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On Some Nonlinear Dispersive Systems

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To memory of my father

To my mother

Abstract

We study local and global well-posedness of the initial value problem (IVP) associated to the coupled Schrödinger-Korteweg-de Vries equation and Schrödinger-Debye systems. We also consider the Benney system and discuss some ill-posedness issues regarding this system. For the coupled Schrödinger-Korteweg-de Vries equation we obtain a local result for weak initial data that allows to use the conserved quantities in the energy space to prove global well-posedness in that space. Both results considerably improve the previous ones [2, 44]. Concerning the Schrödinger-Debye systems we also obtain local and global results improving the ones given in [10]. As a consequence of our study of the Benney system we show that the best local result for that system, in the focusing case, is for data in $L^2(\mathbb{R}) \times H^{-1/2}(\mathbb{R})$. The techniques used to prove our results are recents argument introduced by Bourgain, Kenig, Ponce and Vega to study general nonlinear dispersive equations.

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Contents

Introduction

In this work we study the initial value problem (IVP) associated to the nonlinear coupled system

$$\begin{cases} i\partial_t u + \delta\partial_x^2 u = \alpha uv + \beta|u|^2u, & t, x \in \mathbb{R}, \\ \tau\partial_t v + \lambda\partial_x v + \nu P(\partial_x)v + \mu v\partial_x v = \gamma\partial_x(|u|^2) + \epsilon|u|^2, \end{cases} \tag{1}$$

where $\alpha,\ \beta,\ \delta,\ \gamma,\ \lambda,\ \mu,\ \nu$ and ϵ are real constants and $P(\partial_x)$ is a linear differential operator with constant coefficients. This model describes various phenomena of physics and fluid mechanics. For example,

(e_1) the internal gravity wave packet [25] and the capillary-gravity interaction wave [18] when $\gamma < 0,\ \delta = \tau = 1$ and $\beta = \lambda = \nu = \mu = \epsilon = 0$,

(e_2) the capillary-gravity interaction wave [20, 30, 40, 50] when $\delta = \nu = \mu = 1,\ \lambda = \epsilon = 0$ and $P(\partial_x) = \partial_x^3$,

(e_3) the sonic-Langmuir wave interaction in plasma physics [28, 49] when $\delta = \tau = 1,\ \lambda = -1$ and $\beta = \nu = \mu = \epsilon = 0$,

(e_4) the general theory of water wave interaction in a nonlinear medium [4] when $\delta = \tau = 1,\ \nu = 0,\ \mu = 0$ or $\mu = 1$ and $\epsilon = 0$,

(e_5) the motion of two fluids under capillary-gravity waves in deep water flow [20] when $\delta = \tau = \nu = 1,\ \beta = \lambda = \mu = \epsilon = 0,\ \alpha > 0,\ \gamma > 0$ and $P(\partial_x) = \partial_x H\partial_x$,

(e_6) the motion of two fluids under shallow water flow [20] when $\delta=\tau=1,\ \beta=\lambda=\nu=\mu=\gamma=\epsilon=0,\ \alpha>0$ and $\gamma>0$,

and

(e_7) the nonlinear optics [10] when $\delta=1/2,\ \tau>0,\ \alpha=\nu=1,\ \beta=\lambda=\mu=\gamma=0,\ \epsilon=\pm1$ and $P(\partial_x)=1$.

The objective of this work is to consider the well-posedness of the initial value problem (IVP) for the interaction equation(1). The notion of "local well-posedness" to be used here is in the sense of Kato, that is, the solution uniquely exists in certain time interval (unique existence), the solution describes a continuous curve in X (Banach space) in certain time interval whenever initial data belong to X (persistence), and the solution varies continuously depending upon the initial data (continuous dependence). Global well-posedness requires that the same properties hold for all time $t>0$.

In chapter 1 we study IVP for the coupled Schrödinger-Korteweg-de Vries equation

$$\begin{cases} i\partial_t u+\partial_x^2 u=\alpha uv+\beta|u|^2u, & t,x\in\mathbb{R},\\ \partial_t v+\partial_x^3 v+\frac{1}{2}\partial_x(v^2)=\gamma\partial_x(|u|^2), & \\ u(x,0)=u_0(x),\ v(x,0)=v_0(x), & \end{cases} \tag{2}$$

where $u=u(x,t)$ is a complex-valued function, $v=v(x,t)$ is a real-valued function and $\alpha,\ \beta,\ \gamma$ are real constants.

This system governs the interactions between long-wave, $v=v(x,t)$, and short-wave, $u=u(x,t)$, and arises in fluid mechanics as well as plasma physics. The case $\beta=0$ appears in the study of resonant interaction between long and short capillary-gravity waves on water of uniform finite depth [20], in plasma physics [40] and in a diatomic lattice system [50].

The coupled Schrödinger-Korteweg-de Vries equation (2) has been shown not to be a completely integrable system (Benilov and Burtsev [12]). Therefore the solvability of (2) is dependent upon the method of the evolution equations. M. Tsutsumi [44] showed that for $(u_0, v_0) \in H^{k+1/2}(\mathbb{R}) \times H^k(\mathbb{R})$ for $k = 1, 2, 3...$ the coupled system (2) is globally well-posed in $H^{k+1/2}(\mathbb{R}) \times H^k(\mathbb{R})$ using the conservation laws

$$M(t) = \int_{-\infty}^{+\infty} |u(x,t)|^2 dx = M(0), \tag{3}$$

$$K(t) = \int_{-\infty}^{+\infty} \left(\alpha v^2(x,t) + 2\gamma Im(u(x,t)\overline{u_x(x,t)} \right) dx = K(0), \tag{4}$$

and

$$\begin{aligned} E(t) = \int_{-\infty}^{+\infty} \Big(& \alpha\gamma v(x,t)|u(x,t)|^2 + \gamma |u_x(x,t)|^2 + \tfrac{\alpha}{2}|v_x(x,t)|^2 \\ & - \tfrac{\alpha}{6}v^3(x,t) + \tfrac{\beta\gamma}{2}|u(x,t)|^4 \Big) dx = E(0). \end{aligned} \tag{5}$$

The single out nonlinear Schrödinger (NLS) and Korteweg-de Vries (KdV) equations have been extensively studied. For instance, Ginibre-Velo, Cazenave-Weissler, Y. Tsutsumi, more recently Bourgain and Kenig-Ponce-Vega have obtained several results regarding NLS. In particular Y. Tsutsumi established the local well-posedness of the IVP associated to the cubic NLS for data in $L^2(\mathbb{R})$.

For NLS equation with appropriate nonlinearity, well-posedness in $H^s(\mathbb{R})$ with $s \geq 0$ has been shown by Y. Tsutsumi [47], Kato [29], Cazenave-Weissler [16], Ginibre-Velo [22] and for negative Sobolev spaces we can see the works of Kenig-Ponce-Vega [36] and A. Grünrock [26]. For KdV equation, L^2-well posedness was shown by Bourgain [14] and for negative Sobolev spaces by Kenig-Ponce-Vega [34, 35].

In [37] Kenig, Ponce and Vega showed that the best result for local well-posedness for nonlinear Schrödinger equation with cubic nonlinear term ($|u|^2u$) is for data in $L^2(\mathbb{R})$. They have proved that this equation is ill-posed below $L^2(\mathbb{R})$ in the sense that the mapping data-solution ($u_0 \longmapsto u(t)$) is not uniformly continuous. On the other hand the well-posedness for KdV equation in the Sobolev space with negative exponents has been obtained up to

$H^{-3/4+\epsilon}(\mathbb{R})$, $\epsilon > 0$ [14, 34, 35]. In a recent paper [17], M. Christ, J. Colliander, and T. Tao have proved that the KdV equation is locally well-posed in $H^{-3/4}$, in the minimal sense*, from the theory of the modified-KdV equation in $H^{1/4}$ by using a variant of the Miura transform $u \longmapsto u_x + u^2$, which maps solutions of defocusing modified-KdV to real KdV. They also have proved that the KdV equation is not locally well-posed in H^s for any $-1 \leq s < -3/4$; more precisely, the solution operator fails to be uniformly continuous with respect to the H^s norm.

A coupled system like equation (2) is more difficult to handle in the same spaces as in the single equation is solved; the difficulty stems from antisymmetric nature of the characteristics of each linear part. In [2] Bekiranov, Ogawa and Ponce showed that the coupled system (2) is locally well-posed in $H^s(\mathbb{R}) \times H^{s-1/2}(\mathbb{R})$ with $s \geq 0$. These results include and extend the previous results obtained by M. Tsutsumi in [44] for the local well-posedness cases. The question arises is whether the coupled system (2) is well-posed in $L^2(\mathbb{R}) \times H^{-3/4+\epsilon}(\mathbb{R})$. In this work we answer affirmatively this question. Indeed we obtain local well-posedness for weak initial data $(u_0, v_0) \in H^k(\mathbb{R}) \times H^l(\mathbb{R})$ for various values of (k, l) where the lowest admissible values are $(k, l) = (0, -3/4 + \epsilon)$. To obtain our results, we use the Fourier restriction norm method introduced by Bourgain [13, 14] and further developed by Kenig-Ponce-Vega [34, 35, 36] and Ginibre-Tsutsumi-Velo [24]. For an instructive description of this method we refer to [23]. Moreover our results cover the case $(k, l) = (1, 1)$ and hence we obtain global well-posedness in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ using the conservation laws improving previous results obtained by M. Tsutsumi in [44] for global well-posedness.

*This alternative notion of local well-posedness is given in [17] to provide meaning to rough solutions obtained through a limiting procedure of smooth functions, but the uniqueness of solutions may not be guaranteed.

In chapter 2 we study the Cauchy problem for the one dimensional Schrödinger-Debye system

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u = uv, \quad x \in \mathbb{R},\ t \geq 0, \\ \tau\partial_t v + v = \epsilon|u|^2, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{cases} \tag{6}$$

where $\tau > 0$ and $\epsilon = \pm 1$.

This system is derived from Maxwel-Debye equations

$$\begin{cases} i\partial_t A + \frac{c}{2k\eta_0}\triangle A = \frac{w_0}{\eta_0}\nu A, \\ \tau\partial_t \nu + \nu = \eta_2|A|^2, \\ A(x,0) = A_0(x), \quad \nu(x,0) = \nu_0(x), \end{cases}$$

which describes the non resonant delayed interaction of an electromagnetic wave with a media. In these equations A denotes the envelope of a light wave that goes through a media which response is non resonant. This wave induces a change ν of refractive index in the material (initially η_0 for an electromagnetic wave of frequency w_0) with a slight delay τ. The magnitude and the sign of the nonlinear coupling of the matter with the wave is described by the parameter η_2. The light velocity in the vacuum is denoted by c and k denotes the wave vector of the incident electromagnetic wave.

Local solutions in time for the IVP (6) in Sobolev spaces have been obtained by B. Bidégaray [9, 10]; more precisely local well-posedness in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ with $s > 1/2$. In these references, it is shown also that as τ tends to 0 solutions to the system (6) converge to those of the cubic nonlinear Schrödinger equation, namely

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = \epsilon|u|^2 u \tag{7}$$

at least on certain time interval and for compatible initial data $v_0 = \epsilon|u_0|^2$. If $\epsilon = -1$, solutions to (7) exists for all time. We may expect to find similar behavior for the Schrödinger-Debye equations. In this direction we obtain results concerning local and global well-posedness for initial data $(u_0, v_0) \in H^s(\mathbb{R}) \times H^{s-1/2+\delta}(\mathbb{R})$ with $1/2 \le s \le 1$ improving the results in [10]. The conservation law in L^2 for the solution u of the IVP (6),

$$\int |u(x,t)|^2 dx = \int |u_0(x)|^2 dx, \tag{8}$$

is the main argument used to obtain our global results.

In chapter 3 we study the Cauchy problem associated to the most typical case in the theory of wave interaction

$$\begin{cases} i\partial_t u + \partial_x^2 u = \alpha u\eta + \beta|u|^2 u, \quad t, x \in \mathbb{R}, \\ \partial_t \eta + \lambda \partial_x \eta = \gamma \partial_x |u|^2, \\ u(x,0) = u_0, \eta(x,0) = \eta_0, \end{cases} \tag{9}$$

where u is complex valued function, η is a real valued function, $\lambda = \pm 1$ and α, β and γ are real constants.

This system appears in general theory of water wave interaction in a nonlinear medium and was introduced by Benney [4, 5]. The solvability of the system (9) has been studied by several authors. Yajima and Oikawa [49] applied the inverse scattering method and found N-soliton solutions of (9) when $\lambda = 1$, $\gamma = -1$ and $\beta = 0$. Ma [39] proposed a simpler approach of the inverse scattering method. Laurençot [38] considered the orbital stability for a weak solution in $H^1(\mathbb{R})$ with $\beta = 0$. Tsutsumi and Hatano [45] showed local well-posedness for a resonant case ($\lambda = 0$) in $H^{k+1/2}(\mathbb{R}) \times H^k(\mathbb{R})$ with $k = 0$ for $\beta = 0$ and with k positive integer, for $\beta \neq 0$. They also obtained global well-posedness in similar spaces for $\lambda = 0$ and $\alpha = \gamma = 1$ via the conservation laws

$$I_1(t) = \int_{-\infty}^{+\infty} |u(x,t)|^2 dx = I_1(0), \tag{10}$$

$$I_2(t) = \int_{-\infty}^{+\infty} \big(\, \eta(x,t)|u(x,t)|^2 + |u_x(x,t)|^2 + \tfrac{\beta}{2}|u(x,t)|^4 \,\big)\, dx = I_2(0), \tag{11}$$

and

$$I_3(t) = \int_{-\infty}^{+\infty} \big(\, \eta^2(x,t) + 2Im(u(x,t)\overline{u_x(x,t)} \,\big) dx = I_3(0). \tag{12}$$

Moreover using Gauge transformation they also extended these results to the case when the system is not necessarily resonant [46]. Bekiranov, Ogawa and Ponce [1] showed well-posedness for initial data $(u_0, \eta_0) \in H^k(\mathbb{R}) \times H^{k-1/2+\epsilon}(\mathbb{R})$ with $1/2 \leq k < 1$ and $\epsilon > 0$ when $\beta \neq 0$ and $(u_0, \eta_0) \in H^k(\mathbb{R}) \times L^{1/k}(\mathbb{R})$ with $0 < k < 1/2$ when $\beta = 0$. The best result obtained for local well-posedness is in $L^2(\mathbb{R}) \times H^{-1/2}(\mathbb{R})$, this result has been proved recently by Ginibre, Tsutsumi and Velo [24] and Bekiranov, Ogawa and Ponce [3].

In this work, we give an example to show that the IVP (9) is ill-posed in $H^k \times H^l$ with $-1/3 \leq k < 0$ and $k(2l+3)+1 \geq 0$ in the focusing case †, which justifies, in this case, that $L^2(\mathbb{R}) \times H^{-1/2}(\mathbb{R})$ is the best result for local well-posedness as was suggested by Bekiranov, Ogawa and Ponce in [3]. The proof of this result is based on the ideas used by Kenig, Ponce and Vega [37] to show ill-posedness for the nonlinear Schrödinger, Korteweg de Vries and modified Korteweg-de Vries equations (see also Biagioni and Linares [7, 8]). The notion of local well-posedness used in the proof of the above result is the following: the existence, uniqueness, persistence property and instead of continuous dependence of the solution upon data we will require the mapping data $(u_0, \eta_0) \longmapsto (u(t), \eta(t))$, be uniformly continuous, where $(u(t), \eta(t))$ is the solution associated to the initial value problem. In the case when this last requirement is not satisfied we will say that the IVP is ill-posed.

† Similar to the theory of the cubic NLS we say the Benney system is "focusing" in the case $\beta < 0$.

Notations

- $\mathbb{N}$ (natural numbers)
- $\mathbb{R}$ (real numbers)
- $\mathbb{C}$ (complex numbers)
- $\partial_x^k u$ or $u_{x\dots x}$ (partial derivate of u in the variable x of order k)
- $B_r(x) = \{y \in \mathbb{R} : |y| \leq r\}$
- $\mathcal{F}(u)(\xi) := \hat{f}(\xi) := (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ix\xi} u(x) dx$ (Fourier transform of u)
- $\mathcal{F}^{-1}(u)(x) := \check{f}(x) := (2\pi)^{-1/2} \int_{\mathbb{R}} e^{ix\xi} u(\xi) dx$ (inverse Fourier transform of u)
- $D_x^s f := c_s \mathcal{F}^{-1}(|\xi|^s \mathcal{F}(f)(\xi))$ (Riesz potential)
- $Hf := \mathcal{F}^{-1}(-i\ sgn(\xi) \mathcal{F}(f)(\xi))$ (Hilbert transform)
- $S(\mathbb{R}^n)$ (Schwartz space on $\mathbb{R}^n$)
- $\|f\|_s := \left(\int (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi \right)^{1/2}$
- $H^s(\mathbb{R}) := H^s$ (Sobolev space of order s with norm $\|f\|_s$)
- $C([0,T] : X)$ (continuous functions from $[0,T]$ into X)

- $\|f\|_{L^p(X,\mu)} := \left(\int_X |f|^p d\mu\right)^{1/p}$
- $\|f\|_{L^p} := \|f\|_{L^p(\mathbb{R})}$
- $\|f\|_{L^q_T L^p_x} = \left(\int_0^T \|f(\cdot,t)\|^q_{L^p} dt\right)^{1/q}$ ($\|f\|_{L^q_t L^p_x} := \|f\|_{L^q_T L^p_x}$ if $T = \infty$)
- $\|f\|_{L^p_x L^q_T} = \left\| \left(\int_0^T |f(\cdot,t)|^q dt\right)^{\frac{1}{q}} \right\|_{L^p}$ ($\|f\|_{L^p_x L^q_t} := \|f\|_{L^p_x L^q_T}$ if $T = \infty$)
- χ_I (characteristic function of the set I)
- $a+$ (number slightly larger than a)
- $f(x) \lesssim g(x)$ (exists constant $c > 0$ such that $f(x) \leq cg(x)$ for all x)
- $f(x) \simeq g(x)$ ($f(x) \lesssim g(x)$ and $g(x) \lesssim f(x)$)

Chapter 1

Local and global theory of a coupled Schrödinger-Korteweg-de Vries equation

We consider de Cauchy problem for a coupled Schrödinger-KdV equation

$$\begin{cases} i\partial_t u + \partial_x^2 u = \alpha u v + \beta |u|^2 u, & t, x \in \mathbb{R}, \\ \partial_t v + \partial_x^3 v + \frac{1}{2}\partial_x(v^2) = \gamma \partial_x(|u|^2), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is a complex-valued function, $v = v(x,t)$ is a real-valued function and $\alpha,\ \beta,\ \gamma$ are real constants.

Before stating the results we give the following notations. Let $U(t) = e^{it\partial_x^2}$ and $V(t) = e^{-t\partial_x^3}$ be the unitary groups associated with the linear Schrödinger and the linear KdV equations respectively. Now we introduce the function spaces for constructing the local solutions. For $s \in \mathbb{R}$ and $b \in (0,1)$ we let $X^{s,b}$ and $Y^{l,b}$ be the completion of $S(\mathbb{R}^2)$ with respect to norms

$$\begin{aligned}\|f\|_{X^{s,b}} &= \left(\iint \langle\xi\rangle^{2s}\langle\tau+\xi^2\rangle^{2b}\big|\widehat{f}(\tau,\xi)\big|^2 d\tau d\xi\right)^{\frac{1}{2}}\\ &= \|U(-t)f\|_{H_t^b(\mathbb{R},H_x^s)},\end{aligned}$$

$$\begin{aligned}\|g\|_{Y^{l,b}} &= \left(\iint \langle\xi\rangle^{2l}\langle\tau-\xi^3\rangle^{2b}\big|\widehat{g}(\tau,\xi)\big|^2 d\tau d\xi\right)^{\frac{1}{2}}\\ &= \|V(-t)g\|_{H_t^b(\mathbb{R},H_x^l)},\end{aligned}$$

where $\langle\cdot\rangle = 1+|\cdot|$ and $\widehat{f}$ denote the Fourier transform of f in both x and t variables

$$\widehat{f}(\tau,\xi) = (2\pi)^{-1}\iint_{\mathbb{R}^2} e^{-it\tau-ix\xi} f(t,x)dtdx.$$

In what follows ψ denotes a cut off function in $C_0^\infty(\mathbb{R})$ such that $0\le\psi(t)\le 1$,

$$\psi(t) = \begin{cases} 1 & \text{if } |t|\le 1,\\ 0 & \text{if } |t|\ge 2,\end{cases}$$

and let $\psi_T(t) := \psi(\frac{t}{T})$ for $0\le T\le 1$. Various constants are denoted by C.

In [2] Bekiranov, Ogawa and Ponce showed the following local well-posedness result regarding IVP (1.1)

Theorem 1.1 *For any $s\ge 0$, $(u_0,v_0)\in H^s(\mathbb{R})\times H^{s-1/2}(\mathbb{R})$ and $b\in(1/2,7/12)$, there exist $T=T(\|u_0\|_s,\|v_0\|_{s-1/2})>0$ and a unique solution $(u(t),v(t))$ of the initial value problem (1.1), satisfying*

$$\psi_T(t)u\in X^{k,b} \quad \textit{and} \quad \psi_T(t)v\in Y^{s-1/2,b}, \tag{1.2}$$

$$u\in C([0,T]:H^s(\mathbb{R})) \quad \textit{and} \quad v\in C([0,T]:H^{s-1/2}(\mathbb{R})). \tag{1.3}$$

Moreover, the map $(u_0,v_0)\longrightarrow(u(t),v(t))$ is locally Lipschitz from $H^s(\mathbb{R})\times H^{s-1/2}(\mathbb{R})$ into $C([0,T]:H^s(\mathbb{R}))\times C([0,T]:H^{s-1/2}(\mathbb{R}))$.

We note that the best result given by the above theorem is in $L^2(\mathbb{R}) \times H^{-1/2}(\mathbb{R})$. Our purpose here is to establish local well-posedness in $L^2(\mathbb{R}) \times H^{-3/4+\epsilon}(\mathbb{R})$. The method of the proof we will use to obtain our results will be a combination of estimates and the contraction mapping principle . We follow analogous argument introduced by Bourgain [13, 14] to study the KdV and NLS equations in the periodic case; extensively improved by Kenig-Ponce-Vega [34, 35, 36] to establish their results for the KdV and NLS equations and by Ginibre-Tsutsumi-Velo [24] for the Zakharov system.

The main ingredient is the use of space time weighted norms in the phase space to see the smoothing effect of two dispersive linear equations and smoothing effects of the quadratic nonlinearities which is seen as terms of a convolution of weight potentials. Since the quadratic nonlinearities can be written as a form of convolution and the different nature of each characteristic of the linear part of the Schrödinger and KdV equations, we are able to avoid the difficulty of derivative loss which commonly appears to construct weak solution.

1.1 Main results

In this section we give the statements of our main results concerning well-posedness for the IVP (1.1).

The local well-posedness result is as follows:

Theorem 1.2 *For any* $(u_0, v_0) \in H^k(\mathbb{R}) \times H^l(\mathbb{R})$ *with* $k \geq 0$ *and* $l > -3/4$, *provided:*

(i) $k - 1 \leq l \leq 2k - 1/2$ *for* $k \in [0, 1/2]$,

(ii) $k - 1 \leq l < k + 1/2$ *for* $k > 1/2$,

there exist $T = T(\|u_0\|_k, \|v_0\|_l) > 0$ *and a unique solution* $(u(t), v(t))$ *of the initial value problem (1.1), satisfying*

$$\psi_T(t)u \in X^{k,1/2+} \quad and \quad \psi_T(t)v \in Y^{l,1/2+}, \tag{1.4}$$

$$u \in C([0,T] : H^k(\mathbb{R})) \quad and \quad v \in C([0,T] : H^l(\mathbb{R})). \tag{1.5}$$

Moreover, the map $(u_0, v_0) \longrightarrow (u(t), v(t))$ *is locally Lipschitz from* $H^k(\mathbb{R}) \times H^l(\mathbb{R})$ *into* $C([0,T] : H^k(\mathbb{R})) \times C([0,T] : H^l(\mathbb{R}))$.

The key point in the proof of the Theorem 1.2 is the deduction and use of new bilinear estimates in Section 1.3 for the coupling terms in system (1.1).

The following corollary is an immediate consequence of Lemma 1.3 in Section 1.2.

Corollary 1.1 *For any* $(u_0, v_0) \in H^k(\mathbb{R}) \times H^l(\mathbb{R})$ *with* $k \geq 0$ *and* $l \geq 0$*, the solution given by Theorem 1.2 satisfies*

$$\|u\|_{L^r_T L^q_x} < \infty, \quad for \quad 2/r = 1/2 - 1/q, \quad q \in [2, +\infty], \tag{1.6}$$

$$\|v\|_{L^r_T L^q_x} < \infty, \quad for \quad 3/r = 1/2 - 1/q, \quad q \in [2, +\infty]. \tag{1.7}$$

Concerning global well-posedness we have the following result

Theorem 1.3 *Let* $\alpha,\ \gamma \in \mathbb{R}$ *with* $\alpha\gamma > 0$*. Then for* $(u_0, v_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ *the unique solution provided by Theorem 1.2 extends to any time interval* $[0,T]$

Remark 1.1 *Figure 1.1 shows the region* $\mathcal{R}$ *of indices* (k,l) *for which local well-posedness is achieved in Theorem 1.2. This region contains the line* $r : l = k - 1/2$ *with* $k \geq 0$ *corresponding to the results proved by Bekiranov, Ogawa and Ponce in [2]. Moreover, the results corresponding to the line segment* $[p_0, p_1) = \{(0,l) : -3/4 < l \leq -1/2\}$*, improve the results in [2].*

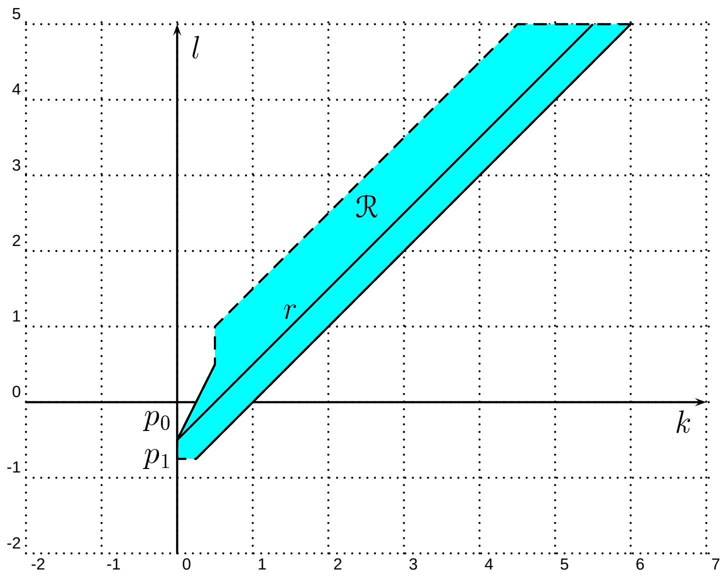

Figure 1.1: Well-posedness' region for IVP (1.1).

Remark 1.2 *If $u \equiv 0$, the system (1.1) becomes into the single KdV equation which is ill-posed in H^l with $-1 \leq l < -3/4$ (see [17]). This observation and the fact that the best result for local well-posedness for the cubic NLS equation is for data in $L^2(\mathbb{R})$ (see [47, 37, 17]) suggests us that Theorem 1.2 in some sense is the best possible except for the limit case, $(k, l) = (0, -3/4)$, which remains open.*

1.2 Preliminary results

We consider the equation of the form

$$i\partial_t \omega - \phi(-i\partial_x)\omega = F(\omega), \tag{1.8}$$

where ϕ is a measurable real-valued function and F some nonlinear function.

The Cauchy Problem for (1.8) with initial data $\omega(0) = \omega_0$ is rewritten as the integral equation

$$\omega(t) = W_\phi(t)\omega_0 - i\int_0^t W_\phi(t-t')F(\omega(t'))dt', \tag{1.9}$$

where $W_\phi(t) = e^{-it\phi(-i\partial_x)}$ is the unitary group that solves the linear part of the equation (1.8).

Let $X_\phi^{s,b}$ be the completion of $S(\mathbb{R}^2)$ with respect to norm

$$\begin{aligned}\|f\|_{X_\phi^{s,b}} &\equiv \|W_\phi(-t)f\|_{H_t^b(\mathbb{R},H_x^s)}\\ &= \|\langle\xi\rangle^s\langle\tau\rangle^b\mathcal{F}(e^{it\phi(-i\partial_x)}f)(\tau,\xi)\|_{L^2_\tau L^2_\xi}\\ &= \|\langle\xi\rangle^s\langle\tau+\phi(\xi)\rangle^b\widehat{f}(\tau,\xi)\|_{L^2_\tau L^2_\xi}.\end{aligned}$$

The following lemma has been proved while establishing the local well-posedness of the Zakharov system by Ginibre, Tsutsumi and Velo in [24].

Lemma 1.1 *Let $-1/2 < b' \le 0 \le b \le b'+1$ and $T \in [0,1]$, then for $F \in X_\phi^{s,b'}$ we have*

$$\|\psi_1(t)W_\phi(t)\omega_0\|_{X_\phi^{s,b}} \le C\|\omega_0\|_{H^s}, \tag{1.10}$$

$$\|\psi_T(t)\int_0^t W_\phi(t-t')F(t',\cdot)dt'\|_{X_\phi^{s,b}} \le CT^{1-b+b'}\|F\|_{X_\phi^{s,b'}}. \tag{1.11}$$

Proof. See Lemma 2.1 in [24]. □

In our case we shall use the spaces $X_\phi^{s,b}$ for the phase functions $\phi_1(\xi) = \xi^2$ and $\phi_2(\xi) = -\xi^3$. Indeed we can rewrite the system (1.1) in the form

$$\begin{cases} i\partial_t u - \phi_1(-i\partial_x)u = \alpha uv + \beta|u|^2u,\\ i\partial_t v - \phi_2(-i\partial_x)v = i\gamma\partial_x(|u|^2) - i\frac{1}{2}\partial_x(v^2).\end{cases} \tag{1.12}$$

Then we have

$$X_{\phi_1}^{k,b} = X^{k,b}, \quad W_{\phi_1} = U(t) = e^{it\partial_x^2},$$

and

$$X^{l,b}_{\phi_2} = Y^{l,b}, \quad W_{\phi_2} = V(t) = e^{-t\partial_x^3},$$

where $U(t)$ and $V(t)$ are the linear Schrödinger and Airy unitary groups, respectively.

On the other hand, if $b > 1/2$, Sobolev lemma implies

$$X^{k,b} \subset C(\mathbb{R} : H^k_x(\mathbb{R})) \quad \text{and} \quad Y^{l,b} \subset C(\mathbb{R} : H^l_x(\mathbb{R})). \tag{1.13}$$

Now, we give some well known Strichartz type estimates for the Schrödinger and KdV linear equations in terms of $X^{s,b}$-norms and $Y^{s,b}$-norms respectively.

Lemma 1.2 *For $b > 1/2$, the following estimates hold:*

$$\|f\|_{L^r_T L^q_x} \le C\|f\|_{X^{0,b}}, \quad for \quad 2/r = 1/2 - 1/q, \quad q \in [2, +\infty], \tag{1.14}$$

$$\|f\|_{X^{0,-b}} \le C\|f\|_{L^{r'}_T L^{q'}_x}, \tag{1.15}$$

and

$$\|g\|_{L^r_T L^q_x} \le C\|g\|_{Y^{0,b}}, \quad for \quad 3/r = 1/2 - 1/q, \quad q \in [2, +\infty], \tag{1.16}$$

$$\le C\|g\|_{Y^{0,-b}} \le C\|g\|_{L^{r'}_T L^{q'}_x}, \tag{1.17}$$

where $1/r + 1/r' = 1$ and $1/q + 1/q' = 1$.

Proof. See [24], Lemma 2.4 with $\nu = 1$, for the proof of (1.14). Using duality we have (1.15). For the proof of (1.16) and (1.17), see Lemma 2 in [27] plus duality. □

The next inequalities will be used to estimate the nonlinear terms in Section 1.3.

Lemma 1.3 *For $p, q > 0, r = \min\{p, q\}$ with $p+q > 1+r,\ s > 1$ and $t > 1/3$ there exists $C > 0$ such that*

$$\int_{-\infty}^{\infty} \frac{dx}{(\mu + |x-a|)^p(\mu + |x-b|)^q} \leq C \frac{\mu^{(1+r-p-q)}}{(\mu + |a-b|)^r}, \quad for \quad \mu > 0, \tag{1.18}$$

$$\int_{-\infty}^{\infty} \frac{dx}{\langle ax - b\rangle^s} \leq \frac{C}{|a|}, \quad for \quad a \neq 0, \tag{1.19}$$

$$\int_{-\infty}^{\infty} \frac{dx}{\langle a_0 + a_1 x + a_2 x^2 + x^3\rangle^t} \leq C. \tag{1.20}$$

Proof. The inequalities (1.18) with $\mu = 1$ and (1.19) follow from simple calculus. They are given in [2, 3, 35]. We prove (1.18) for any $\mu > 0$ as follows

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx}{(\mu + |x-a|)^p(\mu + |x-b|)^q} &= \int_{-\infty}^{\infty} \frac{\mu\, dy}{(\mu + |\mu y-a|)^p(\mu + |\mu y-b|)^q} \\
&= \mu^{1-p-q} \int_{-\infty}^{\infty} \frac{dy}{(1 + |y-\mu^{-1}a|)^p(1 + |y-\mu^{-1}b|)^q} \\
&\leq \mu^{1-p-q} \frac{C}{(1 + \mu^{-1}|a-b|)^r} = C \frac{\mu^{1+r-p-q}}{(\mu + |a-b|)^r},
\end{aligned}$$

where in the last inequality we have used the case $\mu = 1$.

Finally, for the proof of (1.20) we can see Lemma 2.5 in [2]. □

We use the generalization of (1.18) with $\mu > 0$ in the proof of Lemma 1.8 in next section.

The following result well be useful

Lemma 1.4 *Let $x,\ y > 0,\ \beta > 0$ and $\alpha \in [0, 2)$. Then for $\eta > 0$ there exists $c(\eta) > 0$ such that*

$$x^\alpha y^\beta \leq \eta x^2 + c(\eta) y^\gamma \tag{1.21}$$

where

$$\gamma = \frac{2\beta}{2-\alpha} \quad and \quad c(\eta) = \frac{2-\alpha}{2}\left(\frac{\alpha}{2\eta}\right)^{\frac{\alpha}{2-\alpha}}. \tag{1.22}$$

Finally, we recall here the well-known Gagliardo-Nirenberg inequality

Proposition 1.1 *Let $q,\ r \in [1,\infty]$ and $j,\ m \in \mathbb{N} \cup \{0\}$, such that $0 \leq j < m$. Then*

$$\|\partial_x^j u\|_{L^p} \leq C(j,m,q,r,\theta)\|\partial_x^m u\|_{L^r}^{\theta}\|u\|_{L^2}^{1-\theta} \tag{1.23}$$

for all $\theta \in \left[\frac{j}{m}, 1\right]$, where $\frac{1}{p} = j + \theta(\frac{1}{r} - m) + (1-\theta)\frac{1}{q}$.

Proof. See Friedman [19]. □

1.3 Nonlinear estimates

Here we give estimates for the nonlinear terms that are needed in the proof of Theorem 1.2. We begin with the cubic nonlinear term.

Lemma 1.5 *Let $u, \tilde{u} \in X^{k,b}$ with $b \in (1/2, 1)$ and $k \geq 0$. Then for $a \geq 0$ we have that*

$$\||u|^2 u\|_{X^{k,-a}} \leq C\|u\|_{X^{k,b}}^3, \tag{1.24}$$

$$\||u|^2 u - |\tilde{u}|^2\tilde{u}\|_{X^{k,-a}} \leq C(\|u\|_{X^{k,b}}^2 + \|\tilde{u}\|_{X^{k,b}}^2)\|u - \tilde{u}\|_{X^{k,b}}. \tag{1.25}$$

Proof. See Lemma 3.1 in [3]. □

The following lemma is due to Kenig, Ponce and Vega in [35], (see also Bourgain [14]).

Lemma 1.6 *Let $v, \tilde{v} \in Y^{l,b}$. Then there exist $C > 0$ such that*

$$\|\partial_x(v^2)\|_{Y^{l,-a}} \leq C\|v\|_{Y^{l,b}}^2, \tag{1.26}$$

$$\|\partial_x(v^2) - \partial_x(\tilde{v}^2)\|_{Y^{l,-a}} \leq C(\|v\|_{Y^{l,b}} + \|\tilde{v}\|_{Y^{l,b}})\|v - \tilde{v}\|_{Y^{l,b}}, \tag{1.27}$$

hold in the following cases:

(i) $1-a-b \leq \min\{-(2l+1)/2, (4l+3)/12\}$, $b \in (1/2, 1-a]$ *with* $l \in (-3/4, -1/2)$,

(ii) $a \in (5/12, 1/2)$, $b \in (1/2, 7/12)$ *with* $l \in [-1/2, 0)$,

(iii) $a \in [1/4, 1/2)$, $b \in (1/2, 1-a]$ *with* $l \geq 0$.

Proof. See references [34] and [35]. □

Next we prove new bilinear estimates for the interaction terms. Our results improve the estimates given by Bekiranov, Ogawa and Ponce for these terms in [2].

Lemma 1.7 *Let* $u, \tilde{u} \in X^{k,b}$ *and* $v, \tilde{v} \in Y^{l,b}$ *with* $b > 1/2$, $k \geq 0$ *and* $a \in (1/6, 1/2)$. *Then for* $k - l \leq \min\{1, 3a\}$ *there is a constant* $C = C(a, b, k, l) > 0$ *such that*

$$\|uv\|_{X^{k,-a}} \leq C\|u\|_{X^{k,b}}\|v\|_{Y^{l,b}}, \tag{1.28}$$

$$\|uv - \tilde{u}\tilde{v}\|_{X^{k,-a}} \leq C(\|u - \tilde{u}\|_{X^{k,b}}\|v\|_{Y^{l,b}} + \|\tilde{u}\|_{X^{k,b}}\|v - \tilde{v}\|_{Y^{l,b}}). \tag{1.29}$$

Lemma 1.8 *Let* $u_j, \tilde{u}_j \in X^{k,b}, j = 1, 2$, *with* $b > 1/2$ *and* $k \geq 0$. *Then for* $0 \leq a \leq b$ *there is a constant* $C = C(a, b, k, l) > 0$ *such that*

$$\|\partial_x(u_1\overline{u_2})\|_{Y^{l,-a}} \leq C\|u_1\|_{X^{k,b}}\|u_2\|_{X^{k,b}}, \tag{1.30}$$

$$\|\partial_x(|u_1|^2) - \partial_x(|\tilde{u}_1|^2)\|_{Y^{l,-a}} \leq C(\|u_1\|_{X^{k,b}} + \|\tilde{u}_1\|_{X^{k,b}})\|u_1 - \tilde{u}_1\|_{X^{k,b}}), \tag{1.31}$$

hold in the following cases:

(i) $l - 2k \leq \min\{3a - 2b - 1/2, -1/2\}$, *for* $k \in [0, 1/2]$,

(ii) $l - k \leq 3a - b - 1/2$, *for* $k > 1/2$ *and* $b \in (1/2, k]$.

Remark 1.3 *The estimates (1.29) and (1.31) can be deduced from the same argument used to show estimates (1.28) and (1.30) respectively.*

Now we give the proofs of the statements above. We follow closely the argument in [2].

Proof of Lemma 1.7: We let

$$f(\tau,\xi)=\langle\tau+\xi^2\rangle^b\langle\xi\rangle^k\widehat{u}(\tau,\xi)\quad\text{and}\quad g(\tau,\xi)=\langle\tau-\xi^3\rangle^b\langle\xi\rangle^l\widehat{v}(\tau,\xi)$$

to obtain the folowing:

$$\begin{aligned}
\|uv\|_{X^{k,-a}}&=\|\langle\tau+\xi^2\rangle^{-a}\langle\xi\rangle^k\widehat{uv}(\tau,\xi)\|_{L^2_\tau L^2_\xi}\\
&=\Big\|\frac{\langle\xi\rangle^k}{\langle\tau+\xi^2\rangle^a}\widehat{u}*\widehat{v}(\tau,\xi)\Big\|_{L^2_\tau L^2_\xi}\\
&=\sup_{\|\varphi\|_{L^2_{\tau,\xi}}\le 1}\left|\left\langle\!\!\left\langle\frac{\langle\xi\rangle^k}{\langle\tau+\xi^2\rangle^a}\widehat{u}*\widehat{v},\varphi\right\rangle\!\!\right\rangle\right|\\
&:=\sup_{\|\varphi\|_{L^2_{\tau,\xi}}\le 1}|W(u,v,\varphi)|
\end{aligned}$$

where

$$\begin{aligned}
W(u,v,\varphi)&=\left\langle\!\!\left\langle\frac{\langle\xi\rangle^k}{\langle\tau+\xi^2\rangle^a}\left(\frac{f}{\langle\tau+\xi^2\rangle^b\langle\xi\rangle^k}*\frac{g}{\langle\tau-\xi^3\rangle^b\langle\xi\rangle^l}\right),\varphi\right\rangle\!\!\right\rangle\\
&=\iiiint_{\mathbb{R}^4}\frac{\langle\tau+\xi^2\rangle^{-a}\langle\xi\rangle^k g(\tau_1,\xi_1)f(\tau-\tau_1,\xi-\xi_1)\overline{\varphi}(\tau,\xi)}{\langle\tau_1-\xi_1^3\rangle^b\langle\xi_1\rangle^l\langle\tau-\tau_1+(\xi-\xi_1)^2\rangle^b\langle\xi-\xi_1\rangle^k}d\tau_1d\xi_1d\tau d\xi\\
&=\iiiint_{\mathcal{R}_1}+\iiiint_{\mathcal{R}_2}+\iiiint_{\mathcal{R}_3}\equiv I_1\quad+\quad I_2\quad+\quad I_3,
\end{aligned}$$

with $\mathbb{R}^4=\mathcal{R}_1\cup\mathcal{R}_2\cup\mathcal{R}_3$ and $\mathcal{R}_i,\ i=1,2,3$, are defined as follows.

First we split $\mathbb{R}^4$ into three regions $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$,

$$\begin{aligned}
\mathcal{A}&=\big\{(\tau,\tau_1,\xi,\xi_1)\in\mathbb{R}^4:\ |\xi_1|\le 2\big\},\\
\mathcal{B}&=\big\{(\tau,\tau_1,\xi,\xi_1)\in\mathbb{R}^4:\ |\xi_1|>2\text{ and }|3\xi_1^2-2\xi_1+2\xi|\ge\tfrac12|\xi_1|^2\big\},\\
\mathcal{C}&=\big\{(\tau,\tau_1,\xi,\xi_1)\in\mathbb{R}^4:\ |\xi_1|>2\text{ and }|\xi_1^2-\xi_1+2\xi|\ge\tfrac12|\xi_1|^2\big\}.
\end{aligned}$$

Since

$$\mathcal{D}=\big\{(\tau,\tau_1,\xi,\xi_1)\in\mathbb{R}^4:|3\xi_1^2-2\xi_1+2\xi|<\tfrac12|\xi_1|^2,\ |\xi_1^2-\xi_1+2\xi|<\tfrac12|\xi_1|^2\text{ and }|\xi_1|>2\big\}$$

is empty, we have that $\mathbb{R}^4 = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$. Indeed if $(\tau, \tau_1, \xi, \xi_1) \in \mathcal{D}$, then

$$|\xi_1|^2 > |3\xi_1^2 - 2\xi_1 + 2\xi| + |\xi_1^2 - \xi_1 + 2\xi| \geq |2\xi_1^2 - \xi_1| = |\xi_1||2\xi_1 - 1|$$

and hence $|\xi_1| > |2\xi_1 - 1|$, which is a contradiction with the condition $|\xi_1| > 2$.

Note that for any points in $\mathcal{C}$ we have that

$$|\tau + \xi^2| + |\tau_1 - \xi_1^3| + |\tau - \tau_1 + (\xi - \xi_1)^2| \geq |\xi_1^3 - \xi_1^2 + 2\xi\xi_1| \geq \tfrac{1}{2}|\xi_1|^3. \tag{1.32}$$

Now we separate $\mathcal{C}$ into three parts,

$$\begin{aligned}
\mathcal{C}_1 &= \big\{(\tau, \tau_1, \xi, \xi_1) \in \mathcal{C} : \ |\tau_1 - \xi_1^3|,\ |\tau - \tau_1 + (\xi - \xi_1)^2| \leq |\tau + \xi^2|\big\},\\
\mathcal{C}_2 &= \big\{(\tau, \tau_1, \xi, \xi_1) \in \mathcal{C} : \ |\tau + \xi^2|,\ |\tau - \tau_1 + (\xi - \xi_1)^2| \leq |\tau_1 - \xi_1^3|\big\},\\
\mathcal{C}_3 &= \big\{(\tau, \tau_1, \xi, \xi_1) \in \mathcal{C} : \ |\tau + \xi^2|,\ |\tau_1 - \xi_1^3| \leq |\tau - \tau_1 + (\xi - \xi_1)^2|\big\},
\end{aligned}$$

so that one of the following $|\tau + \xi^2|$, $|\tau_1 - \xi_1^3|$ or $|\tau - \tau_1 + (\xi - \xi_1)^2|$ is larger than $\frac{1}{6}|\xi_1|^3$.

We can now define the sets $\mathcal{R}_i$, $i = 1, 2, 3$, as follows :

$$\mathcal{R}_1 = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}_1, \quad \mathcal{R}_2 = \mathcal{C}_2, \quad \mathcal{R}_3 = \mathcal{C}_3$$

and it is clear that $\mathbb{R}^4 = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$.

Now to estimate I_1 we integrate over τ_1 and ξ_1 first and use Cauchy-Schwarz's and Hölder's inequality to obtain

$$\begin{aligned}
|I_1|^2 &\leq \|\varphi\|^2_{L^2_\tau L^2_\xi} \left\| \frac{\langle \xi \rangle^k}{\langle \tau + \xi^2 \rangle^a} \iint \frac{g(\tau_1, \xi_1) f(\tau - \tau_1, \xi - \xi_1) \chi_{\mathcal{R}_1} d\tau_1 d\xi_1}{\langle \tau_1 - \xi_1^3 \rangle^b \langle \xi_1 \rangle^l \langle \tau - \tau_1 + (\xi - \xi_1)^2 \rangle^b \langle \xi - \xi_1 \rangle^k} \right\|^2_{L^2_\tau L^2_\xi}\\
&\leq \iint \frac{\langle \xi \rangle^{2k}}{\langle \tau + \xi^2 \rangle^{2a}} \left| \iint \frac{g(\tau_1, \xi_1) f(\tau - \tau_1, \xi - \xi_1) \chi_{\mathcal{R}_1} d\tau_1 d\xi_1}{\langle \tau_1 - \xi_1^3 \rangle^b \langle \xi_1 \rangle^l \langle \tau - \tau_1 + (\xi - \xi_1)^2 \rangle^b \langle \xi - \xi_1 \rangle^k} \right|^2 d\tau d\xi\\
&\leq \iint \frac{\langle \xi \rangle^{2k}}{\langle \tau + \xi^2 \rangle^{2a}} \Bigg(\iint \frac{\chi_{\mathcal{R}_1} d\tau_1 d\xi_1}{\langle \tau_1 - \xi_1^3 \rangle^{2b} \langle \xi_1 \rangle^{2l} \langle \tau - \tau_1 + (\xi - \xi_1)^2 \rangle^{2b} \langle \xi - \xi_1 \rangle^{2k}} \times\\
&\quad \times \iint |g(\tau_1, \xi_1)|^2 |f(\tau - \tau_1, \xi - \xi_1)|^2 d\tau_1 d\xi_1 \Bigg) d\tau d\xi
\end{aligned} \tag{1.33}$$

$$
\begin{aligned}
&\le \|f\|^2_{L^2_\tau L^2_\xi}\|g\|^2_{L^2_{\tau_1}L^2_{\xi_1}}\\
&\quad\times\left\|\frac{\langle\xi\rangle^{2k}}{\langle\tau+\xi^2\rangle^{2a}}\iint\frac{\chi_{\mathcal{R}_1}d\tau_1 d\xi_1}{\langle\tau_1-\xi_1^3\rangle^{2b}\langle\xi_1\rangle^{2l}\langle\tau-\tau_1+(\xi-\xi_1)^2\rangle^{2b}\langle\xi-\xi_1\rangle^{2k}}\right\|_{L^\infty_\tau L^\infty_\xi}\\
&= \|u\|^2_{X^{k,b}}\|v\|^2_{Y^{l,b}}\\
&\quad\times\left\|\frac{\langle\xi\rangle^{2k}}{\langle\tau+\xi^2\rangle^{2a}}\iint\frac{\chi_{\mathcal{R}_1}d\tau_1 d\xi_1}{\langle\tau_1-\xi_1^3\rangle^{2b}\langle\xi_1\rangle^{2l}\langle\tau-\tau_1+(\xi-\xi_1)^2\rangle^{2b}\langle\xi-\xi_1\rangle^{2k}}\right\|_{L^\infty_\tau L^\infty_\xi}.
\end{aligned}
$$

For I_2 we put $\widetilde{f}(\tau,\xi)=f(-\tau,-\xi)$, integrate over τ and ξ first and folow the same steps as above to get

$$
\begin{aligned}
|I_2|^2 &\le \|g\|^2_{L^2_{\tau_1}L^2_{\xi_1}}\left\|\frac{1}{\langle\xi_1\rangle^{l}\langle\tau_1-\xi_1^3\rangle^{b}}\iint\frac{\langle\xi\rangle^{k}\widetilde{f}(\tau_1-\tau,\xi_1-\xi)\overline{\varphi}(\tau,\xi)\chi_{\mathcal{R}_2}d\tau d\xi}{\langle\tau+\xi^2\rangle^{a}\langle\tau-\tau_1+(\xi-\xi_1)^2\rangle^{b}\langle\xi-\xi_1\rangle^{k}}\right\|^2_{L^2_{\tau_1}L^2_{\xi_1}}\\
&\le \|\widetilde{f}\|^2_{L^2_{\tau_1}L^2_{\xi_1}}\|g\|^2_{L^2_{\tau_1}L^2_{\xi_1}}\\
&\quad\times\left\|\frac{1}{\langle\xi_1\rangle^{2l}\langle\tau_1-\xi_1^3\rangle^{2b}}\iint\frac{\langle\xi\rangle^{2k}\chi_{\mathcal{R}_2}d\tau d\xi}{\langle\tau+\xi^2\rangle^{2a}\langle\tau-\tau_1+(\xi-\xi_1)^2\rangle^{2b}\langle\xi-\xi_1\rangle^{2k}}\right\|^2_{L^\infty_{\tau_1}L^\infty_{\xi_1}} \qquad (1.34)\\
&= \|u\|^2_{X^{k,b}}\|v\|^2_{Y^{l,b}}\\
&\quad\times\left\|\frac{1}{\langle\xi_1\rangle^{2l}\langle\tau_1-\xi_1^3\rangle^{2b}}\iint\frac{\langle\xi\rangle^{2k}\chi_{\mathcal{R}_2}d\tau d\xi}{\langle\tau+\xi^2\rangle^{2a}\langle\tau-\tau_1+(\xi-\xi_1)^2\rangle^{2b}\langle\xi-\xi_1\rangle^{2k}}\right\|_{L^\infty_{\tau_1}L^\infty_{\xi_1}}.
\end{aligned}
$$

Note that $\widetilde{f}(\tau,\xi)=\langle\xi\rangle^k\langle\tau-\xi^2\rangle^b\widehat{u}(-\tau,-\xi)$ and $\|\widetilde{f}\|_{L^2_\tau L^2_\xi}=\|f\|_{L^2_\tau L^2_\xi}=\|u\|_{X^{k,b}}$.

Now using the change of variables $\tau=\tau_1-\tau_2$ and $\xi=\xi_1-\xi_2$ the third region, $\mathcal{R}_3$, is transformed into the set $\widetilde{\mathcal{R}}_3$ such that

$$\widetilde{\mathcal{R}}_3\subseteq\left\{(\tau_1,\tau_2,\xi_1,\xi_2)\in\mathbb{R}^4:\ \tfrac{1}{2}|\xi_1|^3\le|\xi_1^3+\xi_1^2-2\xi_1\xi_2|\le 3|\tau_2-\xi_2^2|\quad\text{and}\quad|\xi_1|>2\right\}.$$

Then I_3 can be estimated as follows

$$\begin{aligned}
|I_3|^2 &\leq \|\widetilde{f}\|^2_{L^2_{\tau_2}L^2_{\xi_2}} \\
&\quad \times \left\| \frac{1}{\langle \xi_2\rangle^k \langle \tau_2-\xi_2^2\rangle^b} \iint \frac{\langle \xi_1-\xi_2\rangle^k g(\tau_1,\xi_1)\widetilde{\overline{\varphi}}(\tau_2-\tau_1,\xi_2-\xi_1)\chi_{\widetilde{\mathcal{R}}_3} d\tau_1 d\xi_1}{\langle \tau_1-\tau_2+(\xi_1-\xi_2)^2\rangle^a \langle \tau_1-\xi_1^3\rangle^b \langle \xi_1\rangle^l} \right\|^2_{L^2_{\tau_2}L^2_{\xi_2}} \\
&\leq \|\widetilde{f}\|^2_{L^2_{\tau_2}L^2_{\xi_2}} \|g\|^2_{L^2_{\tau_1}L^2_{\xi_1}} \\
&\quad \times \left\| \frac{1}{\langle \xi_2\rangle^{2k} \langle \tau_2-\xi_2^2\rangle^{2b}} \iint \frac{\langle \xi_1-\xi_2\rangle^{2k}\chi_{\widetilde{\mathcal{R}}_3} d\tau_1 d\xi_1}{\langle \tau_1-\tau_2+(\xi_1-\xi_2)^2\rangle^{2a} \langle \tau_1-\xi_1^3\rangle^{2b} \langle \xi_1\rangle^{2l}} \right\|^2_{L^\infty_{\tau_2}L^\infty_{\xi_2}} \qquad (1.35) \\
&= \|u\|^2_{X^{k,b}} \|v\|^2_{Y^{l,b}} \\
&\quad \times \left\| \frac{1}{\langle \xi_2\rangle^{2k} \langle \tau_2-\xi_2^2\rangle^{2b}} \iint \frac{\langle \xi_1-\xi_2\rangle^{2k}\chi_{\widetilde{\mathcal{R}}_3} d\tau_1 d\xi_1}{\langle \tau_1-\tau_2+(\xi_1-\xi_2)^2\rangle^{2a} \langle \tau_1-\xi_1^3\rangle^{2b} \langle \xi_1\rangle^{2l}} \right\|_{L^\infty_{\tau_2}L^\infty_{\xi_2}}
\end{aligned}$$

Reviewing the estimates (1.33), (1.34) and (1.35) it suffices show that the following expressions are bounded by a constant C:

$$\left\| \frac{\langle \xi\rangle^{2k}}{\langle \tau+\xi^2\rangle^{2a}} \iint \frac{\chi_{\mathcal{R}_1} d\tau_1 d\xi_1}{\langle \tau_1-\xi_1^3\rangle^{2b} \langle \xi_1\rangle^{2l} \langle \tau-\tau_1+(\xi-\xi_1)^2\rangle^{2b} \langle \xi-\xi_1\rangle^{2k}} \right\|_{L^\infty_{\tau}L^\infty_{\xi}}, \qquad (1.36)$$

$$\left\| \frac{1}{\langle \xi_1\rangle^{2l} \langle \tau_1-\xi_1^3\rangle^{2b}} \iint \frac{\langle \xi\rangle^{2k}\chi_{\mathcal{R}_2} d\tau d\xi}{\langle \tau+\xi^2\rangle^{2a} \langle \tau-\tau_1+(\xi-\xi_1)^2\rangle^{2b} \langle \xi-\xi_1\rangle^{2k}} \right\|^2_{L^\infty_{\tau_1}L^\infty_{\xi_1}}, \qquad (1.37)$$

$$\left\| \frac{1}{\langle \xi_2\rangle^{2k} \langle \tau_2-\xi_2^2\rangle^{2b}} \iint \frac{\langle \xi_1-\xi_2\rangle^{2k}\chi_{\widetilde{\mathcal{R}}_3} d\tau_1 d\xi_1}{\langle \tau_1-\tau_2+(\xi_1-\xi_2)^2\rangle^{2a} \langle \tau_1-\xi_1^3\rangle^{2b} \langle \xi_1\rangle^{2l}} \right\|_{L^\infty_{\tau_2}L^\infty_{\xi_2}}. \qquad (1.38)$$

According to (1.18) with $\mu = 1$ in Lemma 1.3, noting that $\langle \xi\rangle \leq \langle \xi_1\rangle\langle \xi-\xi_1\rangle$ and $\langle \xi_1-\xi_2\rangle \leq \langle \xi_1\rangle\langle \xi_2\rangle$ and using that $k \geq 0$ it suffices to get bounds for:

$$J_0(\tau,\xi) = \frac{1}{\langle \tau+\xi^2\rangle^{2a}} \int \frac{\langle \xi_1\rangle^{2k-2l} d\xi_1}{\langle \tau+\xi^2-\xi_1^3+\xi_1^2-2\xi\xi_1\rangle^{2b}} \quad \text{on} \quad \mathcal{R}_1, \qquad (1.39)$$

$$J_1(\tau_1,\xi_1)=\frac{\langle\xi_1\rangle^{2k-2l}}{\langle\tau_1-\xi_1^3\rangle^{2b}}\int\frac{d\xi}{\langle\tau_1-\xi_1^2+2\xi\xi_1\rangle^{2a}}\quad \text{on}\quad \mathcal{R}_2, \tag{1.40}$$

$$J_2(\tau_2,\xi_2)=\frac{1}{\langle\tau_2-\xi_2^2\rangle^{2b}}\int\frac{\langle\xi_1\rangle^{2k-2l}d\xi_1}{\langle\tau_2-\xi_2^2-\xi_1^3-\xi_1^2+2\xi_1\xi_2\rangle^{2a}}\quad \text{on}\quad \widetilde{\mathcal{R}}_3, \tag{1.41}$$

where we have used that $\min\{a,b\}=a$ and $b>1/2$.

We begin estimating J_0 on $\mathcal{R}_1=\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}_1$. In region $\mathcal{A}$, using $|\xi_1|\le 2,\ a>0$ and $b>1/2$ it is easy to see that

$$|J_0(\tau,\xi)|\le C_{k,l}\int\frac{d\xi_1}{\langle\tau+\xi^2-\xi_1^3+\xi_1^2-2\xi\xi_1\rangle^{2b}}\le C_{k,l}. \tag{1.42}$$

In region $\mathcal{B}$, by the change of variables $\eta=\tau+\xi^2-\xi_1^3+\xi_1^2-2\xi\xi_1$ and the condition $|3\xi_1^2-2\xi_1+2\xi|\ge\frac{1}{2}|\xi_1|^2$ we obtain

$$\begin{aligned}|J_0(\tau,\xi)|&\le\frac{1}{\langle\tau+\xi^2\rangle^{2a}}\int\frac{\langle\xi_1\rangle^{2k-2l}}{|3\xi_1^2-2\xi_1+2\xi|\langle\eta\rangle^{2b}}d\eta\\&\le\frac{1}{\langle\tau+\xi^2\rangle^{2a}}\int\frac{\langle\xi_1\rangle^{2k-2l}}{|\xi_1|^2\langle\eta\rangle^{2b}}d\eta\\&\le C.\end{aligned} \tag{1.43}$$

Here we have used $a>0,\ k-l\le\min\{1,3a\},\ |\xi_1|>2$ and Lemma 1.3-(1.19).

In region $\mathcal{C}_1$, by (1.32) we have that

$$\tfrac{1}{2}|\xi_1|^3\le 3|\tau+\xi^2|<3\langle\tau+\xi^2\rangle$$

and consequently using $a>0$ we obtain

$$\langle\tau+\xi^2\rangle^{-2a}\le C_a|\xi_1|^{-6a}.$$

Then use $k-l\le\min\{1,3a\}$ combined with Lemma 1.3-(1.20) to get

$$\begin{aligned}|J_0(\tau,\xi)|&\le\int\frac{\langle\xi_1\rangle^{2k-2l}}{\langle\tau+\xi^2\rangle^{2a}\langle\tau+\xi^2-\xi_1^3+\xi_1^2-2\xi\xi_1\rangle^{2b}}d\xi_1\\&\le C_a\int\frac{\langle\xi_1\rangle^{2k-2l}}{|\xi_1|^{6a}\langle\tau+\xi^2-\xi_1^3+\xi_1^2-2\xi\xi_1\rangle^{2b}}d\xi_1\\&\le C_a.\end{aligned} \tag{1.44}$$

Next we estimate J_1. Making the change of variables, $\eta = \tau_1 - \xi_1^2 + 2\xi\xi_1$, using (1.32) and the restriction in region $\mathcal{C}_2$, we have

$$|\eta| \leq |\tau_1 - \xi_1^3| + |\xi_1^3 - \xi_1^2 + 2\xi\xi_1| \leq 4|\tau_1 - \xi_1^3| \leq 4\langle \tau_1 - \xi_1^3 \rangle.$$

Moreover in $\mathcal{C}_2$, since $a + b - 1/2 > 0$, we have

$$\tfrac{1}{2}|\xi_1|^3 \leq 3|\tau_1 - \xi_1^3| < 3\langle \tau_1 - \xi_1^3 \rangle$$

and hence

$$\langle \tau_1 - \xi_1^3 \rangle^{-(2a+2b-1)} \leq C_b |\xi_1|^{-3(2a+2b-1)}.$$

Then we can estimate J_1 as follows

$$\begin{aligned}
|J_1(\tau_1, \xi_1)| &\leq \frac{\langle \xi_1 \rangle^{2k-2l}}{\langle \tau_1 - \xi_1^3 \rangle^{2b}} \int_{|\eta| \leq 4\langle \tau_1 - \xi_1^3 \rangle} \frac{(1 + |\eta|)^{-2a}}{2|\xi_1|} d\eta \\
&= \frac{\langle \xi_1 \rangle^{2k-2l}}{\langle \tau_1 - \xi_1^3 \rangle^{2b} |\xi_1|} \frac{1}{(1-2a)} \left(\left(4\langle \tau_1 - \xi_1^3 \rangle \right)^{1-2a} - 1 \right) \\
&\leq C_{a,b} \frac{\langle \xi_1 \rangle^{2k-2l}}{|\xi_1| \langle \tau_1 - \xi_1^3 \rangle^{2a+2b-1}} \\
&\leq C_{a,b} \frac{\langle \xi_1 \rangle^{2k-2l}}{|\xi_1|^{6a+6b-2}} \\
&\leq C_{a,b},
\end{aligned} \tag{1.45}$$

where in the last inequality we have used that $k - l \leq 3a + 3b - 1$ which follows from the conditions $k - l \leq \min\{1, 3a\}$ and $b > 1/2$.

Finally in the region $\widetilde{\mathcal{R}}_3$ we note that

$$\tfrac{1}{2}|\xi_1|^3 < 3\langle \tau_2 - \xi_2^2 \rangle \Longrightarrow \langle \tau_2 - \xi_2^2 \rangle^{-2b} \leq C_b |\xi_1|^{-6b}$$

and from the conditions $k - l \leq \min\{1, 3a\}$ and $1/6 < a < 1/2 < b$ coupled with Lemma 1.3-(1.20), we have that

$$\begin{aligned}
|J_2(\tau_2, \xi_2)| &\leq C_b \int \frac{\langle \xi_1 \rangle^{2k-2l} d\xi_1}{|\xi_1|^{6b} \langle \tau_2 - \xi_2^2 - \xi_1^3 - \xi_1^2 + 2\xi_1\xi_2 \rangle^{2a}} \\
&\leq C_b,
\end{aligned} \tag{1.46}$$

and hence the proof of Lemma 1.7 is completed.

Proof of Lemma 1.8: We let

$$\begin{cases} \tau=\tau_1-\tau_2, \quad \xi=\xi_1-\xi_2, \\ \sigma=\tau-\xi^3, \quad \sigma_1=\tau_1+\xi_1^2, \quad \sigma_2=\tau_2+\xi_2^2, \end{cases} \tag{1.47}$$

and define

$$f(\tau_1,\xi_1)=\langle\xi_1\rangle^k\langle\sigma_1\rangle^b\widehat{u}_1(\tau_1,\xi_1) \quad \text{and} \quad g(\tau_2,\xi_2)=\langle\xi_2\rangle^k\langle\sigma_2\rangle^b\widehat{\overline{u}}_2(-\tau_2,-\xi_2). \tag{1.48}$$

Hence we have

$$\|u_1\|_{X^{k,b}}=\|f\|_{L^2_{\tau_1}L^2_{\xi_1}} \quad \text{and} \quad \|u_2\|_{X^{k,b}}=\|g\|_{L^2_{\tau_2}L^2_{\xi_2}}. \tag{1.49}$$

Now using (1.47), (1.48) and (1.49) we estimate the left hand side of (1.30) as in the proof of Lemma 1.7 to obtain:

$$\begin{aligned} \|\partial_x u_1\overline{u}_2\|_{Y^{l,-a}} &= \|\langle\sigma\rangle^{-a}\langle\xi\rangle^l\widehat{\partial_x(u_1\overline{u}_2)}\|_{L^2_\tau L^2_\xi} \\ &= \|i\xi\langle\sigma\rangle^{-a}\langle\xi\rangle^l\widehat{u}_1*\widehat{\overline{u}}_2(\tau,\xi)\|_{L^2_\tau L^2_\xi} \\ &= \sup_{\|\varphi\|_{L^2_{\tau,\xi}}\le 1}\left|\left\langle\!\!\left\langle \frac{|\xi|\langle\xi\rangle^l}{\langle\sigma\rangle^a}\widehat{u_1}*\widehat{\overline{u}}_2,\varphi\right\rangle\!\!\right\rangle\right| \\ &:= \sup_{\|\varphi\|_{L^2_{\tau,\xi}}\le 1}|W(u_1,u_2,\varphi)| \end{aligned} \tag{1.50}$$

where

$$\begin{aligned} W(u_1,u_2,\varphi) &= \iiiint_{\mathbb{R}^4}\frac{|\xi|\langle\xi\rangle^l}{\langle\sigma\rangle^a}\widehat{u}_1(\tau_1,\xi_1)\widehat{\overline{u}}_2(\tau-\tau_1,\xi-\xi_1)\overline{\varphi}(\tau,\xi)d\tau_1d\xi_1d\tau d\xi \\ &= \iiiint_{\mathbb{R}^4}\frac{|\xi|\langle\xi\rangle^l f(\tau_1,\xi_1)g(\tau_2,\xi_2)\overline{\varphi}(\tau,\xi)}{\langle\sigma\rangle^a\langle\xi_1\rangle^k\langle\sigma_1\rangle^b\langle\xi_2\rangle^k\langle\sigma_2\rangle^b}d\tau_1d\xi_1d\tau d\xi \end{aligned} \tag{1.51}$$

and hence by applying the Cauchy-Schwarz's inequality, we have that

$$
\begin{aligned}
|W(u_1,u_2,\varphi)|^2 &\le \|\varphi\|^2_{L^2_\tau L^2_\xi}\left\| \frac{|\xi|\langle\xi\rangle^l}{\langle\sigma\rangle^a}\iint \frac{f(\tau+\tau_2,\xi+\xi_2)g(\tau_2,\xi_2)}{\langle\xi_1\rangle^k\langle\sigma_1\rangle^b\langle\xi_2\rangle^k\langle\sigma_2\rangle^b}d\tau_2 d\xi_2\right\|_{L^2_\tau L^2_\xi}\\
&\le \|f\|^2_{L^2_\tau L^2_\xi}\|g\|^2_{L^2_{\tau_2}L^2_{\xi_2}}\left\| \frac{|\xi|^2\langle\xi\rangle^{2l}}{\langle\sigma\rangle^{2a}}\iint \frac{d\tau_2 d\xi_2}{\langle\xi_1\rangle^{2k}\langle\sigma_1\rangle^{2b}\langle\xi_2\rangle^{2k}\langle\sigma_2\rangle^{2b}}\right\|_{L^\infty_\tau L^\infty_\xi} \qquad (1.52)\\
&= \|u_1\|^2_{X^{k,b}}\|u_2\|^2_{X^{k,b}}\left\| \frac{|\xi|^2\langle\xi\rangle^{2l}}{\langle\sigma\rangle^{2a}}\iint \frac{d\tau_2 d\xi_2}{\langle\xi_1\rangle^{2k}\langle\sigma_1\rangle^{2b}\langle\xi_2\rangle^{2k}\langle\sigma_2\rangle^{2b}}\right\|_{L^\infty_\tau L^\infty_\xi}.
\end{aligned}
$$

Then it suffices to get bounds for

$$J(\tau,\xi) := \frac{|\xi|^2\langle\xi\rangle^{2l}}{\langle\sigma\rangle^{2a}}\iint \frac{d\tau_2 d\xi_2}{\langle\xi_1\rangle^{2k}\langle\sigma_1\rangle^{2b}\langle\xi_2\rangle^{2k}\langle\sigma_2\rangle^{2b}}.$$

In order to estimate J we consider several cases:

Case-a: $|\boldsymbol{\xi}| < \mathbf{2}$.

Using $k\ge 0$, $b>1/2$, $a\ge 0$, Lemma 1.3-(1.18) with $\mu=1$ and Lemma 1.3-(1.19) we have

$$
\begin{aligned}
J(\tau,\xi) &\le \frac{|\xi|^2\langle\xi\rangle^{2l}}{\langle\sigma\rangle^{2a}}\int d\xi_2\int \frac{d\tau_2}{\langle\tau_2+\tau+(\xi_2+\xi)^2\rangle^{2b}\langle\tau_2+\xi_2^2\rangle^{2b}}\\
&\le C\frac{|\xi|^2\langle\xi\rangle^{2l}}{\langle\sigma\rangle^{2a}}\int \frac{d\xi_2}{\langle\tau+\xi^2+2\xi\xi_2\rangle^{2b}} \qquad (1.53)\\
&\le C\frac{|\xi|^2\langle\xi\rangle^{2l}}{\langle\sigma\rangle^{2a}}\frac{1}{|\xi|}\le C|\xi|\langle\xi\rangle^{2l}\le C_l.
\end{aligned}
$$

Case-b: $|\boldsymbol{\xi}| \ge \mathbf{2}$ and $|\boldsymbol{\xi_1}| \ge \mathbf{2}|\boldsymbol{\xi_2}|$.

In this situation we have

$$|\xi_1|\ge 2|\xi_2| \Longrightarrow |\xi|\le|\xi_1|+|\xi_2|\le\tfrac{3}{2}|\xi_1| \Longrightarrow \langle\xi\rangle^{2k}\le C_k\langle\xi_1\rangle^{2k}. \qquad (1.54)$$

$$|\xi_1|\ge 2|\xi_2| \Longrightarrow |\xi|\ge|\xi_1|-|\xi_2|\ge|\xi_2|. \qquad (1.55)$$

$$|\xi|\ge 2 \Longrightarrow |\xi|^2\le\tfrac{1}{2}|\xi|^3 \Longrightarrow |\xi^3\pm\xi^2|\ge|\xi|^3-|\xi|^2\ge\tfrac{1}{2}|\xi|^3 \Longrightarrow \langle\xi^3\pm\xi^2\rangle\ge\tfrac{1}{2}\langle\xi^3\rangle. \qquad (1.56)$$

$$|\xi| \geq 2 \Longrightarrow \frac{1}{2|\xi|} < 1. \tag{1.57}$$

Now we consider two situations for parameters k and b:

(b_1) $k > 1/2$ and $b \in (1/2, k]$.

Apply (1.18) with $\mu = 1$ in Lemma 1.3 and (1.54) to get

$$\begin{aligned} J(\tau,\xi) &\leq C_k \frac{\langle \xi \rangle^{2l} |\xi|^2 \langle \xi \rangle^{-2k}}{\langle \sigma \rangle^{2a}} \int \frac{d\xi_2}{\langle \xi_2 \rangle^{2k}} \int \frac{d\tau_2}{\langle \tau + \tau_2 + (\xi+\xi_2)^2 \rangle^{2b} \langle \tau_2 + \xi_2^2 \rangle^{2b}} \\ &\leq C_k \frac{\langle \xi \rangle^{2l-2k+2}}{\langle \sigma \rangle^{2a}} \int \frac{d\xi_2}{\langle \xi_2 \rangle^{2k} \langle \tau + \xi^2 + 2\xi\xi_2 \rangle^{2b}} \\ &:= C_k \frac{\langle \xi \rangle^{2l-2k+2}}{\langle \sigma \rangle^{2a}} J^*(\tau,\xi). \end{aligned} \tag{1.58}$$

Now using $1/2 < b \leq k$, Lemma 1.3-(1.18) with $\mu(\xi) = \frac{1}{2|\xi|}$ and (1.57) we compute J^* in the following way

$$\begin{aligned} J^*(\tau,\xi) &= \int \frac{d\xi_2}{(1+|\tau+\xi^2+2\xi\xi_2)^{2b}(1+|\xi_2|)^{2k}} \\ &\leq \int \frac{d\xi_2}{(1+|\tau+\xi^2+2\xi\xi_2|)^{2b}(1+|\xi_2|)^{2b}} \\ &= (2|\xi|)^{-2b} \int \frac{d\xi_2}{\left(\frac{1}{2|\xi|} + |\xi_2 + \frac{1}{2\xi}(\tau+\xi^2)|\right)^{2b} \left(1+|\xi_2|\right)^{2b}} \\ &\leq (2|\xi|)^{-2b} \int \frac{d\xi_2}{\left(\frac{1}{2|\xi|} + |\xi_2 + \frac{1}{2\xi}(\tau+\xi^2)|\right)^{2b} \left(\frac{1}{2|\xi|}+|\xi_2|\right)^{2b}} \\ &\leq C(2|\xi|)^{-2b} \frac{(2|\xi|)^{-(1-2b)}}{\left(\frac{1}{2|\xi|} + \frac{1}{2|\xi|}|\tau+\xi^2|\right)^{2b}} \\ &= C\frac{(2|\xi|)^{2b-1}}{(1+|\tau+\xi^2|)^{2b}} = C\frac{(2|\xi|)^{2b-1}}{\langle \tau+\xi^2 \rangle^{2b}}. \end{aligned} \tag{1.59}$$

Combining (1.58) and (1.59) and using $0 \leq a \leq b$, $\langle \xi^3 + \xi^2 \rangle \leq \langle \tau - \xi^3 \rangle \langle \tau + \xi^2 \rangle$ and

(1.56) we have

$$\begin{aligned}
J(\tau,\xi) &\leq C_{k,b}\frac{\langle\xi\rangle^{2l-2k+2}(2|\xi|)^{2b-1}}{\langle\sigma\rangle^{2a}\langle\tau+\xi^2\rangle^{2b}} \leq C_{k,b}\frac{\langle\xi\rangle^{2l-2k+2b+1}}{\langle\tau-\xi^3\rangle^{2a}\langle\tau+\xi^2\rangle^{2a}}\\
&\leq C_{k,b}\frac{\langle\xi\rangle^{2l-2k+2b+1}}{\langle\xi^3+\xi^2\rangle^{2a}} \leq C_{k,b}\frac{\langle\xi\rangle^{2l-2k+2b+1}}{\langle\xi^3\rangle^{2a}}\\
&\leq C_{k,b}\frac{\langle\xi\rangle^{2l-2k+2b+1}}{\langle\xi\rangle^{6a}}\\
&\leq C_{k,b},
\end{aligned}\tag{1.60}$$

where it was used the inequality $l-k+b+1/2\leq 3a$ which holds from hypothesis (ii).

$\boldsymbol{(b_2)}$ $k\in[0,1/2]$ and $b>1/2$.

We note that in the previous situation to estimate J^* we obtained the following estimate

$$\int\frac{d\xi_2}{(1+|\tau+\xi^2+2\xi\xi_2)^{2b}(1+|\xi_2|)^{2b}} \leq C\frac{(2|\xi|)^{2b-1}}{\langle\tau+\xi^2\rangle^{2b}}.\tag{1.61}$$

On the other hand using $0\leq k\leq 1/2$, $b>1/2$ and (1.55) we have

$$\langle\xi_2\rangle^{2b-2k}\leq\langle\xi\rangle^{2b-2k}.\tag{1.62}$$

Then by (1.61), (1.62) and (1.56) we have

$$\begin{aligned}
J(\tau,\xi) &\leq C_k\frac{\langle\xi\rangle^{2l-2k+2}}{\langle\sigma\rangle^{2a}}\int\frac{d\xi_2}{\langle\xi_2\rangle^{2k}\langle\tau+\xi^2+2\xi\xi_2\rangle^{2b}}\\
&= C_k\frac{\langle\xi\rangle^{2l-2k+2}}{\langle\sigma\rangle^{2a}}\int\frac{\langle\xi_2\rangle^{2b-2k}d\xi_2}{(1+|\tau+\xi^2+2\xi\xi_2)^{2b}(1+|\xi_2|)^{2b}}\\
&\leq C_k\frac{\langle\xi\rangle^{2l-4k+2b+2}}{\langle\tau-\xi^3\rangle^{2a}}\frac{(2|\xi|)^{2b-1}}{\langle\tau+\xi^2\rangle^{2b}}\\
&\leq C_{k,b}\frac{\langle\xi\rangle^{2l-4k+4b+1}}{\langle\xi^3\rangle^{2a}} \leq C_{k,b}\frac{\langle\xi\rangle^{2l-4k+4b+1}}{\langle\xi\rangle^{6a}}\\
&\leq C_{k,b},
\end{aligned}\tag{1.63}$$

where it was used the inequality $l-2k+2b+1/2\leq 3a$ which holds from hypothesis (i).

Case-c: $|\boldsymbol{\xi}| \geq \mathbf{2}$ and $|\boldsymbol{\xi_2}| \geq \mathbf{2}|\boldsymbol{\xi_1}|$.

Clearly, we have (1.56) and (1.57). If we interchange the positions of ξ_1 and ξ_2 in (1.54), (1.55) we get

$$\langle \xi \rangle^{2k} \leq c_k \langle \xi_2 \rangle^{2k}, \tag{1.64}$$

$$|\xi| \geq |\xi_1|. \tag{1.65}$$

Now using $\xi_2 = \xi_1 - \xi$ and $\tau_2 = \tau_1 - \tau$ we have

$$\iint \frac{d\xi_2 d\tau_2}{\langle \xi_1 \rangle^{2k} \langle \sigma_1 \rangle^{2b} \langle \xi_2 \rangle^{2k} \langle \sigma_2 \rangle^{2b}} = \iint \frac{d\xi_1 d\tau_1}{\langle \xi_1 \rangle^{2k} \langle \sigma_1 \rangle^{2b} \langle \xi_2 \rangle^{2k} \langle \sigma_2 \rangle^{2b}}. \tag{1.66}$$

Similarly to case-b we consider two situations for parameters k and b

$(\boldsymbol{c_1})$ $k > 1/2$ and $b \in (1/2, k]$.

Again we apply (1.18) with $\mu = 1$ in Lemma 1.3, (1.64) and (1.66) to get

$$\begin{aligned} J(\tau, \xi) &\leq C_k \frac{\langle \xi \rangle^{2l} |\xi|^2 \langle \xi \rangle^{-2k}}{\langle \sigma \rangle^{2a}} \int \frac{d\xi_1}{\langle \xi_1 \rangle^{2k}} \int \frac{d\tau_1}{\langle \tau_1 - \tau + (\xi_1 - \xi)^2 \rangle^{2b} \langle \tau_1 + \xi_1^2 \rangle^{2b}} \\ &\leq C_k \frac{\langle \xi \rangle^{2l-2k+2}}{\langle \sigma \rangle^{2a}} \int \frac{d\xi_1}{\langle \xi_1 \rangle^{2k} \langle \tau - \xi^2 + 2\xi\xi_1 \rangle^{2b}} \\ &:= C_k \frac{\langle \xi \rangle^{2l-2k+2}}{\langle \sigma \rangle^{2a}} J_*(\tau, \xi). \end{aligned} \tag{1.67}$$

Now using $1/2 < b \leq k$, Lemma 1.3-(1.18) with $\mu(\xi) = \frac{1}{2|\xi|}$ and (1.57) we estimate $J_*(\tau, \xi)$ similarly to $J^*(\tau, \xi)$ in (1.59) to get

$$J_*(\tau, \xi) \leq \frac{C(2|\xi|)^{2b-1}}{\langle \tau - \xi^2 \rangle^{2b}} \tag{1.68}$$

Combining (1.67) and (1.68) and using $0 \leq a \leq b$, $\langle \xi^3 - \xi^2 \rangle \leq \langle \tau - \xi^3 \rangle \langle \tau - \xi^2 \rangle$ and

(1.56) we have

$$
\begin{aligned}
J(\tau,\xi) &\leq C_{k,b}\frac{\langle\xi\rangle^{2l-2k+2}(2|\xi|)^{2b-1}}{\langle\sigma\rangle^{2a}\langle\tau-\xi^2\rangle^{2b}} \leq C_{k,b}\frac{\langle\xi\rangle^{2l-2k+2b+1}}{\langle\tau-\xi^3\rangle^{2a}\langle\tau-\xi^2\rangle^{2a}} \\
&\leq C_{k,b}\frac{\langle\xi\rangle^{2l-2k+2b+1}}{\langle\xi^3-\xi^2\rangle^{2a}} \leq C_{k,b}\frac{\langle\xi\rangle^{2l-2k+2b+1}}{\langle\xi^3\rangle^{2a}} \\
&\leq C_{k,b}\frac{\langle\xi\rangle^{2l-2k+2b+1}}{\langle\xi\rangle^{6a}} \\
&\leq C_{k,b},
\end{aligned}
\tag{1.69}
$$

where we have used the inequality $l-k+b+1/2 \leq 3a$ which holds from hypothesis (ii).

$\boldsymbol{(c_2)}$ $k \in [0,1/2]$ and $b > 1/2$.

Here the calculations are the same as in the subcase $\boldsymbol{(b_2)}$.

Case-d: $\boldsymbol{|\xi| \geq 2}$ and $\boldsymbol{\frac{1}{2}|\xi_2| \leq |\xi_1| \leq 2|\xi_2|}$.

Now we have

$$|\xi| = |\xi_1-\xi_2| \leq |\xi_1|+|\xi_2| \leq \min\{3|\xi_1|, 3|\xi_2|\}$$

and hence $k \geq 0$ implies

$$\langle\xi_1\rangle^{-2k}\langle\xi_2\rangle^{-2k} \leq C_k\langle\xi\rangle^{-4k}. \tag{1.70}$$

Then Lemma 1.3-(1.18) with $\mu = 1$, (1.70) and $a \geq 0$ gives

$$
\begin{aligned}
J(\tau,\xi) &= \frac{|\xi|^2\langle\xi\rangle^{2l}}{\langle\sigma\rangle^{2a}}\int d\xi_2\int\frac{d\tau_2}{\langle\xi_1\rangle^{2k}\langle\xi_2\rangle^{2k}\langle\tau_2+\tau+(\xi_2+\xi)^2\rangle^{2b}\langle\tau_2+\xi_2^2\rangle^{2b}} \\
&\leq C_k\frac{\langle\xi\rangle^{2l-4k}|\xi|^2}{\langle\sigma\rangle^{2a}}\int\frac{d\xi_2}{(1+|\tau+\xi^2+\xi\xi_2|^2)^{2b}} \\
&\leq C_k\frac{\langle\xi\rangle^{2l-4k}|\xi|^2}{\langle\sigma\rangle^{2a}2|\xi|} \leq C_k\frac{\langle\xi\rangle^{2l-4k+1}}{\langle\tau-\xi^3\rangle^{2a}} \\
&\leq C_k\langle\xi\rangle^{2l-4k+1} \\
&\leq C_k
\end{aligned}
\tag{1.71}
$$

where we have used the inequality $l-2k+1/2 \le 0$.

We note that (1.56) and (1.57) holds in this case. Moreover $|\xi| \le |\xi_1|+|\xi_2| \le 3|\xi_1|$ and $k \ge 0$ imply $\langle \xi \rangle^{2k} \le C_k \langle \xi_1 \rangle^{2k}$ as in (1.54). Consequently if $k > 1/2$ and $b \in [1/2, k]$ we can compute $J(\tau,\xi)$ similar to case (b_1) to obtain $J(\tau,\xi) \le C_{k,b}$ for $l-k+b+1/2 \le 3a$.

Collecting the estimates given in the above cases, we complete the proof of Lemma 1.8.

1.4 Proof of main results

In this section we prove Theorems 1.2 and 1.3 regarding local and global well-posedness for the IVP (1.1) respectively.

The Cauchy problem (1.1) is rewritten in a standard way as the integral system

$$\begin{aligned} u(t) &= U(t)u_0 - i\int_0^t U(t-t')\big\{\alpha uv(t') + \beta |u|^2 u(t')\big\}dt', \\ v(t) &= V(t)v_0 + \int_0^t V(t-t')\big\{\gamma \partial_x(|u|^2(t')) - \tfrac{1}{2}\partial_x(v^2(t'))\big\}dt'. \end{aligned} \tag{1.72}$$

One replaces the system (1.72) by the cut off system

$$\begin{aligned} u(t) &= \psi_1(t)U(t)u_0 - i\psi_T(t)\int_0^t U(t-t')\big\{\alpha uv(t') + \beta |u|^2 u(t')\big\}dt', \\ v(t) &= \psi_1(t)V(t)v_0 + \psi_T(t)\int_0^t V(t-t')\big\{\gamma \partial_x(|u|^2(t')) - \tfrac{1}{2}\partial_x(v^2(t'))\big\}dt'. \end{aligned} \tag{1.73}$$

Solving the system (1.73) for all $t \in \mathbb{R}$ solves the system (1.72) locally in time for $|t| \le T$, so that T will be the time of local resolution of (1.72).

1.4.1 Proof of Theorem 1.2

We follow similar arguments as the ones given in [2, 3, 24].

We consider the following function space where we seek our solution

$$Z_\delta(M,N) := \big\{(u,v) \in X^{k,1/2+\delta} \times Y^{l,1/2+\delta} : \|u\|_{X^{k,1/2+\delta}} \le M, \ \ \|v\|_{Y^{l,1/2+\delta}} \le N\big\}$$

where $0 < \delta \ll 1$ and $M, N > 0$ will be chosen below.

$Z_\delta(M,N)$ is a complete metric space with norm

$$\|(u,v)\|_{Z_\delta(M,N)} \equiv \|u\|_{X^{k,1/2+\delta}} + \|v\|_{Y^{l,1/2+\delta}}.$$

For $(u,v) \in Z_\delta(M,N)$ we define the maps

$$\Phi(u,v) = \psi_1(t)U(t)u_0 - i\psi_T(t)\int_0^t U(t-t')\big\{\alpha uv(t') + \beta|u|^2u(t')\big\}dt',$$
$$\Psi(u,v) = \psi_1(t)V(t)v_0 + \psi_T(t)\int_0^t V(t-t')\big\{\gamma\partial_x(|u|^2(t')) - \tfrac{1}{2}\partial_x(v^2(t'))\big\}dt'.$$

Let $a = 1/2 - 2\delta$, $b = 1/2 + \delta$ and δ satisfying the following conditions:

(a) For $k \in [0, 1/2]$, we take

$a_1)$ $0 < \delta \leq \min\{1/12,\ -(2l+1)/2,\ (4l+3)/12\}$, for $l \in (-3/4, -1/2)$.

$a_2)$ $0 < \delta < 1/24$, for $l \geq -1/2$.

(b) For $k > 1/2$ ($i.e$, we have $l > -1/2$ from hypothesis (ii) in Theorem 1.2), we take $0 < \delta \leq \min\{1/24,\ k - 1/2,\ (2k-2l+1)/14\}$.

Then according to Lemma 1.1 and Lemmas 1.5—1.8 we have

$$\begin{aligned}
\|\Phi(u,v)\|_{X^{k,1/2+\delta}} &\leq c_0\|u_0\|_{H^k} + c_1T^\delta\big(\|uv\|_{X^{k,-1/2+2\delta}} + \||u|^2u\|_{X^{k,-1/2+2\delta}}\big)\\
&\leq c_0\|u_0\|_{H^k} + c_1T^\delta\big(\|u\|_{X^{k,1/2+\delta}}\|v\|_{Y^{l,1/2+\delta}} + \|u\|^3_{X^{k,1/2+\delta}}\big)\\
&\leq c_0\|u_0\|_{H^k} + c_1T^\delta\big(MN + M^3\big),
\end{aligned}$$

$$\begin{aligned}
\|\Psi(u,v)\|_{Y^{l,1/2+\delta}} &\leq c_0\|v_0\|_{H^l} + c_2T^\delta\big(\|\partial_x v^2\|_{Y^{l,-1/2+2\delta}} + \|\partial_x|u|^2\|_{Y^{l,-1/2+2\delta}}\big)\\
&\leq c_0\|v_0\|_{H^l} + c_2T^\delta\big(\|v\|^2_{Y^{l,1/2+\delta}} + \|u\|^2_{X^{k,1/2+\delta}}\big)\\
&\leq c_0\|v_0\|_{H^l} + c_2T^\delta\big(M^2 + N^2\big).
\end{aligned}$$

Now taking $M = 2c_0\|u_0\|_{H^k}$ and $N = 2c_0\|v_0\|_{H^l}$ we have that

$$\|\Phi(u,v)\|_{X^{k,1/2+\delta}} \leq \tfrac{M}{2} + c_1T^\delta(M^3 + MN),$$

$$\|\Psi(u,v)\|_{Y^{l,1/2+\delta}} \leq \tfrac{N}{2} + c_2T^\delta(M^2 + N^2).$$

Then $(\Phi(u,v), \Psi(u,v)) \in Z_\delta(M,N)$ for

$$T^\delta \leq \tfrac{1}{2}\min\left\{\tfrac{1}{c_1(M^2+N)},\ \tfrac{N}{c_2(M^2+N^2)}\right\}. \tag{1.74}$$

Similarly we have that

$$\|\Phi(u,v) - \Phi(\tilde{u},\tilde{v})\|_{X^{k,1/2+\delta}} \leq c_3T^\delta(M^2 + M + N)(\ \|u-\tilde{u}\|_{X^{k,1/2+\delta}} + \|v-\tilde{v}\|_{Y^{l,1/2+\delta}}\ ),$$

$$\|\Psi(u,v) - \Psi(\tilde{u},\tilde{v})\|_{Y^{l,1/2+\delta}} \leq c_4T^\delta(M + N)(\ \|u-\tilde{u}\|_{X^{k,1/2+\delta}} + \|v-\tilde{v}\|_{Y^{l,1/2+\delta}}\ ).$$

Then

$$\|(\Phi(u,v),\Psi(u,v)) - (\Phi(\tilde{u},\tilde{v}),\Psi(\tilde{u},\tilde{v}))\|_{Z_\delta(M,N)} \leq \tfrac{1}{2}\|(u,v)-(\tilde{u},\tilde{v})\|_{Z_\delta(M,N)}$$

for

$$T^\delta \leq \tfrac{1}{4}\min\left\{\tfrac{1}{c_3(M^2+M+N)},\ \tfrac{1}{c_4(M+N)}\right\}. \tag{1.75}$$

Therefore the map $\Phi \times \Psi : Z_\delta(M,N) \longrightarrow Z_\delta(M,N)$ is a contraction mapping and we obtain a unique fixed point which solves the equation for any T that satisfies (1.74) and (1.75).

Next, following similar arguments as in [3], we show the uniqueness of the solutions in the the class defined by the conditions (1.4) and (1.5) in Theorem 1.2. For this purpose, we introduce the following auxiliary norms. For $T > 0$, we let

$$\|u\|_{X_T} := \inf_w\ \left\{\|w\|_{X^{k,1/2+\delta}}:\ w \in X^{k,1/2+\delta} \text{ and } w(t) = u(t) \text{ in } H^k(\mathbb{R}),\ \text{for } t \in [0,T]\ \right\},$$

$$\|v\|_{Y_T} := \inf_\phi\ \left\{\|\phi\|_{Y^{l,1/2+\delta}}:\ \phi \in Y^{l,1/2+\delta} \text{ and } \phi(t) = v(t) \text{ in } H^l(\mathbb{R}),\ \text{for } t \in [0,T]\ \right\}.$$

Then, if $\|u_1-u_2\|_{X_T}+\|v_1-v_2\|_{Y_T}=0$, we have $u_1\equiv u_2$ in $H^k(\mathbb{R})$ and $v_1\equiv v_2$ in $H^l(\mathbb{R})$ for $t\in[0,T]$.

Now let $(u_1,v_1)\in Z_\delta(M,N)$ the solution obtained above for the cut off system (1.73) and $(u_2,v_2)\in X^{k,b}\times Y^{l,b}$ be a solution of the same system with the same initial data (u_0,v_0) and we assume $T<1$.

We take

$$\begin{aligned} M_1 &:= \max\,\{M,\ \|u_2\|_{X^{k,b}}\}\,,\\ N_2 &:= \max\,\{N,\ \|v_2\|_{Y^{l,b}}\}\,. \end{aligned}$$

For some $T_1<T$ which will be fixed later, we have

$$\begin{aligned} u_2(t) &= \psi_1(t)U(t)u_0 - i\psi_{T_1}(t)\int_0^t U(t-t')\big\{\alpha u_2v_2(t')+\beta|u_2|^2u_2(t')\big\}dt',\\ v_2(t) &= \psi_1(t)V(t)v_0 + \psi_{T_1}(t)\int_0^t V(t-t')\big\{\gamma\partial_x(|u_2|^2(t'))-\tfrac{1}{2}\partial_x(v_2^2(t'))\big\}dt', \end{aligned}$$

for $t\in[0,T_1]$.

Consider the difference u_1-u_2 and v_1-v_2. For any $\epsilon>0$, there exists $(w_\epsilon,\phi_\epsilon)\in X^{k,b}\times Y^{l,b}$ such that

$$\begin{aligned} \|w_\epsilon\|_{X^{k,b}} &\le \|u_1-u_2\|_{X_{T_1}}+\epsilon,\\ \|\phi_\epsilon\|_{Y^{l,b}} &\le \|u_1-u_2\|_{Y_{T_1}}+\epsilon, \end{aligned} \tag{1.76}$$

and such that for $t\in[0,T_1]$ hold

$$\begin{aligned} w_\epsilon(t) &= u_1(t)-u_2(t),\\ \phi_\epsilon(t) &= v_1(t)-v_2(t). \end{aligned} \tag{1.77}$$

Set $(\tilde{w}_\epsilon,\tilde{\phi}_\epsilon)$ satisfying

$$\begin{aligned} \tilde{w}_\epsilon(t) &= -i\psi_{T_1}(t)\int_0^t U(t-t')\left\{\alpha(w_\epsilon v_1+u_2\phi_\epsilon)+\beta(w_\epsilon|u_1|^2+w_\epsilon\overline{u_1}u_2+\overline{w_\epsilon}u_2^2)\right\}dt',\\ \tilde{\phi}_\epsilon(t) &= \psi_{T_1}(t)\int_0^t V(t-t')\left\{\gamma\partial_x(w_\epsilon\overline{u_1}+\overline{w_\epsilon}u_2)-\tfrac{1}{2}\partial_x(\phi_\epsilon v_1+\phi_\epsilon v_2)\right\}dt'. \end{aligned}$$

By (1.77) we have $\tilde{w}_\epsilon(t) = w_\epsilon(t) = u_1(t) - u_2(t)$ and $\tilde{\phi}_\epsilon(t) = \tilde{\phi}_\epsilon(t)$ for $t \in [0, T_1]$.

We take δ as the beginning of the proof. Then according to Lemma 1.1 and Lemmas 1.5—1.8, we have

$$\|u_1 - u_2\|_{X_{T_1}} \le \|\tilde{w}_\epsilon\|_{X^{k,1/2+\delta}} \le c_3 T_1^\delta (M_1^2 + M_1 + N_1)(\ \|w_\epsilon\|_{X^{k,1/2+\delta}} + \|\phi_\epsilon\|_{Y^{l,1/2+\delta}}\ ),$$
$$\|v_1 - v_2\|_{Y_{T_1}} \le \|\tilde{\phi}_\epsilon\|_{Y^{l,1/2+\delta}} \le c_4 T_1^\delta (M_1 + N_1)(\ \|w_\epsilon\|_{X^{k,1/2+\delta}} + \|\phi_\epsilon\|_{Y^{l,1/2+\delta}}\ ).$$

Hence, if $T_1^\delta \le \frac{1}{4}\ \min\left\{\frac{1}{c_3(M_1^2+M_1+N_1)}, \frac{1}{c_4(M_1+N_1)}\right\}$, we have

$$\|u_1 - u_2\|_{X_{T_1}} + \|v_1 - v_2\|_{Y_{T_1}} \le \tfrac{1}{2}(\|w_\epsilon\|_{X^{k,1/2+\delta}} + \|\phi_\epsilon\|_{Y^{l,1/2+\delta}}),$$

and by (1.76) we conclude

$$\|u_1 - u_2\|_{X_{T_1}} + \|v_1 - v_2\|_{Y_{T_1}} \le \epsilon,$$

for all $\epsilon > 0$.

This proves $u_1 \equiv u_2$ and $v_1 \equiv v_2$ on $[0, T_1]$. Repeating this procedure, we obtain the uniqueness result for any existence interval.

1.4.2 Proof of Theorem 1.3

Let $\alpha\gamma > 0$ and $t > 0$. From (3) we have that $\|u(t)\|_{L^2} = \|u_0\|_{L^2} = \sqrt{M(0)}$, and from (4) we obtain

$$\begin{aligned} \|v(t)\|_{L^2}^2 &= K(0) - \tfrac{2\gamma}{\alpha} \int_{-\infty}^{+\infty} Im(u(t)\overline{u_x(t)})dx \\ &\le K(0) + \tfrac{2\gamma}{\alpha}\sqrt{M(0)}\|u_x(t)\|_{L^2}. \end{aligned} \tag{1.78}$$

Let $\mu = \min\left\{|\gamma|, \frac{|\alpha|}{2}\right\}$. Then using (5), (1.78) and (1.22) we deduce

$$
\begin{aligned}
\|u_x(t)\|_{L^2}^2+\|v_x(t)\|_{L^2}^2 &\le \tfrac{1}{\mu}\left(|\gamma|\|u_x(t)\|_{L^2}^2+\tfrac{|\alpha|}{2}\|v_x(t)\|_{L^2}^2\right)\\
&\le \tfrac{|E(0)|}{\mu}+\tfrac{\alpha\gamma}{\mu}\int_{-\infty}^{+\infty}|v(t)||u(t)|^2dx\\
&\quad+\tfrac{|\alpha|}{6\mu}\int_{-\infty}^{+\infty}|v(t)|^3dx+\tfrac{|\beta\gamma|}{2\mu}\int_{-\infty}^{+\infty}|u(t)|^4dx\\
&\le \tfrac{|E(0)|}{\mu}+\tfrac{\alpha\gamma}{\mu}\|v(t)\|_{L^2}\|u(t)\|_{L^4}^2+\tfrac{|\alpha|}{6\mu}\|v(t)\|_{L^3}^3+\tfrac{|\beta\gamma|}{2\mu}\|u(t)\|_{L^4}^4\\
&\le \tfrac{|E(0)|}{\mu}+\tfrac{\alpha\gamma}{2\mu}\|v(t)\|_{L^2}^2+\tfrac{|\alpha|}{6\mu}\|v(t)\|_{L^3}^3+\tfrac{\alpha\gamma+|\beta\gamma|}{2\mu}\|u(t)\|_{L^4}^4\\
&\le \tfrac{|E(0)|}{\mu}+\tfrac{\alpha\gamma}{2\mu}K(0)+\tfrac{\gamma^2}{\mu}\sqrt{M(0)}\|u_x(t)\|_{L^2}\\
&\quad+\tfrac{|\alpha|}{6\mu}\|v(t)\|_{L^3}^3+\tfrac{\alpha\gamma+|\beta\gamma|}{2\mu}\|u(t)\|_{L^4}^4\\
&\le \tfrac{|E(0)|}{\mu}+\tfrac{\alpha\gamma}{2\mu}K(0)+\tfrac{3\gamma^4}{2\mu^2}M(0)+\tfrac{1}{6}\|u_x(t)\|_{L^2}^2\\
&\quad+\tfrac{|\alpha|}{6\mu}\|v(t)\|_{L^3}^3+\tfrac{\alpha\gamma+|\beta\gamma|}{2\mu}\|u(t)\|_{L^4}^4
\end{aligned}
\tag{1.79}
$$

Using Gagliardo-Nirenberg's type inequalities and (1.22) we obtain

$$
\begin{aligned}
\tfrac{|\alpha|}{6\mu}\|v(t)\|_{L^3}^3 &\le \tfrac{|\alpha|}{6\mu}\|v_x(t)\|_{L^2}^{1/2}\|v(t)\|_{L^2}^{5/2}\\
&\le C_0K(0)^{5/4}\|v_x(t)\|_{L^2}^{1/2}+C_1M(0)^{5/8}\|v_x(t)\|_{L^2}^{1/2}\|u_x(t)\|_{L^2}^{5/4}\\
&\le \tfrac{1}{4}\|v_x(t)\|_{L^2}^2+C_2K(0)^{5/3}+\tfrac{1}{6}\|u_x(t)\|_{L^2}^2+C_3M(0)^{5/3}\|v_x(t)\|_{L^2}^{4/3}\\
&\le \tfrac{1}{4}\|v_x(t)\|_{L^2}^2+C_2K(0)^{5/3}+\tfrac{1}{6}\|u_x(t)\|_{L^2}^2+C_4M(0)^5+\tfrac{1}{4}\|v_x(t)\|_{L^2}^2\\
&= \tfrac{1}{2}\|v_x(t)\|_{L^2}^2+\tfrac{1}{6}\|u_x(t)\|_{L^2}^2+C_2K(0)^{5/3}+C_4M(0)^5.
\end{aligned}
\tag{1.80}
$$

and

$$
\begin{aligned}
\tfrac{\alpha\gamma+|\beta\gamma|}{2\mu}\|u(t)\|_{L^4}^4 &\le \tfrac{\alpha\gamma+|\beta\gamma|}{2\mu}\|u_x(t)\|_{L^2}\|u(t)\|_{L^2}^3\\
&= \tfrac{\alpha\gamma+|\beta\gamma|}{2\mu}M(0)^{3/2}\|u_x(t)\|_{L^2}\\
&\le \tfrac{1}{6}\|u_x(t)\|_{L^2}^2+C_5M(0)^3.
\end{aligned}
\tag{1.81}
$$

where $C_0, ..., C_5$ are positive constants depending only on α, β, and γ.

Then by (1.79), (1.80) and (1.81) we have

$$\begin{aligned}\|u_x(t)\|_{L^2}^2+\|v_x(t)\|_{L^2}^2 \leq \tfrac{1}{2}\|u_x(t)\|_{L^2}^2+\tfrac{1}{2}\|v_x(t)\|_{L^2}^2+\tfrac{|E(0)|}{\mu}+\tfrac{\alpha\gamma}{2\mu}K(0)+\tfrac{3\gamma^4}{2\mu^2}M(0)\\ +C_2K(0)^{5/3}+C_4M(0)^5+C_5M(0)^3,\end{aligned} \tag{1.82}$$

and hence from (1.78), (1.82) and L^2-norm conservation for u we have

$$\|u(t)\|_{H^1}^2+\|v(t)\|_{H^1}^2 \leq G(M(0),K(0),E(0)).$$

Since the last quantity is constant, we can repeat the argument of local existence of solution at time T arriving to a solution for any positive time. The same holds for negative time.

Chapter 2

Local and global theory of a coupled Schrödinger-Debye equation

We consider the IVP associated to the Schrödinger-Debye system,

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u = uv, \quad x \in \mathbb{R},\ t \geq 0, \\ \tau\partial_t v + v = \epsilon|u|^2, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{cases} \tag{2.1}$$

where $\tau > 0$ and $\epsilon = \pm 1$.

We can simplify the system (2.1) by eliminating $v(x,t)$ to obtain the decoupled integro-differential equation

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u = e^{-t/\tau} u v_0(x) + \frac{\epsilon}{\tau} u \int_0^t e^{-(t-t')/\tau} |u(t')|^2 dt', \quad x \in \mathbb{R},\ t \geq 0, \\ u(x,0) = u_0(x). \end{cases} \tag{2.2}$$

where $\tau > 0$ and $\epsilon = \pm 1$.

Using the Duhamel formulation of this equation we have

$$u(t) = S(t)u_0 - i\int_0^t S(t-t')\big(F_0(u(t')) + F_1(u(t'))\big)dt', \tag{2.3}$$

where $S(t) = e^{\frac{it}{2}\partial_x^2} = \mathcal{F}_x^{-1}e^{-\frac{it}{2}\xi^2}\mathcal{F}_x$ is the unitary group associated to the linear equation $i\partial_t u + \frac{1}{2}\partial_x^2 u = 0$ and

$$F_0(u(t)) = e^{-t/\tau}uv_0(x), \quad F_1(u(t)) = \tfrac{\epsilon}{\tau}u\int_0^t e^{-(t-t')/\tau}|u(t')|^2dt'. \tag{2.4}$$

In [10] B. Bidégaray showed the following result

Theorem 2.1 *For $(u_0, v_0) \in H^r(\mathbb{R}) \times H^r(\mathbb{R})$ with $r > 1/2$ the system (2.1) has a unique solution in $C([0,T] : H^r(\mathbb{R}))$ for a small enough T and solutions depend continuously on the initial data.*

Our purpose here is to establish local and global well-posedness results for thr IVP (2.1) in the spaces $H^s(\mathbb{R}) \times H^{s-1/2+\delta}(\mathbb{R})$ with $\delta = 0$ for $s = 1/2$ and $0 < \delta \le 1/4$ for $1/2 < s \le 1$. To obtain our results we will use the so called $L^p - L^q$ estimates. These type of estimates were first established by Strichartz [43] for solutions of the linear Schrödinger equation, i.e,

$$\begin{cases} i\partial_t u + \triangle u = 0, & x \in \mathbb{R}^n,\ t \ge 0, \\ u(x,0) = u_0(x). \end{cases} \tag{2.5}$$

He showed that solutions of (2.5) satisfy

$$\left(\int_{\mathbb{R}}\int_{\mathbb{R}^n} \left|e^{it\triangle}u_0(x)\right|^{2(n+2)/n} dxdt\right)^{n/2(n+2)} \le c\|u_0\|_{L^2}.$$

Generalizations of this result have been obtained by several authors. For instance, Ginibre-Velo [21] and Kenig-Ponce-Vega [31].

We will also use the smoothing effect obtained by Kenig, Ponce and Vega [32, 33] for the non-homogeneous term in one-dimensional case, that is,

$$\|D_x^{1/2}\int_0^t e^{it(t-t')\partial_x^2}G(\cdot,t')dt'\|_{L_T^\infty L_x^2} \le C\|G\|_{L_x^1L_T^2}. \tag{2.6}$$

We proceed as follows. Instead of working with the system (2.1) we use its equivalent integral form (2.3). Then we use the $L^p - L^q$ estimates and the smoothing effect (2.6) to show via the contraction mapping principle that there exists a time $T > 0$ where (2.3) has a unique solution.

To establish global well-posedness we use the same procedure to prove our local theorem combined with the conservation law in L^2 to obtain a priori estimates of the solutions.

2.1 Main results

Concerning well-posedness we have the following results

Theorem 2.2 *Let $1/2 \leq s \leq 1$. Then for any $(u_0, v_0) \in H^s(\mathbb{R}) \times H^{s-1/2+\delta}(\mathbb{R})$ (with $\delta = 0$ for $s = 1/2$ and $0 < \delta \leq 1/4$ for $1/2 < s \leq 1$) there exist $T = T(s, \|u_0\|_s, \|v_0\|_{s-1/2+\delta}) > 0$ and a unique solution $(u(x,t), v(x,t))$ of the IVP (2.1) such that for $q \in [2, \infty]$*

$$(u, v) \in C\big([0,T] : H^s(\mathbb{R})\big) \times C\big([0,T] : H^{s-1/2+\delta}(\mathbb{R})\big), \tag{2.7}$$

$$\|u\|_{L^r_T L^q_x} < \infty, \quad \text{with} \quad 2/r = 1/2 - 1/q, \tag{2.8}$$

$$\|\partial_x u\|_{L^\infty_x L^2_T} < \infty. \tag{2.9}$$

Moreover the map $(u_0, v_0) \longrightarrow (u(t), v(t))$ from $H^s(\mathbb{R}) \times H^{s-1/2+\delta}(\mathbb{R})$ into $C\big([0,T] : H^s(\mathbb{R})\big) \times C\big([0,T] : H^{s-1/2+\delta}(\mathbb{R})\big)$ is locally Lipschitz, that is, for $(u_{0n}, v_{0n}) \in H^s(\mathbb{R}) \times H^{s-1/2+\delta}(\mathbb{R})$ such that $(u_{0n}, v_{0n}) \longrightarrow (u_0, v_0)$ in $H^s(\mathbb{R}) \times H^{s-1/2+\delta}(\mathbb{R})$, the corresponding solutions $\{(u_n(t), v_n(t))\}$ satisfy

$$\begin{aligned} &\|u_n - u\|_{L^\infty_T H^s_x} + \|u_n - u\|_{L^r_T L^q_x} + \|\partial_x (u_n - u)\|_{L^\infty_x L^2_T} \leq C_1(T) \|u_{0n} - u_0\|_{H^s}, \\ &\|v_n - v\|_{L^\infty_T H^{s-1/2+\delta}_x} \leq C_2(T) \big(\|u_{0n} - u_0\|_{H^s} + \|v_{0n} - v_0\|_{H^{s-1/2+\delta}} \big), \end{aligned} \tag{2.10}$$

as $n \longrightarrow \infty$.

Theorem 2.3 *The unique solution provided by Theorem 2.2 extends to any time interval [0, T]. Moreover,*

$$\|u\|_{L^\infty_T H^s_x} + \|u\|_{L^r_T L^q_x} + \|\partial_x u\|_{L^\infty_x L^2_T} \le K_1(T), \tag{2.11}$$
$$\|v\|_{L^\infty_T H^{s-1/2+\delta}_x} \le K_2(T).$$

2.2 Linear estimates

In this section we collect known results on smoothing effect estimates of free Schrödinger evolution group.

Firstly, we use the stronger one-dimensional version of the smoothing effect for the homogeneous term of the linear Schrödinger equation

$$\sup_{x\in\mathbb{R}} \left(\int_{-\infty}^{+\infty} |D_x^{1/2} S(t) u_0|^2 dt \right)^{1/2} \le C\|u_0\|_{L^2} \tag{2.12}$$

and the version for the non-homogeneous term, that is,

$$\sup_{x\in\mathbb{R}} \left(\int_{-\infty}^{+\infty} \left| \partial_x \int_0^t S(t-t')G(x,t')dt' \right|^2 dt \right)^{1/2} \le C\|G\|_{L^1_x L^2_t}. \tag{2.13}$$

Both (2.12) and (2.13) were proved by Kenig, Ponce and Vega [31, 32].

Next, we state other smoothing effect for the non-homogeneous term.

Proposition 2.1 *For any $\theta \in [0,1]$, we have*

$$\|D_x^{\theta/2} \int_0^t S(t-t')G(\cdot,t')dt'\|_{L^\infty_T L^2_x} \le CT^{(1-\theta)/2}\|G\|_{L_x^{2/(1+\theta)} L^2_T}. \tag{2.14}$$

Proof. Using Stein's theorem of analytic families of operators (See [41]) we obtain (2.14) by interpolation between

$$\|D_x^{i\eta} \int_0^t S(t-t')G(\cdot,t')dt'\|_{L^\infty_T L^2_x} \le CT^{1/2}\|G\|_{L^2_x L^2_T}, \quad \eta \in \mathbb{R} \tag{2.15}$$

and

$$\|D_x^{1/2+i\eta}\int_0^t S(t-t')G(\cdot,t')dt'\|_{L_T^\infty L_x^2} \le C\|G\|_{L_x^1 L_T^2}, \quad \eta\in\mathbb{R}. \tag{2.16}$$

The estimates (2.15) and (2.16) were proved by Kenig, Ponce and Vega in [32, 33]. □

Lastly we give the Strichartz estimates for the homogeneous and non-homogeneous terms.

Proposition 2.2 *For any pairs (r_i, q_i) with $q_i \in [2,\infty]$ and $2/r_i = 1/2 - 1/q_i$ (i=1,2) we have*

$$\|S(t)u_0\|_{L_T^{r_1} L_x^{q_1}} \le C\|u_0\|_2, \tag{2.17}$$

$$\|\int_0^t S(t-t')G(\cdot,t')dt'\|_{L_T^{r_1} L_x^{q_1}} \le C\|G\|_{L_T^{r_2'} L_x^{q_2'}}, \tag{2.18}$$

$$\|\int_0^t S(t-t')G(\cdot,t')dt'\|_{L_T^{r_1} L_x^{q_1}} \le CT^{(1/r_2'-1/2)}\|G\|_{L_x^{q_2'} L_T^2}, \tag{2.19}$$

where $\frac{1}{r_2}+\frac{1}{r_2'}=1$ *and* $\frac{1}{q_2}+\frac{1}{q_2'}=1$.

Proof. See Strichartz [43] and Ginibre-Velo [22] for the proof of the first estimate (2.17). For the proof of (2.18) we can see Ginibre-Velo [22], K. Yajima [48], Cazenave-Weissler [16] and Kato [29]. The last estimate (2.19) is a slight modification of the (2.18). Indeed, we note that

$$q_2 \ge 2 \Longrightarrow r_2 \ge 4 \Longrightarrow r_2' \le 4/3 < 2,$$

$$q_2 \ge 2 \Longrightarrow 2 \ge q_2'.$$

Then, using Hölder's and Minkowski's inequalities we have

$$\|G\|_{L_T^{r_2'} L_x^{q_2'}} \le T^{(1/r_2'-1/2)}\|G\|_{L_T^2 L_x^{q_2'}} \le T^{(1/r_2'-1/2)}\|G\|_{L_x^{q_2'} L_T^2},$$

and hence, (2.19) follows from (2.18). □

2.3 Nonlinear estimates

In this section we give the main results that will be used to estimate the nonlinear terms in the proof of Theorem 2.2.

To handle the nonlinear terms with fractional derivates, we need the following commutators estimates deduced by Kenig, Ponce and Vega in [33].

Proposition 2.3 *Let* $\alpha \in (0,1)$, $\alpha_1, \alpha_2 \in (0,\alpha)$, $\alpha_1 + \alpha_2 = \alpha$ *and* $p, p_1, p_2, q, q_1, q_2 \in (1,\infty)$. *Then*

$$\|D_x^\alpha(fg) - fD_x^\alpha g - gD_x^\alpha f\|_{L_x^p L_T^q} \le C\|D_x^{\alpha_1} f\|_{L_x^{p_1} L_T^{q_1}} \|D_x^{\alpha_2} g\|_{L_x^{p_2} L_T^{q_2}}, \tag{2.20}$$

with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ *and* $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$.

$$\|D_x^\alpha(fg) - fD_x^\alpha g - gD_x^\alpha f\|_{L_x^p L_T^2} \le C\|f\|_{L_x^{p_1} L_T^\infty} \|D_x^\alpha g\|_{L_x^{p_2} L_T^2}, \tag{2.21}$$

with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$.

$$\|D_x^\alpha(fg) - fD_x^\alpha g - gD_x^\alpha f\|_{L_x^p L_T^1} \le C\|D_x^{\alpha_1} f\|_{L_x^{p_1} L_T^{q_1}} \|D_x^{\alpha_2} g\|_{L_x^{p_2} L_T^{q_2}}, \tag{2.22}$$

with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ *and* $\frac{1}{q_1} + \frac{1}{q_2} = 1$.

$$\|D_x^\alpha(fg) - fD_x^\alpha g - gD_x^\alpha f\|_{L_x^1 L_T^2} \le C\|D_x^{\alpha_1} f\|_{L_x^{p_1} L_T^{q_1}} \|D_x^{\alpha_2} g\|_{L_x^{p_2} L_T^{q_2}}, \tag{2.23}$$

with $\frac{1}{p_1} + \frac{1}{p_2} = 1$ *and* $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{2}$.

$$\|D_x^\alpha(fg) - fD_x^\alpha g - gD_x^\alpha f\|_{L_x^p} \le C\|f\|_{L_x^\infty} \|D_x^\alpha g\|_{L_x^p}. \tag{2.24}$$

Proof. See Appendix in [33]. $\square$

We also use the following Sobolev type inequality:

Lemma 2.1 *Let $0 < \alpha < 1$ and $\frac{1}{q} = \frac{1}{2} - \alpha$. Then we have*

$$\|f\|_{L^q(\mathbb{R})} \leq C_\alpha \|D^\alpha f\|_{L^2(\mathbb{R})}. \tag{2.25}$$

This lemma is an immediate consequence of the Hardy-Littlewood-Sobolev Theorem.

Finally we prove the following proposition which will be useful in the proof of the Theorem 2.2.

Proposition 2.4 *Let $1/2 \leq s \leq 1$ and $F_1(u)$ as in 2.4. Then we have*

$$\|D_x^{s-1/2} F_1(u)\|_{L_x^1 L_T^2} \leq \frac{C}{\tau}\big(T^{5/4} + T^{3/2}\big)\|u\|^2_{L_T^\infty H_x^{s-1/2}}\big(\,\|u\|_{L_T^\infty H_x^s} + \|u\|_{L_T^4 L_x^\infty}\big). \tag{2.26}$$

Proof. First we consider the particular case $s = 1/2$ and we get

$$\begin{aligned}
\|F_1(u)\|_{L_x^1 L_T^2} &\leq \frac{C}{\tau}\|u\|_{L_x^2 L_T^2}\Big\|\int_0^t e^{-(t-t')/\tau}|u(t')|^2 dt'\ \Big\|_{L_x^2 L_T^\infty} \\
&\leq \frac{C}{\tau} T^{1/2}\|u\|_{L_T^\infty L_x^2}\||u|^2\|_{L_x^2 L_T^1} \\
&\leq \frac{C}{\tau} T^{1/2}\|u\|_{L_T^\infty L_x^2}\|\overline{u}u\|_{L_T^1 L_x^2} \\
&\leq \frac{C}{\tau} T^{1/2}\|u\|_{L_T^\infty L_x^2}\|u\|_{L_T^{4/3} L_x^2}\|u\|_{L_T^4 L_x^\infty} \\
&\leq \frac{C}{\tau} T^{5/4}\|u\|^2_{L_T^\infty L_x^2}\|u\|_{L_T^4 L_x^\infty} \\
&\leq \frac{C}{\tau}\big(T^{5/4} + T^{3/2}\big)\|u\|^2_{L_T^\infty L_x^2}\big(\,\|u\|_{L_T^4 L_x^\infty} + \|u\|_{L_T^\infty H_x^{1/2}}\,\big).
\end{aligned}$$

Next, for $1/2 < s \leq 1$ we have

$$\|D_x^{s-1/2} F_1(u)\|_{L_x^1 L_T^2} \leq \frac{1}{\tau}(A_1 + A_2 + A_3)$$

with

$$\begin{aligned}
A_1 = \Big\|D_x^{s-1/2}\big(\,u\int_0^t e^{-(t-t')/\tau}|u(t')|^2 dt'\,\big) &- u\int_0^t e^{-(t-t')/\tau} D_x^{s-1/2}|u(t')|^2 dt' \\
&- \big(D_x^{s-1/2}u\big)\int_0^t e^{-(t-t')/\tau}|u(t')|^2 dt'\ \Big\|_{L_x^1 L_T^2},
\end{aligned}$$

$$A_2 = \big\|\big(D_x^{s-1/2}u\big)\int_0^t e^{-(t-t')/\tau}|u(t')|^2dt'\ \big\|_{L^1_xL^2_T},$$

$$A_3 = \big\|u\int_0^t e^{-(t-t')/\tau}D_x^{s-1/2}|u(t')|^2dt'\ \big\|_{L^1_xL^2_T}.$$

Proposition 2.3-(2.22, 2.23), Minkowski's, Hölder's and Sobolev's inequalities yield

$$\begin{aligned}
A_1 &\le C\big\|D_x^{s/2-1/4}u\big\|_{L_x^{4/(3-2s)}L_T^{4/(3-2s)}}\big\|D_x^{s/2-1/4}\int_0^t e^{-(t-t')/\tau}|u(t')|^2dt'\ \big\|_{L_x^{4/(1+2s)}L_T^{4/(2s-1)}}\\
&\le CT^{(2s-1)/4}\big\|D_x^{s/2-1/4}u\big\|_{L_T^{4/(3-2s)}L_x^{4/(3-2s)}}\big\|\int_0^t e^{-(t-t')/\tau}D_x^{s/2-1/4}|u(t')|^2dt'\ \big\|_{L_x^{4/(1+2s)}L_T^{\infty}}\\
&\le CT^{(2s-1)/4}\big\|D_x^{s/2-1/4}u\big\|_{L_T^{4/(3-2s)}L_x^{4/(3-2s)}}\big\|D_x^{s/2-1/4}|u|^2\ \big\|_{L_x^{4/(1+2s)}L_T^{1}}\\
&\le CT^{1/2}\big\|D_x^{s/2-1/4}u\big\|_{L_T^{\infty}L_x^{4/(3-2s)}}\big(\ 2\|u\|_{L_x^2L_T^2}\big\|D_x^{s/2-1/4}u\big\|_{L_x^{4/(2s-1)}L_T^2}\\
&\quad +\ C\big\|D_x^{s/4-1/8}u\big\|_{L_x^{8/(5-2s)}L_T^2}\big\|D_x^{s/4-1/8}u\big\|_{L_x^{8/(6s-3)}L_T^2}\ \big)\\
&\le CT^{1/2}\big\|D_x^{s-1/2}u\big\|_{L_T^{\infty}L_x^2}\big(\ 2\|u\|_{L_T^2L_x^2}\big\|D_x^{1/2}u\big\|_{L_T^2L_x^2}+C\|D_x^{s/2-1/4}u\big\|_{L_T^2L_x^2}\|D_x^{3/4-s/2}u\big\|_{L_T^2L_x^2}\ \big)\\
&\le CT^{3/2}\|u\|_{L_T^{\infty}H_x^{s-1/2}}\big(\ \|u\|_{L_T^{\infty}L_x^2}\|u\|_{L_T^{\infty}H_x^s}+\|u\|_{L_T^{\infty}H_x^{s-1/2}}\|u\|_{L_T^{\infty}H_x^s}\ \big)\\
&\le CT^{3/2}\|u\|^2_{L_T^{\infty}H_x^{s-1/2}}\|u\|_{L_T^{\infty}H_x^s}\\
&\le C\big(T^{5/4}+T^{3/2}\big)\|u\|^2_{L_T^{\infty}H_x^{s-1/2}}\big(\ \|u\|_{L_T^4L_x^{\infty}}+\|u\|_{L_T^{\infty}H_x^s}\ \big).
\end{aligned}$$

From $1/2 < s \le 1$ we have $2 \le 4/(2s-1)$, $2 \le 8/(5-2s)$ and $2 \le 8/(6s-3)$. We have used this fact to apply Minkowski's and Sobolev's inequalities. Next by Hölder's and Minkowski's inequalities it follows that

$$
\begin{aligned}
A_2 &\leq \left\| D_x^{s-1/2} u \right\|_{L_x^2 L_T^2} \| |u|^2 \|_{L_x^2 L_T^1} \\
&\leq C T^{1/2} \|u\|_{L_T^\infty H_x^{s-1/2}} \|u\|_{L_T^{4/3} L_x^2} \|u\|_{L_T^4 L_x^\infty} \\
&\leq C T^{5/4} \|u\|_{L_T^\infty H_x^{s-1/2}} \|u\|_{L_T^\infty L_x^2} \|u\|_{L_T^4 L_x^\infty} \\
&\leq C \big(T^{5/4} + T^{3/2} \big) \|u\|_{L_T^\infty H_x^{s-1/2}}^2 \big(\|u\|_{L_T^4 L_x^\infty} + \|u\|_{L_T^\infty H_x^s} \big).
\end{aligned}
$$

Finally using Hölder's and Minkowski's inequalities and Proposition 2.3-(2.24) we have that

$$
\begin{aligned}
A_3 &\leq \|u\|_{L_x^2 L_T^2} \left\| D_x^{s-1/2} |u|^2 \right\|_{L_x^2 L_T^1} \\
&\leq T^{1/2} \|u\|_{L_T^\infty L_x^2} \left\| D_x^{s-1/2} |u|^2 \right\|_{L_T^1 L_x^2} \\
&\leq C T^{1/2} \|u\|_{L_T^\infty L_x^2} \|u\|_{L_T^4 L_x^\infty} \left\| D_x^{s-1/2} u \right\|_{L_T^{4/3} L_x^2} \\
&\leq C T^{5/4} \|u\|_{L_T^\infty L_x^2} \|u\|_{L_T^\infty H_x^{s-1/2}} \|u\|_{L_T^4 L_x^\infty} \\
&\leq C \big(T^{5/4} + T^{3/2} \big) \|u\|_{L_T^\infty H_x^{s-1/2}}^2 \big(\|u\|_{L_T^4 L_x^\infty} + \|u\|_{L_T^\infty H_x^s} \big).
\end{aligned}
$$

Collecting the estimates for A_1, A_2 and A_3 we obtain the desired estimate (2.26). □

2.4 Proof of main results

In this section we prove the theorems enunciated in the beginning of this chapter. We use the point fixed method for the proof of local result and we obtain a priori estimates for the proof of global results.

2.4.1 Proof of Theorem 2.2

We define the complete metric space X_T^a for $T > 0$ and $a > 0$ as

$$
X_T^a = \{ u \in C([0,T] : H^s(\mathbb{R})) \ / \ |\!|\!| u |\!|\!|_T \leq a \}, \tag{2.27}
$$

where

$$\|u\|_T \equiv \|u\|_{L^\infty_T H^s_x} + \|u\|_{L^4_T L^\infty_x}. \tag{2.28}$$

We also define

$$X_T = \{u \in C([0,T] : H^s(\mathbb{R})) \;/\; \|u\|_T < \infty\}, \tag{2.29}$$

and we consider the map

$$\Phi(u) = S(t)u_0 - i\int_0^t S(t-t')\big(F_0(u(t')) + F_1(u(t'))\big)dt'. \tag{2.30}$$

It will be established that for appropriate choices of a and T, depending only on s, $\|u_0\|_{H^s}$ and $\|v_0\|_{H^{s-1/2+\delta}}$, that if $u \in X_T^a$ then $w = \Phi(u)$ belongs to X_T^a and $\Phi : X_T^a \to X_T^a$ is a contraction map. Thus most of what follows is the estimation of $\|\Phi(u)\|_T$. We need to bound the non linear term in (2.30).

Now by Proposition 2.2-(2.17) we have that

$$\|S(t)u_0\|_T \le C_0\|u_0\|_{H^s}, \tag{2.31}$$

and for $0 < \delta \le 1/4$ we apply (2.14), (2.19), (2.21) and Minkowski's, Hölder's and Sobolev's inequalities to obtain

$$
\begin{aligned}
\|\!| \int_0^t S(t-t')F_0(u(t'))dt' \|\!|_T &\le C_1\Big(\| \int_0^t S(t-t')F_0(u(t'))dt'\|_{L_T^\infty L_x^2} \\
&\quad + \| \int_0^t S(t-t')F_0(u(t'))dt'\|_{L_T^4 L_x^\infty} + \|D_x^s \int_0^t S(t-t')F_0(u(t'))dt'\|_{L_T^\infty L_x^2}\Big) \\
&\le C_1\Big(T^{1/4}\|e^{-t/\tau}v_0 u\|_{L_x^1 L_T^2} + \|D_x^{1/2-\delta}\int_0^t D_x^{s-1/2+\delta}(e^{-t/\tau}v_0 u)dt'\|_{L_T^\infty L_x^2}\Big) \\
&\le C_1\Big(T^{1/4}\|e^{-t/\tau}v_0\|_{L_x^2 L_T^\infty}\|u\|_{L_x^2 L_T^2} + T^\delta \|D_x^{s-1/2+\delta}(e^{-t/\tau}v_0 u)\|_{L_x^{1/(1-\delta)} L_T^2}\Big) \\
&\le C_1\Big(T^{3/4}\|v_0\|_{L^2}\|u\|_{L_T^\infty L_x^2} + T^\delta\|D_x^{s-1/2+\delta}u\|_{L_x^{2/(1-2\delta)}L_T^2}\|e^{-t/\tau}v_0\|_{L_x^2 L_T^\infty} \\
&\quad + T^\delta \|u\|_{L_x^{2/(1-2\delta)}L_T^2}\|D_x^{s-1/2+\delta}(e^{-t/\tau}v_0)\|_{L_x^2 L_T^\infty}\Big) \\
&\le C_1(T^{3/4} + T^{\delta+1/2})\|v_0\|_{H^{s-1/2+\delta}}\|u\|_{L_T^\infty H_x^s} \\
&\le C_1(T^{3/4} + T^{\delta+1/2})\|v_0\|_{H^{s-1/2+\delta}}\|\!|u\|\!|_T.
\end{aligned} \tag{2.32}
$$

Further by (2.14), (2.19) and Proposition 2.4 we have

$$
\begin{aligned}
\|\!| \int_0^t S(t-t')F_1(u(t'))dt' \|\!|_T &\le C_2\Big(\| \int_0^t S(t-t')F_1(u(t'))dt'\|_{L_T^\infty L_x^2} \\
&\quad + \| \int_0^t S(t-t')F_1(u(t'))dt'\|_{L_T^4 L_x^\infty} + \|D_x^s \int_0^t S(t-t')F_1(u(t'))dt'\|_{L_T^\infty L_x^2}\Big) \\
&\le \tfrac{C_2}{\tau}\Big(T^{1/4}\|F_1(u)\|_{L_x^1 L_T^2} + \|D_x^{s-1/2}F_1(u)\|_{L_x^1 L_T^2}\Big) \\
&\le \tfrac{C_2}{\tau}(T^{3/2} + T^{7/4} + T^{5/4} + T^{3/2})\|u\|^2_{L_T^\infty H_x^{s-1/2}}\|\!|u\|\!|_T \\
&\le \tfrac{C_2}{\tau}(T^{3/2} + T^{7/4} + T^{5/4})\|u\|^2_{L_T^\infty H_x^{s-1/2}}\|\!|u\|\!|_T.
\end{aligned} \tag{2.33}
$$

Therefore we see from (2.31)-(2.33) that

$$\begin{aligned}\|\!|\Phi(u)\|\!|_T &\le C_0\|u_0\|_{H^s} + C_1(T^{3/4}+T^{\delta+1/2})\|v_0\|_{H^{s-1/2+\delta}}\|\!|u\|\!|_T\\ &\quad + \tfrac{C_2}{\tau}(T^{3/2}+T^{7/4}+T^{5/4}+T^{3/2})\|u\|^2_{L^\infty_T H^{s-1/2}_x}\|\!|u\|\!|_T\\ &\le C_0\|u_0\|_{H^s} + aC_1(T^{3/4}+T^{\delta+1/2})\|v_0\|_{H^{s-1/2+\delta}}\\ &\quad + a^3\tfrac{C_2}{\tau}(T^{3/2}+T^{7/4}+T^{5/4}).\end{aligned} \tag{2.34}$$

Thus we first choose $a = 2C_0\|u_0\|_{H^s}$ and then T satisfying the following conditions

$$C_1(T^{3/4}+T^{\delta+1/2})\|v_0\|_{H^{s-1/2+\delta}} \le \tfrac{1}{4} \tag{2.35}$$

and

$$\tfrac{C_2}{\tau}(T^{3/2}+T^{7/4}+T^{5/4})a^2 \le \tfrac{1}{4}. \tag{2.36}$$

It is easy to see that if $u \in X^a_T$ then $w = \Phi(u) \in C([0,T] : H^s)$ (see [32]). Then we conclude that $\Phi : X^a_T \to X^a_T$.

Using

$$\begin{aligned}F_1(u) - F_1(\tilde u) =& \tfrac{\epsilon}{\tau}(u-\tilde u)\int_0^t e^{-(t-t')/\tau}|u|^2 dt'\\ &+ \tfrac{\epsilon}{\tau}\tilde u\int_0^t e^{-(t-t')/\tau}\left(\,\overline{u}(u-\tilde u) + \tilde u\left(\overline{u}-\overline{\tilde u}\,\right)\right)dt',\end{aligned} \tag{2.37}$$

similar arguments show that

$$\begin{aligned}\|\!|\Phi(u)-\Phi(\tilde u)\|\!|_T &\le C_1(T^{3/4}+T^{\delta+1/2})\|v_0\|_{H^{s-1/2+\delta}}\|\!|u-\tilde u\|\!|_T\\ &+ \tfrac{C_2}{\tau}(T^{3/2}+T^{7/4}+T^{5/4})(\|\!|u\|\!|^2_T + \|\!|u\|\!|_T\|\!|\tilde u\|\!|_T + \|\!|\tilde u\|\!|^2_T)\|\!|u-\tilde u\|\!|_T.\end{aligned} \tag{2.38}$$

Consequently $\Phi : X^a_T \to X^a_T$ is a contraction map and hence there exists a unique $u \in X^a_T$ with $\Phi(u) = u$.

Now we let (r,q) with $q \in [2,\infty]$ and $2/r = 1/2 - 1/q$. Using (2.12), (2.13), Proposition 2.2-(2.17, 2.19) and that the solution u satisfies $u = \Phi(u)$, we have

$$\begin{aligned} \|\partial_x u\|_{L_x^\infty L_T^2} + \|u\|_{L_T^r L_x^q} \le C_0\|u_0\|_{H^{s-1/2}} + C_1\left(1+T^{1/4}\right)\|F_0(u)\|_{L_x^1 L_T^2} \\ + \tfrac{C_2}{\tau}\left(1+T^{1/4}\right)\|F_1(u)\|_{L_x^1 L_T^2}. \end{aligned} \tag{2.39}$$

Hence, the additional regularities (2.8) and (2.9) hold.

On the other hand the solution $v(x,t)$ satisfies

$$v(t) = e^{-t/\tau}v_0 + \tfrac{\epsilon}{\tau}\int_0^t e^{-(t-t')/\tau}|u(t')|^2dt', \quad t \in [0,T]. \tag{2.40}$$

Then for $1/2 < s \le 1$ and $0 < \delta \le 1/4$, using (2.20) and Lemma 2.1 we have that

$$\begin{aligned} \|v(t)\|_{L^2} &+ \|D_x^{s-1/2+\delta}v(t)\|_{L^2} \le \|v_0\|_{L^2} + \|D_x^{s-1/2+\delta}v_0\|_{L^2} \\ &+ \tfrac{1}{\tau}\int_0^t e^{-(t-t')/\tau}\||u|^2\|_{L^2}dt' + \tfrac{1}{\tau}\int_0^t e^{-(t-t')/\tau}\|D_x^{s-1/2+\delta}|u|^2\|_{L^2}dt' \\ &\le \|v_0\|_{L^2} + \|D_x^{s-1/2+\delta}v_0\|_{L^2} + \tfrac{1}{\tau}\|u\overline{u}\|_{L_T^1L_x^2} + \tfrac{1}{\tau}\|D_x^{s-1/2+\delta}(u\overline{u})\|_{L_T^1L_x^2} \\ &\le \|v_0\|_{L^2} + \|D_x^{s-1/2+\delta}v_0\|_{L^2} + \tfrac{1}{\tau}\|u\|_{L_T^4L_x^\infty}\|u\|_{L_T^{4/3}L_x^2} + \tfrac{1}{\tau}T^{1/2}\|D_x^{s-1/2+\delta}(u\overline{u})\|_{L_x^2L_T^2} \\ &\le \|v_0\|_{L^2} + \|D_x^{s-1/2+\delta}v_0\|_{L^2} + \tfrac{1}{\tau}T^{3/4}\|u\|_{L_T^4L_x^\infty}\|u\|_{L_T^\infty L_x^2} \\ &+ \tfrac{1}{\tau}\|D_x^{s/2-1/4+\delta/2}u\|_{L_x^4L_T^4}^2 + \tfrac{1}{\tau}\|u\|_{L_T^4L_x^\infty}\|D_x^{s-1/2+\delta}u\|_{L_T^4L_x^2} \\ &\le \|v_0\|_{L^2} + \|D_x^{s-1/2+\delta}v_0\|_{L^2} + \tfrac{1}{\tau}T^{3/4}\|u\|_{L_T^4L_x^\infty}\|u\|_{L_T^\infty L_x^2} \\ &+ \tfrac{1}{\tau}T^{1/2}\|D_x^{s/2+\delta/2}u\|_{L_T^\infty L_x^2}^2 + \tfrac{1}{\tau}T^{1/4}\|u\|_{L_T^4L_x^\infty}\|D_x^{s-1/2+\delta}u\|_{L_T^\infty L_x^2} \\ &\le \|v_0\|_{H^{s-1/2+\delta}} + \tfrac{1}{\tau}\left(T^{3/4}+T^{1/2}+T^{1/4}\right)|\!|\!|u|\!|\!|_T^2. \end{aligned} \tag{2.41}$$

Hence $v(t) \in H^{s-1/2+\delta}$ in $[0,T]$.

Finally we explain how to extend our uniqueness result in X_T^a to the class X_T defined in (2.29). Suppose $w \in X_{T_1}$ for some $T_1 \in (0,T)$ is a strong solution of the IVP (2.1). We take $b = \max\{a, |\!|\!|w|\!|\!|_{T_1}\}$. For $T_2 < T_1$ will be fixed later; the argument used to obtain that Φ is a contraction map shows that

$$\begin{aligned}\|u-w\|_{T_2} &\leq C_1(T_2^{3/4}+T_2^{\delta+1/2})\|v_0\|_{H^{s-1/2+\delta}}\|u-w\|_{T_2}\\ &+\tfrac{C_2}{\tau}(T_2^{3/2}+T_2^{7/4}+T_2^{5/4})(\|u\|_{T_2}^2+\|u\|_{T_2}\|w\|_{T_2}+\|w\|_{T_2}^2)\|u-w\|_{T_2}\\ &\leq C_1(T_2^{3/4}+T_2^{\delta+1/2})\|v_0\|_{H^{s-1/2+\delta}}\|u-w\|_{T_2}\\ &+3b^2\tfrac{C_2}{\tau}(T_2^{3/2}+T_2^{7/4}+T_2^{5/4})\|u-w\|_{T_2}.\end{aligned}$$

Then taking T_2 satisfying

$$C_1(T_2^{3/4}+T_2^{\delta+1/2})\|v_0\|_{H^{s-1/2+\delta}} \leq \tfrac{1}{4}$$

and

$$3b^2\tfrac{C_2}{\tau}(T_2^{3/2}+T_2^{7/4}+T_2^{5/4}) \leq \tfrac{1}{4},$$

we have $\|u-w\|_{T_2}=0$. Hence $u(t)\equiv w(t), t\in[0,T_2]$. By reapplying this argument this result can be extended to the whole interval $[0,T]$. This yields the uniqueness result in X_T and the proof of local theorem is completed.

Remark 2.1 *If $s>1/2$, we have $C([0,T]:H^s(\mathbb{R}))\bigcap L_T^4L_x^\infty = C([0,T]:H^s(\mathbb{R}))$ from Sobolev's lemma. In this case, our solution satisfies the "most natural" definition for local well-posedness: For any $R>0$ there exists $T=T(R)>0$ such that the data–solution map is uniformly continuous and uniquely defined from the ball $\{u_0\in H^s(\mathbb{R})<R\}$ to the space $C([0,T]:H^s(\mathbb{R}))$.*

2.4.2 Proof of Theorem 2.3

In order to prove global well-posedness we give the a priori estimate for the solutions given by Theorem 2.2.

Let $[0,T^*)$ be the maximal time interval on which the Cauchy problem (2.2) has a unique solution $u\in X_T$ for any $T<T^*$. Suppose that $T^*<\infty$ and we will show that it leads to the contradiction.

First we note that the solution $u(x,t)$ of the IVP (2.2) satisfies $\Phi(u)=u$ and from (2.34) we see that

$$\begin{aligned}\|u\|_T \le C_0\|u_0\|_{H^s} + C_1(T^{3/4}+T^{\delta+1/2})\|v_0\|_{H^{s-1/2+\delta}}\|u\|_T \\ + \tfrac{C_2}{\tau}(T^{3/2}+T^{7/4}+T^{5/4})\|u\|^2_{L^\infty_T H^{s-1/2}_x}\|u\|_T.\end{aligned} \tag{2.42}$$

for any $T<T^*$.

Now we consider two cases:

(i) $s=1/2$ and $\delta=0$.

Using the conservation law (8) in L^2 and (2.42) we have that

$$\|u\|_T \le C_0\|u_0\|_{H^{1/2}} + \mu_1(T)\|u\|_T + \mu_2(T)\|u\|_T \tag{2.43}$$

where

$$\mu_1(T) = C_1(T^{3/4}+T^{1/2})\|v_0\|_{L^2}, \tag{2.44}$$

$$\mu_2(T) = \tfrac{C_2}{\tau}(T^{3/2}+T^{7/4}+T^{5/4})\|u_0\|^2_{L^2}. \tag{2.45}$$

The functions μ_1 and μ_2 are continuous and $\mu_1(0)=\mu_2(0)=0$. Hence we can take $\tilde{T}\in[0,T^*]$ so that

$$\mu_1(\tilde{T}) \le 1/4 \quad \text{and} \quad \mu_2(\tilde{T}) \le 1/4$$

with $\tilde{T}$ depending only on $\|u_0\|_{L^2}$ and $\|v_0\|_{L^2}$. Then from (2.43) we obtain

$$\|u\|_{T'} \le 2C_0\|u_0\|_{H^{1/2}} \tag{2.46}$$

for any $T'\in[0,\tilde{T}]$.

If $\tilde{T} = T^*$, we have that the solution u of the IVP (2.2) can be extended to the time interval $[0, T^*]$ with

$$\sup_{t\in[0,T^*]} \|u(t)\|_{H^{1/2}} \leq 2C_0\|u_0\|_{H^{1/2}},$$

and we see that it contradicts the maximality of T^*. Therefore, suppose that $\tilde{T} < T^*$. Let $m \in \mathbb{N}$ be such that $T^* \leq m\tilde{T}$ and replace $\tilde{T}$ by $\tilde{T} = T^*/m$.

Now consider the Cauchy problem

$$\begin{cases} i\partial_t u^{(1)} + \frac{1}{2}\partial_x^2 u^{(1)} = e^{-t/\tau}u^{(1)}v_0(x) + \frac{\epsilon}{\tau}u\int_0^t e^{-(t-t')/\tau}|u^{(1)}|^2 dt', \\ u^{(1)}(x,\tilde{T}) = u(x,\tilde{T}), \quad x \in \mathbb{R}. \end{cases} \tag{2.47}$$

Uniqueness of solutions yields that

$$\begin{cases} u(x,t), \quad t \in [0,\tilde{T}], \\ u^{(1)}(x,t), \quad t \in [\tilde{T}, 2\tilde{T}], \end{cases} \tag{2.48}$$

is a solution of IVP (2.2) in $[0, 2\tilde{T}]$.

Using that $\|u_0\|_{L^2} = \|u(\tilde{T})\|_{L^2}$ and the same procedure to obtain (2.46), we have

$$\begin{aligned} |\!|\!| u |\!|\!|_{2\tilde{T}} &\leq \max\left\{2C_0\|u_0\|_{H^{1/2}},\ 2C_0\|u(\tilde{T})\|_{H^{1/2}}\right\} \\ &\leq \max\left\{2C_0\|u_0\|_{H^{1/2}},\ 4C_0^2\|u_0\|_{H^{1/2}}\right\}. \end{aligned} \tag{2.49}$$

Then, repeating this process m times, we see that

$$|\!|\!| u |\!|\!|_{T^*} \leq \max\left\{2C_0\|u_0\|_{H^{1/2}},\ 4C_0^2\|u_0\|_{H^{1/2}}, ...,\ (2C_0)^m\|u_0\|_{H^{1/2}}\right\} \tag{2.50}$$

which contradicts the maximality of T^*. Hence $T^* = +\infty$.

We also note that for any $T > 0$ we have

$$|\!|\!| u |\!|\!|_T \leq K(T) := \max\left\{2C_0\|u_0\|_{H^{1/2}},\ 4C_0^2\|u_0\|_{H^{1/2}}, ...,\ (2C_0)^{m(T)}\|u_0\|_{H^{1/2}}\right\}$$

where $m(T) = \left[\frac{T}{\tilde{T}}\right] + 1$. Consequently

$$\|u\|_{L^\infty_T H^{1/2}} \leq K(T). \tag{2.51}$$

(ii) $1/2 \leq s \leq 1$ and $0 < \delta \leq 1/4$.

Since $H^s \hookrightarrow H^{1/2}$, we may regard the solution as being contained in $H^{1/2}$. Moreover

$$\|u\|_{L^\infty_T H^{s-1/2}} \leq \|u\|_{L^\infty_T H^{1/2}} \leq K(T). \tag{2.52}$$

Again, we suppose that $T^*_s < \infty$, where $[0, T^*_s)$ is the maximal time interval of existence of the solution.

Now we put

$$K_0 := \sup \ \{K(T) : T \in [0, T^*_s] \} . \tag{2.53}$$

Then from (2.42) and (2.52) for any $T \in [0, T^*_s)$ we have

$$|\!|\!| u |\!|\!|_T \leq C_0 \|u_0\|_{H^s} + \mu_1(T) |\!|\!| u |\!|\!|_T + \mu_2(T) |\!|\!| u |\!|\!|_T \tag{2.54}$$

where

$$\mu_1(T) = C_1(T^{3/4} + T^{1/2}\ )\|v_0\|_{H^{s-1/2+\delta}}, \tag{2.55}$$

$$\mu_2(T) = \tfrac{C_2}{\tau}(T^{3/2} + T^{7/4} + T^{5/4}\ )K_0^2. \tag{2.56}$$

Now using (2.54), (2.55) and (2.56), we can choose $\tilde{T}$, depending only on $\|v_0\|_{H^{s-1/2+\delta}}$, sufficiently small such that

$$\mu_1(\tilde{T}) \leq 1/4 \quad \text{and} \quad \mu_2(\tilde{T}) \leq 1/4$$

and consequently

$$|\!|\!| u |\!|\!|_{T'} \leq 2C_0 \|u_0\|_{H^s} \tag{2.57}$$

for any $T' \in [0, \tilde{T}]$.

Similar to case (**i**) we get $T_s^* = +\infty$ and $\|u\|_T \le K_1(T)$ for any $T > 0$. Then the proof of Theorem 2.3 is completed.

Chapter 3

Ill-posedness for the Benney System

We consider the IVP associated to the Benney system, that is,

$$\begin{cases} i\partial_t u + \partial_x^2 u = \alpha u\eta + \beta|u|^2 u, \quad x \in \mathbb{R},\ t \in \mathbb{R}, \\ \partial_t \eta + \lambda \partial_x \eta = \gamma \partial_x(|u|^2), \\ u(x,0) = u_0(x),\ \eta(x,0) = \eta_0(x), \end{cases} \tag{3.1}$$

where u is a complex valued function, η is a real valued function, $\lambda = \pm 1$ and α, β and γ are real constants.

The following result is due to Ginibre-Tsutsumi-Velo [24].

Theorem 3.1 *The Benney System (3.1) is locally well-posed for initial data* $(u_0, \eta_0) \in H^k(\mathbb{R}) \times H^l(\mathbb{R})$ *provided*

$$-1/2 < k - l \leq 1 \quad \text{and} \quad 0 \leq l + 1/2 \leq 2k. \tag{3.2}$$

The solution satisfies:

$$u \in C([0,T], H^k(\mathbb{R})), \quad \eta \in C([0,T], H^l(\mathbb{R})). \tag{3.3}$$

Note that if $k-l$ is fixed the lowest allowed values of (k,l) are attained for $k-l=\frac{1}{2}$ and are given by $(k,l)=(0,-1/2)$. Moreover local well-posedness was shown by Bekiranov, Ogawa and Ponce [3] in the line $l=k-1/2$ with $k\geq 0$.

In the work of both, Ginibre-Tsutsumi-Velo [24] and Bekiranov-Ogawa-Ponce [3], the best result obtained for local well-posedness for IVP (3.1) is in the space $L^2(\mathbb{R})\times H^{-1/2}(\mathbb{R})$. Since scaling argument can not be applied to the Benney system to obtain a criticality notion it is not clear that whether this result is optimal. In this work we are going to prove that this result is in fact the best possible to get local well-posedness. For this, we prove the following theorem concerning ill-posedness for IVP (3.1).

Theorem 3.2 *The Benney System (3.1) is ill posed in $H^k(\mathbb{R})\times H^l(\mathbb{R})$ for $\beta<0$ provided*

$$-1/3\leq k<0 \quad \textit{and} \quad k(2l+3)+1\geq 0. \tag{3.4}$$

To prove the Theorem 3.2 we follow the argument used by Kenig-Ponce-Vega [37] to show ill-posedness for some canonical nonlinear dispersive models. The main ingredient in our proof is the use of the properties of the solitary wave solutions of the system (3.1). The existence of such special functions for any speed of propagation, $c>0$, and the exponential decay are strongly applied.

Remark 3.1 *For $\beta\geq 0$, it is not possible to apply the same argument used in the proof of Theorem 3.2. Nevertheless, we can give a criticality notion for the special case $\beta=\lambda=0$. Indeed, if (u,η) is a solution of the system (3.1) with initial data $(u_0(x),\eta_0(x))$ then*

$$u_\mu(x,t)=\mu^{3/2}u(\mu x,\mu^2 t),$$

$$\eta_\mu(x,t)=\mu^2\eta(\mu x,\mu^2 t),$$

solves (3.1) with initial data $u_{\mu 0} = \mu^{3/2} u_0(\mu x)$ *and* $\eta_{\mu 0} = \mu^2 \eta_0(\mu x)$*. Now taking the homogeneous derivative of order* k *and* l *in* L^2 *for* u_μ *and* η_μ *respectively yields the following*

$$\|D_x^k u_\mu\|_{L^2}^2 = \mu^{2+2k} \|D_x^k u\|_{L^2}^2,$$

$$\|D_x^l \eta_\mu\|_{L^2}^2 = \mu^{3+2l} \|D_x^l \eta\|_{L^2}^2.$$

Hence, the notion of criticality is well defined for the Benney system with initial data $(u_0, \eta_0) \in H^k(\mathbb{R}) \times H^l(\mathbb{R})$*, and the critical values turn out to be* $k = -1$ *and* $l = -3/2$*. We note that the optimal relation between* k *and* l *is* $k - l = 1/2$*.*

In Figure 3.1 we compare the results for local well-posedness given by Theorem 3.1 with our results for ill-posedness in Theorem 3.2.

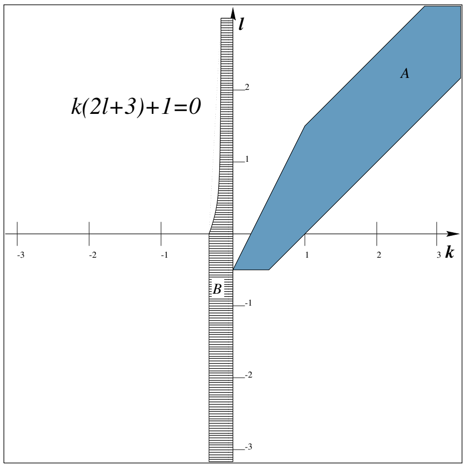

Figure 3.1: The region A contains indices (k,l) where local well-posedness was shown in [3, 24]. Ill-posedness is shown by our example for indices (k,l) in region B

3.1 Solitary waves

In this section we obtain solitary wave solutions for the Benney system.

We will look for solutions of equation (3.1) of the form:

$$u(x,t) = e^{i\omega t}\phi(x-ct) \quad \text{and} \quad \eta(x,t) = \psi(x-ct) \tag{3.5}$$

where $\omega > 0$, $c > 0$ and ϕ and ψ are two smooth functions of L^2 which decrease rapidly to zero at infinity. See [38].

Substituting (3.5) in (3.1) we have the following system of ordinary differential equations for ϕ and ψ

$$\begin{cases} -ic\phi' - \omega\phi + \phi'' = \alpha\phi\psi + \beta|\phi|^2\phi \\ (\lambda - c)\psi' = \gamma(|\phi|^2)'. \end{cases} \tag{3.6}$$

Taking $c > |\lambda| = 1$, we obtain

$$-ic\phi' - \omega\phi + \phi'' = (\beta + \frac{\alpha\gamma}{\lambda - c})|\phi|^2\phi. \tag{3.7}$$

Setting $\phi(x) = e^{\frac{icx}{2}}h(x)$, where h is a real valued function and using (3.7) we have

$$h'' - (\omega - \frac{c^2}{4})h - (\beta - \frac{\alpha\gamma}{c-\lambda})h^3 = 0. \tag{3.8}$$

We can see [6] and [42] for the following statements. The equation (3.8) has positive, even, smooth and exponentially decreasing solutions if the conditions

$$\omega - \frac{c^2}{4} > 0 \quad \text{and} \quad \beta(c-\lambda) - \alpha\gamma < 0 \tag{3.9}$$

are satisfied. The solution in this case is given by

$$h(x) = \frac{2\mu\sigma}{e^{-\sigma x} + e^{\sigma x}} = \mu\sigma \operatorname{sech}(\sigma x) \tag{3.10}$$

where,

$$\mu = \sqrt{\frac{2(c-\lambda)}{\alpha\gamma - \beta(c-\lambda)}} \quad \text{and} \quad \sigma = \sqrt{\omega - \frac{c^2}{4}}. \tag{3.11}$$

The set of non trivial solutions of (3.8) in $H^1(\mathbb{R})$ is empty if the condition (3.9) fails.

Remark 3.2 *For $c > 1$ and $\omega > \frac{c^2}{4}$, the condition (3.9) holds in the following cases:*

(i) $\beta < 0, \quad c > \max\left\{1, \lambda + \frac{\alpha\gamma}{\beta}\right\}.$

(ii) $\beta = 0, \quad \alpha\gamma > 0.$

(iii) $\beta > 0, \quad 1 < c < \lambda + \frac{\alpha\gamma}{\beta}.$

We are interested in the case (i). Here the speed of propagation ($c > 1$) is not restricted to a bounded interval and this fact is strongly used in our argument.

Finally, we have the following expressions for the solitary waves:

$$\begin{cases} u_{c,\omega}(x,t) = e^{i\omega t} e^{\frac{ic}{2}(x-ct)} \mu g_\sigma(x-ct), \\ \\ \eta_{c,\omega}(x,t) = -\frac{2\gamma}{\alpha\gamma - \beta(c-\lambda)} g_\sigma^2(x-ct), \\ \\ g_\sigma(x) := \sigma g(\sigma x), \ g(x) = \operatorname{sech}(x). \end{cases} \tag{3.12}$$

3.2 Proof of Theorem 3.2

The idea of the proof is the following: we will take two solitary waves as in (3.12) as our initial data. We will see that under some assumptions the will remain close at initial time and the we will see the evolution of the solutions associated to them to find a contradiction.

Without loss of generality, we may assume $\beta = -1$ and $\alpha = \gamma = \lambda = 1$ in (3.1).

Taking

$$N \gg 1, \quad C = 2N \quad \text{and} \quad \omega = N^2 + \sigma^2, \tag{3.13}$$

and using (3.12) we have that the pair

$$\begin{cases} u_{\sigma,N}(x,t) = e^{-it(N^2-\sigma^2)}e^{iNx}\mu(N)g_\sigma(x-2tN), \\ \\ \eta_{\sigma,N}(x,t) = -\frac{1}{N}g_\sigma^2(x-2tN), \\ \\ \mu(N) = \sqrt{\frac{2N-1}{N}}, \end{cases} \tag{3.14}$$

is the solution of the Benney system (3.1) with initial data $(e^{iNx}\mu(N)g_\sigma(x), -\frac{1}{N}g_\sigma^2(x))$.

Taking Fourier transform we have

$$\widehat{u}_{\sigma,N}(\xi,t) = e^{it(N^2+\sigma^2-2N\xi)}\mu(N)\widehat{g}\Big(\frac{\xi-N}{\sigma}\Big) \tag{3.15}$$

and

$$\widehat{\eta}_{\sigma,N}(\xi,t) = -\frac{\sigma}{N}e^{-2itN\xi}\widehat{g^2}\Big(\frac{\xi}{\sigma}\Big). \tag{3.16}$$

Let us set

$$N_j \simeq N, \quad N_1 < N_2, \quad \omega_j = N_j^2 + \sigma^2, \quad j = 1,2 \tag{3.17}$$

and write

$$u_j(x,t) := u_{\sigma_j,N_j}(x,t) \quad \text{and} \quad \eta_j := \eta_{\sigma_j,N_j}(x,t). \tag{3.18}$$

The fundamental theorem of calculus and the mean value theorem yield the following inequalities

$$\begin{aligned}
|\widehat{u_1}(\xi,0)-\widehat{u_2}(\xi,0)|^2 &= \Big|\mu(N_1)\widehat{g}\Big(\frac{\xi-N_1}{\sigma}\Big)-\mu(N_2)g\Big(\frac{\xi-N_2}{\sigma}\Big)\Big|^2\\
&\lesssim \mu^2(N_1)\Big|\widehat{g}\Big(\frac{\xi-N_1}{\sigma}\Big)-\widehat{g}\Big(\frac{\xi-N_2}{\sigma}\Big)\Big|^2\\
&\quad + |\mu(N_1)-\mu(N_2)|^2\Big|\widehat{g}\Big(\frac{\xi-N_2}{\sigma}\Big)\Big|^2\\
&\simeq \Big|\int_0^1 \widehat{g}\,'\Big(\frac{\xi-N_2+t(N_2-N_1)}{\sigma}\Big)\Big(\frac{N_2-N_1}{\sigma}\Big)dt\Big|^2\\
&\quad + |\mu(N_1)-\mu(N_2)|^2\Big|\widehat{g}\Big(\frac{\xi-N_2}{\sigma}\Big)\Big|^2\\
&\leq |N_2-N_1|^2\sigma^{-2}\Big(\int_0^1\Big|\widehat{g}\,'\Big(\frac{\xi-N_2+t(N_2-N_1)}{\sigma}\Big)\Big|dt\Big)^2\\
&\quad + |\mu'(N_0)|^2|N_1-N_2|^2\Big|\widehat{g}\Big(\frac{\xi-N_2}{\sigma}\Big)\Big|^2
\end{aligned}$$

with $N_0\in(N_1,N_2)$, $\quad \mu'(N_0)=\frac{1}{2N_0^{\frac{3}{2}}\sqrt{2N_0-1}}\simeq\frac{1}{N^2}$. Hence

$$\|u_1(\cdot,0)-u_2(\cdot,0)\|_k^2\lesssim |N_1-N_2|^2\sigma^{-2}I_1+\frac{|N_1-N_2|^2}{N^4}I_2 \tag{3.19}$$

where

$$I_1=\int(1+|\xi|^2)^k\Big(\int_0^1\Big|\widehat{g}\,'\Big(\frac{\xi-N_2+t(N_2-N_1)}{\sigma}\Big)\Big|dt\Big)^2d\xi$$

and

$$I_2=\int(1+|\xi|^2)^k\Big|\widehat{g}\Big(\frac{\xi-N_2}{\sigma}\Big)\Big|^2d\xi.$$

Let

$$\sigma=N^{-2k}. \tag{3.20}$$

Taking $k>-\frac{1}{2}$ $(N^{-2k}<N)$, $\xi\in B_\sigma(tN_1+(1-t)N_2)$ then $|\xi|\simeq N$ for $t\in[0,1]$ and using

that $\widehat{g} \in S(\mathbb{R})$ and $\widehat{g}$ concentrates in $B_1(0)$ we have the following estimates for I_1 and I_2.

$$\begin{aligned} I_1 &\leq \int (1+|\xi|^2)^k \Big(\int_0^1 \Big| \widehat{g}\,' \Big(\frac{\xi - N_2 + t(N_2 - N_1)}{\sigma} \Big) \Big|^2 dt \Big) d\xi \\ &= \int_0^1 \int (1+|\xi|^2)^k \Big| \widehat{g}\,' \Big(\frac{\xi - N_2 + t(N_2 - N_1)}{\sigma} \Big) \Big|^2 d\xi dt \\ &\leq CN^{2k}\sigma \int_0^1 \int \Big| \widehat{g}\,' \Big(y - \frac{tN_1 + (1-t)N_2}{\sigma} \Big) \Big|^2 dy dt \\ &\leq C \|\widehat{g}\,'\|_{L^2}^2, \end{aligned}$$

and

$$\begin{aligned} I_2 &\simeq N^{2k} \int \Big| \widehat{g} \Big(\frac{\xi - N_2}{\sigma} \Big) \Big|^2 d\xi \\ &= N^{2k}\sigma \|\widehat{g}\|_{L^2}^2 = \|g\|_{L^2}^2. \end{aligned}$$

Using (3.19) and the estimates above it follows that

$$\|u_1(\cdot,0) - u_2(\cdot,0)\|_k^2 \lesssim |N_1 - N_2|^2 N^{4k} + \frac{|N_1 - N_2|^2}{N^4}. \tag{3.21}$$

Now we consider the solutions $u_j(x,t)$, $j = 1,2$, at time $t = T$, observe that

$$\|u_j(\cdot,T)\|_s^2 = \|u_j(\cdot,0)\|_s^2 \simeq N^{2s}\sigma \|g\|_{L^2}^2, \quad s \in \mathbb{R}, \quad j = 1,2. \tag{3.22}$$

If $s = k$ in (3.22) then

$$\|u_j(\cdot,T)\|_k^2 \simeq \|g\|_{L^2}^2. \tag{3.23}$$

On the the other hand, the frequencies of $u_j(\cdot,T)$, $j = 1,2$, are localized in $B^* = B_\sigma(N_1) \cup B_\sigma(N_2)$. Hence $|\xi| \simeq N$ and consequently

$$\|u_1(\cdot,T) - u_2(\cdot,T)\|_k^2 \simeq N^{2k} \|u_1(\cdot,T) - u_2(\cdot,T)\|_{L^2}^2. \tag{3.24}$$

Now, $u_j(\cdot,T)$ concentrates in $B_{\sigma^{-1}}(2TN_j)$, $j = 1,2$. Therefore giving $T > 0$, are taking N_1 and N_2 such that

$$T|N_1 - N_2| \gg \sigma^{-1} = N^{2k}. \tag{3.25}$$

We have that there is no interaction of $u_j,\ j=1,2$, at time $t=T$; hence using (3.22) with $s=0$ we obtain

$$\|u_1(\cdot,T)-u_2(\cdot,T)\|_{L^2}^2 \simeq \|u_1(\cdot,T)\|_{L^2}^2+\|u_2(\cdot,T)\|_{L^2}^2 \simeq \sigma. \tag{3.26}$$

Combining (3.24) and (3.26) we obtain

$$\|u_1(\cdot,T)-u_2(\cdot,T)\|_k^2 \geq CN^{2k}\sigma = C. \tag{3.27}$$

Taking

$$N_1 = N \quad \text{and} \quad N_2 = N+\delta N^{-2k} \quad \text{with} \quad \delta>0 \tag{3.28}$$

we get from (3.21)

$$\|u_1(\cdot,0)-u_2(\cdot,0)\|_k^2 \leq C\delta^2(1+N^{-4(k+1)}) \leq C\delta^2, \tag{3.29}$$

here we have used that $k>-\frac{1}{2}$.

Since $k<0$, given $\delta,\ T>0$, we can take N so large that

$$T|N_1-N_2| = T\delta N^{-2k} \gg N^{2k} \iff N^{-4k} \gg \frac{1}{T\delta} \tag{3.30}$$

and hence (3.25), (3.26) and (3.27) hold.

The initial data $\eta_j(x,0),\ j=1,2,$ satisfies

$$\begin{aligned}
\|\eta_j(\cdot,0)\|_l^2 &= \frac{\sigma^2}{N_j^2}\int (1+|\xi|^2)^l |\widehat{g^2}(\frac{\xi}{\sigma})|^2 d\xi \\
&= \frac{\sigma^3}{N_j^2}\int (1+\sigma^2 y^2)^l |\widehat{g^2}(y)|^2 dy \\
&\simeq \frac{\sigma^{3+2l}}{N^2}\int (N^{4k}+y^2)^l |\widehat{g^2}(y)|^2 dy \\
&\leq N^{-2(k(2l+3)+1)} \begin{cases} \|g^2\|_l^2, & l\geq 0 \\ \\ N^{4kl}\|g^2\|_{L^2}^2, & l<0 \end{cases} \\
&\leq C,
\end{aligned}$$

whenever

$$k(2l+3)+1 \geq 0, \text{ for } l \geq 0 \quad \text{and} \quad k \geq -\frac{1}{3}, \text{ for } l < 0. \tag{3.31}$$

On the other hand,

$$\begin{aligned}
\|\eta_1(\cdot,0)-\eta_2(\cdot,0)\|_l^2 &= \sigma^2\Big(\frac{1}{N_1}-\frac{1}{N_2}\Big)^2\int(1+|\xi|^2)^l|\widehat{g^2}\big(\frac{\xi}{\sigma}\big)|^2d\xi \\
&\simeq \frac{\sigma^3}{N^4}(N_1-N_2)^2\int(1+\sigma^2y^2)^l|\widehat{g^2}(y)|^2dy \\
&= \frac{\sigma^{3+2l}}{N^4}N^{-4k}\delta^2\int(N^{4k}+y^2)^l|\widehat{g^2}(y)|^2dy \\
&\leq \delta^2N^{-2(k(2l+5)+2)}\begin{cases} \|g^2\|_l^2, & l \geq 0 \\ \\ N^{4kl}\|g^2\|_{L^2}^2, & l < 0 \end{cases} \\
&\leq C\delta^2,
\end{aligned}$$

where in the last inequality we use that

$$k(2l+5)+2 \geq 0, \text{ for } l \geq 0 \quad \text{and} \quad k \geq -\frac{2}{5}, \text{ for } l < 0. \tag{3.32}$$

Note that for $k < 0$ the condition (3.31) implies the condition (3.32) and the proof is complete.

Concluding remarks

Here we give an account of the principal results obtained in this work and point out some open problems.

In Chapter 1 we proved that the IVP associated to the Schrödinger-KdV system is locally well-posed for the given initial data in $L^2(\mathbb{R}) \times H^{-3/4+\epsilon}(\mathbb{R})$. According to the method we utilized, this result is the best possible because the trilinear estimate in Lemma 1.5 is false for $k < 0$, and the bilinear estimate in Lemma 1.6 is false for $l < -3/4$. But at this point, it is not clear whether we can have the local well-posedness result in the space $L^2(\mathbb{R}) \times H^{-3/4}(\mathbb{R})$.

Moreover, we proved that the system 1.1 is globally well-posed in the energy space $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ using the conservation laws. We believe that this result could be improved to get global solution in $H^k(\mathbb{R}) \times H^l(\mathbb{R})$ with $k,\ l \leq 1$. One example which gives insight in this direction is the following particular case. If we take $\beta = \gamma = 0$, the IVP 1.1 turns to

$$\begin{cases} i\partial_t u + \partial_x^2 u = \alpha uv, \quad x \in \mathbb{R},\ t \in \mathbb{R}, \\ \partial_t v + \partial_x^3 v + \frac{1}{2}\partial_x v^2 = 0, \end{cases} \tag{3.33}$$

with initial data $u(x,0) = u_0(x)$ and $v(x,0) = v_0(x)$. In this case, if we take $(u_0, v_0) \in H^k(\mathbb{R}) \times H^1(\mathbb{R})$ with $1/2 < k \leq 1$, using Theorem 1.2, the fact that $H^k(\mathbb{R})$ with $k > 1/2$ is an algebra and the H^1–conserved quantities for the KdV equation we can obtain global solutions. Also we observe that, using recently introduced Bourgain's techniques to obtain global solution in spaces where there are no conservation laws, it could be possible to obtain global solution to system 1.1 in $H^k(\mathbb{R}) \times H^1(\mathbb{R})$ with $k_0 < k \leq 1$ and $k_0 \in [1/2, 1)$.

In Chapter 2 we proved local and global well-posedness for the IVP associated to the Schrödinger-Debye system for given data in $H^s(\mathbb{R}) \times H^{s-1/2+\delta}(\mathbb{R})$ with $\delta = 0$ for $s = 1/2$ and $0 < \delta \leq 1/4$ for $1/2 < s \leq 1$. We believe that, using the same techniques utilized to prove Theorem 1.2 it is possible to obtain local well-posedness result for IVP (2.1) in the sobolev spaces with negative indices. Also, we think, it is interesting to conduct a similar study of local and global well-posedness for the following modified Schrödinger-Debye system

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u = uv, \quad x \in \mathbb{R}^n,\ n = 2, 3,\ t \geq 0, \\ \tau\partial_t v + v = \epsilon |u|^p, \quad p > 0 \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{cases} \tag{3.34}$$

where $\tau > 0$ and $\epsilon = \pm 1$. This system was proposed by B. Bidégaray in [10], which was inspired from the general NLS equations.

In Chapter 3 we proved that the Benney system is ill-posed in the space $H^k(\mathbb{R}) \times H^l(\mathbb{R})$ for $-1/3 \leq k < 0$ and $k(2l + 3) + 1 \geq 0$ whenever $\beta < 0$. To obtain this result we used the existence of solitary wave solutions to the system (3.1). This result shows that the space $L^2(\mathbb{R}) \times H^{-1/2}(\mathbb{R})$ is the best possible to have local well-posedness. We consider that it would be interesting to resolve the following open problems:

- In the case $\lambda = \beta = 0$, we have the notion of criticality with critical values $(k, l) = (-1, -3/2)$. At this point, we do not know whether or not the system (3.1) is locally well-posed in $H^k(\mathbb{R}) \times H^l(\mathbb{R})$ with $(k, l) \in \Omega = [-1, 0) \times [-3/2, 0)$.
- In the case $\beta > 0$ we can not use the solitary wave solutions to obtain the results as in the case $\beta < 0$. We think it may be possible to get ill-posedness result using argument analogous to the ones used by M. Christ, J. Colliander, and T. Tao in [17] to obtain ill-posedness to NLS-defocusing and KdV.

- Finally, we observe that, in the case $\lambda = \beta = 0$, orbital stability of solitary waves is studied by Laurençot in [38]. But in general case, we don't know whether this result still holds.

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