Regularity properties of the diffusion coefficient for a mean zero exclusion process.

Aluna: Raquel Mariela Sued
Orientador: Cláudio Landim

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Resumo

O comportamento hidrodinâmico para o processo de exclusão simples de média zero é governado por uma equação não linear parabólica. No presente trabalho mostramos que o coeficiente de difusão é uma função regular.

Apresentação do Problema e Resultados

Estamos interessados em estudar a evolução de um sistema composto por uma grande quantidade de componentes. Um gás, por exemplo, confinado em um recipiente. Conforme a dinâmica clássica, as partículas (da ordem de 10^{23}) se movimentam segundo as leis de Newton. Posição e a velocidade de cada partícula em cada instante determinam os estado microscópicos do sistema.

Seguindo o enfoque da mecânica estatística introduzido por Boltzmann-Gibbs, sem preocuparmo-nos com os estados microscópicos, estudaremos os estados de equilíbrio do sistema. Estes estão caracterizados por uma quantidade finita de parâmetros macroscópicos, $p = (p_1, \cdots, p_a)$, que não se alteram ao longo do tempo. Desta forma, parâmetros macroscópicos ficam definidos para sistemas em equilíbrio. A pressão, densidade e temperatura constantes, nosso gás encontra-se em equilíbrio.

Mesmo quando não podemos determinar os parâmetros macroscópicos do sistema total, podemos supor que na vizinhança de cada ponto u pertencente ao recipiente temos um estado de equilíbrio caracterizado por parâmetros macroscópicos locais que evoluem temporalmente. Temos então que os sistemas fora do equilíbrio são descritos mediante parâmetros macroscópicos $p(t, u)$ (macroestado) variando no tempo e no espaço.

No melhor dos casos, é de se esperar que $p(t, u)$ evolua suavemente sendo solução de uma equação diferencial parcial, chamada equação hidrodinâmica. No entanto provar este fato é bastante difícil e apesar dos esforços realizados, este problema não foi ainda completamente resolvido.

No presente trabalho, a fim de estudar o comportamento macroscópico do sistema, substituimos a evolução microscópica determinística por dinâmicas onde as partículas se movimentam estocásticamente, simplificando o problema. Assumimos que o número total de partículas é conservado e estudamos a evolução da densidade como parâmetro macroscópico. Dependendo do modelo proposto para a evolução microscópica, dado um macroestado inicial (condição inicial), obteremos a equação diferencial que determina a lei de evolução do macroestado. Nosso principal resultado é que para o modelo de exclusão simples de média zero o coeficiente da equação hidrodinâmica é uma função regular. Como consequência temos boa solução para a equação, representando a solução o valor do parâmetro macroscópico, dependendo do tempo e do espaço.

Assumindo regularidade do coeficiente da equação hidrodinâmica junto com as chamadas condições de setor, que são demostradas no Teorema (5.0.5) do presente trabalho, Komoriya prova em [2] o comportamento hidrodinâmico do sistema mediante o método da entropia relativa. Xu prova, no seu trabalho de doutoramento, o comportamento hidrodinâmico do processo de exclusão simples de média zero em dimensão $d = 1$.

As técnicas utilizadas para provar a regularidade do coeficiente de difusão foram desenvolvidas por Landim, Olla e Varadhan [8], utilizando a dualidade generalizada, introduzida por Landim, Yau [9] e Sethuraman, Varadhan e Yau [11].

Contents

Chapter 1

Introduction

1.1 Presentation of the problem and main result

We are interested in studying the evolution of a system compounded by a large number of components, evolving according a deterministic law. A gas, for example, in a recipient where the particles (of order 10^{23}) are evolving according Newton law. In order to know position and velocity of each particle at each time (microscopic states), we should work with systems too big.

Following the statistical mechanics approach introduced by Boltzmann, without caring about the microscopic states of the system, we concentrate our attention in the equilibrium states. They are characterized by a finite number of macroscopic parameters, $p = (p_1, \cdots, p_a)$, that do not change in time. In this way, macroscopic parameters are defined by equilibrium states.

At constant pressure, temperature and density , our gas is in equilibrium, while the atmosphere around it is not.

Nevertheless, we know exactly what we mean when we talk about the temperature in our city. Even when we are not able to characterize the macroscopic parameters of the whole system, around each point u of it we have an equilibrium state, characterized by local macroscopic parameters that evolve in time. In this way, systems out of equilibrium are characterized by macroscopic parameters $p(t, u)$ depending in time and space.

We expect $p(t, u)$ to evolve continuously solving a partial differential equation, called hydrodynamic equation. This is a hard problem and despite many efforts it was not completely resolved.

In the present work, recalling that we focus our attention in the macroscopic behavior of the system, we replace the deterministic microscopic evolution by a dynamic where particles evolve stochastically. Assuming that the total number of particles is conserved by the dynamic, we study the evolution of the density as macroscopic parameter. For each stochastic rule of evolution presented to model the microscopic evolution and given an initial macro-state, we will get a partial differential equation that determines the evolution of the macro-state. Our principal result consist in proving that for the mean zero simple exclusion process the diffusion coefficient of the hydrodynamic equation is a regular function. This implies that there exists regular solution for the hydrodynamic equation, being the value of the macroscopic parameter, as a function of space and time.

We will now formulate mathematically the problem recently presented considering that the microscopic evolution is given by mean zero simple exclusion process. To simplify the notation, we consider the one-dimensional case.

1.2 Macroscopic versus microscopic in space scaling

Consider a system evolving on a one-dimensional torus, denoted by $\mathbb{T} = [0, 1)$. For each $N \in \mathbb{N}$, consider a partition of the torus given by $0, 1/N, 2/N, \ldots N-1/N$. The exclusion process allows at most one particle per site. For $x = 0, \ldots N-1$, consider $\eta(x) = 1$ if in the interval $[x/N, x+1/N)$ there is one particle and $\eta(x) = 0$ if there is non. Then, the configuration $\eta = (\eta(0), \ldots, \eta(N-1)$ belongs to the state space $\chi_N = \{0,1\}^{\mathbb{T}_N}$, where $\mathbb{T}_N$ denotes the discrete one-dimensional torus with N sites, given by $\mathbb{T}_N = \{0, \ldots, N-1\}$. Up to this moment, we have two spatial scales: the macroscopic one, denoted by $\mathbb{T}$, and the microscopic one given by $\mathbb{T}_N$. Positions in the macroscopic scale are denoted by letters u and v and are in correspondence with the microscopic positions $[uN]$ and $[vN]$, respectively, where $[a]$ stands for the integer part of a. In order to denote positions in the microscopic scale, we use letters x, y and z, which are related to the points x/N, y/N and z/N in the macroscopic scale $\mathbb{T}$. Recall that we want to describe the microscopic behavior of the system from the microscopic dynamic, which is describe in the following section.

1.3 Simple exclusion model

The evolution of the exclusion process can be informally described as follows: each particle waits a mean one exponential time. When the clock rings, it chooses a site to jump. The probability that a particle located at x picks the site y is given by $p(y-x)$; for p a probability measure on $\mathbb{Z}$. If the chosen site is free, the particle jumps. Otherwise it remains in its place and waits for a new exponential time. All the particles do the same, independently one of each other. If the probability p is symmetric, $p(x) = p(-x)$, we have a symmetric exclusion process. In our case, we will work with a mean zero probability measure p, $\sum xp(x) = 0$, with finite range meaning that $p(x) = 0$ for $|x|$ large enough. and so we get a mean zero simple exclusion process. In this way, we get a Markov process in the state space χ_N, whose generator is given by formula (2.1.2).

1.4 Equilibrium and non-equilibrium states

For $\alpha \in [0,1]$ denote by ν_α^N the Bernoulli product measure on χ_N with density α. This probability is obtained by placing a particle with probability α at each site x, independently from the other sites. This one-parameter family of probability measures is stationary for the mean zero exclusion process. This means that if we start the process by placing particles according to the measure ν_α^N and let the process evolve, at any later time t we see the same distribution ν_α^N. In this way, we have characterized the equilibrium states of the process.

Once we know the equilibrium states, we want to study the system out of equilibrium. This means that we we start the process with a microscopic measure associated to an initial continuous profile ρ_0, defined on the torus and taking values in $[0,1]$. For this purpose, for N fix, place a particle at each site x with probability $\rho_0(x/N)$, independently from the other sites. This probability, denoted by $\nu_{\rho_0}^N$, is called a product measure probability with slowly varying parameter associated to the profile ρ_0.

Given a point u in the macroscopic scale, around the point $[uN]$ in the microscopic scale the measure $\nu_{\rho_0}^N$ looks like a product measure with constant density given by $\rho_0(u)$, for N large enough. To be more precise, we see that

$$E_{\nu_{\rho_0}^N}[\eta([uN])] = \rho_0([uN]/N) \to \rho_0(u) = E_{\nu_{\rho_0(u)}}[\eta(0)] \quad \text{for } \rho_0 \text{ smooth.}$$

In fact, defining $\tau_x\eta(y) = \eta(x+y)$ and $\tau_x f(\eta) = f(\tau_x\eta)$, for any local function ψ (depends on a finite number of coordinates) and for $u \in \mathbb{T}$, we get that

$$E_{\nu^N_{\rho_0}}[\tau_{[uN]}\psi] \to E_{\nu_{\rho_0(u)}}[\psi] \ ,$$

where $\nu_{\rho_0(u)}$ is a product measure with constant density given by $\rho_0(u)$.

1.5 Evolution of non-equilibrium states

Characterize the state of the process at a macroscopic time $t > 0$ consists in finding a function $\rho(t, u)$ such that

$$E_{\nu^N_{\rho_0}}[\tau_{[uN]}\psi(\eta_t)] \to E_{\nu_{\rho(t,u)}}[\psi] \ , \tag{1.5.1}$$

where the right hand side in the previous expression denotes the expectation for a product measure with constant density $\rho(t, u)$ and, the left hand side, denotes the expectation for the process at time t with initial distribution given by $\nu^N_{\rho_0}$. If (1.5.1) holds for every local function ψ, we say that conservation of local equilibrium holds for the system. In this case, given a continuous function H defined on the torus, we get that

$$\int_{\mathbb{T}} H(u) E_{\nu^N_{\rho_0}}[\tau_{[uN]}\psi(\eta_t)\, du \to \int_{\mathbb{T}} H(u)\tilde{\psi}(\rho(t, u))\, du \ , \tag{1.5.2}$$

where $\tilde{\psi}(\alpha) = E_{\nu_\alpha}[\psi]$. Now, we say that the system satisfies the property of weak conservation of local equilibrium if

$$N^{-1} \sum_{x\in\mathbb{T}_N} H(x/N)\tau_x\psi(\eta_t) \to \int_{\mathbb{T}} H(u)\tilde{\psi}(\rho(t, u))\, du \ , \tag{1.5.3}$$

for all continuous H and for all local function ψ. If we choose $\psi(\eta) = \eta(0)$ and (1.5.3) holds for every continuous function H, we say that the hydrodynamic behavior of the system is characterized.

It remains to present the hydrodynamic equation governing the evolution of the macroscopic parameter. In order to see macroscopic variations, we need to introduce a new temporal scale. In our case, since the mean displacement of each particle is zero, to observe changes in the density we need to speed up the process by N^2. In Chapter 2, studying the equation satisfied by the empirical measures, we deduce the hydrodynamic equation.

In this work we prove that the diffusion coefficient $D(\alpha)$ of the hydrodynamic equation is a smooth function in α. This fact guarantees the existence of regular solutions for the hydrodynamic equation and allows the derivation of weak conservation of local equilibrium through the relative entropy method [15]. Furthermore, since the system is attractive, good dependence on the initial condition for the solution of the hydrodynamic equation allows to prove conservation of local equilibrium [4].

1.6 Previous works in the area

The nongradient method was developed by Varadhan [12] and Quastel [10], for reversible systems. Xu [14] extended nongradient approach to the non reversible setting, considering the asymmetric mean zero simple exclusion process in dimension $d = 1$.

The method used to prove regularity of the diffusion coefficient was developed by Landim, Olla and Varadhan in [8], using the generalized duality techniques introduced by Landim and Yau [9] and Sethuraman,Varadhan and Yau [11]. A crucial step of this machinery consists in controlling asymmetric part of the generator by the symmetric one. This is related to the so-called sector condition, assumed by Komoriya in [2] to prove hydrodynamic behavior of the system and proved in the present work.

1.7 Structure of the work

This work is organized as follows. In Chapter 2 we introduce the notation and state the main theorem. In Chapter 3 we give a finite dimensional version of the techniques used to prove the main result. In Chapter 4 we describe duality tools and describe several spaces and operators that appear in the dual representation. In Chapter 5 we state some results related to the sector conditions and give sufficient conditions for solving resolvent equation. In Chapter 6 we study the main properties of the Hilbert space of fluctuations that allow us to get in Chapter 7 a new expression for the diffusion coefficient. With this new expression, also in Chapter 7,we prove that the diffusion coefficient is a regular function. Finally, in Chapter 8 we perform the computations corresponding to the relative entropy method.

Chapter 2

Notations and Results

2.1 The model

Fix a mean zero probability p on $\mathbb{Z}^d_* = \mathbb{Z}^d \setminus \{0\}$, that vanishes outside a finite set and is irreducible. This last property means that the set $\{x : p(x) > 0\}$ generate the whole group $\mathbb{Z}^d$. The generator of the simple exclusion process on $\mathbb{Z}^d$ associated to p acts on local functions f as

$$(Lf)(\eta) \;=\; \sum_{x,y\in\mathbb{Z}^d} p(y-x)\eta(x)\{1-\eta(y)\}[f(\eta^{x,y}) - f(\eta)] \;, \tag{2.1.1}$$

where $\eta^{x,y}$ stands for the configuration obtained from η by exchanging the occupation variables $\eta(x)$, $\eta(y)$:

$$(\eta^{x,y})(z) \;=\; \begin{cases} \eta(z) & \text{if } z \neq x,\, y \;\;, \\ \eta(x) & \text{if } z = y \;\;, \\ \eta(y) & \text{if } z = x \;\;. \end{cases}$$

For α in $[0,1]$, denote by ν_α the Bernoulli product measure on $\mathcal{X} = \{0,1\}^{\mathbb{Z}^d}$ with density α. This one-parameter family of measures is stationary for the simple exclusion dynamics and in the symmetric case, $p(x) = p(-x)$, these measures are reversible. Expectation with respect to ν_α is represented by $< \cdot >_\alpha$ and the scalar product in $L^2(\nu_\alpha)$ by $< \cdot, \cdot >_\alpha$.

Denote by s and a the symmetric and the anti-symmetric parts of the probability p:

$$s(x) = (1/2)[p(x) + p(-x)] \;, \quad a(x) = (1/2)[p(x) - p(-x)] \,.$$

Let L^s and L^a be the symmetric and the anti-symmetric part of the generator L in $L^2(\nu_\alpha)$, respectively. L^s and L^a are obtained replacing p by s, a in the definition of L. Also consider the probability $p^*(y) = p(-y)$ and let L^* be the generator obtained replacing p by p^* in the definition of L (2.1.1). Observe that L^* is the adjoint operator of L in $L^2(\nu_\alpha)$.

We will work on the torus. For a positive integer N, denote by $\mathbb{T}_N$ the torus with N points $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$ and $\mathbb{T}_N^d = (\mathbb{T}_N)^d$. The continuous d-dimensional torus is denoted by $\mathbb{T}^d$ and is identified with $[0,1)^d$. Consider the exclusion process evolving in the torus $\mathbb{T}_N^d$. This is a Markov process on the state space $\chi_N = \{0,1\}^{\mathbb{T}_N^d}$, whose generator L_N acts on a function f as

$$(L_N f)(\eta) \;=\; \sum_{x,y\in\mathbb{T}_N^d} p(y-x)\eta(x)\{1-\eta(y)\}[f(\eta^{x,y}) - f(\eta)]\,. \qquad (2.1.2)$$

2.2 The hydrodynamic equation

In order to deduce the hydrodynamic equation associated to this system, we look for the equation satisfied by the empirical measures. For a probability measure μ_N on χ_N, denote by $\mathbb{P}_{\mu_N}$ the measure in $D([0,\infty),\chi_N)$ induced by the Markov process with generator L_N speeded up by N^2, with initial distribution given by μ_N. For each smooth function $H : [0,T]\times\mathbb{T}^d \to \mathbb{R}$, let $M^{H,N}(t) = M^H(t)$ be the martingale defined by:

$$M^H(t) = < \pi_t^N, H_t > - < \pi_0^N, H_0 > - \int_0^t (\partial_s + N^2 L_N) < \pi_s^N, H_s > ds,$$

where $\pi_s^N = \pi^N(\eta_s)$ is the empirical measure associated to configuration η_s. In general, $\pi^N(\eta) = N^{-d}\sum_{x\in\mathbb{T}_N^d} \eta(x)\delta_{x/N}$ and δ_u stands for the dirac measure on u.

Observe that

$$L_N\eta(x) = \frac{1}{2}\sum_y \left\{W_{x-y,x} - W_{x,x+y}\right\}\,,$$

where the current $W_{x,x+y}$ between x and $x+y$ in this model is given by

$$W_{x,x+y} = \;\eta(x)[1-\eta(x+y)]p(y) - \eta(x+y)[1-\eta(x)]p(-y)\,. \qquad (2.2.1)$$

A spatial summation by parts and a second order approximation allow us to write

$$N^{2-d} \sum_{x \in \mathbb{T}_N^d} H(x/N) L_N \eta(x) = \sum_i N^{1-d} \sum_{x \in \mathbb{T}_N^d} \partial_{u_i} H(x/N) \tau_x W_i \quad (2.2.2)$$
$$+1/4 \sum_{i,j} N^{-d} \sum_{x \in \mathbb{T}_N^d} \partial_{u_i, u_j} H(x/N) \tau_x G_{i,j} + O(1/N) ,$$

where

$$W_i = 1/2 \sum_y y_i \, W_{0,y} , \quad G_{i,j} = \sum_y y_i \, y_j \, W_{0,y} .$$

Observe that $E_{\nu_\alpha}[G_{i,j}] = 0$ for every α. This implies that the second term in (2.2.2) is negligible when deducing the hydrodynamic equation. Then, it remains to replace W_i by an object that allows us to do a second summation by parts. Following the nongradient method presented in Chapter 7 of [1], developed by Varadhan [12] and Quastel [10], we can prove that there exists a collection of functions $d_{i,j} \colon [0,1] \to \mathbb{R}$ such that

$$\limsup_{\epsilon \to 0} \limsup_{N \to \infty} \mathbb{E}_{\mu_N} \Big[\Big| \int_0^t N^{-d} \sum_{x \in \mathbb{T}_N^d} H(s, x/N) \times \quad (2.2.3)$$
$$\times N \Big\{ \tau_x W_i + \sum_{j=1}^d \Big[d_{i,j}(\eta^{\epsilon N}(x + e_j)) - d_{i,j}(\eta^{\epsilon N}(x)) \Big] \Big\} ds \Big| \Big] = 0 .$$

This implies that the hydrodynamic behavior of this system is governed by the non-linear equation

$$\partial_t \rho \; = \; \sum_{i,j=1}^d \partial^2_{u_i, u_j} d_{i,j}(\rho) \; = \; \sum_{i,j=1}^d \partial_{u_i} \{ D_{i,j}(\rho) \partial_{u_j} \rho \} ,$$

where $D_{i,j}(\alpha) = (d/d\alpha) d_{i,j}(\alpha)$. The following goal is to give an explicit form for the diffusion coefficients $D_{i,j}$ that appears in the previous equation.

2.3 The diffusion coefficient and main result

Let introduce a semi-norm in $\mathcal{C}_0$, the space of local functions with mean zero with respect to all grand canonical measures ν_α. Denote $\chi(\alpha) = \alpha(1-\alpha)$.

For h in $\mathcal{C}_0$, consider

$$\ll h \gg_\alpha \;=\; \sup_{a\in\mathbb{R}^d} \Big\{ 2\sum_{i=1}^d a_i \ll h \gg_{\alpha,i} - \frac{1}{2}\chi(\alpha)\sum_v s(v)(\sum_{i=1}^d a_i v_i)^2 \Big\} \tag{2.3.1}$$

$$+ \sup_{g\in\mathcal{C}_0} \Big\{ 2 \ll h, g \gg_{\alpha,0} - \ll -L^s g, g \gg_{\alpha,0} \Big\} , \tag{2.3.2}$$

where

$$\ll h \gg_{\alpha,i} = \Big\langle \sum_{x\in\mathbb{Z}^d} x_i \eta(x) h \Big\rangle_\alpha , \quad \ll h, g \gg_{\alpha,0} = \Big\langle \sum_{x\in\mathbb{Z}^d} \tau_x h, g \Big\rangle_\alpha \tag{2.3.3}$$

and $\{\tau_x,\, x\in\mathbb{Z}^d\}$ is the group of translations.

It may be proved that $\ll \cdot \gg_\alpha$ verifies the parallelogram identity. Then, there exists a semi-inner product on $\mathcal{C}_0$ associated to the semi-norm. Denote by H_α Hilbert space induced by $\ll \cdot,\cdot \gg_\alpha$ on $\mathcal{C}_0$. The techniques developed to study non gradient systems (see Chapter 7 in [1]), shows that the matrix $D = \{D_{i,j}(\alpha)\}_{1\le i,j\le d}$ is such that

$$W_i + \sum_j D_{i,j}(\alpha)[\eta(e_j) - \eta(0)] \in \overline{L\mathcal{C}_0}$$

in H_α, for $0<\alpha<1$, where W_i are the functions defined in (2.2). In other words, $D_{i,j}(\alpha)$ are such that

$$\inf_{u\in\mathcal{C}_0} \ll W_i + \sum_j D_{i,j}(\alpha)[\eta(e_j) - \eta(0)] - Lu \gg_\alpha = 0 \,.$$

In the symmetric case, a simple computation shows that the hydrodynamic equation is given by

$$\partial_t \rho \;=\; \sum_{i,j=1}^d \partial_{u_i}\{\frac{1}{2}\sigma_{i,j}\, \partial_{u_j}\rho\} \,, \tag{2.3.4}$$

where $\sigma = \{\sigma_{i,j}\}$ is the positive defined matrix given by

$$\sigma_{i,j} = \sum_y s(y)\, y_i\, y_j \,. \tag{2.3.5}$$

As in [5], we can prove that

$$
\begin{aligned}
&\beta^* D^s(\alpha)\beta - \frac{1}{2}\beta^*\sigma\beta = \\
&\frac{1}{\chi(\alpha)} \sup_{g\in\mathcal{C}_0} \Big\{ \sum_j (\sigma\beta)_j \ll \eta(e_j) - \eta(0), L^* g \gg_\alpha \\
&\qquad + \frac{1}{2\chi(\alpha)} \sum_{k,i} \sigma_{k,i} \ll \eta(e_i) - \eta(0), L^* g \gg_\alpha \ll \eta(e_k) - \eta(0), L^* g \gg_\alpha \\
&\qquad - \ll L^* g, L^* g \gg_\alpha \Big\}
\end{aligned}
$$

The main result of this work is the following.

Theorem 2.3.1. *The function $D_{i,j}(\alpha)$ is continuous in $[0,1]$ and C^∞ on $(0,1)$ for $1 \le i, j \le d$.*

In order to prove this result, we need to find an appropriate expression for $D_{i,j}(\alpha)$. This is done in Chapter 7, where we study deeply the structure of the Hilbert space H_α, emphasizing the differences between this model and the symmetric non gradient model. Observe that the first term in (2.3.1) is easy to compute. The next chapters are consecrated to deal with the second term of (2.3.1). Before that, we will enunciate a result concerning the relative entropy method.

2.4 About the relative entropy method

Given a profile ρ: $\mathbb{T}^d \to [0,1]$, we denote by ν_ρ^N the product measure with slowly varying parameter associated to ρ on χ_N:

$$
\nu_{\rho(\cdot)}^N\{\eta, \eta(x) = 1\} = \rho(x/N)\ , \quad \text{for } x \in \mathbb{T}_N^d\ .
$$

Conservation of local equilibrium states that if we start the process with an initial distribution close to the product measure with slowly varying parameter associated to the initial profile ρ_0, the distribution of the process at a microscopic time t should be close to a product measure with slowly varying parameter associated to $\rho(t,\cdot)$, solution of the Cauchy problem

$$
\begin{cases} \partial_t \rho = \sum_{i,j} \partial_{u_i}(D_{i,j}(\rho)\partial_{u_j}\rho)\ , \\ \rho(0,\cdot) = \rho_0(\cdot)\ . \end{cases} \tag{2.4.1}
$$

Consider an initial profile $\rho_0(\cdot)$, strictly contained in $(0,1)$. Regularity of the coefficient of the hydrodynamic equation announced in Theorem 2.3.1 guarantees (see [3]) that if the initial profile ρ_0 is of class $C^{2+e}(\mathbb{T}^d)$ for some $0 < e < 1$, then problem (2.4.1) admits a classic solution $\rho(t,u)$ twice continuously differentiable in space and once continuously differentiable on time.

Given two probability measures μ and ν on the same space, the relative entropy of μ with respect to ν, denoted by $H(\mu/\nu)$ and computed in Chapter 8, gives an idea of proximity between the probability measures involved. We can prove the following result.

Theorem 2.4.1. *Let $(\mu_N)_{n\geq 1}$ be a sequence of probability measures on χ_N whose entropy with respect to $\nu^N_{\rho_0(\cdot)}$ is of order $o(N^d)$:*

$$H(\mu_N/\nu^N_{\rho_0(\cdot)}) = o(N^d) \ .$$

Then, the relative entropy of the state of the process (with initial distribution μ_N) at the macroscopic time t with respect to $\nu^N_{\rho(t,\cdot)}$ is of order $o(N^d)$:

$$H(\mu_t^N/\nu^N_{\rho(t,\cdot)}) = o(N^d) \ , \quad \textit{for every } t \geq 0 \ ,$$

where μ_t^N is the distribution at time t of the process speeded up by N^2, with initial distribution μ_N.

Finally, attractivity of the system and good dependence on the initial profile of the solution of equation (2.4.1) (as in Theorem 4.5, Appendix 2 in [1]) allow to deduce conservation of local equilibrium, as in Chapter 9 of [1].

Assuming regularity of the diffusion coefficient of the hydrodynamic equation and sector conditions, that are proved in Theorem 5.0.5 below, Komoriya proves in [2] the hydrodynamic behavior of this system by the relative entropy method. Xu proves in [14] the hydrodynamic behavior of the mean zero exclusion process in dimension $d = 1$ by the entropy production method.

Chapter 3

Outline of proof

3.1 The finite dimensional case

In this chapter we present a finite dimensional case of the technique used to prove regularity of the coefficients $\{D_{i,j}(\alpha) : 1 \leq i, j \leq d\}$. The method was developed by Landim, Olla and Varadhan in [8], using the generalized duality techniques introduced by Landim and Yau [9] and Sethuraman, Varadhan and Yau [11].

One of the advantages of working in the finite dimensional case is that, for example, linear operators are bounded. In the general case, we deal we densely defined operators and computations becomes more demanding.

Consider the real finite dimensional Hilbert space $(\mathbb{R}^n, < \cdot, \cdot >)$, where $< \cdot, \cdot >$ denote the usual inner product in $\mathbb{R}^n$. Given a symmetric and negative matrix $S \in \mathbb{R}^{n \times n}$, we introduce the semi-norm $\| \cdot \|_1$ in $\mathbb{R}^n$ by

$$\|x\|_1^2 = < -Sx, x > ,$$

and denote by H_1 the Hilbert space induced by $\| \cdot \|$ in $\mathbb{R}^n$. In order to get the dual space of H_1, we look for objects in $\mathbb{R}^n$ that defines a bounded operator with respect to the $\| \cdot \|_1$ norm:

$$\|y\|_{-1} = \min \Big\{ C \ : \ | < y, x > | \leq C \|x\|_1 \Big\}$$

Observe that $| < x, y > | \leq \|x\|_1 \ \|y\|_{-1}$ for $\|y\|_{-1} < \infty$. We use H_{-1} to denote the Hilbert space induced by $\| \cdot \|_{-1}$ in $\mathbb{R}^n$. A simple computation allows to prove the following variational formula for the $\| \cdot \|_{-1}$ norm:

$$\|y\|_{-1}^2 = \sup_{x \in \mathbb{R}^n} \Big\{ 2 < y, x > - < -Sx, x > \Big\} .$$

From the variational formula, taking $x = \lambda y$ and maximizing over λ, we get that if $y \neq 0$ and $Sy = 0$, then $\|y\|_{-1} = \infty$. A simple computation permits to show that if $Sx = y$, then $\|y\|_{-1} = \|x\|_1$.

Consider now an asymmetric matrix $A \in \mathbb{R}^{n\times n}$, $< Ax, y >= - < x, Ay >$, and a real function c: $[0,1] \to \mathbb{R}$, continuous in the close interval and infinitely differentiable in $(0,1)$. Let $L_\alpha \colon \mathbb{R}^n \to \mathbb{R}^n$ be the operator defined by $L_\alpha = c(\alpha)A + S$. Observe that

$$< -L_\alpha x, x >=< -Sx, x >= \|x\|_1^2 \quad \text{for all } \alpha \,.$$

Fix $y \in \mathbb{R}^n$ with $\|y\|_{-1} < \infty$ and consider the resolvent equation

$$\lambda x_\lambda(\alpha) - L_\alpha x_\lambda(\alpha) = y \,. \tag{3.1.1}$$

Since $-L_\alpha$ is a positive bounded operator, we know that there exists a unique solution for the resolvent equation. Fix $z \in \mathbb{R}^n$ with $\|z\|_{-1} < \infty$ and define

$$G_\lambda(\alpha) =< x_\lambda(\alpha), z > \quad .$$

The machinery that we are considering works to prove regularity of functions appearing as $\lim_{\lambda\to 0} G_\lambda(\alpha)$, in the following sense.

Theorem 3.1.1. *Suppose that there exists a finite constant C such that*

$$\|Ax\|_{-1} \leq C \, \|x\|_1 \,. \tag{3.1.2}$$

Then, when λ goes to zero, $G_\lambda(\alpha)$ converges to a regular function $G(\alpha)$.

To prove convergence, see [6]. To prove regularity of the limit, we will show that $G_\lambda(\alpha)$ is regular in α for all λ and get uniform bounds:

$$\sup_\lambda \sup_{\alpha\in[0,1]} G_\lambda(\alpha) < \infty \,.$$

Then, we need to differentiate and bound. Prove that there exists $G_\lambda^{(k)}(\alpha)$, the k-derivate of G_λ, for $\alpha \in (\delta, 1-\delta)$ and then get uniform bounds:

$$\sup_\lambda \sup_{\alpha\in(\delta,1-\delta)} |G_\lambda^{(k)}(\alpha)| < \infty \,.$$

Taking inner product with $x_\lambda(\alpha)$ both sides of the resolvent equation (3.1.1), we get that

$$\lambda\|x_\lambda(\alpha)\|^2 + \|x_\lambda(\alpha)\|_1^2 =< y, x_\lambda(\alpha) >\leq \|x_\lambda(\alpha)\|_1 \, \|y\|_{-1} \,,$$

so

$$\|x_\lambda(\alpha)\|_1 \le \|y\|_{-1} \quad \text{and} \quad \lambda\|x_\lambda(\alpha)\|^2 \le \|y\|_{-1}^2 \,. \tag{3.1.3}$$

In order to get uniform bounds, observe that

$$|G_\lambda(\alpha)| = |<x_\lambda(\alpha), z>| \le \|x_\lambda(\alpha)\|_1 \; \|z\|_{-1} \le \|y\|_{-1} \; \|z\|_{-1} \,.$$

The following step consists in proving continuity in α of the function $G_\lambda(\alpha)$. Observe that

$$|G_\lambda(\alpha) - G_\lambda(\beta)| = |<x_\lambda(\alpha) - x_\lambda(\beta), z>| \le \|x_\lambda(\alpha) - x_\lambda(\beta)\|_1 \; \|z\|_{-1} \,.$$

Then we need to see that $\|x_\lambda(\alpha) - x_\lambda(\beta)\|_1$ goes to zero, when α goes to β. Recall that $x_\lambda(\alpha)$ and $x_\lambda(\beta)$ are, respectively, solution of the resolvent equations

$$\begin{aligned} \lambda x_\lambda(\alpha) - L_\alpha x_\lambda(\alpha) &= y \\ \lambda x_\lambda(\beta) - L_\beta x_\lambda(\beta) &= y \,. \end{aligned}$$

Subtracting the previous expressions, we get that

$$\lambda\Big(x_\lambda(\alpha) - x_\lambda(\beta)\Big) - L_\alpha\Big(x_\lambda(\alpha) - x_\lambda(\beta)\Big) = \Big(L_\alpha - L_\beta\Big)x_\lambda(\beta) \,.$$

From (3.1.3), we get that

$$\|x_\lambda(\alpha) - x_\lambda(\beta)\|_1 \le |c(\alpha) - c(\beta)| \|Ax_\lambda(\beta)\|_{-1} \,.$$

At this point we use the hypothesis of the Theorem. We can control the $\|\cdot\|_{-1}$ norm of $Ax_\lambda(\beta)$ in terms of the $\|\cdot\|_1$ norm of $x_\lambda(\beta)$, to get that

$$|c(\alpha)-c(\beta)| \|Ax_\lambda(\beta)\|_{-1} \le \; C \; |c(\alpha)-c(\beta)| \; \|x_\lambda(\beta)\|_1 \le C \; |c(\alpha)-c(\beta)| \; \|y\|_{-1} \,.$$

Finally, continuity of the function $c(\alpha)$ guarantees that $\|x_\lambda(\alpha)-x_\lambda(\beta)\|_1$ goes to zero, when α goes to β.

The following step is to differentiate the function $G_\lambda(\alpha)$ in α. In order to do that, assume that $x_\lambda(\alpha)$ is differentiable and differentiate formally the resolvent equation (3.1.1) to get that $x'_\lambda(\alpha)$ should satisfy the following equation:

$$\lambda x'_\lambda(\alpha) - L_\alpha x'_\lambda(\alpha) = L'_\alpha x_\lambda(\alpha) = c'(\alpha) A x_\lambda(\alpha) \,.$$

The same kind of computation performed when proving continuity of G_λ allows to prove that

$$\lim_{h\to 0} \| \frac{x_\lambda(\alpha+h) - x_\lambda(\alpha)}{h} - x'_\lambda(\alpha)\|_1 = 0 \; ,$$

and the conclude that $G'_\lambda(\alpha) =< x'_\lambda(\alpha), z >$. Also, as when bounding in the previous case, we get uniform bounds for $G'_\lambda(\alpha)$. Iterating this argument, we conclude the proof.

Observe that the second term appearing in formula (2.3.1) is, somehow, a $\|\cdot\|_{-1}$ norm. In Chapter 4, we use Fourier representation of $L^2(\nu_\alpha)$ and get a "good" formula for the generator in terms of the Fourier coefficients. Some other Hilbert spaces are introduced in the same Chapter. In Chapter 5, we enunciate some results related with condition (3.1.2) and solve resolvent equations. In Chapter 6 we study the structure of the space H_α, defined in Chapter 2, and in Chapter 7 we get that the diffusion coefficient may be express in such a way that the machinery recently developed works.

Chapter 4

Duality

Considering the second line in formula (2.3.1), we examine in this chapter the action of the symmetric part L^s of the generator on the space of local functions endowed with a particular scalar product $\ll \cdot, \cdot \gg_{\alpha,0}$. Some notation and computations of this chapter are taken from [7]. Fix, once for all, a density α in $(0,1)$. All expectations in this chapter are taken with respect to ν_α and we omit all subscripts.

4.1 The dual space

For each $n \geq 0$, denote by $\mathcal{E}_n$ the subsets of $\mathbb{Z}^d$ with n points and let $\mathcal{E} = \cup_{n\geq 0}\mathcal{E}_n$ be the class of finite subsets of $\mathbb{Z}^d$. For each A in $\mathcal{E}$, let Ψ_A be the local function

$$\Psi_A \;=\; \prod_{x\in A} \frac{\eta(x)-\alpha}{\sqrt{\chi(\alpha)}}\ ,$$

where $\chi(\alpha) = \alpha(1-\alpha)$. By convention, $\Psi_\phi = 1$. It is easy to check that $\{\Psi_A,\, A\in\mathcal{E}\}$ is an orthonormal basis of $L^2(\nu_\alpha)$. For each $n\geq 0$, denote by $\mathcal{G}_n$ the subspace of $L^2(\nu_\alpha)$ generated by $\{\Psi_A,\, A\in\mathcal{E}_n\}$, so that $L^2(\nu_\alpha) = \oplus_{n\geq 0}\mathcal{G}_n$. Functions in $\mathcal{G}_n$ are said to have degree n. We use π_n to denote the projection operator from $L^2(\nu_\alpha)$ to the subspace $\mathcal{G}_n$. Then, given a function f in $L^2(\nu_\alpha)$, we may write

$$f = \sum_{n\geq 0} \pi_n f = \sum_{n\geq 0}\sum_{A\in\mathcal{E}_n} \mathfrak{f}(A)\Psi_A\ . \tag{4.1.1}$$

Note that the coefficients $\mathfrak{f}(A)$ depend not only on f but also on the density α: $\mathfrak{f}(A) = \mathfrak{f}(A,\alpha)$. If f is a local function, $\mathfrak{f}\colon \mathcal{E}\to\mathbb{R}$ is a function of finite support. Denote by $\mathcal{C}$ the space of local functions and recall that $\mathcal{C}_0$ is the set of local functions that have mean zero with respect to all grand canonical measure ν_β. We have a simple characterization of $\mathcal{C}_0$ functions in terms of their Fourier coefficients:

$$f \in \mathcal{C}_0 \iff \sum_{A \in \mathcal{E}_n} \mathfrak{f}(A, \beta) = 0 \quad \forall\, n \geq 0, \quad \forall\, \beta. \tag{4.1.2}$$

For local functions u, v in $\mathcal{C}_0$, define the scalar product $\ll \cdot\,, \cdot \gg$ (previously noted by $\ll \cdot\,, \cdot \gg_{\alpha,0}$) by

$$\ll u, v \gg = \sum_{x \in \mathbb{Z}^d} < \tau_x u, v > , \tag{4.1.3}$$

where $\{\tau_x,\, x \in \mathbb{Z}^d\}$ is the group of translations. Since $\ll u - \tau_x u\,,\, v \gg = 0$ for all x in $\mathbb{Z}^d$, this scalar product is only semidefinite positive. Denote by $L^2_{\ll \cdot, \cdot \gg}(\nu_\alpha)$ the Hilbert space generated by the local functions in $\mathcal{C}_0$ and the inner product $\ll \cdot, \cdot \gg$. The scalar product of two local functions u, v can be written in terms of the Fourier coefficients of u, v through a simple formula. To this end, fix two local functions u, v and write them in the basis $\{\Psi_A,\, A \in \mathcal{E}\}$:

$$u \;=\; \sum_{A \in \mathcal{E}} \mathfrak{u}(A) \Psi_A \,, \quad v \;=\; \sum_{A \in \mathcal{E}} \mathfrak{v}(A) \Psi_A \,.$$

An elementary computation shows that

$$\ll u, v \gg = \sum_{x \in \mathbb{Z}^d} \sum_{n \geq 1} \sum_{A \in \mathcal{E}_n} \mathfrak{u}(A)\, \mathfrak{v}(A + x) \,.$$

In this formula, $B + z$ is the set $\{x + z;\, x \in B\}$. The summation starts from $n = 1$ because we are working with functions in $\mathcal{C}_0$.

We say that two finite subsets A, B of $\mathbb{Z}^d$ are equivalent if one is the translation of the other. This equivalence relation is denoted by $\sim$ so that $A \sim B$ if $A = B + x$ for some x in $\mathbb{Z}^d$. Let $\tilde{\mathcal{E}}_n$ be the quotient of $\mathcal{E}_n$ with respect to this equivalence relation: $\tilde{\mathcal{E}}_n = \mathcal{E}_n/_\sim$, $\tilde{\mathcal{E}} = \mathcal{E}/_\sim$. For any summable function $\mathfrak{f}$: $\mathcal{E} \to \mathbb{R}$,

$$\sum_{A \in \mathcal{E}} \mathfrak{f}(A) \;=\; \sum_{A \in \tilde{\mathcal{E}}} \sum_{z \in \mathbb{Z}^d} \mathfrak{f}(A + z) \,.$$

In particular, for two local functions u, v,

$$\ll u,v \gg = \sum_{x,z\in\mathbb{Z}^d}\sum_{n\geq 1}\sum_{A\in\tilde{\mathcal{E}}_n} \mathfrak{u}(A+z)\,\mathfrak{v}(A+x+z) \;=\; \sum_{n\geq 1}\sum_{A\in\tilde{\mathcal{E}}_n} \tilde{\mathfrak{u}}(A)\,\tilde{\mathfrak{v}}(A)\;,$$

where, for a finite set A and a summable function $\mathfrak{u}\colon \mathcal{E}\to\mathbb{R}$,

$$\tilde{\mathfrak{u}}(A) \;=\; \sum_{z\in\mathbb{Z}^d} \mathfrak{u}(A+z)\;. \tag{4.1.4}$$

We say that a function $\mathfrak{f}\colon \mathcal{E}\to\mathbb{R}$ is translation invariant if $\mathfrak{f}(A+x)=\mathfrak{f}(A)$ for all sets A in $\mathcal{E}$ and all sites x of $\mathbb{Z}^d$. Of course, the functions $\tilde{\mathfrak{u}}$ are translation invariant. Fix a subset A of $\mathbb{Z}^d$ with n points. There are n sets in the class of equivalence of A that contain the origin. Therefore, summing a translation invariant function $\mathfrak{f}$ over all sets A in $\tilde{\mathcal{E}}_n$ is the same as summing $\mathfrak{f}$ over all sets B in $\mathcal{E}_n$ that contain the origin divided by n:

$$\sum_{A\in\tilde{\mathcal{E}}_n} \mathfrak{f}(A) \;=\; \frac{1}{n}\sum_{\substack{A\in\mathcal{E}_n\\ A\ni 0}} \mathfrak{f}(A)$$

if $\mathfrak{f}(A)=\mathfrak{f}(A+x)$ for all A, for all x. Let $\mathcal{E}_*$ be the class of all finite subsets of $\mathbb{Z}^d_* = \mathbb{Z}^d\backslash\{0\}$ and let $\mathcal{E}_{*,n}$ be the class of all subsets of $\mathbb{Z}^d_*$ with n points. Then, we may write

$$\begin{aligned}\ll u,v \gg \;&=\; \sum_{n\geq 1}\frac{1}{n}\sum_{\substack{A\in\mathcal{E}_n\\ A\ni 0}} \tilde{\mathfrak{u}}(A)\,\tilde{\mathfrak{v}}(A)\\ &=\; \sum_{n\geq 0}\frac{1}{n+1}\sum_{A\in\mathcal{E}_{*,n}} \tilde{\mathfrak{u}}(A\cup\{0\})\,\tilde{\mathfrak{v}}(A\cup\{0\})\;.\end{aligned}$$

Summarizing, for a finitely supported function $\mathfrak{f}:\mathcal{E}\to\mathbb{R}$, define $\mathfrak{T}\mathfrak{f}\colon \mathcal{E}_*\to\mathbb{R}$ by

$$(\mathfrak{T}\mathfrak{f})(A) \;=\; \tilde{\mathfrak{f}}(A\cup\{0\}) \;=\; \sum_{z\in\mathbb{Z}^d} \mathfrak{f}([A\cup\{0\}]+z)\;, \tag{4.1.5}$$

then we have that

$$\ll u,v \gg = \sum_{n\geq 0}\frac{1}{n+1}\sum_{A\in\mathcal{E}_{*,n}} \mathfrak{T}\mathfrak{u}(A)\,\mathfrak{T}\mathfrak{v}(A)\;. \tag{4.1.6}$$

To state some properties of the transformation $\mathfrak{T}$, we need some notation. For a subset A of $\mathbb{Z}_*^d$ and z in $\mathbb{Z}_*^d$, $S_z A$ is the set defined by

$$S_z A \;=\; \begin{cases} A - z & \text{if } z \notin A, \\ \Big[(A-z)\setminus\{0\}\Big] \cup \{-z\} & \text{if } z \in A\,. \end{cases} \tag{4.1.7}$$

Therefore, to obtain $S_z A$ from A when z belongs to A, we first translate A by $-z$, getting a new set which contains the origin, and we then remove the origin and add site $-z$.

Remark 4.1.1. *(a) Since f belongs to $\mathcal{C}_0$, $\mathcal{E}_1$ is irrelevant for defining $\mathfrak{T}\mathfrak{f}$, because we understand $\mathfrak{T}\mathfrak{f}$ as a function on $\mathcal{E}_*$ and $\mathfrak{T}\mathfrak{f}(\phi) = \sum_{z\in\mathbb{Z}^d} \mathfrak{f}(\{z\}) = 0$.*

(b) Not any function $\mathfrak{f}_* : \mathcal{E}_* \to \mathbb{R}$ is the image by $\mathfrak{T}$ of some function $\mathfrak{f} : \mathcal{E} \to \mathbb{R}$ since

$$(\mathfrak{T}\mathfrak{f})(A) \;=\; (\mathfrak{T}\mathfrak{f})(S_z A) \tag{4.1.8}$$

for all z in A.

(c) Let $\mathfrak{f}_* : \mathcal{E}_* \to \mathbb{R}$ be a finitely supported function with $\mathfrak{f}_*(\phi) = 0$ and satisfying (4.1.8): $\mathfrak{f}_*(A) = \mathfrak{f}_*(S_z A)$ for all z in A. Define $\mathfrak{f} : \mathcal{E} \to \mathbb{R}$ by

$$\mathfrak{f}(B) \;=\; \begin{cases} |B|^{-1}\mathfrak{f}_*(B\setminus\{0\}) & \textit{if } B \ni 0\,, \\ 0 & \textit{otherwise}\,. \end{cases} \tag{4.1.9}$$

An elementary computations shows that $\mathfrak{T}\mathfrak{f} = \mathfrak{f}_*$. This choice, which is not unique, makes $\mathfrak{f}$ vanish on $\mathcal{E}_1$.

(d) The operation that transforms $\mathfrak{f}$ in $\mathfrak{T}\mathfrak{f}$ reduces by one the degree of a function. Thus, morally, the translations in the inner product $\ll \cdot,\cdot \gg$ are reducing by one the degrees and changing the space $\mathbb{Z}^d$ in $\mathbb{Z}_*^d$.

To keep notation simple, most of the times, real functions on $\mathcal{E}$ or on $\mathcal{E}_*$ are indistinctively denoted by the symbols $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{u}$, $\mathfrak{v}$.

4.2 Some Hilbert spaces

For $n \geq 0$, let

$$L^2(\mathcal{E}_{*,n}) = \{\mathfrak{f} \colon \mathcal{E}_{*,n} \to \mathbb{R} : \sum_{A\in\mathcal{E}_{*,n}} \mathfrak{f}^2(A) < \infty\}$$

and for $\mathfrak{f}$, $\mathfrak{g}$ in $L^2(\mathcal{E}_{*,n})$, define $< \mathfrak{f}, \mathfrak{g} >= \sum_{A\in\mathcal{E}_{*,n}} \mathfrak{f}(A)\mathfrak{g}(A)$ for every n. We put $\|\mathfrak{f}\|^2 =< \mathfrak{f}, \mathfrak{f} >$ whenever $\mathfrak{f} \in L^2(\mathcal{E}_{*,n})$. Define, analogously, $L^2(\mathcal{E}_n)$. For $n \geq 0$ consider the following spaces:

$$\mathcal{F}_n = \left\{ \mathfrak{f}\colon \mathcal{E}_n \to \mathbb{R} \ : \ \mathfrak{f}(A) \neq 0 \quad \text{for a finite number of sets } A \text{ and } \sum_{A\in\mathcal{E}_n} \mathfrak{f}(A) = 0 \right\} ,$$

$$\mathcal{F}_{*,n} = \left\{ \mathfrak{f}\colon \mathcal{E}_{*,n} \to \mathbb{R} \ : \ \begin{array}{l} \mathfrak{f}(A) \neq 0 \quad \text{for a finite number of sets } A, \sum_{A\in\mathcal{E}_{*,n}} \mathfrak{f}(A) = 0 \\ \text{and} \quad \mathfrak{f}(S_z B) = \mathfrak{f}(B) \quad \text{for all } B \in \mathcal{E}_{*,n}, \text{ for all } z \in B \end{array} \right\} .$$

From $\sum_{A\in\mathcal{E}_{*,0}} \mathfrak{f}(A) = f(\phi) = 0$, we get that $\mathcal{F}_{*,0} = \{0\}$.

Observe that the operator $\mathfrak{T}$, defined by formula 4.1.5, maps $\mathcal{F}_n$ to $\mathcal{F}_{*,n-1}$. A function $\mathfrak{f} \in \mathcal{F}_n$ or $\mathfrak{f} \in \mathcal{F}_{*,n}$ is called a finite supported function of degree n. Put $\mathcal{I}_n$ and $\mathcal{I}_{*,n}$ for the closure of $\mathcal{F}_n$ and $\mathcal{F}_{*,n}$ as subspaces of $L^2(\mathcal{E}_n)$ and $L^2(\mathcal{E}_{*,n})$, respectively. For $\mathfrak{f} : \mathcal{E}_* \to \mathbb{R}$ define the projection π_n by

$$(\pi_n \mathfrak{f})(A) \ = \ \begin{cases} \mathfrak{f}(A) & \text{if } |A| = n, \\ 0 & \text{otherwise} . \end{cases} \tag{4.2.1}$$

Let

$$\mathcal{I}_* = \{ \mathfrak{f}\colon \mathcal{E}_* \to \mathbb{R} \ : \ \pi_n \mathfrak{f} \in \mathcal{I}_{*,n} \quad \forall\, n \geq 0 \} . \tag{4.2.2}$$

Given a local function in $\mathcal{C}_0$, take its Fourier coefficients in $L^2(\nu_\alpha)$ and apply the operator $\mathfrak{T}$ to the Fourier coefficients. The image of this transformation belongs to the space of finite supported functions given by

$$\mathcal{F}_* \ = \ \left\{ \mathfrak{f}\colon \mathcal{E}_* \to \mathbb{R} \ : \ \begin{array}{l} \pi_n(\mathfrak{f}) = 0 \quad \text{for al } n \geq n_0 \text{ , for some } n_0, \\ \text{and} \quad \pi_n(\mathfrak{f}) \in \mathcal{F}_{*,n} \quad \text{for all } n \geq 0 \end{array} \right\} . \tag{4.2.3}$$

Consider the inner product $\ll \cdot, \cdot \gg_{0,k}$ in the space $\mathcal{F}_*$, given by:

$$\ll \mathfrak{f}, \mathfrak{g} \gg_{0,k} = \sum_{n\geq 0} (n+1)^{2k-1} < \pi_n \mathfrak{f}, \pi_n \mathfrak{g} >$$

so that

$$\|\mathfrak{f}\|_{0,k}^2 = \sum_{n\geq 0} (n+1)^{2k-1} < \pi_n \mathfrak{f}, \pi_n \mathfrak{f} > \ . \tag{4.2.4}$$

The terms corresponding to $n = 0$ in each of the previous expressions is equal to zero. Let $\mathbb{I}_*^k$ be the Hilbert space induced by the inner product $\ll \cdot, \cdot \gg_{0,k}$ and $\mathcal{F}_*$. Observe that we have the following embeddings:

$$\mathbb{I}_*^0 \hookleftarrow \mathbb{I}_*^1 \cdots \hookleftarrow \mathbb{I}_*^s \hookleftarrow \mathbb{I}_*^{s+1} \cdots .$$

An explicit way to construct the spaces $\mathbb{I}_*^k$ is adding the Hilbert spaces $\mathcal{I}_{*,n}$ weighted by $(n+1)^{2k-1}$:

$$\mathbb{I}_*^k \;=\; \{\mathfrak{f} \in \mathcal{I}_* \;:\; \sum_n (n+1)^{2k-1} \|\pi_n \mathfrak{f}\|^2 < \infty\} .$$

With this notation, for local functions f and g, in view of (4.1.6), we have that

$$\ll f, g \gg \, = \, \ll \mathfrak{T}\mathfrak{f}, \mathfrak{T}\mathfrak{g} \gg_{0,0} ,$$

where $\mathfrak{f}$ and $\mathfrak{g}$ are the Fourier coefficients of f and g respectively.

We now examine the action of the symmetric part of the generator L on the basis $\{\Psi_A,\, A \in \mathcal{E}\}$ (see diagram 4.3.4 below as reference). Fix a function $u \in \mathcal{C}_0$ and denote by $\mathfrak{u}$ its Fourier coefficients. A straightforward computation shows that

$$L^s u \;=\; \sum_{A \in \mathcal{E}} (\mathcal{L}_s \mathfrak{u})(A) \Psi_A , \tag{4.2.5}$$

where $\mathcal{L}_s$ is the generator of finite symmetric random walks evolving with exclusion on $\mathbb{Z}^d$:

$$(\mathcal{L}_s \mathfrak{u})(A) \;=\; (1/2) \sum_{x,y \in \mathbb{Z}^d} s(y-x)[\mathfrak{u}(A_{x,y}) - \mathfrak{u}(A)] \tag{4.2.6}$$

and $A_{x,y}$ is the set defined by

$$A_{x,y} \;=\; \begin{cases} (A \backslash \{x\}) \cup \{y\} & \text{if } x \in A,\ y \notin A, \\ (A \backslash \{y\}) \cup \{x\} & \text{if } y \in A,\ x \notin A, \\ A & \text{otherwise} . \end{cases} \tag{4.2.7}$$

Furthermore, an elementary computation, based on the fact that

$$\sum_{z \in \mathbb{Z}^d} \mathfrak{f}([B \cup \{y\}] + z) \;=\; \mathfrak{T}\mathfrak{f}(S_y B)$$

for all subsets B of $\mathbb{Z}^d_*$, sites y not in B and finitely supported functions $\mathfrak{f} : \mathcal{E} \to \mathbb{R}$, shows that for every set B in $\mathcal{E}_*$

$$\mathfrak{T}\mathcal{L}_s\mathfrak{u}\,(B) \;=\; \mathfrak{L}_s\mathfrak{T}\mathfrak{u}\,(B)\;, \tag{4.2.8}$$

where

$$\begin{aligned}(\mathfrak{L}_s\mathfrak{v})\,(B) \;=\; (1/2)\sum_{x,y\in\mathbb{Z}^d_*} s(y-x)[\mathfrak{v}(B_{x,y})-\mathfrak{v}(B)] \;+ \\ +\sum_{y\notin B} s(y)[\mathfrak{v}(S_yB)-\mathfrak{v}(B)]\;.\end{aligned} \tag{4.2.9}$$

This computation should be understood as follows. We introduced an equivalence relation in $\mathcal{E}$ when we decided not to distinguish between a set and its translations. This is the same as assuming that all sets contain the origin. If n particles evolve as exclusion random walks on $\mathbb{Z}^d$, one of them fixed to be at the origin, two things may happen. Either one of the particles which is not at the origin jumps or the particle we assumed to be at the origin jumps. In the first case, this is just a jump on $\mathbb{Z}^d_*$ and is taken care by the first piece of the generator $\mathfrak{L}_s$. In the second case, however, since we are imposing the origin to be always occupied, we need to translate back the configuration to the origin. This part corresponds to the second piece of the generator $\mathfrak{L}_s$.

We are now in a position to define the Hilbert space induced by the local functions in $\mathcal{C}_0$, the symmetric part of the generator L and the scalar product $\ll \cdot,\cdot \gg$. For two local functions u, v in $\mathcal{C}_0$, let

$$\ll u,v \gg_1 = \ll u,(-L^s)v \gg$$

and let $H_1 = H_1(\mathcal{C}_0, L^s, \ll \cdot,\cdot \gg)$ be the Hilbert space generated by mean zero local functions f and the inner product $\ll \cdot,\cdot \gg_1$. By (4.2.5), (4.1.6) and (4.2.8) the previous scalar product is equal to

$$\begin{aligned}-\sum_{n\ge 0}\frac{1}{n+1}\sum_{A\in\mathcal{E}_{*,n}}\mathfrak{T}\mathfrak{u}(A)\,\mathfrak{T}\mathcal{L}_s\mathfrak{v}(A) \;&=\; -\sum_{n\ge 0}\frac{1}{n+1}\sum_{A\in\mathcal{E}_{*,n}}\mathfrak{T}\mathfrak{u}(A)\,(\mathfrak{L}_s\mathfrak{T}\mathfrak{v})(A) \\ &=\; \sum_{n\ge 0}\frac{1}{n+1} < \pi_n\mathfrak{T}\mathfrak{u}\,,\,(-\mathfrak{L}_s)\pi_n\mathfrak{T}\mathfrak{v} >\end{aligned}$$

because $\mathfrak{L}_s$ keeps the degree of the functions mapping $L^2(\mathcal{E}_{*,n})$ in itself.

Now, for each $n \geq 0$, denote by $< \cdot, \cdot >_1$ the scalar product on $\mathcal{F}_{*,n}$ defined by

$$< \mathfrak{f}, \mathfrak{g} >_1 = < \mathfrak{f}, (-\mathfrak{L}_s)\mathfrak{g} >$$

and denote by $\mathfrak{H}_1(\mathcal{F}_{*,n})$ the Hilbert space induced by the scalar product $< \cdot, \cdot >_1$ on $\mathcal{F}_{*,n}$. The associated norm is denoted by $\|\mathfrak{f}\|_1^2 =< \mathfrak{f}, (-\mathfrak{L}_s)\mathfrak{f} >$. Furthermore, for an integer $k \geq 0$, denote by $\mathfrak{H}_{1,k} = \mathfrak{H}_1(\mathcal{F}_*, \mathfrak{L}_s, k)$ the Hilbert space induced by the finitely supported functions $\mathfrak{f}$, $\mathfrak{g} \in \mathcal{F}_*$ with scalar product

$$\ll \mathfrak{f}, \mathfrak{g} \gg_{1,k} = \ll \mathfrak{f}, (-\mathfrak{L}_s)\mathfrak{g} \gg_{0,k} = \sum_{n\geq 0} (n+1)^{2k-1} < \pi_n \mathfrak{f}, (-\mathfrak{L}_s)\pi_n \mathfrak{g} > \ .$$

The associated norm is denoted by $\| \cdot \|_{1,k}$ so that

$$\|\mathfrak{f}\|_{1,k}^2 = \ll \mathfrak{f}, \mathfrak{f} \gg_{1,k} \ .$$

It follows from the previous notation that

$$\|\mathfrak{f}\|_{1,k}^2 \ = \ \sum_{n\geq 0} (n+1)^{2k-1} \|\pi_n \mathfrak{f}\|_1^2 \ . \tag{4.2.10}$$

Observe that for every local function u, $v \in \mathcal{C}_0$,

$$\ll u, v \gg_1 = \ll \mathfrak{T}\mathfrak{u}, \mathfrak{T}\mathfrak{v} \gg_{1,0},$$

where $\mathfrak{u}$ and $\mathfrak{v}$ are the Fourier coefficients of u and v, respectively.

To introduce H_{-1}, the dual space of H_1, consider the functions $u \in \mathcal{C}_0$ that define a bounded operator respect to the $\| \cdot \|_1$ norm and the inner product $\ll \cdot, \cdot \gg$: u such that there exists a constant C with

$$| \ll u, v \gg | \leq \ C \ \|v\|_1 \quad \text{for all } v \in \mathcal{C}_0 \ . \tag{4.2.11}$$

The smallest C satisfying the previous condition is denoted by $\|u\|_{-1}$ and satisfies the following variational formula:

$$\|u\|_{-1}^2 \ = \ \sup_v \Big\{ 2 \ll u, v \gg - \ll v, v \gg_1 \Big\} \ , \tag{4.2.12}$$

where the supremum is taken over all local functions v in $\mathcal{C}_0$. Denote by $H_{-1} = H_{-1}(\mathcal{C}_0,\, L^s, \ll \cdot, \cdot \gg)$ the Hilbert space generated by the local functions and the semi-norm $\| \cdot \|_{-1}$.

Since L^s keeps the degree of a function and since the spaces $\mathcal{G}_n$ are orthogonal, for local functions of degree n, we may restrict the supremum to local functions of the same degree, so that

$$\|f\|_{-1}^2 \;=\; \sum_{n\geq 1} \|\pi_n f\|_{-1}^2 \;.$$

In the same way, for an integer $n \geq 1$ and a finitely supported function $\mathfrak{u} \in \mathcal{F}_{*,n}$, let

$$\|\mathfrak{u}\|_{-1}^2 \;=\; \sup_{\mathfrak{v}} \Big\{ 2 < \mathfrak{u}, \mathfrak{v} > - < \mathfrak{v}, \mathfrak{v} >_1 \Big\} \;,$$

where the supremum is carried over all finitely supported functions $\mathfrak{v} \in \mathcal{F}_{*,n}$. Observe that, as when defining H_{-1}, we have that $\|\mathfrak{u}\|_{-1}$ is the smallest constant verifying

$$| < \mathfrak{v}, \mathfrak{u} > | \leq \; C \, \|\mathfrak{v}\|_1 \quad \text{for all } \mathfrak{v} \in \mathcal{F}_{*,n} \;. \tag{4.2.13}$$

Denote by $\mathfrak{H}_{-1} \;=\; \mathfrak{H}_{-1}(\mathcal{F}_{*,n})$ the Hilbert space induced by the finitely supported functions $\mathfrak{u} \in \mathcal{F}_{*,n}$ and the semi-norm $\| \cdot \|_{-1}$.

For a integer $k \geq 0$, define the $\| \cdot \|_{-1,k}$ norm of a finite supported function $\mathfrak{u} \in \mathcal{F}_*$ by

$$\|\mathfrak{u}\|_{-1,k}^2 \;=\; \sup_{\mathfrak{v}} \Big\{ 2 \ll \mathfrak{u}, \mathfrak{v} \gg_{0,k} - \ll \mathfrak{v}, (-\mathfrak{L}_s)\mathfrak{v} \gg_{0,k} \Big\} \;,$$

where the supremum is carried over all finitely supported functions $\mathfrak{v} \in \mathcal{F}_*$. Denote by $\mathfrak{H}_{-1,k} \;=\; \mathfrak{H}_{-1}(\mathcal{F}_*, \mathfrak{L}_s, k)$ the Hilbert space induced by this semi-norm and the space of finite supported functions. Here again, since $\mathfrak{L}_s$ does not change the degrees of a function, for every finitely supported $\mathfrak{u} \in \mathcal{F}_*$,

$$\|\mathfrak{u}\|_{-1,k}^2 \;=\; \sum_{n\geq 1} (n+1)^{2k-1} \|\pi_n \mathfrak{u}\|_{-1}^2 \tag{4.2.14}$$

and for any local function $u \in \mathcal{C}_0$,

$$\|u\|_{-1} \;=\; \|\mathfrak{T}\mathfrak{u}\|_{-1,0} \;, \tag{4.2.15}$$

where $\mathfrak{u}$ denotes the Fourier coefficient of u.

We end this section summarizing the different norms recently defined. In $\mathcal{C}_0$ we have

$$\ll u, v \gg = \sum_{x\in\mathbb{Z}^d} < \tau_x u, v > \,, \; \|u\|_1^2 = \ll u, -L_s u \gg \;, \tag{4.2.16}$$

$$\|u\|_{-1}^2 \;=\; \sup_{v\in\mathcal{C}_0} \Big\{ 2 \ll u, v \gg - \ll v, v \gg_1 \Big\} \;.$$

In $\mathcal{F}_{*,n}$

$$
\begin{aligned}
< \mathfrak{u}, \mathfrak{v} > & \sum_{A \in \mathcal{E}_{*,n}} \mathfrak{u}(A)\mathfrak{v}(A), \ \|\mathfrak{u}\|_1^2 =< \mathfrak{u}, -\mathfrak{L}_s \mathfrak{u} >, \qquad (4.2.17)\\
\|\mathfrak{u}\|_{-1}^2 &= \sup_{\mathfrak{v} \in \mathcal{F}_{*,n}} \Big\{ 2 < \mathfrak{u}, \mathfrak{v} > - < \mathfrak{v}, \mathfrak{v} >_1 \Big\} .
\end{aligned}
$$

Adding the respective norms with appropriate weights, in $\mathcal{F}_*$ we get that

$$
\begin{aligned}
&\ll \mathfrak{u}, \mathfrak{v} \gg_{0,k} = \sum_n (n+1)^{2k-1} < \pi_n \mathfrak{u}, \pi_n \mathfrak{v} > , \qquad (4.2.18)\\
&\ll \mathfrak{u}, \mathfrak{v} \gg_{1,k} = \sum_n (n+1)^{2k-1} < \pi_n \mathfrak{u}, \pi_n \mathfrak{v} >_1 ,\\
&\|\mathfrak{u}\|_{-1,k}^2 = \sum_n (n+1)^{2k-1} \|\pi_n \mathfrak{u}\|_{-1}^2
\end{aligned}
$$

4.3 The Fourier coefficients of the generator L

We conclude this chapter deriving explicit expressions for the generator L on the basis $\{\Psi_A,\ A \subset \mathbb{Z}^d\}$. A long and simple computation gives the following dual representation: For every local function $u = \sum_{A \in \mathcal{E}} \mathfrak{u}(A)\Psi_A$,

$$
Lu \ = \ \sum_{A \in \mathcal{E}} (\mathcal{L}_\alpha \mathfrak{u})(A)\Psi_A \ , \qquad (4.3.1)
$$

where $\mathcal{L}_\alpha = \mathcal{L}_s + (1-2\alpha)\mathcal{L}_d + \sqrt{\chi(\alpha)}(\mathcal{L}_+ + \mathcal{L}_-)$,

$$
\begin{aligned}
(\mathcal{L}_d \mathfrak{u})(A) &= \sum_{x \in A, y \notin A} a(y-x)\{\mathfrak{u}(A_{x,y}) - \mathfrak{u}(A)\} \ , \qquad (4.3.2)\\
(\mathcal{L}_+ \mathfrak{u})(A) &= 2 \sum_{x \in A, y \in A} a(y-x)\, \mathfrak{u}(A \backslash \{y\}) \ ,\\
(\mathcal{L}_- \mathfrak{u})(A) &= 2 \sum_{x \notin A, y \notin A} a(y-x)\, \mathfrak{u}(A \cup \{y\})
\end{aligned}
$$

and $\mathcal{L}_s$ is defined by (4.2.6). Furthermore, for any function $\mathfrak{u} : \mathcal{E} \to \mathbb{R}$, $\mathfrak{T}\mathcal{L}_\alpha \mathfrak{u} = \mathfrak{L}_\alpha \mathfrak{T}\mathfrak{u}$, provided

$$
\mathfrak{L}_\alpha = \mathfrak{L}_s + (1-2\alpha)\mathfrak{L}_d + \sqrt{\chi(\alpha)}\{\mathfrak{L}_+ + \mathfrak{L}_-\}
$$

and, for $A \in \mathcal{E}_*$, $\mathfrak{v} : \mathcal{E}_* \to \mathbb{R}$ a finitely supported function,

$$\begin{aligned} (\mathfrak{L}_d \mathfrak{v})(A) &= \sum_{\substack{x\in A, y\notin A\\ x,y\neq 0}} a(y-x)\{\mathfrak{v}(A_{x,y}) - \mathfrak{v}(A)) \\ &\quad + \sum_{\substack{y\notin A\\ y\neq 0}} a(y)\{\mathfrak{v}(S_y A) - \mathfrak{v}(A)\}\ , \\ (\mathfrak{L}_+ \mathfrak{v})(A) &= 2 \sum_{x\in A, y\in A} a(y-x)\, \mathfrak{v}(A\backslash\{y\}) \\ &\quad + 2\sum_{x\in A} a(x)\{\mathfrak{v}(A\backslash\{x\}) - \mathfrak{v}(S_x[A\backslash\{x\}])\}\ , \\ (\mathfrak{L}_- \mathfrak{v})(A) &= 2 \sum_{\substack{x\notin A, y\notin A\\ x,y\neq 0}} a(y-x)\, \mathfrak{v}(A\cup\{y\})\ . \end{aligned} \tag{4.3.3}$$

The following commutative diagram illustrate the relation between the operators recently defined. The first arrow down assigns to each function $u \in \mathcal{C}_0$ its Fourier coefficients.

$$\begin{array}{ccc} \mathcal{C}_0 & \xrightarrow{L} & \mathcal{C}_0 \\ \downarrow & & \downarrow \\ \mathcal{F} & \xrightarrow{\mathcal{L}_\alpha} & \mathcal{F} \\ \mathfrak{T}\downarrow & & \mathfrak{T}\downarrow \\ \mathcal{F}_* & \xrightarrow{\mathfrak{L}_\alpha} & \mathcal{F}_* \end{array} \tag{4.3.4}$$

Chapter 5

Sector condition, the resolvent equation and some estimates

We prove in this chapter three important results. They will be useful to understand the structure of the space H_α, defined in Chapter 2. The first one, whose proof is postponed to Appendix A, is related to the the so-called sector condition for the generator L. The second result, Theorem 5.0.6, based on an elementary computation, states that all functions in $\mathcal{C}_0$ have finite $\|\cdot\|_{-1}$ norm. Finally, the last result states that all local function in $\mathcal{C}_0$ may be approximated in H_{-1} by local functions in the range of the generator L.

We start with a result related to the sector condition.

Theorem 5.0.1. *There exists a finite constant C_0, depending only on the probability p, such that*

$$< \mathfrak{L}_d \mathfrak{f}, \mathfrak{g} >^2 \le C_0 < \mathfrak{f}, -\mathfrak{L}_s \mathfrak{f} >< \mathfrak{g}, -\mathfrak{L}_s \mathfrak{g} >$$

for each $\mathfrak{f}, \mathfrak{g}$ in $\mathcal{I}_{*,n}$. The same result remains in force if $\mathfrak{L}_d$ is replaced by $\mathfrak{L}_+$ or $\mathfrak{L}_-$ with $\mathfrak{g}$ in $\mathcal{I}_{*,n+1}$ and $\mathcal{I}_{*,n-1}$, respectively.

The proof of this result, as well as the proofs of some other estimates concerning the operators $\mathfrak{L}_+$, $\mathfrak{L}_-$, $\mathfrak{L}_d$ and $\mathfrak{L}_s$ are presented in Appendix A. We state some corollaries that are repeatedly used in this work. The first one is a simple consequence of the definition of $\|\cdot\|_{-1}$ norm for $\mathfrak{f} \in \mathcal{I}_{*,n}$ given in 4.2.11.

Corollary 5.0.2. *For* $\mathfrak{f} \in \mathcal{I}_{*,n}$ *we have that*

$$\|\mathfrak{L}_* \mathfrak{f}\|_{-1} \leq C_0 \, \|\mathfrak{f}\|_1 \, ,$$

where $\mathfrak{L}_*$ *stands for the operators* $\mathfrak{L}_s$*,* $\mathfrak{L}_d$*,* $\mathfrak{L}_+$ *and* $\mathfrak{L}_-$*.*

Corollary 5.0.3. *There exists a finite constant* C_0*, depending only on the probability* p*, such that for every* $\mathfrak{f} \in \mathcal{I}_{*,n}$

$$\|\mathfrak{L}_* \mathfrak{f}\| \; \leq \; C_0 \, n \, \|\mathfrak{f}\| \, , \tag{5.0.1}$$

where $\|\cdot\|$ *is the usual norm on* $L^2(\mathcal{E}_{*,n})$ *and* $\mathfrak{L}_*$ *stands for the operators* $\mathfrak{L}_s$*,* $\mathfrak{L}_d$*,* $\mathfrak{L}_+$*,* $\mathfrak{L}_-$*. Then, for every* $\mathfrak{f} \in \mathcal{I}_*$*, we have*

$$\begin{aligned} \|\mathfrak{L}_* \mathfrak{f}\|_{0,k} \;&\leq\; C_0 \, \|\mathfrak{f}\|_{0,k+1} \, , \\ \|\mathfrak{L}_* \mathfrak{f}\|_{-1,k} \;&\leq\; C_0 \, \|\mathfrak{f}\|_{1,k} \, . \end{aligned}$$

Proof: A simple computation shows that $\mathfrak{L}_s$ is a bounded operator on each $\mathcal{I}_{*,n}$ and that there exist a constant C, depending only on p, such that

$$\|\mathfrak{L}_s \mathfrak{f}\| \leq \; C \, n \, \|\mathfrak{f}\| \, ,$$

for all $\mathfrak{f} \in \mathcal{I}_{*,n}$. The first statement follows from this observation and Theorem 5.0.1. For the second one, just recall definition of $\|\cdot\|_{0,k}$, $\|\cdot\|_{1,k}$ and $\|\cdot\|_{-1,k}$ done in (4.2.4), (4.2.10) and (4.2.14), respectively. Also observe that $\|\mathfrak{f}\|_{1,k} \leq C_0 \|\mathfrak{f}\|_{0,k+1}$. □

Remark 5.0.4. *The previous corollary is saying that* $\mathfrak{L}_\alpha$ *is a bounded operator from* $\mathbb{I}_*^{k+1}$ *to* $\mathbb{I}_*^k$ *(*$\|\mathfrak{L}_* \mathfrak{f}\|_{0,k} \leq C_0 \|\mathfrak{f}\|_{0,k+1}$*). From* $\|\mathfrak{L}_* \mathfrak{f}\|_{-1,k} \leq C_0 \|\mathfrak{f}\|_{1,k}$ *we get that there exists a bounded extension for the operator* $\mathfrak{L}_\alpha$ *from* $\mathfrak{H}_{1,k}$ *to* $\mathfrak{H}_{-1,k}$*.*

Finally, we get to the last result concerning this kind of estimates. Using the dual representation for the operator L, presented in formula (4.3.1), and performing the same kind of computations presented in the proof of Theorem 5.0.1, we can prove the sector condition.

Theorem 5.0.5. *There exists a constant* C*, depending only on the probability* p*, such that*

$$< Lf, g >_\alpha^2 \leq \; C \; < L^s f, f >_\alpha < L^s g, g >_\alpha \, ,$$

for every local function f *and* g*, for all* $\alpha \in [0, 1]$*.*

We turn now to the space H_{-1}. We first prove that all function in $\mathcal{C}_0$ belong to H_{-1} and then show that they may be approximated by functions in the range of the operator L.

The next result follows after some computations based on the fact the for $\mathfrak{v} \in \mathcal{F}_{*,n}$, $\sum \mathfrak{v}(A) = 0$.

Theorem 5.0.6. *If $\mathfrak{u} \in \mathcal{F}_{*,n}$, we have that*

$$\|\mathfrak{u}\|_{-1} \ < \ \infty$$

and, from identity (4.2.15), we conclude that for $u \in \mathcal{C}_0$

$$\|u\|_{-1} \ < \ \infty \ .$$

The following result states that all local function in $\mathcal{C}_0$ may be approximated in H_{-1} by local functions in the range of the generator.

Theorem 5.0.7. *Given $\mathfrak{G} \in \mathcal{F}_*$ there exists a sequence $\mathfrak{h}_\lambda$ in $\mathfrak{F}_*$ such that*

$$\lim_{\lambda \to 0} \|\mathfrak{G} - \mathfrak{L}_\alpha \mathfrak{h}_\lambda\|_{-1,0} = 0 \ .$$

In view of identity (4.2.15), the previous result is telling that given a local function g in $\mathcal{C}_0$, there exists a sequence h_λ in $\mathcal{C}_0$ such that

$$\lim_{\lambda \to 0} \|g - Lh_\lambda\|_{-1} = 0 \ .$$

The proof of Theorem 5.0.7, presented at the end of this chapter, is based on the analysis of a resolvent equation associated to the operator $\mathfrak{L}_\alpha$.

Proposition 5.0.8. *Given $\mathfrak{g} \in \mathcal{I}_*$ with $\|\mathfrak{g}\|_{-1,k+1} < \infty$ for some $k \geq 0$, for each $\lambda > 0$ there exists a unique function $\mathfrak{f}_\lambda \in \mathcal{I}_*$ with $\|\mathfrak{f}_\lambda\|_{0,k+1} < \infty$, that solves the resolvent equation*

$$\lambda \mathfrak{f}_\lambda - \mathfrak{L}_\alpha \mathfrak{f}_\lambda = \mathfrak{g} \tag{5.0.2}$$

in $\mathbb{I}_*^k$. Furthermore: for each $k \geq 0$, there exist constants C_k, depending on k and the probability p, such that

$$\lambda {\|\mathfrak{f}_\lambda\|^2}_{0,k} \leq C_k \,\|\mathfrak{g}\|_{-1,k}^2 \quad \textit{and} \quad \|\mathfrak{f}_\lambda\|_{1,k}^2 \leq C_k \,\|\mathfrak{g}\|_{-1,k}^2 \ , \tag{5.0.3}$$

with $C_0 = 1$.

The proof of Proposition 5.0.8 requires some lemmas and some estimates on the operators $\mathfrak{L}_+$, $\mathfrak{L}_-$, $\mathfrak{L}_d$ and $\mathfrak{L}_s$ presented in Appendix A. Let $\mathcal{J}_n$ be the subset of functions in $\mathcal{I}_*$ of degree less or equal than n:

$$\mathcal{J}_n = \{\mathfrak{f} \in \mathcal{I}_* : \mathfrak{f}(A) = 0 \text{ if } |A| > n\} \ . \tag{5.0.4}$$

Define Π_n as the projection on $\mathcal{J}_n$.

$$\Pi_n(\mathfrak{f}) = \sum_{i \le n} \pi_i(\mathfrak{f}) \ . \tag{5.0.5}$$

In order to prove Proposition 5.0.8, we start solving the resolvent equation restricted to $\mathcal{J}_n$. Fix α and consider $\mathfrak{L}_n = \Pi_n \mathfrak{L}_\alpha \Pi_n$ as an operator from $\mathcal{J}_n$ into itself. For $\mathfrak{g}_n \in \mathcal{J}_n$, consider the resolvent equation given by

$$\lambda \mathfrak{f}_{\lambda,n} - \mathfrak{L}_n \mathfrak{f}_{\lambda,n} = \mathfrak{g}_n \ . \tag{5.0.6}$$

Lemma 5.0.9. *There exists a unique solution $\mathfrak{f}_{\lambda,n} \in \mathcal{J}_n$ for the equation (5.0.6). Furthermore, if $||\mathfrak{g}_n||_{-1,0}$ is finite, then*

$$\lambda ||\mathfrak{f}_{\lambda,n}||^2_{0,0} \le \ ||\mathfrak{g}_n||^2_{-1,0} \ , \quad ||\mathfrak{f}_{\lambda,n}||^2_{1,0} \le \ ||\mathfrak{g}_n||^2_{-1,0} \ . \tag{5.0.7}$$

Proof: We first show that the operator $\mathfrak{L}_n$ is bounded and negative in $\mathcal{J}_n$ with respect to the scalar product$\ll \cdot, \cdot \gg_{0,0}$. With these results, existence and uniqueness of solutions of equation (5.0.6) is proved in the usual way.

For the first statement, by Corollary 5.0.3, there exists a finite constant $C(n)$ (also depending in α) such that for all $\mathfrak{f} \in \mathcal{J}_n$

$$||\mathfrak{L}_n \mathfrak{f}||_{0,0} \ \le \ C(n) \ ||\mathfrak{f}||_{0,0} \ .$$

This implies that $\mathfrak{L}_n$ is bounded operator in $\mathcal{J}_n$. To see that it is negative, by Corollary A.0.24, we have that for all $\mathfrak{f} \in \mathcal{J}_n$, $\ll \mathfrak{L}_n \mathfrak{f}, \mathfrak{f} \gg_{0,0} = \ll \mathfrak{L}_s \mathfrak{f}, \mathfrak{f} \gg_{0,0}$ $\le \ 0$.

To obtain the bounds, take inner product $(\ll, \gg_{0,0})$ with $\mathfrak{f}_{\lambda,n}$ on both sides of equation (5.0.6) to get that

$$\lambda ||\mathfrak{f}_{\lambda,n}||^2_{0,0} + \ll -\mathfrak{L}_n \mathfrak{f}_{\lambda,n} \ , \ \mathfrak{f}_{\lambda,n} \gg_{0,0} = \ll \mathfrak{g}, \mathfrak{f}_{\lambda,n} \gg_{0,0} \ .$$

It remains to apply Schwarz inequality to conclude. Observe that the symmetric part of the operator $\mathfrak{L}_\alpha$ is $\mathfrak{L}_s$ only when working with $\ll \cdot, \cdot \gg_{0,0}$. $\square$

In fact we can obtain stronger estimates on the solution $\mathfrak{f}_{\lambda,n}$ of the truncated resolvent equation (5.0.6). The following Lemma is taken from [11]. The estimates obtained in Theorem 5.0.1 are crucial in the proof of this result.

Lemma 5.0.10. *Let $\mathfrak{f}_{\lambda,n}$ be the solution of the equation*

$$\lambda \mathfrak{f}_{\lambda,n} - \mathfrak{L}_n \mathfrak{f}_{\lambda,n} = \mathfrak{g}_n \ . \tag{5.0.8}$$

For any $k \geq 1$, there exist a finite constant C_k, depending on k and the probability p, such that

$$\lambda ||\mathfrak{f}_{\lambda,n}||_{0,k}^2 \leq C_k \, ||\mathfrak{g}_n||_{-1,k}^2 \ , \quad ||\mathfrak{f}_{\lambda,n}||_{1,k}^2 \leq C_k \, ||\mathfrak{g}_n||_{-1,k}^2 \ . \tag{5.0.9}$$

We are now in the position to prove Proposition 5.0.8.

Proof of Proposition 5.0.8: The idea of the proof is to solve the resolvent equation projected into $\mathcal{J}_n$ and then to show that the solutions converge to a function $\mathfrak{f}_\lambda$ in the domain of the operator $\mathfrak{L}_\alpha$ that solves the original equation. Recall the definition of the projection Π_n and the operator $\mathfrak{L}_n$ given in (5.0.5) and just before (5.0.6), respectively. By Lemma 5.0.9, for each $n \geq 1$ there exists $\mathfrak{f}_{\lambda,n}$, solution of

$$\lambda \mathfrak{f}_{\lambda,n} - \mathfrak{L}_n \mathfrak{f}_{\lambda,n} = \Pi_n \mathfrak{g} \ . \tag{5.0.10}$$

Since $\|\Pi_n \mathfrak{g}\|_{-1,k} \leq \|\mathfrak{g}\|_{-1,k}$, by Lemma 5.0.10

$$\begin{aligned} ||\mathfrak{f}_{\lambda,n}||_{1,k}^2 &\leq C_k ||\Pi_n \mathfrak{g}||_{-1,k}^2 \leq C_k ||\mathfrak{g}||_{-1,k}^2 \ , \\ \lambda ||\mathfrak{f}_{\lambda,n}||_{0,k}^2 &\leq C_k ||\Pi_n \mathfrak{g}||_{-1,k}^2 \leq C_k ||\mathfrak{g}||_{-1,k}^2 \ . \end{aligned}$$

In particular, for each λ, $\mathfrak{f}_{\lambda,n}$ is a bounded sequence for $\| \, \|_{0,k+1}$ norm. Then, there exists a subsequence $\mathfrak{f}_{\lambda,n_j}$ converging weakly to some function $\mathfrak{f}_\lambda$ with $\|\mathfrak{f}_\lambda\|_{0,k+1}$ finite. We claim that the limit is a solution of the resolvent equation (5.0.2).From Remark 5.0.4, we have that $\mathfrak{L}_\alpha \colon \mathbb{I}_*^{k+1} \to \mathbb{I}_*^k$ is a bounded operator and so preserves weakly convergent sequences. This means that $\mathfrak{L}_\alpha \mathfrak{f}_{\lambda,n_j}$ converges weakly to $\mathfrak{L}_\alpha \mathfrak{f}$ in $\mathbb{I}_*^k$. We also have that $\mathfrak{f}_{\lambda,n_j}$ converges weakly to $\mathfrak{f}$ in $\mathbb{I}_*^k$ and that $\Pi_n \mathfrak{g}$ converges to $\mathfrak{g}$. All the previous convergences implies that $\mathfrak{f}$ is solution of the resolvent equation on $\mathbb{I}_*^k$:

$$\lambda \mathfrak{f}_\lambda - \mathfrak{L}_\alpha \mathfrak{f}_\lambda = \mathfrak{g} \ .$$

Take inner product $\ll \cdot, \cdot \gg_{0,0}$ with $\mathfrak{f}_\lambda$ in the previous expression and considering that $\mathfrak{f}$ belongs to $\mathbb{I}_*^1$, use Remark A.0.25 to get that

$$||\mathfrak{f}_\lambda||_{1,0}^2 \leq ||\mathfrak{g}||_{-1,0}^2 \ , \quad \lambda ||\mathfrak{f}_\lambda||_{0,0}^2 \leq |||\mathfrak{g}||_{-1,0}^2 \ .$$

Uniqueness of solution follows from the fact that $\lambda - \mathfrak{L}_\alpha$ is a strictly positive operator on $\mathbb{I}_*^k$, for $k \geq 1$. To conclude the proof of Proposition 5.0.8, it remains to get the bounds announced. Once we have solution for the resolvent equation, the prove of Lemma 5.0.10 works in the space $\mathcal{I}_*$. Then we get that, for $k \geq 1$,

$$\begin{aligned} \lambda ||\mathfrak{f}_\lambda||_{0,k}^2 &\leq C_k \, ||\mathfrak{g}||_{-1,k}^2 \, , \\ ||\mathfrak{f}_\lambda||_{1,k}^2 &\leq C_k \, ||\mathfrak{g}||_{-1,k}^2 \, , \end{aligned}$$

where C_k are the constants appearing in Lemma 5.0.10. □

We conclude this chapter with the proof of Theorem 5.0.7.

Proof of Theorem 5.0.7: Given $\mathfrak{G} \in \mathcal{F}_*$, we know by Theorem 5.0.6 that $||\mathfrak{G}||_{-1,k} < \infty$ for all k. Then, by Proposition 5.0.8, there exists $\mathfrak{f}_\lambda$ solution of the resolvent equation

$$\lambda \mathfrak{f}_\lambda - \mathfrak{L}_\alpha \mathfrak{f}_\lambda = -\mathfrak{G} \, . \tag{5.0.11}$$

We will see that

$$\lim_{\lambda \to 0} ||\mathfrak{G} - \mathfrak{L}_\alpha \mathfrak{f}_\lambda||_{-1,0} = 0 \, . \tag{5.0.12}$$

Then approximate $\mathfrak{L}_\alpha \mathfrak{f}_\lambda$ by $\mathfrak{L}_\alpha \mathfrak{h}_\lambda$ in $|| \cdot ||_{-1,0}$ with $\mathfrak{h}_\lambda$ in $\mathcal{F}_*$ close to $\mathfrak{f}_\lambda$ in the $|| \cdot ||_{1,0}$ norm.

To prove convergence (5.0.12), we start showing that $\mathfrak{L}_\alpha \mathfrak{f}_\lambda$ is bounded for the $|| \cdot ||_{-1,0}$ norm. Then we characterize weak limits. Finally we prove that $\mathfrak{L}_\alpha \mathfrak{f}_\lambda$ is Cauchy for the $|| \cdot ||_{-1,0}$ norm.

Take inner product ($\ll \cdot, \cdot \gg_{0,0}$) with $\mathfrak{f}_\lambda$ on both sides of equation (5.0.11) and apply Schwarz inequality to get that

$$\begin{gathered} ||\mathfrak{f}_\lambda||_{1,0} \leq ||\mathfrak{G}||_{-1,0} \, , \\ \lambda \ll \mathfrak{f}_\lambda, \mathfrak{f}_\lambda \gg_{0,0} \leq ||\mathfrak{G}||_{-1,0}^2 \, . \end{gathered}$$

Observe, in particular, that $\lambda \mathfrak{f}_\lambda$ converges to 0 in the $|| \cdot ||_{0,0}$ norm. As $\mathfrak{L}_\alpha$ is a linear combination of $\mathfrak{L}_*$, for $* = s, d, +, -$, use Corollary 5.0.3 and Proposition 5.0.8, to get that

$$||\mathfrak{L}_\alpha \mathfrak{f}_\lambda||_{-1,0} \; \leq \; C(\alpha) ||\mathfrak{f}_\lambda||_{1,0} \; \leq \; C(\alpha) ||\mathfrak{G}||_{-1,0} \, .$$

Therefore $\mathfrak{L}_\alpha \mathfrak{f}_\lambda$ is bounded for the $|| \cdot ||_{-1,0}$ norm. As in Lemma 2.8 of [6], we can prove that

1- If $\mathfrak{L}_\alpha \mathfrak{f}_{\lambda_j}$ converges weakly in $\|\cdot\|_{-1,0}$ norm as $\lambda_j \downarrow 0$, then the limit is $\mathfrak{G}$.

2- There exists $\mathfrak{f} \in \mathfrak{H}_{1,0}$ such that $\mathfrak{f}_\lambda$ converges strongly to $\mathfrak{f}$ in $\mathfrak{H}_{1,0}$.

Since, by Corollary 5.0.3,

$$\|\mathfrak{L}_\alpha \mathfrak{f}_\lambda - \mathfrak{L}_\alpha \mathfrak{f}_{\tilde{\lambda}}\|_{-1,0} \ \leq \ C(\alpha)\|\mathfrak{f}_\lambda - \mathfrak{f}_{\tilde{\lambda}}\|_{1,0} \ ,$$

and since $\mathfrak{f}_\lambda$ converges strongly in $\mathfrak{H}_{1,0}$, $\mathfrak{L}_\alpha \mathfrak{f}_\lambda$ is Cauchy for $\|\cdot\|_{-1,0}$. Considering that we have just characterized all weak limit points, it follows that $\mathfrak{L}_\alpha \mathfrak{f}_\lambda$ converges strongly to $\mathfrak{G}$ in $\|\cdot\|_{-1,0}$:

$$\|\mathfrak{L}_\alpha \mathfrak{f}_\lambda - \mathfrak{G}\|_{-1,0} \to 0 \ .$$

Take $\mathfrak{h}_\lambda$ in $\mathfrak{F}_*$ such that $\lim_{\lambda\to 0} \|\mathfrak{f}_\lambda - \mathfrak{h}_\lambda\|_{1,0} = 0$. From Corollary 5.0.3, we get that $\|\mathfrak{L}_\alpha \mathfrak{f}_\lambda - \mathfrak{L}_\alpha \mathfrak{h}_\lambda\|_{-1,0} \ \leq \ C(\alpha)\|\mathfrak{f}_\lambda - \mathfrak{h}_\lambda\|_{1,0}$. Since $\mathfrak{L}_\alpha \mathfrak{f}_\lambda$ converges to $\mathfrak{G}$ in $\mathfrak{H}_{-1,0}$ we can conclude that $\mathfrak{L}_\alpha \mathfrak{h}_\lambda$ also converges to $\mathfrak{G}$ in $\mathfrak{H}_{-10}$:

$$\lim_{\lambda\to 0} \|\mathfrak{L}_\alpha \mathfrak{h}_\lambda - \mathfrak{G}\|_{-1,0} = 0 \ .$$

□

Remark 5.0.11. *Recall that in Remark 5.0.4 we said that the operator $\mathfrak{L}_\alpha$ admits an extension from $\mathfrak{H}_{1,0}$ to $\mathfrak{H}_{-1,0}$. Some how, we are saying that we can solve the equation $\mathfrak{L}_\alpha \mathfrak{f} = \mathfrak{G}$ with $\mathfrak{f}$ in $\mathfrak{H}_{1,0}$.*

Chapter 6

The Space H_α

We prove in this chapter a structure theorem for the Hilbert space of variances, H_α, that allows to derive, in the next chapter, an explicit formula for the diffusion coefficient $D_{i,j}(\alpha)$. Recall that $\sigma = (\sigma_{i,j})$ for $1 \le i,j \le d$, is the matrix defined by $\sigma_{i,j} = \sum_y s(y) y_i y_j$ and that $\chi(\alpha) = \alpha(1-\alpha)$. For $\alpha \in (0,1)$ and $h \in \mathcal{C}_0$ consider

$$\begin{aligned} \ll h \gg_\alpha &= \sup_{a \in \mathbb{R}^d} \Big\{ 2 \sum a_i \ll h \gg_{\alpha,i} - \frac{1}{2} \chi(\alpha)\, a * \sigma\, a \Big\} \\ &\quad + \sup_{g \in \mathcal{C}_0} \Big\{ 2 \ll g, h \gg_{\alpha,0} - \ll -L^s g, g \gg_{\alpha,0} \Big\} , \end{aligned} \tag{6.0.1}$$

where

$$\ll h \gg_{\alpha,i} = \Big\langle \sum_{x \in \mathbb{Z}^d} x_i \eta(x) h \Big\rangle_\alpha , \qquad \ll h, g \gg_{\alpha,0} = \Big\langle \sum_{x \in \mathbb{Z}^d} \tau_x h, g \Big\rangle_\alpha$$

and $a^* \sigma\, a$ is matrix product with a^* for a line vector in $\mathbb{R}^d$.

We prove in Lemma 6.0.13 that $\ll h \gg_\alpha < \infty$ for every h in $\mathcal{C}_0$. Recall that H_α the Hilbert space induced by the semi-norm $\ll \cdot \gg_\alpha^{1/2}$ on $\mathcal{C}_0$. In this chapter we show that every element in H_α can be approximated by $\sum D_j[\eta(e_j) - \eta(0)] + Lu$ for D in $\mathbb{R}^d$ and $u \in \mathcal{C}_0$. The main result is the following.

Theorem 6.0.12.

$$H_\alpha = \overline{\{\eta(e_j) - \eta(0)\, ,\, 1 \le j \le d\} \oplus L\mathcal{C}_0} \, . \tag{6.0.2}$$

In fact, given $g \in \mathcal{C}_0$, there exist unique $\{D_j(\alpha),\ 1 \le j \le d\}$ such that

$$\inf_{u \in \mathcal{C}_0} \ll g + \sum_{j=1}^{d} D_j(\alpha)[\eta(e_j) - \eta(0)] - Lu \gg_\alpha = 0 \ .$$

Furthermore,consider $\mathfrak{m}_j \in \mathcal{F}_{*,1}$ given by

$$\mathfrak{m}_j(\{x\}) = 2\ a(x)\ x_j \ . \tag{6.0.3}$$

Then

$$D_j(\alpha) = \frac{-1}{\chi(\alpha)} \ll \pi_1 g \gg_{\alpha,j} + \lim_{\lambda \to 0} \ll \mathfrak{f}_\lambda, \mathfrak{m}_j \gg_{0,0} ,$$

where $\mathfrak{f}_\lambda$ solves the resolvent equation

$$\lambda \mathfrak{f}_\lambda - \mathfrak{L}_\alpha \mathfrak{f}_\lambda = -\mathfrak{T}\mathfrak{g}$$

in $\mathcal{I}_*$, $\mathfrak{g}$ denotes the Fourier coefficients of g and $\mathfrak{T}$ is the operator defined in (4.1.5).

We start proving that $\ll h \gg_\alpha$ is finite for $h \in \mathcal{C}_0$.

Lemma 6.0.13. *If $h \in \mathcal{C}_0$ then $\ll h \gg_\alpha < \infty$.*

Proof: We need to see that each term appearing in definition (6.0.1) is finite if $h \in \mathcal{C}_0$. Put $|\!|\!| h |\!|\!|_\alpha$ for the first one:

$$|\!|\!| h |\!|\!|_\alpha = \sup_{a \in \mathbb{R}^d} \Big\{ 2 \sum a_i \ll h \gg_{\alpha,i} - \frac{1}{2} a^*\ \sigma\ a\ \alpha\{1-\alpha\} \Big\} \ . \tag{6.0.4}$$

This term may be computed since the matrix σ has an inverse. Put $\mathbb{H}$ for the column vector in $\mathbb{R}^d$ whose coordinates are given by $\mathbb{H}_i = \ll h \gg_{\alpha,i}$. Denote by $\mathbb{H}^*$ the transposition of $\mathbb{H}$ and by σ^{-1} the inverse of the matrix σ. Then we have that

$$|\!|\!| h |\!|\!|_\alpha = 2/\chi(\alpha)\ \mathbb{H}^*\ \sigma^{-1}\ \mathbb{H} \, .$$

The second term appearing in definition (6.0.1) is $||h||_{-1}^2$, defined in (4.2.12). In Theorem 5.0.6 we claimed that $||h||_{-1} < \infty$ if $h \in \mathcal{C}_0$. This completes the proof of the present lemma. □

Observe that the semi norm $\ll \cdot \gg_\alpha$ depends only on the symmetric part of the generator. It may be proved, as in Chapter 7 of [1], that given a cylinder function h in $\mathcal{C}_0$ and a sequence of positive integers K_ℓ such that $0 \le K_\ell \le (2\ell+1)^d$ and $\lim_{\ell\to\infty} K_\ell/(2\ell)^d = \alpha$, then

$$\lim_{\ell\to\infty} (2\ell)^{-d} \Big\langle (-L^s_{\Lambda_\ell})^{-1} \sum_{|x|\le \ell_h} \tau_x h \, , \sum_{|x|\le \ell_h} \tau_x h \Big\rangle_{\ell, K_\ell} \;=\; \ll h \gg_\alpha \, , \tag{6.0.5}$$

where $L^s_{\Lambda_\ell}$ represents the symmetric part of the generator restricted to the box $\Lambda_\ell = [-\ell,\ell]^d \cap \mathbb{Z}^d$, ℓ_h is such that $\sum_{|x|\le \ell_h} \tau_x h$ is measurable with respect to $\{\eta(x);\, x \in \Lambda_\ell\}$, $\ell_h/\ell \to 1$ and we are considering the uniform measure on the space of configurations in the box Λ_ℓ with K_ℓ particles.

Recall from (2.2.1) that $W_{x,x+y}$ and $W^*_{x,x+y}$ stand for the current of the process and the dual process, respectively. For $1 \le i \le d$, let

$$\begin{aligned} \nabla_i(\eta) &= \textstyle\sum_y y_i[\eta(0) - \eta(y)] s(y) \\ W_i^*(\eta) &= \tfrac{1}{2} \textstyle\sum_y y_i W^*_{0,y} \\ W_i(\eta) &= \tfrac{1}{2} \textstyle\sum_y y_i W_{0,y} \end{aligned} \tag{6.0.6}$$

$$. \tag{6.0.7}$$

As in [5], we apresent some identities that can be formally derived from (6.0.5), and the relations $L[\sum x_i \eta(x)] = \sum \tau_x W_i$ and $L^s[\sum x_i\eta(x)] = 1/2 \sum \tau_x \nabla_i$.

$$\begin{array}{ll} \ll L^* g, \nabla_i \gg_\alpha = -2 \ll W_i, g \gg_{\alpha,0} & \ll Lg, \nabla_i \gg_\alpha = -2 \ll W_i^*, g \gg_{\alpha,0} \\ \ll \nabla_i, h \gg_\alpha = -2 \ll h \gg_{\alpha,i} & \ll \nabla_i, L^s g \gg_\alpha = 0 \\ \ll \nabla_i, \nabla_k \gg_\alpha = 2\chi(\alpha)\sigma_{i,k} & \ll L^s f, h \gg_\alpha = - \ll f, h \gg_{\alpha,0} \ . \end{array} \tag{6.0.8}$$

A crucial difference between symmetric non gradient systems and asymmetric ones appears when we want to compute $\ll Lh, Lh \gg_\alpha$. In the symmetric case, the last line in (6.0.8) gives us an explicit formula for this object. In the asymmetric case, the sector condition proved in Theorem 5.0.5 allows to control $\ll Lh, Lh \gg_\alpha$ in terms of $\ll L^s h, L^s h \gg_\alpha$. The following Proposition is a consequence of Theorem 5.0.5 and (6.0.5).

Proposition 6.0.14. *There exists a constant C depending only on the probability p such that*

$$\ll L^* h, L^* h \gg_\alpha \le \; C \ll L^s h, L^s h \gg_\alpha$$

and

$$\ll Lh, Lh \gg_\alpha \le\ C\ \ll L^s h, L^s h \gg_\alpha ,$$

for any function $h \in \mathcal{C}_0$.

We start studying the spaces involved in decomposition 6.0.2.

Lemma 6.0.15. $\{[\eta(e_i) - \eta(0)],\ 1 \le i \le d\}$ *are linearly independent in H_α.*

Proof: From $\ll \nabla_i, \nabla_k \gg_\alpha = 2\chi(\alpha)\sigma_{i,k}$ we get that $\{\nabla_i,\ 1 \le i \le d\}$ are linearly independent in H_α. On the other hand, since $h = \tau_x h$ in H_α for any $h \in \mathcal{C}_0$ and $x \in \mathbb{Z}^d$, we get that

$$\nabla_i = \sum_y y_i[\eta(0) - \eta(y)]s(y) = \sum_{y,j} y_i y_j [\eta(0) - \eta(e_j)]s(y) = \sum_j \sigma_{i,j}[\eta(0) - \eta(e_j)] \ .$$

This means that $\{\nabla_i,\ 1 \le i \le d\}$ and $\{[\eta(e_i) - \eta(0)],\ 1 \le i \le d\}$ generate the same linear space in H_α. □

The following result is taken from [2]. It states that the spaces generating H_α in (6.0.2) are in direct sum.

Lemma 6.0.16. *The linear space generated by $\{\nabla_i,\ 1 \le i \le d\}$ in H_α does not intersect the closure of $L\mathcal{C}_0$:*

$$\{\nabla_i,\ 1 \le i \le d\}\ \cap\ \overline{L\mathcal{C}_0} = \{0\} \ .$$

To complete this chapter, it remains to prove that the spaces presented in the decomposition (6.0.2) generate the space H_α. Recall that in the space of Fourier coefficients, we have defined the projections operators $\pi_n \colon L^2(\mathcal{E}) \to L^2(\mathcal{E}_n)$ in (4.2.1). We also use π_n to denote the projection operator from $L^2(\nu_\alpha)$ to the subspace $\mathcal{G}_n$, defined in Chapter 4. If $f = \sum_{A \in \mathcal{E}} \mathfrak{f}(A)\Psi_A$, we use $\pi_n f$ for $\sum_{A \in \mathcal{E}_n} \mathfrak{f}(A)\Psi_A$. With this notation, we have that $f = \sum \pi_n f$. Observe that $\pi_0 h = E_{\nu_\alpha}[h] = 0$ since we work with functions $h \in \mathcal{C}_0$. We say that a cylinder function has degree n if all its Fourier coefficients are zero, except those of degree n: $f = \sum_{A \in \mathcal{E}_n} \mathfrak{f}(A)\Psi_A$.

The following lemma shows that $|\!|\!| h |\!|\!|_\alpha$ is relates to $\pi_1 h$, the degree one part of the function, while the remainder is related to $h - \pi_1 h$.

Lemma 6.0.17. *For every $h \in \mathcal{C}_0$, we have that*

$$\ll h \gg_\alpha = \|\pi_1 h\|_\alpha + \ll \mathfrak{T}(\mathfrak{h} - \pi_1 \mathfrak{h} \gg_{-1,0} ,$$

where $\mathfrak{h}$ are the Fourier coefficients of h and $\mathfrak{T}$ is the operator defined in (4.1.5).

Proof: Functions of different degrees are orthogonal in H_α since the operator L^s preserves degree. Then,

$$\ll h \gg_\alpha = \ll \pi_1 h \gg_\alpha + \ll h - \pi_1 h \gg_\alpha \ .$$

We claim that $\ll \pi_1 h \gg_\alpha = \|\pi_1 h\|_\alpha$ and $\ll h - \pi_1 h \gg_\alpha = \ll \mathfrak{T}\mathfrak{h} \gg_{-1,0}$. For the first identity, observe that if $\mathfrak{h}$ and $\mathfrak{g}$ denote the Fourier coefficients of f and g respectively, we have that

$$\ll g, \pi_1 h \gg_{\alpha,0} = \sum_{x \in \mathbb{Z}^d} \mathfrak{h}(x, \alpha) \sum_{y \in \mathbb{Z}^d} \mathfrak{g}(y, \alpha) = 0 ,$$

by (4.1.2). Then, the second term in (6.0.1) is equal to zero if h has degree one. So that $\ll \pi_1 h \gg_\alpha = \|\pi_1 h\|_\alpha$.

For the second identity, observe that for Ψ_A with $|A| \geq 2$, we have

$$\ll \Psi_A \gg_{\alpha,i} = \Big\langle \sum_{x \in \mathbb{Z}^d} x_i \eta(x) \Psi_A \Big\rangle_\alpha = 0 \ .$$

Since $h - \pi_1 h = \sum_{n \geq 2} \sum_{A \in \mathcal{E}_n} \mathfrak{h}(A) \Psi_A$, we get that $\ll h - \pi_1 h \gg_{\alpha,i} = 0$. In particular, $\ll h - \pi_1 h \gg_\alpha = \|\mathfrak{T}(\mathfrak{h} - \pi_1 \mathfrak{h})\|_{-1,0} = \|\mathfrak{T}\mathfrak{h}\|_{-1,0}$ since $\mathfrak{T}\pi_1\mathfrak{h} = 0$, as we observed at the first point of Remark 4.1.1. □

Proof of Theorem 6.0.12 We need to prove that given $g \in \mathcal{C}_0$, there exist unique $\{D_j(\alpha),\ 1 \leq j \leq d\}$ such that

$$\inf_{u \in \mathcal{C}_0} \ll g + \sum_{j=1}^d D_j(\alpha)[\eta(e_j) - \eta(0)] - Lu \gg_\alpha = 0 \ .$$

Uniqueness for D_j follows from Lemma 6.0.15 and Lemma 6.0.16. From Lemma 6.0.17, we know that

$$\ll g + \textstyle\sum_{j=1}^d D_j(\alpha)[\eta(e_j) - \eta(0)] - Lu \gg_\alpha =$$
$$\|\pi_1 g + \textstyle\sum_{j=1}^d D_j(\alpha)[\eta(e_j) - \eta(0)] - \pi_1 Lu\|_\alpha + \|\mathfrak{T}\mathfrak{g} - \mathfrak{L}_\alpha \mathfrak{T}\mathfrak{u}\|_{-1,0} , \quad (6.0.9)$$

where, as usual, $\mathfrak{u}$ and $\mathfrak{g}$ are the Fourier coefficients of u and g respectively. At this point we realize that is convenient to work in the space $\mathcal{I}_*$. Put $\mathfrak{G} = \mathfrak{T}\mathfrak{g}$ and $\mathfrak{f} = \mathfrak{T}\mathfrak{u}$. From Theorem 5.0.7, we know that there exists $\mathfrak{h}_\lambda$ in $\mathcal{F}_*$ such that $\|\mathfrak{G} - \mathfrak{L}_\alpha \mathfrak{h}_\lambda\|_{-1,0}$ goes to zero as λ goes to zero.

Take $u_\lambda \in \mathcal{C}_0$ such that $\mathfrak{T}(\mathfrak{u}_\lambda) = \mathfrak{h}_\lambda$, as in (4.1.1). We have that

$$\begin{array}{c} \lim_{\lambda\to 0} \ll g + \sum D_j(\alpha)[\eta(e_j) - \eta(0)] - Lu_\lambda \gg_\alpha = \\ \lim_{\lambda\to 0} \interleave \pi_1 g + \sum D_j(\alpha)[\eta(e_j) - \eta(0)] - \pi_1 L u_\lambda \interleave_\alpha \end{array} ,$$

so we need to find $D_j(\alpha)$ such that

$$\lim_{\lambda\to 0} \interleave \pi_1 g + \sum D_j(\alpha)[\eta(e_j) - \eta(0)] - \pi_1 L u_\lambda \interleave_\alpha = 0 . \tag{6.0.10}$$

Looking at the explicit formula for $\interleave \cdot \interleave_\alpha$ given in in Lemma 6.0.13, we reduce the problem to find $D_j(\alpha)$ such that $\ll \pi_1 g + \sum D_j[\eta(e_j) - \eta(0)] - \pi_1 L u_\lambda \gg_{\alpha,k}$ goes to zero for $k = 1, \cdots, d$, as λ goes to 0. Since $\ll \eta(e_j) - \eta(0) \gg_{\alpha,k} = \delta_{k,j}\chi(\alpha)$, we need to prove that

$$D_j(\alpha) = \frac{1}{\chi(\alpha)}\Big[- \ll \pi_1 g \gg_{\alpha,j} + \lim_{\lambda\to 0} \ll \pi_1 L u_\lambda \gg_{\alpha,j}\Big]$$

is well defined. From the dual representation for the operator L obtained in Chapter 4, we have that

$$\begin{aligned} \pi_1(Lu) &= \sum_{x\in\mathbb{Z}^d} \mathcal{L}_s \mathfrak{u}_1(x)\Psi_x + (1-2\alpha)\sum_{x\in\mathbb{Z}^d} \mathcal{L}_d \mathfrak{u}_1(x)\Psi_x \\ &+ \sqrt{\chi(\alpha)}\sum_{x\in\mathbb{Z}^d} \mathcal{L}_- \mathfrak{u}_2(x)\Psi_x + \sqrt{\chi(\alpha)}\sum_{x\in\mathbb{Z}^d} \mathcal{L}_+ \mathfrak{u}_0(x)\Psi_x , \end{aligned}$$

where the operators involved in the previous expression where defined in (4.3.2).

By construction, $\mathfrak{u}_\lambda(x) = 0$ for every x in $\mathbb{Z}^d$ (see remark (4.1.1)) and also

$$\mathfrak{u}_\lambda(x,y) = \begin{cases} 1/2\ \mathfrak{h}_\lambda(x) & \text{if } y = 0 ,\ x \neq 0 \ , \\ 1/2\ \mathfrak{h}_\lambda(y) & \text{if } x = 0 ,\ y \neq 0 \ . \end{cases}$$

Then, an elementary computation gives that

$$\pi_1 L u_\lambda = \sqrt{\chi(\alpha)}\sum_{x\in\mathbb{Z}^d} \mathcal{L}_- \mathfrak{u}_{\lambda,2}(x)\Psi_x = \sum a(x)\mathfrak{h}_\lambda(x)\{\eta(x) - \alpha\} .$$

Since

$$\ll \eta(x) - \alpha \gg_{\alpha,j} = x_j \; \chi(\alpha)$$

we have that

$$\ll \pi_1 L u_\lambda \gg_{\alpha,j} = \chi(\alpha) \sum_{x \in \mathbb{Z}^d} a(x) \mathfrak{h}_\lambda(x) x_j \ ,$$

so that

$$\frac{1}{\chi(\alpha)} \lim_{\lambda \to 0} \ll \pi_1 L u_\lambda \gg_{\alpha,j} = \lim_{\lambda \to 0} \ll \mathfrak{h}_\lambda, \mathfrak{m}_j \gg_{0,0} \ ,$$

with $\mathfrak{m}_j$ defined in (6.0.3). Recall, from Theorem 5.0.7, that we took $\mathfrak{h}_\lambda \in \mathcal{F}_*$ such that $\|\mathfrak{h}_\lambda - \mathfrak{f}_\lambda\|_{1,0}$ goes to zero as λ goes to zero, with $\mathfrak{f}_\lambda$ solution of the resolvent equation (5.0.11). Since $\mathfrak{m}_j$ belongs to $\mathfrak{H}_{-1,0}$, we have that

$$\lim_{\lambda \to 0} \ll \mathfrak{h}_\lambda, \mathfrak{m}_j \gg_{0,0} = \lim_{\lambda \to 0} \ll \mathfrak{f}_\lambda, \mathfrak{m}_j \gg_{0,0} \ . \tag{6.0.11}$$

This last limit exists since $\mathfrak{f}_\lambda$ is converging in $\|\cdot\|_{1,0}$ (as was shown in the proof of Theorem 5.0.7) and $\mathfrak{m}_j$ belongs to $\mathfrak{H}_{-1,0}$. Finally,

$$D_j(\alpha) = \frac{-1}{\chi(\alpha)} \ll \pi_1 g \gg_{\alpha,j} + \lim_{\lambda \to 0} \ll \mathfrak{f}_\lambda, \mathfrak{m}_j \gg_{0,0} \ ,$$

with $\mathfrak{f}_\lambda$ solution of (5.0.11), is well defined and solves the problem. $\square$

We finish this chapter presenting the proofs of some facts mentioned alone it. We start proving Proposition 6.0.14.

Proof of Proposition 6.0.14:

$$\begin{aligned}
\ll Lh, Lh \gg_\alpha = \lim_{\ell \to \infty} (2\ell)^{-d} \Big\langle (-L^s_{\Lambda_\ell})^{-1} \sum_{|x| \le \ell_h} \tau_x Lh \, , \sum_{|x| \le \ell_h} \tau_x Lh \Big\rangle_{\ell, K_\ell} &= \\
\lim_{\ell \to \infty} (2\ell)^{-d} \sup_{g \in \mathcal{F}_\ell} \{ 2 < \sum \tau_x Lh, g > - < -Lg, g > \} &\le \\
\lim_{\ell \to \infty} (2\ell)^{-d} \sup_{g \in \mathcal{F}_\ell} \{ 2C \| \sum \tau_x h \|_1 \|g\|_1 - \|g\|_1^2 &\le \\
\lim_{\ell \to \infty} (2\ell)^{-d} \tilde{C} \| \sum \tau_x h \|_1^2 &= \\
\tilde{C} \ll -L_s h, h \gg_{\alpha,0}
\end{aligned}$$

Where at this point, $\|f\|_1^2 = < -Lf, f >$ for the usual inner product in $L^2(\nu_\alpha)$ and $\|g\|_{-1}$ the dual norm. $\square$

Proof of Lemma 6.0.16: If $\sum a_i \nabla_i = Lu$ with $u \in \mathcal{C}_0$, take inner product with $L_s u$ on both sides of the previoes expression and use (6.0.8) to get that

$$0 = \ll \sum a_i \nabla_i, L_s u \gg_\alpha = \ll Lu, L_s u \gg_\alpha = \ll L_s u, L_s u \gg_\alpha$$

Since $\ll Lu, Lu \gg_\alpha \leq C \ll L_s u, L_s u \gg_\alpha$, we get that $Lu = 0$. $\square$

Chapter 7

Regularity of the Diffusion Coefficient

The goal of this chapter is to prove Theorem 2.3.1. In order to do that, we start deriving a convenient expression for the diffusion coefficient. This new formulation, together with an appropriate way of differentiating, allow to prove the regularities properties of the diffusion coefficient.

As we mentioned in Chapter 2, the techniques developed to prove hydrodynamic behavior of non gradient systems show that the diffusion coefficient $D_{i,j}(\alpha)$ of the hydrodynamic equation for the mean zero simple exclusion process is characterized by

$$\inf_{u\in\mathcal{C}_0} \ll W_i + \sum_j D_{i,j}(\alpha)[\eta(e_j) - \eta(0)] - Lu \gg_\alpha = 0 \ ,$$

where W_i and its Fourier coefficients, $\mathfrak{W}_i(A) = \mathfrak{W}_i(A,\alpha)$, are given by

$$W_i(\eta) = 1/2 \sum_y y_i W_{0,y} = 1/2 \sum_y y_i\, \eta(-y)\, p(y) - \sum_y \eta(0)\, \eta(y)\, a(y)\, y_i \quad (7.0.1)$$

and

$$\mathfrak{W}_i(A,\alpha) \ = \ \begin{cases} -\sqrt{\chi(\alpha)}\; y_i \Big(1/2p(-y) + \alpha\, a(y)\Big) & \text{if } A = \{y\} \ , \\ -\chi(\alpha)\; a(y)\; y_i \ , & \text{if } A = \{0,y\} \\ 0 & \text{otherwise .} \end{cases} \quad (7.0.2)$$

Furthermore, as in Chapter 7 in [1], with the help of Proposition 6.0.14, we can prove that

$$\inf_{u \in \mathcal{C}_0} \sup_{\alpha \in [0,1]} \ll W_i + \sum_j D_{i,j}(\alpha)[\eta(e_j) - \eta(0)] - Lu \gg_\alpha = 0 \, . \tag{7.0.3}$$

Also, as in Lemma 5.2 of [5], we can prove that

$$\inf_{u \in \mathcal{C}_0} \sup_{\alpha \in [0,1]} \ll W_i^* + \sum_j D_{i,j}(\alpha)[\eta(e_j) - \eta(0)] - L^* u \gg_\alpha = 0 \, , \tag{7.0.4}$$

where W_i^* is obtained replacing $p(y)$ by $p(-y)$ in formula (7.0.1).

To get an explicit formula for the diffusion coefficient, go back to Theorem 6.0.12 to get that

$$D_{i,j}(\alpha) = \frac{-1}{\chi(\alpha)} \ll \pi_1 W_i \gg_{\alpha,j} + \lim_{\lambda \to 0} \ll \tilde{\mathfrak{f}}_\lambda^i, \mathfrak{m}_j \gg_{0,0} \, ,$$

where $\mathfrak{m}_j$ is given in (6.0.3) and where $\tilde{\mathfrak{f}}_\lambda^i$ is the solution of the resolvent equation

$$\lambda \tilde{\mathfrak{f}}_\lambda^i - \mathfrak{L}_\alpha \tilde{\mathfrak{f}}_\lambda^i = -\mathfrak{T}\mathfrak{W}_i \, .$$

Elementary computations give that

$$\mathfrak{T}\mathfrak{W}_i = - \, \chi(\alpha) \, \mathfrak{m}_i \, ,$$

$$\ll \pi_1 W_i \gg_{\alpha,j} = -\frac{1}{2}\chi(\alpha) \sum y_i \, y_j \, p(y) = -\frac{1}{2}\chi(\alpha) \, \sigma_{i,j} \, ,$$

so that

$$D_{i,j}(\alpha) = \frac{1}{2} \, \sigma_{i,j} + \lim_{\lambda \to 0} \chi(\alpha) \ll \frac{\tilde{\mathfrak{f}}_\lambda^i}{\chi(\alpha)}, \mathfrak{m}_j \gg_{0,0} \, .$$

Let $\mathfrak{f}_\lambda^i = \tilde{\mathfrak{f}}_\lambda^i / \chi(\alpha)$ to obtain that $\mathfrak{f}_\lambda^i$ is the solution of the resolvent equation

$$\lambda \mathfrak{f}_\lambda^i - \mathfrak{L}_\alpha \mathfrak{f}_\lambda^i = \mathfrak{m}_i$$

and that

$$D_{i,j}(\alpha) = \frac{1}{2} \, \sigma_{i,j} + \lim_{\lambda \to 0} \chi(\alpha) \ll \mathfrak{f}_\lambda^i, \mathfrak{m}_j \gg_{0,0} \, . \tag{7.0.5}$$

Remark 7.0.18. *As we observed in Remark 5.0.11, we can find $\mathfrak{f}^i$ in $\mathfrak{H}_{1,0}$ such that $\mathfrak{L}_\alpha \mathfrak{f}^i = -\mathfrak{m}_i$. Then the limit for defining the diffusion coefficient is equal to $\ll \mathfrak{f}^i, \mathfrak{m}_j \gg$, the value of $\mathfrak{m}_j$ at $\mathfrak{f}^i$. This allows us to prove that*

$D(\alpha) \geq 1/2\,\sigma$ in the sense of matrix. It is enough to prove that $\sum a_i \ll \mathfrak{f}^i, \mathfrak{m}_j, \gg a_j \geq 0$ for every $a_1, \cdots, a_d$. Since $\mathfrak{L}_\alpha(\sum a_i \mathfrak{f}^i) = -\sum a_i \mathfrak{m}_i$, we get that

$$\sum a_i \ll \mathfrak{f}^i, \mathfrak{m}_j, \gg a_j = \ll \sum a_i \mathfrak{f}^i, \sum a_k \mathfrak{w}_k \gg = \| \sum a_i \mathfrak{f}^i \|_{1,0}^2 \geq 0 \ .$$

Proof of Theorem 2.3.1: Considering the formula presented in (7.0.5) for the diffusion coefficient of the hydrodynamic equation, the proof of its regularities properties is a simple consequence of the following lemma.

Lemma 7.0.19. *Take $\mathfrak{r}$ and $\mathfrak{S}$, finite supported functions in $\mathcal{F}_*$ with values not depending in α. Consider the resolvent equation*

$$\lambda \mathfrak{f}_\lambda(\alpha) - \mathfrak{L}_\alpha \mathfrak{f}_\lambda(\alpha) = \mathfrak{r} \ . \tag{7.0.6}$$

For each $\lambda > 0$, consider the function $\mathfrak{G}_\lambda \colon [0,1] \to \mathbb{R}$ defined by

$$\mathfrak{G}_\lambda(\alpha) = \ll \mathfrak{f}_\lambda(\alpha) \ , \ \mathfrak{S} \gg_{0,0} \ . \tag{7.0.7}$$

Then, there exists a subsequence $\lambda_k \downarrow 0$ such that $\mathfrak{G}_{\lambda_k}$ converges uniformly to a smooth function on $[0,1]$. Furthermore: the limit is continuous in the whole interval and C^∞ in its interior.

Proof:

To prove the existence of such subsequence we will show that the functions $\mathfrak{G}_\lambda$ are smooth for each $\lambda > 0$ and we will get uniform bounds, in $\lambda > 0$, for the L^∞ norm of the derivatives:

$$\sup_\lambda \sup_{\alpha \in [\varepsilon, 1-\varepsilon]} |\mathfrak{G}_\lambda^k(\alpha)| \leq A_k \ \forall \, \varepsilon, \ \forall \, k \ ,$$

where the upper index indicate the k-th derivate. For $k = 0$ we need to show that the functions $\mathfrak{G}_\lambda$ are continuous and uniformly bounded in $[0,1]$. In order to get the announced bound for $k = 0$ bound, take inner product $\ll \cdot \ , \ \cdot \gg_{0,0}$ with $\mathfrak{f}_\lambda$ on both sides of equation (7.0.6) and use that $| \ll \mathfrak{u}, \mathfrak{v} \gg_{0,0} | \leq \|\mathfrak{u}\|_{1,0} \|\mathfrak{v}\|_{-1,0}$ to get that

$$\begin{aligned} \ll \lambda \mathfrak{f}_\lambda(\alpha), \mathfrak{f}_\lambda(\alpha) \gg_{0,0} - \ll \mathfrak{L}_\alpha \mathfrak{f}_\lambda(\alpha), \mathfrak{f}_\lambda(\alpha) \gg_{0,0} \ &= \ \ll \mathfrak{r}, \mathfrak{f}_\lambda(\alpha) \gg_{0,0} \ , \\ \lambda \|\mathfrak{f}_\lambda(\alpha)\|_{0,0}^2 + \|\mathfrak{f}_\lambda(\alpha)\|_{1,0}^2 \ &\leq \ \|\mathfrak{f}_\lambda(\alpha)\|_{1,0} \|\mathfrak{r}\|_{-1,0} \ , \\ \|\mathfrak{f}_\lambda(\alpha)\|_{1,0} \ &\leq \ \|\mathfrak{r}\|_{-1,0} \ . \end{aligned}$$

Then, for every $\lambda > 0$, we have that

$$|\mathfrak{G}_\lambda(\alpha)| = | \ll \mathfrak{f}_\lambda(\alpha), \mathfrak{S} \gg_{0,0} | \leq \|\mathfrak{r}\|_{-1,0}\|\mathfrak{S}\|_{-1,0}$$

where, by hypothesis, the last term does not depend on α.

The following step is to differentiate (and also prove continuity on the whole interval $[0, 1]$). This is the content of Lemma 7.0.20, below. It says that we can differentiate $\mathfrak{f}_\lambda(\alpha)$ in $\| \cdot \|_{1,k}$. Furthermore: the derivate, $\mathfrak{f}_l'ambda(\alpha)$ satisfies the resolvent equation

$$\lambda\mathfrak{f}'_\lambda(\alpha) \ - \ \mathfrak{L}_\alpha\mathfrak{f}'_\lambda(\alpha) \ = \ \mathfrak{L}'(\alpha)\,\mathfrak{f}_\lambda(\alpha)\ ,$$

with $\mathfrak{L}'(\alpha)$ defined below (formula (7.0.12). Then $\mathfrak{G}'_\lambda(\alpha) = \ll \mathfrak{f}'_\lambda(\alpha), \mathfrak{S} \gg_{0,0}$. Once we have differentiated, we need to bound. For that, recall Proposition 5.0.8. and Corollary 5.0.3 to get that

$$\|\mathfrak{g}'_\lambda(\alpha)\|_{1,0} \ \leq \ \|\mathfrak{L}'(\alpha)\mathfrak{f}_\lambda(\alpha)\|_{-1,0} \ \leq \ C(\alpha)\|\mathfrak{f}_\lambda(\alpha)\|_{1,0}.$$

Collecting all this estimates, we get that

$$\begin{aligned} |\mathfrak{G}'_\lambda(\alpha)| = | \ll \mathfrak{f}'_\lambda(\alpha), \mathfrak{S} \gg_{0,0} | \ &\leq \ \|\mathfrak{f}'_\lambda(\alpha)\|_{1,0}\,\|\mathfrak{S}\|_{-1,0} \\ &\leq \ C(\alpha)\|\mathfrak{r}\|_{-1,0}\|\mathfrak{S}\|_{-1,0} \end{aligned}$$

for $C(\alpha)$ continuous in $(0, 1)$. Now, applying to Corollary 5.0.3, we can check that $\mathfrak{L}'(\alpha)\,\mathfrak{f}_\lambda(\alpha)$ satisfies the hypothesis of Lemma7.0.20. So, iterating the previous argument, we can differentiate and bound. □

We end this chapter with the announced result that gives sense to differentiate.

Lemma 7.0.20. *Consider $\mathfrak{g}(\alpha)$ with $\|\mathfrak{g}(\alpha)\|_{-1,k} < \infty$ for every k. For each $\lambda > 0$, let $\mathfrak{f}_\lambda(\alpha)$ be the solution of the resolvent equation*

$$\lambda\mathfrak{f}_\lambda(\alpha) - \mathfrak{L}_\alpha\mathfrak{f}_\lambda(\alpha) = \mathfrak{g}(\alpha)\ .$$

Fix $\alpha \in (0, 1)$. If

$$\lim_{h\to 0} \|\mathfrak{g}(\alpha + h) - \mathfrak{g}(\alpha)\|_{-1,k} = 0$$

for all $k \geq 0$, then we get that

$$\lim_{h\to 0} \|\mathfrak{f}_\lambda(\alpha + h) - \mathfrak{f}_\lambda(\alpha)\|_{1,k} \ \to \ 0 \qquad (7.0.8)$$

$$\lim_{h\to 0} \|\mathfrak{f}_\lambda(\alpha + h) - \mathfrak{f}_\lambda(\alpha)\|_{0,k} \ \to \ 0\ , \qquad (7.0.9)$$

for all k. Furthermore: suppose that there exists a function $\mathfrak{G}(\alpha)$, with $\|\mathfrak{G}(\alpha)\|_{-1,k} < \infty$ for every k, such that

$$\left\|\frac{\mathfrak{g}(\alpha+h)-\mathfrak{g}(\alpha)}{h} - \mathfrak{G}(\alpha)\right\|_{-1,k} \to 0 \tag{7.0.10}$$

as $h \to 0$ for every k. Then, for $\alpha \in (0,1)$ and λ fixed, there exist $\mathfrak{F}_\lambda(\alpha)$ solution of the resolvent equation

$$\lambda\mathfrak{F}_\lambda(\alpha) - \mathfrak{L}_\alpha\mathfrak{F}_\lambda(\alpha) = \mathfrak{G}(\alpha) + \mathfrak{L}'_\alpha\mathfrak{f}_\lambda(\alpha)\,. \tag{7.0.11}$$

where

$$\mathfrak{L}'_\alpha = -2\mathfrak{L}_d + \sqrt{\chi(\alpha)}'\{\mathfrak{L}_+ + \mathfrak{L}_-\}\,, \tag{7.0.12}$$

such that

$$\begin{aligned}\left\|\frac{\mathfrak{f}_\lambda(\alpha+h)-\mathfrak{f}_\lambda(\alpha)}{h} - \mathfrak{F}_\lambda(\alpha)\right\|_{1,k} &\to 0\\ \left\|\frac{\mathfrak{f}_\lambda(\alpha+h)-\mathfrak{f}_\lambda(\alpha)}{h} - \mathfrak{F}_\lambda(\alpha)\right\|_{0,k} &\to 0\end{aligned}$$

as $h \to 0$, for every k.

Proof: We have fixed $\alpha \in (0,1)$ and $\lambda > 0$. Call $R(\alpha) = \sqrt{\chi(\alpha)}$. By Proposition 5.0.8, in order to guarantee existence of solution for the equation (7.0.11), we need to prove that $\|\mathfrak{G}(\alpha) + \mathfrak{L}'_\alpha\mathfrak{f}_\lambda(\alpha)\|_{-1,1} < \infty$. By hypothesis, $\|\mathfrak{G}(\alpha)\|_{-1,1} < \infty$. For the other term, use Corollary 5.0.3 and Lemma 5.0.10, for each of the following inequalities

$$\|\mathfrak{L}_*\mathfrak{f}_\lambda(\alpha)\|_{-1,1} \le C_0\,\|\mathfrak{f}_\lambda(\alpha)\|_{1,1} \le C_1\,\|\mathfrak{g}(\alpha)\|_{-1,1}\,,$$

where the operator $\mathfrak{L}_*$ stands for $\mathfrak{L}_s$, $\mathfrak{L}_d$, $\mathfrak{L}_+$, $\mathfrak{L}_-$. Then, we get that

$$\|\mathfrak{L}'_\alpha\mathfrak{f}_\lambda(\alpha)\|_{-1,1} \le C(\alpha)\|\mathfrak{g}(\alpha)\|_{-1,1}.$$

Let $\mathfrak{F}_\lambda(\alpha)$ be the unique solution of (7.0.11). We want to see that the incremental quotient of $\mathfrak{f}_\lambda(\alpha)$ (in α) converges to $\mathfrak{F}_\lambda(\alpha)$.

Consider the following resolvent equations:

$$\begin{aligned}\lambda\mathfrak{f}_\lambda(\alpha) - \mathfrak{L}_\alpha\mathfrak{f}_\lambda(\alpha) &= \mathfrak{g}(\alpha)\\ \lambda\mathfrak{f}_\lambda(\alpha+h) - \mathfrak{L}_{(\alpha+h)}\mathfrak{f}_\lambda(\alpha+h) &= \mathfrak{g}(\alpha+h)\end{aligned}$$

Subtracting them we get that

$$\lambda[\mathfrak{f}_\lambda(\alpha+h) \, - \mathfrak{f}_\lambda(\alpha)] \, - \mathfrak{L}_\alpha[\mathfrak{f}_\lambda(\alpha+h) \, - \mathfrak{f}_\lambda(\alpha)] = \tag{7.0.13}$$
$$\mathfrak{g}(\alpha+h) \, - \mathfrak{g}(\alpha) - 2\,h\,\mathfrak{L}_d\mathfrak{f}_\lambda(\alpha+h) \tag{7.0.14}$$
$$+(R(\alpha+h) - R(\alpha))(\mathfrak{L}_+ + \mathfrak{L}_-)\mathfrak{f}_\lambda(\alpha+h)\,.$$

At this point, using the bounds obtained in Proposition 5.0.8 and computing the $\|\cdot\|_{-1,k}$ norm of the right hand side of (7.0.13), we get the convergence in (7.0.8).

Consider the following objects

$$\begin{aligned} \mathfrak{f}^*_\lambda(\alpha,h) &= \frac{\mathfrak{f}_\lambda(\alpha+h) - \mathfrak{f}_\lambda(\alpha)}{h} - \mathfrak{F}_\lambda(\alpha)\,, \\ \mathfrak{g}^*(\alpha,h) &= \frac{\mathfrak{g}(\alpha+h) - \mathfrak{g}(\alpha)}{h} - \mathfrak{G}(\alpha)\,, \\ R^*(\alpha,h) &= \frac{R(\alpha+h) - R(\alpha)}{h} - R'(\alpha)\,. \end{aligned}$$

Subtract equation (7.0.11) from equation (7.0.13) divided by h to get that

$$\begin{aligned} \lambda\mathfrak{f}^*_\lambda(\alpha,h) - \mathfrak{L}_\alpha\mathfrak{f}^*_\lambda(\alpha,h) &= \mathfrak{g}^*(\alpha,h) - 2\mathfrak{L}_d[\mathfrak{f}_\lambda(\alpha+h) - \mathfrak{f}_\lambda(\alpha)] \\ &+ R^*(\alpha,h)(\mathfrak{L}_+ + \mathfrak{L}_-)\mathfrak{f}_\lambda(\alpha+h) \\ &+ R'(\alpha)(\mathfrak{L}_+ + \mathfrak{L}_-)[\mathfrak{f}_\lambda(\alpha+h) - \mathfrak{f}_\lambda(\alpha)]\,. \end{aligned}$$

Using the hypothesis concerning $\mathfrak{g}(\alpha)$ and $\mathfrak{G}(\alpha)$, Lemma 5.0.10 and Corollary 5.0.3, we can see that the $\|\cdot\|_{-1,k}$ norm of each term on the right hand side of the previous expression vanishes as $h \downarrow 0$. Then, applying to Lemma 5.0.10 we conclude the result. □

Chapter 8

Relative Entropy Method.

In this chapter we prove Theorem 2.4.1, announced in Section 2. Recall the the initial profile ρ_0 is required to be strictly contained in $(0,1)$. If $0 < K_1$ and $K_2 < 1$ are such that $K_1 \le \rho_0(u) \le K_2$ for all $u \in \mathbb{T}^d$, the maximum principle implies that $K_1 \le \rho(t,u) \le K_2$, for any time t, for all $u \in \mathbb{T}^d$.

Fix $\alpha \in (0,1)$. Let ν_α^N denote the product measure with density α on $\chi_N = \{0,1\}^{\mathbb{T}_N^d}$. Given the probabilities measures μ and ν on χ_N with densities f and g relative to ν_α^N, respectively, we have that the relative entropy of a measure μ with respect to the measure ν, denoted by $H(\mu/\nu)$, is given by

$$\int_{\chi_N} \log(f/g)\; f\; d\nu_\alpha^N\ . \tag{8.0.1}$$

Denote by $\phi_t^N(\eta)$ and f_t^N the density of $\nu_{\rho(t,\cdot)}^N$ and μ_t^N with respect to the product measure ν_α^N, respectively.

$$\phi_t^N = \frac{d\nu_{\rho(t,\cdot)}^N}{d\nu_\alpha^N}\ , \qquad f_t^N = \frac{d\mu_t^N}{d\nu_\alpha^N}\ . \tag{8.0.2}$$

The non-gradient method used to deduce the hydrodynamic equation requires to consider a small perturbation of $\nu_{\rho(t,\cdot)}^N$. For local functions $F_1, \cdots, F_d \in \mathcal{C}_0$ and real regular functions $G_1(t,u), \cdots, G_d(t,u)$ to be determined later, define the density with respect to the reference measure ν_α^N by:

$$\psi_{t,F}^N(\eta) = \frac{1}{Z_{t,F}} \exp\Big\{\frac{1}{N}\sum_{x\in\mathbb{T}_N^d}\sum_{i=1}^{d} G_i(t,x/N)\,\tau_x F_i\Big\}\phi_t^N(\eta)\ , \tag{8.0.3}$$

where $Z_{t,F}$ is a normalizing constant. Let $\nu^N_{t,F}$ be the measure corresponding to the density $\psi^N_{t,F}$: $d\nu^N_{t,F} = \psi^N_{t,F}\, d\nu^N_\alpha$.

Theorem 2.4.1 is a simple consequence of the following two results.

Lemma 8.0.21. *For $F \in (\mathcal{C}_0)^d$,*

$$\lim_{N\to\infty} N^{-d}\Big| H(\mu^N_t/\nu^N_{\rho(t,\cdot)}) - H(\mu^N_t/\nu^N_{t,F})\Big| = 0 \ ,$$

uniformly for t in any compact set.

Proof: Considering the formula for the relative entropy given at (8.0.1), we get that

$$N^{-d}\Big| H(\mu^N_t/\nu^N_{\rho(t,\cdot)}) - H(\mu^N_t/\nu^N_{t,F})\Big| = N^{-d}\Big|\int \log(\frac{\psi^N_{t,F}}{\phi^N_t}) f^N_t d\nu^N_\alpha\Big| \ .$$

From (8.0.3), we get that

$$\log\Big(\frac{\psi^N_{t,F}(\eta)}{\phi^N_t(\eta)}\Big) = 1/N \sum_{x\in\mathbb{T}^d_N}\sum_{i=1}^d G_i(t,x/N)\,\tau_x F_i - \log(Z_{t,F}) \ ,$$

so the result holds once we prove that $\log(Z_{t,F}) = O(N^{d-1})$. To see this, write

$$Z_{t,F} = E_{\nu_{\rho(t,\cdot)}}[\exp\{N^{-1}\sum_{x\in\mathbb{T}^d_N}\sum_{i=1}^d G_i(t,x/N)\,\tau_x F_i\}] \ . \tag{8.0.4}$$

□

Proposition 8.0.22. *Recall that $d\nu^N_{t,F} = \psi^N_{t,F}\, d\nu^N_\alpha$. Then,*

$$\inf_{F\in(\mathcal{C}_0)^d}\lim_{N\to\infty} N^{-d} H(\mu^N_t/\nu^N_{t,F}) = 0 \ .$$

The rest of this chapter is consecrate to prove Proposition 8.0.22. Denote by $H^F_N(t)$ the relative entropy of μ^N_t with respect to $\nu^N_{t,F}$: $H^F_N(t) = H(\mu^N_t/\nu^N_{t,F})$. Our goal is to estimate the relative entropy $H^F_N(t)$ by a term of order $o(N^d,F)$ and by the time integral of the entropy multiplied by a constant: $H^F_N(t) \le o(N^d,F) + \gamma^{-1}\int_0^t H^F_N(s)ds$, with $\inf_{F\in(\mathcal{C}_0)^d}\lim_{N\to\infty} N^{-d} o(N^d,F) = 0$. In this case, Proposition 8.0.22 follows from Gromwall lemma.

By Lemma 8.0.21, it is enough to prove that for some $\gamma > 0$

$$\frac{H_N^F(t)}{N^d} - \frac{H_N^F(0)}{N^d} \le \; N^{-d} o(N^d, F) + \gamma^{-1} \int_0^t \frac{H(\mu_s^N / \nu^N_{\rho(s,\cdot)})}{N^d} ds \; . \qquad (8.0.5)$$

Consider $H_t(u) = H(t,u) = \log \rho(t,u)/(1-\rho(t,u))$. At this point we need the initial condition $\rho_o(\cdot)$ strictly conteined in $(0,1)$ and use the maximum principle to guarantee that the function $H(t,u)$ is well defined. A straightforward computation shows that if $G_i(t,u) = \partial_i H_t(u)$, then

$$\begin{aligned}
\tfrac{H_N^F(t)}{N^d} - \tfrac{H_N^F(0)}{N^d} \;\; & \le \mathbb{E}_{\mu_N} \Big[\int_0^t N^{-d} \sum_{i=1}^d \sum_{x \in \mathbb{T}_N^d} \partial_i H(s, x/N) \; N \; \{ \tau_x W_i^* + L_N^* \tau_x F_i + \\
& \qquad \sum_j D_{i,j}(\eta^{\epsilon N}(x)) [\eta^{\epsilon N}(x+e_j) - \eta^{\epsilon N}(x)] \} ds \Big] \qquad (8.0.6) \\
& + \mathbb{E}_{\mu_N} \Big[\int_0^t N^{-d} \sum_{1 \le i,j \le d} \sum_{x \in \mathbb{T}_N^d} \partial_{i,j} H(s, x/N) \; \tau_x h_{1,i,j}(\eta) ds \Big] \\
& + \mathbb{E}_{\mu_N} \Big[\int_0^t N^{-d} \sum_{1 \le i,j \le d} \sum_{x \in \mathbb{T}_N^d} \partial_{i,j} H(s, x/N) \; d_{i,j}(\eta^{\epsilon N}(x)) ds \Big] \\
& + \mathbb{E}_{\mu_N} \Big[\int_0^t N^{-d} \sum_{1 \le i,j \le d} \sum_{x \in \mathbb{T}_N^d} \partial_i H(s, x/N) \partial_j H(s, x/N) \; \tau_x h_{F,i,j}(\eta) ds \Big] \\
& - \mathbb{E}_{\mu_N} \Big[\int_0^t N^{-d} \sum_{x \in \mathbb{T}_N^d} \frac{\partial_s \rho}{\rho(1-\rho)} [\eta(x) - \rho(s, x/N)] ds \Big] \\
& + o_N(\epsilon, 1) \; ,
\end{aligned}$$

where $o_N(\epsilon, 1) \to 0$ as $N \to \infty$ for all ϵ; $h_{1,i,j}$ belongs to $\mathcal{C}_0$ for all i,j; $W_i^* = \frac{1}{2} \sum_y y_i W_{0,y}^*$ with $W_{0,y}^*$ denoting the currents for the dual process; $d'_{i,j}(\alpha) = D_{i,j}(\alpha)$; $\eta^\ell(x) = (2\ell+1)^d \sum_{\|y-x\| \le \ell} \eta(y)$ and

$$\tilde{h}_{F,i,j}(\beta) = E_{\nu_\beta}[h_{F,i,j}] = \ll -L^s F_i, F_j \gg_{\beta,0} + 1/2 \; \sigma_{i,j}^2 \beta(1-\beta) \; .$$

The non gradient techniques developed when deducing the hydrodynamic equation allow to prove that for $1 \le i \le d$,

$$\inf_{F_i\in\mathcal{C}_0} \lim_{\epsilon\to 0} \lim_{N\to\infty} \mathbb{E}_{\mu_N}\Big[\int_0^t N^{-d} \sum_{x\in\mathbb{T}_N^d} \partial_i H(s,x/N)\; N\; \{\tau_x W_i^* + L_N^*\tau_x F_i + \sum_{j=1}^d D_{i,j}(\eta^{\epsilon N}(x))[\eta^{\epsilon N}(x+e_j) - \eta^{\epsilon N}(x)]\} ds\Big] = 0\ .$$

Since $= E_{\nu_\beta}[h_{1,i,j}] = 0$ for all β in $[0,1]$, the second term on the right hand side of (8.0.6) is negligible. For the remainder, use one and two blocks estimates to get that

$$\begin{aligned} &\frac{H_N^F(t)}{N^d} - \frac{H_N^F(0)}{N^d} \le \\ &\mathbb{E}_{\mu_N}\Big[\int_0^t N^{-d} \sum_{1\le i,j\le d}\sum_{x\in\mathbb{T}_N^d} \partial_{i,j} H(s,x/N)\; d_{i,j}(\eta^\ell(x)) ds\Big] \\ &+\mathbb{E}_{\mu_N}\Big[\int_0^t N^{-d} \sum_{1\le i,j\le d}\sum_{x\in\mathbb{T}_N^d} \partial_i H(s,x/N)\partial_j H(s,x/N)\; \tilde{h}_{F,i,j}(\eta^\ell(x)) ds\Big] \\ &-\mathbb{E}_{\mu_N}\Big[\int_0^t N^{-d} \sum_{x\in\mathbb{T}_N^d} \frac{\partial_s \rho}{\rho(1-\rho)}[\eta(x) - \rho(s,x/N)] ds\Big] \\ &+C_1(\epsilon, N, \ell, F)\ , \end{aligned} \tag{8.0.7}$$

with

$$\inf_{F\in\mathcal{C}_0} \lim_{\ell\to\infty}\lim_{\epsilon\to 0}\lim_{N\to\infty} C_1(\epsilon,N,\ell,F) = 0\ .$$

The following step consists in replacing $\tilde{h}_{F,i,j}$ by $\tilde{h}_{i,j}$, given by

$$\tilde{h}_{i,j}(\beta) = \frac{1}{2}[D_{i,j}(\beta) + D_{j,i}(\beta)]\beta(1-\beta)\ , \tag{8.0.8}$$

where $D_{i,j}$ are the coefficients appearing in the hydrodynamic equation. The two identities presented in (7.0.3) and (7.0.4), respectively, together with the table of computations developed in (6.0.8), allow us to perform the desired replacement.

An integration by parts together with the fact that $\partial_i H(s,u)\rho(s,u)(1-\rho(s,u)) = \partial_i \rho(s,u)$, allow to write

$$\int_{\mathbb{T}_N^d} \sum_{1\le i,j\le d} \partial_{i,j} H(s,u)\; d_{i,j}(\rho(s,u))du =$$
$$-\int_{\mathbb{T}_N^d} \sum_{1\le i,j\le d} \partial_i H(s,u)\partial_j H(s,u)\; \tilde h_{i,j}(\rho(s,u))du\ .$$

Computing $\partial_{i,j}H(s,u)$ and recalling that $\rho(s,u)$ is solution of the Cauchy problem (2.4.1), we get that

$$\sum_{i,j}\Big\{\partial_{i,j}H(s,u)d'_{i,j}(\rho(s,u)) + \partial_i H(s,u)]\partial_j H(s,u)\tilde h'_{i,j}(\rho(s,u))\Big\} = \partial_s\rho(s,u)/[\rho(1-\rho)]\ .$$

Considering all the previous observations, we get that

$$\begin{aligned}
&\frac{H_N^F(t)}{N^d} - \frac{H_N^F(0)}{N^d} \le \\
&\tilde C_1\Big\{\mathbb{E}_{\mu_N}\Big[\int_0^t N^{-d}\sum_{1\le i,j\le d}\sum_{x\in\mathbb{T}_N^d}\partial_{i,j}H(s,x/N)]\ \{d_{i,j}(\eta^\ell(x)) \\
&\qquad -d_{i,j}(\rho(s,x/N) - d'_{i,j}(\rho(s,x/N)[\eta^\ell(x)-\rho(s,x/N)]\}ds\Big] \\
&+\mathbb{E}_{\mu_N}\Big[\int_0^t N^{-d}\sum_{1\le i,j\le d}\sum_{x\in\mathbb{T}_N^d}\partial_i H(s,u)]\partial_j H(s,u)\ \{\tilde h_{i,j}(\eta^\ell(x)) - \\
&\qquad \tilde h_{i,j}(\rho(s,x/N) - \tilde h'_{i,j}(\rho(s,x/N)[\eta^\ell(x)-\rho(s,x/N)]\}ds\Big]\Big\}\ .
\end{aligned}$$

If $d_{i,j}$ and $\tilde h_{i,j}$ were regular functions in the whole interval $[0,1]$, we could replace the last two lines in the previous expression by $C(D,\tilde h)\mathbb{E}_{\mu_N}\Big[\int_0^t N^{-d}\sum[\eta^\ell(x) - \rho(s,x/N)]^2\}ds\Big]$, and conclude the result in the usual way (see Chapter 6 [1]). The same bound holds by coupling arguments even knowing regularity of the functions only in the open interval $(0,1)$, and this concludes the proof.

Appendix A

Estimates on the operators $\mathfrak{L}_d$, $\mathfrak{L}_+$ and $\mathfrak{L}_-$

In this Appendix we prove some results involving the operators $\mathfrak{L}_s$, $\mathfrak{L}_d$, $\mathfrak{L}_+$ and $\mathfrak{L}_-$. Most of them were presented in Chapter5.

Recall that for $\mathfrak{f} \in \mathcal{F}_*$ $\mathfrak{f}(S_z A) = \mathfrak{f}(A)$ for all z in A.

A simple computation shows that the operators $\mathfrak{L}_s$ and $\mathfrak{L}_d$ send $\mathcal{I}_{*,n}$ into them self, while $\mathfrak{L}_+$ and $\mathfrak{L}_-$ map $\mathcal{I}_{*,n}$ into $\mathcal{I}_{*,n+1}$ and $\mathcal{I}_{*,n-1}$, respectively.

The following identity illustrate the fact that the space $\mathcal{I}_*$ enjoys some special properties. For every $\mathfrak{f} : \mathcal{E}_{*,1} \to \mathbb{R}$,

$$(\mathfrak{L}_- \mathfrak{f})(\phi) \;=\; -2 \sum_{x \neq 0} a(x) \mathfrak{f}(\{x\}) \;.$$

In particular, $(\mathfrak{L}_- \mathfrak{f})(\phi) = 0$ for all $\mathfrak{f}$ in $\mathcal{I}_{*,1}$ because in this space $\mathfrak{f}(\{x\}) = \mathfrak{f}(\{-x\})$ and $a(\cdot)$ is asymmetric. In contrast, $(\mathfrak{L}_+ \mathfrak{g})(\{x\}) = 0$ for all functions $\mathfrak{g} : \mathcal{E}_{*,0} \to \mathbb{R}$ so that, for all $\mathfrak{f}$ in $\mathcal{I}_{*,1}$ and all $\mathfrak{g} : \mathcal{E}_{*,0} \to \mathbb{R}$,

$$\mathfrak{L}_- \mathfrak{f} \;=\; 0 \;, \quad \mathfrak{L}_+ \mathfrak{g} \;=\; 0 \;. \tag{A.0.1}$$

Other important consequences of working on the space $\mathcal{I}_*$ are stated in the following lemma.

Lemma A.0.23. *For every $n \geq 1$ and every finitely supported functions $\mathfrak{u}$, $\mathfrak{v} : \mathcal{E}_{*,n} \to \mathbb{R}$*

$$< \mathfrak{L}_d \mathfrak{u}, \mathfrak{v} > = \; - < \mathfrak{u}, \mathfrak{L}_d \mathfrak{v} > \;\; .$$

For every finitely supported functions $\mathfrak{f}$, $\mathfrak{g}$ in $\mathcal{I}_{*,n-1}$, $\mathcal{I}_{*,n}$ respectively,

$$\frac{1}{n+1} < \mathfrak{L}_+ \mathfrak{f}, \mathfrak{g} > = -\frac{1}{n} < \mathfrak{f}, \mathfrak{L}_- \mathfrak{g} > .$$

Proof: The first identity relies on the fact that $\sum_{x,y\in A} a(y-x) = 0$. Note, however, that both pieces of the operator are needed.

The proof of the second statement is more demanding. Fix finitely supported functions $\mathfrak{f}$, $\mathfrak{g}$ in $\mathcal{I}_{*,n-1}$, $\mathcal{I}_{*,n}$, respectively. By the explicit form of $\mathfrak{L}_+$,

$$\begin{aligned} < \mathfrak{g}, \mathfrak{L}_+ \mathfrak{f} > \;&=\; 2 \sum_{A\in\mathcal{E}_{*,n}} \sum_{x,y\in A} a(y-x)\, \mathfrak{g}(A) \mathfrak{f}(A\setminus\{y\}) \\ &+\; 2 \sum_{A\in\mathcal{E}_{*,n}} \sum_{x\in A} a(x)\, \mathfrak{g}(A) \Big\{ \mathfrak{f}(A\setminus\{x\}) - \mathfrak{f}(S_x[A\setminus\{x\}]) \Big\} . \end{aligned}$$

Considering that $S_x[A\setminus\{x\}] = S_x A \setminus \{-x\}$ and since $\mathfrak{g}(S_x A) = \mathfrak{g}(A)$ for x in A because $\mathfrak{g}$ belongs to $\mathcal{I}_{*,n}$, a change of variables $B = S_x A$, $x' = -x$ in $2\sum_{A\in\mathcal{E}_{*,n}} \sum_{x\in A} a(x) \mathfrak{g}(A) \mathfrak{f}(S_x[A\setminus\{x\}])$ permits to rewrite the second term on the right hand side as

$$4 \sum_{x\neq 0} a(x) \sum_{\substack{A\in\mathcal{E}_{*,n}\\ A\ni x}} \mathfrak{g}(A) \mathfrak{f}(A\setminus\{x\})$$

because $a(-x) = -a(x)$. We claim that

$$\sum_{x\neq 0} a(x) \sum_{\substack{A\in\mathcal{E}_{*,n}\\ A\ni x}} \mathfrak{g}(A)\mathfrak{f}(A\setminus\{x\}) \;=\; \frac{1}{n-1} \sum_{x,y\neq 0} a(y-x) \sum_{\substack{A\in\mathcal{E}_{*,n}\\ A\ni x,y}} \mathfrak{g}(A) \mathfrak{f}(A\setminus\{y\}) . \tag{A.0.2}$$

We conclude the proof of the lemma assuming (A.0.2), whose proof is presented at the end. It follows from identity (A.0.2) and the previous expression for $< \mathfrak{g}, \mathfrak{L}_+ \mathfrak{f} >$ that

$$\begin{aligned} < \mathfrak{g}, \mathfrak{L}_+ \mathfrak{f} > \;&=\; 2\Big(1 + \frac{1}{n}\Big) \sum_{A\in\mathcal{E}_{*,n}} \sum_{x,y\in A} a(y-x)\, \mathfrak{g}(A) \mathfrak{f}(A\setminus\{y\}) \\ &\quad + 2\Big(1+\frac{1}{n}\Big) \sum_{A\in\mathcal{E}_{*,n}} \sum_{x\in A} a(x)\, \mathfrak{g}(A) \mathfrak{f}(A\setminus\{x\}) . \end{aligned}$$

The first term of the right hand side, which can be written as

$$2\Big(1+\frac{1}{n}\Big)\sum_{y\neq 0}\sum_{\substack{A\in\mathcal{E}_{*,n}\\ A\ni y}}\mathfrak{g}(A)\mathfrak{f}(A\setminus\{y\})\sum_{x\in A}a(y-x)\ ,$$

is equal to

$$\begin{aligned}&-2\Big(1+\frac{1}{n}\Big)\sum_{y\neq 0}a(y)\sum_{\substack{A\in\mathcal{E}_{*,n}\\ A\ni y}}\mathfrak{g}(A)\mathfrak{f}(A\setminus\{y\})\\ &\qquad-2\Big(1+\frac{1}{n}\Big)\sum_{x,y\neq 0}a(y-x)\sum_{\substack{A\in\mathcal{E}_{*,n}\\ A\ni y, A\not\ni x}}\mathfrak{g}(A)\mathfrak{f}(A\setminus\{y\})\end{aligned}$$

because $\sum_{x\in A}a(y-x)=-a(y)-\sum_{x\neq 0,x\not\in A}a(y-x)$. The first term of this formula cancels with the second one in the last expression for $<\mathfrak{g},\mathfrak{L}_+\mathfrak{f}>$. Therefore,

$$<\mathfrak{g},\mathfrak{L}_+\mathfrak{f}>=\ -2\Big(1+\frac{1}{n}\Big)\sum_{x,y\neq 0}a(y-x)\sum_{\substack{A\in\mathcal{E}_{*,n}\\ A\ni y, A\not\ni x}}\mathfrak{g}(A)\mathfrak{f}(A\setminus\{y\})\ .$$

To conclude the proof of the lemma, it remains to change variables $B=A\setminus\{y\}$ and to recall the definition of the operator $\mathfrak{L}_-$.

We turn now to the proof of Claim (A.0.2). Since for y in A, $\mathfrak{g}(A)=\mathfrak{g}(S_yA)$ and since $|A|=n$, the left hand side of (A.0.2) is equal to

$$\begin{aligned}&\frac{1}{n}\sum_{x\neq 0}a(x)\sum_{\substack{A\in\mathcal{E}_{*,n}\\ A\ni x}}\sum_{\substack{y\in A\cup\{0\}\\ y\neq x}}\mathfrak{g}(S_yA)\mathfrak{f}(A\setminus\{x\})\\ &\quad=\ \frac{1}{n}\sum_{\substack{x,y\neq 0\\ y\neq x}}a(x)\sum_{\substack{A\in\mathcal{E}_{*,n}\\ A\ni x,y}}\mathfrak{g}(S_yA)\mathfrak{f}(A\setminus\{x\})\ +\ \frac{1}{n}\sum_{x\neq 0}a(x)\sum_{\substack{A\in\mathcal{E}_{*,n}\\ A\ni x}}\mathfrak{g}(A)\mathfrak{f}(A\setminus\{x\})\ .\end{aligned}$$

Notice that the second term on the right hand side is precisely the original one. Consider the first term. Perform a change of variables $B=S_yA$, rewrite $(S_{-y}A)\setminus\{x\}$ as $S_{-y}(A\setminus\{x-y\})$ and recall that $\mathfrak{f}(S_{-y}(A\setminus\{x-y\})=\mathfrak{f}(A\setminus\{x-y\})$ if $-y$ belongs to A because $\mathfrak{f}$ is in $\mathcal{I}_{*,n-1}$, to rewrite this expression as

$$\frac{1}{n}\sum_{\substack{x,y\neq 0\\ y\neq x}}a(x)\sum_{\substack{A\in\mathcal{E}_{*,n}\\ A\ni x-y,-y}}\mathfrak{g}(A)\mathfrak{f}(A\setminus\{x-y\})\ .$$

A change of variables $x' = x - y$, $y' = -y$, shows that this expression is equal to
$$\frac{1}{n} \sum_{x,y \neq 0} a(x-y) \sum_{\substack{A \in \mathcal{E}_{*,n} \\ A \ni x,y}} \mathfrak{g}(A) \mathfrak{f}(A \setminus \{x\}) \ .$$
To prove (A.0.2), it remains to recollect all previous identities. □

Corollary A.0.24. *The operators $\mathfrak{L}_+ + \mathfrak{L}_-$ and $\mathfrak{L}_d$ are anti-symmetric with respect to the inner product $\ll, \gg_{0,0}$:*

$$\ll \mathfrak{f}, (\mathfrak{L}_+ + \mathfrak{L}_-)\mathfrak{g} \gg_{0,0} = \ - \ll (\mathfrak{L}_+ + \mathfrak{L}_-)\mathfrak{f}, \mathfrak{g} \gg_{0,0}$$
$$\ll \mathfrak{f}, \mathfrak{L}_d \mathfrak{g} \gg_{0,0} = \ - \ll \mathfrak{L}_d \mathfrak{f}, \mathfrak{g} \gg_{0,0}$$

for all finitely supported functions $\mathfrak{f}$, $\mathfrak{g}$ in $\mathcal{F}_*$. The same statement remains in force if $\mathfrak{L}_+ + \mathfrak{L}_-$ and $\mathfrak{L}_d$ are replaced by $\Pi_n(\mathfrak{L}_+ + \mathfrak{L}_-)\Pi_n$ and $\Pi_n \mathfrak{L}_d \Pi_n$, respectively, for every $n \geq 1$ with $\mathfrak{f}$ and $\mathfrak{g}$ in $\mathcal{J}_n$, defined in (5.0.4).

Remark A.0.25. *From the previous Corollary, we get that $\ll \mathfrak{L}_\alpha \mathfrak{f}, \mathfrak{f} \gg_{0,0} = \ll \mathfrak{L}_s \mathfrak{f}, \mathfrak{f} \gg_{0,0}$ for every $\mathfrak{f}$ in $\mathcal{F}_*$. Consider $k \geq 1$. Given $\mathfrak{f} \in \mathbb{I}_*^k$, take $\mathfrak{f}_n \in \mathcal{F}_*$ such that $\|\mathfrak{f}_n - \mathfrak{f}\|_{0,k}$ goes to zero as $n \uparrow \infty$. Since $\mathfrak{L}_\alpha$ and $\mathfrak{L}_s$ are bounded operators from $\mathbb{I}_*^k$ to $\mathbb{I}_*^0$, we get that $\ll \mathfrak{L}_\alpha \mathfrak{f}, \mathfrak{f} \gg_{0,0} = \ll \mathfrak{L}_s \mathfrak{f}, \mathfrak{f} \gg_{0,0}$.*

Recall Theorem 5.0.1 from Chapter 5. Hereafter the constant C_0 may change from line to line.

Theorem A.0.26. *There exists a finite constant C_0, depending only on the probability p, such that*

$$< \mathfrak{L}_d \mathfrak{f}, \mathfrak{g} >^2 \leq C_0 < \mathfrak{f}, -\mathfrak{L}_s \mathfrak{f} >< \mathfrak{g}, -\mathfrak{L}_s \mathfrak{g} >$$

for each $\mathfrak{f}, \mathfrak{g}$ in $\mathcal{I}_{*,n}$. The same result remains in force if $\mathfrak{L}_d$ is replaced by $\mathfrak{L}_+$ or $\mathfrak{L}_-$ with $\mathfrak{g}$ in $\mathcal{I}_{*,n+1}$ and $\mathcal{I}_{*,n-1}$, respectively.

The proof of Theorem A.0.26 is divided in several lemmas. Before starting, we need to introduce some definitions and recall some results. Since the $\| \cdot \|_1$ norm plays a crucial role, we give its explicit form. For $\mathfrak{f} \in \mathcal{I}_{*,n}$, from the definition of the operator $\mathfrak{L}_s$ given in (4.2.9), we get that

$$< \mathfrak{f}, -\mathfrak{L}_s \mathfrak{f} > \;=\; 1/4 \sum_{x,y \in \mathbb{Z}^d_*} s(y-x) \sum_{B \in \mathcal{E}_{*,n}} [\mathfrak{f}(B_{x,y}) - \mathfrak{f}(B)]^2 \quad \text{(A.0.3)}$$
$$+ \;\; 1/2 \sum_y s(y) \sum_{B \notin y} [\mathfrak{f}(S_y B) - \mathfrak{f}(B)]^2 \,,$$

where $B_{x,y}$ was defined in (4.2.7). Observe that first term of the previous expression, may be written as

$$\frac{1}{2} \sum_{b \in \mathbb{Z}^d} s(b) \sum_{x \in \mathbb{Z}^d_*} \sum_{\substack{A \in \mathcal{E}_{*,n-1} \\ A \cap \{x, x+b\} = \phi}} [\mathfrak{f}(A \cup \{x+b\}) - \mathfrak{f}(A \cup \{x\})]^2 \,. \qquad \text{(A.0.4)}$$

Theorem A.0.26 will be proved for loop probabilities. A probability π is said to be a loop probability of length m if it is of the form

$$\pi(x) \;=\; \sum_{i=1}^{m} \frac{1}{m} \, I_{\{x = y_i - y_{i-1}\}}$$

where $0 = y_0, y_1, \ldots, y_m = y_0 \in \mathbb{Z}^d$. It will be denote by $\pi = \{y_0, \cdots, y_m\}$. Consider the jumps $a_i = y_i - y_{i-1}$ and think that π assigns mass $1/m$ to each a_i. Observe that they do not need to be different so π is not necessarily a uniform probability measure. The symmetric part of π assigns mass $1/2m$ to $\pm a_i$, for $i = 1, \ldots, m$. The value of the anti-symmetric part of the probability π is $1/2m$ for a_i and $-1/2m$ for $-a_i$ for $i = 1, \ldots, m$. We prove in Appendix B that every mean zero probability p may be decomposed as a convex combination of loop probabilities. Considering formula (A.0.3), there is no loss of generality in the proof of Theorem A.0.26, assuming that p is a loop probability.

For a loop probability we prove Theorem A.0.26 by induction on the length of the loop. For the inductive step, we need to relate the $\| \cdot \|_1$ norms corresponding to different probabilities. For this purpose, we define the following objects. Given a probability p, we say that x is in the support $\mathbb{S}_p$ of p if $p(x) > 0$.

$$\mathbb{S}_p = \{x : p(x) > 0\} \,. \qquad \text{(A.0.5)}$$

We say that x is attainable if it may be connected with the origin in the following sense: there exists a sequence $z_0 = 0, z_1, \ldots, z_n = x$ with

$p(z_{i+1} - z_i) > 0$. We say that x is attainable after m steps if m is the length of a shortest path connecting x with the origin. We note by $\mathbb{A}_p$ the set of attainable points for the probability p. Observe that for mean zero probabilities, $\mathbb{A}_p = \mathbb{A}_s$ where s is the symmetric part of the probability p. This result is clear for a loop probability and then, by the decomposition result, the same holds for every mean zero probability.

A straightforward computation considering that we are working in $\mathcal{I}_{*,n}$, shows that Dirichlet forms associated to different probabilities are related in the following way:

Remark A.0.27. *Given two mean zero probabilities p_1 and p_2 with $\mathbb{S}_{p_1} \subseteq \mathbb{A}_{p_2}$, there exists a finite constant C such that*

$$< \mathfrak{f}, -\mathfrak{L}_s^{p_1} \mathfrak{f} > \leq \; C \; < \mathfrak{f}, -\mathfrak{L}_s^{p_2} \mathfrak{f} > ,$$

where $\mathfrak{L}_s^{p_i}$ are the operators defined in (4.2.9), corresponding to the probabilities p_i for $i = 1, 2$.

In what follows, we deal with the operators defined in (4.3.3), associated to different probabilities. In order to avoid confusions, we will use $\mathfrak{L}_*^{\pi}$ for $* = s, d, +, -$, to denote the corresponding operators related to the probability π. We are now able to start proving Theorem A.0.26. Almost all the computations are obtained performing some change of variables and considering that we are working with functions in the space $\mathcal{I}_*$.

Lemma A.0.28. *Given a loop probability π there exists a finite constant C_0, depending only on the probability π, such that*

$$< \mathfrak{g}, \mathfrak{L}_+^{\pi} \mathfrak{f} >^2 \leq \; C_0 \; < \mathfrak{f}, -\mathfrak{L}_s^{\pi} \mathfrak{f} > < \mathfrak{g}, -\mathfrak{L}_s^{\pi} \mathfrak{g} > ,$$

for each $\mathfrak{f} \in \mathcal{I}_{*,n-1}$, $\mathfrak{g} \in \mathcal{I}_{*,n}$.

Proof: The proof of this result is by induction in the length of the loop. We show the inductive step and prove the result for a loop of length three. Note that the constants C and C_0 may change from line to line.

Fix a loop probability $\pi = \{y_0, \cdots, y_m\}$. An elementary computation shows that

$$\begin{aligned} < \mathfrak{g}, \mathfrak{L}_+^{\pi} \mathfrak{f} > \;&=\; 2 \sum_{A \in \mathcal{E}_{*,n}} \sum_{x,y \in A} a(y-x)\, \mathfrak{g}(A) \mathfrak{f}(A \setminus \{y\}) \\ &+\; 2 \sum_{A \in \mathcal{E}_{*,n}} \sum_{x \in A} a(x)\, \mathfrak{g}(A) \Big[\mathfrak{f}(A \setminus \{x\}) - \mathfrak{f}(S_x(A \setminus \{x\})) \Big] . \end{aligned} \tag{A.0.6}$$

Observe that $a(y-x) = \pm 1/m$ for $y-x = \pm a_i$ and $a(y-x) = 0$ otherwise. Then, except for $y-x = a_i$ or $y-x = -a_i$, $a(y-x) = 0$. Thus,the first term in the right hand side of (A.0.6) is equal to

$$\frac{1}{m}\sum_{i=1}^{m}\sum_{x\neq 0,-a_i}\sum_{\substack{A\in\mathcal{E}_{*,n-2}\\ A\cap\{x,x+a_i\}=\phi}} \mathfrak{f}(A\cup\{x\})\mathfrak{g}(A\cup\{x,x+a_i\})$$
$$-\frac{1}{m}\sum_{i=1}^{m}\sum_{x\neq 0,a_i}\sum_{\substack{A\in\mathcal{E}_{*,n-2}\\ A\cap\{x,x-a_i\}=\phi}} \mathfrak{f}(A\cup\{x\})\mathfrak{g}(A\cup\{x,x-a_i\})\ .$$

We perform a change of variables in the second term of the previous expression and get that the difference is equal to

$$\frac{1}{m}\sum_{i=1}^{n}\sum_{x\neq 0,-a_i}\sum_{\substack{A\in\mathcal{E}_{*,n-2}\\ A\cap\{x,x+a_i\}=\phi}} \Big[\mathfrak{f}(A\cup\{x\}) - \mathfrak{f}(A\cup\{x+a_i\})\Big]\mathfrak{g}(A\cup\{x,x+a_i\})\ .$$

Recall that $a(x) = 1/2m$ for $x = a_i$, $a(x) = -1/2m$ for $x = -a_i$ and use this fact to write the second term on the right hand side of (A.0.6) as

$$\frac{1}{m}\sum_{i=1}^{m}\sum_{\substack{A\in\mathcal{E}_{*,n-1}\\ A\cap\{a_i\}=\phi}} \Big[\mathfrak{f}(A) - \mathfrak{f}(S_{a_i}A)\Big]\mathfrak{g}(A\cup\{a_i\})$$
$$-\frac{1}{m}\sum_{i=1}^{m}\sum_{\substack{A\in\mathcal{E}_{*,n-1}\\ A\cap\{-a_i\}=\phi}} \Big[\mathfrak{f}(A) - \mathfrak{f}(S_{-a_i}A)\Big]\mathfrak{g}(A\cup\{-a_i\})$$
$$= 2\,\frac{1}{m}\sum_{i=1}^{m}\sum_{\substack{A\in\mathcal{E}_{*,n-1}\\ A\cap\{a_i\}=\phi}} \Big[\mathfrak{f}(A) - \mathfrak{f}(S_{a_i}A)\Big]\mathfrak{g}(A\cup\{a_i\})\ ,$$

where we performed a change of variables in the second line ($B = S_{-a_i}B$) and used that for $\mathfrak{g}\in\mathcal{I}_*$ we get that $\mathfrak{g}(S_{a_i}B\cup\{-a_i\}) = \mathfrak{g}(S_{a_i}(B\cup\{a_i\})) = \mathfrak{g}(B\cup\{a_i\})$ to obtain the last identity. Let

$$N_b = \sum_{x\neq 0,-b}\sum_{\substack{A\in\mathcal{E}_{*,n-2}\\ A\cap\{x,x+b\}=\phi}} \Big[\mathfrak{f}(A\cup\{x\}) - \mathfrak{f}(A\cup\{x+b\})\Big]\mathfrak{g}(A\cup\{x,x+b\})\ , \quad \text{(A.0.7)}$$

$$M_b = \sum_{\substack{A\in\mathcal{E}_{*,n-1}\\ A\cap\{b\}=\phi}} \Big[\mathfrak{f}(A) - \mathfrak{f}(S_b A)\Big]\mathfrak{g}(A\cup\{b\})\ , \tag{A.0.8}$$

so that

$$< \mathfrak{g}, \mathfrak{L}_+^\pi \mathfrak{f} >= \frac{1}{m}\sum_{i=1}^m N_{a_i} + \frac{2}{m}\sum_{i=1}^m M_{a_i}\ .$$

We are ready to perform the inductive step. Let $z = \sum_{i=3}^m a_i$ and consider the loop probabilities $\pi_1 = \{y_0, y_1, y_2, y_0\}$ and $\pi_2 = \{y_0, y_2, y_3, \cdots, y_m\}$ corresponding to jumps a_1, a_2, z and $-z, a_3, \cdots, a_m$, respectively. Observing that $N_{-b} = -N_b$ and $M_{-b} = -M_b$ we get that

$$\begin{aligned} < \mathfrak{g}, \mathfrak{L}_+^\pi \mathfrak{f} > \quad &= 1/m\ [(N_{a_1} + N_{a_2} + N_z) + N_{-z} + \textstyle\sum_{i=3}^m N_{a_i}] \\ &\ +1/m\ [(M_{a_1} + M_{a_2} + M_z) + M_{-z} + \textstyle\sum_{i=3}^m M_{a_i}] \\ &= 3/m\ \ < \mathfrak{g}, \mathfrak{L}_+^{\pi_1}\mathfrak{f} > +m - 1/m < \mathfrak{g}, \mathfrak{L}_+^{\pi_2}\mathfrak{f} >\ \ . \end{aligned}$$

Since $\mathbb{S}_{\pi_i} \subset \mathbb{A}_\pi$, by Remark A.0.27 we get that $< \mathfrak{f}, -\mathfrak{L}_s^{\pi_i}\mathfrak{f} > \le C < \mathfrak{f}, -\mathfrak{L}_s^\pi \mathfrak{f} >$ for $i = 1, 2$. This fact and the inductive hypothesis give that

$$\begin{aligned} < \mathfrak{g}, \mathfrak{L}_+^\pi \mathfrak{f} >^2 \ \ &\le\ C\ \Big\{ < \mathfrak{f}, \mathfrak{L}_s^{\pi_1}\mathfrak{f} >< \mathfrak{g}, \mathfrak{L}_s^{\pi_1}\mathfrak{g} > + < \mathfrak{f}, \mathfrak{L}_s^{\pi_2}\mathfrak{f} >< \mathfrak{g}, \mathfrak{L}_s^{\pi_2}\mathfrak{g} > \Big\} \\ &\le\ C\ \ < \mathfrak{f}, \mathfrak{L}_s^\pi \mathfrak{f} >< \mathfrak{g}, \mathfrak{L}_s^\pi \mathfrak{g} >\ , \end{aligned}$$

which concludes the inductive step.

Now, we need to prove the estimate for a loop probability of length three. Consider π corresponding to jumps a_i for $i = 1, 2, 3$ with $a_1 + a_2 + a_3 = 0$. We start showing that $\{\sum_{i=1}^3 N_{a_i}\}^2 \le C < \mathfrak{f}, \mathfrak{L}_s^\pi \mathfrak{f} >< \mathfrak{g}, \mathfrak{L}_s^\pi \mathfrak{g} >$ and then we prove the same kind of bound for $\{\sum_{i=1}^3 M_{a_i}\}^2$. To keep notation simple, let $d = 1$ and take $a_1 = 1$, $a_2 = 2$ and $a_3 = -3$. According to definition (A.0.7) for N_b, we get that

$$
\begin{aligned}
\sum_{i=1}^3 N_{a_i} = & \sum_{x\neq 0,-1} \sum_{\substack{A\in\mathcal{E}_{*,n-1}\\ A\cap\{x,x+1\}=\phi}} \Big[\mathfrak{f}(A\cup\{x\}) - \mathfrak{f}(A\cup\{x+1\})\Big]\mathfrak{g}(A\cup\{x,x+1\}) \\
+ & \sum_{x\neq -1,-3} \sum_{\substack{A\in\mathcal{E}_{*,n-1}\\ A\cap\{x+1,x+3\}=\phi}} \Big[\mathfrak{f}(A\cup\{x+1\}) - \mathfrak{f}(A\cup\{x+3\})\Big]\mathfrak{g}(A\cup\{x+1,x+3\}) \\
+ & \sum_{x\neq 0,-3} \sum_{\substack{A\in\mathcal{E}_{*,n-1}\\ A\cap\{x,x+3\}=\phi}} \Big[\mathfrak{f}(A\cup\{x+3\}) - \mathfrak{f}(A\cup\{x\})\Big]\mathfrak{g}(A\cup\{x,x+3\})\ ,
\end{aligned}
\tag{A.0.9}
$$

where for the second and last line we performed a change of variable. We will decompose this sum in three terms: $\sum_{i=1}^3 N_{a_i} = T_1 + T_2 + T_3$. The decomposition appears because we want to add over the same values of x and then we work for adding in the same sets A. For T_1, take from (A.0.9) $x = -3$ in the first line, $x = 0$ in the second line and $x = -1$ in the last one. T_2 is obtained taking $x \neq 0, -1, -3$ in the three sums of (A.0.9) and imposing A to contain $x+3$, x and $x+1$ in each line, respectively. Finally, T_3 is obtained taking $x \neq 0, -1, -3$ in all the sums and imposing A not to contain each of the previous elements. Recall that we are working with finite sets A in $\mathbb{Z}^d_*$. Sometimes we omit from the notation the cardinal of A. Some others we put in evidence that $A\cap\{0\} = \phi$. For T_1, we get

$$
\begin{aligned}
T_1 = & \sum_{A\cap\{-3,-2,0\}=\phi} \Big[\mathfrak{f}(A\cup\{-3\}) - \mathfrak{f}(A\cup\{-2\}\Big]\mathfrak{g}(A\cup\{-2,-3\}) \\
+ & \sum_{A\cap\{0,1,3\}=\phi} \Big[\mathfrak{f}(A\cup\{1\}) - \mathfrak{f}(A\cup\{3\})\Big]\mathfrak{g}(A\cup\{1,3\}) \\
+ & \sum_{A\cap\{-1,0,2\}=\phi} \Big[\mathfrak{f}(A\cup\{2\}) - \mathfrak{f}(A\cup\{-1\})\Big]\mathfrak{g}(A\cup\{-1,2\})\ .
\end{aligned}
\tag{A.0.10}
$$

Let $S_3A = B$ in the second line of (A.0.10) and $S_2A = B$ in the third one to get that

$$
\begin{aligned}
T_1 \;=\; & \sum_{A\cap\{-3,-2,0\}=\phi} \Big[\mathfrak{f}(A\cup\{-3\}) - \mathfrak{f}(A\cup\{-2\}\Big]\mathfrak{g}(A\cup\{-2,-3\}) \\
+ & \sum_{A\cap\{-3,-2,0\}=\phi} \Big[\mathfrak{f}(S_{-3}A\cup\{1\}) - \mathfrak{f}(S_{-3}A\cup\{3\})\Big]\mathfrak{g}(S_{-3}A\cup\{1,3\}) \\
+ & \sum_{A\cap\{-3,-2,0\}=\phi} \Big[\mathfrak{f}(S_{-2}A\cup\{2\}) - \mathfrak{f}(S_{-2}A\cup\{-1\})\Big]\mathfrak{g}(S_{-2}A\cup\{-1,2\})\ .
\end{aligned}
$$

Recall that, by definition of S_z, $S_{-2}A\cup\{-1,2\} = S_{-2}(A\cup\{-2,-3\})$. In this case, since $\mathfrak{f}$ and $\mathfrak{g}$ belong to $\mathcal{I}_*$, we get that $\mathfrak{g}(S_{-2}A\cup\{-1,2\}) = \mathfrak{g}(S_{-2}(A\cup\{-2,-3\})) = \mathfrak{g}(A\cup\{-2,-3\})$. It is not difficult to check, using this kind of identities, that the previous expression vanishes. For T_2, we get that

$$
\begin{aligned}
T_2 \;& = \sum\sum \Big[\mathfrak{f}(A\cup\{x\,x+3\}) - \mathfrak{f}(A\cup\{x+1,x+3\})\Big]\mathfrak{g}(A\cup\{x,x+1,x+3\}) \\
& + \sum\sum \Big[\mathfrak{f}(A\cup\{x,x+1\}) - \mathfrak{f}(A\cup\{x,x+3\})\Big]\mathfrak{g}(A\cup\{x,x+1,x+3\}) \\
& + \sum\sum \Big[\mathfrak{f}(A\cup\{x+1,x+3\}) - \mathfrak{f}(A\cup\{x,x+1\})\Big]\mathfrak{g}(A\cup\{x,x+1,x+3\})\ ,
\end{aligned}
$$

where the first sum in each line is for $x \neq 0,-1,-3$ and the second one is for $A \in \mathcal{E}_{*,n-3}$ such that $A\cap\{x,x+1,x+3\} = \phi$. Then T_2 vanishes too. Finally, for T_3, we get

$$
\begin{aligned}
T_3 = \;& \sum\sum \Big[\mathfrak{f}(A\cup\{x\ \}) - \mathfrak{f}(A\cup\{x+1\})\Big]\mathfrak{g}(A\cup\{x,x+1\}) \qquad \text{(A.0.11)} \\
& + \sum\sum \Big[\mathfrak{f}(A\cup\{x+1\}) - \mathfrak{f}(A\cup\{x+3\})\Big]\mathfrak{g}(A\cup\{x+1,x+3\}) \\
& + \sum\sum \Big[\mathfrak{f}(A\cup\{x+3\}) - \mathfrak{f}(A\cup\{x\})\Big]\mathfrak{g}(A\cup\{x,x+3\})\ ,
\end{aligned}
$$

where the first sum in each line is for $x \neq 0,-1,-3$ and the second one is for $A \in \mathcal{E}_{*,n-3}$ such that $A\cap\{x,x+1,x+3\} = \phi$. Add and subtract $\mathfrak{f}(A\cup\{x+1\})$ in the first factor of the last line in (A.0.11) to get that

$$
\begin{aligned}
T_3 = \sum\sum \; & \Big[\mathfrak{f}(A\cup\{x+3\}) - \mathfrak{f}(A\cup\{x+1\})\Big] \\
& \qquad \Big[\mathfrak{g}(A\cup\{x,x+3\}) - \mathfrak{g}(A\cup\{x+1,x+3\})\Big] \\
& + \Big[\mathfrak{f}(A\cup\{x+1\}) - \mathfrak{f}(A\cup\{x\})\Big] \\
& \qquad \Big[\mathfrak{g}(A\cup\{x,x+3\}) - \mathfrak{g}(A\cup\{x,x+1\})\Big].
\end{aligned}
$$

By Schwarz inequality, the previous expression is bounded by $C < \mathfrak{f}, -\mathfrak{L}_s^\pi, \mathfrak{f} >$ $< \mathfrak{g}, -\mathfrak{L}_s^\pi, \mathfrak{g} >$, in view of formula (A.0.3).

We turn now to the expression $\sum_{i=1}^3 M_{a_i}$ (see (A.0.8) for definition of M_b). It may be rewritten as

$$
\begin{aligned}
& \sum_{i=1}^3 \sum_{A\cap\{0,a_i\}=\phi} \Big[\mathfrak{f}(A) - \mathfrak{f}(S_{a_i}A)\Big]\mathfrak{g}(A\cup\{a_i\}) = \\
& \sum_{A\cap\{0,1\}=\phi} \Big[\mathfrak{f}(A) - \mathfrak{f}(S_1 A)\Big]\mathfrak{g}(A\cup\{1\}) \qquad\qquad (A.0.12)\\
& + \sum_{A\cap\{1,3\}=\phi} \Big[\mathfrak{f}(S_1 A) - \mathfrak{f}(S_3 A)\Big]\mathfrak{g}(S_1 A\cup\{2\}) \\
& - \sum_{A\cap\{0,3\}=\phi} \Big[\mathfrak{f}(A) - \mathfrak{f}(S_3 A)\Big]\mathfrak{g}(S_3 A\cup\{-3\}\ ,
\end{aligned}
$$

where the last two terms in the previous expression are obtained after the change of variable $S_2 A = B$ and $S_{-3} A = B$, respectively. We decompose each expression in two, to obtain sums carried over the same sets. In the following expression, the first three terms correspond to the terms obtained by imposing A not to contain 3, 0 and 1, respectively in each of the three last lines of (A.0.12). The sum over the sets A that contain 3, 0 and 1, correspond to the last three lines. Therefore,

$$
\begin{aligned}
\sum_{i=1}^{3} M_{a_i} = \ \textstyle\sum_{A\cap\{0,1,3\}=\phi} \ & \Big[\mathfrak{f}(A) - \mathfrak{f}(S_1A)\Big]\mathfrak{g}(A\cup\{1\}) \\
& +\Big[\mathfrak{f}(S_1A) - \mathfrak{f}(S_3A)\Big]\mathfrak{g}(S_1A\cup\{2\}) \\
& -\Big[\mathfrak{f}(A) - \mathfrak{f}(S_3A)\Big]\mathfrak{g}(S_3A\cup\{-3\}) \\
& +\Big[\mathfrak{f}(A\cup\{3\}) - \mathfrak{f}(S_1A\cup\{2\})\Big]\mathfrak{g}(A\cup\{1,3\}) \\
& +\Big[\mathfrak{f}(S_1A\cup\{-1\}) - \mathfrak{f}(S_3A\cup\{-3\})\Big]\mathfrak{g}(S_1A\cup\{2,-1\}) \\
& -\Big[\mathfrak{f}(A\cup\{1\}) - \mathfrak{f}(S_3A\cup\{-2\})\Big]\mathfrak{g}(S_3A\cup\{-3,-2\}).
\end{aligned}
$$

After some operations recalling the definition of S_z and the fact that $\mathfrak{f}$ and $\mathfrak{g}$ belong to $\mathcal{I}_*$ (as we did when working with T_1), we get that the sum of the last three terms vanishes. For the three remaining, add and subtract $\mathfrak{f}(S_1A)$ in the third line, to get

$$
\begin{aligned}
& \sum_{A\cap\{0,1,3\}=\phi} \Big[\mathfrak{f}(A) - \mathfrak{f}(S_1A)\Big]\Big[\mathfrak{g}(A\cup\{1\}) - \mathfrak{g}(A\cup\{3\})\Big] \\
+ & \sum_{A\cap\{0,1,3\}=\phi} \Big[\mathfrak{f}(S_1A) - \mathfrak{f}(S_3A)\Big]\Big[\mathfrak{g}(S_1(A\cup\{3\})) - \mathfrak{g}(S_3(A\cup\{3\}))\Big].
\end{aligned}
$$

By Schwarz inequality, this expression is bounded by $C\|\mathfrak{f}\|_1\|\mathfrak{g}\|_1$, in view of expression (A.0.3) for $\|\cdot\|_1$. This concludes the proof of the lemma. □

Lemma A.0.29. *Given a loop probability π there exists a finite constant C_0, depending only on the probability π, such that*

$$
< \mathfrak{g}, \mathfrak{L}^{\pi}_{-}\mathfrak{f} >^2 \le C_0 < \mathfrak{f}, \mathfrak{L}^{\pi}_{s}\mathfrak{f} >< \mathfrak{g}, \mathfrak{L}^{\pi}_{s}\mathfrak{g} > ,
$$

for each $\mathfrak{f} \in \mathcal{I}_{n+1}$, $\mathfrak{g} \in \mathcal{I}_n$.

Proof: This result follows from Lemma A.0.23 and Lemma A.0.28. In any case, we can perform the computations. For the inductive step,

$$
N_b = \sum_{x\neq 0,-b} \ \sum_{\substack{A\in\mathcal{E}_{*,n} \\ A\cap\{x,x+b\}=\phi}} \Big[\mathfrak{f}(A\cup\{x+b\}) - \mathfrak{f}(A\cup\{x\})\Big]\mathfrak{g}(A) ,
$$

we get that

$$< \mathfrak{g}, \mathfrak{L}^{\pi}_{-} \mathfrak{f} >= \frac{1}{m} \sum_{i=1}^{m} N_{a_i} \ .$$

As in the previous lemma, define $z = \sum_{i=3}^{m} a_i$. Observing that $N_{-b} = -N_b$, we obtain that

$$< \mathfrak{g}, \mathfrak{L}^{\pi}_{-} \mathfrak{f} >= \frac{3}{m} < \mathfrak{g}, \mathfrak{L}^{\pi_1}_{-} \mathfrak{f} > + \frac{m-1}{m} < \mathfrak{g}, \mathfrak{L}^{\pi_2}_{-} \mathfrak{f} > ,$$

where π_1 and π_2 are concentrated in a_1, a_2, z and $-z, a_3, \dots, a_m$, respectively. This proves that the induction works. To prove the estimate for the case $\pi(a_i) = \frac{1}{3}$ where $a_1 + a_2 + a_3 = 0$, we perform the same kind of estimations done in Lemma A.0.28 to show that $\{\sum_{i=1}^{3} N_{a_i}\}^2 \leq C < \mathfrak{f}, \mathfrak{L}^{\pi}_{s} \mathfrak{f} >< \mathfrak{g}, \mathfrak{L}^{\pi}_{s} \mathfrak{g} >$.

□

Lemma A.0.30. *Given a loop probability π there exists a finite constant C_0, depending only on the probability π, such that*

$$< \mathfrak{g}, \mathfrak{L}^{\pi}_{d} \mathfrak{f} >^2 \leq C_0 < \mathfrak{f}, \mathfrak{L}^{\pi}_{s} \mathfrak{f} >< \mathfrak{g}, \mathfrak{L}^{\pi}_{s} \mathfrak{g} >$$

for each $\mathfrak{f}, \mathfrak{g} \in \mathcal{I}_n$.

Proof: We follow the strategy used in the previous cases. Observe that

$$< \mathfrak{g}, \mathfrak{L}^{\pi}_{d} \mathfrak{f} > \ = \ 1/(2m) \textstyle\sum_{i=1}^{m} N_{a_i} \ + \ 1/(2m) \textstyle\sum_{i=1}^{m} M_{a_i} \ ,$$

where

$$N_b = \textstyle\sum_{x \neq 0, -b} \sum_{\substack{A \in \mathcal{E}_{*,n-1} \\ A \cap \{x, x+b\} = \phi}} \Big[\mathfrak{f}(A \cup \{x+b\}) - \mathfrak{f}(A, \cup \{x\}) \Big] \Big[\mathfrak{g}(A \cup \{x\}) + \mathfrak{g}(A \cup \{x+b\}) \Big] ,$$

$$M_b = \textstyle\sum_{\substack{A \in \mathcal{E}_{*,n} \\ A \cap \{b\} = \phi}} \Big[\mathfrak{f}(S_b A) - \mathfrak{f}(A) \Big] \Big[\mathfrak{g}(A) + \mathfrak{g}(S_b A) \Big] .$$

This decomposition allows us to repeat the same kind of computation performed when proving Lemma A.0.28 and Lemma A.0.29. □

Appendix B

Decomposition of a mean zero probability as a convex combination of loop probabilities

We start explaining what a loop probability is. Given $a_1, \ldots, a_N$ points in $\mathbb{Z}^d$ such that $\sum_{i=1}^N a_i = 0$, consider the probability π that assigns mass $1/N$ over each a_i, for $i = 1, \ldots. N$. As we do not require the points a_i to be different, this is not necesarily an uniform probability. Observe that π is a mean zero probability; we call it a loop probability. In order to motivate the name of this probability, set $y_i = \sum_{j=1}^i a_j$ and observe that $y_n = 0$. This means that starting from the origin, jumping from y_i to y_{i+1} we arrive back to the origin. The $y_1, \ldots, y_n$ form a loop (or cycle). $y_{i+1} - y_i = a_i$ is called a jump.

Definition B.0.31. *A probability π is a loop probability if there exists a closed path $y_1, \ldots, y_n$ in $\mathbb{Z}^d$ such that*

$$\pi(x) = \frac{1}{n} \sum_{i=1}^{n} I_{\{x = y_{i+1} - y_i\}}.$$

We will prove that every compactly supported mean zero probability in $\mathbb{Z}^d$ may be written as a convex combination of loop probabilities.

Lemma B.0.32. *Given a probability P in $\mathbb{Z}^d$ such that $P(x) = 0$ for $|x|$ big enough (compactly supported) and $\sum x \ P(x) = 0$, there exists $\alpha_j > 0$ with*

$\sum \alpha_j = 1$ *and loop probabilities* π_j *such that*

$$P = \sum \alpha_j \; \pi_j .$$

Proof: We will use z_j to denote vectors in different spaces and z_j^i for the i-th coordinate of the vector z_j. Observe that if $P(x_l) \in \mathbb{Q}$ for all x_l, then P is itself a loop probability. In this case we get that $P(x_l) = m_l/b$ with $\sum m_l = b$. This corresponds to a loop probability taking the jump x_l m_l times.

We will prove the lemma by induction in n, the number of point in the support $\mathbb{S}_P$ of the probability P, defined in (A.0.5) ($n = \#\{x : P(x) > 0\}$).

1- : $n = 2$. In this case we get that exist $x_1, x_2 \in \mathbb{Z}_*^d$ such

$$x_1 P(x_1) + x_2 P(x_2) = 0 \text{ and } P(x_1) + P(x_2) = 1 \; .$$

This two equations determine P. Solving for $P(x_1)$, $P(x_2)$, we get that

$$P(x_1) = -x_2^i/(x_1^i - x_2^i) \text{ and } P(x_2) = x_1^i/(x_1^i - x_2^i) \; ,$$

independently of i. This shows that the probability P takes values in $\mathbb{Q}$ and then it is a loop probability.

2- Inductive step: Consider P supported in a set of cardinality n: $\#\mathbb{S}_P = n$. We would like to write

$$P = c_1 p_1 + (1 - c_1)\tilde{P}$$

for same $0 < c_1 < 1$, p_1 a loop probability, $\tilde{P}$ a mean zero probability supported in a set with less than n points. In this case, by the inductive hypothesis, we will be able to decompose $\tilde{P}$ as a convex combination of loop probabilities and therefore, the same holds for P.

The problem is reduced to prove the existence of π_1, a loop probability concentrated in $\mathbb{S}_P = \{x_1, \cdots, x_n\}$. If such π_1 exists, take

$$c_1 = \min_i P(x_i)/\pi_1(x_i).$$

Since both P and π_1 are probabilities, $c_1 \leq 1$. If $c_1 = 1$ then $P = \pi_1$ and P is a loop probability. Otherwise, $\tilde{P}$ defined by

$$\tilde{P}(x_i) = \frac{P(x_i) - c_1 \pi_1(x_i)}{1 - c_1}$$

is a mean zero probability whose support is smaller than the support of P, as we wanted.

We will now prove the existence of a loop probability concentrated in $x_1, \cdots, x_n$. We are looking for a linear combination of x_i with rational positive coefficients that adds up to zero, i.e., $q_i \in \mathbb{Q}$, $q_i > 0$ such that $\sum q_i\, x_i = 0$. Then, we normalize and obtain the desired probability.

Without loss of generality, we may suppose that $\{x_1, \ldots, x_s\}$ is a basis of the linear subspace in $\mathbb{R}^d$ generated by $x_1, \cdots, x_n$.

Set $w_j = x_j$ for $j = 1, \ldots, s$. There exist constants $\beta_l^k \in \mathbb{Q}$ (after all the vectors are in $\mathbb{Z}^d$) for $k = 1, \ldots, s$ and $l = 1, \ldots, n$ such that

$$\sum_{k=1}^{s} \beta_k^l\, w_k \;=\; x_l$$

Consider the matrices $W \in \mathbb{Q}^{d\times s}$ where $W_{ij} = w_j^i$, $B \in \mathbb{Q}^{s\times n}$ given by $B_{kl} = \beta_k^l$ and $X \in \mathbb{Q}^{d\times n}$ with $X_{i,j} = x_j^i$. With this notation, we get that

$$W\,B \;=\; X\,.$$

Let $\alpha \in \mathbb{R}^n$. Since the columns of W are linearly independent,

$$\sum_{i=1}^{n} \alpha_i x_i \;=\; X\alpha \;=\; WB\alpha = 0 \quad\Longleftrightarrow\quad B\alpha = 0 \tag{B.0.1}$$

We have chosen the basis w_i in such a way that $w_i = x_i$, so the first $s\times s$ block of the matrix B is the identity in $\mathbb{R}^{s\times s}$. We have $B = [I_s, A]$ where A is an $s\times(n-s)$ matrix with rational coefficients. For each $\alpha \in \mathbb{R}^n$, put $\alpha = (\tilde{\alpha}_1, \tilde{\alpha}_2)$ where $\tilde{\alpha}_1 = (\alpha_1, \cdots, \alpha_s)$ and $\tilde{\alpha}_2 = (\alpha_{s+1}, \cdots, \alpha_n)$. With this notation, condition (B.0.1) may be read as

$$\sum_{i=1}^{n} \alpha_i x_i = 0 \quad\Longleftrightarrow\quad \tilde{\alpha}_1 = -A\tilde{\alpha}_2 \tag{B.0.2}$$

Now consider the linear operator $(-A) : \mathbb{R}^{n-s} \to \mathbb{R}^s$. By hypothesis, we know that $\sum P(x_i)x_i = 0$. Then, using condition (B.0.2), we set that

$$(P(x_1), \cdots, P(x_s)) = -A[(P(x_{s+1}), \cdots, P(x_n))]$$

with all the entries of $P(x_i)$ for $1 \leq i \leq n$ positive. By continuity of the linear transformation $(-A)$, we can choose $(q_{s+1}, \cdots, q_n)$ close to $(P(x_{s+1}), \cdots, P(x_n))$, positive and rational, such that $(-A)[(q_{s+1}, \cdots, q_n)]$ is also positive (and clearly it is rational because (-A) is a rational matrix). Then
$(-A)[(q_{s+1}, \cdots, q_n)], (q_{s+1}, \cdots, q_n)$ are the coefficient that we are looking for.

$\square$

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