

because H^ε coincides with H in $\mathbb{T}\backslash(\partial\Lambda)^\varepsilon$. Using the same argument as before, we obtain

$$\begin{aligned}
&\varliminf_{N\to\infty} \mathbb{Q}^{\Lambda,N}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\langle \pi_t, H^\varepsilon\rangle \ - \ \langle \pi_0, H^\varepsilon\rangle \ - \ \int_0^t \langle \pi_s, \mathcal{L}_\Lambda H\rangle \, ds\Big| \ > \ \delta/3\Big] \\
&\le \varlimsup_{N\to\infty} \mathbb{Q}^{\Lambda,N}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\langle \pi_t, H\rangle \ - \ \langle \pi_0, H\rangle \ - \ \int_0^t \langle \pi_s, \mathcal{L}_\Lambda H\rangle \, ds\Big| \ > \ \delta/9\Big] \\
&\qquad +2\varlimsup_{N\to\infty} \mathbb{Q}^{\Lambda,N}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\langle \pi_t, H^\varepsilon - H\rangle\Big| \ > \ \delta/9\Big]\,.
\end{aligned}$$

Again, for small enough ε, the second probability in the sum above is null. From Corollary 8.3.6, we finally conclude that (8.31) holds. Therefore $\mathbb{Q}^*$ is concentrated on absolutely continuous paths $\pi_t(du) = \rho(t,u)du$ with positive density bounded by 1, and $\mathbb{Q}^*$ a.s.

$$\langle\!\langle \rho_t, H\rangle\!\rangle - \langle\!\langle \rho_0, H\rangle\!\rangle \ = \ \int_0^t \langle\!\langle \rho_s\,,\, \mathcal{L}_\Lambda H\rangle\!\rangle \, ds\,,$$

for any $H\in\mathfrak{D}_\Lambda$. Hence we have proved Lemma 8.3.4.

8.3.3 Uniqueness of weak solutions

Now, we prove that the solution of (8.5) is unique. It suffices to check that the only solution of (8.5) with $\rho_0 \equiv 0$ is $\rho \equiv 0$, because of the linearity of $\mathcal{L}_\Lambda$. Let $\rho : \mathbb{R}_+ \times \mathbb{T}^d \to \mathbb{R}$ be a weak solution of the parabolic differential equation

$$\begin{cases} \partial_t \rho \;=\; \mathcal{L}_\Lambda \rho \\ \rho(0,\cdot) \;=\; 0\,. \end{cases}$$

By definition,

$$\langle\!\langle \rho_t, H \rangle\!\rangle \;=\; \int_0^t \langle\!\langle \rho_s, \mathcal{L}_\Lambda H \rangle\!\rangle \, ds \;, \tag{8.34}$$

for all functions H in $\mathcal{H}^1_\Lambda$ and all $t > 0$. From the Theorem 8.1.2, the operator $-\mathcal{L}_\Lambda$ has countable eigenvalues $\{\mu_n : n \geq 0\}$ and eigenvectors $\{F_n\}$. All eigenvalues have finite multiplicity, $0 = \mu_0 \leq \mu_1 \leq \cdots$, and $\lim_{n\to\infty} \mu_n = \infty$. Besides, the eigenvectors $\{F_n\}$ form a complete orthonormal system in the $L^2(\mathbb{T}^d)$. Define

$$R(t) = \sum_{n\in\mathbb{N}} \frac{1}{n^2(1+\mu_n)} \langle\!\langle \rho_t, F_n \rangle\!\rangle^2,$$

for all $t > 0$. Notice that $R(0) = 0$ and $R(t)$ is well defined because ρ_t belongs to $L^2(\mathbb{T}^d)$. Since ρ satisfy (8.34), we have that $\frac{d}{dt}\langle\!\langle \rho_t, F_n \rangle\!\rangle^2 = -2\mu_n \langle\!\langle \rho_t, F_n \rangle\!\rangle^2$. Then

$$(\tfrac{d}{dt} R)(t) = -\sum_{n\in\mathbb{N}} \frac{2\mu_n}{n^2(1+\mu_n)} \langle\!\langle \rho_t, F_n \rangle\!\rangle^2 \,,$$

because $\sum_{n\leq N} \frac{-2\mu_n}{n^2(1+\mu_n)} \langle\!\langle \rho_t, F_n \rangle\!\rangle^2$ converges uniformly to $\sum_{n\in\mathbb{N}} \frac{-2\mu_n}{n^2(1+\mu_n)} \langle\!\langle \rho_t, F_n \rangle\!\rangle^2$, as N increases to infinity. Thus $R(t) \geq 0$ and $(\frac{d}{dt} R)(t) \leq 0$, for all $t > 0$ and $R(0) = 0$. From this, we obtain $R(t) = 0$ for all $t > 0$. Since $\{F_n\}$ is a complete orthonormal system, $\langle\!\langle \rho_t, \rho_t \rangle\!\rangle = 0$, for all $t > 0$, which implies $\rho \equiv 0$.

Part IV

Appendix

Appendix A

A.1 Analysis tools

Proposition A.1.1. *Let $\mathcal{H}$ be a Hilbert space, $f : \mathcal{H} \to \mathbb{R}$ a linear functional. If there exists $K_0 > 0$ and there exists the positive integer number κ such that*

$$\sup_{x \in \mathcal{H}} \{f(x) - \kappa \|x\|^2_{\mathcal{H}}\} \leq K_0, \tag{A.1}$$

then f is bounded.

Proof. The supremum above implies $|f(x)| \leq K_0 + \kappa \|x\|^2_{\mathcal{H}}$, for all $x \in \mathcal{H}$. Thus, $\|f\|_{\mathcal{H}^*} = \sup_{\|x\|_{\mathcal{H}} \leq 1} |f(x)| \leq K_0 + \kappa$. □

Proposition A.1.2. *Let E be a metric space, $F \subseteq E$ a closed set and $g : F \to \overline{\mathbb{R}}$ a lower semi-continuous functional. Then, the extension*

$$f(x) \;=\; \begin{cases} g(x), & \text{if } x \in F, \\ +\infty, & \text{otherwise,} \end{cases}$$

is lower semi-continuous.

Proof. Consider a sequence $x_n \to x$.

If $x \notin F$, since $F^{\complement}$ is open, then $\underline{\lim}_n f(x_n) = +\infty = f(x)$.

If $x \in F$, and only finite x_n belong to F, then $\underline{\lim}_n f(x_n) = +\infty \geq f(x)$.

If $x \in F$, and there are infinite $x_n \in F$, let be x_{n_k} the subsequence of all these terms whose belongs F. Since g is lower semi-continuous, then $\underline{\lim}_n f(x_n) = \underline{\lim}_{n_k} g(x_{n_k}) \geq g(x) = f(x)$. □

Proposition A.1.3. *Given the sequences of real numbers $a_N, b_N \geq 0$ and $c_N \nearrow \infty$,*

$$\overline{\lim_N} \tfrac{1}{c_N} \log(a_N + b_N) \;=\; \max\left\{ \overline{\lim_N} \tfrac{1}{c_N} \log a_N,\, \overline{\lim_N} \tfrac{1}{c_N} \log b_N \right\}$$

Proof. Since the logarithmic function is increasing, the left side above is greater or equal than the left side. On the other hand,

$$\begin{aligned}
\overline{\lim_N} \tfrac{1}{c_N} \log(a_N + b_N) &\leq \overline{\lim_N} \tfrac{1}{c_N} \log 2 \max\{a_N, b_N\} \\
&= \overline{\lim_N} \tfrac{1}{c_N} \log \max\{a_N, b_N\} \\
&= \max\left\{ \overline{\lim_N} \tfrac{1}{c_N} \log a_N,\, \overline{\lim_N} \tfrac{1}{c_N} \log b_N \right\}.
\end{aligned}$$

□

Proposition A.1.4. *Let E be a metric space, and $f, g : E \to \overline{\mathbb{R}}$ two lower semi-continuous functionals. Then, $f \vee g$ and $f + g$ are also lower semi-continuous.*

Proof.

$$(f \vee g)(x) \leq \varliminf_{x_n \to x} f(x_n) \vee \varliminf_{x_n \to x} g(x_n) \leq \varliminf_{x_n \to x} (f \vee g)(x_n)$$

and

$$(f + g)(x) \leq \varliminf_{x_n \to x} f(x_n) + \varliminf_{x_n \to x} g(x_n) \leq \varliminf_{x_n \to x} (f + g)(x_n)\,.$$

□

Proposition A.1.5. *Let $\{f_n\}$ be a sequence of lower semi-continuous functions. Then $\sup_n f_n$ is a lower semi-continuous function.*

Proof. For all $x \in E$,

$$f_n(x) \leq \varliminf_{x_k \to x} f_n(x_k) \leq \varliminf_{x_k \to x} \Big[\sup_n f_n(x_k)\Big]\,, \quad \forall n\,.$$

□

Proposition A.1.6. *Let $\{f_n\}$ be a sequence of convex functions. Then $\sup_n f_n$ is a convex function.*

Proof. For all $x, y \in E$ and for each $\theta \in [0,1]$,

$$f_n\Big(\theta x + (1-\theta) y\Big) \leq \theta f_n(x) + (1-\theta) f_n(y) \leq \theta \Big[\sup_n f_n(x)\Big] + (1-\theta)\Big[\sup_n f_n(y)\Big]\,, \quad \forall n\,.$$

□

Proposition A.1.7. *Assume that L is a reversible generator with respect to an invariant measure ν in a countable space-state E, and $V : \mathbb{R}_+ \times E \to \mathbb{R}$ is a bounded function (clearly, $L + V_t$ will be a symmetric operator in $L^2(\nu)$). Denote by Γ_t the largest eigenvalue of $L + V_t$:*

$$\Gamma_t = \sup_{\|f\|_2 = 1} \Big\{ \langle V_t, f^2 \rangle_\nu + \langle L f, f \rangle_\nu \Big\}\,.$$

Then, the supremum above can be taken over only positive functions f, or else,

$$\Gamma_t = \sup_{f \text{ density}} \Big\{ \langle V_t, (\sqrt{f})^2 \rangle_\nu + \langle L\sqrt{f}, \sqrt{f} \rangle_\nu \Big\}\,.$$

Proof. Follows from the expression for the Dirichlet Form (see [16]),

$$\langle Lf, f\rangle_\nu = -\tfrac{1}{2}\sum_{x,y\in E} \nu(x)\, L(x,y)[f(y)-f(x)]^2\,,$$

and the inequality $|\,|f(y)|-|f(x)|\,| \le |f(y)-f(x)|$. $\square$

Proposition A.1.8. *Denote by $H_N(\mu_N|\nu_\alpha^N)$ the entropy of a probability measure μ_N with respect to a stationary state ν_α^N. We refer to [16, Section A1.8] for a precise definition. Then, there exists a finite constant K_0, depending only on α, such that*

$$H_N(\mu_N|\nu_\alpha^N) \,\le\, K_0 N\,,$$

for all probability measures μ_N.

Proof. Recall that ν_α^N is Bernoulli product of parameter α. By the explicit formula given in [16, Theorem A1.8.3],

$$\begin{aligned}
H_N(\mu_N|\nu_\alpha^N) &= \sum_{\eta\in\{0,1\}^{\mathbb{T}_N}} \mu_N(\eta)\,\log\frac{\mu_N(\eta)}{\nu_\alpha^N(\eta)}\\
&\le \sum_{\eta\in\{0,1\}^{\mathbb{T}_N}} \mu_N(\eta)\,\log\frac{1}{\nu_\alpha^N(\eta)}\\
&\le \sum_{\eta\in\{0,1\}^{\mathbb{T}_N}} \mu_N(\eta)\,\log\frac{1}{[\alpha\wedge(1-\alpha)]^N}\\
&= N\left(-\log[\alpha\wedge(1-\alpha)]\right).
\end{aligned}$$

$\square$

Recall the Definition 2.4.2 of the space $L^2(0,T;\mathcal{H}^1(\mathbb{T}\backslash\{a\}))$: space of all measurable functions $\xi:[0,T]\to\mathcal{H}^1(\mathbb{T}\backslash\{a\})$ with

$$\|\xi\|_{L^2(0,T;\mathcal{H}^1(\mathbb{T}\backslash\{a\}))} \,:=\, \Big(\int_0^T \|\xi_t\|^2_{\mathcal{H}^1(\mathbb{T}\backslash\{a\})}\,dt\Big)^{1/2} \,<\, \infty\,.$$

Lemma A.1.9. *If a function $\xi\in L^2([0,T]\times\mathbb{T})$ is such that there exists a function $\partial\xi\in L^2([0,T]\times\mathbb{T})$ satisfying*

$$\langle\!\langle \partial_u H, \xi\rangle\!\rangle \,=\, -\langle\!\langle H, \partial\xi\rangle\!\rangle\,,$$

for all functions $H\in C^{0,1}([0,T]\times\mathbb{T})$ with compact support in $[0,T]\times(\mathbb{T}\backslash\{a\})$, then $\xi\in L^2(0,T;\mathcal{H}^1(\mathbb{T}\backslash\{a\}))$.

Proof. The function $\xi:[0,T]\to\mathcal{H}^1(\mathbb{T}\backslash\{a\})$ is mensurable, if:

(*i*) The function $\xi:[0,T]\to\mathcal{H}^1(\mathbb{T}\backslash\{a\})$ is weakly measurable, i.e., for all $G\in L^2(\mathbb{T}\backslash\{a\})$, the function $t\mapsto\langle G,\partial\xi_t\rangle$ is Lebesgue mensurable.

(*ii*) The function $\xi : [0,T] \to \mathcal{H}^1(\mathbb{T}\backslash\{a\})$ is almost separably valued, i.e., there exists a subset $N \subset [0,T]$, with $|N| = 0$, such that the set $\{\xi_t; t \in [0,T]\backslash N\}$ is separable.

In this case, $\mathcal{H}^1(\mathbb{T}\backslash\{a\})$ is separable, since any subset of a separable Banach space is itself separable, one can take N above to be empty, and it follows that we need verify the weak measurability.

We know that $t \mapsto \langle G, \partial\xi_t\rangle$ is Lebesgue mensurable, for all $G \in C^1(\mathbb{T}\backslash\{a\})$ with compact support. By density, one can conclude that the function $\xi : [0,T] \to \mathcal{H}^1(\mathbb{T}\backslash\{a\})$ is weakly measurable.

To show that the norm $\|\xi\|_{L^2(0,T;\mathcal{H}^1(\mathbb{T}\backslash\{a\}))}$ is finite, use that $\partial\xi \in L^2([0,T]\times\mathbb{T})$. □

Lemma A.1.10. *Let ρ be a function in $L^2(0,T;\mathcal{H}^1(\mathbb{T}\backslash\{a\}))$. Then, for any $F \in C^{0,1}([0,T]\times(\mathbb{T}\backslash\{a\}))$:*

$$\begin{aligned}&\int_0^T\int_0^1 \rho_s(u)\,\partial_u F_s(u)\,du\,ds\\&= -\int_0^T\int_0^1 \partial_u\rho_s(u)\,F(u,s)\,du\,ds + \int_0^T \Big\{\rho_s(1)\,F_s(1) - \rho_s(1)\,F_s(0)\Big\}\,ds\,.\end{aligned}$$

Notice the partial derivative in ρ is the weak derivative, while the partial derivative in H is the usual one. Besides that, the function F is smooth, but possibly not null at the boundary $[0,T]\times\{0,1\}$, and therefore is not valid the integration by parts in the sense of the Sobolev space $L^2(0,T;\mathcal{H}^1(\mathbb{T}\backslash\{a\}))$, which has no boundary integrals.

This lemma is proved in Lemma 7.6.1.

A.2 Skorohod space

Proposition A.2.1. *$\mathcal{M}_0$ is a closed subset of $\mathcal{M}$ endowed with the weak* topology.*

Proof. Let $\pi_n \in \mathcal{M}_0$, $\pi_n \stackrel{\omega^*}{\to} \pi$. It is enough to prove that $0 \le \pi([a,b]) \le b-a$, for all $[a,b] \subset \mathbb{T}$. Take a continuous function $f_\varepsilon : \mathbb{T} \to \mathbb{R}$ such that $f_\varepsilon(u) = 1$ if u belongs to the interval $[a,b]$, $f_\varepsilon(u) = 0$, if $x \in \mathbb{T}\backslash[a-\varepsilon, b+\varepsilon]$ and is linearly interpolated in the other regions. Thus,

$$b-a+2\varepsilon \ge \lim_{n\to\infty}\int_{\mathbb{T}} f_\varepsilon(u)\rho_n(u)du = \int_{\mathbb{T}} f_\varepsilon(u)d\pi(u) \ge \pi([a,b])\,.$$

□

Proposition A.2.2. *$\mathcal{D}\big([0,T],\mathcal{M}_0\big)$ is a closed subset of $\mathcal{D}\big([0,T],\mathcal{M}\big)$ for the Skorohod topology.*

Proof. Let $\{\pi_n(t,\cdot); t \in [0,T]\} \subset \mathcal{D}\big([0,T],\mathcal{M}_0\big)$, $\{\pi_n(t,\cdot); t \in [0,T]\}$ converges to $\{\pi(t,\cdot); t \in [0,T]\}$ in the Skorohod topology. By Skorohod metric, for all $t \in [0,T]$ and $\varepsilon > 0$ fixed, there exists $t_\varepsilon \in [0,t]$ such that $|t_\varepsilon - t| < \varepsilon$ and $\pi_n(t_\varepsilon,\cdot) \stackrel{\omega^*}{\to} \pi(t,\cdot)$, as n increases to infinity. Each $\pi_n(t_\varepsilon,\cdot) \in \mathcal{M}_0$ and $\mathcal{M}_0$ is a closed subset of $\mathcal{M}$, then $\pi(t,\cdot) \in \mathcal{M}_0$. □

Lemma A.2.3. *Let* $\{\pi_n(t,\cdot);\, t \in [0,T]\}$ *converging to* $\{\pi(t,\cdot);\, t \in [0,T]\}$ *in the space* $\mathcal{D}\big([0,T],\mathcal{M}\big)$ *with the Skorohod topology. Then* $\pi_n(t,\cdot) \stackrel{\omega^*}{\to} \pi(t,\cdot)$*, as* n *increases to infinity, for almost surely* $t \in (0,T)$*, for* $t = 0$ *and for* $t = T$.

Proof. First, we recall the definition of Skorohod metric,

$$d(\pi_n,\pi) = \inf_{\lambda\in\Lambda} \max\left\{ \|\lambda\|,\, \sup_{0\le t\le T} \delta(\pi_n(t,\cdot),\pi(\lambda(t),\cdot)) \right\},$$

where Λ is a set of strictly increasing continuous functions λ of $[0,T]$ onto itself, $\|\lambda\| = \sup_{s\neq t}\left|\log\frac{\lambda(t)-\lambda(s)}{t-s}\right|$ and δ is the metric wich metrize the convergence ω^* in $\mathcal{M}$. Since $\{\pi(t,\cdot);\, t \in [0,T]\} \in \mathcal{D}\big([0,T],\mathcal{M}\big)$, the function $t \to \pi(t,\cdot)$ has at most countably many points of discontinuity. Let $t \in (0,T)$ point of continuity of $t \to \pi(t,\cdot)$,

$$\delta(\pi_n(t,\cdot),\pi(t,\cdot)) \le \delta(\pi_n(t,\cdot),\pi(\lambda(t),\cdot)) + \delta(\pi(\lambda(t),\cdot),\pi(t,\cdot))\,.$$

Choosing a suitable $\lambda \in \Lambda$, the last two terms above are small, when n is large. Thus the convergence almost surely is true.

The convergence in $t = 0$ and $t = T$ is obtained from the fact that functions $\lambda \in \Lambda$ must to satisfy $\lambda(0) = 0$ and $\lambda(T) = T$. $\square$

Proposition A.2.4. *Let* $G^i \in C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$*, for all* $i = 1,2,3$*. The functional*

$$\pi \in \mathcal{D}([0,T],\mathcal{M}_0) \mapsto \langle \pi_T, G^1_T\rangle - \langle \pi_0, G^2_0\rangle - \int_0^T \langle \pi_t, G^3_t\rangle\, dt$$

is continuous in Skorohod topology.

Proof. This statement follows from A.2.3. $\square$

Proposition A.2.5. *Let* $H \in C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$*, the linear functional defined by* $\ell^{int}_H : \mathcal{D}([0,T],\mathcal{M}_0) \to \mathbb{R}$*, defined in* (4.2)*, is continuous in Skorohod topology.*

Proof. By definition of ℓ^{int}_H and the fact that $H \in C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$, we have that the functions H_t and $\partial_t H_t + \Delta H_t$ belongs to $C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$, $\forall t \in [0,T]$. Proposition A.2.4 concludes the proof. $\square$

Proposition A.2.6. *Let* $H \in C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$*, the functional*

$$\pi(\cdot,du) = \rho(\cdot,u)du \in \mathcal{D}([0,T],\mathcal{M}_0) \mapsto \int_0^T \langle \chi(\rho_t), (\partial_u H_t)^2\rangle\, dt$$

is upper semi-continuous in Skorohod topology.

Proof. Let $\{\pi_n(t,\cdot);\, t\in[0,T]\}$ converging to $\{\pi(t,\cdot);\, t\in[0,T]\}$ in the space $\mathcal{D}\big([0,T],\mathcal{M}_0\big)$ with the Skorohod topology. By Proposition A.2.3, $\pi_n(t,\cdot)\overset{\omega^*}{\to}\pi(t,\cdot)$, as n increases to infinity, for almost surely $t\in[0,T]$. Consider the approximation of the identity $\iota_\varepsilon(u)=\frac{1}{2\varepsilon}\mathbf{1}_{(-\varepsilon,\varepsilon)}(u)$. Since $\rho*\iota_\varepsilon$ converges, as $\varepsilon\downarrow 0$, to ρ in $L^1([0,T]\times\mathbb{T})$ and $|\rho_t^2-(\rho_t*\iota_\varepsilon)^2|\le 2|\rho_t-\rho_t*\iota_\varepsilon|$, then

$$\int_0^T\langle\chi(\rho_t),(\partial_u H_t)^2\rangle\,dt=\lim_{\varepsilon\to 0}\int_0^T\langle\chi(\rho_t*\iota_\varepsilon),(\partial_u H_t)^2\rangle\,dt\,,$$

By Portmanteau Theorem, $(\rho^n*\iota_\varepsilon)(t,u)=\frac{1}{2\varepsilon}\int_{[u-\varepsilon,u+\varepsilon]}\rho_t^n(v)\,dv$ converves, as $n\to\infty$, to $(\rho*\iota_\varepsilon)(t,u)=\frac{1}{2\varepsilon}\int_{[u-\varepsilon,u+\varepsilon]}\rho_t(v)\,dv$, for almost all (t,u). And, since χ is concave, the right side above is equal to

$$\begin{aligned}&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_0^T\Big\langle\chi\Big(\tfrac{1}{2\varepsilon}\int_{[\cdot-\varepsilon,\cdot+\varepsilon]}\rho_t^n(v)\,dv\Big),(\partial_u H_t)^2\Big\rangle\,dt\\ &\ge\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_0^T\Big\langle\tfrac{1}{2\varepsilon}\int_{[\cdot-\varepsilon,\cdot+\varepsilon]}\chi(\rho_t^n(v))\,dv,(\partial_u H_t)^2\Big\rangle\,dt\,.\end{aligned}$$

Since ρ^n is uniformly bounded by 1 and $H\in C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$, it is easy to see that we may interchange limits. This shows that the last expression is equal to

$$\lim_{n\to\infty}\int_0^T\langle\chi(\rho_t^n),(\partial_u H_t)^2\rangle\,dt\,.$$

□

Proposition A.2.7. *If G_1, G_2, G_3 are continuous functions defined in the torus $\mathbb{T}$, the application from $D([0,T],\mathcal{M})$ to $\mathbb{R}$ that associates to a trajectory $\{\pi_t:0\le t\le T\}$ the number*

$$\sup_{0\le t\le T}\Big|\,\langle\pi_t,G_1\rangle\;-\;\langle\pi_0,G_2\rangle\;-\;\int_0^t\langle\pi_s,G_3\rangle\,ds\,\Big|$$

is continuous for the Skorohod metric in $D([0,T],\mathcal{M})$.

Proof. If G is a continuous function in the torus, the application $\pi\mapsto\langle\pi,G\rangle$ is a continuous application from $\mathcal{M}$ to $\mathbb{R}$ in the weak topology. From this observation and the definition of the Skorohod metric as an infimum under reparametrizations (c.f. [16]), the statement follows. □

A.3 Properties of weak solutions of (1.7)

In Section 2.4, we prove that the weak solution of hydrodynamic equation belongs to a Sobolev space, then these properties may follow from this fact. But here we present different proofs. In the first two lemmata we use only the definition of weak solution.

Lemma A.3.1. *Let $\rho : [0,T] \times \mathbb{T} \to \mathbb{R}$ the solution of* (1.7)*. Then, the function*

$$\psi_a(t) = \lim_{\varepsilon \downarrow 0} \int_0^t \Big[\frac{1}{\varepsilon}\int_{(a-\varepsilon,a)} \rho_s(u)\,du\Big]\,ds$$

is well defined and it is absolutely continuous with respect to Lebesgue.

Proof. Let's choose the auxiliar function $h_\varepsilon(z) = \frac{1}{\varepsilon}\mathbf{1}_{(a-\varepsilon,a)}(z) - 1$, which belongs to $L^2(\mathbb{T})$ and satisfies $\int_{\mathbb{T}} h_\varepsilon(z)\,dz = 0$. Now we define

$$H_\varepsilon(x) \;=\; \int_{(0,x]} \left(\beta_\varepsilon + \int_0^y h_\varepsilon(z)\,dz\right)(dy + \delta_1(dy))\,,$$

for $\beta_\varepsilon \in \mathbb{R}$ such that $\int_{(0,1]}\big(\beta_\varepsilon + \int_0^y h_\varepsilon(z)\,dz\big)(dy+\delta_1(dy)) = 0$. Notice that H_ε, defined in this way, belongs to $\mathcal{D}_W \subset C^{1,2}\big([0,T]\times\overline{\mathbb{T}\backslash\{a\}}\big)$. Since $\rho : [0,T]\times\mathbb{T}\to\mathbb{R}$ is a integral solution of (1.7),

$$\begin{aligned}
\int_0^t ds\,\Big[\frac{1}{\varepsilon}\int_{(a-\varepsilon,a)} \rho_s(u)\,du\Big] \;&=\; \int_0^t ds\,\langle \rho_s, h_\varepsilon\rangle + \int_0^t ds\,\langle\rho_s, 1\rangle\\
&=\; \langle \rho_t, H_\varepsilon\rangle - \langle\rho_0, H_\varepsilon\rangle + \int_0^t ds\,\langle \rho_s, 1\rangle.
\end{aligned}$$

By the choose of h_ε, we have that $\int_0^y h_\varepsilon(z)\,dz = \frac{1}{\varepsilon}\int_{(a-\varepsilon,a)}\mathbf{1}_{[0,y]}(z)\,dz - y$ converges to $\mathbf{1}_{[0,y]}(a) - y$, as $\varepsilon \downarrow 0$. This fact and Dominated Convergence Theorem implies that H_ε converges to H, where H is equal to

$$H(x) \;=\; \int_{(0,x]} \big(\beta + \mathbf{1}_{[0,y]}(a) - y\big)\,(dy + \delta_1(dy))\,,$$

for $\beta \in \mathbb{R}$ such that $\int_{(0,1]}\big(\beta + \mathbf{1}_{[0,y]}(a) - y\big)(dy + \delta_1(dy)) = 0$. Thus,

$$\int_0^t ds\,\Big[\frac{1}{\varepsilon}\int_{(a-\varepsilon,a)}\rho_s(u)\,du\Big] \to \langle\rho_t, H\rangle - \langle\rho_0, H\rangle + \int_0^t ds\,\langle\rho_s, 1\rangle,$$

uniformly in t, as $\varepsilon \downarrow 0$, because ρ is a bounded function. So, we have proved that ψ_a is well-defined. We will prove now that ψ_a is lipschitz, in particular, it is absolutely continuous. For all $t < t' \in [0,T]$,

$$\begin{aligned}
|\langle\rho_t, H\rangle - \langle\rho_{t'}, H\rangle| \;&=\; \lim_{\varepsilon\downarrow 0} |\langle\rho_t, H_\varepsilon\rangle - \langle\rho_{t'}, H_\varepsilon\rangle|\\
&=\; \lim_{\varepsilon\downarrow 0} \Big|\int_t^{t'} \langle\rho_s, h_\varepsilon\rangle\,ds\Big|\\
&\le\; 2|t'-t|\,.
\end{aligned}$$

where, in last inequality, we have used that $|\rho_t| \le 1$, Lebesgue almost surely, for all $t \in [0,T]$, and $\int_{\mathbb{T}} |h_\varepsilon(z)|dz \le 2$, for all $\varepsilon > 0$. □

One can observe that, fixed $a \in \mathbb{T}$, any redefinition of $\rho_t(a)$ for all values of $t \in [0,T]$ does not change the fact ρ_t is a integral solution of (1.7). We will do that in the following way: the values of $\rho_t(a)$ will be chosen as $\frac{d\psi_a}{d\lambda}(t)$, the Radon-Nykodyn derivative of ψ_a with respect to Lebesgue.

Lemma A.3.2. *Let $f : [0,T] \to \mathbb{R}$ be a continuous function and $\rho : [0,T] \times \mathbb{T} \to \mathbb{R}$ the solution of (1.7) redefined in $a \in \mathbb{T}$, as said before. Then,*

$$\lim_{\varepsilon \downarrow 0} \int_0^t f(s) \Big[\frac{1}{\varepsilon} \int_{(a-\varepsilon,a)} \rho_s(u)\, du \Big]\, ds = \int_0^t f(s) \rho_s(a)\, ds\,.$$

Proof. By previous Lemma, $\lim_{\varepsilon\downarrow 0} \int_0^t \big[\frac{1}{\varepsilon} \int_{(a-\varepsilon,a)} \rho_s(u)\, du\big]\, ds = \int_0^t \rho_s(a)\, ds$. Because f is continuous, the result follows from an uniform approximation of f by simple functions. The second limit is analogous. $\square$

In what follows, we obtain a natural consequence that any function in $\mathcal{H}^1(\mathbb{T}\backslash\{a\})$, almost surely in time, will be continuous in $\mathbb{T}\backslash\{a\}$.

Proposition A.3.3. *If $\xi : [0,T] \times \mathbb{T} \to \mathbb{R}$ belongs to $L^2(0,T;\mathcal{H}^1(\mathbb{T}\backslash\{a\}))$ then, almost surely in $t \in [0,T]$,*

$$\xi_t(v) - \xi_t(u) = \int_{(u,v)} \partial_u \xi_t(z)\, dz, \quad \forall u, v \in (a, 1+a)\,,$$

where $\partial_u \xi$ is given in the Definition 2.4.2 .

Proof. From Definition 2.4.2 ,

$$\int_0^T \int_{\mathbb{T}} \partial_u H(s,r)\, \xi(s,r)\, dr\, ds = - \int_0^T \int_{\mathbb{T}} H(s,r) \partial_u \xi(s,r)\, dr\, ds\,,$$

for all $H \in C^{0,1}([0,T] \times \mathbb{T})$ with compact support contained in $[0,T] \times (\mathbb{T}\backslash\{a\})$. Using approximation of indicators functions by continuous functions, it implies, Lebesgue almost surely in time,

$$\int_{\mathbb{T}} \partial_u H(r)\, \xi_t(r)\, dr = - \int_{\mathbb{T}} H(r)\, \partial_u \xi_t(r)\, dr, \tag{A.2}$$

for all $H \in C^1$ with compact support contained in $\mathbb{T}\backslash\{a\}$. Recall the notation that $f_t(u)$ is equal to $f(t,u)$, for all function $f : [0,T] \times \mathbb{T} \to \mathbb{R}$. Fixed a time $t \in [0,T]$ for which is valid the equality above.

Define the sequence of step functions $f_n : \mathbb{T} \to \mathbb{R}$ by

$$f_n(z) = n[\mathbf{1}_{(v,v+\frac{1}{n})}(z) - \mathbf{1}_{(u,u+\frac{1}{n})}(z)]$$

and the sequence of $H_n : \mathbb{T} \to \mathbb{R}$ by

$$H_n(r) = \int_{(0,r)} f_n(z)\, dz\,.$$

Since f_n is not a continuous function, $H_n \notin C^1$. However, the equality (A.2) is still valid for such f_n and H_n, through the approximation of f_n and H_n by the continuous functions $f_n^\varepsilon : \mathbb{T} \to \mathbb{R}$ and functions $H_n^\varepsilon(r) = \int_{(0,r)} f_n^\varepsilon(z)\,dz$, respectively. These functions f_n^ε are defined by $f_n^\varepsilon(z) = 0$, for all $z \in \mathbb{T}\backslash\{(u-\varepsilon, u+\frac{1}{n}+\varepsilon) \cup (v-\varepsilon, v+\frac{1}{n}+\varepsilon) \cup \{a\}\}$, $f_n^\varepsilon(z) = c_n^\varepsilon$, for all $z \in (u, u+\frac{1}{n})$, $f_n^\varepsilon(z) = d_n^\varepsilon$, for all $z \in (v, v+\frac{1}{n})$, in the other intervals f_n^ε is defined by linear interpolation. The constants c_n^ε and d_n^ε are choosen in such way that $\int_{(u-\varepsilon,u+\frac{1}{n}+\varepsilon)} f_n^\varepsilon(z)\,dz = -1$ and $\int_{(v-\varepsilon,v+\frac{1}{n}+\varepsilon)} f_n^\varepsilon(z)\,dz = 1$.

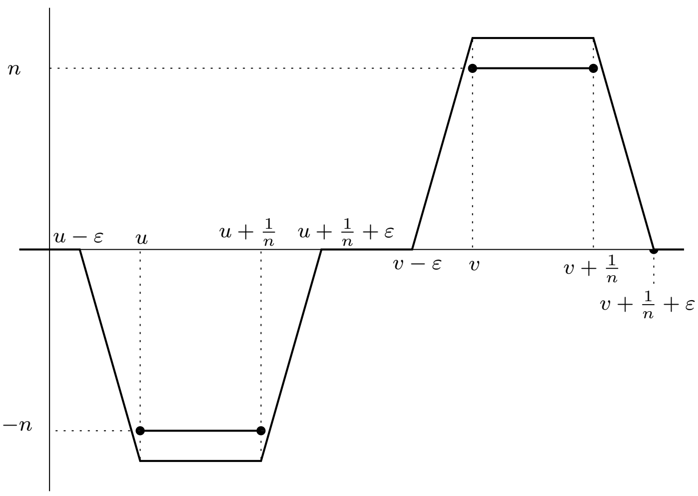

Figure A.1: Functions f_n^ε and f_n

Notice that $H_n(r) \overset{n\to\infty}{\longrightarrow} -\mathbf{1}_{(u,v)}(r)$, almost surely. Using Cauchy-Schwarz, $\langle -\partial_u \xi_t, H_n \rangle$ converges to $\langle \partial_u \xi_t, \mathbf{1}_{[u,v)} \rangle$, as $n \to \infty$. Denote by $\langle \cdot, \cdot \rangle$ the intern product in $L^2(\mathbb{T})$. By the definition of f_n, we have that

$$\langle \xi_t, f_n \rangle = n \int_{(v,v+1/n)} \xi_t(z)dz - n \int_{(u,u+1/n)} \xi_t(z)dz\,.$$

Take the limit when n increases to infinity, by the Lebesgue-Besicovitch Differentiation Theorem (c.f. [5]) we obtain that $\langle \xi_t, f_n \rangle$ converges to $\xi_t(v) - \xi_t(u)$, almost surely in $u, v \in (a, 1+a)$, which finishes the proof. □

Proposition A.3.4. *Suppose that there exists a function $\partial_u \xi : [0,T] \times \mathbb{T} \to \mathbb{R}$ such that, almost surely in $t \in [0,T]$,*

$$\xi_t(v) - \xi_t(u) = \int_{(u,v)} \partial_u \xi_t(z)\,dz, \quad \forall u, v \in (a, 1+a)\,,$$

almost surely in $t \in [0,T]$ and $\int_{\mathbb{T}} \partial_u \xi_t(z)\,dz = 0$. Then, $\xi \in L^2(0,T;\mathcal{H}^1(\mathbb{T}\backslash\{a\}))$ and $\partial_u \xi$ corresponds to that one in Definition 2.4.2 .

Proof. By the hypothese about ξ, we get

$$\langle\!\langle \partial_u H, \xi \rangle\!\rangle = \int_0^T \int_{\mathbb{T}} \partial_u H_s(r) \Big[\int_{(0,r)} \partial_u \xi_t(z)\, dz + \xi_s(0) \Big]\, dr\, ds\,.$$

Since H belongs to $C^{0,1}$, the second term above is null. In the first integral we use Fubini's Theorem, then the expression above is equal to

$$\int_0^T \int_{\mathbb{T}} \int_{(z,1)} \partial_u H_s(r)\, dr\, \partial_u \xi_t(z)\, dz\, ds = \int_0^T \int_{\mathbb{T}} [H(t,1) - H(t,z)] \partial_u \xi_t(z)\, dz\, ds\,.$$

Now, we use again the hypothese about ξ and the last expression becomes,

$$-\int_0^T \int_{\mathbb{T}} H(t,z) \partial_u \xi_t(z)\, dz\, ds = -\langle\!\langle H, \partial_u \xi \rangle\!\rangle\,.$$

□

Proposition A.3.5. *Are equivalent:*

(i) $\xi \in L^2(0,T;\mathcal{H}^1(\mathbb{T}\backslash\{a\}))$;

(ii) *Almost surely in* $t \in [0,T]$,

$$\xi_t(v) - \xi_t(u) = \int_{(u,v)} \partial_u \xi_t(z)\, dz\,, \qquad \forall u, v \in (a, 1+a)\,,$$

where $\partial_u \xi \in L^2([0,T] \times \mathbb{T})$ *and, a.s. in* $t \in [0,T]$, $\int_{\mathbb{T}} \partial_u \xi_t(z)\, dz = 0$;

(iii) *There exists a positive integer number* κ *such that* $\sup_H \langle\!\langle \partial_u H, \xi \rangle\!\rangle - \kappa \langle\!\langle H, H \rangle\!\rangle < \infty$ *, where the supremum is carried over all functions* H *in* $C^{0,1}([0,T] \times \mathbb{T})$ *with compact support contained in* $[0,T] \times (\mathbb{T}\backslash\{a\})$.

Proof. $(i) \Rightarrow (ii)$ has been proved in A.3.3 and $(ii) \Rightarrow (i)$ has been proved in A.3.4. $(iii) \Rightarrow (i)$ follows by the Riesz Representation Theorem and A.1.1. It just remains to prove $(i) \Rightarrow (iii)$. Suppose that (i) is valid. By Cauchy-Schwarz and Young's inequality,

$$\begin{aligned} \langle\!\langle \partial_u H, \xi \rangle\!\rangle &= -\langle\!\langle H, \partial_u \xi \rangle\!\rangle \leq \sqrt{\langle\!\langle H, H \rangle\!\rangle \langle\!\langle \partial_u \xi, \partial_u \xi \rangle\!\rangle} \\ &\leq \kappa \langle\!\langle H, H \rangle\!\rangle + \tfrac{1}{2} \langle\!\langle \partial_u \xi, \partial_u \xi \rangle\!\rangle\,, \end{aligned}$$

from what we conclude (iii). □

Lemma A.3.6. *Let* $\xi \in L^2(0,T;\mathcal{H}^1(\mathbb{T}\backslash\{a\}))$ *such that there exists a positive integer number* κ *that satisfies* $\sup_H \langle\!\langle \partial_u H, \xi \rangle\!\rangle - \kappa \langle\!\langle H, H \rangle\!\rangle < \infty$, *where the supremum is taken over all* $H \in C^{0,1}([0,T] \times \mathbb{T})$ *with compact support contained in* $[0,T] \times (\mathbb{T}\backslash\{a\})$. *Then,*

$$\sup_H \Big\{ \langle\!\langle \partial_u H, \xi \rangle\!\rangle - \kappa \langle\!\langle H, H \rangle\!\rangle \Big\} = \tfrac{1}{4\kappa} \int_0^T \|\partial_u \xi_t\|_{L^2}^2\, dt\,, \tag{A.3}$$

where the supremum is taken over all $H \in C^{0,1}([0,T] \times \mathbb{T})$ *with compact support contained in* $[0,T] \times (\mathbb{T}\backslash\{a\})$.

Proof. Using that $\xi \in L^2(0,T;\mathcal{H}^1(\mathbb{T}\backslash\{a\}))$ and the Young's inequality,

$$\langle\!\langle \partial_u H, \xi \rangle\!\rangle - \kappa \langle\!\langle H, H \rangle\!\rangle \;=\; -\langle\!\langle H, \partial_u \xi \rangle\!\rangle - \kappa \langle\!\langle H, H \rangle\!\rangle \leq \tfrac{1}{4\kappa} \int_0^T \|\partial_u \xi_t\|_{L^2}^2 \, dt \,,$$

for all $H \in C^{0,1}([0,T]\times\mathbb{T})$ with compact support contained in $[0,T]\times(\mathbb{T}\backslash\{a\})$. For the other hand, there exists a sequence $\{H^n\}_n \subset C^{0,1}([0,T]\times\mathbb{T})$ with compact support contained in $[0,T]\times(\mathbb{T}\backslash\{a\})$ such that $-H^n$ converges to $\partial_u\xi$ in $L^2(0,T;L^2(\mathbb{T}))$. Thus,

$$\tfrac{1}{4\kappa} \int_0^T \|\partial_u \xi_t\|_{L^2}^2 \, dt = \lim_{n\to\infty} \Big\{ \tfrac{1}{2\kappa} \langle\!\langle -H^n, \partial_u \xi \rangle\!\rangle - \tfrac{\kappa}{4\kappa^2} \langle\!\langle -H^n, -H^n \rangle\!\rangle \Big\} \leq \sup_H \langle\!\langle H, \partial_u \xi \rangle\!\rangle - \kappa \langle\!\langle H, H \rangle\!\rangle \,,$$

where also the supremum is taken over all $H \in C^{0,1}([0,T]\times\mathbb{T})$ with compact support contained in $[0,T]\times(\mathbb{T}\backslash\{a\})$. One can conclude this proof. □

Lemma A.3.7. *Let $f \in L^\infty(\mathbb{T})$. Then, $f * \iota_\varepsilon^a$ is a Lipschitz function in the open interval $(a,1+a)$. Here, we identify the $\mathbb{T}\backslash\{a\}$ with the open interval $(a,1+a)$.*

Proof. Let $u,v \in (a,1+a)$ and analyze that

$$|(f*\iota_\varepsilon^a)(u) - (f*\iota_\varepsilon^a)(v)| \leq \|f\|_\infty \int_{\mathbb{T}} |\iota_\varepsilon^a(z,u) - \iota_\varepsilon^a(z,v)| \, dz \leq \|f\|_\infty \tfrac{2}{\varepsilon} |u-v| \,.$$

□

Lemma A.3.8. *For any γ, ε and $\pi \in \mathcal{D}([0,T],\mathcal{M})$,*

$$\mathcal{E}((\pi * \iota_\gamma) * \iota_\varepsilon^a) < \infty \,.$$

Proof. This proof follows by Lemma A.3.7, Definition 1.4.1 and Proposition A.3.5. □

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