4° PDPETRO, Campinas, SP
21st–24th October 2007 2.4.0087-1

A REDUCED MODEL FOR INTERNAL WAVES INTERACTING WITH SUBMARINE STRUCTURES AT INTERMEDIATE DEPTH

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In the context of ocean dynamics, a reduced strongly nonlinear one-dimensional model for the evolution of internal waves over an arbitrary seabottom with submerged structures is derived. The reduced model is aimed at obtaining an efficient numerical method for this two-dimensional problem. Two layers containing inviscid, immiscible, irrotational fluids of different densities are defined. The upper layer is shallow compared with the characteristic wavelength at the interface of the two-fluid system, while the bottom region's depth is comparable to the characteristic wavelength. The non-linear evolution equations describe the behaviour of the internal wave elevation and mean upper-velocity for this water configuration. The system is a generalization of the one proposed by Choi and Camassa for the flat bottom case in the same physical settings. Due to the presence of topography a variable coefficient accompanies each space derivative. These Boussinesq-type equations contain the Intermediate Long Wave (ILW) equation and the Benjamin-Ono (BO) equation when restricted to the unidirectional wave regime. We intend to use this model to study the interaction of the wave with the bottom profile. The dynamics include wave scattering, dispersion and attenuation among other phenomena. The research is relevant in oil recovery in deep ocean waters, where salt concentration and differences in temperature generate stratification in such a way that internal waves can affect offshore operations and submerged structures. Important properties of the model will be discussed. A hierarchy of one-dimensional models is derived from this strongly nonlinear model by considering the different regimes (linear, weakly nonlinear or strongly nonlinear) as well as the flat or corrugated bottom cases. Numerical schemes based on the method of lines for all of them will be described. The numerical results from the Matlab implementations will be shown including periodic topography experiments and solitary waves solutions.

Internal waves, Inhomogeneous media, Asymptotic theory.

1 INTRODUCTION

Modelling internal waves is of great interest in the study of ocean dynamics. Internal ocean waves appear when salt concentration and differences in temperature generate stratification. They can interact with the bottom topography and submerged structures

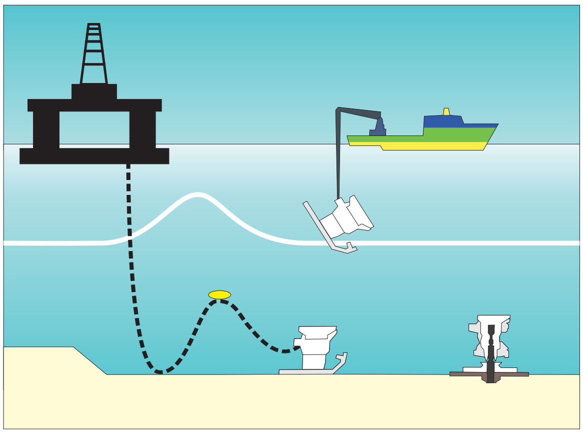

Figure 1: Offshore operations, internal waves and submerged structures.

as well as with surface waves. In particular, in oil recovery in deep ocean water, internal waves can affect offshore operations and submerged structures, see Figure 1. Accurate reduced models are a first step in producing efficient computational methods for engineering problems in oceanography. This was the goal in [16, 1].

To describe this nonlinear wave phenomenon in deep waters there are several bidirectional models containing the Intermediate Long Wave (ILW) equation and the Benjamin-Ono (BO) equation, starting from works such as [2, 6, 17, 10, 12] to more recent papers such as [13, 3, 4, 5, 9]. In these models two fundamental mechanisms, nonlinearity and dispersion, are responsible for the main features of the propagating wave. One of the most interesting behaviours observed is the existence of solitary wave solutions with permanent shape. They are observed when the steepening of a given wave front due to the nonlinearity and the flattening and attenuation promoted by the dispersion are balanced on a particular scale. Usually the contribution of nonlinearity is quantified by the non-dimensional nonlinearity parameter α, which is the ratio between the wave amplitude and the fluid layer thickness. It appears as a small non-zero parameter in the so-called weakly nonlinear regime, and accompanies the nonlinear terms. On the other hand, the dispersion parameter β is the squared ratio between the fluid layer thickness and the typical wavelength. It appears in the dispersion relation, making the phase velocity a function of the wavenumber k. The balance that creates a solitary wave is commonly obtained through a scaling relation between α and β, in the form of a power law, for asymptotic values $\alpha \ll 1$ and $\beta \ll 1$. In the water configuration considered here, it is the scaling $\alpha = O(\sqrt{\beta})$ that leads to the ILW [10, 12]. In the limit when one layer thickness tends to infinity, the ILW equation becomes the BO equation [2, 6, 17].

For all these models, the dependence on the vertical coordinate has been eliminated by focusing on specific regimes and using systematic asymptotic expansion methods in small parameters. This results in a considerable simplification of the original Euler equations that leads to more efficient computational methods than the integration of the Euler system in the presence of a free interface. However, the approximation needs to be accurate even for large values of the parameters α and β. In other words, the model needs to be robust enough to cover several regimes in which the viscosity effects are negligible, justifying the use of the Euler equations. In [5], the authors compared

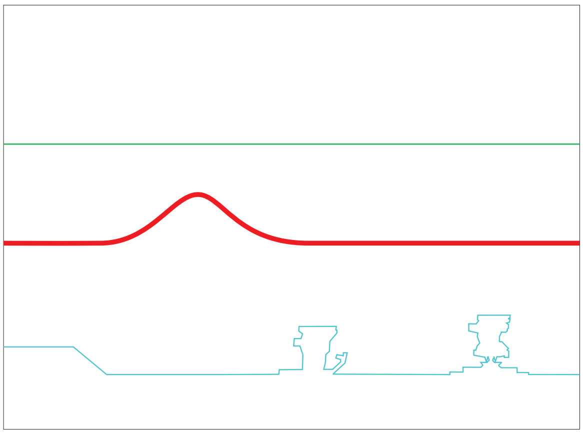

Figure 2: Two-fluid system configuration. Horizontal rigid lid (green), interface (red), bottom topography (cyan).

weakly nonlinear models with experimental data obtained by Koop and Butler in [11]. They found a divergence. This motivated them to propose a strongly nonlinear model for flat bottom that shares the simplicity of the weakly nonlinear ones and extends its domain of validity. The numerical results agree very well with the experimental data. This model is generalized in the present work to consider an arbitrary sea bottom. We remark that the new model support bidirectional wave propagation, so it is able to capture the reflected wave from the propagation over a nonuniform sea bottom.

The models found in the literature consider flat or slowly varying bottom topography. Here, the model of Choi and Camassa is generalized to the case of an arbitrary bottom topography by using the conformal mapping technique described in [16]. We obtained a strongly nonlinear long-wave model like Choi and Camassa's which is able to describe large amplitude internal solitary waves.

2 METHODS

2.1 Governing Equations & Reduced Model

A system of two layers of fluids constrained to a region limited by a horizontal rigid lid at the top and an arbitrary bottom topography is considered, as described in Figure 2. Define the layer density of each inviscid, immiscible, irrotational fluid as ρ_1 for the upper layer and ρ_2 for the lower layer. For a stable stratification, $\rho_2 > \rho_1$. Similarly, (u_i, w_i) denotes the velocity components and p_i the pressure, where $i = 1, 2$. The upper layer is assumed to have an undisturbed thickness h_1, much smaller than the characteristic wavelength of the perturbed interface $L > 0$, hence the upper layer will be in the shallow water regime. At the lower layer the irregular bottom is described by $z = h_2(h(x/l) - 1)$. The function h needs not to be continuous neither univalued, see for example Figure 2 where a polygonal shaped topography is sketched. We can assume that h has compact support so the roughness is confined to a finite interval. Moreover h_2 is the undisturbed thickness of the lower layer outside the irregular bottom region and it is comparable with the characteristic wavelength L, that characterizes an intermediate

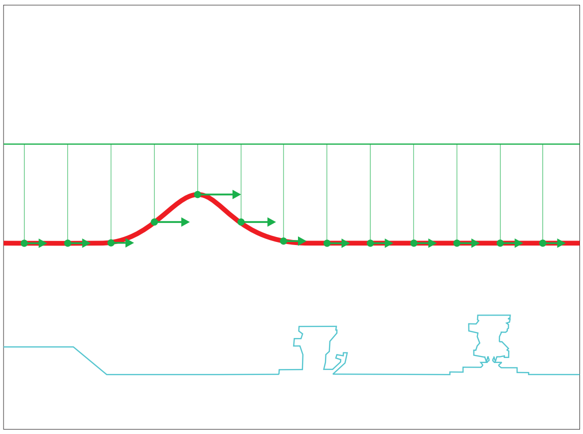

Figure 3: Vertical averaging of physical quantities involved in the upper layer equations. The green arrows indicate the mean-layer horizontal upper velocity $\overline{u_1}$.

depth regime. In the slowly varying bottom case we define $\varepsilon = L/l \ll 1$; when a more rapidly varying bottom is of concern, the horizontal length scale for bottom irregularities l is such that $h_1 < l \ll L$. The coordinate system is positioned at the undisturbed interface between layers. The displacement of the interface is denoted by $\eta(x,t)$ and we may assume that initially it has compact support.

The corresponding Euler equations are

$$\begin{aligned} u_{i\,x} + w_{i\,z} &= 0, \\ u_{i\,t} + u_i u_{i\,x} + w_i u_{i\,z} &= -\frac{p_{i\,x}}{\rho_i}, \\ w_{i\,t} + u_i w_{i\,x} + w_i w_{i\,z} &= -\frac{p_{i\,z}}{\rho_i} - g, \end{aligned}$$

for $i = 1, 2$. Subscripts x, z and t stand for partial derivatives with respect to spatial coordinates and time. The continuity condition at the interface $z = \eta(x,t)$ demands that

$$\eta_t + u_i \eta_x = w_i, \qquad p_1 = p_2,$$

namely, a kinematic condition for the material curve and no pressure jumps allowed.

At the top we impose a rigid lid condition,

$$w_1(x, h_1, t) = 0,$$

commonly used in ocean and atmospheric models, while at the irregular impermeable bottom

$$-\frac{h_2}{l}\, h'\left(\frac{x}{l}\right) u_2 + w_2 = 0.$$

By averaging in the vertical direction (see Figure 3) we reduced the two-dimensional (2D) Euler equations for the upper layer ($i = 1$) to a one-dimensional (1D) system. To close that system in terms of the perturbation of the interface η and the mean-layer horizontal upper velocity $\overline{u_1}$, an asymptotic expansion in the small dispersion parameter

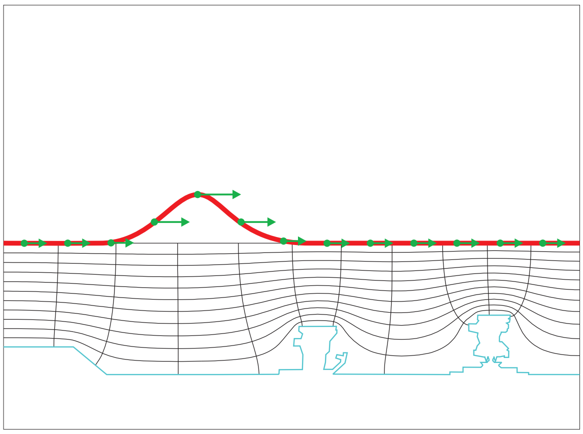

Figure 4: Conformal mapping of the undisturbed bottom layer.

$\beta = \left(\frac{h_1}{L}\right)^2$ was done. The resulting dimensionless equations are

$$\begin{cases} -\eta_t + ((1-\eta)\overline{u_1})_x = 0, \\ \overline{u_1}_t + \overline{u_1} \cdot \overline{u_1}_x = -\eta_x - \Big(p_2(x, \eta(x,t), t)\Big)_x + O(\beta). \end{cases}$$

Details can be found in [18, 19].

Actually, we still need to get an expression for p_2 in order to close the system and also to establish a connection with the lower intermediate depth layer. To get an approximation for $\Big(p_2(x, \eta(x,t), t)\Big)_x$ from the Euler equations for the lower fluid layer, a conformal mapping of the undisturbed bottom layer is performed, see Figure 4. Then the approximate potential problem is solved in the terrain-following coordinates (ξ, ζ), giving rise to a Hilbert transform on the strip term. Substituting in the curvilinear coordinates system we obtain the following,

$$\begin{cases} \eta_t - \dfrac{1}{M(\xi)}[(1-\eta)\overline{u_1}]_\xi = 0, \\ \overline{u_1}_t + \dfrac{1}{M(\xi)}\overline{u_1}\,\overline{u_1}_\xi + \dfrac{1}{M(\xi)}\left(1 - \dfrac{\rho_2}{\rho_1}\right)\eta_\xi = \sqrt{\beta}\,\dfrac{\rho_2}{\rho_1}\dfrac{1}{M(\xi)}\mathcal{T}\left[((1-\eta)\overline{u_1})_{\xi t}\right]. \end{cases} \tag{1}$$

This is a Boussinesq-type system with variable (time independent) coefficients $1/M(\xi)$. $\mathcal{T}$ is the Hilbert transform on the strip. The terrain coefficient $M(\xi)$ comes from the conformal mapping and it contains the bottom information. Since $M(\xi)$ is an analytic function, a highly complex boundary profile has been converted into a smooth variable coefficient in the system above. No smallness assumption was made on the wave amplitude, thus the model derived is strongly nonlinear. It involves a Hilbert transform on the strip characterizing the presence of harmonic functions (hence the potential flow) below the interface. Efficient computational methods can be produced for this accurate reduced model which governs, to leading order, a complex two-dimensional problem. If the bottom is flat, then $M(\xi) = 1$ and the same system derived in [5] is obtained, which is a nice consistency check.

	Linear	Weakly nonlinear	Strongly nonlinear
Flat bottom	FFT	FFT	Explicit matrices
Rough bottom	Explicit matrices	Explicit matrices	Explicit matrices

Table 1: Hierarchy of one-dimensional models. Computation of the dispersive term.

2.2 Numerical Method

From system (1), a hierarchy of one-dimensional models can be derived by considering the different regimes (linear, weakly nonlinear or strongly nonlinear) as well as the flat or corrugated bottom cases. To find the solution for the initial value problem of these systems is a nontrivial task. That is why we resorted to numerical methods to find approximate solutions. For numerical implementations, we worked on the periodic domain $\xi \in \Pi[0, 2\ell]$. The choice of a computational periodic domain was made to be able to use spectral methods to compute the Hilbert transform.

According to the method of lines, we discretized in space and solved a coupled system of ODEs by a finite difference formula in t like, for example, the fourth order Runge-Kutta integration scheme (RK4). First, an approximation scheme for the ξ-derivatives must be used for the discretization in space. Our choice was to approximate the ξ-derivative of a function $f(\xi)$ by the fourth order, five point formula

$$f_\xi(\xi_j) = \frac{8(f_{j+1} - f_{j-1}) + f_{j-2} - f_{j+2}}{12\Delta\xi} + O(\Delta\xi^4). \tag{2}$$

where $f_j = f(\xi_j)$, $\xi_j = j\Delta\xi$, $\Delta\xi = 2\ell/N$, $j = 1, \ldots, N$. Finally, to compute the dispersive term involving the Hilbert transform, we used the Fast Fourier Transform (FFT) for the linear and constant coefficients cases. When the dispersive term has a nonlinear dependence between η and $\overline{u_1}$, or it has a variable coefficient accompanying it, the use of the FFT is no longer straightforward, so we use a matrix formulation. By using a spectral matrix instead of an FFT, we are only paying a price in complexity but not in accuracy. Table 1 sumarizes the strategy used for each model. Details can be found in [19].

3 RESULTS & DISCUSSION

3.1 Periodic topography experiment

Here we present an example of a wave propagating over a periodic topography bottom. We avoid the computation of $M(\xi)$ from the variable depth bottom, which can be costly even using Driscoll's package [7]. Let us assume that it is a function of the form

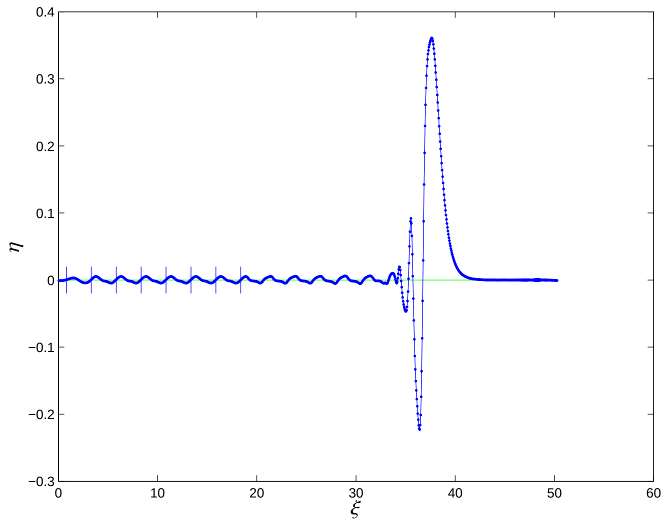

Figure 5: Pulse propagating over a synthetic periodic slowly-varying topography. Dotted line: numerical solution for $t = 32.3977$, vertical bars mark spatial intervals of size 2.5133 that fall together with the end of each period of the reflected signal.

$M(\xi) = 1 + n(\xi)$ where $n(\xi)$ describes periodic fluctuations. This choice is not far from the real coefficient that comes from mapping a periodic piecewise linear topography, see [14, 16, 15]. So, let us consider a periodic slowly-varying coefficient $M(\xi)$ defined on the domain $[0, 16\pi]$ as

$$M(\xi) = \begin{cases} 1 + 0.5\sin(5\xi), & \text{for } 6\pi \leq \xi \leq 12\pi, \\ 1, & \text{elsewhere.} \end{cases}$$

The initial perturbation of the interface is the Gaussian function

$$\eta_0(\xi) = 0.5e^{-a(\xi-\pi)^2/64}$$

with $a = 200$ and effective width $L = 2.4$. The ratio inhomogeneities/wavelength is about 0.5236. The physical parameters are $\rho_1 = 1$, $\rho_2 = 2$, $\beta = 0.0001$, $\alpha = 0.01$. We employed 1024 grid points, spatial discretization $\Delta\xi = 0.0491$, time discretization $\Delta t = 0.0491$. The numerical solution for the weakly nonlinear corrugated bottom model (WNCM) using RK4 for time $t = 32.3977$ is shown in Figure 5. As expected from Bragg's phenomenon theory [8], twice the period of the bottom oscillations (2.5133) is in very good agreement with the reflected wavelength. In Figure 5 vertical bars marking spatial intervals of size 2.5133 fall together with the end of each period of the reflected signal.

3.2 Approximate solitary wave solution

Now we present an example of an internal solitary wave from the Regularized ILW equation evolving according to the weakly nonlinear flat bottom model (WNFM). That is, we take as initial condition for the WNFM a solitary wave from its unidirectional reduction. We expect the wave to behave almost like a solitary wave. In particular,

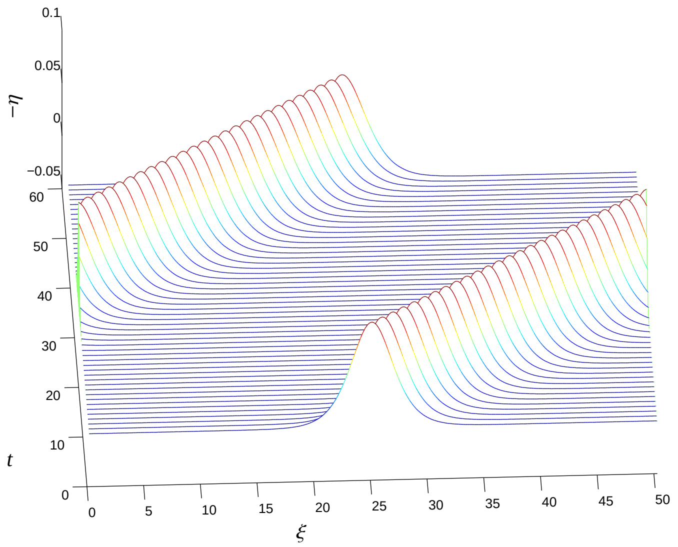

Figure 6: Propagation of a single solitary wave until $t = 50.8545$.

the balance between nonlinearity and dispersion should be maintained and the wave should travel without a significant change of shape. The velocity of propagation should be similar to that in the ILW equation. The numerical solution in the interval $[0, 16\pi]$ is obtained by the RK4 numerical solver for $\rho_1 = 1$, $\rho_2 = 2$, $\beta = 0.0001$, $\alpha = 0.01$, $N = 256$, $\Delta\xi = 0.1963$, $\Delta t = 0.1963$. The expected behaviour of the wave is captured by the numerical method for long times as shown in Figure 6. The pulse propagates with an approximate velocity of 0.9884 in conformity with its propagation velocity $c = 0.9961$ in the Regularized ILW equation. The shape of the solitary wave is preserved for long times. The error between the initial condition and the solution that returns to the original position at approximate time $t = 50.8545$ is 0.0047. Taking into account that the initial condition came from the unidirectional case, the result is satisfactory.

A more extensive set of examples can be found in [19]. It suggests that the model proposed can be implemented numerically and that its basic qualitative properties are well captured by the numerical solutions.

4 CONCLUSION

In the present work, a one-dimensional strongly nonlinear variable coefficient Boussinesq-type model for the evolution of internal waves in a two-layer system is shown. The regime considered is a shallow water configuration for the upper layer and an intermediate depth for the lower layer. The bottom has an arbitrary, not necessarily smooth nor single-valued profile generalizing the flat bottom model derived in [5]. This arbitrary topography is dealt with by performing a conformal mapping as in [16]. In the unidirectional propagation regime the model reduces to an ILW equation when a slowly varying topography is assumed. The adjustment for the periodic wave case and its computational implementation is also performed. We study the interaction of internal waves with periodic bottom profiles and the evolution of approximate solitary wave solutions. The expected qualitative behaviour is captured. We intend to use the strongly nonlinear model to study the interaction of large amplitude internal waves with multiscale topography profiles. The refocusing and stabilization of solitary waves for the large levels of nonlinearity allowed by this model is the goal of current research.

5 ACKNOWLEDGEMENTS

The first author was supported by an ANP/PRH-32 scholarship. This work was supported by CNPq/Brazil under Grant 300368/96-8.

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UM MODELO REDUZIDO PARA ONDAS INTERNAS INTERAGINDO COM ESTRUTURAS SUBMARINAS A PROFUNDIDADES INTERMEDIRIAS

No contexto da dinâmica oceânica, é obtido um modelo reduzido unidirecional fortemente não linear para a evolução de ondas internas sobre topografias de fundo arbitrário. Com o modelo reduzido busca-se obter métodos numéricos eficientes para resolver o problema bidimensional. São consideradas duas camadas contendo dois fluidos invíscidos, imiscíveis e irrotacionais de densidades diferentes. A camada superior é delgada se comparada à longitude de onda característica entanto que a profundidade da camada inferior é da mesma ordem da longitude de onda característica. As equações de evolução não lineares obtidas descrevem o comportamento da elevação da onda interna e a velocidade superior média para esta configuração da água. O sistema é uma generalização daquele proposto por Choi e Camassa para o caso de fundo plano nas mesmas condições físicas. Devido à presença da topografia, cada derivada espacial está acompanhada por um coeficiente variável. Estas equações de Boussinesq contêm a equação da Onda Longa Intermediária (*Intermediate Long Wave*, ILW) e a equação de Benjamin-Ono (BO) se restritas ao regime unidirecional de propagação de ondas. Pretendemos utilizar este modelo para estudar a interação das ondas com o perfil do fundo. A dinâmica inclui reflexão, dispersão e atenuação das ondas entre outros fenômenos. A pesquisa é de importância na recuperação de petróleo em águas profundas oceânicas onde a concentração de sal e as diferenças de temperatura geram estratificação de tal forma que as ondas internas podem afetar as operações *offshore* e as estruturas submersas. Propriedades importantes do modelo serão discutidas. Uma hierarquia de modelos unidimensionais é obtida a partir do modelo fortemente não linear ao considerar os diferentes regimes (linear, fracamente não linear, fortemente não linear) assim como os casos de fundo plano e variável. Métodos numéricos baseados no método das linhas serão descritos. Mostraremos os resultados obtidos com a implementação em Matlab, incluindo experimentos numéricos com topografia periódica e soluções de ondas viajantes.

Ondas internas, meio não homogêneo, teoria assintótica.