APPROXIMATE SOLUTIONS FOR THE FILTRATION PROBLEM IN RADIAL GEOMETRIES.

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Abstract. *The oil recovery process goes through different stages during the life of a reservoir. For a small fraction of this time the natural pressure of the compressed oil, under the top layers of the terrestrial crust, suffices to expel the oil under production. However, during most of the time, the production is performed by the injection of water in one well, forcing the oil to be expelled at another well. The ratio between the injected flux and the pressure required to sustain this flux is called injectivity. The loss of injectivity by the deposition of particles brought by the water in the porous medium is called deep bed formation damage. To remedy this damage is costly, therefore it is important to predict the damage. Mathematical models are used for this purpose. We describe an approximate solution for the filtration problem in radial geometries, constructed using perturbation analysis on the equations governing the model. The availability of simple explicit solutions is very important for engineers, which have to make decisions without waiting for numerical computations. Until now, exact solutions have been reported for two cases: the first one corresponds to one-dimensional flows; the second occurs for cylindrical or spherical geometry when there is a linear relation between the filtration function and the suspended particle mass. We have extended the solution for the latter geometries in a model where the deposition depends linearly on the suspended particle mass and weakly on the deposited mass. We also developed a stable implicit second order finite difference scheme. This is the first such scheme for the filtration problem. We show results for flow in cylindrical coordinates.*

Keywords: *deep bed formation, perturbed explicit solutions, high order box scheme*

1. INTRODUCTION

Oil wells go through different production stages before they are abandoned. Primary oil recovery is due to natural reservatory pressure, which causes the oil to rise to the surface naturally through the production well. In secondary oil recovery, water is injected into the reservatory to push out the oil through another well. This procedure accounts for most of the oil a reservoir produces. The efficiency of an injection well is measured in terms of its *injectivity*: the ratio between the injection flow rate and the injection pressure required to maintain this rate. The injection rate is proportional to the recovery rate, and the injection pressure represents operational cost.

Off-shore wells often use sea water for injection. This water contains numerous suspended particles, both mineral and organic, and therefore it is unclean. This causes the well injectivity to decline, because the porous medium acts as a filter to the particles suspended in the water. The capture of suspended particles inside the porous medium and consequent loss of permeability characterize a phenomenon called *deep bed formation damage*. Experience shows that another phenomenon takes place on the injection surface of the porous medium, called *cake formation*. The "cake" is an aglomeration of particles outside the porous medium, and thus it is altogether separate from the formation damage. It causes injectivity decline as well.

Many theoretical and laboratorial studies were carried out to understand the filtration process (e.g. Espedal et al. (1981), Herzig et al. (1970)). We utilize the model for deep bed filtration presented in the fundamental work of Herzig et al. (1970), which consists of equations expressing the particle mass conservation and the particle retention process (Wennberg and Sharma (1997), Bedrikovetsky et al. (2001), de Zwart et al. (2006)). They form a quasi-linear system of equations containing the empirical *filtration function* $\widehat{\Lambda}(\sigma, c; u)$, which represents the kinetics of particle retention.

Works such as Bedrikovetsky et al. (2004) and Alvarez et al. (2007) describe methods to determine the filtration function through the solution of an inverse problem. However there is a high cost to solve an inverse problem, because it requires the repeated solution of a direct problem. This is what motivates the quest for accurate direct methods to solve the problem. Our work proposes two direct solutions in radial geometries. The first is an approximate solution constructed using perturbation analysis. The availability of simple explicit solutions may be very important for engineers, who have to make decisions without waiting for numerical computations. The second is a stable implicit second order finite difference scheme, which is the first such scheme for the filtration problem, see Mitchell et al. (2006), Silva and Marchesin (2006).

We present in Section 2 the physical formulation of the filtration problem. In Section 3 we develop the approximate equation and finalize in Section 4 with the box scheme and simulations in cylindrical coordinates.

2. PHYSICAL MODEL

We present a physical model for the phenomenon called deep bed filtration. In the vicinity of production wells, the classical theory has to be cast into radial coordinates (de Zwart et al., 2006). Such model aims to provide the profile of deposition $\sigma(r, t)$ along the porous rock, where σ is the pore volume filled by trapped particles. The model provides the profile occupied by suspended particles $c(r, t)$ as well.

We write the mass conservation law:

$$\frac{\partial}{\partial t}(\phi c + \sigma) + u(r)\frac{\partial}{\partial r} c = 0, \tag{1}$$

where the rock porosity ϕ is assumed to be uniform; $\sigma(r,t)$ and $c(r,t)$ are concentrations (mass per pore volume) of trapped particles in the pores and suspended particles in the liquid phase, respectively; $r_i \leq r \leq r_e$ is the position relative to the symmetry center in the sample rock; $t \geq 0$ is the time past since the beginning of injection and $u(r)$ is the average intrinsic flow velocity.

Usually $c \cong 10^{-4}$ while σ grows up to to 10^{-2}, i.e., except at initial times, it turns out that c and $\frac{\partial c}{\partial t}$ are much smaller then σ and $\frac{\partial \sigma}{\partial t}$, respectively. Because of this fact, a simplified mass conservation equation was proposed by Herzig et al. (1970):

$$\frac{\partial \sigma}{\partial t} + u(r)\frac{\partial}{\partial r}c = 0. \tag{2}$$

The simplified equation (2) is appropriate for our purposes since usually there is no accurate data measurements at initial times. In fact, at such times both σ as c are so small that experimental error predominates in the measured data, at least using current state-of-the-art measurement techniques.

We assume that there are no particles in the rock at the beginning, i.e.

$$\sigma(r,0) \equiv 0 \quad \text{for} \quad r \in [r_i, r_e]. \tag{3}$$

For simplicity, we consider constant injection rate of solid particles as boundary condition,

$$c(r_i, t) = 1. \tag{4}$$

The key element in this model is the empirical filtration function $\widehat{\Lambda}(\sigma, c; u)$, given in units of inverse length, which cannot be measured directly. The law for deposition rate of particles is written as

$$\frac{\partial \sigma}{\partial t} \ = \widehat{\Lambda}(\sigma, c; u). \tag{5}$$

Furthermore, since it is not possible to determine the filtration function by physical means, equations (2) and (5) are heuristic in nature.

Remark 1. In the case that $\widehat{\Lambda}$ does not vanish as time increases, equation (5) tells us that σ grows up indefinitely, which is not physical. So the model proposed by equations (2) and (5) is only suited for simulations where $\sigma \ll \phi$.

We are interested in two formulations of the filtration function. The first is linear and most suitable to the development of numerical methods. The second is nonlinear and takes into account the flow velocity as well as the already deposited particles, which turns out to be more useful in engineering aplications.

We follow de Zwart et al. (2006) who, based on literature, field observations and laboratory experiments, admit a nonlinear dependence of the filtration function on the Darcy velocity. We also assume a dependence on the entrapped particles, so the deposition rate diminishes as the porous medium becomes clogged. We write

$$\widehat{\Lambda}(\sigma, c; u) = \lambda_0 |u|^{\delta+1}(1 - \varepsilon\sigma)c, \tag{6}$$

where λ_0 is the filtration coefficient of the medium, the δ-power expresses the non-linearity of the velocity dependence and the parameter ε is positive and small in the sense $\max|\varepsilon\sigma(r,t)| \ll 1$. Note that for $\delta = 0$ and $\varepsilon = 0$ the filtration function reduces to that found in the classical literature.

The Darcy velocity is written

$$u(r) = \frac{q}{S_n(r)},$$

where q is the flux injected, which we assume constant, and S_n is the area of the n-dimensional flow iso-surface: analogously, we will call its volume V_n. With the change of variables

$$x = \frac{r^n}{r_i^n}, \quad \tilde{t} = \frac{q}{V_n} t \quad \text{and} \quad \gamma = (\delta + 1)\left(\frac{n-1}{n}\right),$$

we obtain from eqs. (2) and (6), dropping the tilde,

$$\begin{cases} \dfrac{\partial c}{\partial x} = -\dfrac{\partial \sigma}{\partial t} & \text{(a);} \\ \dfrac{\partial \sigma}{\partial t} = \lambda_0 x^{-\gamma}(1 - \varepsilon\sigma)c & \text{(b).} \end{cases} \tag{7}$$

Although this filtration function may be useful in engineering aplications, for the development of the theory we need a simpler function to which we can apply the well established method of finite differences (Strikwerda, 2004) to determine the major properties of the solution. As it was done in Silva and Marchesin (2006), we propose a linear filtration function as follows. We take a first order approximation of $\widehat{\Lambda}(\sigma, c; u) = \Lambda(\sigma, c)|u|$ at the origin, writing

$$\Lambda(\sigma, c) = \Lambda(0,0) + \frac{\partial}{\partial \sigma}\Lambda(0,0)\sigma + \frac{\partial}{\partial c}\Lambda(0,0)c + \mathcal{O}((\sigma, c)^2) : \tag{8}$$

there is no deposition in the absence of suspended particles, so we must have $\Lambda(0,0) = 0$. It is also clear that deposition should increase when we increase the amount of suspended particles ($\frac{\partial}{\partial c}\Lambda(0,0) = \beta > 0$) and should decrease as the porous medium fills ($-\frac{\partial}{\partial \sigma}\Lambda(0,0) = \alpha > 0$). From these considerations and equation (8) we can write

$$\Lambda(\sigma, c) = \beta c - \alpha\sigma. \tag{9}$$

Remark 2. Note that the non-linear filtration function does not allow entrapped particles to return to suspension, but the linear filtration function does. Herzig's model assumes that the filtration function should be non-negative: therefore, for physical consistency, we must have $\beta \gg \alpha$.

For the numerical method, we will focus on the cylindrical coordinates, the most suited geometry to simulate deposition in the field. The problem reduces to

$$\begin{cases} \dfrac{\partial c}{\partial r} = -\Lambda(\sigma, c) & \text{(a),} \\ \dfrac{\partial \sigma}{\partial t} = \dfrac{\Lambda(\sigma, c)}{r} & \text{(b).} \end{cases} \tag{10}$$

3. APPROXIMATE SOLUTION

We aproach the problem (2), (6), (3) and (4) as a perturbation of the case $\varepsilon = 0$, which was solved in de Zwart et al. (2006): for $\gamma \neq 1$ we have

$$\begin{cases} c_0(x,t) = \exp\left\{\dfrac{\lambda_0}{(1-\gamma)}\left[x^{1-\gamma} - 1\right]\right\} & \text{(a),} \\ \sigma(x,t)_0 = \lambda_0 t \exp\left\{\dfrac{\lambda_0}{(1-\gamma)}\left[x^{1-\gamma} - 1\right]\right\} & \text{(b).} \end{cases} \tag{11}$$

A more complete discussion, for water production, can be found in J. M. Silva (2008). We will restrict our analysis to this case, as the case $\gamma = 1$ is analogous. We write the solution as an asymptotic series in ε

$$\begin{cases} c(x,t) &= c_0(x,t) - \varepsilon\, c_1(x,t) + \mathcal{O}(\varepsilon^2) \qquad \text{(a)}, \\ \sigma(x,t) &= \sigma_0(x,t) - \varepsilon\, \sigma_1(x,t) + \mathcal{O}(\varepsilon^2) \qquad \text{(b)}, \end{cases} \tag{12}$$

where c_1 and σ_1 are first order corrections, c_0 and σ_0 are the solutions given in (11) and the minus sign is for convenience. Substituting (12) in the system (2), (6) we get a new system for the first order corrections:

$$\begin{cases} \partial_x c_1 &= -\lambda_0\, x^{-\gamma}\, [c_0\sigma_0 + c_1] \qquad \text{(a)}, \\ \partial_t \sigma_1 &= \lambda_0\, x^{-\gamma}\, [c_0\sigma_0 + c_1] \qquad \text{(b)}. \end{cases} \tag{13}$$

Note that now eq. (13a) can be solved directly. As boundary conditions for the system (13) we have from (12a) that $c(1,t) = c_0(1,t) - \varepsilon\, c_1(1,t) + \mathcal{O}(\varepsilon^2)$; equation (4) says that both $c(1,t) = 1$ and $c_0(1,t) = 1$ hold for all $\varepsilon \geq 0$, because they both are solutions of (2), (6); then we must have $c_1(1,t) \equiv 0$, and adding a continuity argument to a similar derivation for $\sigma_1(1,t)$, we arrive at $\sigma_1(1,t) \equiv 0$.

Equation (13a) is a family of ODEs in the parameter t. We simply compute

$$\frac{d}{dx}\left[\exp\left(\int_1^x \lambda_0 s^{-\gamma} ds\right) c_1(x,t)\right] = -\exp\left(\int_1^x \lambda_0 s^{-\gamma} ds\right) \lambda_0 x^{-\gamma} c_0(x,t)\sigma_0(x,t). \tag{14}$$

Its solution clearly bifurcates in the cases $\gamma = 1$ and $\gamma \neq 1$. Only the second case is general and useful in practice, so it is the only case we shall study. Integrating (14), using (11) and the boundary condition for the first order approximation, we find

$$c_1(x,t) = -\lambda_0^2 t \left[e^{\frac{-\lambda_0}{1-\gamma}(x^{1-\gamma}-1)} \int_1^x s^{-2\gamma} e^{\frac{-\lambda_0}{1-\gamma}(x^{1-\gamma}-1)} ds \right]. \tag{15}$$

We use the boundary condition for the first order approximation in (15) and write

$$\sigma_1(x,t) = \lambda_0 x^{-\gamma} \int_0^t \left[c_0(x,s)\sigma_0(x,s) + c_1(x,s)\right] ds. \tag{16}$$

4. NUMERICAL SCHEME

We now introduce the grid notation that will be used from here on. Consider for the Or axis a discretization of the interval $[r_i, r_e]$ such that for $h > 0$ we write $\{r_m = r_i + mh,\ m = 0, 1, 2, \ldots, M\}$ and $\{r_{m+1/2} = r_i + (m + 1/2)h,\ m = 0, 1, 2, \ldots, M-1\}$ where,

$$h = \frac{r_e - r_i}{M}. \tag{17}$$

Analogously for the Ot axis we write a discretization of the interval $\mathbb{R}^+$, such that for $k > 0$ we write $\{t_n = nk,\ n = 0, 1, 2, \ldots\}$. This allows us to index $c(x,t)$ and $\sigma(x,t)$ in the simpler notation

$$(c)^n_m = c(r_m, t_n), \quad \text{and} \quad (\sigma)^n_m = \sigma(r_m, t_n). \tag{18}$$

We reserve the notation c_m^n, σ_m^n for the discrete functions on the (r_m, t_n) gridpoint. Likewise, we use the notation $(\Lambda)_m^n = \Lambda((\sigma)_m^n, (c)_m^n)$, which allows us to reserve Λ_m^n to its discrete counterpart.

Due to the nature of the PDE system, the numerical problem reduces to the composition of two simpler problems. The first is to find the solution of the PDE system itself inside the computational domain. The second is to solve a pair of ODEs at the boundary. We will construct the solution in the interior first, where the real interest lies, and then make some remarks as to what happens at the boundary.

4.1 The box scheme

For the problem (10) we propose the following scheme

$$\begin{cases} \left(\sigma_{m+1}^{n+1} + \sigma_m^{n+1}\right) - \left(\sigma_{m+1}^n + \sigma_m^n\right) + \lambda \dfrac{\left(c_{m+1}^{n+1} + c_{m+1}^n\right) - \left(c_m^{n+1} + c_m^n\right)}{r_{m+1/2}} = 0 \quad \text{(a);} \\ \dfrac{c_{m+1}^{n+1} - c_m^{n+1}}{h} = -\dfrac{1}{2}\left[\Lambda_{m+1}^{n+1} + \Lambda_m^{n+1}\right] \quad \text{(b).} \end{cases} \tag{19}$$

This scheme is second order accurate as will be shown. Applying central differences to (10a) multiplied by $1/r$ at $r_{m+1/2}$ we find, for the left hand side, the expression

$$\frac{1}{r_{m+1/2}} \frac{(c)_{m+1}^n - (c)_m^n}{h} = \left(\frac{1}{r}\frac{\partial}{\partial r} c\right)_{m+1/2}^n + \mathcal{O}(h^2); \tag{20}$$

for the right side, we discretize Λ/r as

$$\frac{1}{r_{m+1/2}} \frac{(c)_{m+1}^n - (c)_m^n}{h} = \frac{1}{2}\left(\frac{\Lambda((c)_{m+1}^n, (\sigma)_{m+1}^n)}{r_{m+1}} + \frac{\Lambda((c)_m^n, (\sigma)_m^n)}{r_m}\right) + \mathcal{O}(h^2). \tag{21}$$

For (10b) we use a centered discretization for the time derivative, which gives us

$$\frac{(\sigma)_m^{n+1} - (\sigma)_m^n}{k} = \frac{1}{2r_m}\left[\Lambda((c)_m^{n+1}, (\sigma)_m^{n+1}) + \Lambda((c)_m^n, (\sigma)_m^n)\right] + \mathcal{O}(k^2). \tag{22}$$

Taking the mean in r of (22) between r_m and r_{m+1}, the mean in t of (21) between t_n and t_{n+1} and adding, we arrive at

$$\left[(\sigma)_{m+1}^{n+1} + (\sigma)_m^{n+1}\right] - \left[(\sigma)_{m+1}^n + (\sigma)_m^n\right] + \lambda \frac{\left[(c)_{m+1}^{n+1} + (c)_{m+1}^n\right] - \left[(c)_m^{n+1} + (c)_m^n\right]}{r_{m+1/2}} = O(k^2) + O(h^2), \tag{23}$$

which is the discretization found in eq. (19a).

To derive (19b), we discretize (10a) as

$$\frac{(c)_{m+1}^n - (c)_m^n}{h} = -\frac{1}{2}\left[(\Lambda)_{m+1}^n + (\Lambda)_m^n\right] + \mathcal{O}(h^2). \tag{24}$$

4.2 Stability of the box scheme, a heuristic approach

We proceed with the linear filtration function, following the approach taken by Silva and Marchesin (2006), where it was shown that the box scheme is unconditionally stable in linear geometry with this linear filtration function. Also, the results of simulations based on the box scheme showed no oscillation at all, a remarkable fact considering that it relies heavily on means over the grid. Here, we furnish a heuristic argument regarding the stability of this scheme and (3) in radial geometry. We take eqs. (19) and (9) and apply at the boundary the discretization which follows. Using (24), (9) and (3), we get for the Or axis

$$c^0_{m+1} = c^0_m - h\left[\frac{\left(c^0_{m+1} + c^0_m\right)}{2}\right], \tag{25}$$

This is a simple stable second order implicit ODE numerical scheme, which we rewrite to obtain the recurrence

$$c^0_{m+1} = \frac{2-h}{2+h}c^0_m, \tag{26}$$

which represents a scheme that is oscillation free if

$$h \leq 2; \tag{27}$$

in the Ot axis, we begin with eq. (22), writing:

$$\sigma^{n+1}_0 = \sigma^n_0 - \frac{k}{2r_0}\left[\beta\left(c^{n+1}_0 + c^n_0\right) - \alpha\left(\sigma^{n+1}_0 + \sigma^n_0\right)\right], \tag{28}$$

which is the discretization on the Ot axis with c^n_0 given by (4). Again, we get a well known numerical method in ODE's theory: from the recurrence

$$\sigma^{n+1}_0 = \frac{2r_0 - \alpha k}{2r_0 + \alpha k}\sigma^n_0 + \frac{\beta k}{2r_0 + \alpha k}\left(c^{n+1}_0 + c^n_0\right), \tag{29}$$

we see it is oscillation free if

$$k \leq \frac{2r_0}{\alpha}. \tag{30}$$

Using the linear filtration function, we can make the linear box scheme explicit

$$\left\{\begin{array}{rcll} c^{n+1}_{m+1} & = & Bc^{n+1}_m + A\left(\sigma^{n+1}_{m+1} + \sigma^{n+1}_m\right) & \text{(a);} \\ \sigma^{n+1}_{m+1} & = & -\sigma^{n+1}_m + \dfrac{r_{m+1/2}}{r_{m+1/2} + \lambda A}\left\{\sigma^n_{m+1} + \sigma^n_m - \lambda/r_{m+1/2}\left[c^n_{m+1} + (B-1)c^{n+1}_m - c^n_m\right]\right\} & \text{(b),} \end{array}\right. \tag{31}$$

where

$$A = \frac{\alpha h}{2(1 + h\beta/2)} \quad \text{and} \quad B = \frac{1 - h\beta/2}{1 + h\beta/2}.$$

Using this scheme, we have perfomed simulations with many values for the parameters α and β, and found clear evidence of (probably unconditional) stability. There was no case of divergence in the solution as the grid was refined. A case example is illustrated in figure 1, with parameters chosen for their physical relevance.

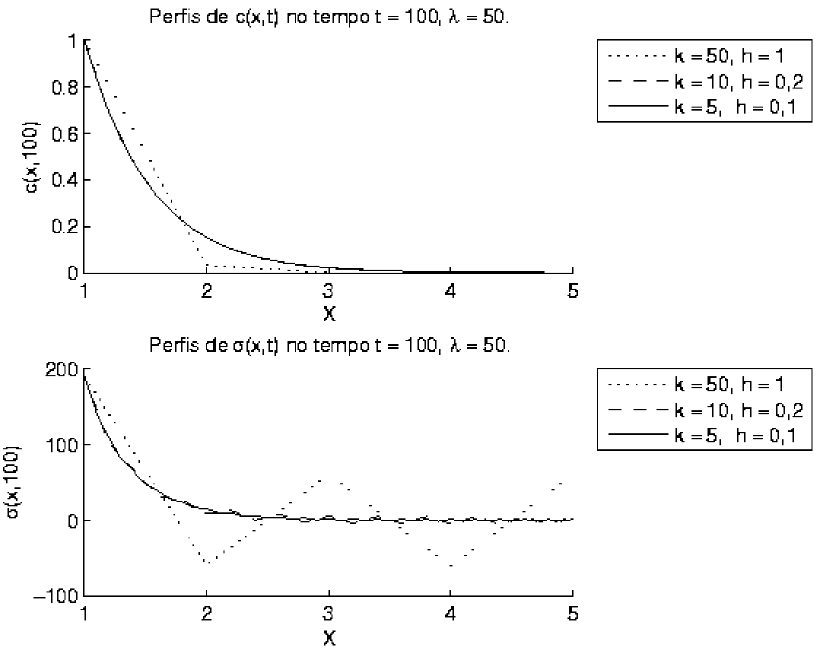

Figure 1: Radial box scheme convergence behavior, case $\alpha = 0.001$, $\beta = 2$. The grid refining shows clear evidence of convergence.

5. CONCLUSION

Based on the fundamental work of Herzig et al. (1970) we have proposed two approximate solutions for the filtration problem in radial geometries. The first was constructed using perturbation analysis on the equations governing the model. This solution allows one to make decisions without waiting for numerical computations. The second was a stable implicit second order box finite difference scheme that furnishes better results in the long run. We have shown results for flow in cylindrical coordinates.

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