Statistical stability for diffeomorphisms with dominated splitting

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1. Introduction

A central topic in dynamical systems is the study of statistical properties of the system. In this direction, we can consider the average along the orbits and then compare it with the average of the system in the ambient space. Given any ergodic invariant measure for the system it is well-known that for almost every point with respect to this measure the temporal and spatial averages coincide. In many cases, the invariant measure is a singular measure, so it may be physically very difficult to find a point satisfying the property above. An SRB measure is an invariant measure for the system for which the time average coincides with the spatial average in a positive Lebesgue measure subset of the ambient space.

A program towards a global theory of diffeomorphisms has been proposed a few years ago by Palis [**12**]. The core of his conjecture is that every dynamical system can be approximated by one having only finitely many attractors, all of which have finitely many SRB measures which are robust with respect to small perturbations of the system.

Statistical stability tells us much about how the system varies under small deterministic perturbations.

The question of the existence of SRB measures has an affirmative answer in the setting of uniformly hyperbolic systems [**18, 7, 6, 16**], as well as of systems with certain weak forms of hyperbolicity [**4, 2**].

Uniformly expanding smooth maps are well-known to be statistically stable, as are Axiom A diffeomorphisms [**17**] restricted to the basin of their attractors. On the other hand, [**1, 3**] proved statistical stability for a large class of transformations exhibiting non-uniformly expanding behavior. Statistical stability for a certain open class of diffeomorphisms having partially hyperbolic attractors whose central direction is mostly contracting was proved in [**10, 9**] .

In [**2**] it was proved that SRB measures exist for diffeomorphisms having dominated splitting with mostly expanding center-unstable direction and other technical conditions. The main tool used there is the existence of Gibbs cu-states. Gibbs cu-states are the non-uniform version of the Gibbs u-states introduced by Pesin and Sinai [**14**]. Several other properties of Gibbs u-states are proved in [**5**].

In this paper we extend this properties to Gibbs cu-states, especially with respect to their relationship with SRB measures.

1.1. Statement of results.

1.1. Statement of results. — Let us consider diffeomorphisms $f : M \to M$ defined over a compact Riemanian boundaryless manifold M. We denote by m a fixed normalized Riemannian volume form on M and we call it the *Lebesgue measure* on M.

The *time average* of a continuous function $\varphi : M \to \mathbb{R}$ along the orbit of $x \in M$ is:

$$\tilde{\varphi}(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x)).$$

If μ is an invariant measure, the *basin* of μ is the set

$$B(\mu) = \{x \in M \ : \ \tilde{\varphi}(x) = \int_M \varphi d\mu, \ \forall \varphi \in C(M;\mathbb{R})\}.$$

An invariant measure μ is an *SRB measure* or *physical measure* if $B(\mu)$ has positive Lebesgue measure.

Let $f : M \to M$ be a C^2-diffeomorphism. Let $U \subseteq M$ be a neighborhood such that $\overline{f(U)} \subseteq U$ and $\Lambda = \cap_{n\geq 1} f^n(U)$ is an attractor.

The attractor Λ has a *dominated splitting* if there is a continuous Df-invariant decomposition $T_\Lambda M = E^{cu} \oplus E^{cs}$ of the tangent bundle to M over Λ and some constant $0 < \lambda < 1$ satisfying

$$\|Df^n|E^{cs}_x\| \|(Df^{-n}|E^{cu}_{f^n(x)})\| \leq C\lambda^n,$$

for every $x \in \Lambda$, and for every $n \geq 1$. The subbundle E^{cs} is *uniformly contracting* if

$$\|Df^n|E^{cs}_x\| \leq C\lambda^n,$$

for every $x \in \Lambda$ and $n \geq 1$. In this case, we denote $E^{cs} = E^s$ and we say that the attractor Λ is *partially hyperbolic.*

The diffeomorphism f has *non-uniform expansion along the center-unstable direction* if there exists a constant $c_0 > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log \|Df^{-1}|E^{cu}_{f^j(x)}\| \leq -c_0 < 0. \tag{1}$$

for all x in a full Lebesgue measure subset of U. Under these conditions Alves, Bonatti and Viana [**2**] proved:

Theorem 1.1. — *If $f \in \mathrm{Diff}^2(M)$ has an attractor Λ which is partially hyperbolic with non-uniformly expansion along the center-unstable direction, then there exist finitely many ergodic SRB measures and the union of their basins covers a full Lebesgue measure subset of the basin of Λ.*

The main tool used in the proof of Theorem 1.1 is the construction of Gibbs cu-states. Denote by $u = \dim E^{cu}$ and $s = \dim E^{cs}$.

Definition 1. — *An invariant measure μ supported in Λ is a* Gibbs cu-state *if the u larger Lyapunov exponents are positive μ-almost everywhere and the conditional measures of μ along the corresponding local strong-unstable manifolds are almost everywhere absolutely continuous with respect to Lebesgue measure on these manifolds.*

Theorem 1.1 is a direct consequence of the following result also proved in [**2**] and the uniformly contracting condition on the center-stable direction.

Theorem 1.2. — *If $f \in \mathrm{Diff}^2(M)$ has an attractor Λ which admits a dominated splitting with non-uniform expansion along the center-unstable direction, then there exist ergodic Gibbs cu-states supported on Λ.*

Alves, Bonatti and Viana give a constructive proof of the existence of Gibbs cu-states (see [**2**] or Subsection 3.1 for more details).

We first study the Gibbs cu-state in the setting of diffeomorphisms with dominated splitting and obtain a general description of such measures.

A *cylinder* $\mathcal{C}$ is a diffeomorphic image of $B^u \times B^s$ where B^u and B^s are balls in $\mathbb{R}^u$ and $\mathbb{R}^s$ respectively. We say that a C^1-disk D *crosses* $\mathcal{C}$ if it is contained in $\mathcal{C}$ and is a graph over B^u.

Theorem A. — *Let $f \in \mathrm{Diff}^2(M)$ exhibit an attractor with dominated splitting and let μ be a Gibbs cu-state for f. Then*

1. *for μ-almost every point x and every $\delta > 0$ small enough, there exists a cylinder containing x, $\mathcal{C}(x,\delta) \subseteq B(x,\delta)$, and a family $\mathcal{K}(x,\delta)$ of disjoint unstable disks crossing the cylinder $\mathcal{C}(x,\delta)$ such that their union $K(x,\delta)$ has positive μ-measure;*
2. *denoting by ρ_z the density of the conditional measure μ_z along the disk through z, then, for μ-almost every $z \in \operatorname{supp}\mu$ and for every x, $y \in W^u_{loc}(z)$, we have*

$$\frac{\rho_z(x)}{\rho_z(y)} = \prod_{k=0}^{\infty} \frac{\det(Df^{-1}|E^{cu}_{f^{-k}(x)})}{\det(Df^{-1}|E^{cu}_{f^{-k}(y)})}. \tag{2}$$

3. *the support of μ contains global unstable manifolds whose union has full μ-measure;*
4. *every ergodic component of μ is a Gibbs cu-state.*

The main tools used in the proof are Pesin theory and distortion properties given by the dominated splitting.

If we add the hypothesis of non-uniform expansion along the center unstable direction, our main result is Theorem C below. But an essential ingredient in its proof is the following fact: the construction of Gibbs cu-states done in [**2**] provides all the possible Gibbs cu-states.

We denote by $\mathcal{G}(f)$ the class of Gibbs cu-states for f constructed in Theorem 1.2.

Theorem B. — *If $f \in \mathrm{Diff}^2(M)$ has a dominated splitting which is non-uniformly expanding along the E^{cu} direction, then every ergodic Gibbs cu-state supported in Λ is in $\mathcal{G}(f)$.*

Another ingredient in the proof of Theorem C are the uniform bounds obtained from [**2**] and the properties of Gibbs cu-states given by Theorem A.

Theorem C. — *Consider the set of pairs (f, μ) where $f \in \mathrm{Diff}^2(M)$ has an attractor with dominated splitting and non-uniform expansion along the center-unstable direction with uniform c_0 and μ is a Gibbs cu-state for f. Then this set is closed.*

As a corollary of Theorem C we obtain the following relationship between SRB measures and Gibbs cu-states:

Corollary D. — *If $f \in \mathrm{Diff}^2(M)$ has an attractor with dominated splitting which is non-uniformly expanding along the E^{cu} direction, then every ergodic SRB measure is a Gibbs cu-state.*

Another consequence of Theorem C is related to the statistical stability of partially hyperbolic diffeomorphisms.

Definition 2. — *We say that $f_0 \in \mathrm{Diff}^r(M)$ is C^r*-statistically stable *if for every sequence $f_n \in \mathrm{Diff}^r(M)$ converging to f_0 in the C^r-topology, and for every sequence μ_n of SRB measures for f_n, the weak* accumulation measures of $(\mu_n)_n$ are in the convex hull of finitely many SRB measures for f_0.*

Corollary E. — *If $f \in \mathrm{Diff}^2(M)$ has an attractor Λ which is partially hyperbolic and non-uniformly expanding along the center-unstable direction with c_0 uniform in a neighborhood of f, then f is C^k-statistically stable, $k \geq 2$.*

An interesting consequence of Theorem C is the following

Theorem F. — *Let $f \in \mathrm{Diff}^2(M)$ have an attractor Λ exhibiting a dominated splitting with non-uniform expansion along the center-unstable direction. Let us suppose that f satisfies*

$$\limsup_{n\to+\infty} \frac{1}{n}\|Df^n|E^{cs}_x\| < 0 \tag{3}$$

for any disk D contained in some unstable local manifold and for Lebesgue almost every point $x \in D$. Then f has finitely many SRB measures and the union of their basins covers a full Lebesgue measure subset of the basin of Λ. In addition, if non-uniform expansion along the central-unstable direction holds in a C^k-neighborhood of f with uniform c_0 and every diffeomorphism in such neighboorhood satisfies (3) , then f is C^k-statistically stable, $k \geq 2$.

This paper is organized as follows. In Section 2 we study the Gibbs cu-states using only the hypothesis of dominated splitting on the attractor and we prove Theorem A. In Section 3 we add the hypothesis of non-uniform expansion along the center-unstable direction. We first outline the proof of Theorem 1.2 and then we prove Theorem B and Theorem C. Finally, Section 4 is dedicated to studying the relationship between Gibbs cu-states and SRB measures. There we prove Corollary D, Corollary E and Theorem F, and we present an example of an open class of diffeomorphisms of the torus $\mathbb{T}^n$, $n \geq 4$, with dominated splitting, but not partially hyperbolic, exhibity non-uniform expansion along the center-unstable direction, and admitting a unique SRB measure whose basin has full Lebegue measure in $\mathbb{T}^n$.

2. Gibbs cu-states

In this section we assume that $f : M \to M$ is a C^2-diffeomorphism exhibiting an attractor with dominated splitting and that there is a Gibbs cu-state μ for f supported on Λ. Our main goal is to prove Theorem A. We first fix some definitions and notations.

Let X, Y be compact metric spaces. Let ν be a Borelean probability measure over X. Let $h : X \to Y$ be a measurable function. We denote by $h_*\nu$ the *push-forward* measure defined on Y by

$$h_*\nu(B) = \nu(h^{-1}(B))$$

for every measurable set $B \subseteq Y$. If $A \subseteq X$ is such that $\nu(A) > 0$, then we denote by $(\nu|A)$ the *restriction measure* defined on A by

$$(\nu|A)(B) = \frac{\nu(A \cap B)}{\nu(A)}$$

for every measurable set $B \subseteq A$.

Denote by $\mathcal{F}$ the partition of $X \times Y$ into "horizontal lines"

$$\mathcal{F} = \{X \times \{y\} \,:\, y \in Y\}.$$

Let $\pi_Y : X \to Y$ be the canonical projection. Given a measure μ on $X \times Y$ we write $\hat{\mu} = \pi_{Y*}\mu$. The measure μ is *absolutely continuous* with respect to ν along the lamination $\mathcal{F}$ if there exists a measurable function $\rho : X \times Y \to [0, +\infty)$ such that

$$\mu(B) = \int_B \rho(x, y)\, d\nu(x)\, d\hat{\mu}(y)$$

for every measurable set $B \subseteq X \times Y$. The measures $\mu_y = \rho(\cdot, y)\nu$ are called *conditional measures* of μ along the "leaf" $\mathcal{F}(y) = X \times \{y\}$ and $\rho(\cdot, y) = \rho_y(\cdot)$ is the *density* of μ_y along the "leaf" $\mathcal{F}(y)$.

Proposition 2.1. — *For μ-almost every point x and every $\delta > 0$ small enough, there exists a cylinder containing x, $\mathcal{C}(x, \delta) \subseteq B(x, \delta)$, and a family $\mathcal{K}(x, \delta)$ of disjoint unstable disks crossing the cylinder $\mathcal{C}(x, \delta)$ such that their union $K(x, \delta)$ has positive μ-measure.*

Proof. — If μ is a Gibbs cu-state for f, then μ-almost every $x \in \Lambda$ has u positive Lyapunov exponents in the E^{cu}_x direction. By Pesin´s theory [**13**], for such x there exists a unique C^1-embedded disk $W^u_{loc}(x)$ tangent to E^{cu}_x at x, and such that the diameter of $f^n(W^u_{loc}(x))$ converges exponentially fast to zero as $n \to \infty$. The C^1-disk $W^u_{loc}(x)$ depends in a measurable way on the point x. This implies that there exists a sequence $(\Lambda_n)_n$ of nested compact subsets with $\mu(\Lambda \setminus \Lambda_n)$ converging to zero as $n \to \infty$, and that there exist continuous maps

$$\Lambda_n \ni x \mapsto W^u_{loc}(x)$$

which associate to every point $x \in \Lambda_n$ an embedded C^1-disk $W^u_{loc}(x)$. The sets Λ_n are called *hyperbolic blocks* of Λ. In particular, there exists a uniform lower bound on the size of $W^u_{loc}(x)$ in Λ_n: there are $\delta_n > 0$ such that the pre-image of $W^u_{loc}(x)$ under the exponential map of M at x contains the graph of a C^1 map defined from $B(0, \delta_n) \subseteq E^{cu}_x$ to E^{cs}_x.

Given any $0 < \delta < \delta_n/4$ and $x \in \Lambda_n$ we can define the tubular neighborhood $\mathcal{C}(x, \delta)$ of $W^u_{loc}(x)$ as the image under the exponential map of M at y of all the vectors of norm less than $\delta > 0$ in the orthogonal complement of E^{cu}_y, for all $y \in W^u_{loc}(x)$. If $\delta > 0$ is small enough then this neighborhood $\mathcal{C}(x, \delta)$ is a cylinder and it comes equipped with the canonical projection π onto $W^u_{loc}(x)$ which is a C^1 map. We denote by $\mathcal{K}(x, \delta)$ the family of local strong-unstable manifolds at points of Λ_n that cross $\mathcal{C}(x, \delta)$.

There exist $y_1, .., y_k \in \Lambda_n$ such that $\Lambda_n \subseteq \cup_{j=1}^k \mathcal{C}(y_j, \delta)$, because Λ_n is compact. We may suppose that each of these cylinders has positive μ-measure, and we obtain a covering (μ mod 0) of Λ_n. As a consequence, for all $j = 1, .., k$ we have $\mu(\Lambda_n \cap \mathcal{C}(y_j, \delta)) > 0$. On the other hand, for each $z \in \Lambda_n \cap \mathcal{C}(y_j, \delta)$, we have that $W^u_{\delta_n}(z)$ crosses $\mathcal{C}(y_j, \delta)$, because $\delta < \delta_n/4$. Then, for all $j = 1, .., k$,

$$\mu(K(y_j, \delta)) > \mu(\Lambda_n \cap \mathcal{C}(y_j, \delta)) > 0.$$

We consider the set of $x \in \operatorname{supp} \mu$ such that $x \in \Lambda_n$ for some $n \geq 1$. This set has full μ-measure. For $\delta > 0$, there exists $y \in \Lambda_n$ such that $x \in \mathcal{C}(y, \delta)$. Since x is in $\operatorname{supp} \mu$, $\mathcal{C}(y, \delta)$ must have positive μ measure. It is clear that δ can be chosen arbitrary small. To obtain the statement, we write $\mathcal{C}(x, \delta) = \mathcal{C}(y, \delta)$ and $\mathcal{K}(x, \delta) = K(y, \delta)$.

$\square$

If μ is a Gibbs cu-state and $x \in \operatorname{supp}\mu$, then Proposition 2.1 implies for each $\delta > 0$ small enough, there exists a cylinder $\mathcal{C}(x,\delta)$ and a family $\mathcal{K}(x,\delta)$ of disks crossing $\mathcal{C}(x,\delta)$. Let B^s be the image under the exponential map of M at x of all vectors of norm less than or equal to $\delta > 0$ contained in the orthogonal complement of E^{cu}_x and let $\pi_s : \mathcal{C}(x,\delta) \to B^s$ be the projection on B^s along the center-unstable leaves. We can induce a measure $\hat{\mu}$ on B^s given by

$$\hat{\mu}(A) = \mu(\pi_s^{-1}(A) \cap K(x,\delta))$$

for measurable $A \subseteq B^s$, where $K(x,\delta)$ is the union of all disks in $\mathcal{K}(x,\delta)$. Of course $\hat{\mu}$ is locally f-invariant, i.e. $\hat{\mu}(B \cap K(x,\delta)) = \hat{\mu}(f^{-1}(B \cap K(x,\delta)))$ for every Borelean set B, because μ is f-invariant.

Let μ be a Gibbs cu-state for f, $z \in \operatorname{supp}\mu$ and μ_z be the conditional measure of μ in $W^u_{loc}(z)$. The conditional measure μ_z is f-invariant if

$$\mu_{f^{-1}(z)}(f^{-1}(A)) = \mu_z(A)$$

for all Borelean sets $A \subseteq W^u_{loc}(z)$.

Lemma 2.1. — *The conditional measure μ_z is f-invariant for μ-almost every $z \in \operatorname{supp}\mu$.*

Proof. — Let $B = \{z : \mu_z \text{ is not } f\text{-invariant}\}$ and assume that $\mu(B) > 0$. By Proposition 2.1 there exist $z' \in \operatorname{supp}\mu$, $\delta > 0$, a cylinder $C(z',\delta)$ and a family of unstable disks $\mathcal{K}(z',\delta)$ such that $\mu(B \cap K(z',\delta)) > 0$ and $\hat{\mu}(B \cap K(z',\delta)) > 0$, where $K(z',\delta)$ is the union of the disks in the family $\mathcal{K}(z',\delta)$.

For each $z \in B \cap K(z',\delta)$, let $A_z \in W^u_{loc}(z)$ be such that $\mu_z(A_z) > 0$ and assume that $\mu_{f^{-1}(z)}(f^{-1}(A_z)) > \mu_z(A_z)$ (the case $\mu_{f^{-1}(z)}(f^{-1}(A_z)) < \mu_z(A_z)$ is analogous). If we put $E = \cup_{z \in B \cap K(z',\delta)} A_z$, the disintegration of the measure implies that $\mu(f^{-1}(E)) > \mu(E)$, which contradicts the f-invariance of μ.

In fact, by disintegration,

$$\mu(E) = \int_{B \cap K(z',\delta)} \int_{A_z} d\mu_z \, d\hat{\mu}(z) = \int_{B \cap K(z',\delta)} \mu_z(A_z) d\hat{\mu}(z).$$

On the other hand, we have

$$\mu(f^{-1}(E)) = \int_{f^{-1}(B \cap K(z',\delta))} \int_{f^{-1}(A_z)} d\mu_z \, d\hat{\mu}(z) = \int_{f^{-1}(B \cap K(z',\delta))} \mu_z(f^{-1}(A_z)) d\hat{\mu}(z).$$

Since $\hat{\mu}(B \cap K(z',\delta)) = \hat{\mu}(f^{-1}(B \cap K(z',\delta)))$, and since we are assuming we have the inequality $\mu_{f^{-1}(z)}(f^{-1}(A_z)) > \mu_z(A_z)$, it follows that $\mu(f^{-1}(E)) > \mu(E)$.

□

An essential ingredient here is the Holder control of the Jacobian given by the domination.

Lemma 2.2. — *There exists $\xi > 0$ such that, given $L > 0$ and any C^2 disk $D \subseteq U$ transverse to the center stable direction E^{cs}, then there exists $C_1 > 0$ such that*

$$x \mapsto \log|\det(Df|T_x f^n(D))|$$

is (C_1, ξ)-Holder on every domain of diameter L inside any $f^n(D)$, $n \geq 1$.

We refer the reader to the proof in [**2**], Section 2. We observe that this constant depends only on the diffeomorphism f.

We denote by ρ_z the density of the conditional measure μ_z of μ along the unstable disk through z. Our next goal is to characterize this density.

Lemma 2.3. — *For every x and y in the same local unstable manifold, the product*

$$\prod_{k=0}^{\infty} \frac{\det(Df^{-1}|E^{cu}_{f^{-k}(x)})}{\det(Df^{-1}|E^{cu}_{f^{-k}(y)})}$$

converges and is bounded away from zero and infinity.

Proof. — Let $x, y \in W^u_\varepsilon(z)$ and set $J^u_k(x) = |\det Df^{-1}|E^{cu}_{f^k(x)}|$, $k \geq 0$. Lemma 2.2 implies that the map $x \to \log(J^u_k(x)^{-1})$ is (C_1, ξ)-Holder. Let $\lambda > 0$ be the smallest Lyapunov exponent for z in the E^{cu}-direction. Then, for $N \geq 1$,

$$\begin{aligned}
\left|\log \prod_{k=0}^{N} \frac{J^u_k(x)}{J^u_k(y)}\right| &\leq \sum_{k=0}^{N} |\log J^u_k(x) - \log J^u_k(y)| \\
&\leq \sum_{k=0}^{N} C_1 \mathrm{dist}_{f^k(W^u_\varepsilon(z))}(f^k(x), f^k(y))^\xi \\
&\leq \sum_{k=0}^{N} C_1 C^\xi e^{-k\lambda\xi} \mathrm{dist}_{W^u_\varepsilon(z)}(x, y)^\xi
\end{aligned}$$

The symmetry of the product (we may exchange x and y) implies that the product converges for all $x, y \in W^u_\varepsilon(z)$ and is non-zero. Moreover the convergence is absolute and Holder with respect to x and y, so the product is bounded away from zero and infinity.

□

Remark 1 : The convergence of the product depends only on f and on the smallest Lyapunov exponent along the disk.

Proposition 2.2. — *For μ-almost every $z \in \operatorname{supp}\mu$ and for every $x,\ y \in W^u_{loc}(z)$,*

$$\frac{\rho_z(x)}{\rho_z(y)} = \prod_{k=0}^{\infty} \frac{\det(Df^{-1}|E^{cu}_{f^{-k}(x)})}{\det(Df^{-1}|E^{cu}_{f^{-k}(y)})}.$$

This implies that the densities are bounded away from zero and infinity.

Proof. — We fix z and $W^u_{loc}(z) = D$. Since μ_z is absolutely continuous with respect to Lebesgue measure in D, there exists some $\rho : D \to \mathbb{R}$ which is measurable and positive μ_z-almost everywhere such that

$$\mu_z(B) = \int_B \rho \, dm_D$$

for all Borelean subsets $B \subseteq D$. Let ρ_n be the density of $\mu_{f^{-n}(z)}$. By change of variables and by the invariance of conditional measures we have for $x \in W^u_{loc}(z)$ that

$$\rho(x) = C\rho_n(f^{-n}(x)) \prod_{k=0}^{n-1} J^u_k(x) \tag{4}$$

for any $n \geq 0$ where $C > 0$ is a constant of normalization depending of z and n. Then, for every $x,\ y \in W^u_{loc}(z)$,

$$\frac{\rho(x)}{\rho(y)} = \frac{\rho_n(f^{-n}(x))}{\rho_n(f^{-n}(y))} \prod_{k=0}^{n-1} \frac{J^u_k(x)}{J^u_k(y)}.$$

By Lemma 2.3 the right hand product converges to a non-zero value, so the quotient $\rho_n(f^{-n}(x))/\rho_n(f^{-n}(y))$ also converges. On the other hand, for all $\varepsilon > 0$ there exists a compact subset $K_\varepsilon \subseteq W^u_{loc}(z)$ with $m(W^u_{loc}(z) \setminus K_\varepsilon) < \varepsilon$ such that $\rho_n | f^n(K_\varepsilon)$ is continuous; moreover the continuity is uniform with respect to n due to (4). Then for $x, y \in K_\varepsilon$ and taking n large enough, we have

$$\frac{\rho_n(f^{-n}(x))}{\rho_n(f^{-n}(y))} \to 1$$

when $n \to \infty$. This completes the proof.

□

If x has a local strong unstable manifold $W^u_{loc}(x)$, the *global unstable manifold* of x is the set

$$W^u(x) = \bigcup_{n \geq 0} f^n(W^u_{loc}(f^{-n}(x)).$$

Next result follows from the fact that the densities ρ are f-invariant and bounded away from zero and infinity.

Corollary 2.1. — *If μ is a Gibbs cu-state of f, then the support of μ contains global unstable manifolds whose union has full μ-measure.*

Proof. — Let Λ_n be a hyperbolic block. Then for all $x \in \Lambda_n$ there exists $W^u_\delta(x)$ with $\delta > 0$ uniform. Moreover $0 < \mu(\Lambda_n) < 1 - \varepsilon_n$, with $\varepsilon_n \to 0$ as $n \to \infty$. Let μ_n be the restriction of μ to Λ_n. It is sufficient to prove that for μ_n-almost every $x \in \Lambda_n$, one has $W^u_\delta(x) \subseteq \operatorname{supp} \mu_n$.

For each $x \in \operatorname{supp} \mu_n$, we can construct a cylinder $\mathcal{C}$ that contains $W^u_\delta(x)$ and such that if $z \in \Lambda_n \cap \mathcal{C}$, then $W^u_\delta(z)$ crosses $\mathcal{C}$. Suppose there is $y \in W^u_\delta(x)$ such that $y \notin \operatorname{supp} \mu_n$. Then there exists a small neighborhood $y \in V \subseteq \mathcal{C}$ such that $\mu_n(V) = 0$. But by the disintegration of μ_n we have

$$\mu_n(V) = \int \mu_{n,z}(V \cap W^u_\delta(z))\, d\hat{\mu}_n(z),$$

but each $\mu_{n,z}$ has strictly positive density. Then there exists a neighborhood of x having zero $\hat{\mu}_n$-measure, which contradicts the fact that x is in the support of μ_n.

□

Let $R(f)$ be the set of regular points of f, that is the set of points in M such that the Birkhoff average exists and

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \lim_{n \to -\infty} \frac{1}{n} \sum_{k=1}^{n+1} \varphi(f^k(x))$$

for all $\varphi \in C^0(M; \mathbb{R})$. It is well-known that this set has full measure with respect to any f-invariant measure μ. Consider $x \in \operatorname{supp}\mu$ such that $W^u(x) \subseteq \operatorname{supp}\mu$ and $\mu_x(W^u(x) \cap R(f)) = 1$. Then Lebesgue almost every point of $W^u(x)$ is contained in a unique ergodic component of μ.

Let μ be a Gibbs *cu*-state for f and let μ^* be an ergodic component of μ. Since the support of μ^* consists of entire global manifolds (by Corollary 2.1) μ^* must be a Gibbs cu-state.

Corollary 2.2. — *If μ is a Gibbs cu-state of f, then every ergodic component of μ is a Gibbs cu-state.*

Here we conclude the proof of Theorem A. Proposition 2.1 and Proposition 2.2 correspond to statements 1 and 2 of Theorem A and Corollary 2.1 and Corollary 2.2 correspond to statements 3 and 4. Next lemma plays an important role in the following sections:

Lemma 2.4. — *Let μ be a Gibbs cu-state for f. For μ-almost every $x \in \Lambda$ and every $\delta > 0$ small enough, there exists a cylinder $\mathcal{C}(x,\delta)$ such that $\hat{\mu}$-almost every disk $D \in \mathcal{K}(x,\delta)$ satisfies $B(\mu) \cap R(f) \cap D$ has full Lebesgue measure in D.*

Proof. — We observe that $\mu(B(\mu) \cap R(f)) = 1$. For almost every $x \in B(\mu) \cap R(f) \cap \operatorname{supp}\mu$ and every $\delta > 0$ small enough, there exists a cylinder $\mathcal{C}$ and a family $\mathcal{K}$ of disk crossing $\mathcal{C}$ such that the union of those disks has positive μ-measure. Let us consider the probability measure $(\mu|K)$. Then

$$(\mu|K)(B(\mu) \cap R(f) \cap K) = 1.$$

By disintegration, $\hat{\mu}|K$-almost every disk $D \in \mathcal{K}$ satisfies

$$\mu_D(D \cap B(\mu) \cap R(f)) = \mu_D(D).$$

Because μ_D is absolutely continuous with respect to Lebesgue measure and the density ρ_D is bounded from zero and infinity, it follows that $m_D(B(\mu) \cap R(f) \cap D) = m_D(D)$. □

3. Gibbs cu-states and the non-uniform expansion condition

3.1. Building Gibbs cu-states. — Let $f : M \to M$ be a C^2-diffeomorphism having an attractor Λ with a dominated splitting and non-uniform expansion along the E^{cu} direction. The goal of this subsection is to briefly review the construction of Gibbs cu-states (cf. Theorem 1.2 [**2**]).

A disk $D \subset U$ is *tangent to the center-unstable cone field C^{cu}* if the tangent subspace to D at each point $x \in D$ is contained in the corresponding cone $C^{cu}(x)$. We fix a C^2 disk D tangent to the center-unstable cone field such that:

1. The set of points in D having non-hyperbolic behavior has full Lebesgue measure in the disk. This is possible because we assume that almost every point in U satisfies (1).
2. There are fixed $\xi > 0$ and $C_1 > 0$ as in Lemma 2.2 such that the functions J_k defined on $f^k(D) \subset U$ by $J_k(x) = \log|\det Df|T_x f^k(D)|$, for $k = 1, .., n$ are (C_1, ξ)-Holder. These constants depend only on f.

Definition 3. — *Given $\sigma < 1$, we say that n is a σ-*hyperbolic time *for a point $x \in U$ if*

$$\prod_{j=n-k+1}^{n} \|Df^{-1}|E^{cu}_{f^j(x)}\| \leq \sigma^k$$

for all $1 \leq k \leq n$.

Conditions 1 and 2 above imply that there exist many (positive density at infinity) σ-hyperbolic times for points $x \in D$ satisfying (1) with $\sigma < e^{-c_0/3}$. The rate depends on c_0 and f. This follows from an adapted version of Pliss lemma [**15**] also proved in [**11**] and [**2**]:

Proposition 3.1. — *Given any $x \in \tilde{D}$ and any sufficient large $N \geq 1$, there exists σ-hyperbolic times $1 \leq n_1 < .. < n_l \leq N$ for x with $l \geq \frac{-|\log \sigma|}{\sup |\log \|Df^{-1}|E^{cu}\| - 2|\log \sigma|} N$.*

Remark 2: Hyperbolic times can not be continuous with respect to the diffeomorfism f, but the rate

$$\theta = \frac{-|\log \sigma|}{\sup |\log \|Df^{-1}|E^{cu}\| - 2|\log \sigma|}$$

depends continuously on f (in the C^1-topology and c_0.

As a consequence of the existence of σ-hyperbolic times, we obtain backward uniform contraction and bounded distortion properties. More precisely (see [**2**]) :

Proposition 3.2. — *There exist $C_2 > 0$ and $\delta_1 > 0$ such that for all $x \in D$, for all σ-hyperbolic times n and for every $y \in D$ such that $\mathrm{dist}_{f^n(D)}(f^n(x), f^n(y)) \leq \delta_1$, we have*

$$\mathrm{dist}_{f^{n-k}(D)}(f^{n-k}(x), f^{n-k}(y)) \leq \sigma^{k/2} \mathrm{dist}_{f^n(D)}(f^n(x), f^n(y)), \tag{5}$$

$$\frac{1}{C_2} \leq \frac{|\det Df^n|T_y D|}{|\det Df^n|T_x D|} \leq C_2. \tag{6}$$

The constant C_2 in (6) above depends on σ, δ_1 and depends on the Hölder constant C_1. We remark that (6) is similar to the quotient factor in Proposition 2.2.

For each $j \geq 1$, let $\hat{H}_j$ be a finite set of $x \in D$ such that j is an σ-hyperbolic time for x. For $\delta = \delta_1/4$, we denote by $\Delta_j(x, \delta)$ the δ-neighborhood of $f^j(x)$ inside $f^j(D)$. We choose $\hat{H}_j$ such that the balls $\Delta_j(x, \delta)$ are pairwise disjoint. We denote by Δ_j the union of such balls.

We can choose $\hat{H}_j$ satisfying the following (see [**2**] Proposition 3.3 and Lemma 3.4): there exists a constant $\tau > 0$, depending only on f, such that for any j

$$f^j_* m_D(\Delta_j \cap f^j(U)) \geq f^j_* m_D(\Delta_j \cap f^j(\hat{H}_j)) \geq \tau m_D(\hat{H}_j)$$

Consider the set of accumulation points of $(\Delta_j)_j$:

$$\Delta_\infty = \bigcap_{n=1}^{\infty} \overline{\bigcup_{j \geq n} \Delta_j}.$$

Observe that $\Delta_j \subseteq f^j(D) \subseteq f^j(U)$. Then, since U is positively invariant,

$$\overline{\bigcup_{j\geq n} \Delta_j} \subseteq \overline{f^n(U)} \subseteq f^{n-1}(U)$$

and so $\Delta_\infty \subseteq \Lambda$.

Given $y \in \Delta_\infty$ there exist a sequence $(j_i)_i \to \infty$, disks $D_i = \Delta(x_i, \delta) \subseteq \Delta_{j_i}$ and points $y_i \in D_i$, $y_i \to y$ as $i \to \infty$. By passing to a subsequence if necessary, we may suppose that the centers x_i converge to some point x and, by Arzela-Ascoli theorem, that the D_i converge to a disk $D(x)$ of radius δ around x. Then y is in the closure $\overline{D(x)}$ of $D(x)$, and $\overline{D(x)} \subseteq \Delta_\infty$ and the points x are in the set

$$\hat{H}_\infty = \bigcap_{n=1}^{\infty} \overline{\bigcup_{j\geq n} f^j(\hat{H}_j)}.$$

Observe that D_i is contained in the j_i-iterate of D, which was taken tangent to the center-unstable cone field. So the domination property implies that the angle between D_i and E^{cu} goes to zero as $i \to \infty$. By Proposition 3.2, given $k \geq 1$ then f^{-k} is a $\sigma^{k/2}$-contraction on D_i, for every large i. Passing to the limit, we get that every f^{-k} is a $\sigma^{k/2}$-contraction on $D(x)$, and that $D(x)$ is tangent to the center-unstable subbundle at every point of $D(x) \subseteq \Lambda$, including x.

In particular we have shown that the subspace E^{cu}_x is indeed uniformly expanding for Df. The domination property means that any expansion that Df exhibits along the complementary direction is weaker than this. Then, see [**13**], there exists a unique strong-unstable manifold $W^u_{loc}(x)$ tangent to E^{cu} which is contracted by negative iterates of f at a rate of at least $\sigma^{k/2}$, when k gets large. Moreover $D(x)$ is contained in $W^u(x)$ because it is contracted by every f^{-k}, $k \geq 1$, and all its negative iterates are tangent to the center-unstable cone field. Summing up, we have

Proposition 3.3. — *The family of disks $D(x)$, with $x \in \hat{H}_\infty$, constructed as above satisfies:*

1. *the radius of $D(x)$ is $\delta_1/4$ uniformly in $x \in \hat{H}_\infty$;*
2. *for every $y \in \Delta_\infty$ there exists $x \in \hat{H}_\infty$ such that $y \in \overline{D(x)}$;*
3. *for all $x \in \hat{H}_\infty$, the subspace E^{cu}_x satisfies*

$$\|Df^{-k}|E^{cu}_x\| \leq \sigma^{k/2}, \quad \textit{for all} \ \ k \geq 0;$$

4. *$D(x)$ is contained in the corresponding strong-unstable manifold $W^u_{loc}(x)$;*
5. *$D(x)$ is tangent to the center-unstable subbundle at every point of $\Lambda \cap D(x)$.*

We now consider the sequence of averages of push-forwards of Lebesgue measure restricted to such a disk D

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j\, m_D.$$

Remark 3: The argument that follows does not change if we consider φm_D instead of m_D, where φ is a measurable function bounded away from zero and infinity, m_D- almost everywhere.

We decompose μ_n as a sum of two measures ν_n and η_n, where

$$\nu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j\, m_D|\Delta_j$$

and $\eta_n = \mu_n - \nu_n$. Observe that the support of ν_n is $\cup_{j=0}^{n-1}\Delta_j$.

Now, we consider any subsequence $(n_k)_k$ such that μ_{n_k} and ν_{n_k} converge to μ and ν respectively. Then the support of ν is contained in the set

$$\Delta_\infty = \bigcap_{n=1}^{\infty} \overline{\bigcup_{j\geq n} \Delta_j} \subseteq \Lambda$$

of accumulation points of $(\Delta_j)_j$. Proposition 3.3 gives a characterization of the support of ν. Moreover ([**2**] Proposition 3.5 and Remark 3.6), there is $\alpha_1 = \alpha_1(c_0, f) > 0$ such that, for all $n \geq 1$ and $k \geq n$ large enough,

$$\nu_k(f^n(U)) \geq \nu_k(f^k(U)) \geq \alpha_1.$$

This is because $\overline{f^k(U)} \subseteq f^n(U)$. Then, $\nu(f^n(U)) \geq \alpha_1$ and so

$$\nu(\Lambda) = \nu(\bigcap_{n\geq 1} f^n(U)) = \liminf_{n\to\infty} \nu(f^n(U)) \geq \alpha_1.$$

Recall from Proposition 3.3 that, given any $y \in \Delta_\infty$, there exist a point $x \in \hat{H}_\infty$ and a disk $D(x)$ of size $\delta_1/4$ around x such that $y \in \overline{D(x)} \subseteq \Delta_\infty$. For any such x and $r > 0$ small, let $C_r(x)$ be the tubular neighborhood of $\overline{D(x)}$, defined as the union of the images under the exponential map at each point $z \in \overline{D(x)}$ of all vectors orthogonal to $\overline{D(x)}$ at z with norm less than or equal to r. We take r to be sufficiently small, so that $C_r(x)$ is a cylinder endowed with the canonical projection $\pi : C_r(x) \to \overline{D(x)}$. We may suppose that the boundary of $C_r(x)$ has zero ν-measure (observe that r depends on the size of the domain of the exponential map, and so depends continuously on f).

For any $\varepsilon > 0$, we can fix a cover of $\overline{D(x)}$ by finitely many domains $D_{x,l} \subseteq \overline{D(x)}$, $l = 1, .., N(\varepsilon)$, small enough so that the intersection of each $C_{x,l} = \pi^{-1}(D_{x,l})$ with any smooth disk γ tangent to the center-unstable cone field has diameter less than ε inside γ. We choose the cover with the least possible $N(\varepsilon)$ and take the $D_{x,l}$ diffeomorphic to the compact ball B^u, so that every $C_{x,l}$ is a cylinder.

We say that a disk γ crosses $C_{x,l}$ if π maps $\gamma \cap C_{x,l}$ diffeomorphicaly onto $D_{x,l}$. For each $j \geq 0$, let $K_j(x,l)$ be the union of the intersections of $C_{x,l}$ with all the disks in Δ_j that cross $C_{x,l}$ and let $K_\infty(x,l)$ be the union of the intersections of $C_{x,l}$ with all the disks in Δ_∞ that cross $C_{x,l}$. Fixing a small enough ε for at least one of the cylinders $C_{x,l}$ the part of the measure ν that is carried by the disks in $K_\infty(x,l)$ has positive mass $\alpha > 0$, depending on the rate of hyperbolic times, and so depending on c_0 and f (See [**2**] Lemma 4.2 and Lemma 4.3).

In the following we write $\mathcal{C} = C_{x,l}$, $\tilde{D} = D_{x,l}$ and $\mathcal{K}_j$, $0 \leq j \leq \infty$, the family of disks whose union is $K_j = K_j(x,l)$.

Observe that from the construction of μ_n the measure $f_*^j m_D|\Delta_j$ is absolutely continuous with respect to Lebesgue measure along $f^j(D)$. Moreover, from Proposition 3.2(6) the density of the normalization of this measure is uniformly bounded from below and from above. The construction preserves this property for ν_n and ν.

Let us introduce $\hat{K} = \cup_{0\leq j\leq\infty} K_j \times \{j\}$. In this space, we consider the sequence of finite measures $\hat{\nu}_n$ defined by

$$\hat{\nu}_n(B_0 \times \{0\} \cap .. \cap B_{n-1} \times \{n-1\}) = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j m_D(B_j),$$

and $\hat{\nu}_n(B_n) = 0$ whenever B is in $\cup_{n\leq j\leq\infty} K_j \times \{j\}$. We also consider a sequence of partitions $\mathcal{P}_k$ in $\hat{K}$ constructed as follows. Fix an arbitrary point $z \in \tilde{D}$ and let V be the inverse image $\pi^{-1}(z)$ under the canonical projection. Fix a sequence $\mathcal{V}_k$, $k \geq 1$, of increasing partitions of V with diameter going to zero. Then, by definition, two points (x,m), $(y,n) \in \hat{K}$ are in the same atom of the partition $\mathcal{P}_k$ if

- the disk in Δ_m containing x and the disk in Δ_n containing y intersect some common element of $\mathcal{V}_k$;
- either $m \geq k$ and $n \geq k$, or $m = n < k$.

It is clear from the construction that for any point $\xi \in K_j$ and every $0 \leq j \leq \infty$, one has

$$\mathcal{P}_1(\xi) \supset .. \supset \mathcal{P}_k(\xi) \supset \dots,$$

and $\cap_{k=1}^\infty \mathcal{P}_k(\xi)$ coincides with the intersection of the cylinder $\mathcal{C}$ with the disk in Δ_j that contains ξ. We define $\hat{\pi} : \hat{K} \to \tilde{D}$ by $\hat{\pi}(x,j) = \pi(x)$.

Clearly, any weak* accumulation measure of the sequence $\hat{\nu}_n$ must be supported in $K_\infty \times \{\infty\}$. We have chosen a sequence $(n_k)_k$ such that ν_{n_k} converges to the measure ν. It is easy to see that this is just the same as saying that $\hat{\nu}_k$ converges to the measure $\hat{\nu}$ defined by $\hat{\nu}(B \times \{\infty\}) = \nu(B)$ for any Borel set $B \subseteq \mathcal{C}$, so ν and $\hat{\nu}$ are naturally identified.

Proposition 3.4. — *There exist $C_3 > 1$, depending on f only, and a family of conditional measures $(\nu_\gamma)_\gamma$ of $\nu|\mathcal{K}_\infty$ along the disks $\gamma \in \mathcal{K}_\infty$ such that ν_γ is absolutely continuous with respect to the Lebesgue measure m_γ on γ, with*

$$\frac{1}{C_3} m_\gamma(B) \leq \nu_\gamma(B) \leq C_3 m_\gamma(B) \tag{7}$$

for any Borel set $B \subseteq \gamma$.

The reader can see the proof in [**2**] Section 4. The constant C_3 depends on the Lebesgue measure along the disks in the cylinder (so depends on f) and depends on the constant C_2 obtained in Proposition 3.2.

The construction of Gibbs cu-states concludes as follows: there exists an ergodic component μ_z of μ having positive measure on $\mathcal{K}_\infty$ which is absolutely continuous along the disks.

Each disk $D \in \mathcal{K}_\infty$ is completely contained in some ergodic component because it is contained in some local-unstable manifold. In particular, all Lyapunov exponents of μ_z in the center unstable direction are larger than $-\log\sigma > 0$. The domination condition implies that all the other exponents are less than $-\log\sigma + \log\lambda < -\log\sigma$. Again, by Pesin theory, μ_z-almost every point has a local strong-unstable manifold which is an embedded disk whose backward orbits contract at the exponential rate $\log\sigma$. Moreover the disks $D \in \mathcal{K}_\infty$ contain the local strong-unstable manifolds of points in its interior.

Summing up this section, we have the following

Theorem 3.1. — [Alves, Bonatti, Viana [**2**]] *Any diffeomorphism f with a dominated splitting which is non-uniformly expanding along the center unstable direction has an ergodic Gibbs cu-state. More precisely: there exist a cylinder $\mathcal{C} \subseteq M$ and a family $\mathcal{K}_\infty$ of disjoint disks contained in $\mathcal{C}$ which are graphs over B^u, and a ergodic invariant probability measure μ supported on Λ such that:*

1. *the cylinder contains a ball whose radius is uniformly bounded away from zero, depending continuously on the diffeomorphism f;*
2. *there exists $\alpha > 0$ such that the union of all disks in K_∞ has μ-measure larger than α, depending on f and c_0;*
3. *μ has absolutely continuous conditional measures along the disk in $\mathcal{K}_\infty$. The densities of the conditional measures are bounded away from zero and infinity by a constant depending on f and c_0;*
4. *the $u = \dim E^{cu}$ largest Lyapunov exponents are larger than $-\log\sigma > 0$.*

3.2. Proofs of Theorems B and C. — We start by proving Theorem B. Let $f \in \mathrm{Diff}^2(M)$ and Λ be an attractor having a dominated splitting which is mostly expanding along the E^{cu} direction. Let $\mathcal{G}(f)$ be the class of Gibbs cu-states constructed in Subsection 3.1. Theorem B can be reformulated in the following words: every ergodic Gibbs cu-state supported in Λ must be in $\mathcal{G}(f)$.

Proof of Theorem B: Let μ be an ergodic Gibbs cu-state for f supported on Λ. Let also $D \subseteq W^u_{loc}(x)$ be in the support of μ, such that $D \cap B(\mu)$ has full μ_D-measure in D (cf. Lemma 2.4). We may assume that D satisfies condition 2 in Subsection 3.1 taking an iterate of D if necessary ([**2**] Corollary 2.4, Proposition 2.9). Condition 1 is satisfied by ergodicity: consider the function $\varphi(x) = \log \|Df^{-1}|E^{cu}_x\|$. Birkhoff´s Ergodic Theorem implies

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log \|Df^{-1}|E^{cu}_{f^j(x)}\| = \int \log \|Df^{-1}|E^{cu}_y\| \, d\mu(y) \leq -c_0 < 0$$

for μ_D-almost every point in D, where c_0 depends on the Lyapunov exponents of D in the E^{cu} direction. But $\mu_D = \rho_D m_D$ where ρ_D is a measurable function bounded away from zero and infinity, so the claim above holds Lebesgue-almost everywhere in D.

Let $\tilde{\mu}$ be a ergodic Gibbs cu-state obtained as a weak* accumulation measure of

$$\frac{1}{n}\sum_{j=0}^{n-1} f^j_* \left(\frac{\mu_D}{\mu_D(D)}\right).$$

Of course, $\tilde{\mu} \in \mathcal{G}(f)$ because μ_D is absolutely continuous with respect to Lebesgue measure on D.

Observe that for every continuous $\varphi : M \to \mathbb{R}$ we have

$$\frac{1}{n}\sum_{j=0}^{n-1} f^j_* \left(\frac{\mu_D(\varphi)}{\mu_D(D)}\right) = \frac{1}{\mu_D(D)} \int_D \frac{1}{n}\sum_{j=0}^{n-1} \varphi \circ f^j(x) \, d\mu_D. \tag{8}$$

Denote by F_n the average $\frac{1}{n}\sum_{j=0}^{n-1} \varphi \circ f^j$. Each F_n is μ_D-integrable and bounded by $\|\varphi\|$. Also F_n converges pointwise to $\int \varphi \, d\mu$ because $B(\mu)$ has full μ_D-measure on the disk D. The dominated convergence theorem implies that the right hand side of (8) converges to $\int \varphi \, d\mu$.

On the other hand the left hand side of (8) was assumed to have an accumulation measure $\tilde{\mu}$, so it converges to $\int \varphi \, d\tilde{\mu}$. As a consequence $\mu = \tilde{\mu}$.

□

Let (f_n) be a sequence of diffeomorphisms converging to f in the C^k-topology, $k \geq 2$. We assume that each f_n exhibits a dominated splitting with non-uniform expansion along the $E^{cu}(f_n)$ direction with constants C, α and c_0 not depending on $n \geq 0$ (cf. Subsection 1.1. Let μ_n be an ergodic Gibbs cu-states of f_n. We will assume that μ_n tends to a probability measure μ^* in the weak* topology (taking a subsequence if necessary). To prove Theorem C, we need to prove that μ^* is a cu-Gibbs state for f.

Is clear that μ^* is f-invariant. By Theorem B each μ_n is a Gibbs cu-state in $\mathcal{G}(f)$. Then, for each $n \geq 1$, there exist $(\mathcal{C}_n)_n$ and $(\mathcal{K}^n_\infty)_n$ cylinders and families of disks associated to (f_n, μ_n). From Subsection 3.1 we may assume that:

(a) the size of the disks is uniformly bounded from below;

(b) there exists $\alpha > 0$ such that, for all $n \geq 0$, we have

$$\mu_n(K^n_\infty) \geq \alpha > 0, \tag{9}$$

where K^n_∞ is the union of the disks in $\mathcal{K}^n_\infty$, since we are assuming that c_0 is uniform (cf. Proposition 3.1;

(c) there exist $C_3 > 1$ such that for all $n \geq 1$ the family of conditional measures $(\mu_{n,D})_{D\in\mathcal{K}^n_\infty}$ of $\mu_n|K^n_\infty$ along the disks $D \in \mathcal{K}^n_\infty$ satisfies:

$$\frac{1}{C_3} m_D(B) \leq \mu_{n,D}(B) \leq C_3 m_D(B) \tag{10}$$

for any Borel set $B \subseteq D$.

We prove that μ^* is an Gibbs cu-state by completing the following steps.

1. We construct a cylinder $\mathcal{C}^*$ and a family $\mathcal{K}^*_\infty$ of disjoint disks contained in $\mathcal{C}^*$ which are graphs over B^u such that all the disks in $\mathcal{K}^*_\infty$ are local uniformly expanding manifolds under f.
2. The union K^*_∞ of all disks in $\mathcal{K}^*_\infty$ has positive μ^*-measure.
3. The restriction of μ^* to that union has absolutely continuous conditional measures along the disks in $\mathcal{K}^*_\infty$.
4. Almost every ergodic component of μ^* is a Gibbs cu-state.

Of course, by the ergodic decomposition theorem, μ^* must be a Gibbs cu-state, because all of its ergodic components are Gibbs cu-states.

We prove these steps in the following lemmas:

Lemma 3.1. — *There exist a cylinder $\mathcal{C}^*$ and a family $\mathcal{K}^*_\infty$ of disjoint disks contained in $\mathcal{C}^*$ which are graphs over B^u such that all disks in $\mathcal{K}^*_\infty$ are local uniformly expanding manifolds.*

Proof. — Let $(\mathcal{C}_n)_n$ and $(\mathcal{K}^n_\infty)_n$ be the sequences of cylinders and families of disks associated to μ_n respectively. By the compactness of M and considering a subsequence if necessary, we may suppose that $\mathcal{C}_n$ converges to $\mathcal{C}^*$.

We claim that $\mathcal{C}^*$ is a cylinder. Indeed, the $\mathcal{C}_n$ are diffeomorphic images of $B^u \times B^s$ where B^u and B^s are compact balls in $\mathbb{R}^u$ and $\mathbb{R}^s$ respectively corresponding to $\mathcal{C}_n$, $n \geq 1$. Let

$(B^u_n)_n$ and $(B^s_n)_n$ be the diffeomorphic images of B^u and B^s in M, respectively. By the Arzela-Ascoli Theorem $(B^u_n)_n$ converges to a disk B^u_* and $(B^s_n)_n$ converges to a disk B^s_*. So, for $\mathcal{C}^*$ to be a cylinder, it must satisfy:

(i) the diameters of B^u_n and B^s_n do not go to zero, when n tends to infinity.

(ii) the angle between B^u_n and B^s_n does not go to zero, when n tends to infinity.

On the one hand by construction , each $\mathcal{C}_n$ contains balls with radius uniformly bounded away from zero from Theorem 3.1, so (i) is fullfilled. On the other hand, by the domination property at the family (f_n), (ii) must hold.

Now, we considered the family $\mathcal{K}^*_\infty$ of disks D^u contained in $\mathcal{C}^*$ which are accumulated by sequences $(D^u_n)_n$ of disk, $D^u_n \in \mathcal{K}^n_\infty$, $n \geq 1$. Observe that every disk $D^u_n \in \mathcal{K}^n_\infty$ is tangent to the center-unstable cone field for f_n; by continuity of the splitting with respect to the diffeomorphism, $D^u \in \mathcal{K}^*_\infty$ must be tangent to the center-unstable cone field of f. For any $x, y \in D^u$ let $(x_n)_n$ and $(y_n)_n$ be two sequences of points in D^u_n converging to x and y respectively. By Proposition 3.3, for all $k \geq 0$ fixed we have

$$\text{dist}(f_n^{-k}(x_n), f_n^{-k}(y_n)) \leq \sigma^{-k/2}\text{dist}(x_n, y_n).$$

Passing to the limit when $n \to \infty$, we obtain

$$\text{dist}(f^{-k}(x), f^{-k}(y)) \leq \sigma^{-k/2}\text{dist}(x, y),$$

for all $x, y \in D^u$ and all $k \geq 0$. We conclude that every f^k is an $\sigma^{k/2}$-contraction on $D(x)$, and $D(x)$ is tangent to the center-unstable subbundle at every point in $\Lambda \cap D(x)$ (including x).

In particular we have shown that the subspace E^{cu}_x is indeed uniformly expanding for Df. The domination property means that any expansion Df may exhibit along the complementary direction is weaker than this. Then, there exists a unique strong-unstable manifold $W^u_{loc}(x)$ tangent to E^{cu} which is contracted by negative iterates of f by a rate of at least $\sigma^{k/2}$, when k gets large, see [**13**]. Moreover $D(x)$ is contained in $W^u(x)$ because it is contracted by every f^{-k}, $k \geq 1$, and all its negative iterates are tangent to the center-unstable cone field.

□

Lemma 3.2. — *The union of all disks in $\mathcal{K}^*_\infty$ has positive μ^*-measure.*

Proof. — Recall from Subsection 3.1 that there exists $\alpha > 0$ such that, for all $n \geq 0$, we have

$$\mu_n(K^n_\infty) \geq \alpha > 0.$$

Let $\delta > 0$. Then there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$

$$K_\infty^n \subseteq B(K_\infty^*, \delta).$$

On the other hand,

$$K_\infty^* = \bigcap_{\delta>0} B(K_\infty^*, \delta).$$

Choosing $\delta > 0$ such that $\partial\mu^*(B(K_\infty^*, \delta)) = 0$, we have

$$\mu^*(B(K_\infty^*, \delta)) = \lim_{n\to\infty} \mu_n(B(K_\infty^*, \delta)) \geq \alpha > 0,$$

and so,

$$\mu^*(K_\infty^*) = \liminf_{\delta\to 0} \mu^*(B(K_\infty^*, \delta)) \geq \alpha > 0.$$

□

Lemma 3.3. — *There exist a constant $C_1 > 0$ and a family of conditional measures $(\mu_D^*)_D$ of $\mu^*|K_\infty^*$ along the disks $D \in \mathcal{K}_\infty^*$ such that μ_D^* is absolutely continuous with respect to Lebesgue measure m_D on D, with*

$$\frac{1}{C_3} m_D(B) \leq \mu_D^*(B) \leq C_3 m_D(B)$$

for every Borel set $B \subseteq D$.

Proof. — By the compactness of $\mathcal{C}^*$, for any $\xi \in D$ where D is any disk in $\mathcal{K}_\infty^*$, the exponential map is well defined in a ball of radius $\tilde{r} > 0$ around ξ. For any Borel set $B \subset D$ we define the set $\tilde{B}$ as the tubular neighborhood of B, that is, the union of the images under the exponential map at each point $\xi \in B$ of all vectors orthogonal to D at ξ.

We fix a sequence of partitions $\mathcal{P}_k$ on $\mathcal{K}_\infty^*$ constructed as follows. Let V be the inverse image of B_*^s under the diffeomorphism between $B_*^u \times B_*^s$ and $\mathcal{C}^*$. Fix a sequence $\mathcal{V}_k$, $k \geq 1$, of increasing partitions of V with positive diameter less than $\tilde{r}$ and going to zero. Then, we say that two points x, $y \in \hat{K}$ are in the same atom of the partition $\mathcal{P}_k$ if the disk in D_1 containing x and the disk in D_2 containing y intersect the same element of $\mathcal{V}_k$. It is clear from the construction that for any point $\xi \in K_\infty^*$,

$$\mathcal{P}_1(\xi) \supset .. \supset \mathcal{P}_k(\xi) \supset \dots$$

and $\cap_{k=1}^\infty \mathcal{P}_k(\xi)$ coincides with the disk in D that contains ξ.

For any Borelean set $B \subseteq D$ we have, from Proposition 3.4,

$$\frac{1}{C_3} m(B_n)\mu_n(\mathcal{P}_k(\xi)) \le \mu_n(\tilde{B} \cap \mathcal{P}_k(\xi)) \le C_3 m(B_n)\mu_n(\mathcal{P}_k(\xi))$$

where C_3 does not depend on n, D_n is a disk in $\mathcal{K}^n_\infty$ near D and $B_n = \tilde{B} \cap D_n$, $n \ge 1$. By construction $m(B_n)$ converges to $m(B)$ and so passing to the limit when $n \to \infty$ we have

$$\frac{1}{C_3} m(B)\mu^*(\mathcal{P}_k(\xi)) \le \mu^*(\tilde{B} \cap \mathcal{P}_k(\xi)) \le C_3 m(B)\mu^*(\mathcal{P}_k(\xi)).$$

Now, by the Radon-Nikodym Theorem, we have that the disintegration of μ^* along the disk $\cap_{k=1}^\infty \mathcal{P}_k(\xi)$ is absolutely continuous with respect to Lebesgue measure in this disk, and the densities are almost everywhere bounded from above by C_3 and from below by $1/C_3$.

□

Remark 4: The densities of μ^*_D are uniformly (with respect to D) bounded away from zero and infinity.

Let ξ be an invariant measure for f. Given a point x let us denote by ξ_x the probability measure given by the time average along the orbit of x

$$\int \varphi \, d\xi_x = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x)$$

for every continuous $\varphi : M \to \mathbb{R}$. According to the Ergodic Decomposition Theorem (cf. [**11**]) ξ_x is well defined and ergodic for every x in a set $\Sigma(f) \subseteq M$ that has full measure with respect to any invariant measure. Moreover, for every bounded measurable function $\varphi : M \to \mathbb{R}$ we can write

$$\int \varphi \, d\xi = \int \int \varphi \, d\xi_x \, d\xi(x),$$

and for every such φ the integral $\int \varphi d\xi$ coincides with the time average ξ-almost everywhere.

Fix B a measurable subset of M such that

$$m_\gamma(B \cap \gamma) = 0 \quad \text{for every} \quad \gamma \in \mathcal{K}^*_\infty,$$

and $\mu(B)$ is maximal among all measurable sets with this property. Observe that $\mu^*(B) = 0$, because μ^* is absolutely continuous along the leaves on $\mathcal{K}^*_\infty$ (cf. Lemma 3.3). Let $Z_\infty = K^*_\infty \cap \Sigma(f) \cap R(f) \setminus B$. Then $\mu^*(Z_\infty) > 0$ and let $(\mu^*|Z_\infty)$ be the restriction of μ^* to Z_∞.

Let A be any measurable subset of Z_∞ such that $m_\gamma(A \cap \gamma) = 0$ for every $\gamma \in \mathcal{K}^*_\infty$. Then $\mu^*(A)$ must be zero, since we took $\mu^*(B)$ maximal. This means that $(\mu^*|Z_\infty)$ is absolutely continuous with respect to the product $m_\gamma \times \hat{\mu}^*$, where $\hat{\mu}^*$ stands for the quotient measure induced by $(\mu^*|Z_\infty)$ on $\mathcal{K}^*_\infty$. As a consequence, the conditional measures $\tilde{\mu}^*_\gamma$ of $(\mu^*|Z_\infty)$

on the disks $\gamma \in \mathcal{K}_\infty^*$ are absolutely continuous with respect to Lebesgue measure m_γ for $\hat{\mu}^*$-almost all $\gamma \in \mathcal{K}_\infty^*$. On the other hand, for any measurable set $A \subseteq Z_\infty$,

$$\mu^*(A) = \int \mu_x^*(A)\, d\mu^*(x),$$

where the integral is taken over $\Sigma(f) \subseteq M$.

Let us denote by $\mathbf{1}_A$ the characteristic function of the measurable subset A. Then we have (as already mentioned)

$$\mu_x^*(A) = \int \mathbf{1}_A\, d\mu_x^* = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbf{1}_A(f^j(x))$$

μ^*-almost everywhere. So $\mu_x^*(A)$ can be non-zero only if x has some iterate in $A \subseteq Z_\infty$, for μ^*-almost every point x. Let $k(z)$ denote the first backward return time to Z_∞ of a point $z \in Z_\infty$, this means $k(z)$ is the smallest positive integer such that $f^{-k(z)}(z) \in Z_\infty$. This is defined μ^*-almost everywhere, by Poincare's recurrence theorem. Observing also that $\mu_z^* = \mu_{f^j(z)}^*$ for every z and every integer $j \in \mathbb{Z}$, we have

$$\mu^*(A) = \int \mu_x^*(A)\, d\mu^*(x) = \int_{Z_\infty} k(z)\mu_z^*(A)\, d\mu^*(z)$$

for any measurable subset A of Z_∞.

Lemma 3.4. — *Let λ be a finite measure on a measure space Z, with $\lambda(Z) > 0$. Let $\mathcal{K}$ be a measurable partition of Z, and $(\lambda_z)_{z\in Z}$ be a family of finite measures on Z such that*

1. *the function $z \to \lambda_z(A)$ is measurable, and constant on each element of $\mathcal{K}$, for any measurable set $A \subset Z$.*
2. *$\{w : \lambda_z = \lambda_w\}$ is a measurable set with full λ_z-measure, for every $z \in Z$.*

Assume that $\lambda(A) = \int l(z)\lambda_z(A)\, d\lambda$ for some measurable function $l : Z \to \mathbb{R}_+$ and any measurable subset A of Z. Let $\{\tilde{\lambda}_\gamma : \gamma \in \mathcal{K}\}$, and $\{\tilde{\lambda}_{z,\gamma} : \gamma \in \mathcal{K}\}$, be the disintegrations of λ and λ_z, respectively, into conditional probability measures along the elements of the partition $\mathcal{K}$. Then

$$\tilde{\lambda}_{z,\gamma} = \tilde{\lambda}_\gamma$$

for λ-almost every $z \in Z$ and $\hat{\lambda}_z$-almost every γ, where $\hat{\lambda}_z$ is the quotient measure induced by λ_z on $\mathcal{K}$.

The reader can see the proof in [**2**], Lemma 6.2.

Lemma 3.5. — *The measure μ_x^* is a Gibbs cu-state, for μ^*-almost every point $x \in K_\infty^*$.*

Proof. — Let $x \in K \cap \Sigma \cap R(f)$. Observe that $K^*_\infty \cap \Sigma \cap R(f)$ has full μ^*-measure on K^*_∞ and that

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \log \|Df^{-n}|E^{cu}_{f^n(x)}\| &\leq \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df^{-1}|E^{cu}_{f^j(x)}\| \\
&= \int \log \|Df^{-1}|E^{cu}_y\| \, d\mu^*_x(y) \\
&\leq -c_0 < 0.
\end{aligned}$$

This implies that the u largest Lyapunov exponents are positive.

Now, if we take $Z = Z_\infty$, $\lambda = (\mu^*|Z_\infty)$, $\mathcal{K} = \mathcal{K}^*_\infty$, $\lambda_z = (\mu^*_z|Z_\infty)$ and $l(z) = k(z)$ for each $z \in Z_\infty$ in Lemma 3.4, the conditional probability measures $\tilde{\mu}^*_{z,D}$ of $(\mu^*_z|Z_\infty)$ along the disks $D \in \mathcal{K}^*_\infty$ coincide almost everywhere with the corresponding conditional measures $\tilde{\mu}^*_D$ of $(\mu^*|Z_\infty)$. Recall that we had already shown that the latter are almost everywhere absolutely continuous with respect to Lebesgue measure on the corresponding disks $D \in \mathcal{K}^*_\infty$.

□

As we observed above, $\mu^*_x = \mu^*_{f^j(x)}$, for μ^*-almost every x and for all $j \in \mathbb{Z}$. Now, we define the measure ν^* by

$$\nu^* = \int_{K^*_\infty} \mu^*_x \, d\mu(x).$$

For μ^*-almost every $x \in K^*_\infty$, μ^*_x is a Gibbs cu-state (cf. Lemma 3.5) and so is $\frac{\nu^*}{\nu^*(M)}$. Of course, by definition, ν^* is a component (not necessarily ergodic) of μ^*, but $\nu^*(K^*_\infty) > 0$.

Writing $K_0 = \operatorname{supp} \nu^*$, observe that $K^*_\infty \subseteq K_0$ and by the invariance of the support we have

$$K_0 = \overline{\bigcup_{j\in\mathbb{Z}} f^j(K^*_\infty)}.$$

Now let $A \subseteq K_0$ be measurable. Then

$$\begin{aligned}
\mu^*(A) &= \int_M \mu^*_x(A) \, d\mu^*(x) \\
&= \int_{K_0} \mu^*_x(A) \, d\mu^*(x) + \int_{M\setminus K_0} \mu^*_x(A) \, d\mu^*(x) \\
&= \nu^*(A) + \int_{M\setminus K_0} \mu^*_x(A) \, d\mu^*(x).
\end{aligned}$$

This proves that ν^* is a component of μ^*. In order to prove that $\nu^* = \mu^*$ it is sufficient to prove that $\mu^*(M \setminus K_0) = 0$.

For $N \in \mathbb{N}$, let us denote by K_n^N the set

$$K_n^N = \bigcup_{j=-N}^{N} f_n^j(K_\infty^n),$$

and by K_0^N the set

$$K_0^N = \bigcup_{j=-N}^{N} f^j(K_\infty^*).$$

For all $N \in \mathbb{N}$, we define $\mu_n^N = (\mu_n|K_n^N)$ and $\nu_N^* = (\nu^*|K_0^N) = (\mu^*|K_0^N)$. It is clear that, for any $N \in \mathbb{N}$ fixed, μ_n^N converges to ν_N^* when n goes to infinity. On the other hand, for any $n \geq 0$ fixed, μ_n^N converges to μ_n and ν_N^* converges to ν^* when N goes to infinity.

Lemma 3.6. — $\mu^*(M \setminus K_0) = 0$

Proof. — Suppose otherwise. Let $B \subseteq M \setminus K_0$ be an open set such that $\mu^*(B) \geq \beta > 0$ and assume that $\mu^*(\partial B) = 0$. For a fixed $\varepsilon > 0$ we have $\mu_n(B) \geq \beta - \varepsilon$ for $n \geq 0$ large enough . Taking $\delta > 0$ small and fixed, then there exists $N(n, \delta) = N$ such that $\mu_n(B) - \mu_n^N(B) < \delta$. If N does not depend on $n \geq 0$, for instance if the set $\{N(n, \delta) \ : \ n \geq 0\}$ is bounded, then B is in the support of each μ_n^N and, passing to the limit when $n \to \infty$, a positive measure subset of B intersects $K_0^N \subseteq K_0$, which is a contradiction. Then we may assume that $N(n, \delta) = N(n)$ goes to infinity when $n \to \infty$.

It is no restriction to assume also that $\mu_n^N(f_n^{-j}(B)) = 0$, for $j = 0, \ldots, N(n) - 1$, with $N(n) > 0$ (the case $N(n)$ is analogous) and

$$0 < \beta - \varepsilon - \delta \leq \mu_n(f_n^{-N}(B)).$$

This implies that $f_n^{-N}(B)$ intersects K_∞^n in a positive $\hat{\mu}_n$-measure subset of disks D in $\mathcal{K}_\infty^n$, and on each of these disks D the Lebesgue measure of $f_n^{-N}(B) \cap D_n$ is positive. More precisely

$$0 < \beta - \varepsilon - \delta \leq \mu_n(f_n^{-N}(B)) = \int_{\mathcal{K}_\infty^n} \int_D \mathbf{1}_{f_n^{-N}(B)}(x) \rho_D^n(x) \, dm_D(x) \, d\hat{\mu}_n(D)$$

where ρ_D^n is the density of μ_n along the disk D in $\mathcal{K}_\infty^n$. For each disk D in a positive $\hat{\mu}_n$-measure subset of $\mathcal{K}_\infty^n$ we have

$$0 < \int_D \mathbf{1}_{f_n^{-N}(B)}(x) \rho_D^n(x) \, dm_D(x) \leq C(n) m_D(f_n^{-N}(B)).$$

But for each $n \geq 0$, $B \cap f_n^N(D) \cap f_n^N(\mathcal{K}_\infty^n)$ is contained in some disk whose diameter is exponentially contracted for the past (cf. Proposition 3.3). This implies that $m_D(f_n^{-N}(B)) \to 0$

when $n \to \infty$. So $C(n)$ must go to infinity. This is a contradiction because the densities are uniformly bounded (cf. Proposition 3.4 and Lemma 3.3). $\square$

4. Gibbs cu-states and SRB measures

In this section we study the relationships between Gibbs cu-states and SRB measures and conclude with some applications of our result to the study of statistical stability for partially hyperbolic systems. First, our goal is to prove Corollary D.

We assume that f is a C^2-diffeomorphism with a topological attractor Λ with a dominated splitting which is non-uniformly expanding along the center-unstable direction. Consider a disk D transverse to the center-stable direction. First, we prove that the constructions of the previous section can be done replacing the disk D by a positive Lebesgue measure subset E.

Proposition 4.1. — *Given a center-unstable domain D and any positive Lebesgue measure set $E \subseteq D$, every weak* accumulation point of*

$$\mu_{n,E} = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j\left(\frac{m_E}{m_D(E)}\right)$$

has an ergodic component which is a Gibbs cu-state of f.

Proof. — Given any $\delta > 0$ we may find pairwise disjoint domains $D_1, ..., D_s$ in D such that the relative Lebesgue measure on E inside each D_i is larger than $1-\delta$, and the total measure of E outside the union of the D_i is less than $\delta m_D(E)$. Then, for any $j \geq 1$, we have

$$\begin{aligned} f_*^j\left(\frac{m_E}{m_D(E)}\right) &= \sum_{i=1}^{s} \frac{m_D(D_i)}{m_D(E)} f_*^j\left(\frac{m_{D_i}}{m_D(D_i)}\right) \\ &\quad + \frac{1}{m_D(E)} f_*^j m_{(E\setminus \cup_{i=1}^s D_i)} - \frac{1}{m_D(E)} \sum_{i=1}^{s} f_*^j m_{D_i \setminus E}. \end{aligned}$$

The total masses of both the second and the third term do not depend on j, and are less than δ. Therefore, every accumulation point of $\mu_{n,E}$ differs from an accumulation point of

$$\sum_{i=1}^{s} \frac{m_D(D_i)}{m_D(E)} \frac{1}{n} \sum_{j=0}^{n-1} f_*^j\left(\frac{m_{D_i}}{m_D(D_i)}\right)$$

by a measure whose total mass is less than δ. Applying Theorem 1.2 to each domain D_i, every point of accumulation of this last sequence has an ergodic component which is a Gibbs cu-state whose densities are uniformly bounded and satisfy the ratio relation of Proposition 2.2. Making δ go to zero and applying Theorem A and Theorem C we get that every weak* accumulation point of $\mu_{n,E}$ has an ergodic component which is a Gibbs cu-state. $\square$

Proof of Corollary D: Let μ be an ergodic SRB measure for f supported in Λ. Consider any disk D inside U, where U is a neighborhood of Λ as in Subsection 1.1. Let us suppose the D is transverse to the center-stable subbundle and intersecting the basin of μ on a full Lebesgue measure subset D_0. On one hand,

$$\frac{1}{n}\sum_{j=0}^{n-1} f_*^j\left(\frac{m_{D_0}}{m_D(D_0)}\right) = \frac{1}{m_D(D_0)}\int_{D_0}\frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(x)}\,dm_D(x)$$

converges to μ when $n\to\infty$. On the other hand, by Proposition 4.1 and the hypothesis of ergodicity it follows that μ must be a Gibbs cu-state.

$\square$

4.1. Statistical Stability. — Now we present several applications of our results to the study of statistical stability for systems with a weak form of hyperbolicity.

4.1.1. The partially hyperbolic case. — Now we assume that f has a partially hyperbolic attractor Λ with splitting $T_\Lambda M = E^{cu}\oplus E^s$. The measure μ constructed in Section 3 has an ergodic component μ_*, with support contained in Λ, which is a Gibbs cu-state. Then there exists a disk $D_\infty \in \mathcal{K}_\infty$ such that $m_{D_\infty}(B(\mu_*)) > 0$ ([**2**], lemma 4.5). Because the strong-stable foliation is absolutely continuous [**8**], $m(B(\mu_*))$ must be positive, so μ_* is an SRB measure. Moreover, a full Lebesgue measure subset of U is contained in the union of finitely many SRB measures supported in Λ ([**2**], corollary 4.6)).

Proof of Corollary E: Let $(\mu_n)_n$ be a sequence of ergodic SRB probability measures for f_n, converging to μ. By Theorem A, there exist a cylinder $\mathcal{C}^*$ and a family $\mathcal{K}^*_\infty$ of disjoint disks contained in $\mathcal{C}^*$ which are graphs over B^u and local uniformly expanding manifolds. Moreover, $\mu(\mathcal{K}^*_\infty)\geq\alpha>0$ and μ is absolutely continuous with respect to Lebesgue measure along these disks.

Now if we take a tubular neighborhood of D_∞ given by Lemma 2.4 using the stable foliation, then since this foliation is Hölder continuous we have a positive Lebesgue measure set of points in $B(\mu)$, so μ is an SRB measure. By Theorem 1.1 there are finitely many ergodic SRB measures and the union of their basins covers a full Lebesgue measure subset of U, so μ must be in the convex hull of such measures. $\square$

4.1.2. The dominated splitting case. — In the setting where f has an attractor with dominated splitting $E^{cs}\oplus E^{cu}$ and with $s=\dim E^{cs}$ Lyapunov exponent of μ all negatives, μ is in fact a SRB measure. This is a consequence of the absolute continuity property of μ and the absolute continuity of the stable lamination [**13**]: the union of the stable manifolds through the point whose time averages are given by μ is a positive Lebesgue measure subset of M.

Proof of Theorem F: First we prove the existence of SRB measures. Let μ be an ergodic Gibbs cu-state for f. It exists by Theorem 1.2. Let D_∞ be a disk such that $D_\infty \cap B(\mu) \cap R(f)$ has full Lebesgue measure on D_∞. Such a disk exists by Lemma 2.4, and is contained in some local unstable manifold. By hypothesis, Lebesgue almost every point in D_∞ satisfies (3). So the set A of points in $D_\infty \cap B(\mu) \cap R(f)$ satisfying (3) has full Lebesgue measure on D_∞.

For $\varepsilon > 0$, we denote by $D_\infty(\varepsilon)$ the tubular neighborhood of radius ε around D_∞, defined as the image under the exponential map of M of all the vectors of norm less that $\varepsilon > 0$ in the orthogonal complement of E^{cu}_x, for all $x \in D_\infty$. If $\varepsilon > 0$ is small enough then $D_\infty(\varepsilon)$ is a cylinder.

For every point in $x \in A$, there exists a C^1 embedded disk $W^s_{loc}(x)$ tangent to E^{cs}_x at x such that the diameter of $f^n(W^s_{loc}(x))$ converges exponentially fast to zero as $n \to \infty$. These disks $W^s_{loc}(x)$ depend in a measurable way on the point x, and the lamination $\{W^s_{loc}(x) : x \in A\}$ is absolutely continuous. Since $A \in B(\mu)$ every $y \in W^s_{loc}(x)$ is in $B(\mu)$ also.

The domination condition on the splitting together the absolute continuity of the stable lamination implies that every disk D tangent to the E^{cu} direction crossing $D_\infty(\varepsilon)$, and close enough to D_∞, intersects the lamination $\{W^s_{loc}(x) : x \in A\}$ in a positive Lebesgue measure subset. Finally, Fubini's Theorem implies that the Lebesgue measure of $B(\mu)$ is positive.

Now we prove that there are finitely many ergodic SRB measures. Suppose otherwise. Let (μ_n) be a sequence of ergodic SRB measures of f converging in the weak* topology to a measure μ. By Corollary D, each μ_n must be a Gibbs cu-state for f. Theorem C implies that μ is a Gibbs cu-state. By the argument used above, μ must be an SRB measure also.

Observe from Theorem B that $\mu_n \in \mathcal{G}(f)$ for each n, so there is a sequence $\mathcal{C}_n$ of hyperbolic blocks associated to μ_n converging to $\mathcal{C}$, a hyperbolic block associated to μ. Moreover, the size of the disks crossing the cylinder is uniformly bounded from below.

Let D_∞ crossing $\mathcal{C}$, $D_\infty(\varepsilon)$ and A be the sets defined above for μ. Let D_n and A_n be the corresponding subsets defined for μ_n in the block $\mathcal{C}_n$. For $n \geq 1$ large enough, the disk D_n crosses $D_\infty(\varepsilon)$. The argument above implies that D_n intersects the lamination $\{W^s_{loc}(x) : x \in A\}$ in a subset with positive Lebesgue measure on D_n. Each D_n is contained in some local unstable manifold, so if there exists some point in the basin of μ then every point in these manifolds is in the basin too. But there exists a positive Lebesgue measure subset of D_n contained in the basin of μ_n, so $B(\mu) = B(\mu_n)$ for all $n > 1$ large enough, and then $\mu = \mu_n$.

Let $\mu_1, \dots \mu_n$ be the finitely many SRB measures for f supported in Λ. Now we prove that $m(B(\Lambda) \setminus \cup_{i=1}^n B(\mu_i)) = 0$. Suppose that $m(U \setminus \cup_{i=1}^n B(\mu_i)) > 0$. Then there exists a C^2-disk D tangent to the center-unstable cone field such that conditions 1 and 2 of Section 3.1 hold and $m_D(D \cap \cup_{i=1}^n B(\mu_i)) = 0$. Let $\mu = \mu_i$ be a Gibbs cu-state constructed from the iterates of Lebesgue measure on D as in Section 3.1.

From this construction, given $k \geq 1$ large enough, the Lebesgue measure of $f^n(D) \cap B(\mu)$ on $f^n(D)$ is bounded from below away from zero, thus the Lebesgue measure of $D \cap B(\mu)$ on D is also bounded from below and away from zero. This is a contradiction.

In order to prove statistical stability, consider $(f_n)_n$ a sequence of C^2-diffeomorphisms converging to f in the C^k-topology, $k \geq 2$. Assume that $(\mu_n)_n$ is a sequence of ergodic SRB measures for $(f_n)_{n\geq 1}$ and that μ is a weak* accumulation measure of this sequence. By Corollary D, each μ_n must be a Gibbs cu-state for f_n. Theorem C implies that μ is a Gibbs cu-state. By Theorem A every ergodic component of μ is a Gibbs cu-state. Applying to each ergodic component of μ the argument used above, every ergodic component of μ must also be an SRB measure, and so it must be in the convex hull of finitely many ergodic SRB measures.

□

Example : Bonatti and Viana [**4**] constructed an open class of diffeomorphisms $\mathcal{N}$ defined on $\mathbb{T}^n$, $n \geq 4$ such that every $f \in \mathcal{N}$ satisfies:

(a) f has a dominated splitting but is not partially hyperbolic,
(b) f is non-uniformly expanding in the center-unstable direction.

They also proved there exist SRB measures for such f. After this, Tahzibi [**19**] proved that

(c) the SRB measure is unique.

In this case, the SRB measure corresponds to a unique Gibbs cu-state and by Theorem C this SRB measure is C^k-statistically stable, $k \geq 2$.

Remark 5: However, it is not known whether there are general conditions ensuring the uniqueness of the SRB measure for partial hyperbolic diffeomorphisms or for diffeomorphisms with dominated splitting.

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