Hénon-like attractors in arbitrary dimensions: SRB measures and basin problem

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Abstract

We prove that Hénon-like strange attractors of diffeomorphisms in any dimensions, such as considered in [BC91],[MV93],[Via93] support a unique Sinai-Ruelle-Bowen (SRB) measure and have the no-holes property: Lebesgue almost every point in the basin of attraction is generic for the SRB measure. This extends two-dimensional results of Benedicks-Young [BY93] and Benedicks-Viana [BV01], respectively.

Introduction

Extending the framework of [BC91], Mora and Viana [MV93] were able to show that strange attractors (or repellers) of Hénon type occur, with positive probability in parameter space, in the generic unfolding of homoclinic tangencies of surface diffeomorphisms. Indeed, a renormalization scheme permits to construct, from the original family of diffeomorphisms unfolding the tangency, a family $(f_a)_a$ of diffeomorphisms of the plane arbitrarily close (in the C^k topology, any $k \geq 3$) to the family of quadratic endomorphisms

$$Q_a(x,y) = (1 - ax^2, 0).$$

And Mora and Viana [MV93] extend the conclusions of Benedicks and Carleson [BC91] to such *Hénon-like families*.

Then Benedicks and Young [BY93] proved that all these Hénon-like strange attractors support a unique *SRB measure*, that is, an invariant ergodic probability measure μ such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(z))=\int\varphi\,d\mu$$

for every continuous function $\phi:\mathbb{R}^2\to\mathbb{R}^2$, and for a set of points z with positive Lebesgue measure. This set is denoted by $B(\mu)$ and called the *ergodic basin* of μ.

More recently, Benedicks and Viana [BV01] proved that Hénon-like strange attractors have the no-hole property, thus solving a problem raised in the early eighties by Sinai and by Ruelle, for this class of systems. The no-hole property means that

$$\mathrm{Leb}(B(\Lambda)\setminus B(\mu))=0,$$

that is, the measure μ describes the time-averages of Lebesgue *almost every orbit* that converges to the attractor. As a main step of the proof, they show that almost every orbit in $B(\Lambda)$ belongs to the stable set of some orbit of Λ.

Back in the early nineties, Viana [Via93] extended the results of [MV93] to homoclinic bifurcations of diffeomorphisms on any manifold. In this setting one assumes that the saddle-point associated to the homoclinic tangency is sectionally dissipative: the product of any two eigenvalues is less than 1 in norm. Then a renormalization scheme permits to find from that original family unfolding the tangency, families $(f_a)_a$, arbitrarily close to

$$Q_a(x,y_1,\dots,y_q)=(1-ax^2,0,\dots,0),$$

which Viana [Via93] proves to display strange attractors Λ_a for a positive Lebesgue measure set of parameters.

In the present paper we prove that these high-dimensional strange attractors share the nice ergodic properties of their two-dimensional counterparts. Indeed our first main theorem is

Theorem A. *Let $(f_a)_a$ be a Hénon-like family of diffeomorphisms of $\mathbb{R}^{q+1}$ as above. Then, for a positive Lebesgue measure set of parameters a, the diffeomorphism f_a has a unique SRB measure μ_a and the support of this measure coincides with the strange attractor Λ_a.*

The positive Lebesgue measure set of parameters in this and in the next theorem is the one that was constructed en [Via93], for which Λ_a is known to exist.

In [Via93] it is also shown that these Hénon-like families one obtains in the context of sectionally dissipative homoclinic bifurcations are *partially hyperbolic*, in the sense that there exists an open set K, containing the attractor Λ_a for each a, and there exists a continuous splitting

$$T_k\mathbb{R}^{q+1} = E^{ss} \oplus E^{cu}, \quad \dim E^{cu} = 2,$$

of the tangent bundle restricted to K satisfying for some $\lambda > 1$,

- $Df_z.E_z^{ss} = E_{f(z)}^{ss}$ and $Df_z.E_z^{cu} = E_{f(z)}^{cu}$ *(invariance)* for every $z \in K \cap f^{-1}(K)$;
- $\left\|Df_z|E_z^{ss}\right\| \le \lambda$ *(uniform contraction)*
- $\left\|Df_z|E_z^{ss}\right\| \left\|(Df_z|E_z^{cu})^{-1}\right\| \le \lambda$ *(dominating property)*

In fact E_a^{ss} and E_a^{cu} are defined for every parameter a (not just a positive measure subset) and they may be taken uniformly close to constant (i.e. parallel) subbundles of $\mathbb{R}^{q+1}$.

Our next main theorem is

Theorem B. *Let $(f_a)_a$ be a partially hyperbolic Hénon-like family. For the same set of parameters as in Theorem A, the attractor Λ_a has the no-hole property:*

(1) $B(\Lambda) = \cup_{\zeta\in\Lambda} W^s(\zeta)$ *up to a zero Lebesgue set;*

(2) $\mathrm{Leb}(B(\Lambda) \setminus B(\mu)) = 0$

The proofs of these results occupies Sections 2 through 14. While we exploit several ideas from the previous paper, a lot of new difficulties have to be dealt with in this extension to high dimensions. Most important, the arguments of [BV01] to solve the basin problem on surfaces, especially their construction of a sequence of pseudo-Markov partitions into rectangles, makes crucial use of the topology of the plane. A seemingly natural generalization of this strategy to higher dimensions soon runs into trouble: the geometry of successive partition elements becomes more and more complicated (in 2 dimension they are all just rectangles), a difficult which seems unsurmountable.

Instead, our strategy has been to carry out all our constructions on surfaces transverse to the strong-stable bundle E^{ss}. Partial hyperbolicity ensures that a convenient family of such surfaces is preserved under forward iterates (we do not need to assume existence of an invariant “central" manifold).

1. Notations, definitions and preliminary results

Let b be a small and positive constant. In the sequel, we will suppose that b is much smaller than any other constants appearing in the text. Let $a \in [1,2]$ and $\phi_a : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R} \times \mathbb{R}^q$ be the quadratic endomorphism given by $\phi_a(x,y) = (1 - ax^2, 0)$. We consider parametrized families of diffeomorphisms $(f_a)_a$ in $\mathbb{R}^q$ such that $\|f_a - \phi_a\|_{C^3} \le K\sqrt{b}$ for all $a \in [1,2]$, where $K > 0$ is fixed but arbitrary.

Hence the derivative of f_a at z has the form,

$$Df_a(z) = \begin{bmatrix} -2a\pi_x(z) + R_1(z) & R_2(z) \\ R_3(z) & R_4(z) \end{bmatrix} \tag{1.1}$$

with $\|R_i\|_{C^2} \le K\sqrt{b}$ for $i = 1, \ldots, 4$.

We fix $\delta > 0$ and $[a_1, a_2] \subset [1,2]$ satisfying $b \ll 2 - a_2 < 2 - a_1 \ll \delta$ and $(a_2 - a_1) \ge (2 - a_1)/10$. For these parameters, f_a has a unique fixed point P_a that is a continuation for the fixed point of ϕ_a with x-coordinate positive. For generic families like that, we know that $\Lambda = \overline{W^u(P_a)}$ (where $W^u(P_a)$ is the unstable manifold associated to P_a) is a strange attractor for each a in a set $E \subset [a_1, a_2]$ with positive Lebesgue measure.

From now on we fix $a \in E$ and write $f = f_a$ and write $P = P_a$.

In order to describe precisely a context for what follows, let us introduce some terminology and notation, besides a few results mostly present in [BC91],[MV93] and [Via93].

- Having fixed b and δ as above, we choose real numbers $\alpha, \beta, \sigma_1, \sigma_2$ satisfying $\sqrt{e} < \sigma_1 < \sigma_2 < 2$ and $\delta \ll \alpha \ll \beta \ll 1$.
- The strip $\mathbf{I}(\delta) = (-\delta, \delta) \times \mathbb{R}^q$ will be referred to as the *critical region*.
- For $z \in W^u$, let $\boldsymbol{t}(z)$ represent a unit vector tangent to W^u at z.
- Throughout the text, we will use consistently C (and c) to represent an arbitrary large (respectively small) constant not depending on δ or b.
- We represent a generic point in $\mathbb{R} \times \mathbb{R}^q$ as (x, y) and the projection on the first coordinate by $\pi_x(\cdot)$.
- The slope of a vector $(x, \boldsymbol{y}) \in \mathbb{R} \times \mathbb{R}^q$ with respect to the horizontal direction (corresponding to the first coordinate) is defined as
slope$(x, \boldsymbol{y}) = \|\boldsymbol{y}\| \,/\, |x|$.

- The vector $(1,0,0,\cdots,0)$ will be represented by $\boldsymbol{w}_0$.
- Given points z and w we write $z \prec w$ whenever $\pi_x(z) < \pi_x(w)$.
- The notation $\mathrm{dist}_{\mathrm{H}}(z,w)$ stands for the horizontal distance from z to w.
- Given a point z in W^u and a parametrization γ of W^u near z, the notation $\gamma(z,\varepsilon)$ stands for the piece of unstable manifold extending ε around z.

Definition 1.1 ($C^2(\varepsilon)$ curve). Given $\varepsilon > 0$, a curve $x \mapsto (x,y(x)) \in \mathbb{R}\times\mathbb{R}^q$ is said to be $C^2(\varepsilon)$ if $\|y(x)\|$, $\|y'(x)\|$ and $\|y''(x)\|$ are bounded by ε for all x.

Definition 1.2 (Tangential position). Let z and w be points in $\mathbb{R}\times\mathbb{R}^q$ associated to respective tangent vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. One says that $(z,\boldsymbol{u})$ is in *tangential position* with respect to $(w,\boldsymbol{v})$ if there exists a $C^2(1/10)$ curve containing both points and tangent to $\boldsymbol{u}$ at z and to $\boldsymbol{v}$ at w.

Definition 1.3 (Expanding points). Given $\lambda > 0$, a point z is said to be *λ-expanding up to time n* if $\left\|Df^j(z)\boldsymbol{w}_0\right\| \geq \lambda^j$, for all $1 \leq j \leq n$. If n can be taken arbitrarily large we say simply that z is *λ-expanding*.

Proposition 1.4 (Critical set).
Let f be as above. There exists a countable set $\mathcal{C} \in W^u(P)\cap\mathbf{I}(\delta)$ whose members ζ are called critical points *and satisfy:*

(a) $\pi_x(\zeta) \leq C\sqrt{b}$ *;*

(b) $\mathrm{slope}(\boldsymbol{t}(\zeta)) \leq C\sqrt{b}$*;*

(c) $\mathrm{slope}(\boldsymbol{t}(f(\zeta)) \geq c/\sqrt{b}$*;*

(d) $\|Df^n(f(\zeta)).\boldsymbol{t}(f(\zeta))\| \leq (Cb)^n$*;*

(e) $\|Df^n(f(\zeta)).\boldsymbol{w}_0\| \geq \sigma_1^n$*;*

(f) For each positive integer n such that $f^n(\zeta)$ is in $\mathbf{I}(\delta)$ there exists a critical point $\widetilde{\zeta} \in \mathcal{C}$ such that $(f^n(\zeta),\boldsymbol{w}_0)$ is in tangential position with respect to $(\widetilde{\zeta},\boldsymbol{t}(\widetilde{\zeta}))$.

Moreover there is a critical point $\zeta \in \mathcal{C}$ whose orbit is dense in the attractor.

We write W^u as union of compact segments G_k, $k \geq 0$, which are inductively defined: G_0 is the segment in W^u joining $f^2(\bar{z})$ to $f(\bar{z})$ where $\bar{z}$ is the point in $W^u \cap \{(x,y)\ ;\ x=0\}$ closest to P_a (in W^u). For $k>0$ we put $G_k = f^k(G_0) \setminus f^{k-1}(G_0)$.

Now we collect some facts relating the critical set $\mathcal{C}$ to the segments G_k.

Proposition 1.5 (Critical points and generations).
Let $\mathcal{C}_k = \mathcal{C} \cap G_k$. Then the following holds:

(a) *$\mathcal{C}_k$ is finite for every $k \geq 0$. In fact $\mathcal{C}_k$ contains exactly one point for $k = 0, 1$;*

(b) *There exists a real number $\rho > 0$ such that for all $k \geq 0$, if $\zeta \in \mathcal{C}_k$ then $\gamma(\zeta, \delta\rho^k)$ is a $C^2(C\sqrt{b})$ curve;*

(c) *For all $k>0$, if $\zeta \in \mathcal{C}_k$ then there exists $\widetilde{k} < k$ such that $\mathcal{C}_{\widetilde{k}}$ contains a critical point $\widetilde{\zeta}$ with $\operatorname{dist}(\zeta, \widetilde{\zeta}) \leq b^{k/10}$.*

Remark 1.6. We can suppose that every $z \in G_k$ that is σ_1-expanding and such that $\gamma(z, \delta\rho^k)$ is a $C^2(C\sqrt{b})$ curve is effectively in $\mathcal{C}$ (see [MV93], section 4).

If z is a λ-expanding point up to time m then sufficiently near z all points have the same property. Then in a neighbourhood of z it is possible to define a vector field tangent to the most expanding direction at each point. Such a field will be represented by $\boldsymbol{f}^{(m)}(\cdot)$ and in view of the next proposition, it will often be more convenient to write it as $(1, \widetilde{\boldsymbol{f}}^{(m)}(\cdot))$. Furthermore, we will use $\boldsymbol{E}^{(m)}(\cdot)$ to represent the hyperplane $\{\boldsymbol{f}^{(m)}(\cdot)\}^{\perp}$ which naturally contains the maximal contracting directions at the respective point.

Proposition 1.7. *Let ζ be λ-expanding up to time n. Let $\sqrt{b} < \tau < \lambda^4$. If ξ satisfies*

$$\operatorname{dist}(f^j(\xi), f^j(\zeta)) \leq \tau^j \quad \textit{for all} \quad 0 \leq j \leq n-1,$$

then, for any point z in the τ^n-neighbourhood of ξ and for every $1 \leq k \leq n$ we have

(a) *$\boldsymbol{f}^{(k)}(z)$ is defined and $\operatorname{slope}(\boldsymbol{f}^k(z)) \leq C\sqrt{b}$;*

(b) *$\operatorname{angle}(\boldsymbol{f}^{(j)}(z), \boldsymbol{f}^{(k)}(z)) \leq (Cb)^j$ for all $1 \leq j \leq k$;*

(b) *$\left\|Df^j(z)\boldsymbol{e}\right\| \leq (Cb)^j$ for all unit vector $\boldsymbol{e} \in \boldsymbol{E}^{(k)}$ and $1 \leq j \leq k$;*

(c) *$\left\|D\boldsymbol{f}^{(k)}(z)\right\| \leq C\sqrt{b}$ and $\left\|D^2\boldsymbol{f}^{(k)}(z)\right\| \leq C\sqrt{b}$;*

(d) $\left\|D(Df^j \boldsymbol{f}^{(k)})(z)\right\| \leq (Cb)^j$ *for all* $1 \leq j \leq k$*;*

(e) $1/10 \leq \|Df^n(\xi)\boldsymbol{w}_0\| / \|Df^n(z)\boldsymbol{w}_0\| \leq 10$*;*

(f) $\operatorname{angle}(Df^n(\xi)\boldsymbol{w}_0, Df^n(z)\boldsymbol{w}_0) \leq (\sqrt{C\tau})^n$.

The preceding result implies that for points expanding for all times the contractive hyperplanes $\boldsymbol{E}^{(k)}(z)$ converges exponentially to a limit hyperplane $\boldsymbol{E}(z) = \boldsymbol{E}^{\infty}(z)$. In such a case according the next proposition, there exists a (exponentially) contracting hypersurface tangent to $\boldsymbol{E}$ at each point. In the two dimensional setting similar hypersurfaces (curves) exist even for points that are only expanding up to some time $n < \infty$. Unfortunately, this is not the case in higher dimensions: in general distributions of k-dimensional hyperplanes with $k > 1$ are not integrable.

Proposition 1.8. *If z is an expanding point then there exists a neighbourhood $\Gamma(z)$ of z in its stable set $W^s(z)$ that can be parametrized by $\mathbb{R}^q \ni y \mapsto (x(y), y)$, with $\|y\| \leq C\sqrt{b}$ and $\|x'\|, \|x''\| \leq C\sqrt{b}$. Moreover, we have*

$$\operatorname{dist}(f^n(z_1), f^n(z_2)) \leq (Cb)^n \operatorname{dist}(z_1, z_2), \quad \textit{for all } z_1, z_2 \in \Gamma(z) \textit{ and } n \geq 1.$$

If z_1, z_2 are expanding points, then

$$\operatorname{angle}(t_\Gamma(\xi_1), t_\Gamma(\xi_2)) \leq C\sqrt{b}\operatorname{dist}(\xi_1, \xi_2), \quad \textit{for every } \xi_1 \in \Gamma(z_1) \textit{ and } \xi_2 \in \Gamma(z_2),$$

where $t_\Gamma(\xi_i)$ stands for a unit vector tangent to $\Gamma(z_i)$ at $\xi_i, i = 1, 2$.

If we were able to construct hypersurfaces tangent everywhere to the contractive hyperplanes of each finite order, by a limit process we could get the proof of the previous result. However, as explained before integral hypersurfaces for the distribtion $\{\boldsymbol{E}^{(n)}(z)\}_z$ can not be expected to exist in general and so we have to follow another path. What we are going to do is to construct hypersurfaces that are almost-tangent to such a distribution. These hypersurfaces will be contractive too (for the same number of iterates) and will tend, as time goes to infinity, to a hypersurface contractive for all iterates proving the previous proposition.

We postpone the results we have in mind to Section 6 when we will have already defined some necessary notions.

Given a point z which is expanding up to some time m and a tangent vector $\boldsymbol{v}$ at z it is often useful to split this vector into contracting and horizontal direction. Next two lemmas states this procedure in a precise way.

Lemma 1.9. *Let $U \subset \mathbf{I}(\delta)$ be an open and conex set such that in $f(U)$ the λ-expanding directions of order m are defined. Let $\boldsymbol{f}^{(m)}$ be the vector field of these directions normalized in such a way that $\boldsymbol{f}^{(m)} = (1, \widetilde{\boldsymbol{f}}^{(m)})$. Let $\boldsymbol{V}$ be a unit vector field defined in U. Then we can write*

$$Df(z).\boldsymbol{V}(z) = \boldsymbol{e}(z) + \beta(z)\boldsymbol{w}_0 \tag{1.2}$$

with uniquely determined $\beta(z) \in \mathbb{R}$ and $\boldsymbol{e}(z) \in \{\boldsymbol{f}^{(m)}(f(z))\}^{\perp}$ satisfying, for all $z, \widetilde{z} \in U$,

1. $|\beta(z) - \beta(\widetilde{z})| \leq \|\boldsymbol{V}(z) - \boldsymbol{V}(\widetilde{z})\| + 5\,\|z - \widetilde{z}\|$
2. $\|\boldsymbol{e}(z) - \boldsymbol{e}(\widetilde{z})\| \leq \|V(z) - V(\widetilde{z})\| + 10\,\|z - \widetilde{z}\|$

Proof. We write $\boldsymbol{f}(z)$ for $\boldsymbol{f}^{(m)}(f(z))$ and observe that we can always write down (1.2) simply putting

$$\beta(z) = \langle Df(z).\boldsymbol{V}(z), \boldsymbol{f}(z)\rangle$$

and

$$\boldsymbol{e}(z) = Df(z).\boldsymbol{V}(z) - \beta(z)\boldsymbol{w}_0.$$

Naturally we have,

$$\begin{aligned}|\beta(z) - \beta(\widetilde{z})| &= |\langle Df(z).V(z), \boldsymbol{f}(z)\rangle - \langle Df(\widetilde{z}).V(\widetilde{z}), \boldsymbol{f}(\widetilde{z})\rangle| \\ &\leq \|Df(z).V(z) - Df(\widetilde{z}).V(\widetilde{z})\|\,\|\boldsymbol{f}(z)\| + \|\boldsymbol{f}(z) - \boldsymbol{f}(\widetilde{z})\|\end{aligned}$$

But,

$$\|Df(z).V(z) - Df(\widetilde{z}).V(\widetilde{z})\| \leq \|Df(z) - Df(\widetilde{z})\| + \|Df(z)\|\,\|V(z) - V(\widetilde{z})\|$$

Since $\left\|D^2 f\right\| \leq (4 + C\sqrt{b})$ and $z \in \mathbf{I}(\delta)$, we get

$$\|Df(z).V(z) - Df(\widetilde{z}).V(\widetilde{z})\| \leq (4 + C\sqrt{b})\,\|z - \widetilde{z}\| + 5\delta\,\|V(z) - V(\widetilde{z})\|$$

which implies the first item. In a similar way, we get,

$$\|\boldsymbol{e}(z) - \boldsymbol{e}(\widetilde{z})\| \leq \|Df(z).V(z) - Df(\widetilde{z}).V(\widetilde{z})\| + |\beta(z) - \beta(\widetilde{z})|$$

where we can use the bounds just obtained to finish the proof. ☐

We can now restrict the previous reasoning to almost-horizontal curves inside the critical region.

Lemma 1.10. *Let U and $\boldsymbol{f}^{(m)}$ be as before and let $\gamma(s) = f(s, y(s))$, where $(s, y(s))$ is a $C^2(1/5)$ smooth curve on U. Then we can write*

$$\gamma'(s) = \boldsymbol{e}(s) + \beta(s)\boldsymbol{w}_0 \tag{1.3}$$

where $\beta(s) \in \mathbb{R}$ and $\boldsymbol{e}(s) \in \{\boldsymbol{f}^{(m)}(\gamma(s))\}^{\perp}$ are uniquely determined and satisfy, for all s:

1. $\|\boldsymbol{e}(s)\| \leq C\sqrt{b}$ *and* $\|\boldsymbol{e}'(s)\| \leq C\sqrt{b}$
2. $|\partial_s \|\boldsymbol{e}(s)\|| \leq C\sqrt{b}$
3. $|\beta(s) + 2as| \leq C\sqrt{b}$
4. $|\beta'(s) + 2a| \leq C\sqrt{b}$.

Proof. As in the previous lemma, we put

$$\beta(s) = \langle \gamma'(s), \boldsymbol{f}(z) \rangle \tag{1.4}$$

and

$$\boldsymbol{e}(s) = \gamma'(s) - \beta(s)\boldsymbol{w}_0.$$

Let us write $\gamma(s) = (\xi(s), \eta(s))$ and $\boldsymbol{f}(s) = \boldsymbol{f}^{(m)}(\gamma(s))$. In view of our hypothesis on f (see 1.1) we easily can check that $\|\eta(s)\|$, $\|\eta'(s)\|$, $\|\eta''(s)\|$, $|\xi'(s) - 2as|$ and $|\xi''(s) - 2a|$ are all bounded by $C\sqrt{b}$.

In view of (1.3) we must have

$$\beta(s)\langle \boldsymbol{w}_0, \boldsymbol{f}(s) \rangle = \langle \gamma'(s) - \boldsymbol{e}(s), \boldsymbol{f}(s) \rangle = \langle \gamma'(s), \boldsymbol{f}(s) \rangle.$$

But $\boldsymbol{f}(s) = (1, \widetilde{\boldsymbol{f}}(s))$ and $\gamma'(s) = (\xi'(s), \eta'(s))$ and so

$$\xi'(s) - \beta(s) = \langle \eta'(s), \widetilde{\boldsymbol{f}}(s) \rangle.$$

This last relation yields to

$$\left|\xi'(s) - \beta(s)\right| \leq C^2 b \qquad \text{and} \qquad \left|\xi''(s) - \beta'(s)\right| \leq C^2 b.$$

which implies immediately (3) and (4). Since $\boldsymbol{e}(s) = (\xi'(s) - \beta(s), \eta'(s))$, we have (1) and (2) as well. □

Remark 1.11. If the curve above contains a critical point $\zeta = (s_0, y(s_0))$ and is tangent to W^u at this point then the splitting with respect to the contracting hyperplane of order m will give us $|\beta(s_0)| \leq (Cb)^m$. This is a consequence of (1.4) and the angle estimates in Proposition 1.7.

2. Estimates for the critical region

Proposition 2.1. *If ζ and ξ are distinct critical points then*

$$\frac{\mathrm{dist}_{\mathrm{H}}(\zeta,\xi) - C\sqrt{b}\,\mathrm{angle}(\boldsymbol{t}(\zeta),\boldsymbol{t}(\xi))}{\mathrm{dist}(\zeta,\xi)} \leq C\sqrt{b}$$

for some constant C.

Proof. We choose vectors $\boldsymbol{u} = (1, \widetilde{\boldsymbol{u}})$ and $\boldsymbol{v} = (1, \widetilde{\boldsymbol{v}})$, respectively collinear with $\boldsymbol{t}(\zeta)$ and $\boldsymbol{t}(\xi)$. Naturally, $\|\widetilde{\boldsymbol{u}}\|$ and $\|\widetilde{\boldsymbol{v}}\|$ are bounded by $C\sqrt{b}$.

Let $\gamma(t)$ be a parametrization by arc length of the linear segment joining ζ to ξ, that is to say,

$$\gamma(t) = \frac{d-t}{d}\zeta + \frac{t}{d}\xi$$

where $d = \mathrm{dist}(\zeta,\xi)$. Similarly, we define a vector field $\boldsymbol{V}$ on γ putting

$$\boldsymbol{V}(t) = \frac{d-t}{d}\boldsymbol{u} + \frac{t}{d}\boldsymbol{v} = \left(1, \frac{d-t}{d}\widetilde{\boldsymbol{u}} + \frac{t}{d}\widetilde{\boldsymbol{v}}\right) := (1, \widetilde{\boldsymbol{V}}(t))$$

In view of (1.1) we have

$$Df(\gamma(t)).\boldsymbol{V}(t) = \begin{bmatrix} -2a\pi_x(\gamma(t)) + R_1(\gamma(t)) + R_2(\gamma(t)).\widetilde{\boldsymbol{V}}(t) \\ R_3(\gamma(t)) + R_4(\gamma(t)).\widetilde{\boldsymbol{V}}(t) \end{bmatrix} := \begin{bmatrix} W_1(t) \\ W_2(t) \end{bmatrix}$$

and since $\partial_t(\pi_x(\gamma(t))) = \partial_t(\pi_x(\zeta) + td^{-1}(\pi_x(\xi) - \pi_x(\zeta)) = \mathrm{dist}_{\mathrm{H}}(\zeta,\xi)/d$, we get

$$\begin{aligned} W_1'(t) = {} & -2a(d_H/d) + \\ & DR_1(\gamma(t)).\gamma'(t) + DR_2(\gamma(t)).\gamma'(t).\widetilde{V}(t) + R_2(\gamma(t)).\widetilde{\boldsymbol{V}}'(t) \\ W_2'(t) = {} & DR_3(\gamma(t)).\gamma'(t) + DR_4(\gamma(t)).\gamma'(t).\widetilde{V}(t) + R_4(\gamma(t)).\widetilde{\boldsymbol{V}}'(t). \end{aligned}$$

where $d_H = \mathrm{dist}_{\mathrm{H}}(\zeta,\xi)$.

Since ζ and ξ are critical points, we can choose a maximal integer m such that all points on $f(\gamma)$ are expanding up to time m. From Proposition 1.7 we know that $d \approx \tau^m \gg (Cb)^m$. Let $\boldsymbol{f}^{(m)} = (1, \widetilde{\boldsymbol{f}}^{(m)})$ be the field of expanding directions defined on a neighbourhood of $f(\gamma)$. Let us write $\boldsymbol{f}(t)$ for $\boldsymbol{f}^{(m)}(f(\gamma(t)))$ and $\widetilde{\boldsymbol{f}}(t)$

for $\widetilde{\boldsymbol{f}}^{(m)}(f(\gamma(t)))$. By splitting $Df(\gamma(t)).\boldsymbol{V}(t)$ into contractive and expanding directions, we get

$$Df(\gamma(t)).\boldsymbol{V}(t) = \boldsymbol{e}(t) + \beta(t)\boldsymbol{w}_0 \quad \text{with } \boldsymbol{e}(t) \in \{\boldsymbol{f}(t)\}^{\perp}$$

and so we must have

$$0 = \langle Df(\gamma(t)).\boldsymbol{V}(t) - \beta(t)\boldsymbol{w}_0, \boldsymbol{f}(t)\rangle = W_1(t) - \beta(t) + W_2(t).\widetilde{\boldsymbol{f}}(t)$$

Hence,

$$\beta(t) = W_1(t) + W_2(t).\widetilde{\boldsymbol{f}}(t) \tag{2.1}$$

$$\beta'(t) = W_1'(t) + W_2'(t).\widetilde{\boldsymbol{f}}(t) + W_2(t).D\widetilde{\boldsymbol{f}}(t). \tag{2.2}$$

We have, for all $t \in [0,d]$,

$$\begin{aligned}
\left\|\widetilde{\boldsymbol{V}}(t)\right\| &\leq C\sqrt{b} \\
\left\|\widetilde{\boldsymbol{V}}'(t)\right\| &\leq \frac{\|\boldsymbol{u}-\boldsymbol{v}\|}{\|\zeta-\xi\|} = d_A/d \\
\left\|\widetilde{\boldsymbol{f}}(t)\right\| &\leq C\sqrt{b}, \\
\left\|D\widetilde{\boldsymbol{f}}(t)\right\| &\leq C\sqrt{b}.
\end{aligned}$$

where $d_A = \|\boldsymbol{u}-\boldsymbol{v}\| \approx \text{angle}(\boldsymbol{t}(\zeta), \boldsymbol{t}(\xi))$. Furthermore, using the bounds for $R_j, j = 1,2$ and their derivatives, we finally get

$$\left|\beta'(t) + 2a(d_H/d)\right| \leq 2C\sqrt{b}(3 + d_A/d)$$

and so

$$\min_{t\in[0,d]} \left|\beta'(t)\right| \geq (d_H/d) - C\sqrt{b}(d_A/d) - 3C\sqrt{b}$$

The fact that ζ and ξ are critical points implies that $|\beta(\zeta)|$ and $|\beta(\xi)|$ are bounded by $(Cb)^m$ and so,

$$2(Cb)^m \geq |\beta(\zeta) - \beta(\xi)| \geq \min_{t\in[0,d]} \left|\beta'(t)\right| |\zeta - \xi|$$

Since $|\zeta - \xi| = d \gg (Cb)^m$ we can suppose that $2d^{-1}(Cb)^m < C\sqrt{b}$. Putting these estimates together, we conclude that

$$\frac{d_H - d_A C\sqrt{b}}{d} - 3C\sqrt{b} \leq C\sqrt{b}$$

which implies immediately the Lemma.

□

Corollary 2.2. *Let z_0 and z_1 be critical points. Then we can not have a smooth $C^2(1/3)$ curve joining z_0 to z_1 and tangent at these points respectively to $\boldsymbol{t}(z_0)$ and $\boldsymbol{t}(z_1)$.*

Proof. The hypothesis implies

$$\operatorname{angle}(\boldsymbol{t}(z_0), \boldsymbol{t}(z_1)) \leq (2/3)\operatorname{dist}_{\mathrm{H}}(z_0, z_1)$$

and

$$\operatorname{dist}(z_0, z_1) \leq (4/3)\operatorname{dist}_{\mathrm{H}}(z_0, z_1).$$

So, we get, using the same notations as in the previous proposition,

$$\frac{d_H - d_A C\sqrt{b}}{d} \geq (3 - C\sqrt{b})/4 \gg C\sqrt{b},$$

which contradicts the previous result.

□

Corollary 2.3. *Let z be a point and $\boldsymbol{v}$ be a tangent vector at z. If $(z, \boldsymbol{v})$ is simultaneously in tangential position with respect to $(z_0, \boldsymbol{t}(z_0))$ and $(z_1, \boldsymbol{t}(z_1))$, where $z_0 \prec z_1$ are critical points, then we can not have $z_0 \prec z \prec z_1$.*

From Proposition 1.7 it is easy to verify that every accumulation point of $\mathcal{C}$ is also expanding for all times. But this is not sufficient to guarantee that such a point belongs to $\mathcal{C}$. This is intimately related to the (inductive) way how $\mathcal{C}$ is constructed (see [BC91],[MV93] and [Via93]). Next Lemma states a sufficient condition however, to guarantee that a point in W^u arbitrarily close to $\mathcal{C}$ actually be in $\mathcal{C}$. Recall that for points $z \in W^u$ we write $\gamma(z, \varepsilon)$ for a neighbourhood of z in W^u extending ε to each side of z.

Lemma 2.4. *Let ρ be the constant appearing in Proposition 1.5. Let $z \in G_n \cap \operatorname{clos}(\mathcal{C})$ be a point of W^u of generation n. If $\gamma(z, \rho^n)$ is a $C^2(C\sqrt{b})$ curve then z is in $\mathcal{C}$.*

Proof. First note that the hypothesis implies z expanding for all times. Choose m sufficient large in order to have $\delta\rho^n < 2\tau^m$ where τ is like in Proposition 1.7 (with

$\lambda = 1$). Therefore all points in $f(\gamma)$ are expanding up to time m. Now we choose $\xi \in \mathcal{C}$ satisfying

$$\|z - \xi\| \leq \tau^m/100 \quad \text{and} \quad \|\boldsymbol{t}(z) - \boldsymbol{t}(\xi)\| \leq \tau^m/100$$

Splitting $\boldsymbol{t}(z)$ and $\boldsymbol{t}(\xi)$ into contracting and horizontal directions as introduced in Lemma 1.9 we get

$$|\beta(z) - \beta(\xi)| \leq 5(\|z - \xi\| + \|\boldsymbol{t}(z) - \boldsymbol{t}(\xi)\|) \leq \tau^m/10 \tag{2.3}$$

But ξ is a critical point (see Remark 1.11) an so $|\beta(\xi)| \leq (Cb)^m \ll \tau^m$ which yields to

$$|\beta(z)| \leq \tau^m/5$$

We know from Lemma 1.10 that $|\beta'(t)| > 2$ for all t. This implies the existence of $\zeta \in \gamma$ with $\beta(\zeta) = 0$ what means that ζ is a critical approximation of order m. Since m is arbitrarily large, we conclude that z is limit of critical approximations located on γ. Hence z is a critical point. (See Remark 1.6). $\square$

Lemma 2.5. *If $z \in W^u \cap \mathbf{I}(\delta)$ and $\gamma(z, \rho^n)$ is a $C^2(C\sqrt{b})$ curve, then*

$$\mathcal{B}(z) := \{\xi \in \mathcal{C}\ ;\ (z, \boldsymbol{t}(z)) \text{ is in tangential position with respect to } (\xi, \boldsymbol{t}(\xi))\}$$

is not empty.

Proof. Let $d = \mathrm{dist}(z, \mathcal{C})$. If $d = 0$ then by Lemma 2.4, $z \in \mathcal{C}$ and $\mathcal{B}(z)$ is trivially not empty. If $d > 0$ then let $\xi \in \mathcal{C}$ be a critical point with dense orbit in the attractor. If $n \geq 1$ is sufficiently large we will have $\mathrm{dist}(\xi_n, z) \ll d$ and $\|\boldsymbol{t}(\xi_n) - \boldsymbol{t}(z)\| \ll d$. Let $\widetilde{\zeta}(\xi_n)$ be the binding critical point for ξ_n assured by Proposition 1.4. Then $\mathrm{dist}(\xi_n, \widetilde{\zeta}(\xi_n)) \geq d/2$ which implies $\widetilde{\zeta}(\xi_n) \in \mathcal{B}(z)$.

$\square$

The tangential position property will be the fundamental fact in order to extend to generic orbits the estimates of growth that holds for critical orbits.

3. Itineraries of points

Given a point z in the domain of our system, almost surely (in a Lebesgue sense) the orbit of z visit the strip $\mathbf{I}(\delta)$ infinitely many times. If n is a integer such that $f^n(z) \in \mathbf{I}(\delta)$, we say that it is a *return time* for z. When this happens, the orbit of z experiences some loss of hyperbolicity. If z is a critical point then we know - see Proposition 1.4, item(f) - that there exists some critical point ζ, near $f^n(z)$, such that $(\zeta, \boldsymbol{t}(\zeta))$ is in tangential position with respect to $(f^n(z), \boldsymbol{v})$, where $\boldsymbol{v}$ is an appropriate almost-horizontal vector. We say that ζ is a *binding (critical) point* for $f^n(z)$ an the existence of such a point is a key fact that permits us by means of a inductive argument to control that loss of hyperbolicity.

The previous section has enabled us to extend this argument to almost all points on W^u.

We are often using the notation $z_n = f^n(z)$ and, if z_n is a return iterate with binding critical point ζ, we write $d_{\mathcal{C}}(z_n) = \text{dist}(z_n, \zeta)$.

We can now describe the itinerary of a typical point $z \in W^u$ as a sequence of a

▪ **Free returns** - corresponding to return iterates occurring outside all bound periods (defined below);

▪ **Bound period** - the maximal piece of orbit after a return while the iterates of z follow closely that of their binding critical point;

▪ **Free period** - the iterates after the end of the bound period associated to a free return and before the next (free) return.

Let us state in a more precise way the notion of bound period. Given a return time n for z with binding critical point ζ we define the bound period of z_n as

$$p(z_n, \zeta) = \max\{p \in \mathbb{N}\,;\, \text{dist}(z_{n+j}, \zeta_j) \leq e^{-\beta j}, \text{ for all } 0 \leq j \leq p\} \tag{3.1}$$

Note that we can have new returns inside bound periods, and this must be (and in fact was) considered while proving the basic results that were referred before. In the context of what follows, we can simply ignore these bound returns.

We finish this section with a well-known result.

Lemma 3.1 (Hyperbolicity outside the critical region).
Let $z \in \mathbf{R}$ be a point whose positive orbit remains outside $\mathbf{I}(\delta)$ up to time n. If $\boldsymbol{v}$ is a unit tangent vector at z with slope less than $1/5$ then

(a) $\left\|Df^j(z)\boldsymbol{v}\right\| \geq c\delta\sigma_2^j$ *for all* $1 \leq j \leq n$;

(b) $\text{slope}(Df^j(z)\boldsymbol{v}) \leq \delta^{-1}C\sqrt{b}$ *for all* $1 \leq j \leq n$.

(c) If either $z \in f(\mathbf{I}(\delta))$ *or* $f^n(z) \in \mathbf{I}(\delta)$ *then* $\|Df^n(z)\boldsymbol{v}\| \geq \sigma_2^n$ *and* $\text{slope}(Df^n(z)\boldsymbol{v}) \leq C\sqrt{b}$.

4. Bound period estimates

This section is devoted to discuss the role of tangential position in achieving the estimates in later sections.

In the sequence we write $w_k(z)$ for $Df^k(f(z))\boldsymbol{w}_0$.

Let ξ be a point in W^u having the following properties:

(a) Expansiveness up to some time p: $\|w_k(\xi)\| \geq \sigma_1^k$ for all $1 \leq k \leq p$;

(b) All returns up to p are tangential. More precisely, for every $0 \leq k \leq p$ such that $\xi_k \in \mathbf{I}(\delta)$ there exists a critical point $\zeta = \widetilde{\zeta}(\xi_k)$ such that $(\xi_k, \boldsymbol{t}(\xi_k))$ is in tangential position with respect to $(\zeta, \boldsymbol{t}(\zeta))$.

(c) The returns have a kind of recurrence control: $\operatorname{dist}(\xi_k, \zeta) \geq e^{-\alpha k}$ for all returns $1 \leq k \leq p$.

Among the points satisfying the requirements above, clearly we have the critical points themself. Nevertheless the properties stated here are all that it is necessary to get the following construction.

Fix a point $\xi \in W^u$ satisfying the three properties above. Let γ be a $C^2(1/10)$ curve passing through ξ and tangent to $\boldsymbol{t}(\xi)$. Let us parametrize γ by $\gamma(s) = (s, y(s))$, with $s \in [s_0, s_1]$ and suppose that $\gamma(s_0) = \xi$. To simplify notations we write $\boldsymbol{w}_j(s) = \boldsymbol{w}_j(f(\gamma(s)))$, for any $j \geq 0$.

We extend the notion of binding not requiring that the binding point be a critical one. Accordingly we define the bound period of points in γ to ξ exactly in the same way as (3.1).

Let us assume that all points in γ remains bounded to ξ for p iterates. This implies that at each return $1 \leq k \leq p$ we have

$$\operatorname{dist}(f^k(\gamma(s), f^k(\xi)) \leq Ce^{-\beta j}$$

and on the other hand

$$\operatorname{dist}(f^k(\xi), \widetilde{\zeta}(f^k(\xi))) \geq e^{-\alpha j} \geq Ce^{-\beta j}.$$

These facts and the three properties above are the fundamental ingredients that permit us to write

$$\boldsymbol{w}_j(s) = \lambda(s)(\boldsymbol{w}_j(s_0) + \varepsilon_j(s)), \tag{4.1}$$

with

$$c \leq \lambda(s) \leq C, \text{ and } \left\|\varepsilon_j(s)\right\| \ll \left\|\boldsymbol{w}_j(s_0)\right\|. \tag{4.2}$$

for all $s \in [s_0, s_1]$

The arguments used to get (4.1) can be found in [BC91, Lemma 7.8] and [MV93, Lemma 10.5].

It follows that $w_j(s) \approx w_j(0) \geq \sigma_1^j$, by item (a) above, and therefore all points $\gamma(s)$ are expanding up to time p.

Let $\boldsymbol{t}(s) = \boldsymbol{t}(f(\gamma(s)))$. Using the splitting algorithm of Lemma 1.10 we get

$$\boldsymbol{t}(s) = \boldsymbol{e}(s) + \beta(s)\boldsymbol{w}_0(s),$$

with $\boldsymbol{e}(s) \in \{f^{(p)}(f(\gamma(s)))\}^{\perp}$.

Iterating by f we get

$$\boldsymbol{t}_j(s) := Df^j(f(\gamma(s)))\boldsymbol{t}(s) = \boldsymbol{e}_j(s) + \beta(s)w_j(s), \tag{4.3}$$

where $\boldsymbol{e}_j(s) = Df^j(f(\gamma(s)))\boldsymbol{e}(s)$.

Fix $\bar{s} \in (s_0, s_1]$ and write $z = \gamma(\bar{s})$. We can estimate $\operatorname{dist}(z_{j+1}, \xi_{j+1})$ as

$$\int_{s_0}^{\bar{s}} \left\|\boldsymbol{t}_j(s)\right\| ds.$$

Using (4.3) and (4.1), we get

$$\begin{aligned}
\operatorname{dist}(z_{j+1}, \xi_{j+1}) &= \int_{s_0}^{\bar{s}} \left\|\boldsymbol{e}_j(s) + \beta(s)\lambda(s)(\boldsymbol{w}_j(s_0) + \varepsilon_j(s))\right\| ds \\
&\leq \int_{s_0}^{\bar{s}} \left\|\beta(s)\lambda(s)(\boldsymbol{w}_j(s_0) + \varepsilon_j(s))\right\| ds + (Cb)^j
\end{aligned}$$

This last integral is bounded by

$$\begin{aligned}
&\left\|\boldsymbol{w}_j(s_0)\right\| \int_{s_0}^{\bar{s}} |\lambda(s)|\,|\beta(s) - \beta(s_0)|\, ds + \\
&\qquad\qquad \int_{s_0}^{\bar{s}} |\lambda(s)|\,|\beta(s_0)|\,\left\|\boldsymbol{w}_j(s_0)\right\| ds + \\
&\qquad\qquad\qquad\qquad \int_{s_0}^{\bar{s}} |\lambda(s)|\,|\beta(s)|\,\left\|\varepsilon_j(s)\right\| ds
\end{aligned}$$

Recall that $|\beta(s) - \beta(s_0)| \approx 2a|s - s_0|$. In particular, if ξ is a critical point, we know that $|\beta(s_0)| \leq (Cb)^p$. With this in mind, and taking into account the lower and upper bounds of $\lambda(s)$ as well as the fact that $\left\|\varepsilon_j(s)\right\| \ll \left\|\boldsymbol{w}_j(s_0)\right\|$ we can conclude that

$$\mathrm{dist}(z_{j+1}, \xi_{j+1}) \approx \|w_j(s_0)\| \,\mathrm{dist}(z, \xi)^2.$$

This key fact is basic for the proof of the next fundamental result. Details can be found in [BV01, Section 2].

Lemma 4.1 (Bound period estimates).
Let $z \in W^u$. If n is a return time for z with $p = p(f^n(z))$ as the corresponding bound period, then

(a) $(1/5)\log(1/d_{\mathcal{C}}(z_n)) \le p \le 5\log(1/d_{\mathcal{C}}(z_n))$;

(b) $\|\boldsymbol{w}_{n+p}(z)\| \ge \sigma_1^{(p+1)/3}\|\boldsymbol{w}_{n-1}(z)\|$ *and* $\operatorname{slope}\boldsymbol{w}_{n+p}(z) \le (C/\delta)\sqrt{b}$;

(c) $\|\boldsymbol{w}_{n+p}(z)\| d_{\mathcal{C}}(z_n) \ge c e^{-\beta(p+1)}\|\boldsymbol{w}_{n-1}(z)\|$;

(d) $\|\boldsymbol{w}_j(z_n)\| \ge \sigma_1^j$ *for* $1 \le j \le p$, *and* $\operatorname{slope}\boldsymbol{w}_p(z_n) \le (C/\delta)\sqrt{b}$.

5. Proof of Theorem A

The proof of Theorem A relies on these fundamental results.

Lemma 5.1 (Orbits ending in free returns).
Let $z \in W^u$ be a free return iterate. Then we have

$$\|Df^{-j}(z).\boldsymbol{t}(z)\| \le e^{-cj}$$

for some $c \ge 0$ and for all $j \ge 0$.

Proof. See [BY93, Lemma 3], [Via93, Lemma 6.2(f)], [MV93, Lemma 9.4] and [BC91, Lemma 7.13]. □

Lemma 5.2 (Maximal free segments).
Maximal free segments of W^u intersecting $\mathbf{I}(\delta)$ are $C^2(C\sqrt{b})$ curves.

Proof. See [BY93, Section 2.2.1] and the arguments in [Via93, Lemma 7.3]. □

Proposition 5.3 (Main Proposition of [BY93]). *If γ is a $C^2(C\sqrt{b})$ curve on W^u intersecting $\mathbf{I}(\delta)$ then there exists a critical point ζ such that for all $z \in \gamma$ we have $(z, \boldsymbol{t}(z))$ is in tangential position with respect to $(\zeta, \boldsymbol{t}(\zeta))$.*

Proof. Observe that it suffices to prove the claim when γ is a maximal free segment. The previous lemma says that int this case γ is an almost-horizontal curve. Furthermore, if some extreme point of γ is in $\mathbf{I}(\delta)$, then it is in a bound return state. Suppose that $|\gamma| \leq \rho^n$ and that z is a extreme point of γ in $\mathbf{I}(\delta)$. Let $\widetilde{\zeta}$ be the critical binding point of z. Let n be the generation of γ and $j < n$ be the minimal integer such that $\hat{z} = f^{-j}(z)$ is in a return time. Let $\hat{\zeta}$ be the binding critical point of $\hat{z}$. Then we have

$$\operatorname{dist}(f^j(\hat{\zeta}), \widetilde{\zeta}) \geq e^{-\alpha j}$$

and

$$\operatorname{dist}(f^j(\hat{\zeta}), f^j(\hat{z})) \leq e^{-\beta j} \ll e^{-\alpha j}.$$

Hence

$$\operatorname{dist}(z, \widetilde{\zeta}) \geq e^{-\alpha j} \geq e^{-\alpha n} \gg \rho^n.$$

This implies that all points in γ are in tangential position with respect to $\widetilde{\zeta}$.

On the other hand, if $|\gamma| \geq \rho^n$ then Lemma 2.5 applies and so, writing

$$\gamma_L = \{z \in \Gamma\,;\ \text{all points } \xi \in \mathcal{B}(z) \text{ satisfy } \xi \prec z\}$$
$$\gamma_R = \{z \in \Gamma\,;\ \text{all points } \xi \in \mathcal{B}(z) \text{ satisfy } z \prec \xi\}$$

and using Corollary 2.3 we have $\gamma = \gamma_L \cup (\gamma \cap \mathcal{C}) \cup \gamma_R$.

If there exists $\zeta \in \gamma \cap \mathcal{C}$ then the assertion is trivial, since γ is $C^2(C\sqrt{b})$ and all points in the curve are in tangential position with ζ.

Otherwise, we have $\gamma = \gamma_L \cup \gamma_R$. It is easy to see that γ_L and γ_R are disjoint open sets in γ and so one of them must be empty. If $\gamma = \gamma_L$ it suffices to choose $\zeta \in \mathcal{B}(z)$ where z is the extreme point to the right of γ. The other case is analogous.

□

With these facts we can construct the SRB measure for f in exactly the same way as in [BY93, Section 3]. Let us briefly sketch the necessary steps.

▪ Let $\zeta_0 = (x_0, y_0)$ be a critical point lying on a $C^2(C\sqrt{b})$ curve γ. Consider the exponential partition

$$(-\delta, \delta) = \cup_{|\mu| \geq \mu_0} I_\mu, \qquad \mu_0 \approx \log(\delta)$$

where $I_\mu = (e^{-\mu-1}, e^{-\mu})$. Each I_μ is at your turn subdivided into μ^2 intervals $I_{\mu j}$ of equal length.

Since we have $|x_0| \ll \delta$ and γ is almost horizontal, the partition above induces easily another partition $\mathcal{P}_{[\zeta_0]}$ on γ centered around ζ_0.

▪ We fix $\mathcal{P}_0 = \mathcal{P}_{[\xi_0]}|\Delta$ where ξ_0 is the critical point of generation zero and Δ is the component of $\mathcal{P}_0$ corresponding to $[e^{-(\mu_0+1)}, e^{-\mu_0})$.

▪ Next define a sequence of partitions $\mathcal{P}_1, \mathcal{P}_2, \cdots$ which are successive refinements of $\mathcal{P}_0$ and are obtained in such a way that points in the same element of $\omega \in \mathcal{P}_n$ have the same itinerary up to time n - that is to say, the same sequence of returns, critical binding points, bound and free periods, an so on. More precisely, to each $\omega \in \mathcal{P}_n$ there exists a sequence of return times $n_1 \leq n_2 \leq \cdots \leq n_k \leq n$ and, among these returns, some escape times, which are those return times t_i corresponding to iterates where $f^{t_i}(\omega)$ contains some $I_{\mu j}$. For details, see [BY93, Section 3.1].

▪ At this point we need a bounded distortion result:

Lemma 5.4 (Distortion estimate).
If z_1 and z_2 are points of $W^u \cap \mathbf{I}(\delta)$ having the same itinerary up to time n, as induced by $\mathcal{P}_n$, then

$$\frac{\|Df^n(z_1)\boldsymbol{t}(z_1))\|}{\|Df^n(z_2)\boldsymbol{t}(z_2))\|} \leq C$$

for some constant $C > 0$.

Proof. See [BY93, Section 2.5] □

▪ Define

$$m_n = \frac{1}{n}\sum_{k=0}^{n-1} f_*^k(\mathrm{Leb}\,|_\Delta)$$

and

$$\hat{m}_n = \frac{1}{n}\sum_{\substack{\omega\in\mathcal{P}_i\\ t_i(\omega)<n}} f_*^{t_i}(\mathrm{Leb}\,|_\omega)$$

and put $\hat{m}_n^+ = \hat{m}_n|_{\Delta+}$

▪ Next we note that there exists a subsequence $(n_j)_j$ such that $(\hat{m}_{n_j}^+)_j$ and $(m_{n_j})_j$ converge respectively to measures $\hat{\lambda}$ and $\hat{\mu}$. Let us write $\boldsymbol{I}(\Delta^+) = \Delta^+ \times \mathbb{R}^q$ and $W_{\Delta^+}^u(z)$ the component of $W^u(z) \cap \boldsymbol{I}(\Delta^+)$. Let $X \subset \boldsymbol{I}(\Delta^+)$ be such that for all $z \in X$ we have $W_{\Delta^+}^u(z) \subset X$ and $\mathrm{Leb}(\mathbb{R}^q - X) = 0$. Let $\mathcal{Q}$ be a partition of X into $W_{\Delta^+}^u$-leaves and $\{\hat{\lambda}_z^{\mathcal{Q}}\}$ be a canonical family of conditional measures of $\hat{\lambda}$ with respect to the partition $\mathcal{Q}$. Let $\{m_z^{\mathcal{Q}}\}$ be the analogous family for the Lebesgue measure m.

▪ The measures $\hat{\lambda}_z^{\mathcal{Q}}$ and $m_z^{\mathcal{Q}}$ are equivalent for $\hat{\lambda}$-a.e. z.

▪ The measures $(\hat{\mu}|X)_z^{\Omega}$ and m_z^{Ω} are equivalent for $\hat{\lambda}$-a.e. z.

Define $\widetilde{X} = \{z\,;\, d\hat{\lambda}/d\hat{\mu} > 0\}$ and let $\widetilde{\mu}$ be the saturation of $\hat{\mu}|\widetilde{X}$ under f. That is,

$$\widetilde{\mu} = \sum_{n=0}^{\infty} f_*^n(\hat{\mu}|(\widetilde{X} \cap \{R > n\}))$$

where $R : \widetilde{X} \to \mathbb{Z}^+$ is the first return time to $\widetilde{X}$ under f.

▪ Both measures $\hat{\mu}|\widetilde{X}$ and $\widetilde{\mu}$ have absolutely continuous conditional measures on W^u-leaves.

All we have to do is to choose μ as one of the ergodic components of $\widetilde{\mu}$.

Uniqueness of μ follows easily from the fact that, by construction, Lebesgue almost all points of W^u are generic for μ. Since, for each SRB measure ν supported in Λ there must exists a Lebesgue positive set of points ν-generic in W^u we are forced to conclude that $\nu = \mu$.

6. Stable leaves

Our aim now is to construct contracting hypersurfaces which can play the role of stable curves of finite order in dimension two.

Lemma 6.1. *Let n be a free return for a point ξ which is λ-expanding up to time n. Then there exists a hypersurface $\Gamma(\xi)$ passing through ξ that is exponentially contracted up to time n at a rate of $(C_1 b)$ for some constant $C_1 > 0$. Furthermore this hypersurface admits a parametrization of the form $(x(y), y)$, with $\|y\| \le C\sqrt{b}$ and $\|D_y x(y)\| \le C\sqrt{b}$.*

Proof. Let $U \subset \mathbb{R}^{q+1}$ be the set of all points that are expanding for f up to time n. Let $H = \{x = \pi_x(f^n(\xi))\}$ be the vertical hyperplane passing through $f^n(\xi)$. We define $\Gamma(\xi) = f^{-n}(H) \cap U(\xi)$ where $U(\xi)$ is the conex component of U containing ξ.

Let $\boldsymbol{v}_0$ be a tangent vector to $\Gamma(\xi)$ at a point z and let $\boldsymbol{f}^{(n)}(z)$ be its maximal n-expanding direction. Let us split $\boldsymbol{v}_0$ into contracting and expanding components as

$$\boldsymbol{v}_0 = \boldsymbol{e}_0 + \beta \boldsymbol{f}_0$$

where $\boldsymbol{f}_0 = \boldsymbol{f}^{(n)}$ and $\boldsymbol{e}_0$ is a unit vector in $\{\boldsymbol{f}_0\}^{\perp}$. Let $\boldsymbol{v}_j = Df^j(z).\boldsymbol{v}_0$, $\boldsymbol{f}_j = Df^j(z).\boldsymbol{f}_0$ and $\boldsymbol{e}_j = Df^j(z).\boldsymbol{e}_0$ for $0 \le j \le n$. Note that $\boldsymbol{e}_n = \{\boldsymbol{f}_n\}^{\perp}$ and so the slope of $\boldsymbol{v}_n$ with respect to the direction of $\boldsymbol{e}_n$ is given by $|\beta|\,\|\boldsymbol{f}_n\| / \|\boldsymbol{e}_n\|$. Since n is a free return, by Lemma 3.1 we have slope$(\boldsymbol{f}_n) \le C\sqrt{b}$. This turns out to be the bound for the angle between $\boldsymbol{e}_n$ and the vertical hyperplane $\{x = \pi_x(f^n(z))\}$. Hence if $\boldsymbol{v}_n$ is tangent to such a hyperplane then we must have

$$|\beta| \leq \frac{\|\boldsymbol{e}_n\|}{\|\boldsymbol{f}_n\|} C\sqrt{b}.$$

Since

$$\|\boldsymbol{v}_j\| \leq \|\boldsymbol{e}_j\| + |\beta| \, \|\boldsymbol{f}_j\|$$

we get

$$\begin{aligned}
\|\boldsymbol{v}_j\| &\leq (Cb)^j + 4^j C\sqrt{b}(\lambda^{-1}Cb)^n \\
&\leq 4^{-j}(4Cb)^j + C\sqrt{b}(4\lambda^{-1}Cb)^{n-j}(4\lambda^{-1}Cb)^j \\
&\leq (4\lambda^{-1}Cb)^j
\end{aligned}$$

for all $1 \leq j \leq n$.

This proves that all tangent vectors to $\Gamma(\xi)$ are expanding up to time n. A parametrization like in the claim can be found as a direct consequence of the slope estimates in the argument above.

□

7. Symbolic dynamics

We will construct now the equivalent to the partitions defined in [BV01]. There, the authors work with *rectangles*, i.e., regions bounded by two segments of $W^u(P)$ and two stable leaves, in that case one-dimensional manifolds. Here we have to circumvent the problem that with pieces of $W^u(P)$ and almost-vertical codimension-one manifolds we can not determine a canonical region. The approach we are going to follow is to choose, with some flexibility, two-dimensional surfaces bounded by segments of the unstable manifold and by contractive curves. With some machinery we are able to proof that these rectangles will have the same dynamical behaviour as your counterpart in dimension two.

First of all, we need to establish the existence of abundant long stable leaves in order to be able of doing any partitioning of the described region.

We know from Proposition 1.8 that there exists stable leaves passing through each expanding point. Hence our goal now is to find expanding points near critical values. Next lemma states sufficient condition in order to get expansiveness.

Lemma 7.1. *Let $z \in \mathbf{I}(\delta)$ and $k \geq 1$. Let $1 \leq n_1 \leq \cdots \leq n_s \leq k$ be the return times of z up to k. Suppose that for every $1 \leq j \leq s$ there exists a critical point ξ_j such that $(f^{n_j}(z), w_{n_j-1}(f(z)))$ is in tangential position with respect to $(\xi_j, \boldsymbol{t}(\xi_j))$. Then*

(a) If $\operatorname{dist}(f^{n_j}(z), \xi_j) \geq e^{-2\beta n_j}$ for all $1 \leq j \leq s$ then $f(z)$ is $\sqrt[5]{\sigma}_1$-expanding up to time k;

(b) If $\operatorname{dist}(f^{n_j}(z), \xi_j) \geq e^{-5n_j}$ for all $1 \leq j \leq s$ then $f(z)$ is λ-expanding up to time k, with $\lambda \geq 10^{-20}$.

Proof. See Lemma 3.4 and Remark 3.2 in [BV01]. □

We call a point satisfying the hypotheses of the preceding lemma as having *controlled returns*.

Next proposition will establish the existence of a large number of such points nearby every critical point. This will enables us to construct a family of long stable leaves accumulating exponentially the long stable leave passing through a given critical value.

Proposition 7.2 (Families of stable leaves).
Let ζ be a critical point and let $\Gamma = \{(x(y), y) \; ; \; \|y\| \leq C\sqrt{b}\}$ be its long stable leave. There exists a family of long stable leaves $\{\Gamma_{r,l} \; ; \; r \geq \Delta, 0 \leq l \leq r^2\}$ satisfying:

(a) $\operatorname{dist}_H(\Gamma_{r,l}, \Gamma) \approx e^{-2r}$ for every $0 \leq l \leq r^2$;

(b) $\operatorname{dist}_H(\Gamma_{r,l-1}, \Gamma_{r,l}) \approx e^{-2r}/r^2$ for every $1 \leq l \leq r^2$;

(c) For $l > 0$ $\Gamma_{r,l}$ is to the right of $\Gamma_{r,l-1}$.

(d) Each $\Gamma_{r,l}$ intersects W^u in a point z which has all its returns controlled in the sense of Lemma 7.1. More precisely,

$$d_{\mathcal{C}}(f^n(z)) \geq e^{-\beta n}$$

for each free return n of z.

The proof of this Proposition is an easy consequence of the following

Lemma 7.3. *Let γ be a free segment of W^u intersecting $\mathbf{I}(\delta)$ and fix $0 < \varepsilon < 1/100$. If $\operatorname{length}(\gamma) \geq \varepsilon d_{\mathcal{C}}(z)$ for all $z \in \gamma$ then there exists a point $z_\gamma \in \gamma$ with controlled returns.*

Proof. See Lemma 3.1 of [BV01] including Remark 3.1. □

8. The rectangles

In order to proof Theorem B, the strategy we are going to follow is closely related to that present in [BV01]. According, given an arc γ of W^u, we consider a good set of points in a neighbourhood of γ and try to get dynamical properties of such points mirroring the behaviour of that arc. Unfortunately, however, the explosion of combinatorial complexity in the geometry of iterates of arbitrary sets leads us to restrict our efforts in stating conditions where the context, while being an almost two-dimensional one, permits us to mimic the arguments of the referred paper.

Recall that we are assuming that $\Lambda = \overline{W^u}$ is a partially hyperbolic set for f. Let us be more precise about that.

Let A be a real number larger than 3. We suppose that in a open neighborhood $V = (-A,A)^{q+1}$ of Λ there exists two continuous subbundles E^{ss} and E^{cu} of TV, such that $T_zV = E^{ss}(z) \oplus E^{cu}(z)$ for all z in V, and there exists a real number $0 < \lambda < 1$ such that

(a) $Df_z.E^{ss}_z = E^{ss}_{f(z)}$ for all $z \in V \cap f^{-1}(V)$;

(b) $Df_z.E^{cu}_z = E^{cu}_{f(z)}$ for all $z \in V \cap f^{-1}(V)$;

(c) $\left\|Df_z|E^{ss}_z\right\| \leq \lambda$;

(d) $\left\|Df_z|E^{ss}_z\right\| \left\|(Df_z|E^{cu}_z)^{-1}\right\| \leq \lambda$.

Furthermore we assume that E^{ss} admits a integral foliation $\mathcal{F}^{ss}$ and is an almost constant subbundle, that is to say, there exists a parallel subbundle (of codimension 2) $\hat{E}$ defined in V and a small real number $\varepsilon_1 > 0$ such that

$$\text{angle}(E^{ss}(z), \hat{E}) \leq \varepsilon_1 \qquad \text{for all } z \in V.$$

Analogously, we admit that E^{cu} is uniformly close to a parallel subbundle $\hat{F}$ of dimension two.

All these assumptions are satisfied for a large class of examples observed while considering families $(\varphi_\mu)_\mu$ unfolding a homoclinic tangency associated to a hyperbolic fixed point p of φ_0 that is sectionally dissipative and whose eigenvalues satisfy: $|\sigma| > 1 > |\lambda_1| > |\lambda_3| \geq \cdots \geq \left|\lambda_q\right|$. For details about this, see [Via93] sections 2 and 3.

Observe that in this context, we are supposing that $D\varphi(p)$ has a unique least contracting eigenvalue. Naturally the existence of exactly one or two least contracting eigenvalues is a robust assumption. Nevertheless Palis and Viana proved in [PV94, Section 5] the following fact: given a C^2 one-parameter family of dissipative diffeomorphisms like φ_μ above which at $\mu = 0$ goes through a homoclinic tangency associated to a hyperbolic fixed (or periodic) point p there exists a sequence of parameters $\widetilde{\mu}_j \to 0$ and points $\widetilde{p}_j \to p$ such that each $\widetilde{p}_j$ is a hiperbolic periodic point for $\widetilde{\varphi}_j$ with an unique weakest contracting eigenvalue.

With this scenario in mind let us describe the construction of our rectangles.

Consider the critical point $\xi_0 \in G_0$. We know that $f(G_0)$ intersects the stable leave Γ^s of the fixed point P exactly in two points. These points together with the critical value $f(\zeta_0)$ determine an arc γ_1 of W^u as shown in Figure 1.

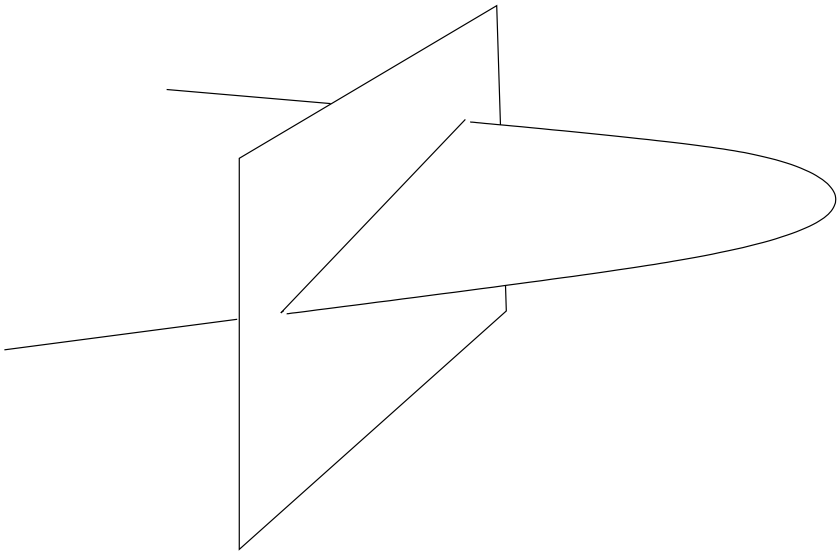

Figure 1: The surface S.

Let S be a compact surface having as border $\gamma_1 \cup \gamma_2$, where γ_2 is a curve on Γ. For all z in γ we can assume that $\boldsymbol{t}(z) \in E^{cu}(z)$ and since this subbundle is almost constant it is easy to get S transversal to $\mathcal{F}^{ss}$. Moreover, we can admit that S is the graph of a function $\psi : \hat{F}(0) \to \mathbb{R}^{q+1}$ satisfying $\|D\psi\| \leq c$ for some small constant c.

The domination property (d) above permits us to conclude that all iterates $f^n(S)$ will be similarly graphs of functions $\psi_n : \hat{F}(0) \to \mathbb{R}^{q+1}$ with the same bounds in their derivatives.

Consider the critical point $\xi_0 \in G_0$. Using Proposition 7.2 we can divide the region between the stable manifold of the fixed point P and the stable leave associated to $f(\zeta_0)$ as shown in Figure 2.

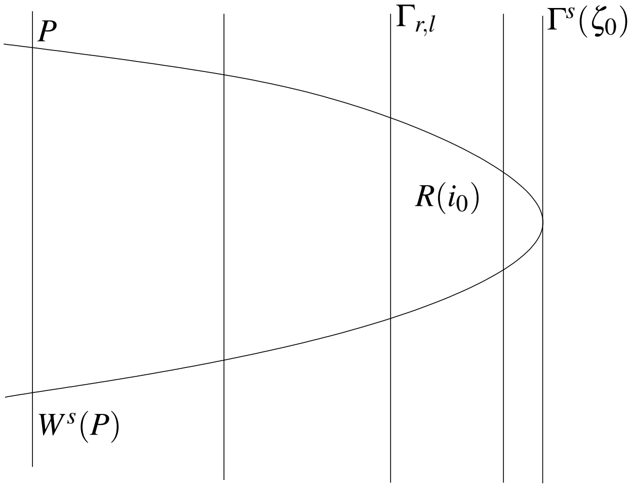

Figure 2: Initial partition.

Observe that two adjacent stable leaves are connected by two segments of W^u. Each pair of adjacent stable leaves determine over the surface S in a natural way a region which we treat as a typical rectangle. Observe that such a region is bounded by two arcs of W^u and two arcs in these stable leaves.

To each rectangle that can be constructed this way we attach an identifying symbol i_0 consisting of the tuple (r, l, ζ_0) associated to the leave $\Gamma_{r,l}$ that defines the left border of the rectangle.

9. Itineraries of rectangles

Let $R(i_0)$ be one of the rectangles whose construction has been just described. We consider the points of $R(i_0)$ as bounded to the critical point ξ_0. During a certain amount of time $R(i_0)$ will follow the orbit of ξ_0. Let us consider the bound period p_1 of $R(i_0)$ as the minimal bound period among all of its points:

$$p_1 = \min\{p(z, \xi_0)\ ;\ z \in R(i_0)\}$$

Let $n_1 > p_1$ be the first time when $\widetilde{R} := f^{n_1}(R(i_0))$ intersects $\mathbf{I}(\delta)$, thats to say, the first free return of $R(i_0)$.

Assume for a while that the following holds:

Claim. The unstable borders of $\widetilde{R}$ are almost horizontal curves and each one contains a critical point or both are simultaneously in tangential position with respect to a same critical point.

Let us describe how to proceed with the inductive partitioning of $R(i_0)$ considering separately the two possibilities.

Case 1. Each unstable border contains a critical point.

The picture looks like in Figure 3. The left side shows the situation at return time n_1. The right side, one iterate after the return.

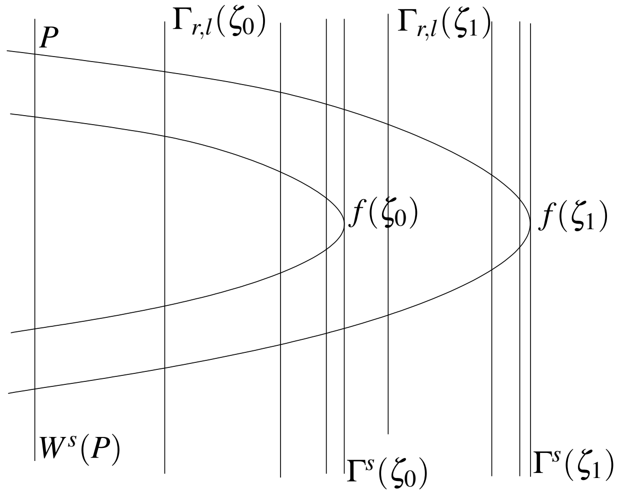

Figure 3: Partition after returns - Two critical points.

Let us call γ_0 and γ_1 the two unstable borders of $\widetilde{R}$ and ζ_0 and ζ_1 their respective critical points. We can suppose we have $f(\zeta_1)$ to the right of $f(\zeta_0)$.

We consider the family of stable leaves

$$\{\Gamma_{r,l}(\zeta_0)\} \cup \{\Gamma_{r,l}(\zeta_1)\ ;\ \Gamma_{r,l}(\zeta_1) \text{ is to the right of } f(\zeta_0)\}.$$

- The strip between the stable leave of the fixed point P and $\Gamma_{\Delta,0}(\zeta_0)$ is identified as the triple $(0,0,\zeta_0)$;

- The strip formed by the leaves $\Gamma_{r,l-1}(\zeta_*)$ and $\Gamma_{r,l}(\zeta_*)$ is identified as the triple (r,l,ζ_*), where $*$ can be 0 or 1;

At your hand, these strips induces a natural partition of $\widetilde{R}$ and so of $R(i_0)$ itself. Namely we put $i_1(z) =$ the triple corresponding to the strip where $f^n(z)$ lies. Accordingly, we call $R(i_0, i_1)$ the subrectangle of $R(i_0)$ formed by these points.

Case 2. There is a common critical point with respect to which the two unstable borders are in tangential position.

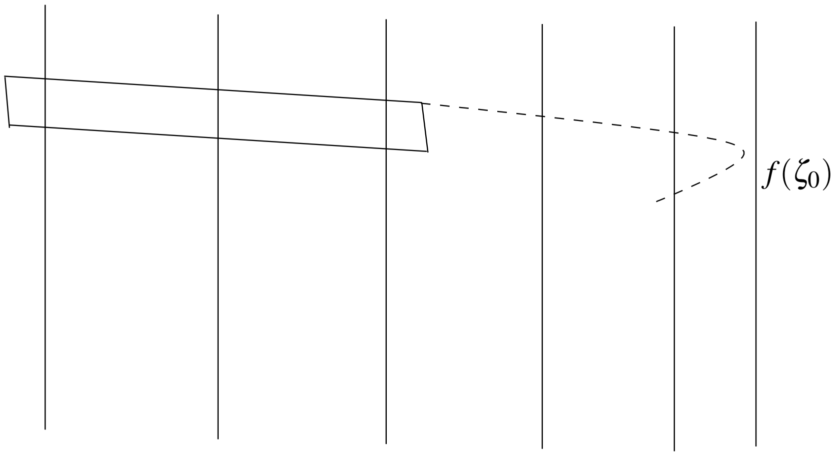

Figure 4: Partition after returns - One critical point.

The respective picture is shown in Figure 4. Let us call ζ_0 the critical point with respect to which the borders of the rectangle $\widetilde{R}$ is in tangential position.

- The strip between the stable leave of the fixed point P and $\Gamma_{\Delta,0}(\zeta_0)$ is identified as the triple $(0,0,\zeta_0)$;

- The strip formed by the leaves $\Gamma_{r,l-1}(\zeta_0)$ and $\Gamma_{r,l}(\zeta_0)$ is identified as the triple (r,l,ζ_0);

In a similar fashion as in the previous case, these triples will be used as the identifying symbols of distinguished itineraries, given place to a collection of subrectangles $R(i_0, i_1)$ of $R(i_0)$.

Let us proceed now by induction. Assume the itinerary of the rectangle $R :=$ $R(i_0, ..i_{k-1})$ has been assigned up to time n_{k-1}. Suppose the triple identified as i_{k-1} be (r,l,ζ). Then we follows the same steps before:

- Consider all points in R as bounded to the critical point ζ.
- Determine the bound period

$$p_{k-1} = \min\{p(z,\zeta)\ ;\ z \in R\}$$

- Determine the next free return

$$n_k = \min\{n > p_{k-1}\ ;\ f(R) \text{ intersects } \mathbf{I}(\delta)\}$$

- Use algorithm above to get the new partitioning of R.

This algorithmic procedure will be completely legitimated when we prove the claim, which we are going to get as a consequence of the next lemma.

Lemma 9.1. *Let γ_0^u and γ_u^1 be the unstable borders of $f^{n_k}(R(i_0,\cdots,i_k)$. Let z_0^+ and z_1^+ be its respective right extreme points. Then the following holds.*

(a) γ_i^u *is a* $C^2(C\sqrt{b})$ *curve;*

(b) $\text{angle}(\boldsymbol{t}(z_0^+),\boldsymbol{t}(z_1^+)) \leq (1/10)\,\text{dist}(z_0^+,\zeta)$*;*

(c) $\text{length}(\gamma_i^s) \leq (1/10)\,\text{dist}(z_0^+,\zeta)$*;*

Items (b) and (c) holds if we replace z_0^+ and z_1^+ by z_0^- and z_1^-, the left extrem points of γ_i.

The proof of this lemma will be given later. Let us show how to proof the claim with the aid of this Lemma. Let ξ_i be the binding critical point of γ_i. Write $d_i = \text{dist}(z_i^+,\xi_i)$ and put

$$d_{max} = \max\{d_0,d_1\} \quad \text{and} \quad d_{min} = \min\{d_0,d_1\}.$$

Let us suppose that $\xi_0 \prec z_0^+$ and $z_1^+ \prec \xi_1$ Let m be an integer such that

$$\tau^{m+1} < d_{max} < \tau^m.$$

Note that $(Cb)^m \ll \tau^{m+1} \leq d_{max}$. Furthermore, all points in $f(\gamma_i)$ are expanding up to time m. Using the sppliting argument of Lemma 1.10, we get

$$\left|\beta(z_0^+)\right| < (Cb)^m - d_0(2a + C\sqrt{b})$$

and

$$\left|\beta(z_1^+)\right| > -(Cb)^m + d_0(2a - C\sqrt{b}).$$

Hence

$$\left|\beta(z_1^+) - \beta(z_0^+)\right| > -2(Cb)^m + 4a\,d_{min} > 2d_{min}.$$

On the other hand, by Lemma 1.9 we know that

$$\left|\beta(z_1^+) - \beta(z_0)\right| \le \left\|\boldsymbol{t}(z_0^+) - \boldsymbol{t}(z_1^+)\right\| + 5\left\|z_0^+ - z_1^+\right\| \le (2/3)d_{min}.$$

These two estimates contradict each other, and so we conclude the claim.

10. Height of the rectangles at return times

Proposition 10.1. *Let $k \ge 1$. Let γ^u be any of the unstable borders of $f^{n_k}(R(i_0, \cdots, i_{k-1}))$ and let ζ be a point in this rectangle. Then there exists a point ξ in γ^u such that*

(a) $\operatorname{dist}(\zeta, \xi) \le (Cb)^{n_k/2}$;

(b) $\operatorname{dist}(f^{-m_{k-1}}(\zeta), f^{-m_{k-1}}(\xi)) \le \min\{10(Cb)^{n_{k-1}/2}e^{r_{k-1}}, 10(Cb)^{n_{k-1}/4}\}$.

Proof. Recall that $n_k = n_{k-1} + m_{k-1} + 1$. Let $R = f^{n_k}(R(i_0, \cdots, i_{k-1}))$. Let $\widetilde{\zeta} = f^{-m_{k-1}}(\zeta)$. Since $\widetilde{\zeta}$ is expanding for the next m_{k-1} iterates and the return n_k is free, we can use Lemma 6.1 and construct a stable hypersurface Γ of order m passing trhough $\widetilde{\zeta}$. This hypersurface will determine a point $\widetilde{\xi}$ in $f^{-m_{k-1}}(\gamma^u)$ as follows. If Γ intersects $f^{-m_{k-1}}(\gamma^u)$ then let $\widetilde{\xi}$ be the intersection point. Otherwise Γ must intersect one of the stable borders of $f^{-m_{k-1}}(R)$ and we let $\widetilde{\xi}$ be the intersection point between this stable border and $f^{-m_{k-1}}(\gamma^u)$. In any case, note that during the next iterates $\widetilde{\xi}$ and $\widetilde{\zeta}$ will get closer at a rate of (Cb). After m_{k-1} iterates (at time n_k) we will have

$$\operatorname{dist}(\zeta, f^{m_{k-1}}(\widetilde{\xi})) = \operatorname{dist}(f^{m_{k-1}}(\widetilde{\zeta}), f^{m_{k-1}}(\widetilde{\xi})) \le \operatorname{dist}(\widetilde{\zeta}, \widetilde{\xi})(Cb)^{m_{k-1}}.$$

We wish to state that it is sufficient to put $\xi = f^{m_{k-1}}(\widetilde{\xi})$ in order to prove part (a). Therefore all we have to do is to find a good upper bound for $\operatorname{dist}(\widetilde{\zeta}, \widetilde{\xi})$ which is precisely the content of part (b). In fact assuming that (b) is true, we get

$$\begin{aligned}\operatorname{dist}(\zeta, \xi) &\le 10(Cb)^{n_{k-1}/2}e^{r_{k-1}}(Cb)^{m_{k-1}} \\ &\le 10(Cb)^{n_k/2}e^{r_{k-1}}(Cb)^{(m_{k-1}-1)/2} \le (Cb)^{n_k/2},\end{aligned}$$

where we use that $r_{k-1} < m_{k-1}$ which implies $e^{r_{k-1}} \ll (Cb)^{(m_{k-1}-1)/2}$.

Now we prove item (b) by induction. At time n_1 we have that $\widetilde{\zeta}$ and $\widetilde{\xi}$ are points in $R(i_0)$, whose height is bounded by $C\sqrt{b}$ and so item (b) is trivially verified. As for $k > 1$, let us to divide the proof into two cases depending on the relative sizes of the rectangle and the distance to its critical binding point at time n_{k-1}.

Case 1. $e^{-r_{k-1}} \leq (Cb)^{n_{k-1}/4}$

Let us parametrize the unstable border $f^{-m_{k-1}-1}(\gamma^u)$ of $f^{-m_{k-1}-1}(R)$ as $\widetilde{\gamma}^u(x) = (x, y(x))$. First of all observe that the induction hypotheses implies the existence of some x_0 such that

$$\operatorname{dist}(f^{-1}(\widetilde{\xi}), \widetilde{\gamma}^u(x_0)) \leq (Cb)^{n_{k-1}/2}$$

Therefore for all x we have,

$$\operatorname{dist}(f^{-1}(\widetilde{\xi}), \widetilde{\gamma}^u(x)) \leq \operatorname{dist}(f^{-1}(\widetilde{\xi}), \widetilde{\gamma}^u(x_0)) + \operatorname{dist}(\widetilde{\gamma}^u(x_0), \widetilde{\gamma}^u(x)).$$

But all points in $f^{-m_{k-1}}(R)$ are at a distance at most $e^{r_{k-1}}$ of the binding critical point ζ_{k-1} and since $\widetilde{\gamma}^u$ is $C^2(C\sqrt{b})$, we get

$$\operatorname{dist}(f^{-1}(\widetilde{\xi}), \widetilde{\gamma}^u(x)) \leq (Cb)^{n_{k-1}/2} + (1 + C\sqrt{b}) \operatorname{dist}(x_0, x) \leq (Cb)^{n_{k-1}/2} + 4e^{-r_{k-1}}.$$

This last inequality implies immediately item(b).

Case 2. $e^{-r_{k-1}} > (Cb)^{n_{k-1}/4}$

See [BV01, Lemma 3.10]. ☐

11. Close returns

From earlier results, we know that whenever a rectangle returns (freely), say at time n_k, and we are faced up with the case when neither unstable border contains a critical point then both borders are in tangential position with respect to a common critical binding point. Naturally the same happens with every point inside the rectangle.

On the other hand, when there are two critical points, one on each unstable border, we can expect that some points in the rectangle be in tangential position with respect to one of these critical point only if the length of the rectangle is substantially bigger than its height. According to the result in the previous section, we have to show that in this case the length is much greater than $(Cb)^{n_k/2}$. Since our aim is to get expansiveness and we already know that tangential position and control of recurrence are the two fundamental conditions for that, we are lead to distinguish returns that are too close of the binding critical point. More precisely we have,

Definition 11.1. Let $(i_0, i_1, \cdots, i_k, \cdots)$ be an itinerary with associated returns $n_1, n_2, \cdots, n_k, \cdots$. Some of these returns will be called *close returns* of this itinerary as follows. We put $\nu_0 = n_0$ and define, inductively, for $k > 0$, the close return ν_k as

$$\nu_k = \min\{n_j > \nu_{k-1} \; ; \; r_j > 5(n_j - \nu_{k-1})\}$$

Note that we can have $\nu_k = \infty$ for some k.

The notion of close returns will help us to characterize when and in which extent subrectangles arising from a repartitioning of a given rectangle having a return time will be at a bad position with respect to to its binding point. More precisely, we are going to show that points following a given itinerary are expanding between successive close returns.

Proposition 11.2. *Let ν_k be a close return for a rectangle $R = R(i_0, \ldots, i_k)$. Then for all iterates $\nu_k < j \leq \nu_{k+1}$ all points in R remain expanding and their returns are tangential. More precisely, if $z \in R$, then*

$$\left\|Df^i(f^{\nu_k+1}(z))\boldsymbol{w}_0\right\| \geq \lambda^i \quad \textit{for } 1 \leq i \leq \nu_{k+1} - \nu_k - 1$$

and for each return time n of z occurring between ν_k and ν_{k+1}, with associated binding critical point $\widetilde{\zeta}$, we have that $(f^n(z), Df^{n-\nu_k-1}(f^{\nu_k+1}(z))\boldsymbol{w}_0)$ is in tangential position with respect to $(\widetilde{\zeta}, \boldsymbol{t}(\widetilde{\zeta}))$.

See [BV01] Section 3, for details. The proof of this proposition is an application of Lemma 7.1. The previous considerations on the height of the rectangles on return times and its length imply the recurrence control present in item (b) of that lemma. As for the tangential position requirement, this is precisely the contents of Lemma 3.11 of [BV01] and is proved much the same way in this context. $\square$

Consider now a rectangle $R = R(i_0, \ldots, i_{k-1})$. Each stable border γ^s of R has (typically) some preimage in a stable leave Γ constructed as in Proposition 7.2. Recall that one such a stable leave cross W^u in a point which have all its returns controlled in the sense of Lemma 7.1(b). If the unstable borders of $f^n(R)$ have a common binding critical point then this recurrence control permits us state that n is not a close return for R.

If there are two critical points, however, some points inside R will necessarily have a close return at time n. On the other side, in a measure theoretical sense, this is a rare phenomenon as will be shown in the next sessions.

12. Angle estimates

In order to get probabilistic estimates about close returns, will state now some Lipschitz bounds on the angles of tangent vectors to the unstable borders of a rectangle at return times.

Proposition 12.1. *There exists a unit vector field Φ defined in $f^{n_k}(R(i_0,\cdots,i_{k-1}))$ tangent to its unstable borders and with $\|D\Phi\| \le Cb^{-1}e^{4(n_k-\nu_s)}$ where ν_s is the last close return strictly before n_k.*

The proposition implies that if $\widetilde{z}_0$ and $\widetilde{z}_1$ are points in distinct unstable borders of $f^{n_k}(R(i_0,\cdots,i_{k-1}))$ then $\|\boldsymbol{t}(\widetilde{z}_0)-\boldsymbol{t}(\widetilde{z}_1)\| \le Cb^{-1}e^{4(n_k-\nu_s)}$.

The strategy of proof is as follows. Let ν_s be the last close return before n_k. We construct a field Φ_0 with good properties in $f^{\nu_s+1}(R(i_0,\cdots,i_{k(s)}))$ and let the derivative of f push forward this field to the return n_k. The field Φ which we are looking for will be the normalized version of $f^{n_k-\nu_s-1}(\Phi_0)$. Lemma 12.2 will establish the existence of Φ_0 taking special attention to the case $s=0$ corresponding to the first return n_1. This enables us to use the Proposition in a inductive way. Under Df the field Φ_0 is expanding according Lemma 12.3. This fact permits us to conclude the proof.

Let $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ be the unstable borders of $f^{\nu_s}(R(i_0,\cdots,i_{k(s)}))$ and let $\widetilde{z}_i(x) = (x, y_i(x))$ be a parametrization of γ_i, for i=0,1. Let $z_i(x) = f(\widetilde{z}_i(x))$.

Let us write $R_{\nu_s+1} = f^{\nu_s+1}(R(i_0,\cdots,i_{k(s)}))$. Then every point in R_{ν_s+1} is expanding up to time $m = m_{k(s)}$. Hence we can use the splitting algorithm and write, for $i=0,1$,

$$z_i'(x) = E_i(x) + \beta_i(x)\boldsymbol{w}_0, \quad \text{with } E_i(x) \in \boldsymbol{E}^{(m)}(z_i(x))$$

Since $|\beta'| \approx 2a$ and cf. Remark 1.11, we can state that

$$|\beta_i(x)| \ge e^{-r} \text{ for all } x. \tag{12.1}$$

We define Φ_0 for points on the unstable borders γ_0 and γ_1 of R_{ν_s+1} putting

$$\Phi_0(z_i(x)) = \frac{E_i(x)}{\beta_i(x)} + w_0.$$

Lemma 12.2. *The field Φ_0 defined above admits an extension to the whole rectangle R_{ν_s+1} satisfying*

(1) $\|\Phi_0(z)\| \leq Ce^r$ *for all $z \in R_{\nu_s+1}$;*

(2) $\|D\Phi_0(z_i(x)))\cdot z_i'(x)\| \leq C\sqrt{b}e^{2r}$ *for all x.*

Proof. First we check that Φ_0 satisfies (1) and (2) for all points in the union of the unstable borders of $f^{\nu_s+1}(R(i_0,\cdots,i_{k(s)}))$. We get $\|\Phi_0(z_0(x))\| \leq Ce^r$ as an immediate consequence of Lemma 1.10.1. and (12.1). Moreover, the derivative of Φ_0 on the tangent directions to the borders satisfies

$$\begin{aligned}\left\|D\Phi_0(z_i(x))\cdot z_i'(x)\right\| &= \frac{\|\beta_i(x)E_i'(x) - \beta_i'(x)E(x)\|}{\beta_i^2(x)} \\ &\leq e^{2r}(4\,|x|\left\|E_i'(x)\right\| + 4\,\|E_i(x)\|) \leq C\sqrt{b}e^{2r}.\end{aligned}$$

In order to extend Φ_0 to the whole rectangle, let us establish the Lipschitz condition

$$\|\Phi_0(z_0(x)) - \Phi_0(z_1(x))\| \leq Cb^{-1}e^{3r}\,\|z_0(x) - z_1(x)\|\,.$$

To prove this, note that Lemma 1.9 gives us

$$\begin{aligned}\left\|\frac{E_0(x)}{\beta_0(x)} - \frac{E_0(x)}{\beta_0(x)}\right\| &\leq e^{2r}\,\|\beta_1(x)(E_0(x) - E_1(x)) - (\beta_0(x) - \beta_1(x))E_1(x)\| \\ &\leq e^{2r}((4\,|x| + C\sqrt{b})\,\|E_0(x) - E_1(x)\| + C\sqrt{b}\,|\beta_0(x) - \beta_1(x)|)\end{aligned}$$

and Lemma 1.10 yields to

$$\|\Phi_0(z_0(x)) - \Phi_0(z_1(x))\| \leq Ce^{2r}(\left\|z_0'(x) - z_1'(x)\right\| + \|\tilde{z}_0(x) - \tilde{z}_1(x)\|). \tag{12.2}$$

Next we find a bound to the angles after a return. Let us write

$$f(x,y) = (1 - ax + R_0(x,y), R_1(x,y), \cdots, R_q(x,y))$$

and observe that

$$\left\|z_0'(x)-z_1'(x)\right\| \le \sum_{j=0}^{j=q} \left\|\partial_x R_j(x,y_0(x)) - \partial_x R_j(x,y_1(x))\right\| + \\ \sum_{j=0}^{j=q} \left\|\partial_y R_j(x,y_0(x))y_0'(x) - \partial_y R_j(x,y_1(x))y_1'(x)\right\|.$$

Each term in the first sum is bounded by $C\sqrt{b}\,\|y_0(x)-y_1(x)\|$. As for the second sum we can bound each term by

$$\left\|\partial_y R_j(x,y_0(x)) - \partial_y R_j(x,y_1(x))\right\| \left\|y_0'(x)\right\| + \left\|\partial_y R_j(x,y_1(x))\right\| \left\|y_0'(x)-y_1'(x)\right\|.$$

By applying the mean value theorem to R_j we can find a point $\widetilde{y}$ in the segment joining $y_0(x)$ to $y_1(x)$ in $\mathbb{R}^q$ satisfying

$$\left\|\partial_y R_j(x,\widetilde{y})\right\| \|y_0(x)-y_1(x)\| = \left\|R_j(y_0(x)) - R_j(y_1(x))\right\|,$$

and so $\left\|\partial_y R_j(x,y_1(x))\right\| \left\|y_0'(x)-y_1'(x)\right\|$ is bounded by

$$\left\|\partial_y R_j(x,y_1(x)) - \partial_y R_j(x,\widetilde{y})\right\| \left\|y_0'(x)-y_1'(x)\right\| + \\ \left\|\partial_y R_j(x,\widetilde{y})\right\| \left\|y_0'(x)-y_1'(x)\right\|. \quad (12.3)$$

Putting these estimates altogether and taking into account that

$$\|y_1(x)-\widetilde{y}\| \le \|y_0(x)-y_1(x)\| = \|\widetilde{z}_0(x)-\widetilde{z}_1(x)\|$$

as well as

$$\left\|R_j(y_0(x)) - R_j(y_1(x))\right\| \le \|z_0(x)-z_1(x)\|,$$

we get

$$\left\|z_0'(x)-z_1'(x)\right\| \le C\sqrt{b}\,\|\widetilde{z}_0(x)-\widetilde{z}_1(x)\| + \\ \left\|y_0'(x)-y_1'(x)\right\| \|y_0(x)-y_1(x)\|^{-1} \|z_0(x)-z_1(x)\|. \quad (12.4)$$

At this point we need to treat the special case when $s=0$. Recall that in this case the unstable borders of $R(i_0)$ are image of segments in G_0 and G_1. Let $d \approx C\sqrt{b}$ be the distance between $G_0 \cap \mathbf{I}(\delta)$ and $G_1 \cap \mathbf{I}(\delta)$. Since $\left\|y_0'(x)-y_1'(x)\right\| \le C\sqrt{b}$ we can state that

$$\left\|y_0'(x)-y_1'(x)\right\| \|y_0(x)-y_1(x)\|^{-1} \le d^{-1}C\sqrt{b} \le b^{-1}e^r.$$

Otherwise if $s > 0$ we assume by induction that Proposition 12.1 holds at time ν_s and this ensures that

$$\left\|y_0'(x) - y_1'(x)\right\| \left\|y_0(x) - y_1(x)\right\|^{-1} \leq Cb^{-1}e^{4(\nu_s - \nu_{s-1})} \leq b^{-1}e^r, \tag{12.5}$$

where we use the fact that $r > 5(\nu_s - \nu_{s-1})$, since ν_s is a close return, and we can make $(\nu_s - \nu_{s-1}) > p_{k(s-1)+1}$ arbitrarily large by decreasing δ.

By applying the mean value theorem to f^{-1} we get $\left\|Df^{-1}\right\| \leq b^{-1}$ and so

$$\left\|\widetilde{z}_0(x) - \widetilde{z}_1(x)\right\| \leq b^{-1} \left\|z_0(x) - z_1(x)\right\|. \tag{12.6}$$

Replacing these last estimates into (12.4) we get

$$\left\|z_0'(x) - z_1'(x)\right\| \leq Cb^{-1}e^r \left\|z_0(x) - z_1(x)\right\|. \tag{12.7}$$

Then combining these last estimates into (12.2) the Lipschitz estimate in (12) is proved.

We can now define the field Φ_0 for points inside the rectangle. Recall that according explained in Section 8, R_{ν_s+1} can be parametrized in such a way that it is almost the inclusion of a compact subset of $\mathbb{R}^2$. This enables us to write, with a slightly abuse of notation,

$$\Phi_0(tz_0(s) + (1-t)z_1(s)) = t\Phi_0(z_0(s)) + (1-t)\Phi_0(z-1(s))$$

where we use tacitly the parametrization referred before.

This extension clearly respect the bounds on the derivative of the field and concludes the proof. □

With the field just constructed we define in a natural way corresponding fields in the next iterates of $R(i_0, \cdots, i_{k(s)})$. Let $m = n_k - \nu_s - 1$. We define $f_*(z,v) = Df(z)v/\left\|Df(z)v\right\|$, and put inductively

$$\Phi_j(z) = f_*(f^{-1}(z), \Phi_{j-1}(f^{-1}(z)))$$

for $1 \leq j \leq m$.

Lemma 12.3. *Given $\zeta \in f^{n_k}(R(i_0, \cdots, i_{k(s)}))$, let $\zeta_i = f^i(\zeta)$ for each $0 \leq i \leq m$. Then*

$$\left\|Df^i(\zeta_i)\Phi_i(\zeta_i)\right\| \geq 1 \quad \textit{for all } 0 \leq i \leq m.$$

Proof. See as Lemma 4.4 of [BV01]. □

To finish the proof of Proposition 12.1, we define $\Phi = \Phi_m$, as defined above and check the bounds on the derivative of Φ. This is accomplished using the previous lemma in exactly the same way as Lemma 4.5 of [BV01].

13. Probabilistic estimates

Our aim now is to establish that close returns are statistically improbable in a sense that will be made precise later.

Lemma 13.1. *Let $R = R(i_0, \ldots, i_{k-1})$ and $S \subset R$ be the set of points in R for which n_k is a close return. If ν_s is the last close return before n_k then there exists $\theta \in (0,1)$ such that*

$$\mathrm{Leb}(S) \leq \min\{\theta^{-1} e^{-(n_k - \nu_s)}, 1 - \theta\} \, \mathrm{Leb}(R)$$

Proof. As remarked before, we have to proof the claim only if $f^{n_k}(R)$ contains two critical points one at each unstable border. The arguments here are identical to those in [BV01], Section 4.3 and so we are very sketch. Let γ^s be any of the stable borders of $\widetilde{R} := f^{n_k}(R)$ and ζ the corresponding critical point. We begin by observing that

$$\mathrm{dist}(\gamma^s, \zeta) \gg e^{-4(n_k - \nu_s)}$$

This is consequence of the recurrence control present in Lemma 7.1(b) and the comments above.

We fix one of the unstable borders of $\widetilde{R}$ and call ζ its critical point. Since the height of $\widetilde{R}$ is much smaller than its length and as a consequence of Corollary 2.2 we conclude that all points of $f^{n_k}(S)$ are in the intersection of the vertical strip extending $e^{-5(n_k - \nu_s)}$ of each side of ζ.

The last paragraph states that from an one-dimensional (more precisely, horizontal) point of view, the proportion of points in $f^{n_k}(S)$ compared with $f^{n_k}(R)$ satisfies a relation as in the claim. However, we need to proof that the distribution of surface area in $f^{n_k}(R)$ is sufficiently well behaved in order to holds this ratio in a two-dimensional setting.

This can be accomplished with the help of the vector field Φ that was constructed over $\widetilde{R}$ in Proposition 12.1. Since this field is Lipschitz, using a Gronwall-type argument we end up with an estimate like

$$\mathrm{Leb}(f^{n_k}(S)) \leq \min\{\theta^{-1} e^{-(n_k - \nu_s)}, 1 - \theta\} \, \mathrm{Leb}(f^{n_k}(R)).$$

To be able to conclude that this relation holds at time 0, we need to know that there is a bounded area distortion for points in R up to time n_k. With this control, which is stated in the next lemma, the claim follows immediately. $\square$

Lemma 13.2 (Area distortion bounds).
If z and w are points of a same rectangle $R = R(i_0, i_1, \cdots, i_k)$ then

$$\frac{\left|\det Df^j|T_zR\right|}{\left|\det Df^j|T_wR\right|} \leq C$$

for every $0 \leq j \leq n_k$.

Proof. Recall that the rectangle R is contained in the initial surface S that was constructed at the beginning of Section 8. This surface is transverse to the strong stable foliation $\mathcal{F}^{ss}$. Using [ABV00, Proposition 2.2] we conclude that

$$\log\left|\det Df|T_zf^j(S)\right|, \quad 0 \leq j \leq n_k,$$

is Hölder continuous. The rest of the argument is exactly the same as [BV01, Lemmas 4.7 and 4.8]. $\square$

Using repeatedly the Lemma 13.1 above, we can easily conclude that a fixed proportion of points in each rectangle have only a finite number of close returns. This is the contents of

Lemma 13.3. *There is some $\theta_0 \in (0,1)$ such that for any rectangle $R = R(i_0, \cdots, i_k)$, the set of points $H \subset R$ for which no return n_j with $j \geq k$ is a close return satisfies*

$$\mathrm{Leb}(H) \geq \theta_0 \,\mathrm{Leb}(R)$$

Proof. See Lemma 4.10 of [BV01]. $\square$

14. Proof of Theorem B

As a direct consequence of Lemma 13.3 we conclude that each rectangle $R(i_0)$ contains a full Lebesgue measure set $\hat{R}(i_0)$ of points that have only a finite number of close returns. For a typical $z \in \hat{R}(i_0)$, let n_k be its last close return. Then, in view of Proposition 11.2, we get that $f^{n_k}(z)$ is expanding for all times and so passing through this point there is a stable leave. Furthermore this stable leave is sufficiently long and cross the unstable manifold W^u in a point ξ which implies at your turn that $f^n(z) \in W^s(\xi)$. Hence the point z itself is contained in the stable set of the attractor. So we can write

$$\bigcup_{i_0} \hat{R}(i_0) \subset \bigcup_{\xi \in \Lambda} W^s(\xi).$$

Recall now that $\mathcal{F}^{ss}$ is absolutely continuous with respect to Lebesgue and note that if $z \in R(i_0) \cap W^s(\xi)$ for some $\xi \in W^u$ then $\mathcal{F}^{ss}(z) \subset W^s(\xi)$, where $\mathcal{F}^{ss}(z)$ is the strong-stable leave passing through z. This enables us to write

$$A \subset \bigcup_{\xi \in \Lambda} W^s(\xi) \quad \text{(up to a zero Lebesgue measure set)}$$

where $A = \bigcup_{i_0} \bigcup_{z \in R(i_0)} \mathcal{F}^{ss}(z)$.

To finish the proof of part (1) of Theorem B we have to show that almost surely (in a Lebesgue sense) points in $B(\Lambda)$ visit the set A. This can be carried out with arguments close to [Via96].

As for the part (2) of Theorem B, pick an arbitrary rectangle $R = R(i_0, i_1, \cdots, i_k)$ and let $H \subset R$ be the set of points without close returns after n_k, as in Lemma 13.3. Each point in $f^{n_k}(H)$ is expanding for all times and the collection of stable leaves passing through these points intersect W^u. Let $\widetilde{H} = W^u \cap f^{n_k}(H)$. Since the set $B(\mu)$ of generic points of μ has full (one dimensional) Lebesgue measure in W^u we end up with $\widetilde{H} \cap B(\mu)$ having zero Lebesgue measure. It follows that the set of points in $f^{n_k}(H)$ that are not generic for μ are precisely those points whose stable leaves intersect W^u in the exceptional set $\widetilde{H} \cap B(\mu)$. The Lipschitz control on these stable leaves present in the second part of Proposition 1.8 yields to $\mathrm{Leb}(f^{n_k}(H)) \setminus B(\mu) = 0$ and so $\mathrm{Leb}(H) \setminus B(\mu) = 0$. Taking into account every possible itinerary for points in R and applying the same arguments before we conclude that almost surely points in the set A are generic for μ. Since we have shown that points in $B(\Lambda)$ visit the set A with probability one the proof of Theorem B is complete.

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