EFFICIENT NUMERICAL CALCULATION OF LARGE SHOCKS ARISING FROM SMALL DATA

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Abstract. In a recent paper, stable solutions were proved to arise from small amplitude data for an example of conservation laws with a parabolic term possessing the identity as viscosity matrix. The initial data analyzed therein lies outside the elliptic region, and the viscous solutions are related to a bifurcation of planar vector fields with nilpotent singularities studied by Dumortier, Roussarie and Sotomaior (DRS). For non–identity viscosity matrices, the DRS bifurcation is located at the border of the Majda–Pego instability region. The stability proof does not easily generalize for other viscosity matrices: in this work, this issue is studied using numerical simulation. We use a custom high–performance parallel Newton solver for the non–linear system arising from the Crank–Nicolson finite difference discretization. We present numerical evidence that the stable shocks do not arise from arbitrary small Riemann data with non–identity viscous matrices. However, some of the stable shocks arise from small data if one initial state lies in the elliptic region. In some of these cases, the solution seems to be unaffected by small perturbations.

1. Introduction

A famous theorem by Lax [L] states that, under certain hypotheses, systems of n conservation laws with small data have Riemann solutions consisting of n small waves, rarefactions or shocks, separated by constant states.

Matos and Marchesin showed in [MM1] that, if the hypothesis of strictly hyperbolicity is violated, large amplitude Lax shocks arise from small data for an example of quadratic conservation laws; the second order terms in the flux correspond to type IV in Schaeffer and Shearer's classification [SS]. In [MM2], the same model was studied for shocks that are the limits of travelling waves, using the identity as viscosity matrix. It was proved therein that the Riemann solution consists of two shocks with

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$O(1)$ amplitude no matter how small the datum is, provided it is close to a special point on the boundary of the elliptic region. In [MPa], it was established that this point persists under C^3 Whitney perturbations of the flux functions.

The large amplitude solutions in [MM2] are related to a non–classical kind of shock — known as over–compressive shock — that determines regions of wave compatibility. Over–compressive shocks are related to the bifurcation of one of the codimension-3 nilpotent singularities of planar ODEs studied by Dumortier, Roussarie and Sotomaior in [DRS]. In [AMPZ], Azevedo, Marchesin, Plohr and Zumbrum proved that the DRS bifurcation arises from the traveling wave ODE for systems of two conservation laws, and classified them with respect to the viscosity matrix and flux terms. The DRS bifurcation occurs at the border of an instability region studied by Majda and Pego in [MP]. The relationship between large amplitude solutions and the DRS bifurcation leads to the question: will the system studied in [MM2] present the same large amplitude solutions if the viscosity matrix is definite positive? The goal of this work is to answer this question.

The proof given in [MM2] cannot be easily extended to the case of non–identity viscosity matrices, so in this work this issue is studied using numerical simulation. For several viscosity matrices, we perform numerical simulations for Riemman problems (RPs for short) close to the point where the DRS bifurcation occurs. We use the program [IMP] to locate those over–compressive shocks that play a fundamental role in the large amplitude shocks (see [MM2]): knowing the location of the over–compressive shocks we can determine the states of the RP which lead to the large amplitude solutions. The small RPs we are interested in usually take very long to converge to the asymptotic solution. Therefore, we developed and employed a high–performance parallel Newton solver for the nonlinear system arising from the Crank–Nicolson finite difference discretization (see [H1] and [H2]).

In our simulations, we observed that large amplitude solutions are formed from small data. If one of the states of the initial data lies in the elliptic region, we observe in some cases that the viscous profiles remain unaltered, even after the introduction of small perturbations. However, shrinking the initial data necessarily causes one state to lie inside the Majda–Pego instability region, and the large amplitude profiles remain just for a finite length of simulation time. This is strong numerical evidence that large amplitude solutions for arbitrarily small data are unstable if non–identity viscosity matrix are used.

In Section 2, we review some results for systems of two conservation laws, including the recent results given in [MM2]. In Section 3, we present the quadratic model we in investigate in this work, which is a particular case of those discussed in the works reviewed in Section 2. Section 4 summarizes the experimental results we obtained, and the conclusions are given in Section 5.

2. Background

In this section we review some results for systems of two conservation laws in one space dimension that are fundamental to our work. We are concerned with systems of partial differential equations of the form:

$$U_t + F(U)_x = \epsilon(D\cdot U_x)_x, \tag{2.1}$$

where $U(x,t) = (u;v)^T \in \mathbb{R}^2$ for $x \in \mathbb{R}$ and $t \geq 0$, $F \in C^2(\mathbb{R}^2, \mathbb{R}^2)$, D is a definite positive constant (viscosity) matrix and ϵ is a (viscosity) scalar parameter.

Definition 2.1. The set of U in $\mathbb{R}^2$ is called:
i) *strictly hyperbolic region* if $DF(U)$ has two distinct real eigenvalues;
ii) *elliptic region* if $DF(U)$ has two distinct complex conjugate eigenvalues;
iii) *coincidence locus* if $DF(U)$ has one double real eigenvalue;

In the strictly hyperbolic region, we order the characteristic speeds of $DF(U)$ and denote the lower one as 1-speed $\lambda_1(U)$ and the higher one as 2-speed $\lambda_2(U)$. The corresponding right and left eigenvectors are $r_1(U)$, $r_2(U)$ and $l_1(U)$, $l_2(U)$, normalized such that $l_k(U)\cdot r_k(U) = 1$, for $k \in \{1, 2\}$.

Definition 2.2. The set of U in the strictly hyperbolic region where $\nabla\lambda_k \cdot r_k = 0$, for $k = 1$ or $k = 2$, i.e., where the genuine nonlinearity is lost, is called *k-inflection locus*. The union of the two sets is simply called *inflection* [S].

A Riemann problem (RP) is an initial value problem in one dimension with constant data on each side of the origin called U_L and U_R, that is

$$U(x,0) = U_L \text{ if } x < 0 \quad\text{and}\quad U(x,0) = U_R \text{ if } x > 0. \tag{2.2}$$

Following Gel'fand [G] and Courant–Friedrichs [CF], it is required that the shocks are limits of the traveling waves $U(x,t) = \bar{U}(\eta)$, $\eta = (x - st)/\epsilon$ of the equation (2.1) with $\lim_{\eta\to\pm\infty} \bar{U}(\eta) = U_\pm$ when $\epsilon \to 0$, i.e., we impose that the associated ordinary differential equation

$$\dot{U} = D^{-1}\big(F(U) - F(U_-) - s(U - U_-)\big) \tag{2.3}$$

has an orbit "connecting" the equilibria U_- and U_+: the existence of such an orbit implies that the famous Rankine-Hugoniot relation

$$F(U_+) - F(U_-) - s(U_+ - U_-) = 0 \tag{2.4}$$

is satisfied.

Following [MP], for 2×2 systems with definite positive viscosity matrix D, we denote the (k-)Majda–Pego instability region by k-MPIR and its border by k-BMPIR. We now present a synthetic version of the Majda–Pego theorem [MP, Theo. 3.1], for which we first introduce some definitions.

Definition 2.3. Let U be in the strictly hyperbolic region. The state U is in k-MPIR if and only if $l_k(U)Dr_k(U) < 0$. We define the Majda–Pego instability region MPIR as 1-MPIR $\cup$ 2-MPIR.

Definition 2.4. Let U be in the strictly hyperbolic region. If $l_k(U)Dr_k(U) = 0$ we say that U is on the border $k - \text{BMPIR}$. We define the border of the Majda–Pego instability region BMPIR as $1 - \text{BMPIR} \cup 2 - \text{BMPIR}$.

Remark 2.5. In the strictly hyperbolic region, we have $l_1(U)Dr_1(U) > 0$ or $l_2(U)Dr_2(U) > 0$. So, if U lies in the MPIR, then either $l_1(U)Dr_1(U) < 0$ or $l_2(U)Dr_2(U) < 0$.

Theorem 2.6. *Let U_L be in the k-MPIR, $k \in \{1, 2\}$, outside the inflection locus, and U_R be in a neighborhood of U_L such that (U_L, U_R) form a Lax k-shock. Assuming that D satisfies the non–degeneracy condition*

$$-\xi^2 D + i\xi(DF(U_L) - \lambda(U_L)) \text{ is non singular for all } \xi \neq 0, \tag{2.5}$$

then there exists no trajectory "connecting" U_L to U_R, that is, the weak Lax k-shock (U_L, U_R) does not possess a viscous profile.

This theorem implies that, if the existence of a viscous profile for shocks is imposed, then the classical theorem of Lax is not valid when U_L lies in the MPIR. That is, near U_L the solution of the RP does not contain the classical sequence: weak 1-shock, weak 2-shock. This happens because the stability of the equilibria of the ODE (2.3) is not given by $\lambda_k(U_\pm) - s$, where s is the shock speed, as would naturally arise from the identity viscosity matrix and the theorem of Lax.

The codimension-3 bifurcation of planar vector fields with nilpotent singularities studied in [DRS] by Dumortier, Roussarie and Sotomaior is related to the the viscous solutions studied in [MM2]. In [AMPZ], it was proved that the DRS bifurcations arise in systems with quadratic flux function: these bifurcations were classified depending on the flux and viscosity matrix. The location of the nilpotent singularity lies in the following intersection:

Definition 2.7. The intersection of the k-BMPIR and the k-inflection, for some k, is called the *DRS point*.

In [AMPZ], it was also proved that for the flux function of Type IV (in the classification given in [SS]), the over–compressive shocks arise near the DRS point; in [MM2], it was proved that large amplitude solutions arise for RPs near the DRS point. The authors were concerned with solutions of (2.1) and (2.2) that are sequences of two shocks. That is, in the limit as $\epsilon \to 0$, the solution becomes:

$$U_L \text{ if } x < s_1 t, \quad U_M \text{ if } s_1 t < x < s_2 t \quad \text{and} \quad U_R \text{ if } s_2 t < x. \tag{2.6}$$

Remark 2.8. Other kinds of large amplitude solutions may arise for RPs near the DRS point, for instance, containing rarefactions.

The large amplitude solutions are related to the over–compressive shocks, which are limits of wave compatibility. The structure of the solution depends on which side of the over–compressive shocks the initial data lie. If the right state lies on one side of the over–compressive shocks, the RP has a small amplitude solution; if the right state lies on the other side, the RP has a large amplitude solution (see Fig. 1); for further explanation see [MM2].

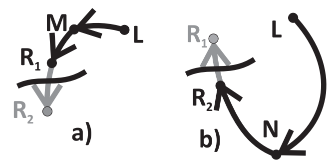

FIGURE 1. (a) The solution (L, M, R_1) is compatible and the solution (L, M, R_2) is incompatible; (b) The solution (L, N, R_1) is incompatible and the solution (L, N, R_2) is compatible.

3. THE SYSTEM OF QUADRATIC CONSERVATION LAWS

We study a system (2.1) of type IV [SS] with flux function

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3u^2 + v^2 \\ 2uv \end{pmatrix} + \begin{pmatrix} cu + 2v \\ cv \end{pmatrix}. \tag{3.1}$$

The linear factor on the right hand side of Equation (3.1) could be multiplied by any positive number, and the results would be similar except for scaling. The parameter c produces a translation in the space coordinate, i.e., all waves will move with an additional speed c: in each simulation, we set c in order to trap the waves near the origin. The quadratic term on the right hand side arises from setting $a = 3$ and $b = 0$ in the normal form given in [SS]. We expect other type IV models with parameters close to these to lead to similar results. The eigenvalues of DF are $\lambda_1 = 2u - \sqrt{u^2 + (v+1)^2 - 1}$ and $\lambda_2 = 2u + \sqrt{u^2 + (v+1)^2 - 1}$: notice that $\lambda_1 = \lambda_2$ along the circle $u^2 + (v+1)^2 = 1$, the coincidence locus. The interior of this circle is the elliptic region in this model. The curve $12u^2 + 9u^2v + 25v + 30v^2 + 9v^3 = 0$ is the inflection locus; if $u > 0$ a maximum of λ_1 is reached, and if $u < 0$ a minimum of λ_2 is reached.

We focus our attention on the simplest large amplitude solutions — 1-shock, 2-shock. We locate the over–compressive shocks using [IMP]: we chose the left and right states of the RP, U_L and U_R, in such way that they are close to the DRS point and the compatible solution has large amplitude. In the numerical solution we observe that, for an over–compressive shock to exist, either U_L or U_R lies outside of the MPIR. We then use [H2] to integrate numerically the RP. The results are shown in Section 4.

4. Simulations

We now present the results of our simulations for the viscosity matrices

$$\text{(a) } D_1 = \begin{bmatrix} 0.6 & 0.2 \\ 0.2 & 0.8 \end{bmatrix} \quad \text{and} \quad \text{(b) } D_2 = \begin{bmatrix} 1.1 & 0.05 \\ -0.1 & 0.85 \end{bmatrix}.$$

We observed similar results with various other non–identity viscosity matrices, and for brevity we present only these two, which suffice for illustrative purposes.

In all simulations, we set the viscosity parameter to $\epsilon = 10$, the CFL number to zero, and the density of space grid to 2, i.e., grid spacing was 0.5.

4.1. Simulations with $D = D_1$.

For the viscosity matrix D_1, straightforward calculations show that the BMPIR is the ellipse

$$u^2 - \frac{uv}{12} - \frac{2u}{3} + \frac{6337v^2}{576} + \frac{793v}{36} + \frac{10}{9} = 0.$$

The DRS point lies approximately at $(0.617; -0.201)$, on the intersection of the 1-BMPIR and the 1-inflection. The point $(-0.919; -0.523)$, which lies in the intersection between the 1-BMPIR and the 2-inflection, is not a DRS point. The non–degeneracy condition (2.5) is satisfied except for a segment on $v = (7u - 2)/2$; therefore Theorem 2.6 holds near the DRS point.

We simulate the small RP near the DRS listed in the Table 4.1.

4.1.1. *Riemann problem 1a.*

The state $U_L = (0.638; -0.183)$ lies inside the strictly hyperbolic region but outside of the MPIR, and the state $U_R = (0.407; -0.109)$ lies in the elliptic region. We set between $c = -1$

Problem	u_L	v_L	u_R	v_R
1a	0.638	-0.183	0.407	-0.109
1b	0.729	-0.257	0.494	-0.167
1c	0.612	-0.187	0.546	-0.161

TABLE 1. The RPs simulated for the viscosity matrix D_1

and $c = -0.973$ (see Equation 2.1), so the waves sometimes are moving left and sometimes are moving right.

The simulation results for Problem 1a are shown in Figure 2. At simulation time $t = 1.5e5$, the shocks are not completely formed; $t = 1.5e5$ is an extremely long time in comparison to the typical convergence time, which is of the order of $t = 3e2$. At $t = 5.0e5$, the shocks are clearly formed and remain stable until $t = 3.0e6$. We conclude that this is probably the asymptotic solution.

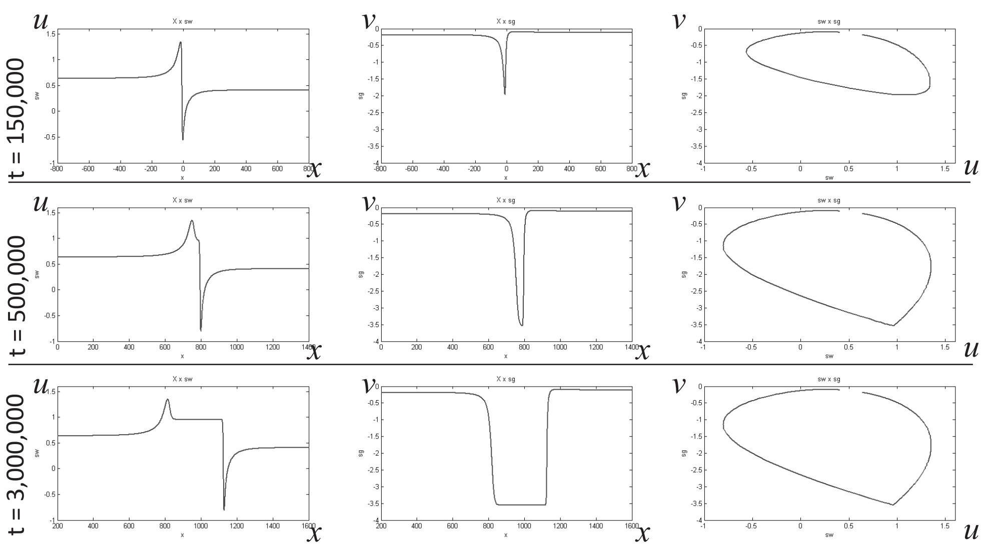

FIGURE 2. The evolution of Problem 1a.

In order to test the stability of the solutions, we introduce at $t = 1.5e6$ a small perturbation on the constant state U_R, which lies in the elliptic region: the results of the simulation with U_R perturbed are shown in Figure 3, where we observe an intriguing phenomenon for which we have no explanation: (i) at first, the small perturbation increases ($t = 1.0e3$); (ii) next, the perturbation and the wave solution strongly interact ($t = 3.4e$); (iii) finally, the solution seems to "swallow" the perturbation; the solution progresses as if no perturbation ever existed ($t = 5.0e3$).

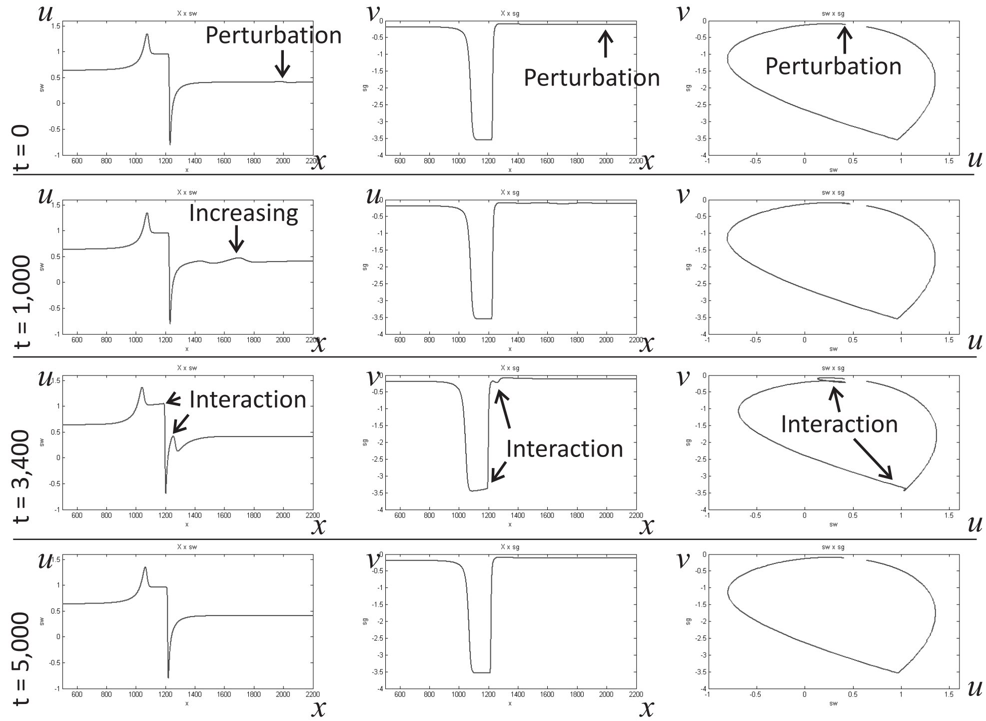

FIGURE 3. The evolution of Problem 1a with perturbation.

4.1.2. *Riemann problem 1b.* The state $U_L = (0.729; -0.257)$ lies in the strictly hyperbolic region but outside of the MPIR, and the state $U_R = (0.494; -0.167)$ lies inside the elliptic region. We set $c = -1.146$ (see Equation 2.1).

The simulation results for Problem 1b are shown in Figure 4. At simulation time $t = 5.0e4$, the shocks are not yet completely formed; at $t = 1.0e5$, the shocks are clearly formed and remain stable until $t = 1.5e6$. We conclude that this is probably the asymptotic solution.

4.1.3. *Riemann problem 1c.* The state $U_L = (0.612; -0.187)$ lies in the strictly hyperbolic region but outside of the MPIR, and the state $U_R = (0.546; -0.161)$ lies inside the 1-MPIR. We set $c = -1.03$ (see Equation 2.1).

The simulation results for Problem 1c are shown in Figure 5. At simulation time $t = 9.0e4$, the shocks are beginning to be formed; suddenly, the shocks increase them size and move faster than expected; oscillations and a strange non constant intermediate state also arise (see $t = 1.3e5$). This behavior is conserved till $t = 5.0e5$.

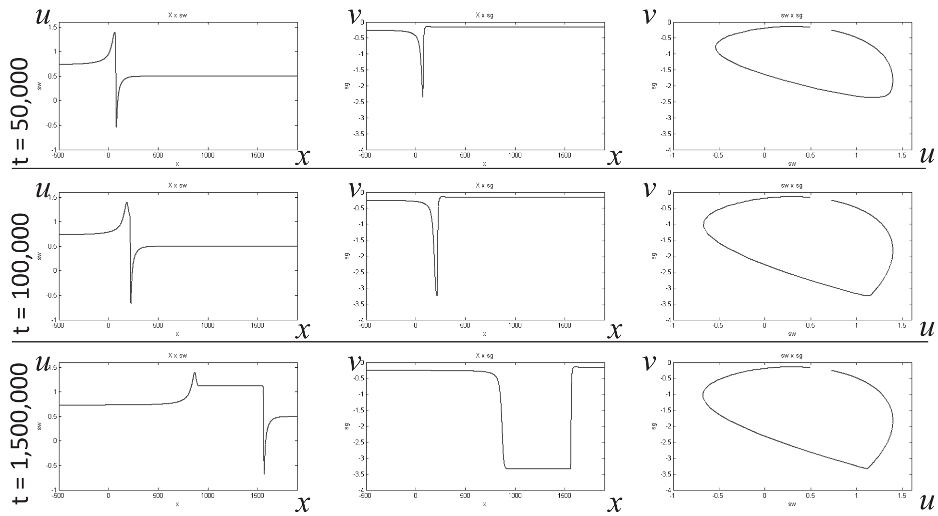

FIGURE 4. The evolution Problem 1b.

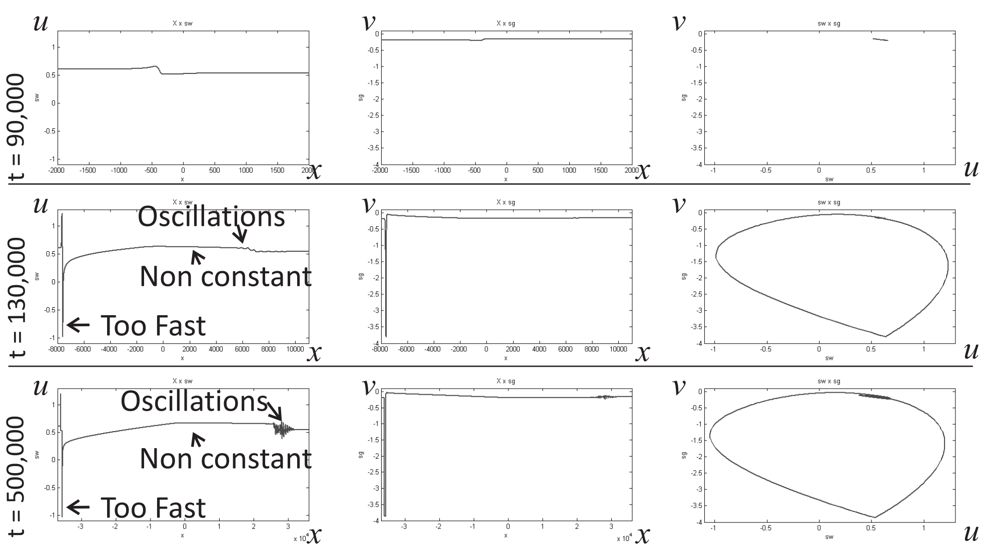

FIGURE 5. The evolution of Problem 1c.

4.2. **Simulations with $D = D_2$.** For the viscosity matrix D_2, straightforward calculations show that the BMPIR is the ellipse

$$u^2 + \frac{5uv}{748} + \frac{190v^2}{187} + \frac{5u}{187} + \frac{1517v}{748} - \frac{2}{187} = 0.$$

the DRS point lies approximately at $(-0.771; -0.334)$ on the intersection of the 2-BMPIR and the 2-inflection; the point $(0.304; -0.045)$ lies in the intersection between the 2-BMPIR and the 1-inflection, and is not a DRS point. The non–degeneracy condition (2.5) is satisfied except for a segment on $v = -39u - 4$.

We simulate the small RP near the DRS listed in Table 2.

Problem	u_L	v_L	u_R	v_R
2a	-0.608	-0.236	-1.086	-0.571
2b	-0.785	-0.373	-0.992	-0.534

TABLE 2. The RP simulated for the viscosity matrix D_2

4.2.1. *Riemann problem 2a.* The state $U_L = (-0.608; -0.236)$ lies in the elliptic region; the state $U_R = (-1.086; -0.571)$ lies in the strictly hyperbolic region but outside of the MPIR. We set $c = 1.422$ (see Equation 2.1).

The simulation results for Problem 2a are shown in Figure 6. At simulation time $t = 4.0e5$, the shocks are not yet completely formed; at $t = 8.0e5$, the shocks are clearly formed and remain stable until $t = 1.0e7$. We conclude that this is probably the asymptotic solution.

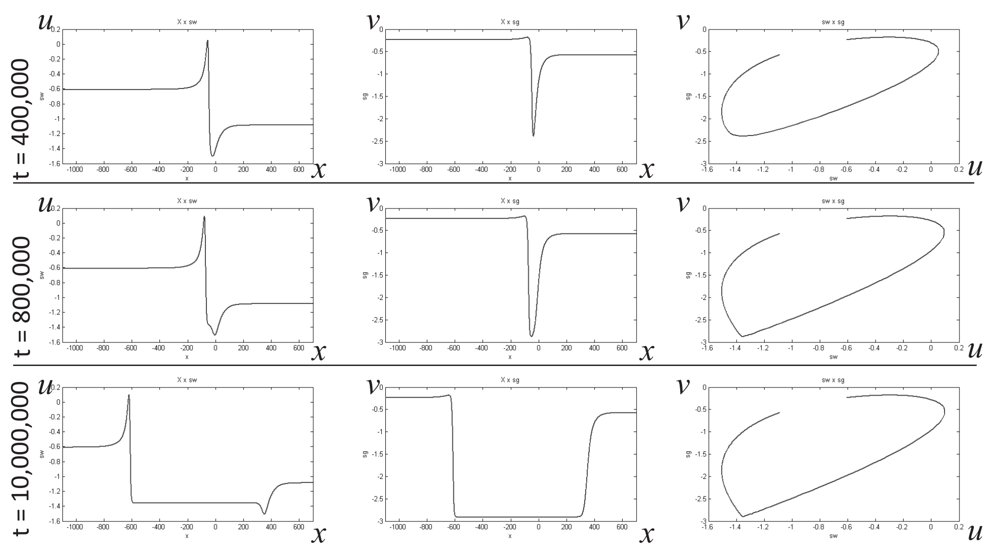

FIGURE 6. The evolution Problem 2a.

4.2.2. *Riemann problem 2b.* The state $U_L = (-0.785; -0.373)$ lies in the 2-MPIR; the state $U_R = (-0.992; -0.534)$ lies in the strictly hyperbolic region but outside of the MPIR. We set $c = 1.44$ (see Equation 2.1).

The simulation results for Problem 2b are shown in Figure 7. At simulation time $t = 5.0e4$, the shocks are not yet completely formed; at $t = 1.0e5$, the shocks are formed and a small instability appears. We observe oscillations up until $t = 1.5e5$.

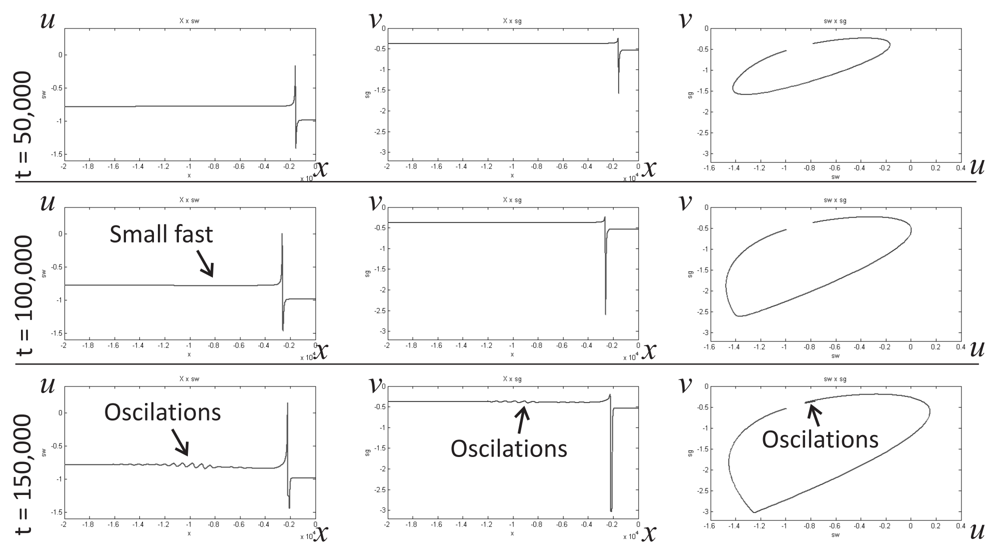

FIGURE 7. The evolution of Problem 2b.

5. CONCLUSIONS

Large amplitude solution for certain small initial Riemann problem data are observed numerically, even when one of the states lies in the elliptic region. These solutions appear to be stable, even though some instabilities that increase in time are "swallowed" by the large amplitude solutions after a while. The solutions are unstable if one of the initial states lies in the Majda-Pego instability region. Thus there are no arbitrarily small initial data giving rise to stable large solution. In some simulations, not presented in this work, long lasting solution with no oscillations are formed after the high oscillations appear. We do not believe that they are asymptotic solutions. This fact may be explored in future work.

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