

# A SIMPLE INVERSE SOLVER FOR THE FILTRATION FUNCTION IN FLOW OF WATER WITH PARTICLES IN POROUS MEDIA

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ABSTRACT. Models for deep bed filtration in the injection of sea water with solid inclusions depend on an empirical filtration function that represents the rate of particle retention in the porous rock. This function must be calculated indirectly from experimental measurements of other quantities. In this work, we determine the filtration function from the retained particle distribution history measured in laboratory tests by means of X-rays. The practical petroleum engineering purpose is to predict injectivity loss in the porous rock around wells.

The recovery procedure employs optimization to estimate filtration function parameters, as well as a fast solver for the direct problem, based on integrating ordinary differential equations. We utilize a simple model of Hertzog et al. because the implementations of the direct and inverse problem solvers are easier, without causing noticeable loss of accuracy, as compared to more detailed models.

In this report, the mathematical foundation of the method and its numerical implementation are discussed in detail. Good numerical results for synthetic data are shown.

*Inverse problem, Parameter estimation, Formation damage, Deep bed filtration, Convection-reaction equations*

## 1. INTRODUCTION

Most of the oil in the world is produced by injecting water in some wells and recovering oil in other wells. The recovered oil comes with reservoir water, which contains oil droplets and solid particles. The produced water must be separated from the oil and discarded taking environmental precautions. In off-shore fields, produced water and sea water are used in the injection. However, the injection of poor quality water in a well curtails its injectivity because of particle retention in the pores, or *deep bed filtration*. In this paper we study the deep bed filtration during injection of water containing solid particles.

Many theoretical and laboratorial studies were carried out to understand the filtration process (e.g. [7], [8]). We utilize the model for deep bed filtration presented in the fundamental work of Hertzog et al. [8], which consists of equations expressing the particle mass conservation and the particle retention process ([4], [8], [10]). They form a quasi-linear system of equations containing the empirical *filtration function*  $\lambda(\sigma)$ , which represents the kinetics of particle retention.

Methods for determining the filtration function from the effluent concentration history at the core outlet were presented in [13] and [15] for constant filtration  $\lambda$ . A recovery method for general  $\lambda(\sigma)$  was presented in [5] and [3], under the assumption that the injected particle concentration is constant. This hypothesis was relaxed in [1], [2], where the knowledge of the time-dependent injected concentration is utilized.

Accurate measurements of the concentration history are hard and tedious to perform, yet they provide data insufficient for recovering the filtration function except in the absence of cake formation. Even in this case, determining the filtration function is an inverse problem that is ill-posed in practice. It has been solved by means of a functional equation provided the

filtration function does not approach zero (see [1], [2]). In the latter case, it has been solved by parameter estimation by optimizing a least square functional with Tikhonov regularization, [1], [11]. At any rate, these methods are delicate or laborious.

In practice, one is often interested in situations where cake formation occurs. This cake consists of particles that do not succeed in penetrating the wall where fluid is injected. Then the effluent concentration ceases to be sufficient for the determination of the filtration function, since it is not possible to determine the amount of suspended particles that is retained in the cake. This poses a quandary, which has only recently been solved by direct measurements of the retained particle distribution history by means of X-rays [Mohammed] or CT-Scans[Firas], for particles consisting of hematite.

In this work we solve the inverse problem of determining  $\lambda(\sigma)$  solely from measurements taken at many different times of the distribution of retained particles along the core (see Fig. 1.1). Unlike methods based on the effluent concentration history, this method does not suffer from any limitations due to cake formation, and it is well posed. The method is validated in this work only for synthetic data.

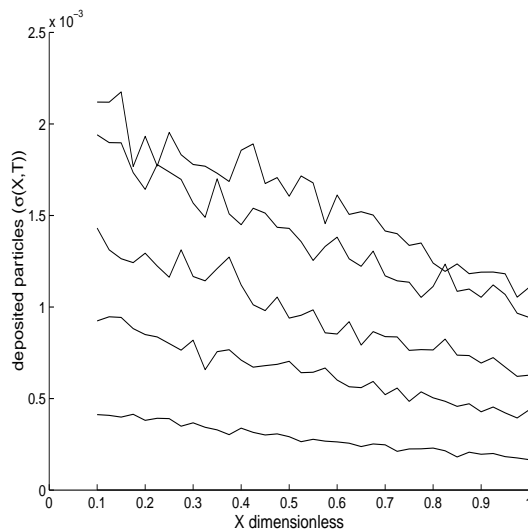


FIGURE 1.1. Typical “experimental” deposited particle distributions at 5 times.

The paper is organized as follows. In Section 2 we present the deep bed filtration model from [8] as a quasi-linear system of partial differential equations, which is reduced to solving ordinary differential equations (ODEs). In Section 3, the inverse problem is presented, transformed into an optimization procedure for parameter estimation, which minimizes a certain functional. In Section 4, the numerical results are shown. In Section 5, cake formation is discussed and in Section 6 a summary of our results is presented. In Appendix A, we discuss the mathematical solution of the PDE’s for our model and the reduction to ordinary differential equations, while in Appendix B we discuss the mathematical well-posedness of the inverse problem. In Appendix C we derive explicit formulae that simplify the evaluation of the functional to be minimized.

## 2. GLOBAL SOLUTION FOR THE DIRECT PROBLEM

We assume that the mass density of solid particles in suspended and entrapped states are both equal. We also assume that water is incompressible and that it is injected at constant

volumetric rate for simplicity. The mass conservation equation proposed by Herzig et al. [8] neglecting diffusive and suspended particle mass accumulation terms can be written as:

$$\phi \frac{\partial \sigma}{\partial t} + U \frac{\partial c}{\partial x} = 0, \quad (2.1)$$

where  $\phi$  is the rock porosity. Here  $c$  and  $\sigma$  are the (nondimensional) concentrations of the particles suspended in the water and entrapped in the rock pores, respectively, at position  $x$  and time  $t$ , while  $U$  is the fluid injection rate. The quantities  $c$  and  $\sigma$  have values between 0 and 1, but usually  $c \cong 10^{-4}$ , while  $\sigma$  grows up to  $10^{-2}$ ; the independent variables satisfy  $0 \leq x \leq L$  and  $t \geq 0$ . Here  $L$  denotes the length of the core. Thus  $\phi c$  and  $\phi \sigma$  are the concentration of suspended and retained particles per unit volume of core rock. X-ray measurements capture the sum of these quantities. However, because  $c$  is much smaller than  $\sigma$  most of the time, we often neglect  $c$  in such a sum. (See Remark 2.1.)

The model ([3], [8]) requires a law for particle deposition rate, usually written as:

$$\frac{\partial \sigma}{\partial t} = \lambda(\sigma) c U. \quad (2.2)$$

Equation (2.2) is heuristic, since the form of the right hand side cannot be determined from first principles; it means that the retention probability is proportional to the available concentration of suspended particles. This concentration is in turn proportional to  $c$  and to the flow rate  $U$ . Physically, (2.2) cannot be valid for large  $c$  or  $\sigma$ . In particular, Eq. (2.2) cannot take into account release of deposited particles. The non-negative  $\lambda(\sigma)$  is an empirical coefficient known as the *filtration function*, which cannot be measured directly. The inverse problem we solve in this work consists in determining the filtration function. (Notice that it has units of inverse length).

**Remark 2.1.** Except at initial times, it turns out that  $c$  is much smaller than  $\sigma$ . Because of this fact, Herzig et al. [8] proposed the simplified mass conservation equation (2.1), while the correct form of the mass conservation equation is the following [3], for negligible dispersive effects:

$$\phi \frac{\partial}{\partial t} (c + \sigma) + U \frac{\partial c}{\partial x} = 0. \quad (2.3)$$

The simplified equation (2.1) is appropriate for our purposes since usually there are no accurate data measurements at early times. Indeed, at such times both  $c$  and  $\sigma$  are so small that experimental error predominates in the measured data.

**2.1. Boundary and initial conditions.** We denote the solid particle concentration entering into the porous medium as follows:

$$x = 0 : \quad c = c_i(t) > 0, \quad t > 0. \quad (2.4)$$

In the absence of cake formation,  $c_i$  is the volumetric solid particle concentration in the injection fluid. As initial data at  $t = 0$ , we assume that the rock contains water with no particles:

$$t = 0 : \quad \sigma = 0 \quad \text{and} \quad c = 0 \quad \text{for} \quad 0 \leq x \leq L. \quad (2.5)$$

Now, we make the following change of variables:

$$X = \frac{x}{L}, \quad T = \frac{U}{\phi L} t. \quad (2.6)$$

The time  $T$  is in a non-dimensional unit (the so-called pore volume or PV). Using Eq. (2.6) the system (2.1)–(2.2) and the definition (2.4) can be rewritten in non-dimensional form as follows, in terms of  $c(X, T)$  and  $\sigma(X, T)$  with  $0 \leq X \leq 1$ , and  $T \geq 0$ :

$$\frac{\partial \sigma}{\partial T} + \frac{\partial c}{\partial X} = 0, \quad (2.7)$$

$$\frac{\partial \sigma}{\partial T} = \lambda(\sigma)c, \quad (2.8)$$

$$c(0, T) = c_i((\phi L/U)T), \quad (2.9)$$

where the non-dimensional  $\lambda(\sigma)$  in (2.8) is the dimensional  $\lambda(\sigma)$  in (2.2) multiplied by  $\phi L$ . Along the line  $X = 0$ , equation (2.8) reads:

$$\frac{d\sigma(0, T)}{dT} = \lambda(\sigma(0, T))c(0, T), \quad \text{and} \quad \sigma(0, 0) = 0. \quad (2.10)$$

Integrating Eq. (2.10) provides  $\sigma(0, T)$ , which is always positive and increasing.

**2.2. Solution of the direct deposition problem.** In Appendix A we prove the following:

**Theorem 2.2.** *Assume that  $\lambda(\sigma)$  is a known function, which is non-negative, piecewise differentiable with uniformly bounded derivative. Assume also that  $c_i(t)$  is known. Then the PDEs (2.7)–(2.8) with data (2.5),(2.9) has a unique solution, which can be obtained by solving the family of ordinary differential equations (2.10),(2.11a) and (2.9),(2.11b). The family is:*

$$\frac{\partial \sigma}{\partial X} = -\lambda(\sigma)\sigma \quad \text{and} \quad \frac{\partial c}{\partial X} = -\lambda(\sigma)c. \quad (2.11)$$

In equation (2.11a), the time  $T$  is fixed, so it should be regarded as an ordinary differential equation in  $X$ , which determines  $\sigma$  in terms of  $X$  provided the value of  $\sigma(0, T)$  is available. This value can be calculated by solving the ODE (2.10a) with data (2.10b).

Once  $\sigma(X, T)$  is known, we notice analogously that for each  $T$ , (2.11b) is an ODE that requires the value of  $c(0, T)$  to determine  $c(X, T)$ . The required value is given by (2.9).

**Remark 2.3.** In this work, we do not need to compute  $c(X, T)$ , and therefore we will not mention (2.11b) any more. However, in case one needs to compute it, the best practical way is to solve the two equations (2.11) simultaneously as a system of ODEs rather than sequentially.

**Remark 2.4.** If the effluent concentration  $c_e(t)$  is measured we do not need  $c_i(t)$  to determine the solution of the deposition problem. Along the line  $X = 1$  we determine  $c(1, T)$  and  $\sigma(1, T)$  from Eqs. (2.6), (2.8) by means of:

$$c(1, T) = c_e((\phi L/U)T), \quad (2.12)$$

$$\frac{d\sigma(1, T)}{dT} = \lambda(\sigma(1, T))c(1, T), \quad \text{and} \quad \sigma(1, 0) = 0. \quad (2.13)$$

As in Theorem 2.2, the unique solution of the PDEs (2.7)–(2.8) with data (2.5),  $c_e(t)$  is obtained by solving the ODEs (2.13),(2.11a) and (2.12),(2.11b).

**2.3. Calculation of the deposition distribution.** In this section we discuss the solutions of Eq. (2.11a). For  $\sigma, \psi$  between 0 and 1, let us introduce the primitive:

$$\chi(\sigma, \psi) = - \int_{\psi}^{\sigma} \frac{d\eta}{\eta\lambda(\eta)}. \quad (2.14)$$

Differentiation of  $\chi(\sigma(X, T), \psi)$  relative to  $X$  with fixed  $T$  shows that  $\chi(\sigma, \psi)$  is the unique solution of (2.11a) such that  $\sigma = \psi$  at  $X = 0$ . In other words, the solution of this ODE is

$$X = \chi(\sigma, \psi). \quad (2.15)$$

Fix  $\psi$ . Because  $\lambda(\sigma)$  is never negative, according to Eq. (2.11a), the relationship (2.15) defines  $\sigma$  as a function of  $X$ , which we denote as

$$\sigma = \sigma(X, \psi). \quad (2.16)$$

Notice that the fixed time  $T$  does not show up explicitly in this expression, because (2.15) only depends on time indirectly, i.e., through the value  $\psi = \sigma(0, T)$ .

### 3. THE RECOVERY METHOD

Here we describe the recovery method for obtaining the filtration function. First an expression for the filtration function is chosen depending on some coefficients or parameters. The values of these coefficients are recovered by using an optimization procedure to minimize a functional that represent the difference between the solution of the direct problem for the chosen filtration function and the available experimental data. To compensate for the fact that parameter estimation in the system of partial differential equations (2.11) is inherently unstable ([1],[6]), i.e., almost always the recovered parameters do not depend on the data in a stable way, we use Tikhonov's regularization, which yields stable approximations of ill-posed inverse problems ([14]). The well-posedness of the direct problem of calculating the deposition distribution history is discussed in Appendix A. The well-posedness of the inverse problem of recovering the filtration function is discussed in Appendix B.

We assume that the experimental deposition  $\sigma_{exp}$  is measured at  $m$  positive times  $T_j$ , i.e., for  $T = T_j, j = 1, \dots, m$ , at all  $n$  positive positions  $X_i$ , i.e., for  $X = X_i, i = 1, \dots, n$ . We also use the notation  $X_0 := 0, T_0 := 0, \Delta X_i = X_i - X_{i-1}$  and  $\Delta T_j = T_j - T_{j-1}$ .

**3.1. Recovering the filtration function.** In this section we formulate our method for solving the inverse problem of finding the filtration function from the retained particles history measured in laboratory experiments.

Before the recovery procedure, a parametrization  $\lambda(\sigma, \psi; \theta)$  must be chosen for the filtration function given in 2.16, where  $\theta = (\theta_1, \dots, \theta_p)$  is the set of parameters for the filtration function and  $p$  is a number of parameters. The form of these parametric functions and their parameter ranges are dictacted by physical properties of the filtration function. Then we minimize in  $\theta$  and  $\psi$  a functional expressing the deviation between the retained particle distribution predicted with the current filtration function parameter values and the retained particle distribution measured in the laboratory, as a discrete approximation of an integral in  $X$  and  $T$ :

$$F^\sigma(\psi_j, \theta; \alpha) = \sum_{j=1}^n \Delta T_j \left( \sum_{i=1}^m \Delta X_i (\sigma(X_i, \psi_j; \theta) - \sigma_{exp}(X_i, T_j))^2 \right) + \alpha^2 \sum_{k=1}^p (\theta_k - \theta_k^*)^2 \quad (3.1)$$

Here  $\sigma_{exp}(X_i, T_j)$  represents the retained particles measured in the laboratory and  $\sigma(X_i, T_j; \theta, \psi_j)$  is obtained by solving Eq. (2.11a) for fixed  $(\theta, \psi_j)$  and  $T_j$ , i.e.,

$$\sigma(X_i, \psi_j; \theta) = - \int_0^{X_i} \lambda(\sigma(X, \psi_j; \theta)) \sigma(X, \psi_j; \theta) dX + \psi_j. \quad (3.2)$$

Notice that we also introduce in  $\sigma$  a secondary set of parameters  $(\psi_1, \dots, \psi_m)$ , where  $\psi_j$  with  $j = 1, \dots, m$  represent the unknown values of  $\sigma$  at  $(0, T_j)$ . The second term on the RHS of (3.1) is Tikhonov's regularization term with regularization parameter  $\alpha$ , and  $(\theta_1^*, \dots, \theta_p^*)$  are reference values of the parameters  $\theta$ , specified by physical considerations.

**Remark 3.1.** Notice that  $\sigma(X_i, \psi_j; \theta)$  can be obtained solving the ODE (2.11a) with initial data  $\psi_j$ . To initiate the optimization procedure, first guesses of  $\psi_j$  can be obtained by simple extrapolation of the experimental data. For the example in Fig. 1.1 we prolong the data curves to the  $X = 0$  axis to find  $\psi_1, \dots, \psi_5$ .

**Remark 3.2.** For many forms of  $\lambda(\sigma)$ , explicit formulae can be obtained for the integral in (3.2), see Appendix C. In this case, it is easy to implement (3.2) analytically, and the methods in the next subsection are unnecessary.

**3.2. Numerical integration.** Examining (3.1), we see that for each  $T_j$  we need the values of  $\sigma$  for all  $i, i = 1, \dots, n$ . In the absence of an analytic formula as mentioned in Remark 3.2 we proceed as follows.

A practical way of calculating the values  $\sigma(X_i, \psi_j; \theta)$  integrates (3.2) as follows. We see that Eq. (3.2) yields for a fixed  $j$ :

$$\sigma(X_i, \psi_j; \theta) = - \int_{X_{i-1}}^{X_i} \lambda(\sigma(X, \psi_j; \theta)) \sigma(X, \psi_j; \theta) dX + \sigma(X_{i-1}, \psi_j; \theta), \quad (3.3)$$

for  $i = 1, \dots, n$ , and (for  $i = 0$ ):

$$\sigma(X_0 = 0, \psi_j; \theta) := \psi_j. \quad (3.4)$$

Let us drop the fixed time indicator  $j$  and introduce the short notations:

$$\sigma_i := \sigma(X_i, \psi_j; \theta), \quad \sigma_0 := \psi_j, \quad \lambda(\sigma_i) := \lambda(\sigma(X_i, \psi_j; \theta); \theta). \quad (3.5)$$

We use the trapezoidal rule for integration, which is second order accurate in  $\Delta X$ , and is the highest order formula we can implement easily with the available information. Then Eq. (3.3) can be approximated by the following expression:

$$\sigma_i = - \frac{\Delta X}{2} (\lambda(\sigma_i) \sigma_i + \lambda(\sigma_{i-1}) \sigma_{i-1}) + \sigma_{i-1}. \quad (3.6)$$

As the calculations are performed with  $i$  increasing from 1 to  $n$ ,  $\sigma_{i-1}$  is already known, and (3.6) is actually the following implicit equation in the unknown  $\sigma_i$ :

$$\left(1 + \frac{\Delta X}{2} \lambda(\sigma_i)\right) \sigma_i = \left(1 - \frac{\Delta X}{2} \lambda(\sigma_{i-1})\right) \sigma_{i-1}. \quad (3.7)$$

For each  $i$ , this equation can be solved by the following fixed point iterative procedure, if  $\Delta X$  is small enough (we use the notation  $\sigma^{(k)} := \sigma_i^{(k)}$ ,  $\sigma^{(0)} := \sigma_{i-1}$ ,  $k = 0, 1, 2, \dots$ ):

$$\sigma^{(k+1)} = \frac{1}{1 + \frac{\Delta X}{2} \lambda(\sigma^{(k)})} \left( \left(1 - \frac{\Delta X}{2} \lambda(\sigma_{i-1})\right) \sigma_{i-1} \right). \quad (3.8)$$

Notice that the term in the second parentheses on the RHS of the equation above does not change in the iterative procedure in  $k$ , so it can be precomputed. Notice also that only a few iterations in  $k$  should be needed, as  $\sigma$  changes very little between contiguous positions. For instance, if only one iteration is used, the scheme becomes:

$$\sigma_i = \frac{1}{1 + \frac{\Delta X}{2}\lambda(\sigma_{i-1})} \left(1 - \frac{\Delta X}{2}\lambda(\sigma_{i-1})\right)\sigma_{i-1}. \quad (3.9)$$

However, we still have the option of using Newton's method if we think the convergence of the iterative procedure fails or is too slow.

#### 4. NUMERICAL RESULTS FOR SYNTHETIC DATA

In this section we present an example where the functional (3.1) is minimized numerically. We use synthetic data to calibrate the model and to test the algorithm. These data are generated by fixing the parameters  $\theta$  and solving the direct problem given by the system of equations (2.11) and (2.9), (2.10).

We simulate observational error in real data by introducing random errors to the exact results. We prescribe the filtration functions  $\lambda(\sigma) = \max\{0, \theta_1 + \theta_2\sigma\}$ , with  $p = 2$ ,  $\theta_1 = 1$ ,  $\theta_2 = -171$  and create a set of synthetic data by adding random perturbations of the order of 0.05.

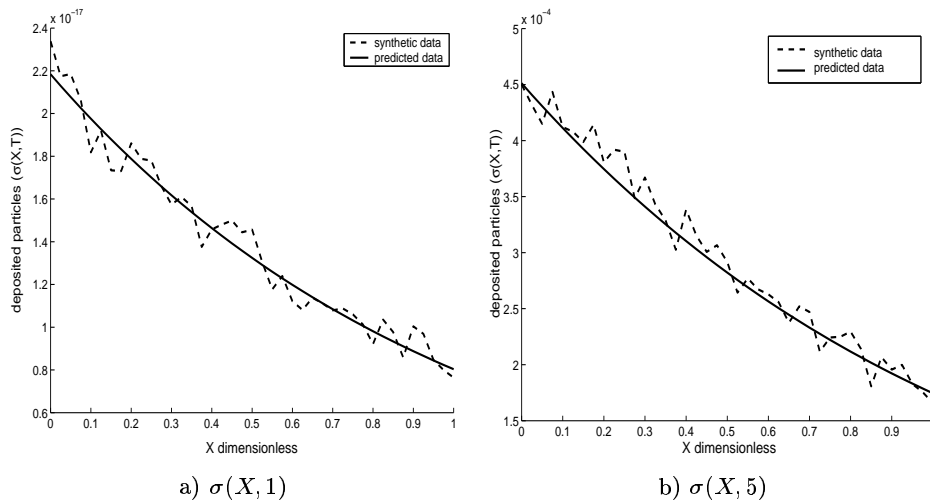


FIGURE 4.1. Synthetic and predicted deposition data at times 1 and 5.

We performed the recovery using  $n = 40$  space intervals, and  $m = 6$ , and the last retained particle distribution corresponds to 425 PVI. As shown in Figures (4.1)–(4.3), we obtained excellent matches at all times between the predicted and synthetic deposition distribution data. Moreover, the recovered filtration function  $\lambda(\sigma) = \max\{0, 1.02 - 193.7\sigma\}$  is obtained accurately with a relative square error of  $4 \times 10^{-4}$ .

In this example we did not need Tikhonov's regularization, i.e., we took  $\alpha = 0$ . Methods to estimate appropriate values for the regularization parameter can be found in [14]. We used the MATLAB routine *lsqcurvefit* for the minimization of the functional in Eq. (3.1).

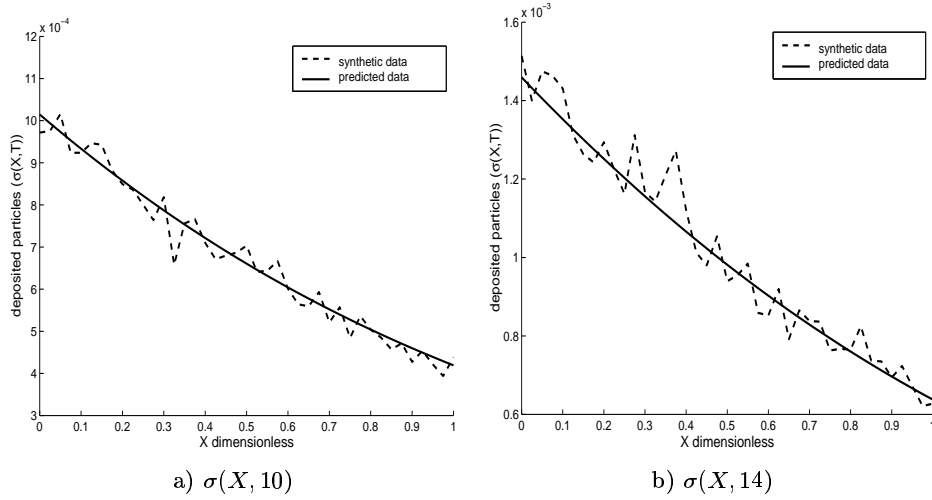


FIGURE 4.2. Synthetic and predicted deposition data at times 10 and 14.

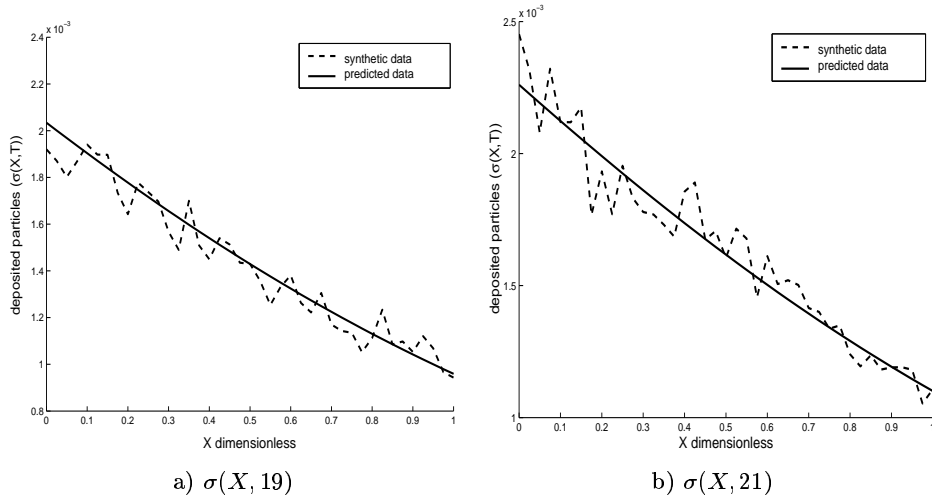


FIGURE 4.3. Synthetic and predicted deposition data at times 19 and 21.

## 5. CAKE FORMATION

Let us assume that it is possible to measure the effluent concentration  $c_e(t)$ , which determines  $c(1, T)$  by means of a change of variables as in (2.9).

The following relationship was proved in [8] along horizontal characteristics:

$$\frac{\sigma(1, T)}{c(1, T)} = \frac{\sigma(0, T)}{c(0, T)}. \quad (5.1)$$

This relationship follows from Corollary A.5.

Then using the values of  $\sigma(1, T)$ ,  $\sigma(0, T)$  obtained from the recovery method, by means of (5.1) we can calculate  $c(0, T)$ , i.e. the injected concentration history  $c_i(t)$ .

Usually, the concentration of suspended particles in the injection liquid is known. We conclude that we can compute the history of the cake mass per unit area. An example of this procedure for determining the cake at the final time based on a post-mortem analysis of the core rather than on X-ray data is found in [12].



When the effluent concentration is too small, this procedure for determining the cake mass history fails. An alternative procedure devised by [Mohammed] works very well in this situation.

## 6. CONCLUSION

The method described here reduces the inverse problem of recovering the empirical filtration coefficient from measurements of particle distribution along the core taken at several different times to parameter estimation implemented by an optimization procedure. It works well for synthetic data.

Of course, if the form of the particle deposition rate given in (2.2) is not a reasonably accurate description of reality, then this method might not work at all for real data. If it works, it provides a  $\lambda(\sigma)$  for the model given by (2.1)–(2.2) that approximates real data.

It would be really interesting to test the method with real data, just to answer the scientific question of how good the model (2.1)–(2.2) really is. This model was proposed 40 years ago and was never actually validated, because of lack of sufficiently high quality data, such as that provided by X-rays.

### APPENDIX A. WELL-POSEDNESS OF THE DIRECT PROBLEM

**A.1. Quasi-linear theory for the partial differential equation.** The system (2.7)–(2.8) has two characteristic directions, which are  $(dX, dT)^T = (1, 0)^T$  (with infinite speed) and  $(dX, dT)^T = (0, 1)^T$  (with zero speed).

**A.2. Solution of the partial differential equation.** Temporarily, we make the following positivity assumption:

**Assumption A.1.** The filtration function is a positive  $C^1$  function of  $\sigma$  in  $0 \leq \sigma < 1$ , such that  $\lambda(\sigma) > 0$  for  $0 \leq \sigma < 1$ .

This assumption is somewhat stringent in practical applications because  $\lambda(\sigma)$  may vanish for colloidal suspensions. From Assumption A.1, we can define the first integral  $\Psi$  of  $1/\lambda$ , i.e., we can choose  $\Psi$  so that  $\Psi(0) = 0$  as follows:

$$\Psi(\sigma) = \int_0^\sigma \frac{d\eta}{\lambda(\eta)}; \quad \text{and introduce} \quad \mu = \int_0^1 \frac{d\eta}{\lambda(\eta)}. \quad (\text{A.1})$$

Depending on the behavior of  $\lambda(\sigma)$  near 1, the range of  $\Psi : [0, 1) \rightarrow [0, \mu)$  is either a finite or infinite interval.

**Lemma A.2.** *Assume that there exists a  $C^1$  solution of (2.7)–(2.8), (2.5), (2.9). Then this solution satisfies the family of ordinary differential equations (2.10), (2.11a) and (2.9), (2.11b).*

Proof: Differentiating Eq. (A.1) and using Eq. (2.8) we obtain

$$\frac{\partial \Psi(\sigma)}{\partial T} = c. \quad (\text{A.2})$$

Since  $\Psi$  is  $C^2$  the derivative of (A.2) relative to  $X$

$$\frac{\partial c}{\partial X} = \frac{\partial^2 \Psi(\sigma)}{\partial T \partial X} \quad (\text{A.3})$$

is well defined. Substituting (A.3) in (2.7) we have

$$\frac{\partial^2 \Psi(\sigma)}{\partial T \partial X} = -\frac{\partial \sigma}{\partial T}. \quad (\text{A.4})$$

Now, we consider (A.4) in the infinite rectangle  $\{(X, T) : 0 \leq X \leq 1, T \geq 0\}$ . Integrating (A.4) in  $T$  along a vertical line from the point  $(X, 0)$  on the lower side of the rectangle to a fixed  $(X, T)$  we obtain:

$$\frac{\partial \Psi(\sigma)}{\partial X}(X, T) - \frac{\partial \Psi(\sigma)}{\partial X}(X, 0) = \sigma(X, T) - \sigma(X, 0). \quad (\text{A.5})$$

Using (2.2) we see that  $\sigma(X, 0) = 0$  and  $\frac{d\Psi(\sigma)}{dX}(X, 0) = 0$ ; using these expressions in (A.5) we obtain (2.11a). Using Eqs. (2.7) and (2.8) we obtain (2.11b). We repeat (2.11) here:

$$\frac{\partial \sigma}{\partial X} = -\lambda(\sigma)\sigma \quad \text{and} \quad \frac{\partial c}{\partial X} = -\lambda(\sigma)c.$$

In summary, we have proved that if (2.7)–(2.8) with data (2.4)–(2.5) has a  $C^1$  solution satisfying (2.10), then this solution must satisfy (2.11).

We eliminate the strict positivity Assumption A.1 in the following converse of Lemma A.2. See [2] for a proof that can be adapted to this model.

**Lemma A.3.** *Assume that  $\lambda(\sigma)$  is non-negative, piecewise differentiable with uniformly bounded derivative. Then the unique solution of (2.10), (2.11a) and (2.9), (2.11b) is a weak solution of (2.7)–(2.8) with data (2.5), (2.9).*

**Remark A.4.** Assume that  $\lambda(\sigma) > 0$  for  $0 \leq \sigma < \sigma_0$  and  $\lambda(\sigma) = 0$  for  $\sigma_0 \leq \sigma \leq 1$ . Then according to Eq.(2.10),  $\sigma(0, T)$  may increase up to  $\sigma_0$  at a finite time  $T_0$ . In this case, from this time on the deposition stays constant with value  $\sigma_0$  and the effluent concentration is the same as the inlet concentration, i.e.  $c(1, T) = c(0, T)$ .

**Corollary A.5.** The ratio  $\sigma/c$  is constant along characteristic lines with fixed  $T$ . Proof: Dividing (2.11a) by (2.11b) for fixed  $T$  and transposing terms we obtain  $dc/c = -d\sigma/\sigma$ . The result follows by integrating this differential equation.

## APPENDIX B. WELL-POSEDNESS OF THE INVERSE PROBLEM

We prove that the inverse problem formulated of determining  $\lambda(\sigma)$  based on idealized experimental data for  $\sigma$  for all  $X$  and  $T$  has a unique solution.

Having the initial data  $\sigma(0, T)$  we observe that the solution  $\sigma$  of Eq. (2.11a) is given by an ordinary differential equation along characteristic lines, and the continuity of the solution is a consequence of the theorem on ODE solution continuity with respect to parameter changes (see [9], pag. 91). Let us denote by  $\sigma(X, T; \lambda)$  the solution of (2.10), (2.11a) corresponding to the filtration function  $\lambda$ . Then, the map

$$\lambda \rightarrow \sigma(X, T; \lambda), \quad (\text{B.1})$$

is continuous in the uniform norm. Moreover, by the uniqueness theorem for ODEs, for each filtration function there exists a unique solution  $\sigma$  in the infinite rectangle  $0 \leq X \leq 1$  and  $T \geq 0$ .

First, we make the ‘‘conforming hypothesis’’: let  $\sigma_1 = \sigma(X_{i_1}, T_{j_1})$ ,  $\sigma_2 = \sigma(X_{i_2}, T_{j_2})$  where  $\sigma$  represents the experimental data from which experimental error has been removed. The conforming hypothesis is the following: if  $\sigma_1 = \sigma_2 > 0$  then  $\frac{\partial \sigma}{\partial X}(X_{i_1}, T_{j_1}) = \frac{\partial \sigma}{\partial X}(X_{i_2}, T_{j_2})$ ;

if  $\sigma_1 = \sigma_2 = 0$  then  $\frac{\partial \sigma}{\partial X}(X_{i_1}, T_{j_1}) = \frac{\partial \sigma}{\partial X}(X_{i_2}, T_{j_2}) = 0$ . Geometrically, this means that the idealized deposited particle distribution curves are just horizontally (or timewise) shifted pieces of a single master curve, as already mentioned in [12].

Now, from Eq. (2.11a) we obtain

$$\sigma(X, T)\lambda(\sigma(X, T)) = \frac{\partial \sigma}{\partial X}(X, T). \quad (\text{B.2})$$

From (B.2) and the conforming hypothesis, we conclude that the filtration function is uniquely determined for positive  $\sigma$  by the values of  $\sigma(X, T)$  on  $0 \leq X \leq 1$  and  $T \geq 0$ . To obtain  $\lambda$  for  $\sigma = 0$  a further continuity hypothesis is made, so we recover  $\lambda(\sigma = 0)$  by a limiting procedure using the values of  $\lambda$  for positive  $\sigma$ .

The conforming hypothesis receives this name because it expresses the fact that experimental data have properties compatible with the mass conservation equation (2.1) and the heuristic equation (2.2).

Other aspects of well-posedness remain to be studied.

## APPENDIX C. SOME EXPLICIT FORMULAE FOR THE DEPOSITION DISTRIBUTION.

In this section we find solutions for Eq. (2.11a) (or more precisely for Eq. (3.2)) for some useful parametrized forms of the filtration function. These formulae are to be used in (3.1).

We recall the definitions in Eqs. (2.14), (2.15), and (2.16).

**C.1. Linear filtration coefficient.** When  $\lambda(\sigma)$  depends linearly on  $\sigma$ , it is convenient to separate the case when  $\lambda$  can vanish from the case when it does not vanish. If  $a$  is large, the initial part of the plot of  $\lambda = \lambda_0 - a\sigma$  is linear, until it becomes zero, and then remains zero. (This vanishing occurs for the injection of certain colloidal suspensions.)

**C.1.1. Partially vanishing filtration.** Here we discuss case with  $a \geq \lambda_0$ . In this case  $\lambda(\sigma)$  is zero for some interval of  $\sigma$ :

$$\begin{cases} \lambda(\sigma) = \lambda_0 - a\sigma > 0 & \text{for } 0 \leq \sigma < \lambda_0/a, \\ \lambda(\sigma) = 0 & \text{for } \lambda_0/a \leq \sigma \leq 1. \end{cases} \quad (\text{C.1})$$

From Eq. (2.15), we obtain for  $0 \leq \sigma \leq \min(1, \lambda_0/a)$  that:

$$X = \frac{1}{\lambda_0} \left( \log \left( \frac{1 - a\sigma/\lambda_0}{\sigma} \right) - \log \left( \frac{1 - a\psi/\lambda_0}{\psi} \right) \right). \quad (\text{C.2})$$

Notice that  $X \rightarrow -\infty$  if  $\sigma \rightarrow \lambda_0/a$  and that  $X \rightarrow \infty$  as  $\sigma \rightarrow 0$ . Solving for  $\sigma$ , we obtain:

$$\sigma(X, \psi; \theta) = \frac{1}{a\lambda_0 + \exp \left( \lambda_0 X + \log \left( \frac{1 - a\psi/\lambda_0}{\psi} \right) \right)}. \quad (\text{C.3})$$

This is the formula for  $\sigma(X, \psi; \theta)$  to be used in Eq. (3.1), where  $\theta_1 = \lambda_0$  and  $\theta_2 = a$ . It can be used for other recoveries similar to that shown in Section 4.

C.1.2. *Non-vanishing filtration.* Consider the case with  $a < \lambda_0$ , which includes the case  $a < 0$ . In this case  $\lambda(\sigma)$  is positive:

$$\lambda(\sigma) = \lambda_0 - a\sigma > 0 \quad \text{for } 0 \leq \sigma \leq 1 \quad (\text{C.4})$$

The formula (C.3) is still valid.

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