Instituto Nacional de Matemática Pura e Aplicada

The Weighted Extended B-Splines Finite Element Method

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To my family and my little king.

Abstract

When using the finite element method based on domain dependent grids to solve numerically elliptic partial differential equations with Dirichlet boundary condition, we have to consider that constructing and refining appropriate grids and triangulations are not an easy task, specially in dimensions $m > 2$. This difficulty can be overcame by using domain independent uniform grids, B-splines and related objects, such as finite element basis functions. In this monography, we present the main properties of uniform B-splines, uniform weighted B-splines and uniform weighted extended B-splines when used as finite element spaces to solve elliptic partial equation with Dirichlet boundary conditions. We include the error analysis using best approximation estimates.

Keywords: Finite elements method, uniform B-splines, approximation.

Resumo

Ao usar o método dos elementos finitos baseado em malhas dependentes do domínio para resolver numericamente equações diferenciais parciais elípticas com condição de fronteira de Dirichlet, temos que considerar que construir e refinar malhas e triangulacões apropriadamente não são tarefas faceis, especialmente para dimensões $m > 2$. Esta dificuldade pode ser contornada usando malhas uniformes independentes do domínio, B-splines e outros conceitos relacionados, como funções bases de elementos finitos. Nesta dissetação, apresentamos as propriedades principais dos B-splines em malhas uniformes, dos B-splines com pesos em malhas uniformes e dos B-splines estendidos com pesos em malhas uniformes quando são usados como espaços de elementos finitos para resolver equações diferenciais parciais elípticas com condições de fronteira de Dirichlet. Incluí-se a análise de erro usando estimativas de melhor aproximação.

Palavras chaves: Método de elementos finitos, B-splines em malhas uniformes, aproximação.

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Introduction

When speaking of numerical approximation of partial differential equation the Finite Element Method is one of the most employed numerical schemes, especially for elliptic equations.

Consider the following elliptic partial differential equation with Dirichlet boundary conditions:

$$\begin{cases} Lu &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $Lu = -\nabla \cdot (a\nabla u) + cu$ with $a(x) \geq a_0 > 0$ in Ω and $c(x) \geq 0$ in Ω.

The Finite Element Method for solving (1) consists basically of two steps. Step 1: Construct a weak formulation of (1), i.e., to pose the problem in an appropriate Hilbert space H. In general the new problem has the form:

$$\begin{cases} \text{Find } u \in H \text{ such that:} \\ A(u, v) = F(v) \quad \forall v \in H, \end{cases} \tag{2}$$

where A is a continuous and elliptic bilinear form (See Definition 4.1 and the example after it) and F is a linear functional on H.
Step 2: Given a weak formulation (2) posed in H, replace H (infinite dimensional) by a finite dimensional space $V_h \subseteq H$, and we obtain a discrete problem of the form:

$$\begin{cases} \text{Find } u_h \in V_h \text{ such that:} \\ A(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h, \end{cases} \tag{3}$$

with solution u_h known as the Galerkin approximation of u (solution of (1)). Once an appropriated basis for V_h is chosen, (3) becomes a well behaved linear system.

How to choose V_h? In general there are two ways of defining V_h. V_h can depend on or be independent of a grid or a triangulation. In the case of a mesh dependence choice of V_h, the mesh can itself depend on the domain Ω or not.

When using domain dependent grids we have to consider that constructing and refining appropriate grids and triangulations are not easy tasks especially

in dimensions $m > 2$. See [1] for a precise definition of "appropriated triangulation". On the other hand in dimension $m = 3$ or more, the data structure needed to handle triangulation information becomes complicated.
This difficulty can be overcame by using domain independent uniform grids, but in this case it is difficult to represent of the domain boundary information.

After choosing a uniform grid independent of the domain Ω we have several alternative choices for V_h and its basis. One of this choices is B-splines or related objects as basis functions and V_h the finite dimensional spaces spanned by them.

The advantages of using B-splines are that we have:

- Simple basis functions.
- No domain dependent grid is required.
- Accurate approximations are possible with relatively low dimensional subspaces.
- Smoothness of the Galerkin approximation can be chosen arbitrarily.
- Approximation can be chosen arbitrarily.

Moreover, even though it is not studied here, we mention that this approach is suitable for parallelization and multi-grid techniques.

In this monograph we organize the material from simpler one dimensional results to higher dimensional situations and then present the application to Finite Element Method for elliptic partial differential equation.

We introduce Sobolev spaces, Hausdorff metric and the weight functions in Chapter 1. In Chapter 2 we present all the material about B-splines in one dimension that we are going to use on the rest of work. All definitions are based in uniform grids, that is, with uniform B-splines. See e.g., [4] for definition of non-uniform B-spines. Working with uniform grids will permit us to use tensor product in order to extend the definition to higher dimension which is done in Chapter 3. Later on, in Chapter 3, we introduce three finite dimensional spaces: the space of B-spline $\mathfrak{B}_h^n(\Omega)$ with support in Ω, the space of weighted B-spline $w\mathfrak{B}_h^n(\Omega)$ with support in Ω, and the space of weighted extended B-spline $web\mathfrak{B}_h^n(\Omega)$ with support in Ω. In principle all of these three spaces can serve as a Finite Element Space V_h. We also are going to show that $w\mathfrak{B}_h^n(\Omega)$ and $web\mathfrak{B}_h^n(\Omega)$ are the ones really useful as V_h. In Chapter 4 we show that $web\mathfrak{B}_h^n(\Omega)$ has the stability and approximation properties required for Finite Element Spaces.

Chapter 1

Preliminaries

In this short chapter we present the notation used throughout this thesis and some basic ideas about Sobolev spaces, Hausdorff metric and signed weight functions.

1.1 Notation

If f is a function defined on $\mathbb{R}^m$ that depends on parameters $a_1, ..., a_p$ we write $f(a_1, ..., a_p; x)$, for its value at $x = (x_1, ..., x_m) \in \mathbb{R}^m$.

The dependence of constants on parameters $a_1, ..., a_p$ is written as

$$\text{const } (a_1, ..., a_p).$$

Dependency on parameters are not always indicated if it is clear from the context.

Multi-indexing is a collection of notational devices, whose main goal is to avoid drowning in a sea of indices. A single multi-index is used to denote dependence on several indices: $\alpha = (\alpha_1, ..., \alpha_m)$.

Given an indexing set S (typically $S = \mathbb{Z}$), a multi-index is an element of S^m. Thus a multi-index in $\mathbb{R}^m$ is a m-tuple $\alpha = (\alpha_1, ..., \alpha_m)$ with positive numbers $\alpha_\nu \in \mathbb{Z}$, $\nu = 1, ..., m$.

The order of a multi-index is the number $|\alpha| := \sum_{\nu=1}^m \alpha_\nu$.
Given multi-index α, β we define:

$$\begin{aligned}
\alpha + \beta &:= (\alpha_1 + \beta_1, ..., \alpha_m + \beta_m), \\
\alpha \leq \beta &\iff \alpha_\nu \leq \beta_\nu, \quad \forall \nu = 1, 2, ..., m, \\
\alpha! &:= \alpha_1! \alpha_2! \cdots \alpha_m!.
\end{aligned}$$

In particular when $n \in \mathbb{R}$, we write $\alpha \leq n$ to indicate $\alpha_\nu \leq n$, $\nu = 1, ..., m$.

If $\mathbf{x} = (x_1, ..., x_m) \in \mathbb{R}^m$ and α is a multi-index, the monomial $\mathbf{x}^\alpha$ is

$$\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}.$$

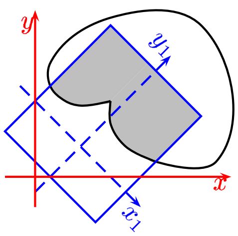
A polynomial in $\mathbb{R}^m$ is a function of the form:

$$p(\mathbf{x}) = \sum_{|\alpha| \le n} c_\alpha \mathbf{x}^\alpha \quad \text{with } c_\alpha \in \mathbb{R}.$$

Given a multi-index $\alpha \in \mathbb{Z}^m$ we define the differential operator ∂^α, as:

$$\partial^\alpha = \prod_{\nu=1}^{m} \partial_\nu^{\alpha_\nu}, \text{ where } \partial_\nu^{\alpha_\nu} = (\frac{\partial}{\partial x_\nu})^{\alpha_\nu}.$$

When $n_1 = n_2 = ... = n_m = n$ we use the same symbol n for the integer n_1 and the multi-index $(n_1, n_1, ...n_1)$.

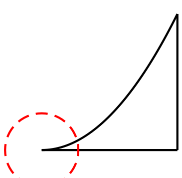


A Lipschitz domain.

1.2 Sobolev Spaces

We use Ω to denote a bounded and open set.

Definition 1.1 (Lipschitz Domain). *Given $\Omega \subset \mathbb{R}^m$, we say that Ω is a Lipschitz domain if $\partial\Omega$ is locally the graph of a Lipschitz continuous function and Ω lies on one side of this graph.*



A Non Lipschitz domain.

Given $\Omega \subset \mathbb{R}^m$, a Lipschitz domain, let $L^2(\Omega)$ be the space of square Lebesgue integrable functions:

$$L^2(\Omega) := \left\{ \psi : \Omega \to \mathbb{R} \ : \ \int_\Omega |\psi|^2 < \infty \right\}$$

with the usual norm given by $\|\psi\|_{L^2(\Omega)} := \left(\int_\Omega |\psi|^2\right)^{\frac{1}{2}}$ which is obtained from the inner product:

$$(\phi, \psi)_{L^2(\Omega)} := \int_\Omega \phi\psi.$$

We denote by $L_0^2(\Omega)$ the subspace of $L^2(\Omega)$ of the functions of zero average.

The space $\mathbf{L}^2(\Omega) := L^2(\Omega)^m$ is the cartesian product of $L^2(\Omega)$ m times with the norm

$$\|f\|^2_{\mathbf{L}^2(\Omega)} := \sum_1^m \|f_i\|^2_{L^2(\Omega)}.$$

Let $C_0^\infty(\Omega)$ be the space of infinitely differentiable functions having compact support in Ω.

We denote by ∂u, ∇u or $\operatorname{grad} u$ the m-tuple of functions $(\partial_1 u, \ldots, \partial_m u)$.

Let $\mathcal{D}(\Omega)$ be the space $C_0^\infty(\Omega)$ with the following sense of convergence: $\{f_n\}$ converges if there exist a function $f \in C_0^\infty(\Omega)$ such that the supports of $\{f_n\}$ are all contained in a compact subset of Ω and their derivatives $\{\partial^\alpha f_n\}$ converge uniformly to $\partial^\alpha f$ for all multi-index α.

Let $\mathcal{D}'(\Omega)$ be the dual space of $\mathcal{D}(\Omega)$, i.e., the space of distributions, which is the space of linear functionals on $\mathcal{D}(\Omega)$ that are continuous with respect to the notion of convergence defined above.

In this section the notation $\langle \cdot, \cdot \rangle$ is used for the duality pairing between $\mathcal{D}(\Omega)$ and its dual, i.e., we write:

$$\langle f, \psi \rangle, \quad \psi \in \mathcal{D}(\Omega), \;\; f \in \mathcal{D}'(\Omega).$$

If f is a distribution and α is a multi-index it is possible to define its derivative (in the sense of distributions) $\partial^\alpha f$ by:

$$\langle \partial^\alpha f, \psi \rangle = (-1)^{|\alpha|} \langle f, \partial^\alpha \psi \rangle, \quad \psi \in \mathcal{D}(\Omega).$$

Now we define a very important family of subspaces of $L^2(\Omega)$, the *Sobolev Spaces.* By definition, a function $f \in L^2(\Omega)$ belongs to $H^\ell(\Omega)$, $\ell \in \mathbb{N}$, if $|\alpha| \leq \ell$ implies $\partial^\alpha f \in L^2(\Omega)$, or more precisely:

$$H^\ell(\Omega) := \Big\{ f \in L^2(\Omega) \; : \; \forall \alpha, |\alpha| \leq \ell, \;\; \exists \, f_\alpha \in L^2(\Omega), \quad \text{s. t. } \langle \partial^\alpha f, \psi \rangle = \int_\Omega f_\alpha \psi, \;\; \forall \psi \in \mathcal{D}(\Omega) \Big\}.$$

In $H^\ell(\Omega)$ we consider the following inner product:

$$(f, g)_{H^\ell(\Omega)} := \sum_{|\alpha| \leq \ell} (\partial^\alpha f, \partial^\alpha g)_{L^2(\Omega)},$$

which gives the norm:

$$\|f\|^2_{H^\ell(\Omega)} := (f, f)_{H^\ell(\Omega)} = \sum_{|\alpha| \leq \ell} \|\partial^\alpha f\|^2_{L^2(\Omega)} = \sum_{|\alpha| \leq \ell} \int_\Omega |\partial^\alpha f|^2.$$

A very important functional on $H^\ell(\Omega)$ is the seminorm, given by:

$$|f|^2_{H^\ell(\Omega)} := \sum_{|\alpha| = \ell} \|\partial^\alpha f\|^2_{L^2(\Omega)} = \sum_{|\alpha| = \ell} \int_\Omega |\partial^\alpha f|^2.$$

For other definitions of Sobolev spaces (and related spaces) see [1] and [7].

Given $\Gamma \subset \partial\Omega$ with non-vanishing $(m-1)$-dimensional measure and relatively open with respect to $\partial\Omega$, we denote by $H_0^1(\Omega, \Gamma)$ the subspace of $H^1(\Omega)$ consisting of functions that vanish on Γ. When $\Gamma = \Omega$ we denote this by $H_0^1(\Omega)$.

Next lemma is very useful in order to look for equivalent norms in Sobolev spaces.

Lemma 1.1 (Poincaré Inequality). *Let $u \in H^1(\Omega)$. Then there exist constants, depending only on Ω, such that*

$$\|u\|^2_{L^2(\Omega)} \leq C_1 |u|^2_{H^1(\Omega)} + C_2 \left(\int_\Omega u \right)^2 .$$

In particular, the seminorm $|\cdot|_{H^1(\Omega)}$ is equivalent to the norm $\|\cdot\|_{H^1(\Omega)}$ in $H^1(\Omega) \cap L^2_0(\Omega)$. We can obtain similar results by integrating over a subset of Ω or even in a part of the boundary by using the Trace Theorem (see [1], [7]).

Lemma 1.2 (Friedrichs Inequality). *Let $\Gamma \subset \partial\Omega$ with non-vanishing $(m-1)$-dimensional measure and relatively open with respect to $\partial\Omega$. Then there exist constants, depending only on Ω and Γ, such that for $u \in H^1(\Omega)$,*

$$\|u\|^2_{L^2(\Omega)} \leq C_1 |u|^2_{H^1(\Omega)} + C_2 \|u\|^2_{L^2(\Gamma)}.$$

In particular, if $u \in H^1_0(\Omega, \Gamma)$ the $H^1(\Omega)$-seminorm is equivalent to the $H^1(\Omega)$-norm.

1.3 A Note on Hausdorff Metric

Now we present a brief review of Hausdorff metric. This is going to be needed in Section 3.4.

Let (M, d) be a metric space. The distance between a point $x \in M$ and a set $A \in M$ is:

$$d(x, A) = \inf_{y \in A} d(x, y), \tag{1.1}$$

where d is the metric of M, i.e., the distance between a point and a set is the minimum distance between the point and every element in the set.

Definition 1.2 (Hausdorff metric). *Let (M, d) be a metric space. Let*

$$N := \left\{ A \subseteq M : \begin{array}{c} A \text{ is bounded, not empty, and such that} \\ d(a, A) = 0 \text{ if and only if } a \in A. \end{array} \right\}.$$

Given A, $B \in N$, let

$$\rho(A, B) := \max\left\{ \sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B) \right\}.$$

Then ρ defines a metric in N, called the Hausdorff metric.

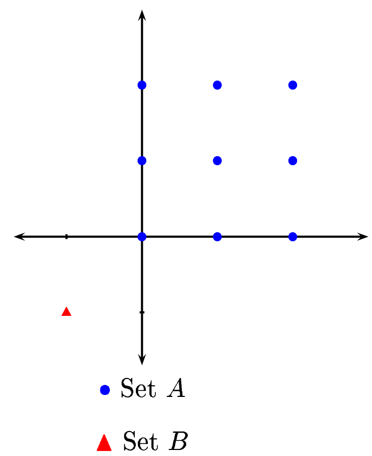
We see that if A and B are within a Hausdorff distance r of each other then every point of A is within distance r of some point of B, and every point of B

is within distance r of some point of A.

An alternative way of defining the Hausdorff metric uses the r-neighborhood,

$$\rho(A,B) := \inf\left\{ r > 0 : A \subseteq B_r(B) \quad \text{and} \quad B \subseteq B_r(A) \right\}$$

where, for $A \in N$ and $r > 0$, $B_r(A) = \{y : d(x,y) < r \text{ for some } x \in A\}$ is an open r-neighborhood of A and by convention the infimum is ∞ for empty sets.



Example 1.1. *Consider $M = (\mathbb{Z}^2, ||\cdot||_\infty)$, and the sets $A = \{0,1,2\}^2$ and $B = \{(-1,-1)\} = \{b\}$. The Hausdorff distance between sets A and B is*

$$\rho(A,B) = \max\{1,3\} = 3.$$

1.4 Signed Weight Functions

R-functions were created by Vladimir Logvinovich Rvachev in Ukranie. The name "R-functions" was given after his father.
In general a real valued function, $\phi : \mathbb{R}^m \to \mathbb{R}$, is an R-function if it has a property that is completely determined by the corresponding property of its arguments, or, equivalently, completely determined by a finite partition of $\mathbb{R}^m$. One of the most useful examples of such a property is the function sign. Define the function $\mathsf{sign} : \mathbb{R} \to \{0,1\}$ such that

$$\mathsf{sign}\ (x) := \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0. \end{cases}$$

An R-function works as a Boolean switching function, changing its sign only when any of its arguments change their signs (see [9], [10]).
This definition is useful because $\mathsf{sign}\ (x_\nu)$ can be regarded as a in - out or on - off boolean variable. Then we can interpret an R-function as a rule saying, for instance, when ϕ is in - out depending on the state of all x_ν's.

Example 1.2. *Consider the function $r_\cap : \mathbb{R}^2 \to \mathbb{R}$, $r_\cap(x,y) = x + y - \sqrt{x^2+y^2}$. Observe that:*

1. *If x and y are positive then*

$$(x+y)^2 = x^2 + y^2 + 2xy > x^2 + y^2 \quad \Rightarrow \quad r_\cap(x,y) > 0.$$

2. *If x or y is negative then*

$$x + y < \max\{|x|,|y|\} \leq \sqrt{x^2+y^2} \quad \Rightarrow \quad r_\cap(x,y) < 0.$$

From 1 and 2 we have that

$$\mathsf{sign}\ r_\cap(x,y) = \min\{\mathsf{sign}\ (x), \mathsf{sign}\ (y)\}.$$

This implies that

$$[r_\cap > 0] = [x > 0] \cap [y > 0]. \tag{1.2}$$

We can use R-functions to describe geometric objects. For instance, suppose that we have the implicit representation

$$\Omega_\nu = [w_\nu > 0], \quad w_\nu : \mathbb{R}^m \to \mathbb{R}, \quad \nu = 1, 2.$$

According to Example 1.2 we can represent $\Omega := \Omega_1 \cap \Omega_2$ by $r_\cap(w_1, w_2) > 0$. We have the following

Lemma 1.3. *Let $\Omega_\nu \subseteq \mathbb{R}^m$ be represented by $\Omega_\nu = [w_\nu > 0]$, with functions $w_\nu : \mathbb{R}^m \to \mathbb{R}$, $\nu = 1, 2$. If we define:*

$$\begin{aligned} r_c(w_1) &:= -w_1 \\ r_\cap(w_1, w_2) &:= w_1 + w_2 - \sqrt{w_1^2 + w_2^2} \\ r_\cup(w_1, w_2) &:= w_1 + w_2 + \sqrt{w_1^2 + w_2^2} \end{aligned} \tag{1.3}$$

we have that

$$\begin{aligned} [r_c(w_1) > 0] &= \Omega_1^c \\ [r_\cap(w_1, w_2) > 0] &= \Omega_1 \cap \Omega_2 \\ [r_\cup(w_1, w_2) > 0] &= \Omega_1 \cup \Omega_2. \end{aligned}$$

In other to match certain boundary conditions we need weight functions.

Definition 1.3. *A weight function w of order $\gamma \in \mathbb{N}$ on $\Omega \subseteq \mathbb{R}^m$ is a function $w : \overline{\Omega} \to \mathbb{R}$ continuous on $\overline{\Omega}$ such that, there exist constants c_w, C_w such that*

$$c_w \ \ d(x, \Gamma)^\gamma \le w(x) \le C_w \ \ d(x, \Gamma)^\gamma, \quad \forall x \in \Omega,$$

where $\Gamma \subseteq \partial\Omega$ has positive measure with respect to $\partial\Omega$.

The weight function w of order γ is ℓ-regular for some $\ell \in \mathbb{N}$, if the partial derivatives up to order ℓ are bounded and

$$|\partial^\alpha w(x)| \le C_w \ \ d(x, \Gamma)^{\gamma - |\alpha|}, \quad |\alpha| \le \min\{\gamma, \ell\}.$$

If w is defined on $\mathbb{R}^m$, it is continuous, $\Omega = [w > 0]$ and $\Omega^c = [w < 0]$ we say that w is a signed weight function.
Observe that $w = 0$ on Γ.

If w_ν is a signed weight function for Ω_ν, $\nu = 1, 2$ then, from Lemma 1.3,

$$w = r_c(w_1); \quad w = r_\cap(w_1, w_2); \quad w = r_\cup(w_1, w_2)$$

are signed weight functions for Ω_1^c; $\Omega_1 \cap \Omega_2$; $\Omega_1 \cup \Omega_2$.

Chapter 2

One Dimensional B-splines

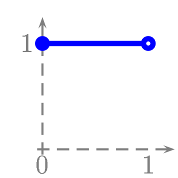
In this chapter we introduce some background about splines that we will need in the rest of this monograph.
In Section 2.1 we define uniform B-splines and present some basic results. We extend these results for scaled and translated B-splines in Section 2.2. We show how to generate polynomials in terms of B-spline functions, the formulas are called Marsden's identity, which is proved in Section 2.3. In particular the family of scaled and translated B-splines are partitions of unity, Section 2.4.

2.1 Uniform B-splines

There are several ways to introduce and define B-splines and related concepts (see [2],[3],[8]). Here we follow the presentation in [6], which is a fast and simple way to do it.

Definition 2.1. *Let*

$$\hat{b}(0;x) = \mathbb{1}_{[0,1)}(x) := \begin{cases} 1, & 0 \leq x < 1 \\ 0, & \textit{otherwise}; \end{cases} \tag{2.1}$$



B-spline $\hat{b}(0;\cdot)$ given by (2.1)

then the uniform B-splines $\hat{b}(n;\cdot)$, $n \geq 1$, $n \in \mathbb{Z}^+$, *are defined inductively by*

$$\hat{b}(n;x) := \int_{x-1}^{x} \hat{b}(n-1;t)dt, \quad \forall x \in \mathbb{R}, \quad n \geq 1.$$

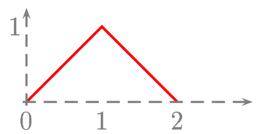
An easy consequence of Definition 2.1 is that

$$\frac{d}{dx}\hat{b}(n;x) = \hat{b}(n-1;x) - \hat{b}(n-1;x-1), \quad \forall x \in \mathbb{R}. \tag{2.2}$$

Note that (2.1) and (2.2) together with $\hat{b}(n;0) = 0$, $n \neq 0$, characterize the uniform B-splines.

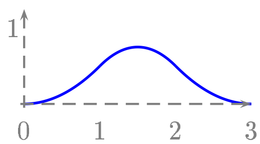
Example 2.1. *For $n = 1$ and $n = 2$, $\hat{b}(n; \cdot)$ have the form*

$$\hat{b}(1; x) = \begin{cases} x, & 0 \leq x \leq 1, \\ 2 - x, & 1 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$



B-spline $\hat{b}(1; \cdot)$ given by (2.3)

$$\hat{b}(2; x) = \begin{cases} x^2/2, & 0 \leq x \leq 1, \\ -x^2 + 3x - 3/2, & 1 \leq x \leq 2, \\ x^2/2 - 3x - 9/2, & 2 \leq x \leq 3, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$



B-spline $\hat{b}(2; \cdot)$ given by (2.4)

In Figure 2.1 we can observe uniform B-splines $\hat{b}(n; \cdot)$, for $n = 0, 1, ..., 10$.

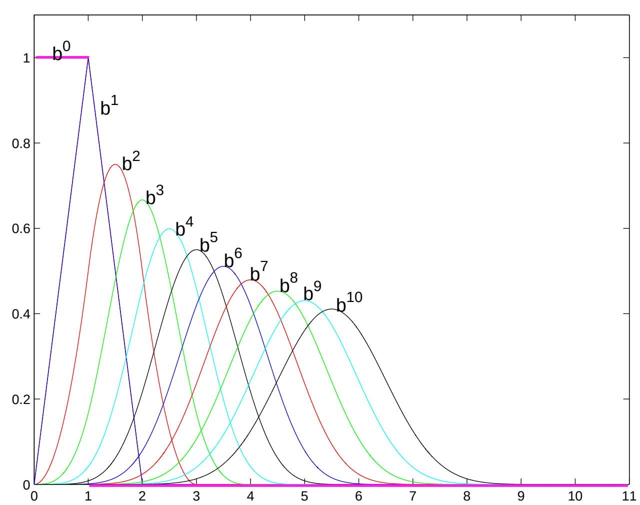


Figure 2.1: B-splines of degree n= 0,...,10, where $b^j = \hat{b}(j; \cdot)$, $j = 1, .., 10$.

A uniform B-spline of degree n has the following properties.

Lemma 2.1 (Positivity and local support). *A B-spline of degree n is positive on $(0, n + 1)$ and vanishes outside this interval.*

Proof. This fact follows by induction directly from Definition 2.1, because we are taking an integral over an interval of width one centered at $x - 1/2$ of a positive function supported in $(0, n)$. $\square$

Lemma 2.2 (Piecewise polynomial structure). *$\hat{b}(n; \cdot)$ is a polynomial of degree n on each interval $[k, k + 1]$, $k = 0, ..., n$.*

Lemma 2.3 (Smoothness). *$\hat{b}(n;\cdot)$ is $(n-1)$-times continuously differentiable.*

Proof. From Example 2.1, $\hat{b}(1;\cdot)$ is continuous. Also note that

$$\hat{b}(n;x) = \int_{x-1}^{x} \hat{b}(n-1;t)dt = \int_0^1 \hat{b}(n-1;s+x-1)ds,$$

hence, the continuous differentiability of $\hat{b}(n;\cdot)$ up to order $n-1$ follows from the continuous differentiability of $\hat{b}(n-1;\cdot)$ up to order $n-2$. □

Lemma 2.4. *$\hat{b}(n;\cdot)|_{(k,k+1)}$, $k = 0, 1, ..., n$, is n times continuously differentiable with constant n-th derivatives.*

Proof. This is true for $n = 1$, see Example 2.1.
If this holds for $n-1$, then from (2.2) it holds also for n. □

Corollary 2.1. *Let $\Omega \subseteq \mathbb{R}$, be an interval such that supp $\hat{b}(n;\cdot) \subseteq \Omega$, then $\hat{b}(n;\cdot) \in H^n(\Omega)$, where $H^n(\Omega)$ is the Sobolev space of order n.*

Lemma 2.5 (Convolution). *The convolution of B-splines $\hat{b}(m;\cdot)$ and $\hat{b}(n;\cdot)$, is a B-spline of degree $m+n+1$, i.e.,*

$$\hat{b}(m;\cdot) * \hat{b}(n;\cdot)(x) = \hat{b}(m+n+1;x), \quad \forall x \in \mathbb{R}.$$

Proof. This fact is easily proved by induction on m.

$$\hat{b}(0;\cdot) * \hat{b}(n;\cdot)(x) = \int_{\mathbb{R}} \hat{b}(0;x-y)\hat{b}(n;y)dy = \int_{x-1}^{x} \hat{b}(n;y)dy = \hat{b}(n+1;x).$$

Suppose that the lemma holds for $m-1$, and by using (2.2) it follows that

$$\begin{aligned}
\frac{d}{dx}\hat{b}(m+n+1;x) &= \hat{b}(m+n;x) - \hat{b}(m+n;x-1) \\
&= \int_{\mathbb{R}} \Big(\hat{b}(m-1;x-y) - \hat{b}(m-1;x-y-1)\Big)\hat{b}(n;y)dy \\
&= \int_{\mathbb{R}} \Big(\frac{d}{dx}\hat{b}(m;x-y)\Big)\hat{b}(n;y)dy \\
&= \Big(\frac{d}{dx}\hat{b}(m;\cdot)\Big) * \hat{b}(n;\cdot)(x) \\
&= \frac{d}{dx}\Big(\hat{b}(m;\cdot) * \hat{b}(n;\cdot)(x)\Big). \quad \square
\end{aligned}$$

Remark 2.1. Lemma 2.5 tells us that uniform B-splines $\hat{b}(n;\cdot)$, $n \geq 1$, could be defined by the recursion formula:

$$\hat{b}(n;\cdot) := \hat{b}(n-1;\cdot) * \hat{b}(0;\cdot), \quad \forall n \geq 1.$$

Lemma 2.6 (Symmetry). *A B-spline of degree n is symmetric with respect to $x = (n+1)/2$, i.e.,*

$$\hat{b}(n; x) = \hat{b}(n; n+1-x), \quad \forall x \in \mathbb{R}. \tag{2.5}$$

Proof. We proceed by induction on n. Observe that for $n = 1$, (2.5) holds at $x = 0, 1$ and $x = 1/2$. Then since $\hat{b}(1; \cdot)$ is piecewise linear we have that (2.5) holds for all $x \in [0, 2]$. For $x \notin [0, 2]$, $2 - x \notin [0, 2]$, then $\hat{b}(1; x) = \hat{b}(1; 2-x) = 0$.

If (2.5) holds for $n-1$, i.e., $\hat{b}(n-1; x) = \hat{b}(n-1; n-x)$, we have that

$$\hat{b}(n; n+1-x) = \int_{n-x}^{n+1-x} \hat{b}(n-1; t)dt = \int_{x-1}^{x} \hat{b}(n-1; s)ds = \hat{b}(n; x)$$

this ends the proof. □

For $f,\ g \in L^2(\mathbb{R})$ define $\langle f, g \rangle := \int_{\mathbb{R}} f(x)g(x)dx$.

We denote as $\hat{s}(n, k-l)$ the scalar product of $\hat{b}(n; \cdot - k)$ and $\hat{b}(n; \cdot - l)$, and by $\hat{d}(n, k-l)$, the scalar product of their derivatives, i.e.,

$$\hat{s}(n, k-l) := \int_{\mathbb{R}} \hat{b}(n; x-k)\hat{b}(n; x-l)dx = \langle \hat{b}(n; \cdot - k), \hat{b}(n; \cdot - l) \rangle$$
$$\hat{d}(n, k-l) := \left\langle \frac{d}{dx}\hat{b}(n; \cdot - k), \frac{d}{dx}\hat{b}(n; \cdot - l) \right\rangle.$$

Lemma 2.7. *Given $k, l \in \mathbb{Z}$ we have*

$$\hat{s}(n, k-l) = \hat{b}(2n+1; n+1+k-l),$$
$$\hat{d}(n, k-l) = -\hat{s}(n-1, k-l-1) + 2\hat{s}(n-1, k-l) - \hat{s}(n-1, k-l+1).$$

Proof. Using the symmetry (Lemma 2.6) and the convolution (Lemma 2.5) we have

$$\begin{aligned}\hat{s}(n, k-l) &= \int_{\mathbb{R}} \hat{b}(n; x-k)\hat{b}(n; x-l)dx = \int_{\mathbb{R}} \hat{b}(n; y+l-k)\hat{b}(n; y)dy \\ &= \int_{\mathbb{R}} \hat{b}(n; (n+1-l+k)-y)\hat{b}(n; y)dy = \hat{b}(2n+1; n+1+k-l).\end{aligned}$$

To prove the second statement we note that

$$\frac{d}{dx}\hat{b}(n; x-k) = \hat{b}(n-1; x-k) - \hat{b}(n-1; x-k-1)$$

hence,

$$\begin{aligned}&\int_{\mathbb{R}} \frac{d}{dx}\hat{b}(n-1; x-k)\frac{d}{dx}\hat{b}(n-1; x-l)dx \\ &\quad = \hat{b}(2n-1; n-l+k) - \hat{b}(2n-1; n-(l+1)+k) \\ &\quad\quad - \hat{b}(2n-1; n-l+(k+1)) + \hat{b}(2n-1; n-(l+1)+(k+1)) \\ &\quad = 2\hat{b}(2n-1; n-l+k) - \hat{b}(2n-1; n-l-1+k) - \hat{b}(2n-1; n-l+k+1) \\ &\quad = 2\hat{s}(n-1, k-l) - \hat{s}(n-1, k-l-1) - \hat{s}(n-1, k+1-l). \quad \square\end{aligned}$$

Lemma 2.8 (Monotonicity). *A B-spline of degree $n > 0$ is strictly monotone on $[0, (n+1)/2]$ and $[(n+1)/2, n+1]$.*

Proof. For $n = 1$, it is obvious, see (2.3). Suppose that the lemma holds for $n-1$. Take $x \in [0, (n+1)/2]$. Observe that since $\hat{b}(n-1; \cdot)$ is increasing in $[0, n/2]$ we have that for all $x \in [0, n/2]$

$$\frac{d}{dx}\hat{b}(n; x) = \hat{b}(n-1; x) - \hat{b}(n-1; x-1) > 0.$$

Then $\hat{b}(n; \cdot)$ is increasing in $[0, n/2]$.

If $x \in [n/2, (n+1)/2]$ we have that $n-x, x-1 \in [0, n/2]$, and $x - 1 < n - x$, then using the symmetry (Lemma 2.6) we have that

$$\frac{d}{dx}\hat{b}(n; x) = \hat{b}(n-1; x) - \hat{b}(n-1; x-1) = \hat{b}(n-1; n-x) - \hat{b}(n-1; x-1) > 0,$$

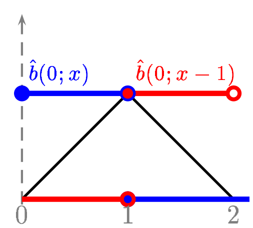
which proves that $\hat{b}(n; \cdot)$ is increasing in $[0, (n+1)/2]$. By symmetry $\hat{b}(n; \cdot)$ is decreasing in $[(n+1)/2, n+1]$. □

Lemma 2.9 (Recurrence Relation). *A B-spline of degree n is a linear combination of B-splines of degree $n-1$, more precisely,*

$$\hat{b}(n; x) = \frac{x}{n}\hat{b}(n-1; x) + \frac{n+1-x}{n}\hat{b}(n-1; x-1), \quad \forall x \in \mathbb{R}. \tag{2.6}$$

Proof. For the case $n = 1$, see the figure in the margin. Assuming that the recurrence relation is valid up to degree $n-1$, i.e.,

$$\begin{aligned} \hat{b}(n-1; x) &= \frac{x}{n-1}\hat{b}(n-2; x) + \frac{n-x}{n-1}\hat{b}(n-2; x-1), \\ \hat{b}(n-1; x-1) &= \frac{x-1}{n-1}\hat{b}(n-2; x-1) + \frac{n-x+1}{n-1}\hat{b}(n-2; x-2), \end{aligned}$$



(2.6) with $n = 1$.

then,

$$\begin{aligned} \frac{n-1}{n}\Big(\hat{b}(n-1; x) - \hat{b}(n-1; x-1)\Big) &= \frac{x}{n}\Big(\hat{b}(n-2; x) - \hat{b}(n-2; x-1)\Big) \\ &\quad + \frac{n+1-x}{n}\Big(\hat{b}(n-2; x-1) - \hat{b}(n-2; x-2)\Big). \end{aligned}$$

Hence

$$\begin{aligned}
\frac{d}{dx}\hat{b}(n;x) &= \hat{b}(n-1;x) - \hat{b}(n-1;x-1) \\
&= \frac{n-1}{n}\Big(\hat{b}(n-1;x) - \hat{b}(n-1;x-1)\Big) + \frac{1}{n}\Big(\hat{b}(n-1;x) - \hat{b}(n-1;x-1)\Big) \\
&= \frac{1}{n}\hat{b}(n-1;x) + \frac{x}{n}\Big(\hat{b}(n-2;x) - \hat{b}(n-2;x-1)\Big) \\
&\quad - \frac{1}{n}\hat{b}(n-1;x-1) + \frac{n+1-x}{n}\Big(\hat{b}(n-2;x-1) - \hat{b}(n-2;x-2)\Big) \\
&= \frac{d}{dx}\Big(\frac{x}{n}\hat{b}(n-1;x)\Big) + \frac{d}{dx}\Big(\frac{n+1-x}{n}\hat{b}(n-1;x-1)\Big).
\end{aligned}$$

It is enough to check that (2.6) holds at some point and we know that both sides vanish at $x = 0$. □

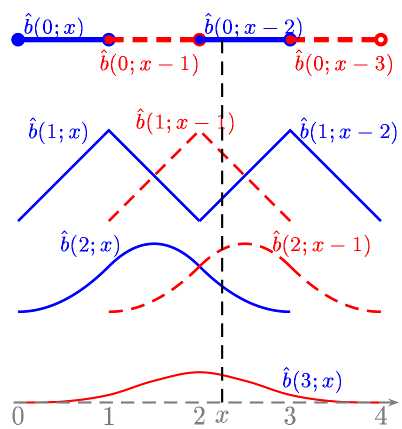


Figure 2.2: Recurrence relation for $n = 3$.

The recurrence relation is an easy way to compute (low order) B-splines. (see Figure 2.2). We have used the formula (2.6) to obtain Figure 2.1 in MatLab by using BS.m (see Figure 2.3).

As noted before, $\hat{b}(n;\cdot)$ is a polynomial of degree n when restricted to interval $[k, k+1]$, $k = 0, 1, ..., n$. Then we can write,

$$\hat{b}(n;x) = a_0 + a_1(x-k) + a_2(x-k)^2 + ... + a_n(x-k)^n.$$

To indicate the dependence on n and k we write $a_l(n,k)$ instead of a_l. With this notation we have for $t = x - k \in [0,1]$

$$\hat{b}(n;x) = a_0(n,k) + a_1(n,k)t + a_2(n,k)t^2 + ... + a_n(n,k)t^n. \qquad (2.7)$$

BS.m

```
function y=BS(x,n)
if n==0
  y=(x>=0).*(x<1);
else
  y=(x/n).*BS(x,n-1)+((n+1-x)/n).*BS(x-1,n-1);
end
```

Figure 2.3: Program for calculating low degree B-splines.

Lemma 2.10 (Taylor Coefficients). *The coefficients in (2.7) can be computed with the following recursion formula:*

$$\begin{aligned} a_l(n,k) &= \frac{k}{n}a_l(n-1,k) + \frac{1}{n}a_{l-1}(n-1,k) \\ &\quad + \frac{n+1-k}{n}a_l(n-1,k-1) - \frac{1}{n}a_{l-1}(n-1,k-1), \end{aligned}$$

for $k, l \in \{0, ..., n\}$ *starting with* $a_0(0,0) = 1$.

Proof. By restricting $x = k + t$ at the interval $[k, k+1]$, and by using the recurrence relation (2.6) we have

$$\begin{aligned} \frac{x}{n}\hat{b}(n-1;x) &= \frac{k+t}{n}\Big(a_0(n-1,k) + a_1(n-1,k)t + ... + a_n(n-1,k)t^n\Big) \\ &= ... + \left(\frac{k}{n}a_l(n-1,k) + \frac{1}{n}a_{l-1}(n-1,k)\right)t^l + ... \end{aligned}$$

and also

$$\begin{aligned} &\frac{n+1-x}{n}\hat{b}(n-1;x-1) \\ &\qquad = \frac{n+1-k-t}{n}\Big(a_0(n-1,k) + a_1(n-1,k)t + ... + a_n(n-1,k)t^n\Big) \\ &\qquad = ... + \left(\frac{n+1-k}{n}a_l(n-1,k) - \frac{1}{n}a_{l-1}(n-1,k)\right)t^l + ..., \end{aligned}$$

if we add up these, we obtain the result. □

Example 2.2. *From Example 2.1 the Taylor coefficients of the B-splines of degree* $n = 2$ *on the intervals* $[1,2]$ *are* $a_0(2,1) = \frac{1}{2}$, $a_1(2,1) = 1$, $a_2(2,1) = -1$, *hence*

$$\hat{b}(2;x) = \frac{1}{2} + (x-1) - (x-1)^2 = -\frac{3}{2} + 3x - x^2, \quad \text{for } x \in [1,2].$$

2.2 Scaled and Translated B-splines

Now we introduce a family of B-splines associated with a uniform grid on $\mathbb{R}$. This family consists of translations and scalings of our uniform B-splines introduced in Section 2.1.

Definition 2.2. *Given $h \in \mathbb{R}$, $h > 0$ and $k \in \mathbb{Z}$, the B-spline of degree n scaled by h and translated kh to the right is called the k-th B-spline of degree n on the grid $h\mathbb{Z}$. It is given by*

$$b(n,k,h;x) := \hat{b}(n; \tfrac{x}{h} - k), \quad \forall x \in \mathbb{R}.$$

Note that the parameters are degree, translation factor and grid size, in this order.

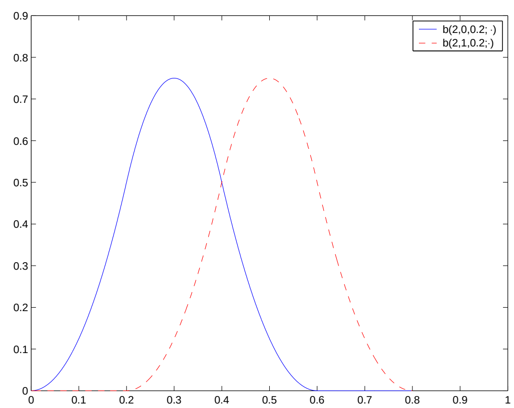


Figure 2.4: $b(2,k,0.2;\cdot)$, $k = 0, 1$.

The properties of a scaled translated B-spline of degree n are generalizations of the properties of the uniform B-spline.

Lemma 2.11 (Positivity and local support). *The B-spline $b(n,k,h;\cdot)$ is positive on $[0, n+1]h + kh$ and vanishes outside this interval.*

Lemma 2.12. $||b(n,k,h;\cdot)||_{L^2} = h^{1/2}||\hat{b}(n;\cdot)||_{L^2}$*, for all $k \in \mathbb{Z}$, $h \in \mathbb{R}$.*

Lemma 2.13 (Piecewise polynomial structure). *$b(n,k,h;\cdot)$ is a polynomial of degree n in each interval $[k, k+1]h$.*

Lemma 2.14 (Smoothness). *$b(n,k,h;\cdot)$ is of class C^{n-1}.*

Note that

$$\frac{d}{dx}b(n,k,h;x) = \frac{1}{h}\Big(b(n-1,k,h;x) - b(n-1,k,h;x-1)\Big). \tag{2.8}$$

We denote as $s(n, k-l)$ the scalar product of $b(n,k,h;\cdot)$ and $b(n,l,h;\cdot)$, and by $d(n,k-l)$, the scalar product of their derivatives, i.e.,

$$s(n,k-l) := \langle b(n,k,h;\cdot), b(n,l,h;\cdot)\rangle$$
$$d(n,k-l) := \Big\langle \frac{d}{dx}b(n,k,h;\cdot), \frac{d}{dx}b(n,l,h;\cdot)\Big\rangle.$$

Lemma 2.15. *Given $k, l \in \mathbb{Z}$ we have*

$$s(n,k-l) = h\hat{b}(2n+1; n+1+k-l)$$
$$d(n,k-l) = \frac{1}{h}\Big(-s(n-1,k-l-1) + 2s(n-1,k-l) - s(n-1,k-l+1)\Big).$$

Proof. This proof is the analogous that of the Lemma 2.7. By symmetry and convolution properties of the B-splines we have

$$\begin{aligned} s(n,k-l) &= \int_{\mathbb{R}} \hat{b}(n; \tfrac{x}{h}-k)\hat{b}(n;\tfrac{x}{h}-l)dx \\ &= h\int_{\mathbb{R}} \hat{b}(n;(n+1+k-l)-y)\hat{b}(n;y)dy \\ &= h\hat{b}(2n+1; n+1+k-l). \end{aligned}$$

To prove the second statement we note that

$$\begin{aligned} &\frac{d}{dx}b(n,k,h;x)\frac{d}{dx}b(n,l,h;x) \\ &\quad = \frac{1}{h^2}\Big(b(n-1,k,h;x) - b(n-1,k+1,h;x)\Big)\Big(b(n-1,l,h;x)b(n-,l+1,h;x)\Big) \end{aligned}$$

then,

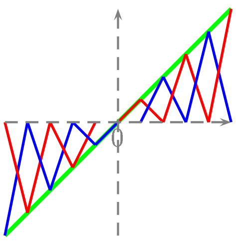
$$\begin{aligned} &h\int_{\mathbb{R}} \frac{d}{dx}b(n-1,k,h;x)\frac{d}{dx}b(n-1,l,h;x)dx \\ &\qquad = 2s(n-1,k-l) - s(n-1,k-l-1) - s(n-1,k+1-l), \end{aligned}$$

this ends the proof. ◻

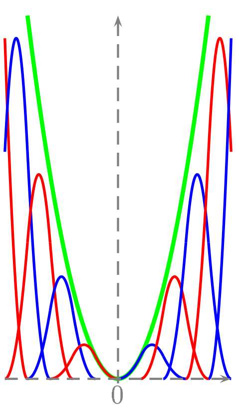
2.3 Marsden's Identity in One Dimension

In this section we study the spaces spanned by the scaled and translated B-splines. We prove Marsden's identity, which plays a central role when studying independence, stability and error estimates.

Definition 2.3. *A cardinal spline of degree less than or equal to n with grid width h is a linear combination of scaled translated B-splines, $\sum_{k\in\mathbb{Z}} c_k b(n,k,h;\cdot)$.*



$x = \sum_{k\in\mathbb{Z}}(k+1)b(1,k,1;x)$



$x^2 = \sum_{k\in\mathbb{Z}}(k^2+3k+2)b(2,k,1;x)$

Example 2.3. *Set $h = 1$, we have*

$$x - t = \sum_{k\in\mathbb{Z}}(k-t+1)b(1,k,1;x) = \sum_{k\in\mathbb{Z}}(k-t+1)\hat{b}(1;x-k).$$

For $t = 0$ see the figure at the margin.

The next result implies that any polynomial can be represented by cardinal splines. It plays an important role when studying error estimates.

Lemma 2.16 (Marsden's Identity in $\mathbb{R}$). *Given x, t real numbers, we define*

$$\Psi^n_{k,h}(t) := h^n \prod_{i=1}^{n}\left(i + k - \frac{t}{h}\right). \tag{2.9}$$

Then, we have

$$(x-t)^n = \sum_{k\in\mathbb{Z}} \Psi^n_{k,h}(t) b(n,k,h;x). \tag{2.10}$$

Proof. By considering the change of variables

$$x = hy, \quad t = hs \quad \text{and} \quad \Phi^n(s) := \prod_{i=1}^{n}(i+s), \text{ with } s \in \mathbb{R},$$

the formula (2.10) is transformed into

$$(y-s)^n = \sum_{k\in\mathbb{Z}} \Phi^n(k-s)\hat{b}(n;y-k).$$

Thus proving Marsden's Identity is equivalent to proving the following lemma. □

Lemma 2.17. *For any $x, t \in \mathbb{R}$,*

$$(x-t)^n = \sum_{k\in\mathbb{Z}} \Phi^n(k-t)\hat{b}(n;x-k), \quad \textit{with} \quad \Phi^n(t) := \prod_{i=1}^{n}(i+t).$$

Proof. For case $n = 1$, see Example 2.3. Suppose that the lemma holds for $n-1$, i.e.,

$$(x-t)^{n-1} = \sum_{k\in\mathbb{Z}} \Phi^{n-1}(k-t)\hat{b}(n-1;x-k), \tag{2.11}$$

by using recurrence relation it follows that

$$\sum_{k\in\mathbb{Z}} \Phi^n(k-t)\hat{b}(n;x-k) = \sum_{k\in\mathbb{Z}} \frac{x-k}{n}\Phi^n(k-t)\hat{b}(n-1;x-k) + \sum_{k\in\mathbb{Z}} \frac{n-x+k}{n}\Phi^n(k-1-t)\hat{b}(n-1;x-k).$$

From the fact that for any $k \in \mathbb{Z}$, $t \in \mathbb{R}$, $\Phi^n(k-1-t) = (k-t)\Phi^{n-1}(k-t)$ holds, and from equation (2.11) we have that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \Phi^n(k-t)\hat{b}(n; x-k) \\
&\quad = \sum_{k\in\mathbb{Z}} \Phi^{n-1}(k-t)\hat{b}(n-1; x-k)\left(\frac{x-k}{n}(n+k-t) + \frac{n-x+k}{n}(k-t)\right) \\
&\quad = \sum_{k\in\mathbb{Z}} \Phi^{n-1}(k-t)\hat{b}(n-1; x-k)(x-t) \\
&\quad = (x-t)^{n-1}(x-t) = (x-t)^n,
\end{aligned}$$

this ends the proof. □

From (2.9) we have that $\Psi^n_{k,h}(t)$ it is a polynomial of degree n in t and k. Denote $\Psi^n_{k,h}(t)$ by $q(k,t)$; in this way Marsden's Identity can by written as

$$(x-t)^n = \sum_{k\in\mathbb{Z}} q(k,t)b(n,k,h;x), \quad \forall x, t \in \mathbb{R}. \tag{2.12}$$

By taking $t = 0$ in (2.12) we obtain:

Corollary 2.2. *For any polynomial p, there exists a polynomial $\tilde{q}$, independent of h such that*

$$p(x) = \sum_{k\in\mathbb{Z}} h^n \tilde{q}(k) b(n,k,h;x).$$

Lemma 2.18. *Given a grid cell $Q = kh + (0, n+1)h$ there exist exactly $(n+1)$ non vanishing B-splines of degree n on this interval, these are*

$$b(n,k,h;\cdot), \quad k = \left\lfloor \frac{x}{h} \right\rfloor - n, \left\lfloor \frac{x}{h} \right\rfloor - n + 1, \ldots, \left\lfloor \frac{x}{h} \right\rfloor. \tag{2.13}$$

where $\lfloor z \rfloor$ denotes the integer part of z.

Since each $x \in \mathbb{R}$ belongs to exactly $n+1$ intervals of the form $kh+(0,n+1)h$, then there is exactly $(n+1)$ non vanishing B-splines (of degree n) at x. These are given in (2.13).

Lemma 2.19 (Linear Independence). *B-splines given in (2.13) are linearly independent.*

2.4 B-splines as Partitions of Unity

In this section we introduce partitions of unity and we prove, using the Marsden identity, that the scaled and translated B-splines form a partition of unity. In particular we prove the constant function 1 belongs to the span of the scaled and translated B-splines.

Definition 2.4. *Let $\{K_\lambda\}_{\lambda\in L}$ a family of subsets of $\mathbb{R}$ (or $\mathbb{R}^m$), we say that this family is locally finite if any $x \in \mathbb{R}$ belongs only to finite numbers of K_λ's, i.e., if for each $x \in \mathbb{R}$ there is a finite subset F of L such that*

$$x \notin K_\lambda, \quad \forall \lambda \in L \setminus F.$$

Definition 2.5. *A family of functions $\{f_\lambda\}_{\lambda\in L}$ is called locally finite if the family of their supports $\{supp\ f_\lambda\}_{\lambda\in L}$ is locally finite.*

Definition 2.6 (Partition of Unity). *A partition of unity of class C^k is a collection of functions $\{f_\lambda\}_{\lambda\in L}$ of class C^k such that:*

1. *For any $\lambda \in L$ and $x \in \mathbb{R}$, $f_\lambda(x) \geq 0$,*
2. *The family $\{supp\ f_\lambda\}_{\lambda\in L}$ is locally finite in $\mathbb{R}$ (or $\mathbb{R}^m$),*
3. *$\sum_{\lambda\in L} f_\lambda(x) = 1$ for any $x \in \mathbb{R}$.*

Note that the sum appearing in item 3 in Definition 2.6 is finite for each $x \in \mathbb{R}$.

Lemma 2.20. *The scaled and translated B-splines $b(n,k,h;\cdot)$ form a partition of unity.*

Proof. Items 1 and 2 of the Definition 2.6 follow from Lemma 2.11. Therefore we only have to prove item 3. Given $x, t \in \mathbb{R}$ and $k \in \mathbb{Z}$ we have

$$\begin{aligned} q(k,t) = \Psi^n_{k,h}(t) &:= h^n \prod_{i=1}^{n} \left(i + k - \frac{t}{h} \right) \\ &= h^n \left(k - \frac{t}{h} \right)^n + \text{polynomial of degree} \leq n-1. \end{aligned}$$

Hence

$$\left. \frac{d^n}{dt^n} q(k,t) \right|_{t=0} = (-1)^n n!.$$

By taking derivatives n-times the equation (2.12) with respect to t and setting $t = 0$ we have that

$$(-1)^n n! = \sum_{k\in\mathbb{Z}} (-1)^n n! b(n,k,h;x)$$

i.e.

$$1 = \sum_{k\in\mathbb{Z}} b(n,k,h;x). \quad \square$$

Chapter 3

Higher Dimensional B-splines and Bounded Domains

In the previous chapter, we worked in the one dimensional euclidean space, now in this chapter we will extend all the theory of Chapter 2 to the m-dimensional euclidean space.
Later we will restrict the collection of all B-splines in $\mathbb{R}^m$ to a subfamily of B-splines supported in a bounded subdomain. This will be used in Chapter 4 when studying finite element with B-splines. The way we shall restrict the support of a B-spline to a bounded domain is via multiplication by a weight function (see Section 1.4 for the definition of weight function). This choice has some peculiarities, in special we have to pay attention to small support B-splines because we want to express general polynomials restricted to a bounded subdomain as a linear combination of B-splines with support in this subdomain. To avoid this we introduce the weighted B-splines, the extended B-splines and the weighted extended B-splines.

3.1 B-splines in $\mathbb{R}^m$

As in the one dimensional case there are different ways of defining B-splines. We want a definition that preserves most of the results already mentioned. Here the tensor product appears naturally.

Definition 3.1 (Tensor Product of B-splines). *Let $\eta = (n_1, n_2, ..., n_m)$, $\eta \in (\mathbb{N} \cup \{0\})^m$, $k = (k_1, k_2, ..., k_m) \in \mathbb{Z}^m$, and $x = (x_1, x_2, ..., x_m) \in \mathbb{R}^m$. The m-variate tensor product B-spline of coordinate degree n_v, grid width h, is defined as*

$$b(\eta, k, h; x) := \prod_{\nu=1}^{m} b(n_\nu, k_\nu, h; x_\nu).$$

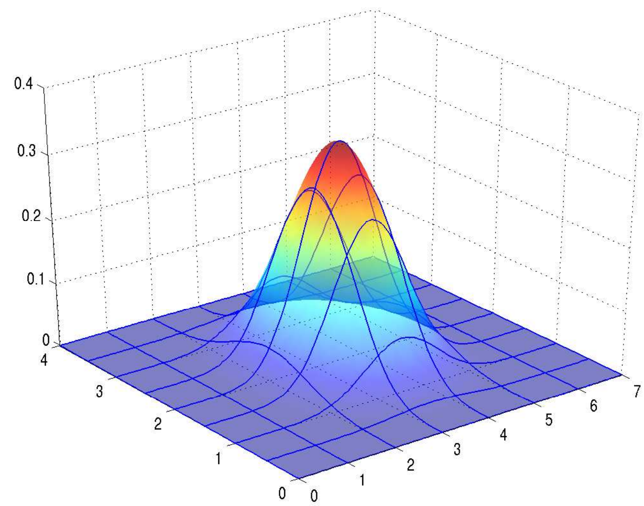


Figure 3.1: B-spline of degree (6,3).

In Figure 3.1 we can see the B-spline $b((6,3),(0,0),1;\cdot)$ obtained from the tensor product of the B-splines $b(6,0,1;\cdot)$ and $b(3,0,1;\cdot)$.

When $n_1 = n_2 = ... = n_m = n$ we use the same symbol n for the integer n_1 and the multi-index $(n_1, n_1, ...n_1)$.

Lemma 3.1 (Positivity and local support). *The B-spline $b(n,k,h;\cdot)$ is positive on $(0, n+1)^m h + (k_1, k_2, ..., k_m)h$ and vanishes outside this m-dimensional cube.*

Proof. This follows from the analogous properties of one dimensional B-splines. See Lemma 2.1. □

Lemma 3.2. $||b(n,k,h;\cdot)||_{L^2} = h^{m/2}||\hat{b}(n;\cdot)||_{L^2}$*, for all* $k \in \mathbb{Z}^m$.

Proof. Let $T_k : \mathbb{R}^m \to \mathbb{R}^m$ be defined by $T_k(x) = hx + kh$; then det $(T_k) = h^m$. In order to simplify the notation we denote $Q_0 = \text{supp } b(n;\cdot) = (0, n+1)^m$ and $Q_k = \text{supp } b(n,k,h;\cdot) = (0, n+1)^m h + kh$.
Then we have that $T_k(Q_0) = Q_k$, and using the change of variables formulas we have

$$||b(n,k,h;\cdot)||^2_{L^2} = \int_{\mathbb{R}^m} \left(\hat{b}(n; \frac{x}{h} - k)\right)^2 dx = \int_{Q_k} \left(\hat{b}(n; \frac{x}{h} - k)\right)^2 dx$$
$$= \int_{Q_0} \left(\hat{b}(n; y)\right)^2 h^m dy = h^m ||\hat{b}(n;\cdot)||^2_{L^2}. \quad \square$$

Lemma 3.3 (Piecewise polynomial structure). *The B-spline $b(n,k,h;\cdot)$ is a polynomial of degree n in the variables $x_1, x_2, ..., x_m$ on each grid cell*

$$Q_l = [0,1]^m h + lh, \quad k \le l \le k+n,$$

i.e., the B-splines can be written as

$$\sum_{\alpha_\nu \le n} c_\alpha x^\alpha, \quad c_{(n,n,\ldots,n)} \neq 0, \quad x \in Q_l.$$

According to Lemma 2.18, we have

Lemma 3.4. *There exists exactly $(n+1)^m$ B-splines of degree n which are nonzero on each grid cell Q_l.*

Let us recall that the derivative of uniform B-splines is given in (2.2) and for scaled translated B-splines is given in (2.8). Now we calculate the first partial derivatives of a B-spline of two variables

$$b(\eta, k, h; x) = b\big((n_1, n_2), (k_1, k_2), h; x_1, x_2\big) = b(n_1, k_1, h; x_1) b(n_2, k_2, h; x_2).$$

$$\begin{aligned}
\frac{\partial}{\partial x_1} b(\eta, k, h; x) &= \left(\frac{\partial}{\partial x_1} b(n_1, k_1, h; x_1)\right) b(n_2; k_2, h)(x_2) \\
&= \frac{1}{h}\Big(b(n_1 - 1, k_1, h; x_1) - b(n_1 - 1, k_1 + 1, h; x_1)\Big) b(n_2, k_2, h; x_2) \\
&= \frac{1}{h}\Big(b(n_1 - 1, k_1, h; x_1) b(n_2, k_2, h; x_2) - b(n_1 - 1, k_1 + 1, h; x_1) b(n_2, k_2, h; x_2)\Big) \\
&= \frac{1}{h}\Big(b\big((n_1 - 1, n_2), (k_1, k_2), h; x_1, x_2\big) - b\big((n_1 - 1, n_2), (k_1 + 1, k_2), h; x_1, x_2\big)\Big),
\end{aligned}$$

taking $\alpha = (1, 0)$ we have:

$$\frac{\partial}{\partial x_1} b(\eta, k, h; x) = \frac{1}{h}\Big(b(\eta - \alpha, k, h; x) - b(\eta - \alpha, k + \alpha, h; x)\Big),$$

with the help of multi-index notation we can compute all the first order partial derivatives:

Lemma 3.5. *The first order partial derivatives of the B-spline $b(\eta, (k_1, \ldots, k_m), h; \cdot)$ is the difference of two B-splines of lower degree, i.e.,*

$$\partial^\alpha b(\eta, k, h; \cdot) = \frac{1}{h}\Big(b(\eta - \alpha, k, h; \cdot) - b(\eta - \alpha, k + \alpha, h; \cdot)\Big) \tag{3.1}$$

for the unit vectors $\alpha = (1, 0, .., 0), (0, 1, \ldots, 0), \ldots, (0, 0, .., 1)$.

Corollary 3.1 (Smoothness). *$b(n, k, h; \cdot)$ is of class C^{n-1} with respect to each variable.*

Corollary 3.2. *$b(n, k, h; \cdot)|_{[0,1]^m h + lk}$, $k \le l \le k + n$, is n times continuously differentiable with constant $\partial^{n\alpha}$ derivatives, and $b(n, k, h; \cdot) \in H^n(\Omega)$, where $H^n(\Omega)$ is the Sobolev space of order n.*

3.2 Splines on Bounded Domains

Up to here we have been working in the whole euclidean space. Since we are interested in applications to elliptic partial differential equations in bounded domains, we now study the properties of B-splines when restricted to a bounded domain. This very important since one of the main advantages of using B-splines to define a finite element space is that we do not need a mesh that depends on the domain.

Consider $\Omega \subset \mathbb{R}^m$, an open Lipschitz domain.

Definition 3.2. *The set of relevant indices of B-splines for* Ω *is*

$$\mathcal{K} = \{k \in \mathbb{Z}^m : \textit{supp } b(n,k,h;\cdot) \cap \Omega \textit{ has positive Lebesgue measure } \}.$$

For a bounded domain $\Omega \subset \mathbb{R}^m$,

$$\mathfrak{B}_h^n(\Omega) := \operatorname*{span}_{k \in \mathcal{K}} \{b(n,k,h;\cdot)\}$$

denotes the linear span of all B-splines which have some non zero measure support in Ω.

From now on we will write $b(k;\cdot)$ instead of $b(n,k,h;\cdot)$, since n and h will be fixed in the rest of this work.

Lemma 3.6 (Marsden's Identity in $\mathbb{R}^m$). *Let* $p(x,t)$ *be a* n *degree polynomial on* x *and* t. *We have*

$$p(x,t) = \sum_{k \in \mathbb{Z}^m} q(k,t) b(k;x) \quad \forall x, t \in \mathbb{R}^m, \quad k \in \mathbb{Z}^m.$$

Proof. We only need to prove the lemma for polynomials of the form

$$p(x,t) = \prod_{\nu=1}^{m} (x_\nu - t_\nu)^n.$$

From Marsden's Identity in $\mathbb{R}$ and (2.12), given x, $t \in \mathbb{R}^m$, we have

$$(x_\nu - t_\nu)^n = \sum_{k_\nu \in \mathbb{Z}} q(k_\nu, t_\nu) b(k_\nu; x_\nu).$$

Forming the tensor product we have[1]

$$p(x,t) = \prod_{\nu=1}^{m} (x_\nu - t_\nu)^n = \sum_{k \in \mathbb{Z}^m} q(k,t) \prod_{\nu=1}^{m} b(k_\nu; x_\nu),$$

where

$$q(k,t) = \prod_{\nu=1}^{m} q(k_\nu, t_\nu). \quad \square$$

[1] Recall that the sum is locally finite.

Considering $t := 0$; $q(k) := q(k,0)$; $p(x) := p(x,0)$ and $x \in \Omega$ in Marsden identity we have:

Corollary 3.3 (Representation of Polynomials). *Any multivariate polynomial* $p(x) = \sum_{\alpha_\nu \leq n} c_\alpha x^\alpha$ *can be represented in the domain* Ω *as a linear combination*

$$p(x) = \sum_{k \in \mathcal{K}} q(k) b(k; x), \quad \forall x \in \Omega,$$

where q *is a multivariate polynomial of degree* $\leq n$ *in each of the variables* k_ν.

Example 3.1. *Let* $h = 1$, $n = 1$, *we want to find the expression of* $p(x) = x_1 x_2$. *Choose* $l = (0,0)$, *then we have*

$$q(0,0) = \Psi^1_{0,1}(0) = 1 \cdot 1 = 1, \qquad q(0,1) = 1 \cdot 2 = 2,$$
$$q(1,0) = 2 \cdot 1 = 2, \qquad q(1,1) = 2 \cdot 2 = 4.$$

Since $q(k) = a + bk_1 + ck_2 + dk_1k_2$, *we have*

$$a = q(0,0) = 1; \quad b = q(0,1) - a = 1; \quad c = 1; \quad d = 1;$$

i.e., $q(k) = k_1k_2 + k_1 + k_2 + 1 = (k_1 + 1)(k_2 + 1)$, *therefore*

$$x_1 x_2 = \sum_{k \in \mathbb{Z}^2} (k_1 + 1)(k_2 + 1) b(k; x).$$

Corollary 3.4. *Any array of coefficients*

$$q(k), \quad k \in l + \{0, 1, ..., n\}^m$$

determines all other B-spline coefficients uniquely.

Proof. This follows from the fact that q is a polynomial of degree less than or equal to n in each variable. □

This is a very important result for doing stabilization. See Section 3.4.

Corollary 3.5 (Local Linear Independence). *For any open subset* $Q \subseteq \Omega$, *the set*

$$\{b(k; \cdot) : k \in \mathcal{K} \quad \text{and} \quad \text{supp } b(k; \cdot) \cap Q \text{ has positive measure } \}$$

is linearly independent.

In order to match certain boundary conditions we need weight functions. See Section 1.4.

Fix a positive weight function w, then $w(\cdot)b(k; \cdot)$ is known as a weighted B-spline.

Definition 3.3. *For a bounded and open Lipschitz domain* $\Omega \subset \mathbb{R}^m$,

$$w\mathfrak{B}^n_h(\Omega) := \underset{k \in \mathcal{K}}{\text{span}} \{wb(k; \cdot)\}$$

denotes the linear span of weighted B-splines.

3.3 Classification of B-splines

Fix $h > 0$, and $\Omega \subseteq \mathbb{R}^m$. A grid cell $Q = [0,1]^m h + lh$ is said to be

$$\begin{aligned} \textsf{interior} &\quad \text{if } Q \subseteq \overline{\Omega}. \\ \textsf{boundary} &\quad \text{if int } Q \cap \partial\Omega \neq \text{ empty} \\ \textsf{exterior} &\quad \text{if } Q \cap \Omega = \text{ empty}. \end{aligned}$$

See Figure 3.2.

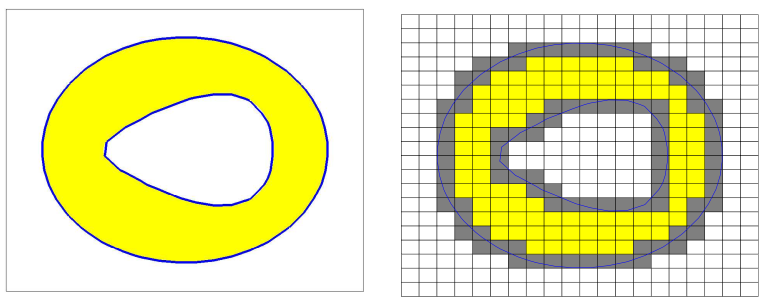


Figure 3.2: We can observe the exterior grid cells (white), the boundary grid cells (gray) and the interior grid cells (yellow); which are classified relative to Ω (left).

Define

$$\mathcal{I} := \{i \in \mathcal{K} : \text{ there exists an interior grid cell } Q \subset \text{supp } b(i;\cdot)\} \text{ and } \mathcal{J} := \mathcal{K}/\mathcal{I}.$$

A B-spline $b(k;\cdot)$ is called inner B-spline if $k + l \in \mathcal{I}$, for some $l \in \{0, 1, ..., n\}^m$ and is called outer B-spline otherwise.

In other words, an inner B-spline $b(i;\cdot)$ has at least an interior grid cell contained in its support; and outer B-spline support consists entirely of boundary and exterior cells. See Figure 3.3.

3.4 Extended B-splines

We start with an illustrative one dimensional example.

Example 3.2. *Consider*

$$\Omega = (2 - \frac{1}{n}, 3); \quad p(x) = x, \quad \forall x \in \Omega.$$

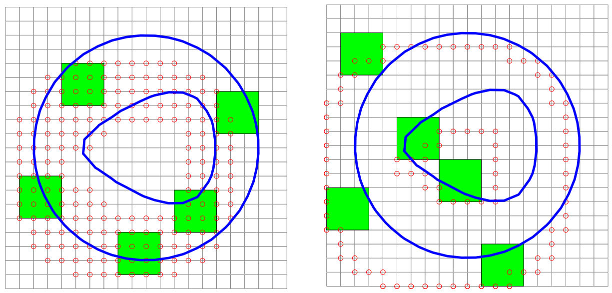


Figure 3.3: We can observe the inner B-splines (left), and outer B-splines (right) for a domain Ω. For each $b(k;\cdot)$ we have marked with "$\circ$" the point $k \in \mathcal{K} \subset \mathbb{R}^2$ which is the lower left corner of its support (green squares). In this case the support of $b(k;\cdot)$ is $k + [0,3]^2$.

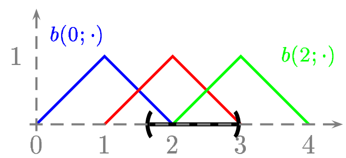
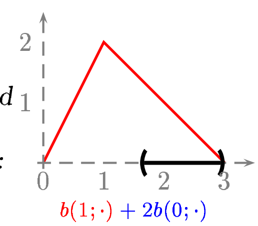


Figure 3.4: $\Omega = (2 - \frac{1}{n}, 3)$

Given $x \in \Omega$ we have the representation

$$\begin{aligned} p(x) =& q(0)b(0;x) + q(1)b(1;x) + q(2)b(2;x) \\ =& 1b(0;x) + 2b(1;x) + 3b(2;x), \end{aligned}$$



We want to modify this expression to exclude small support B-splines restricted to Ω, in this case is $b(0;\cdot)$. See Figure 3.4.

Observe that $q(1) = 2$, $q(2) = 3$ determine the polynomial of coefficients q:

$$q(k) = q(1)\frac{k-2}{1-2} + q(2)\frac{k-1}{2-1}.$$

In particular $q(0) = 2q(1) + q(2)(-1)$. Hence

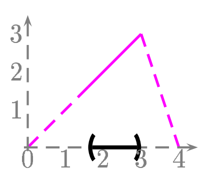
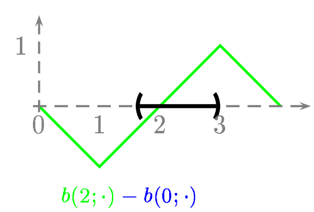


Figure 3.5: Equation (3.2) for $x \in \Omega$.



$$p(x) = q(1)\Big(b(1;x) + 2b(0;x)\Big) + q(2)\Big(b(2;x) - b(0;x)\Big). \tag{3.2}$$

Note that $b(1;\cdot) + 2b(0;\cdot)$ and $b(2;\cdot) - b(0;\cdot)$ does not have small support when restricted to Ω.

Now we discuss the general case. Let p be the restriction to Ω of a polynomial of degree at most n. We know that

$$\begin{aligned} p(x) &= \sum_{k\in\mathcal{K}} q(k)b(k;x), \qquad x \in \Omega, \quad \text{degree}(q) \leq n \\ &= \sum_{i\in\mathcal{I}} q(i)b(i;x) + \sum_{j\in\mathcal{J}} q(j)b(j;x). \end{aligned} \tag{3.3}$$

Definition 3.4. *For each $j \in \mathcal{J}$, choose any array of $(n+1)^m$ inner indices*

$$\mathcal{I}(j) := \ell(j) + \{0,1,...,n\}^m \subset \mathcal{I},$$

i.e., $i \in \mathcal{I}(j)$ if and only if $i = \ell(j) + i' \in \mathcal{I}$, $i' = (i'_1, i'_2, ..., i'_m)$, $i'_\nu \in \{0,1,...,n\}$.

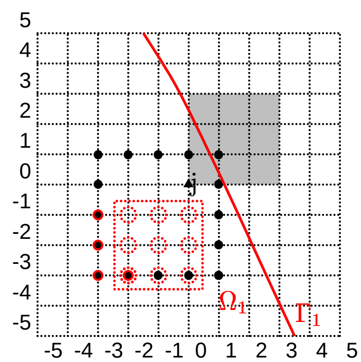


Figure 3.6: For $m = n = 2$ and $j = (0,0)$, the indices of Remark 3.1 are marked with $\bullet$ in black. For Ω_1 the domain on the left of Γ_1 in red. We can choose for $\ell(j)$ any of the indices marked with $\circ$ in red. For $\ell(j) = (-2,-3)$ we have marked $\ell(j) + \{0,1,2\}^2$ with dotted circles in red.

Let $|\partial\Omega|$ the measure of $\partial\Omega$.

Remark 3.1. If $h \ll |\partial\Omega|$, $\partial\Omega$ looks locally as a hyperplane, then one of the indices in (the boundary of) the cube $j - \{-1,0,1,...,n+1\}^m$, can be chosen as $\ell(j)$. See Figures 3.6 and 3.7 for the case $m = n = 2$.

Remark 3.2. If $\ell(j) \in j - \{-1,0,1,...,n+1\}^m$ is one of the indices from Remark 3.1, then the Hausdorff distance (based on the maximun norm) between $\{j\}$ and $\ell(j) + \{0,1,...,n\}^m$ is

$$\rho\Big(\{j\}, \ell(j) + \{0,1,...,n\}^m\Big) = n+1,$$

See Section 1.3 for definition of Hausdorff distance.

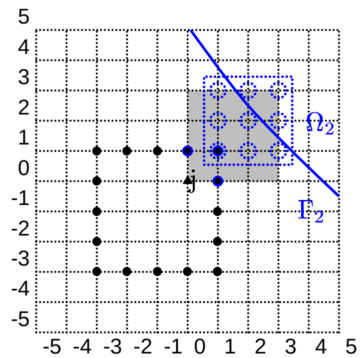


Figure 3.7: For $m = n = 2$ and $j = (0, 0)$, the indices of Remark 3.1 are marked with $\bullet$ in black. For Ω_2 the domain to the right of Γ_2 in blue. We can choose for $\ell(j)$ any of the indices marked with $\circ$ in blue. For $\ell(j) = (1, 1)$ we have marked $\ell(j) + \{0, 1, 2\}^2$ with dotted circles in blue.

Remark 3.3. For each $j \in \mathcal{J}$ we can choose $\mathcal{I}(j)$ as the $(n+1)^m$ dimensional array which is closest to j with respect to the Hausdorff metric based on the maximum norm in $\mathbb{Z}^m$.

Remark 3.4. In the case of $h \ll |\partial\Omega|$, we can assume that if $\ell(j)$ is such that $\rho\big(\{j\}, \ell(j) + \{0, 1, ..., n\}^m\big) > n + 1$ then the B-spline $b(\ell; \cdot)$ is not modified.

Since q is a polynomial of degree at most n in each coordinate variable, it is completely determined by its values at indexes in $\mathcal{I}(j)$ (see Corollary 3.4). Then, the value of q at $k \in \mathbb{Z}^m$ can be computed by the following Lagrange interpolation formula:

$$q(k) = \sum_{i \in \mathcal{I}(j)} q(i) L_i(k) \tag{3.4}$$

where

$$L_i(k) = \prod_{\nu=1}^{m} \prod_{\substack{i' \in \mathcal{I}(j) \\ i'_\nu \neq i_\nu}}^{n} \frac{k_\nu - i'_\nu}{i_\nu - i'_\nu}. \tag{3.5}$$

Note that (3.4) can be written as

$$\tilde{q}(k) = \sum_{i \in \mathcal{I}(j)} \tilde{q}(i) L_i(k)$$

where $\tilde{q}$ is defined in Corollary 2.2. Then $L_i(k)$ does not depend on h. Observe that $L_i(\hat{i}) = \delta_{i,\hat{i}}$, for $\hat{i} \in \mathcal{I}(j)$.

Since $\mathcal{I}(j) = \ell(j) + \{0, 1, ..., n\}^m$ we have $i'_\nu = \ell_\nu + \mu$ with $\mu \in \{0, 1, ..., n\}$, then (3.5) can be written as:

$$L_i(k) = \prod_{\nu=1}^{m} \prod_{\substack{\mu=0 \\ \ell_\nu+\mu\neq i_\nu}}^{n} \frac{k_\nu - (\ell_\nu + \mu)}{i_\nu - (\ell_\nu + \mu)}; \quad i = (i_\nu)_{\nu=1}^m, \quad \ell(j) = (\ell_\nu)_{\nu=1}^m.$$

Define

$$e_{i,j} := L_i(j) = \prod_{\nu=1}^{m} \prod_{\substack{\mu=0 \\ \ell_\nu+\mu\neq i_\nu}}^{n} \frac{j_\nu - \ell_\nu - \mu}{i_\nu - \ell_\nu - \mu} \tag{3.6}$$

and for simplicity set $e_{i,j} = 0$ for $i \notin \mathcal{I}(j)$, then

$$q(j) = \sum_{i\in\mathcal{I}(j)} e_{i,j} q(i),$$

and reordering the terms in (3.3) and defining

$$\mathcal{J}(i) = \{j \in \mathcal{J} : i \in \mathcal{I}(j)\},$$

we get

$$\begin{aligned} p(x) &= \sum_{i\in\mathcal{I}} q(i) b(i;x) + \sum_{j\in\mathcal{J}} \Big(\sum_{i\in\mathcal{I}(j)} e_{i,j} q(i) \Big) b(j;x) \\ &= \sum_{i\in\mathcal{I}} q(i) \Big(b(i;x) + \sum_{j\in\mathcal{J}(i)} e_{i,j} b(j;x) \Big). \end{aligned} \tag{3.7}$$

Note that $\mathcal{J}(i)$ may be empty (and it would be so for most inner B-splines). Observe also that coefficients $e_{i,j}$ do not depend on h.

Lemma 3.7 (Extension coefficients). *There exists constants c and C, independent of h, such that*

$$\begin{aligned} e_{i,j} &= 0, \quad \text{if} \quad ||i - j|| \geq c, \\ |e_{i,j}| &\leq C \quad \text{otherwise.} \end{aligned}$$

Moreover, only $\preceq h^{1-m}$ B-splines near the boundary need to be modified.

Proof. The set of indexes $\mathcal{I}(j)$ is an $(n+1)^m$-array of integers in $\mathcal{I}$ which is closest to j with respect to the Hausdorff metric based on the maximum norm in $\mathbb{Z}^m$. See Remark 3.3. Accordingly, the lattice points $\mathcal{I}(j)$ are close to the point j. We note that, for h sufficiently small, the smooth boundary of Ω is locally close to a hyperplane. Hence, the Hausdorff distance between j and $\mathcal{I}(j)$ can be bounded asymptotically by $n+1$. See Remark 3.1, and 3.2. This yields $e_{i,j} = 0$ for $\rho(\{j\}, \ell(j) + \{0,1,...,n\}^m) > n+1$, see Remark 3.4. On the other hand by definition, the coefficients $e_{i,j}$ are the products of univariate Lagrange polynomials, and by scaling, they are independent of h. This yields the "otherwise" part.
For the second affirmation, it is enough to see Remark 3.4. □

From (3.7) we have

$$p(x) = \sum_{i \in \mathcal{I}} q(i) B(i; x), \quad x \in \Omega$$

where

$$B(i; x) := b(i; x) + \sum_{j \in \mathcal{J}(i)} e_{i,j} b(j; x), \quad x \in \Omega. \tag{3.8}$$

$B(i; \cdot)$ is known an **extended B-spline**.

Note that

$$\text{supp } b(i; \cdot) \subseteq \text{supp } B(i; \cdot).$$

3.5 Web-Splines

Definition 3.5. *Consider the extended B-splines*

$$B(i; \cdot) = b(i; \cdot) + \sum_{j \in \mathcal{J}(i)} e_{i,j} b(j; \cdot).$$

Since supp $b(i; \cdot) = ih + [0, n+1]^m h$, *we can always find an interior grid cell* $\widehat{Q}_{i,\mathcal{J}(i)}$ *such that*

$$\begin{aligned} &\widehat{Q}_{i,\mathcal{J}(i)} \subseteq \textit{supp } b(i; \cdot), \\ &\widehat{Q}_{i,\mathcal{J}(i)} \cap \textit{supp } b(j; \cdot) \textit{ does not have positive measure for } j \in \mathcal{J}, \end{aligned} \tag{3.9}$$

where $\mathcal{J}$ *is the set of outer indices.*

Let $x_i = x_i(\widehat{Q}_{i,\mathcal{J}(i)})$ denote the center of $\widehat{Q}_{i,\mathcal{J}(i)}$

Then, if we multiply the previous extended B-spline $B(i; x)$ by the factor $w(x)/w(x_i)$ where w is a positive weight function (see Section 1.4) and x_i is the center of the interior grid cell of the previous definition, we obtain

$$\mathcal{B}(i; \cdot) := \frac{w(\cdot)}{w(x_i)} B(i; \cdot). \tag{3.10}$$

We refer to it as **weighted extended B-spline** (web-splines).

The factor $w(\cdot)/w(x_i)$ causes the weighted extended B-spline $\mathcal{B}(i; \cdot)$ to vanish at the boundary of Ω and also magnifies functions supported near the boundary for scaling purposes, since $w(\cdot)/w(x_i)$ is close to 1 near x_i.
This fact will important for stability analysis.

Remark 3.5. Observe that supp $\mathcal{B}(i; \cdot) =$ supp $B(i; \cdot) \cap \bar{\Omega}$ and we have

$$\text{supp } \mathcal{B}(i; \cdot) \subseteq \text{supp } b(i; \cdot) \cup \bigcup_{j \in \mathcal{J}(i)} \text{supp } b(j; \cdot).$$

Moreover, when h is small enough we can assume that supp $\mathcal{B}(i;\cdot)$ is connected and supp $b(i;\cdot) \cap$ supp $b(j;\cdot)$, contains at least one edge, $\forall j \in \mathcal{J}(i)$, see Remark 3.1.

Remark 3.6. When h is small enough we observe that

1. supp $\mathcal{B}(i;\cdot) =$ supp $b(i;\cdot) \subseteq x_i + [-(n+1/2),(n+1/2)]^m h$ for inner B-splines that are not modified. See Remark 3.4.

2. Since, from previous remark supp $\mathcal{B}(i;\cdot)$ is connected, given $x \in$ supp $\mathcal{B}(i;\cdot)$ we have
$$||x_i - x||_\infty \leq (n + 1/2 + n + 1)h = (2n + 3/2)h$$
then supp $\mathcal{B}(i;\cdot) \subseteq x_i + [-(2n+3/2),(2n+3/2)]^m h$.

Except for positivity, weighted extended B-splines inherit all essential properties from B-splines.

Lemma 3.8. *For the majority of inner indices i, $\mathcal{B}(i;\cdot) = \frac{w(\cdot)}{w(x_i)} b(i;\cdot)$, as h become small.*

Proof. For weighted extended B-splines with support sufficiently separated from the boundary we have
$$\mathcal{B}(i;x) = (w(x)/w(x_i))b(i;x), \quad \forall x \in \text{supp } \mathcal{B}(i;\cdot).$$
See Remark 3.4. □

Lemma 3.9 (Local support). *The diameter of supp $\mathcal{B}(i;\cdot)$ is of order h.*

Proof. Clearly there exists $c > 0$ such that
$$\text{supp } \mathcal{B}(i;\cdot) \subseteq x_i + [-c,c]^m h,$$
where x_i is the center of interior the grid cell $\widehat{Q}_{i,\mathcal{J}(i)}$ in Definition 3.5, this can be seen easily from Remark 3.6. □

Corollary 3.6. *For h small enough we have $\#(\mathcal{J}(i)) \preceq 1$.*

Proof. Take $j_0 \in \mathcal{J}(i)$ we have
$$\text{supp } b(j_0;\cdot) \subseteq \text{supp } \mathcal{B}(i;\cdot) \subseteq x_i + [-c,c]^m h$$
where $c \leq 2n + 3/2$. In particular $j_0 \in \{k \in \mathbb{Z}^m : k \in x_i + [-c,c]^m h\}$ but this last set can have at most $(4n+3)^m$ grid points. □

Lemma 3.10. *On each set $Q \cap \overline{\Omega}$ only $\preceq 1$ web-splines are nonzero.*

Proof. Define

$$\mathcal{I}(Q) := \{i : \exists x \in Q \text{ such that } \mathcal{B}(i;x) \neq 0\}.$$

If Q is an interior grid cell we have

$$\mathcal{I}(Q) = \{i \in \mathcal{I} : \exists x \in Q \text{ such that } b(i;x) \neq 0\},$$

because in this case $\mathcal{B}(i;\cdot)|_Q = (w(\cdot)/w(x_i))b(i;\cdot)$, since no outer B-splines has an interior grid cell in this support. From Remark 3.4 we know that $\#(\mathcal{I}(Q)) \leq (n+1)^m$.

If Q is a boundary grid cell we have

$$\mathcal{I}(Q) \subset \left\{ i \in \mathcal{I} : \begin{array}{c} \exists x \in Q \text{ such that} \\ b(i;x) \neq 0 \end{array} \right\} \bigcup \left\{ j \in \mathcal{J}(i) : \begin{array}{c} \exists x \in Q \text{ such that} \\ b(j;x) \neq 0 \end{array} \right\},$$

then we can bound

$$\#(\mathcal{I}(Q)) \leq (n+1)^m + \#(\mathcal{J}(i))(n+1)^m.$$

and for Corollary 3.6 we have the result. □

Lemma 3.11. *The collection $\{\mathcal{B}(i;\cdot)\}_{i\in\mathcal{I}}$ is linearly independent.*

Proof. The linear independence of $\mathcal{B}(i;\cdot)$ follows from the fact that the local linear independence for $b(i;\cdot)$, see Corollary 3.5, since $\mathcal{B}(i;\cdot)$ restricted to an interior grid cell is exactly $w(\cdot)/w(x_i)b(i;\cdot)$. □

Definition 3.6. *For a bounded and open Lipschitz domain $\Omega \subset \mathbb{R}^m$,*

$$web\mathfrak{B}_h^n(\Omega) := \underset{i\in\mathcal{I}}{span}\ \{\mathcal{B}(i;\cdot)\}$$

denotes the linear span of weighted extended B-splines $\mathcal{B}(i;\cdot)$.

This way, $\{\mathcal{B}(i;\cdot)\}_{i\in\mathcal{I}}$ is also referred to as a web-basis of $web\mathfrak{B}_h^n(\Omega)$.

Chapter 4

Finite Element Analysis with B-splines

In this chapter we present a Galerkin discretization using B-splines applied to elliptic problems with Dirichlet boundary condition. Recall from the previous chapter the definition of $w\mathfrak{B}_h^n(\Omega)$ (Definition 3.3) and $web\mathfrak{B}_h^n(\Omega)$ (Definition 3.6) which are going to be our finite element spaces with basis functions being the weighted B-splines and weighted extended B-splines, respectively.
We explain this in more detail in Section 4.1, at the end of which we explain the organization of the rest of the chapter.

To avoid proliferation of constants, we will use the notation $A \preceq B$ to represent the inequality $A \leq \text{const}\ B$.

4.1 Finite Elements

Let V be a Hilbert space with inner product $(\cdot,\cdot)_V$ and norm

$$||u||_V := \sqrt{(u,u)_V}, \text{ for all } u \in V$$

Let V' denote the dual space of V, with the dual norm $||\cdot||_{V'}$.
For $a : V \times V \to \mathbb{R}$, continuous (or bounded) bilinear form define

$$||a||_V = \sup_{||u||=1,||v||=1} a(u,v).$$

Note that for all u, $v \in V$ we have $|a(u,v)| \leq ||a||_V ||u||_V ||v||_V$.

Given $f \in V'$ consider the following problem

$$\begin{cases} \text{Find } u \in V \text{ such that:} \\ a(u,v) = f(v) \quad \forall v \in V \end{cases} \tag{4.1}$$

i.e., find u such that the continuous linear functional $v \to a(u, v)$ is precisely f.

Definition 4.1. *We say that a bilinear form $a : V \times V \to \mathbb{R}$ is α-elliptic on V if there exists a real number $\alpha > 0$ such that*

$$a(v, v) \geq \alpha ||v||_V^2 \quad \forall v \in V.$$

Note that if a is elliptic and continuous then $||v||_a := \sqrt{a(v, v)}$ is a norm on V equivalent to $|| \cdot ||_V$,

$$\sqrt{\alpha}\ ||v||_V \leq ||v||_a \leq \sqrt{||a||}\ ||v||_V \quad \forall v \in V.$$

To solve the problem (4.1) we use the following

Lemma 4.1 (Lax-Milgram). *If $a : V \times V \to \mathbb{R}$ is continuous and elliptic then for each $f \in V'$, the problem (4.1) has a unique solution u. Moreover, u satisfies*

$$a(u, v) = f(v), \quad \forall v \in V$$
$$||u||_V \leq \frac{1}{\alpha} ||f||_{V'}.$$

As an example we have a test problem. Solve the equation

$$\begin{cases} -\Delta u &= g \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Consider $V = H_0^1(\Omega)$, and $f \in (H_0^1(\Omega))'$ defined by equation

$$\langle f, v \rangle = f(v) = \int_\Omega g\ v, \quad \forall v \in H_0^1(\Omega).$$

Since, under regular enough conditions we have from Green's formula

$$-\int_\Omega \Delta u\ v = -\int_{\partial\Omega} v\ \partial_\eta u + \int_\Omega \nabla u\ \nabla v = \int_\Omega \nabla u\ \nabla v.$$

Define $a : V \times V \to \mathbb{R}$ by

$$a(u, v) = \int_\Omega \nabla u\ \nabla v.$$

Since u, $v \in H_0^1(\Omega)$, the integral is well defined and a is continuous,

$$|a(u, v)| \leq ||a||\ ||u||_V ||v||_V.$$

To apply Lax-Milgram lemma we need ellipticity of a bilinear from a, this is consequence of Friedrichs Inequality (Lemma 1.2).

We consider approximation schemes for the variational problem just defined. To approximate problem (4.1), we apply the so-called Ritz-Galerkin

method based upon the variational principle. It consists of replacing the infinite-dimensional space V by a finite-dimensional space $V_h \subset V$, $dim V_h = N < \infty$. In the case of these notes, this finite dimensional space is one generated by weighted B-splines (see Section 3.2) or weighted extended B-splines (see Section 3.5).
The Hilbert space V_h is equipped with the same norm $||\cdot||_V$. We assume the following approximation property of V_h

$$\inf_{v_h \in V_h} ||v - v_h|| \to 0 \quad \text{as } h \to 0, \quad \forall v \in V. \tag{4.3}$$

The index h (which is the grid size) will refer to a mesh which our approximations are derived from. The Galerkin approximation to (4.1) is defined as follows

$$\begin{cases} \text{Find } u_h \in V_h \text{ such that:} \\ a(u_h, v) = f(v) \quad \forall v \in V_h. \end{cases} \tag{4.4}$$

Let u be a solution to (4.1), we have:

Lemma 4.2 (Céa's Lemma). *Under the assumptions of Lax-Milgram Lemma there exists a unique solution u_h to problem (4.4), with*

$$||u_h||_V \leq \frac{1}{\alpha} ||f||_{V'} \tag{4.5}$$

$$||u - u_h||_V \leq \frac{||a||}{\alpha} \inf_{v_h \in V_h} ||u - v_h||_V. \tag{4.6}$$

Proof. Subtracting (4.4) form (4.1), we get

$$a(u - u_h, v_h) = 0 \quad \forall v_h \in V_h.$$

Thus

$$\begin{aligned} \alpha \, ||u - u_h||_V^2 &\leq a(u - u_h, u - u_h) \\ &= a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h) \\ &= a(u - u_h, u - v_h) \\ &\leq ||a|| \; ||u - u_h||_V ||u - v_h||_V \quad \forall v_h \in V_h. \end{aligned}$$

Hence (4.6) follows. □

In our case $V = H_0^1(\Omega)$, $V_h = w\mathfrak{B}_h^n(\Omega)$ or $V_h = web\mathfrak{B}_h^n(\Omega)$ and then a Ritz Galerkin approximation is of the form

$$\begin{cases} \text{Find } u_h \in w\mathfrak{B}_h^n(\Omega) \text{ such that:} \\ a(u_h, v) = f(v) \quad \forall v \in w\mathfrak{B}_h^n(\Omega). \end{cases} \tag{4.7}$$

where $u_h = \sum_{i \in \mathcal{K}} a_i wb(i; \cdot)$; or

$$\begin{cases} \text{Find } u_h \in web\mathfrak{B}_h^n(\Omega) \text{ such that:} \\ a(u_h, v) = f(v) \quad \forall v \in web\mathfrak{B}_h^n(\Omega). \end{cases} \tag{4.8}$$

where $u_h = \sum_{i \in \mathcal{I}} a_i \mathcal{B}(i;\cdot)$.

Céa's lemma then becomes

$$\|u - u_h\|_{H^1(\Omega)} \preceq \inf_{v_h \in w\mathfrak{B}_h^n(\Omega)} \|u - v_h\|_{H^1(\Omega)}. \tag{4.9}$$

or

$$\|u - u_h\|_{H^1(\Omega)} \preceq \inf_{v_h \in web\mathfrak{B}_h^n(\Omega)} \|u - v_h\|_{H^1(\Omega)}. \tag{4.10}$$

To estimate the right hand side of (4.9) or (4.10), as usual , we construct a (quasi)interpolation operator with some approximation properties.
In order to construct this interpolation operator and derive the estimates corresponding to its approximation we introduce dual functions in Section 4.2.

After defining the quasi interpolator operator we present its approximations properties, which depends on the regularity of u/w where u is the true solution of our elliptic problem and w is the weight function, see Section 4.3.

4.2 Dual Basis Functions

We will construct dual basis functions for weighted extended B-splines. We first construct a dual function related to $b(0;\cdot)$ on a cube contained in its support.

Lemma 4.3. *Let $h = 1$ and Q an m-dimensional cube with $Q \subset supp\ b(0;\cdot) = [0, n+1]^m$. There exists a function λ such that:*

$$supp\ \lambda \subseteq Q$$

$$\int_Q \lambda(x) b(k;x) dx = \delta_{0,k} := \begin{cases} 1, & if \quad k = 0 \\ 0, & if \quad k \neq 0. \end{cases}$$

Proof. Define

$$I_0 = \{k \in \mathbb{Z}^m : \text{supp } b(k;\cdot) \cap Q \quad \text{has positive measure}\}.$$

Observe that $0 = (0, 0, ..., 0) \in I_0$ and let $V = \text{span } \{b(k;\cdot)|_Q : k \in I_0\}$.

Consider $\rho_0 : V \to \mathbb{R}$ defined by $\rho_0(\sum_{k \in I_0} a_k b(k;\cdot)) = a_0$, since $\{b(k;\cdot)|_Q\}_{k \in I_0}$ are linearly independents, ρ_0 is well defined.
Observe that

$$\langle b, c\rangle_Q = \int_Q b(x)c(x)dx, \quad b, c \in V,$$

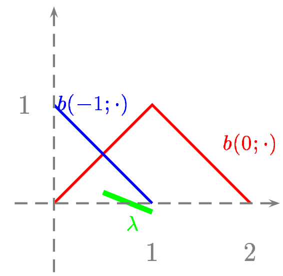
is an inner product on V. Then by Riesz representation theorem there exists $\lambda \in V$ with

$$\langle \lambda, b\rangle_Q = \rho_0(b) \quad \forall b \in V.$$

In particular

$$\langle \lambda, b(k;\cdot)\rangle_Q = \rho_0(b(k;\cdot)) = \delta_{0,k}. \quad \square$$

Example 4.1. *Consider $m = 1$, $h = 1$ and $V = span\,\{b(-1;\cdot), b(0;\cdot)\}$. Let $Q = [1/2, 1]$. There exists $\lambda \in V$ such that*



$\lambda = \frac{1}{45}(14 - 18x)$

$$\langle \lambda, b(-1;\cdot)\rangle_Q = 0, \quad \langle \lambda, b(0;\cdot)\rangle_Q = 1.$$

Put $\lambda = \alpha_1 b(-1;\cdot) + \alpha_2 b(0;\cdot)$, we want to show that

$$\alpha_1\langle b(-1;\cdot), b(-1;\cdot)\rangle_Q + \alpha_2\langle b(0;\cdot), b(-1;\cdot)\rangle_Q = 0,$$
$$\alpha_1\langle b(-1;\cdot), b(0;\cdot)\rangle_Q + \alpha_2\langle b(0;\cdot), b(0;\cdot)\rangle_Q = 1,$$

hence,

$$\frac{1}{12}\begin{pmatrix} 7/2 & 1 \\ 1 & 7/2 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We get $\alpha_1 = -4/45$, $\alpha_2 = 14/45$, i.e., $\lambda = \frac{1}{45}(14 - 18x)$.

Next lemma says that the function λ in Lemma 4.3 can be chosen with bounded $\mathbf{L}^2(\Omega)$ norm. See Section 1.2.

Lemma 4.4. *Let $h = 1$ and $Q_0' \subset supp\ b(0;\cdot) = [0, n+1]^m$. Q_0' an m-dimensional cube of width $\theta \le n+1$. Then there exists a function λ_0 with*

$$supp\ \lambda_0 \subseteq Q_0' \tag{4.11}$$
$$\int_{Q_0'} \lambda_0(x) b(k;x)dx = \delta_{0,k}, \tag{4.12}$$

and $\|\lambda_0\|_{L^2} \le c$, where c is a constant depending only on m, n and θ.

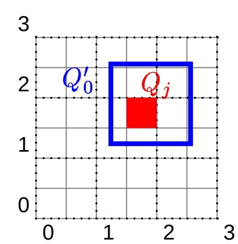
Proof. Since

$$\frac{2}{\theta} < \left\lfloor \frac{2}{\theta} \right\rfloor + 1,$$

the interval $[0, n+1]$ can be partitioned into $(\lfloor \frac{2}{\theta} \rfloor + 1)(n+1)$ intervals of length

$$\frac{1}{\lfloor \frac{2}{\theta} \rfloor + 1} < \frac{\theta}{2}, \tag{4.13}$$

where $\lfloor z \rfloor$ denote the integer part of z.



$\theta = 1.4$, $n = 2$

Choose Q_j

Then $[0, n+1]^m$ can be partitioned in $N = \left(\lfloor \frac{2}{\theta} \rfloor + 1)(n+1)\right)^m$ m-cubes of width $\le \theta/2$.

Let Q_j, $j = 1, 2, .., N$ be these cubes.

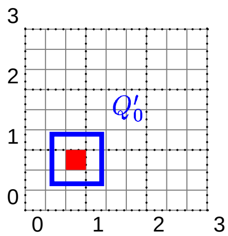
From Lemma 4.3 applied to Q_j we get λ_j, $j = 1, 2, ..., N$ such that

$$\text{supp } \lambda_j \subseteq Q_j$$
$$\langle \lambda_j, b(k;\cdot)\rangle_{Q_j} = \delta_{0,k}.$$

If Q_0' has width θ, then from (4.13) there exists at least an index $j \in \{1, 2, .., N\}$ such that

$$Q_j \subseteq \text{int } Q_0'.$$

Take $\lambda_0 = \lambda_j$ in Q_j and $\lambda_0 = 0$ in Q_0'/Q_j then (4.11) and (4.12) holds.



$n = 2$, $\theta = 0.9$

Choose Q_j

We also have

$$\begin{aligned}\|\lambda_0\|_{L^2}^2 &= \langle \lambda_0, \lambda_0 \rangle_{Q_0'} \\ &= \langle \lambda_j, \lambda_j \rangle_{Q_j} \qquad \text{from definition of } \lambda_0 \\ &\leq \|\lambda_j\|_\infty^2 \text{ volume } (Q_j) \\ &\leq \max_{1\leq j\leq N} \|\lambda_j\|_\infty^2 \text{ volume } (Q_j)\end{aligned}$$

and observe that the last term depends only on θ, m and n. (dependency on n being because λ_j is a linear combinations of $b(k;\cdot)$'s that have degree n). □

Previous lemma can be reformulated for arbitrary h and index i, according to

Lemma 4.5. *Let Q_i' be an m-dimensional cube with width θh, and such that $Q_i' \subset supp\ b(i;\cdot)$, $i \in \mathbb{Z}^m$. There exists a function λ_i with:*

$$supp\ \lambda_i \subseteq Q_i' \tag{4.14}$$

$$\int_{Q_i'} \lambda_i(x) b(k;x) dx = \delta_{i,k} := \begin{cases} 1, & if \quad i = k \\ 0, & if \quad i \neq k \end{cases} \tag{4.15}$$

$$\|\lambda_i\|_{L^2} \leq c h^{-m/2}, \tag{4.16}$$

where c is a constant depending only m (dimension), n (degree of $b(i;\cdot)$) and θ (width factor).

Proof. Let $T_i : \mathbb{R}^m \to \mathbb{R}^m$ defined by $T_i(x) = xh + ih$.
We have det $(T_i) = h^m$, and

$$T_i(\text{supp } b(0;\cdot)) = T_i((0, n+1)^m) = (0, n+1)^m h + ih = \text{supp } b(i;\cdot).$$

Let $Q_i' = T_i(Q_0')$, observe that Q_0' has width θ. From Lemma 4.4 there exists λ_0 such that (4.11) and (4.12) holds.
Define

$$\lambda_i(x) := \frac{1}{h^m} \lambda_0(T_i(x)).$$

Then (4.14) holds and

$$\begin{aligned}\int_{Q_i'} \lambda_i(x) b(k;x) dx &= \int_{T_i(Q_0')} \lambda_i(x) b(k;x) dx \\ &= \int_{Q_0'} \lambda_i(T_i^{-1}(y)) b(k; T_i(y)) \text{ det } (T_i) dy \\ &= \int_{Q_0'} \lambda_0(y) b(k; hy + hi) dy \\ &= \int_{Q_0'} \lambda_0(y) b(k-i, 1; y) dy \\ &= \delta_{0,k-i} = \delta_{k,i},\end{aligned}$$

where

$$b(k; hy + hi) = b(n, k, h; hy + hi) = \hat{b}(n; \frac{hy + hi}{h} - k)$$
$$= \hat{b}(n; y + i - k) = b(n, k - i, 1; y) \quad \text{and } h = 1.$$

Moreover

$$\|\lambda_i\|_{L^2}^2 = \int_{Q_i'} \frac{1}{h^{2m}} \lambda_0^2(T_i(x))dx = \frac{1}{h^m} \int_{Q_0'} \lambda_0^2(y)dy = \frac{1}{h^m} \|\lambda_0\|_{L^2}^2 \leq c\frac{1}{h^m},$$

where c depends only on m, n and θ. (See Lemma 4.4). □

The construction of λ_i related to $b(i; \cdot)$ in the previous lemma can be adapted also for weighted extended B-splines. We have, by recalling the form of weighted extended B-spline, that

$$\mathcal{B}(i; x) = \frac{w(x)}{w(x_i)} B(i; x) = \frac{w(x)}{w(x_i)} \Big(b(i; x) + \sum_{j \in \mathcal{J}(i)} e_{i,j} b(j; x) \Big),$$

where w is a weight function of order γ (see Definition 1.3).

Lemma 4.6. *For each weighted extended B-spline $\mathcal{B}(i; \cdot)$ there exists a function Λ_i such that*

$$\text{supp } \Lambda_i \subseteq Q_i,$$
$$\langle \Lambda_i, \mathcal{B}(i'; \cdot) \rangle_{Q_i} = \delta_{i,i'} \quad i, i' \in \mathcal{I},$$
$$\|\Lambda_i\|_{L^2} \leq ch^{-m/2},$$

where c is a constant depending only on n, w, Ω and $Q_i = \widehat{Q}_{i,\mathcal{J}(i)}$ is the interior grid cell of Definition 3.5, i.e.,

$$Q_i \subseteq \text{supp } b(i; \cdot),$$
$$Q_i \cap \text{supp } b(j; \cdot) \text{ does not have positive measure for } j \in \mathcal{J}. \tag{4.17}$$

Proof. With x_i the center of $Q_i = \widehat{Q}_{i,\mathcal{J}(i)}$. Apply Lemma 4.5 to the cube

$$Q_i' = x_i + \frac{1}{4\sqrt{m}}[-h, h]^m,$$

which has diameter $\theta = \frac{1}{4\sqrt{m}}$ (see Figure 4.1).

We obtain λ_i such that

$$\text{supp } \lambda_i \subseteq Q_i',$$
$$\langle \lambda_i, b(j; \cdot) \rangle_{Q_i'} = \delta_{i,j},$$
$$\|\lambda_i\|_{L^2} \leq c\frac{1}{h^{m/2}}$$

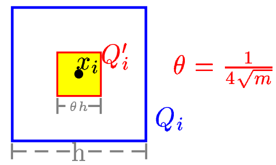


Figure 4.1: Q'_i, and Q_i from Lemma 4.6.

where c is a constant depending only on m, n and θ.

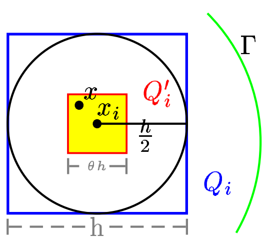
Define

$$\Lambda_i(x) := \frac{w(x_i)}{w(x)}\lambda_i(x), \quad x \in Q'_i \tag{4.18}$$

and extended by zero outside Q'_i.

We have for i, $i' \in \mathcal{I}$, and $x \in Q'_i$

$$\begin{aligned}\int_{Q'_i} \Lambda_i(x)\mathcal{B}(i';x)dx &= \int_{Q'_i} \frac{w(x_i)}{w(x)}\lambda_i(x)\frac{w(x)}{w(x'_i)}\Big(b(i';x) + \sum_{j\in\mathcal{J}(i')} e_{i',j}b(j;x)\Big)dx \\ &= \frac{w(x_i)}{w(x'_i)}\int_{Q'_i}\lambda_i(x)b(i';x)dx \quad \text{because } Q'_i \subseteq Q_i, \text{and eq (4.17)} \\ &= \frac{w(x_i)}{w(x'_i)}\delta_{i,i'} = \delta_{i,i'}.\end{aligned}$$



For $x \in$ supp $\Lambda_i \subseteq Q'_i$ we have

$$d(x_i, x) \leq \sup_{u,v\in Q'_i} d(u,v) = \sqrt{m}\,\theta h = \frac{h}{4},$$

from (1.1) and triangle inequality we have that

$$\begin{aligned}d(x,\Gamma) &\geq d(x_i,\Gamma) - d(x_i,x) \geq d(x_i,\Gamma) - \frac{h}{4} \\ &\geq d(x_i,\Gamma) - \frac{1}{2}d(x_i,\Gamma) \qquad \text{because } Q_i \subseteq \text{ int } \Omega, \text{ and } d(x_i,\Gamma) \geq h/2 \\ &= (1-\frac{1}{2})d(x_i,\Gamma) = \frac{1}{2}d(x_i,\Gamma).\end{aligned}$$

Hence

$$\frac{d(x_i,\Gamma)}{d(x,\Gamma)} \leq 2.$$

Since w is a weighted function of order γ, from definition, we have

$$\frac{w(x_i)}{w(x)} \leq \frac{C_w}{c_w}\left(\frac{d(x_i,\Gamma)}{d(x,\Gamma)}\right)^\gamma \leq 2^\gamma\frac{C_w}{c_w}. \tag{4.19}$$

This implies

$$\max_{x\in Q_i'} \frac{w(x_i)}{w(x)} \leq 2^\gamma \frac{C_w}{c_w} = C.$$

Then

$$\begin{aligned}\|\Lambda_i(\cdot)\|_{L^2}^2 &= \left\| \frac{w(x_i)}{w(\cdot)} \lambda_i(\cdot) \right\|_{L^2}^2 = \int_{Q_i'} \left(\frac{w(x_i)}{w(x)} \right)^2 \lambda_i^2(x) dx \\ &\leq C^2 \int_{Q_i'} \lambda_i^2(x) dx = C^2 \|\lambda_k(\cdot)\|_{L^2}^2 \leq c \frac{1}{h^m}. \quad \square\end{aligned}$$

Lemma 4.7. *If Ω is convex, for any weight function w of order γ there exists a constant C such that*

$$2^{-\gamma} c \leq \frac{w(x)}{w(x_i)} \leq C \left(1 + 2\sqrt{m} \right)^\gamma, \quad \forall x \in \textit{supp } \mathcal{B}(i;\cdot),$$

where x_i is the center of the interior grid cell $\widehat{Q}_{i,\mathcal{J}(i)}$ in Definition 3.5.

Proof. For all x in the support of $\mathcal{B}(i;\cdot)$ we have

$$d(x,\Gamma) \leq d(x_i,\Gamma) + ||x - x_i|| \leq d(x_i,\Gamma) + h\sqrt{m} \leq (1 + 2\sqrt{m}) d(x_i,\Gamma)$$

because $||x - x_i|| \leq \text{diam } \widehat{Q}_{i,\mathcal{J}(i)} = h\sqrt{m}$ and $d(x_i,\Gamma) \geq h/2$. Then,

$$\frac{d(x,\Gamma)}{d(x_i,\Gamma)} \leq 1 + 2\sqrt{m}.$$

Since w is a weight function of order γ there exists constants c_w and C_w such that

$$\frac{w(x)}{w(x_i)} \leq \frac{C_w}{c_w} \left(\frac{d(x,\Gamma)}{d(x_i,\Gamma)} \right)^\gamma \leq \frac{C_w}{c_w} (1 + 2\sqrt{m})^\gamma.$$

In the same way that we did with (4.19) we can show that

$$\frac{w(x)}{w(x_i)} \geq \frac{1}{2^\gamma} \frac{c_w}{C_w},$$

then,

$$2^{-\gamma} c \leq \frac{w(x)}{w(x_i)} \leq C(1 + 2\sqrt{m})^\gamma. \quad \square$$

Lemma 4.8. *For $i \in \mathcal{I}$, the weighted extended B-spline $\mathcal{B}(i;\cdot)$ is uniformly bounded with respect to the grid h*

$$||\mathcal{B}(i;\cdot)||_{L^2} \leq c\, h^{m/2},$$

where c is a constant depending only of n, w and Ω.

Proof. From Lemma 3.7 there exists a constant C independent of h such that $|e_{i,j}| \leq C$, then there exists a real positive constant C_2 independent of h such that

$$\sum_{j\in\mathcal{J}(i)} |e_{i,j}|^2 \leq \sum_{j\in\mathcal{J}(i)} C^2 = C_2. \tag{4.20}$$

From Lemma 4.7 we have that

$$\frac{w(\cdot)}{w(x_i)} \leq C(1+2\sqrt{m})^\gamma = C_1. \tag{4.21}$$

From Lemma 3.2 we get for any $k \in \mathcal{K}$, $||b(k;\cdot)||^2_{L^2} = h^m ||\hat{b}(n;\cdot)||^2_{L^2}$.

From (4.20), (4.21) and the previous statement we have that

$$\begin{aligned}
||\mathcal{B}(i;\cdot)||^2_{L^2} &= \int_{\mathbb{R}^m} \left(\frac{w(x)}{w(x_i)}\right)^2 \left(b(i;x) + \sum_{j\in\mathcal{J}(i)} e_{i,j} b(j;x)\right)^2 dx \\
&\leq C_1^2 \Big(1 + \#(\mathcal{J}(i))\Big) \left[\int_{\mathbb{R}^m} (b(i;x))^2 dx + \sum_{j\in\mathcal{J}(i)} |e_{i,j}|^2 h^m ||\hat{b}(n;\cdot)||^2_{L^2}\right] \\
&= C_1^2 \Big(1 + \#(\mathcal{J}(i))\Big) \left[h^m ||\hat{b}(n;\cdot)||^2_{L^2} + C_2^2 \; h^m ||\hat{b}(n;\cdot)||^2_{L^2}\right] \\
&= C_1^2 \Big(1 + \#(\mathcal{J}(i))\Big) (1 + C_2^2) h^m \; ||\hat{b}(n;\cdot)||^2_{L^2}
\end{aligned}$$

The from Corollary 3.6 i.e., $||\mathcal{B}(i;\cdot)||_{L^2} \leq \text{const } (n, w, \Omega) h^{m/2}$. □

Lemma 4.9 (Stability). *The web-basis $\{\mathcal{B}(i;\cdot)\}_{i\in\mathcal{I}}$ is stable with respect to the grid h,*

$$c\; h^{m/2} ||A|| \leq \left\| \sum_{i\in\mathcal{I}} a_i \mathcal{B}(i;\cdot) \right\|_{L^2} \leq C\; h^{m/2} ||A||,$$

where c and C are constants depending only of n, w, and Ω, and $A = \{a_i\}_{i\in\mathcal{I}}$, $||A||^2 = \sum_{i\in\mathcal{I}} a_i^2$.

Proof. The triangle and the Cauchy-Schwarz inequality and previous lemma imply

$$\left\| \sum_{i\in\mathcal{I}} a_i \mathcal{B}(i;\cdot) \right\|^2_{L^2(Q\cap\Omega)} \leq \left(\sum_{i\in\mathcal{I}(Q)} |a_i| C \;\; h^{m/2} \right)^2 \leq C \;\; h^m \#(\mathcal{I}(Q)) \sum_{i\in\mathcal{I}(Q)} |a_i|^2.$$

Note that

$$i \in \mathcal{I}(Q) \Leftrightarrow Q \in \mathcal{Q}(i), \quad \text{where } \mathcal{Q}(i) := \big\{Q : i \in \mathcal{I}(Q)\big\},$$

then by summing the above inequality over all grid cells Q and interchanging sums we obtain

$$\left\| \sum_{i\in\mathcal{I}} a_i \mathcal{B}(i;\cdot) \right\|^2_{L^2} \leq C h^m \sum_{i\in\mathcal{I}} \sum_{Q\in\mathcal{Q}(i)} |a_i|^2.$$

Since $\#(\mathcal{Q}(i)) \preceq 1$, (see Lemma 3.10), the right-hand side is $\preceq h^m ||A||^2$, proving the upper bound for $\| \sum_{i \in \mathcal{I}} a_i \mathcal{B}(i;\cdot)\|_{L^2}$.

The lower bound is derived with the aid of the weighted dual functions Λ_i. By Lemma 4.6 and the Cauchy-Schwarz inequality we have

$$\begin{aligned} |a_i|^2 &= \left| \left\langle \Lambda_i, \sum_k a_k \mathcal{B}(k;\cdot) \right\rangle_{L^2(Q_i)} \right|^2 \\ &\leq \|\Lambda_i\|^2_{L^2(Q_i)} \left\| \sum_k a_k \mathcal{B}(k;\cdot) \right\|^2_{L^2(Q_i)} \\ &\leq c\, h^{-m} \left\| \sum_k a_k \mathcal{B}(k;\cdot) \right\|^2_{L^2(Q_i)} \end{aligned}$$

Summing this estimate over $i \in \mathcal{I}$, at most $\max_Q \#(\mathcal{I}(Q)) \preceq 1$ repetitions can occur on the right-hand side. Hence,

$$||A||^2 = \sum_{i \in \mathcal{I}} |a_i|^2 \leq c\, h^{-m} \left\| \sum_k a_k \mathcal{B}(k;\cdot) \right\|^2_{L^2}. \quad \square$$

Lemma 4.10 (Bernstein Inequality). *If w is a ℓ-regular weight function of order γ we have*

$$\left\| \sum_{i \in \mathcal{I}} a_i \mathcal{B}(i;\cdot) \right\|_{H^\nu(\Omega)} \leq \ const\ (n, w, \Omega) h^{m/2} h^{-\nu} ||A||, \quad \nu \leq \ell \leq n.$$

Proof. Since the web-splines have local support (see Lemma 3.9) it is enough to estimate the norm of the basis functions.

Let $x \in \operatorname{supp} \mathcal{B}(i;\cdot)$. From Leibniz's rule we have

$$\partial^\alpha \mathcal{B}(i;x) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} \partial^\beta w(x) \frac{1}{w(x_i)} \partial^{\alpha-\beta} \left(b(i;x) + \sum_{j \in \mathcal{J}(i)} e_{i,j} b(j;x) \right).$$

From the definition of ℓ-regularity, it follows that there exists C depending of w such that

$$|\partial^\beta w(x)| \leq C d(x,\Gamma)^{\gamma-|\beta|}, \quad |\beta| \leq \min\{\gamma, \ell\},$$

and from estimations obtained in Lemma 4.7 we have that

$$\begin{aligned} \left| \partial^\beta w(x) \frac{1}{w(x_i)} \right| &\leq C d(x,\Gamma)^{\gamma-|\beta|} \frac{1}{c \quad d(x_i,\Gamma)^\gamma} \leq \frac{C(1+2\sqrt{m})^{\gamma-|\beta|} d(x_i,\Gamma)^{\gamma-|\beta|}}{c \quad d(x_i,\Gamma)^\gamma} \\ &\leq \frac{C(1+2\sqrt{m})^{\gamma-|\beta|}}{c} \frac{1}{d(x,\Gamma)^{|\beta|}} \leq \frac{C(1+2\sqrt{m})^{\gamma-|\beta|}}{c} \frac{2^{|\beta|}}{h^{|\beta|}}. \end{aligned}$$

If $|\beta| \geq \gamma$, let $M = \max_\Omega |\partial^\beta w(x)|$; then

$$\max_\Omega \left| \frac{\partial^\beta w(x)}{w(x_i)} \right| \leq \frac{M}{c} \frac{2^\gamma}{h^\gamma}.$$

Therefore,

$$\max_{x \in \operatorname{supp} \mathcal{B}(i;\cdot)} \left| \frac{\partial^\beta w(x)}{w(x_i)} \right| \leq \max\left\{ \frac{C(1+2\sqrt{m})^{\gamma-|\beta|}}{c}, \frac{M}{c} \right\} \left(\frac{2}{h}\right)^{\min\{|\beta|,\gamma\}}.$$

On the other hand we recall from Lemma 3.2 and Lemma 3.7 that for any $k \in \mathcal{K}$, $\|b(k;\cdot)\|_{L^2}^2 = h^m \|\hat{b}(n;\cdot)\|_{L^2}^2$, and there are only $\preceq 1$ non zero coefficients $e_{i,j}$. Then

$$\left\| \partial^{\alpha-\beta} \left(b(i;\cdot) + \sum_{j \in \mathcal{J}(i)} e_{i,j} b(j;\cdot) \right) \right\|_{L^2}^2 \leq C \frac{h^m}{h^{2|\alpha-\beta|}},$$

and follows that

$$\|\partial^\alpha \mathcal{B}(i;\cdot)\|_{L^2}^2 \leq \frac{h^m}{h^{2|\alpha|}}.$$

Now, by summing all derivatives of order $|\alpha| \leq \nu$, we obtain a bound for the norm $H^\nu(\Omega)$ the $\mathcal{B}(i;\cdot)$. □

4.3 Quasi-Interpolation and Error Estimates

With the aid of the dual functions Λ_i defined in (4.18) we can define the canonical projector

$$\mathcal{P}_h : L^2(\Omega) \to web\mathfrak{B}_h^n(\Omega), \quad \mathcal{P}_h u := \sum_{i \in \mathcal{I}} \langle \Lambda_i, u \rangle_{L^2(\Omega)} \mathcal{B}(i;\cdot). \tag{4.22}$$

Lemma 4.11 (Weighted polynomial precision). *For any polynomial p of coordinate degree $\leq n$*

$$\mathcal{P}_h(wp) = wp.$$

In particular, the spline space $web\mathfrak{B}_h^n(\Omega)$ contains all weighted polynomials of degree $\leq n$ on Ω.

Proof.

$$\begin{aligned} \mathcal{P}_h(wp) &= \sum_{i \in \mathcal{I}} \langle \Lambda_i, wp \rangle_{L^2(\Omega)} \mathcal{B}(i;\cdot) \\ &= \sum_{i \in \mathcal{I}} \langle \Lambda_i, wp \rangle_{L^2(\Omega)} b(i;x) + \sum_{j \in \mathcal{J}} \left(\sum_{i \in \mathcal{I}(j)} e_{i,j} \langle \Lambda_i, wp \rangle_{L^2(\Omega)} \right) b(j;x). \end{aligned}$$

On the other hand

$$w(x)p(x) = \sum_{k\in\mathcal{K}} a_k \mathcal{B}(k;x), \quad x \in \Omega,$$

then, from Lemma 4.6

$$\langle \Lambda_i, wp\rangle_{L^2(\Omega)} = \left\langle \Lambda_i, \sum_{k\in\mathcal{K}} a_k \mathcal{B}(k;\cdot)\right\rangle_{L^2(\Omega)} = a_k\ \delta_{i,k} = a_i.$$

Then

$$\mathcal{P}_h(wp) = wp. \quad \square$$

Lemma 4.12. *If w is a ℓ-regular weight function of order γ, then for any grid cell Q*

$$\|\mathcal{P}_h u\|_{H^\nu(Q\cap\Omega)} \leq const\ (n, w, \Omega) h^{-\nu} \|u\|_{L^2(Q')} \quad \nu \leq \min(\ell, n),$$

where Q' is the union of the supports of all web-splines which are nonzero on $Q \cap \Omega$.

Proof.

$$\|\mathcal{P}_h u\|_{H^\nu(Q\cap\Omega)} \leq \sum_{i\in\mathcal{I}(Q)} \left|\langle \Lambda_i, u\rangle_{L^2(\Omega)}\right| \ \|\mathcal{B}(i;\cdot)\|_{H^\nu(\text{supp } \mathcal{B}(i;\cdot))}.$$

From Lemma 4.6 we have that

$$|\langle \Lambda_i, u\rangle_{L^2(Q_i)}| \leq \|\Lambda_i\|_{L^2(Q_i)}\ \|u\|_{L^2(Q_i)} \leq c(n, w, \Omega) h^{-m/2} \|u\|_{L^2(Q_i)},$$

and also from the estimates in the Bernstein Inequality

$$\|\mathcal{B}(i;\cdot)\|_{H^\nu(\text{supp } \mathcal{B}(i;\cdot))} \leq c h^{m/2} h^{-\nu}.$$

Then

$$|\langle \Lambda_i, u\rangle_{L^2(\Omega)}|\ \|\mathcal{B}(i;\cdot)\|_{H^\nu(\text{supp } \mathcal{B}(i;\cdot))} \leq c h^{-\nu} \|u\|_{L^2(Q_i)}.$$

Define

$$Q' = \bigcup_{i\in\mathcal{I}(Q)} \text{supp } \mathcal{B}(i;\cdot).$$

From Lemma 3.10 we know that $\#(\mathcal{I}(Q))$ is $\preceq 1$, then summing over $\mathcal{I}(Q)$ we get the result. $\square$

The error of web-approximations depends on the regularity of the quotient

$$v := \frac{u}{w}.$$

Lemma 4.13. *If w is a weight function of order $\gamma = 1$ and $u = wv$, then for any subdomain $\Omega_1 \subset \Omega$ with distance δ to the boundary,*

$$\|v\|_{H^\ell(\Omega_1)} \leq const\ (w, \ell)\ \delta^{-1}\Big(\|u\|_{H^\ell(\Omega_1)} + \|v\|_{H^{\ell-1}(\Omega_1)}\Big).$$

Proof. Since $u = vw$, from Leibniz's rule we have

$$\partial^\alpha u(x) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} \partial^\beta v(x) \partial^{\alpha-\beta} w(x)$$

then for $|\alpha| = s$

$$\partial^\alpha v(x) = \frac{1}{w(x)}\Big(\partial^\alpha u(x) - \sum_{\beta < \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} \partial^\beta v(x) \partial^{\alpha-\beta} w(x)\Big).$$

Let $\Omega_1 \subset \Omega$ such that

$$d(x, \Gamma) = \delta \quad \forall x \in \Omega_1, \quad \text{where } \Gamma \subseteq \partial\Omega.$$

Since w is of order 1, there exists a constant c depending of w such that for all $x \in \Omega_1$, we have

$$\frac{1}{w(x)} \leq \frac{1}{c} \frac{1}{d(x,\Gamma)} \leq \frac{1}{c}\frac{1}{\delta}.$$

Then

$$\begin{aligned} |v|_{H^s(\Omega_1)} &\leq \text{const } (w, s)\ \delta^{-1}\Big(|u|_{H^s(\Omega_1)} + \|v\|_{H^{s-1}(\Omega_1)}\Big) \\ &\leq \text{const } (w, s)\ \delta^{-1}\Big(|u|_{H^s(\Omega_1)} + \|v\|_{H^{\ell-1}(\Omega_1)}\Big). \end{aligned}$$

Hence,

$$\|v\|_{H^\ell(\Omega_1)} = \sum_{s=0}^{\ell} |v|_{H^s(\Omega_1)} \leq \text{const } (w, \ell)\ \delta^{-1}\Big(\|u\|_{H^\ell(\Omega_1)} + \|v\|_{H^{\ell-1}(\Omega_1)}\Big).$$

□

Lemma 4.14 (Regularity of Univariate Quotients). *If $p(0) = 0$, and $q(t) := p(t)/t$, $t \in \mathbb{R}$, $p \in H^\ell((0,1))$ then*

$$\|q^{(\ell-1)}\|_{L^2([0,1])} \leq \frac{2}{2\ell - 1}\ \|p^{(\ell)}\|_{L^2([0,1])}.$$

Proof.

$$q(t) = \frac{1}{t}\int_0^t p'(z)dz = \int_0^1 p'(tz)dz.$$

By differentiating $\ell - 1$ times we have that

$$q^{(\ell-1)}(t) = \frac{d^{\ell-1}}{dt^{\ell-1}} q(t) = \int_0^1 p^{(\ell)}(tz) z^{\ell-1} dz.$$

Then, from Minkowsky's inequality

$$\begin{aligned} \|q^{(\ell-1)}\|_{L^2([0,1])} &\le \int_0^1 \|p^{(\ell)}(tz) z^{\ell-1}\|_{L^2([0,1],dt)} dz \\ &= \int_0^1 z^{\ell-1} \|p^{(\ell)}(tz)\|_{L^2([0,1],dt)} dz. \end{aligned}$$

Note that

$$\begin{aligned} \|p^{(\ell)}(tz)\|^2_{L^2([0,1],dt)} &= \int_0^1 |p^{(\ell)}(tz)|^2 dt = \frac{1}{z} \int_0^z |p^{(\ell)}(u)|^2 du \\ &\le \frac{1}{z} \int_0^1 |p^{(\ell)}(u)|^2 du = \frac{1}{z} \|p^{(\ell)}\|^2_{L^2([0,1])}. \end{aligned}$$

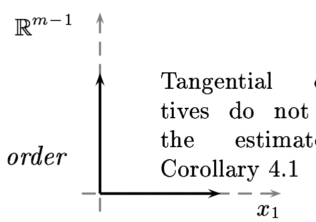
Then

$$\|q^{(\ell-1)}\|_{L^2([0,1])} \le \|p^{(\ell)}\|_{L^2([0,1])} \int_0^1 z^{\ell-3/2} dz = \frac{2}{2\ell - 1} \|p^{(\ell)}\|_{L^2([0,1])}.$$

□

Corollary 4.1. *If p is a function in $\mathbb{R}^m$, $p(0, ..., 0) = 0$ and $q(x) := p(x)/x_1$ we have*

$$\|q\|_{H^{\ell-1}([0,1]^m)} = \sum_{|\alpha| \le \ell-1} \|\partial^\alpha q\|_{L^2([0,1]^m)} \le const(m, \ell)\ \|p\|_{H^\ell([0,1]^m)}$$



Tangential derivatives do not affect the estimate in Corollary 4.1

Lemma 4.15 (Regularity of Quotients). *If w is a weight function of order $\gamma = 1$ and $u = wv$, we have*

$$\|v\|_{H^{\ell-1}(\Omega)} \le const\ (w, \ell, \Omega)\ \|u\|_{H^\ell(\Omega)}. \tag{4.23}$$

Proof. Let $\{f_\lambda\}_{\lambda \in L}$ be a smooth partition of unity. Since $u = \sum_\lambda f_\lambda u$, it is enough to show (4.23) for each of the functions $f_\lambda u$. To see this observe that if (4.23) holds for each one of these functions, i.e., if

$$\|f_\lambda v\|_{H^{\ell-1}(\Omega)} \le \text{const}\ (w, \ell, \Omega)\ \|f_\lambda u\|_{H^\ell(\Omega)} \quad \forall \lambda \in L,$$

we have

$$\begin{aligned} \|v\|_{H^{\ell-1}(\Omega)} &\le \sum_\lambda \|f_\lambda v\|_{H^{\ell-1}(\Omega)} \preceq \sum_\lambda \|f_\lambda u\|_{H^\ell(\Omega)} \\ &\preceq \sum_\lambda \|u\|_{H^\ell(\Omega)} \preceq \|u\|_{H^\ell(\Omega)}. \end{aligned}$$

Consider $u_\lambda = f_\lambda u$. Observe that supp $u_\lambda \subseteq$ supp f_λ.

If $d(\text{supp } u_\lambda, \partial\Omega) = \delta > 0$ then w^{-1} and its derivative are bounded, then we can easily bound derivative of $v = u_\lambda / w$.

If $d(\text{supp } u_\lambda, \partial\Omega) = 0$ we have to work more. Choosing appropriated coordinates we way assume that the support of u_λ is contained in $[0,1]^{m-1} \times [-\epsilon, 1/2]$ and also is contained in

$$\Omega_\lambda = \{x = (\zeta, t) : 0 \leq \zeta_i \leq 1 \quad \psi(\zeta) \leq t \leq 1 + \psi(\zeta), \} \subset \Omega,$$

with $\psi(0) = 0$, $\nabla\psi(0) = 0$, $||\nabla\psi(\zeta)|| \leq \epsilon/m$ and ϵ small. See Figure 4.2.

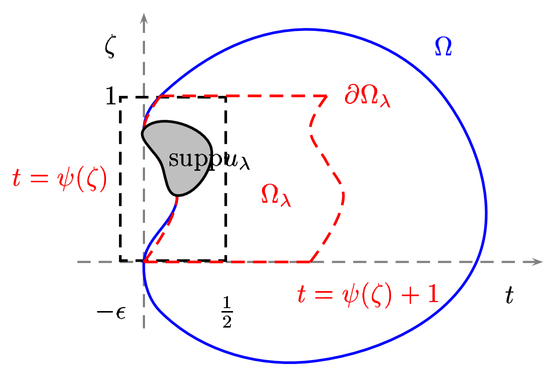


Figure 4.2: Support of Ω_λ

If $\psi \equiv 0$ and $w(x) = t$ we can use the Corollary 4.1.

The general case of $\psi \not\equiv 0$ can be reduced to the case $\psi \equiv 0$ by a change of variables.

We map Ω_λ to the cube $Q = [0,1]^m$ using the transformation $T : \mathbb{R}^m \to \mathbb{R}^m$ given for

$$T(x) = T(\zeta, t) = T(\zeta, s) = y \quad \text{where } s = t - \psi(\zeta). \tag{4.24}$$

See Figure 4.3.

Observe that $T(\Omega_\lambda) = [0,1]^m$ and

$$\begin{aligned} v_\lambda \circ T^{-1}(y) = v_\lambda(x) = \frac{u(\zeta,t)}{w(\zeta,t)} &= \frac{u(\zeta, s+\psi(\zeta))}{w(\zeta, s+\psi(\zeta))} \\ &= \frac{s}{w(\zeta, s+\psi(\zeta))} \frac{u(\zeta, s+\psi(\zeta))}{s} \\ &= \frac{s}{w(\zeta, s+\psi(\zeta))} \frac{u_\lambda \circ T^{-1}(y)}{s} \end{aligned}$$

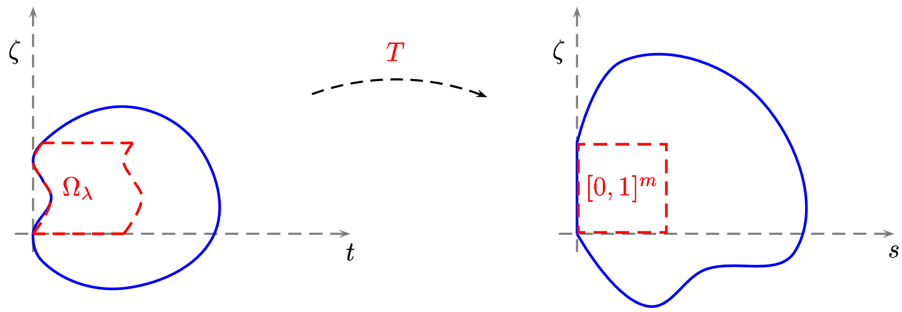


Figure 4.3: The transformation T defined in (4.24) maps Ω_λ to $[0,1]^m$.

where the coefficient

$$\frac{s}{w(\zeta, s+\psi(\zeta))} \preceq 1, \qquad \text{since } w(\cdot) \asymp d(\cdot, \partial\Omega).$$

We have for $v_\lambda = u_\lambda / w$ that

$$\begin{aligned}
\|v_\lambda\|_{H^{\ell-1}(\Omega)} = \|v_\lambda\|_{H^{\ell-1}(\Omega_\lambda)} &\leq const(T)\ \|v_\lambda \circ T^{-1}\|_{H^{\ell-1}(T(\Omega_\lambda))} \\
&\leq const(T) \left\| \frac{u_\lambda \circ T^{-1}}{s} \right\|_{H^{\ell-1}([0,1]^m)} \\
&\leq const(T)\ const(\ell, m)\ \|u_\lambda \circ T^{-1}\|_{H^{\ell}([0,1]^m)} \\
&\leq const(T, \ell, m)\ \|u_\lambda\|_{H^\ell(\Omega_\lambda)}. \quad \square
\end{aligned}$$

Lemma 4.16. *Let $f \in H^\ell(\Omega)$, and w a weight function of order $\gamma = 1$. For any subdomain Ω' of Ω we have*

$$\|w\ f\|_{H^\ell(\Omega')} \preceq (\max_{\Omega'} w)\ \|f\|_{H^\ell(\Omega')} + \|f\|_{H^{\ell-1}(\Omega')}$$

Proof. By Leibniz's rule we have

$$\partial^\mu (w\ f) = \sum_{\beta \leq \mu} \binom{\mu}{\beta} \partial^\beta f\ \partial^{\mu-\beta} w = (\partial^\mu f)\ w + \sum_{\beta < \mu} \binom{\mu}{\beta} \partial^\beta f\ \partial^{\mu-\beta} w.$$

For $\Omega' \subseteq \Omega$

$$\partial^\mu (wf) \leq (\max_{\Omega'} w)\ \partial^\mu f + M \sum_{\beta<\mu} \partial^\beta f,$$

where M depends on the maximum of derivatives of w up order $|\beta|$ which are finite because w is order $\gamma = 1$. Then

$$|w\ f|_{H^p(\Omega')} = \sum_{|\mu|=p} \partial^\mu w\ f \preceq (\max_{\Omega'} w)\ |f|_{H^p(\Omega')} + \|f\|_{H^{p-1}(\Omega')},$$

and

$$\|w\, f\|_{H^\ell(\Omega')} = \sum_{p=0}^{\ell} |w\, f|_{H^p(\Omega')} \preceq (\max_{\Omega'} w)\ \|f\|_{H^\ell(\Omega')} + \|f\|_{H^{\ell-1}(\Omega')}.$$

□

Lemma 4.17 (Jackson Inequality). *If w is a weight function of order $\gamma = 1$, for any function $u \in H^k(\Omega)$ which vanish on $\partial\Omega$, we have*

$$\|u - \mathcal{P}_h u\|_{H^\ell(\Omega)} \leq const\ (n, w, \Omega)\ h^{k-\ell}\ \|u\|_{H^k(\Omega)}, \quad \ell < k \leq n+1.$$

Proof. Define

$$Q' = \bigcup_{i \in \mathcal{I}(Q)} \text{supp}\ \mathcal{B}(i; \cdot) \subseteq \Omega.$$

Let $\pi_n v$ be a polynomial of degree n such that [1]

$$\|v - \pi_n v\|_{H^s(Q')} \preceq h^{\mu - s}\ \|v\|_{H^\mu(Q')} \quad s \leq \mu \leq n+1. \tag{4.25}$$

Recall that if wp is a weighted polynomial then $\mathcal{P}_h(wp) = wp$, see Lemma 4.11. We have

$$\begin{aligned}
\|u - \mathcal{P}_h u\|_{H^\nu(Q\cap\Omega)} &= \|wv - \mathcal{P}_h wv\|_{H^\nu(Q\cap\Omega)} \\
&\leq \|wv - w\pi_n v\|_{H^\nu(Q\cap\Omega)} + \|w\pi_n v - \mathcal{P}_h wv\|_{H^\nu(Q\cap\Omega)} \\
&\leq \|wv - w\pi_n v\|_{H^\nu(Q\cap\Omega)} + \|\mathcal{P}_h(wv - w\pi_n v)\|_{H^\nu(Q\cap\Omega)} \\
&\preceq \|wv - w\pi_n v\|_{H^\nu(Q\cap\Omega)} + h^{-\nu}\|w(v - \pi_n v)\|_{L^2(Q')}.
\end{aligned} \tag{4.26}$$

From Lemma 4.16 we have

$$\begin{aligned}
\|w(v - \pi_n v)\|_{H^\nu(Q\cap\Omega)} \preceq &\left(\max_{Q\cap\Omega} w\right)\|v - \pi_n v\|_{H^\nu(Q\cap\Omega)} \\
&+ \|v - \pi_n v\|_{H^{\nu-1}(Q\cap\Omega)}
\end{aligned} \tag{4.27}$$

and

$$\|w(v - \pi_n v)\|_{L^2(Q')} \preceq \left(\max_{Q'} w\right)\|v - \pi_n v\|_{L^2(Q')}.$$

From the equation (4.26) we have

$$\begin{aligned}
\|u - \mathcal{P}_h u\|_{H^\nu(Q\cap\Omega)} \preceq &\left[\left(\max_{Q\cap\Omega} w\right)\|v - \pi_n v\|_{H^\nu(Q\cap\Omega)} + \|v - \pi_n v\|_{H^{\nu-1}(Q\cap\Omega)}\right] \\
&+ h^{-\nu}\left(\max_{Q'} w\right)\|v - \pi_n v\|_{L^2(Q')} \\
\preceq &\left(\max_{Q'} w\right)\left[\|v - \pi_n v\|_{H^\nu(Q\cap\Omega)} + h^{-\nu}\|v - \pi_n v\|_{L^2(Q')}\right] \\
&+ \|v - \pi_n v\|_{H^{\nu-1}(Q\cap\Omega)}.
\end{aligned} \tag{4.28}$$

Let $\delta = dist(Q', \partial\Omega)$.

[1] We can use the orthogonal projection or other interpolations. See [1], [6].

1. $\delta \leq h$. Since $\operatorname{diam}(Q') \preceq h$ and $w \asymp d(\cdot, \partial\Omega)$, we have that

$$\max_{Q'} w \preceq h.$$

From (4.25)

$$\begin{aligned} \|v - \pi_n v\|_{H^\nu(Q')} &\preceq h^{k-1-\nu} \|v\|_{H^{k-1}(Q')} && (\mu = k-1, s = \nu) \\ \|v - \pi_n v\|_{L^2(Q')} &\preceq h^{k-1} \|v\|_{H^{k-1}(Q')} && (\mu = k-1, s = 0) \\ \|v - \pi_n v\|_{H^{\nu-1}(Q')} &\preceq h^{k-1-\nu+1} \|v\|_{H^{k-1}(Q')} && (\mu = k-1, s = \nu - 1). \end{aligned}$$

Then from (4.28) we deduce that

$$\begin{aligned} \|u - \mathcal{P}_h u\|_{H^\nu(Q\cap\Omega)} \preceq & h\Big[h^{k-1-\nu} \|v\|_{H^{k-1}(Q')} + h^{-\nu} h^{k-1} \|v\|_{H^{k-1}(Q')}\Big] \\ & + h^{k-\nu} \|v\|_{H^{k-1}(Q')} \\ \preceq & \, h^{k-\nu} \, \|v\|_{H^{k-1}(Q')}. \end{aligned}$$

2. $h \leq \delta$. In this case, $Q' \subset \Omega$, and since $\operatorname{diam}(Q') \preceq h$ and $w \asymp d(\cdot, \partial\Omega)$, we have that

$$\max_{Q'} w \preceq \delta + h \preceq \delta, \quad \text{and} \quad \frac{h}{\delta} \preceq 1.$$

Then from (4.25) we have

$$\begin{aligned} \|v - \pi_n v\|_{H^\nu(Q')} &\preceq h^{k-\nu} \|v\|_{H^k(Q')} && (\mu = k, s = \nu) \\ \|v - \pi_n v\|_{L^2(Q')} &\preceq h^{k} \|v\|_{H^k(Q')} && (\mu = k, s = 0) \\ \|v - \pi_n v\|_{H^{\nu-1}(Q')} &\preceq h^{k-\nu+1} \|v\|_{H^k(Q')} && (\mu = k, s = \nu - 1). \end{aligned}$$

Then from (4.28) we deduce that

$$\begin{aligned} \|u - \mathcal{P}_h u\|_{H^\nu(Q\cap\Omega)} &\preceq \delta\Big[h^{k-\nu} \|v\|_{H^k(Q')} + h^k \|v\|_{H^k(Q')}\Big] + h^{k-\nu+1} \|v\|_{H^k(Q')} \\ &\preceq \delta\Big[h^{k-\nu} \|v\|_{H^k(Q')} + h^{k-\nu} \|v\|_{H^k(Q')}\Big] + h^{k-\nu+1} \|v\|_{H^k(Q')} \\ &\preceq \delta \, h^{k-\nu} \|v\|_{H^k(Q')} + h^{k-\nu+1} \|v\|_{H^k(Q')} \\ &\preceq \delta \, h^{k-\nu} \|v\|_{H^k(Q')} + h^{k-\nu} \delta \|v\|_{H^k(Q')}, \quad \text{because } h \leq \delta \\ &\preceq h^{k-\nu} \, \Big(\delta \, \|v\|_{H^k(Q')}\Big) \\ &\preceq h^{k-\nu} \, \delta \, \delta^{-1} \Big(\|u\|_{H^k(Q')} + \|v\|_{H^{k-1}(Q')}\Big) \quad \text{from Lemma 4.13} \\ &\preceq h^{k-\nu} \, \Big(\|u\|_{H^k(Q')} + \|v\|_{H^{k-1}(Q')}\Big). \end{aligned}$$

Summarizing we have that

$$\|u - \mathcal{P}_h u\|_{H^\nu(Q\cap\Omega)} \preceq h^{k-\nu} \left(\|v\|_{H^{k-1}(Q')} + \|u\|_{H^k(Q')} \right) \preceq h^{k-\nu} \|u\|_{H^k(Q')}$$

by Lemma 4.15. Summing on each $Q \cap \Omega$ we get the result. □

Using the Céa's lemma (4.9) or (4.10) we can prove

Lemma 4.18. *Let $u \in H^k(\Omega)$ be the solution of the Dirichlet problem (4.2) and u_h a finite element approximation obtained by solving the Galerkin system (4.7) or (4.8). Then*

$$\|u - u_h\|_{H^1(\Omega)} \preceq h^{k-1} \|u\|_{H^k(\Omega)}.$$

Chapter 5

Numerical Examples and Finals Comments

5.1 Numerical Examples

numericalexamples Consider the problem:

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega \\ \quad u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{5.1}$$

with $\Omega = [w > 0]$, where

$$w(x, y) = 0.09 - (x - 0.5)^2 - (y - 0.5)^2$$

and the right-hand side $f = 4$ so that $u(x, y) = w(x, y)$ solves the problem (5.1).

In figure 5.1 we show the exact solution and its B-spline approximation using $w\mathfrak{B}_h^n(\Omega)$ when $h = 1/64$.

In Figure 5.2 we show the approximation quotient u/w for $web\mathfrak{B}_h^n(\Omega)$.

The sparsity patter of the matrix A using $w\mathfrak{B}_h^n(\Omega)$ and G using $web\mathfrak{B}_h^n(\Omega)$ are showm in Figure 5.3.

The ratio of conditions numbers in this case is $\frac{\kappa(A)}{\kappa(G)} = 1.7\%$.

Observe that G is less sparse that A but we note that A is more ill conditioned that G.

We summarize the numerical experiments in Table 5.1.

5.2 Finals Comments

We proved that the span of weighted B-splines and the span of weighted extended B-splines satisfy the requirements of a finite element space to solve nu-

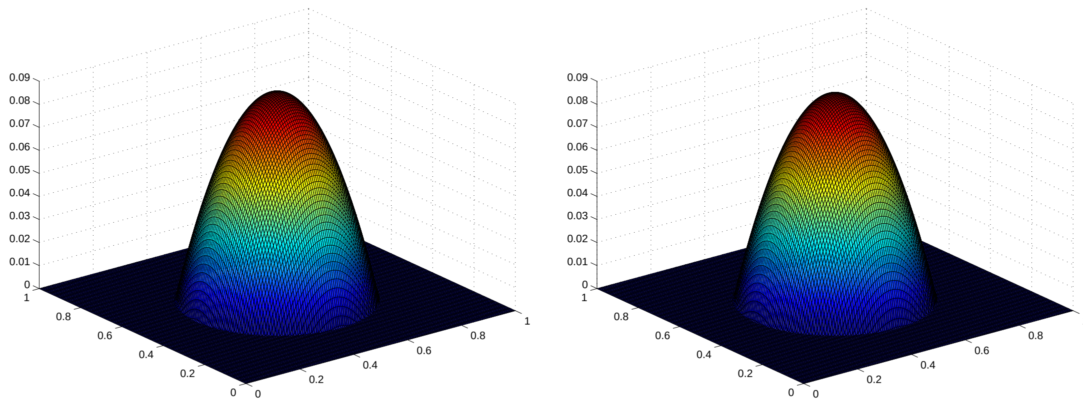


Figure 5.1: Exact solution (left) of problem (5.1) and its B-spline approximation (right) using $w\mathfrak{B}_h^n(\Omega)$.

	h	1/64	1/128	1/256
	$\#\mathcal{I}$	1232	4792	18820
	$\#\mathcal{J}$	164	316	620
	$\# \cup_j \mathcal{I}(j)$	444	900	1812
$w\mathfrak{B}_h^n(\Omega)$	error max	0.0011	4.99×10^{-4}	2.5×10^{-4}
	error L^2	3.25×10^{-4}	1.69×10^{-4}	1.7×10^{-5}
$web\mathfrak{B}_h^n(\Omega)$	error max	0.0021	9.45×10^{-4}	4.8×10^{-4}
	$\dfrac{\kappa(A)}{\kappa(G)}$	1.7%	1.5%	1.7%

Table 5.1: Summary of numericall results

merically a elliptic problem with Dirichlet condition.

We recall that the advantages of using these spaces are

- Simple basis functions.
- No domain dependence grid is required.
- Accurate approximations are possible with relatively low dimensional subspaces.
- Smoothness of the Galerkin approximation can be chosen arbitrary.
- Approximation can be chosen arbitrary.

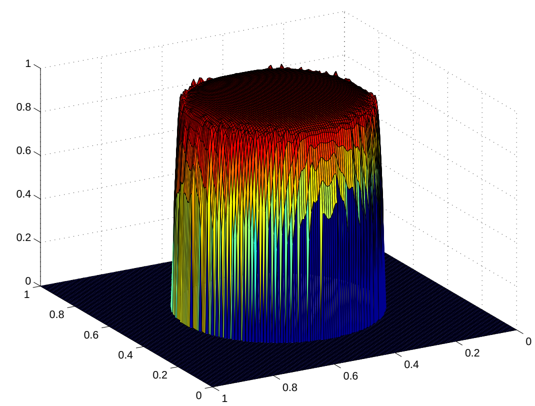


Figure 5.2: Approximation u/w using $web\mathfrak{B}_h^n(\Omega)$.

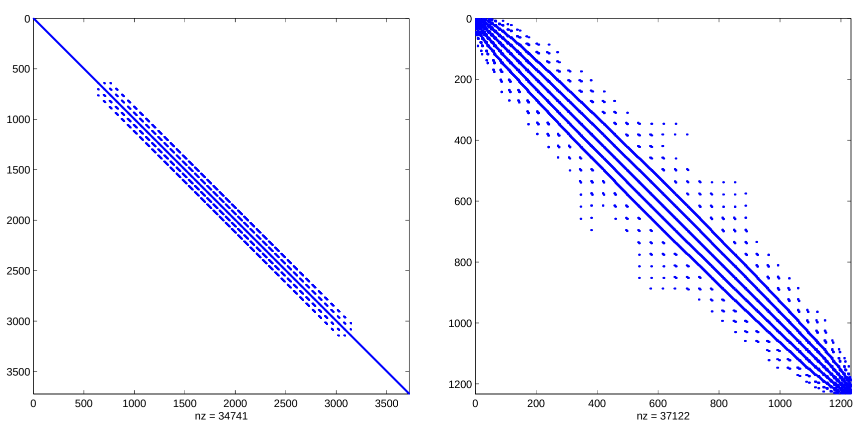


Figure 5.3: Using $w\mathfrak{B}_h^n(\Omega)$ (left) and $web\mathfrak{B}_h^n(\Omega)$ (right)

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