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A REDUCED MODEL FOR INTERNAL WAVES INTERACTING WITH SUBMARINE STRUCTURES AT GREAT DEPTH

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Abstract.

A reduced one-dimensional model for the evolution of internal waves over an arbitrary bottom topography is derived. The reduced model is aimed at obtaining an efficient numerical method for the two-dimensional problem. Two layers containing inviscid, immiscible, irrotational fluids of different densities are defined. The upper layer is shallow compared with the characteristic wavelength at the interface of the two-fluid system, while the bottom region is deeper. The non-linear evolution equations describe the behaviour of the internal wave elevation and mean upper-velocity for this water configuration. These Boussinesq-type equations contain the Intermediate Long Wave (ILW) and the Benjamin-Ono (BO) equations in the unidirectional wave regime. We intend to use this model to study the interaction of the wave with the bottom profile. The dynamics include wave dispersion, reflection and attenuation among other phenomena. The research is relevant in oil recovery in deep ocean waters, where salt concentration and differences in temperature generate stratification in such a way that internal waves can affect offshore operations and submerged structures.

Keywords: *Internal waves, Inhomogeneous media, Asymptotic theory*

1. INTRODUCTION

Modelling internal waves is of great interest in the study of ocean dynamics. These internal waves appear when salt concentration and differences in temperature generate stratification and they can interact with the bottom topography and submerged structures as well as surface waves. In particular, in oil recovery in deep ocean waters, internal waves can affect offshore operations and submerged structures. Accurate reduced models are a first step in producing efficient computational methods in engineering problems. This was the goal in Nachbin (2003); Artiles and Nachbin (2004).

To describe this non-linear wave phenomenon in deep waters there are several bidirectional models containing the Intermediate Long Wave (ILW) and the Benjamin-Ono (BO) equations, starting from works such as Benjamin (1967); Davis and Acrivos (1967); Ono (1975); Joseph (1977); Kubota et al. (1978) to more recent papers such as Matsuno (1993); Choi and Camassa (1996a,b, 1999). The aforementioned bidirectional models consider flat bottom topography. In this paper the model of Choi and Camassa is generalized to the case of an arbitrary bottom topography by using the technique described

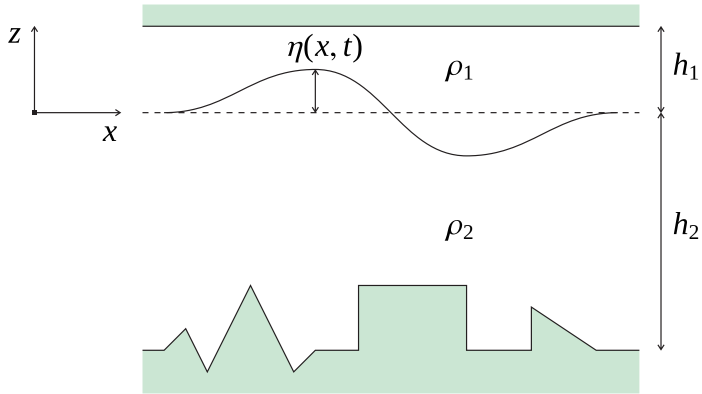

Figure 1: Two-fluid system configuration.

in Nachbin (2003). A system of two layers constrained to a region limited by a horizontal rigid lid at the top and an arbitrary bottom topography is considered, as described in Fig. 1. The upper layer is shallow compared with the characteristic wavelength at the interface of the two-fluid system, while the lower region is deeper. The non-linear evolution equations describe the behaviour of the internal wave elevation and mean upper-velocity for this water configuration. These Boussinesq-type equations contain the ILW and BO equations in the unidirectional wave regime. We intend to use this model to study the interaction of the wave with the bottom profile. The dynamics include wave dispersion, reflection and attenuation among other phenomena.

The paper is organized as follows. In Section 2 the physical setting is presented, along with a set of upper layer averaged equations that will be completed with information from the lower layer to obtain the reduced model. The continuity of pressure at the interface establishes a connection between both layers that is used to add the topography information to the averaged upper layer system. The case when the depth of the bottom topography approaches infinity is also considered. In Section 3 the dispersion relations for the linearized models are computed. In Section 4 the ILW and BO equations are obtained from the reduced model as unidirectional wave propagation models.

2. DERIVATION OF THE EQUATIONS GOVERNING THE DYNAMICS

Define the density of each inviscid, immiscible, irrotational fluid as ρ_1 for the upper fluid and ρ_2 for the lower fluid. For stable stratification, $\rho_2 > \rho_1$. Similarly, (u_i, w_i) denotes the velocity components and p_i the pressure, where $i = 1, 2$. The upper layer is assumed to have an undisturbed thickness h_1, much smaller than the characteristic wavelength of the perturbed interface $L > 0$. The bottom is described by $z = h(x) - h_2$ with roughness confined to the horizontal interval $x \in [0, \mathbf{L}]$. This means that outside this interval the bottom is flat ($h(x) = 0$) and the thickness of the undisturbed lower layer is h_2. It is assumed that $h_2 = O(L)$. The coordinate system is positioned at the undisturbed interface. The displacement of the interface is denoted by $\eta(x,t)$. See Fig. 1.

The corresponding Euler equations are

$$\begin{aligned} u_{ix} + w_{iz} &= 0, \\ u_{it} + u_i u_{ix} + w_i u_{iz} &= -\frac{p_{ix}}{\rho_i}, \\ w_{it} + u_i w_{ix} + w_i w_{iz} &= -\frac{p_{iz}}{\rho_i} - g, \end{aligned}$$

for $i = 1, 2$. Subscripts x, z and t represent partial derivatives with respect to spatial coordinates and time. The continuity condition at the interface $z = \eta(x, t)$ demands that

$$\eta_t + u_i \eta_x = w_i, \qquad p_1 = p_2.$$

At the rigid top,

$$w_1(x, h_1, t) = 0,$$

while at the irregular bottom $z = h(x) - h_2$,

$$h'(x) u_2 + w_2 = 0.$$

Introducing the dimensionless parameter $\beta = \left(\frac{h_1}{L}\right)^2$, it follows from the shallowness of the upper layer that

$$O\left(\sqrt{\beta}\right) = O\left(\frac{h_1}{L}\right) \ll 1.$$

Let $U_0 = \sqrt{g h_1}$ be the characteristic speed. According with these scalings, physical variables involved in the upper layer equations are non-dimensionalized as follows:

$$\begin{aligned} &x = L\tilde{x}, \qquad z = h_1 \tilde{z}, \qquad t = \frac{L}{U_0}\tilde{t}, \qquad \eta = h_1 \tilde{\eta}, \\ &p_1 = (\rho_1 U_0^2)\tilde{p}_1, \qquad u_1 = U_0 \tilde{u}_1, \qquad w_1 = \sqrt{\beta} U_0 \tilde{w}_1. \end{aligned}$$

2.1 Reducing the upper layer dynamics to the interface

The dimensionless equations for the upper layer (the tilde has been removed) are:

$$\begin{aligned} u_{1x} + w_{1z} &= 0, \\ u_{1t} + u_1 u_{1x} + w_1 u_{1z} &= -p_{1x}, \\ \beta \left(w_{1t} + u_1 w_{1x} + w_1 w_{1z}\right) &= -p_{1z} - 1. \end{aligned}$$

The boundary conditions are

$$\begin{aligned} &\eta_t + u_1 \eta_x = w_1 \quad \text{and} \quad p_1 = p_2 \quad \text{at} \quad z = \eta(x, t), \\ &w_1(x, 1, t) = 0. \end{aligned} \tag{1}$$

The bottom layer will be non-dimensionalized according to the scaling $z = L\tilde{z}$.

Focusing on the upper region, consider the following definition: for any function $f(x,z,t)$, let its *mean-layer quantity* $\overline{f}$ be

$$\overline{f}(x,t) = \frac{1}{1-\eta}\int_{\eta}^{1} f(x,z,t)\ dz.$$

Let $\eta_1 = 1-\eta$. From the horizontal momentum equation,

$$\eta_1\overline{u_{1t}} + \eta_1\overline{u_1u_{1x}} + \eta_1\overline{w_1u_{1z}} = -\eta_1\overline{p_{1x}}. \tag{2}$$

In Choi and Camassa (1999), the authors showed how each of these mean-layer quantities could be expressed in terms of $\overline{u_1}$ and η. The difficulty at this stage is breaking up the mean of squared and general quadratic terms. To begin with, note that

$$\begin{aligned}\frac{d}{dt}(\eta_1\overline{u_1}) &= \eta_1\left[\frac{1}{\eta_1}\int_{\eta}^{1} u_{1t}\ dz - \frac{\eta_t}{\eta_1}u_1(x,\eta,t)\right] - \frac{\eta_{1t}}{\eta_1}\int_{\eta}^{1} u_1\ dz - \eta_t\overline{u_1},\\ &= \eta_1\overline{u_{1t}} - \eta_t u_1,\end{aligned}$$

where $u_1(x,z,t)$ is evaluated at the interface $z=\eta(x,t)$. So,

$$\eta_1\overline{u_{1t}} = \frac{d}{dt}(\eta_1\overline{u_1}) + \eta_t u_1. \tag{3}$$

Similarly

$$2\eta_1\overline{u_1u_{1x}} = \eta_x u_1^2 + \left(\eta_1\overline{u_1^2}\right)_x. \tag{4}$$

Therefore at $z=\eta(x,t)$,

$$\eta_1(\overline{u_{1t}} + \overline{u_1u_{1x}}) = (\eta_1\overline{u_1})_t + u_1\eta_t + \frac{1}{2}\eta_x u_1^2 + \frac{1}{2}\left(\eta_1\overline{u_1^2}\right)_x.$$

From Eq. (1), $u_1\eta_t + \frac{1}{2}\eta_x u_1^2 = u_1w_1 - \frac{1}{2}\eta_x u_1^2$ and by substitution,

$$\eta_1(\overline{u_{1t}} + \overline{u_1u_{1x}}) = (\eta_1\overline{u_1})_t + u_1w_1 - \frac{1}{2}\eta_x u_1^2 + \frac{1}{2}\left(\eta_1\overline{u_1^2}\right)_x.$$

On the other side, integration by parts and incompressibility give

$$\eta_1\overline{w_1u_{1z}} = -w_1u_1 - \int_{\eta}^{1} w_{1z}u_1\ dz = -w_1u_1 + \int_{\eta}^{1} u_{1x}u_1\ dz.$$

From Eq. (4),

$$\eta_1\overline{w_1u_{1z}} = -w_1u_1 + \frac{1}{2}\eta_x u_1^2 + \frac{1}{2}\left(\eta_1\overline{u_1^2}\right)_x. \tag{5}$$

Replacing Eqs. (3), (4) and (5) in Eq. (2) and using Eq. (1), the following equation is derived

$$(\eta_1\overline{u_1})_t + \left(\eta_1\overline{u_1^2}\right)_x = -\eta_1\overline{p_{1x}}. \tag{6}$$

Note also that $w_1 = \overline{u_{1x}}\eta_1$ at $z = \eta(x,t)$. This, together with $(\eta_1\overline{u_1})_x = \eta_1\overline{u_{1x}} - u_1\eta_x$, shows that

$$w_1 = u_1\eta_x + (\eta_1\overline{u_1})_x,$$

so that Eq. (1) leads to $\eta_t + u_1\eta_x = u_1\eta_x + (\eta_1\overline{u_1})_x$ and

$$-\eta_{1t} = (\eta_1\overline{u_1})_x. \tag{7}$$

The quantities $\overline{u_1 \cdot u_1}$ and $\overline{p_{1x}}$ prevent the closure of the system of Eqs. (6)–(7). Those quantities will be expressed in terms of η and $\overline{u_1}$ up to a certain order in the dispersion parameter β.

The vertical momentum equation suggests the following asymptotic expansion in powers of β

$$f(x,z,t) = f^{(0)} + \beta f^{(1)} + O(\beta^2)$$

for any of the functions u_1, w_1, p_1.

Since $p_{1z} = -1 + O(\beta)$, integration from η to z leads to

$$p_1(x,z,t) - p_1(x,\eta,t) = -(z-\eta) + O(\beta),$$

and the pressure continuity across the interface gives

$$p_1(x,z,t) = p_2(x,\eta,t) - (z-\eta) + O(\beta).$$

Pressure $p_2(x,\eta,t)$ should be non-dimensionalized in the same fashion as p_1, that is,

$$p_2 = \rho_1 U_0^2 \tilde{p}_2.$$

Define $P(x,t) = p_2(x,\eta(x,t),t)$. Then

$$p_1 = P(x,t) - (z-\eta) + O(\beta),$$

which inmediately yields

$$\begin{aligned}
p_{1x} &= P_x(x,t) + \eta_x + O(\beta),\\
\overline{p_{1x}} &= \frac{1}{\eta_1}\int_\eta^1 P_x(x,t)\, dz + \eta_x + O(\beta)\\
&= P_x(x,t) + \eta_x + O(\beta)\\
&= \Big(p_2(x,\eta(x,t),t)\Big)_x + \eta_x + O(\beta).
\end{aligned}$$

An approximation for P_x will be obtained later from the Euler equations for the lower fluid layer. We now approximate the mean of squared horizontal velocity in terms of $\overline{u_1}$ and η.

In order to express $\overline{u_1 \cdot u_1}$ as a function of $\overline{u_1}$ and η, it should be pointed out that the irrotational condition in non-dimensional variables is

$$\frac{U_0}{h_1} u_{1z} = \sqrt{\beta}\frac{U_0}{L} w_{1x},$$

i. e. $u_{1z} = \beta w_{1x}$. Hence ${u_1^{(0)}}_z = 0$ and $u_1^{(0)}$ is independent from z:

$$u_1^{(0)} = u_1^{(0)}(x,t). \tag{8}$$

By using

$$u_1 = u_1^{(0)} + \beta u_1^{(1)} + O(\beta^2) \tag{9}$$

and Eq. (8) it is straightforward that

$$u_1^2 = u_1^{(0)2} + 2\beta u_1^{(1)} u_1^{(0)} + O(\beta^2),$$

$$\int_\eta^1 u_1^2 \, dz = \int_\eta^1 u_1^{(0)2} \, dz + 2\beta \int_\eta^1 u_1^{(1)} u_1^{(0)} \, dz + O(\beta^2),$$

$$\eta_1 \overline{u_1 \cdot u_1} = u_1^{(0)2}(1-\eta) + 2\eta_1 \beta u_1^{(0)} \overline{u_1^{(1)}} + O(\beta^2),$$

so that

$$\overline{u_1 \cdot u_1} = u_1^{(0)} \cdot u_1^{(0)} + 2\beta u_1^{(0)} \overline{u_1^{(1)}} + O(\beta^2).$$

From Eq. (9),

$$\frac{1}{\eta_1} \int_\eta^1 u_1 \, dz = u_1^{(0)} + \beta \overline{u_1^{(1)}} + O(\beta^2),$$

$$\overline{u_1} = u_1^{(0)} + \beta \overline{u_1^{(1)}} + O(\beta^2),$$

$$\overline{u_1} \cdot \overline{u_1} = u_1^{(0)} \cdot u_1^{(0)} + 2\beta u_1^{(0)} \overline{u_1^{(1)}} + O(\beta^2),$$

thus leading to

$$\eta_1 \overline{u_1 \cdot u_1} = \eta_1 \overline{u_1} \cdot \overline{u_1} + O(\beta^2). \tag{10}$$

Using Eq. (10), Eq. (6) becomes

$$\eta_{1t} \overline{u_1} + \eta_1 \overline{u_1}_t + \left(\eta_1 \overline{u_1} \cdot \overline{u_1} + O(\beta^2)\right)_x = -\eta_1 \overline{p_{1x}},$$

$$\eta_{1t} \overline{u_1} + \eta_1 \overline{u_1}_t + \overline{u_1}(\eta_1 \overline{u_1})_x + \eta_1 \overline{u_1} \cdot \overline{u_1}_x = -\eta_1 \overline{p_{1x}} + O(\beta^2).$$

From Eq. (7) one obtains that

$$\eta_1\overline{u_1}_t + \eta_1\overline{u_1}\cdot\overline{u_1}_x = -\eta_1\overline{p_1}_x + O(\beta^2).$$

Finally, the following set of equations for the upper layer has been deduced:

$$\eta_{1t} + (\eta_1\overline{u_1})_x = 0,$$
$$\overline{u_1}_t + \overline{u_1}\cdot\overline{u_1}_x = -\eta_x + \Big(p_2(x,\eta(x,t),t)\Big)_x + O(\beta),$$

or equivalently,

$$-\eta_t + ((1-\eta)\overline{u_1})_x = 0,$$
$$\overline{u_1}_t + \overline{u_1}\cdot\overline{u_1}_x = -\eta_x + \Big(p_2(x,\eta(x,t),t)\Big)_x + O(\beta).$$

2.2 Connecting the upper and lower layers

The coupling of the upper and lower layers is done through the pressure term. To get an approximation for $P_x(x,t) = \Big(p_2(x,\eta(x,t),t)\Big)_x$ from the Euler equations for the lower fluid layer, observe that in the deep water configuration,

$$\frac{h_2}{L} = O(1),$$

so that the following scaling relation

$$\frac{w_2}{u_2} = O\left(\frac{h_2}{L}\right) = O(1)$$

is valid as a consequence of the continuity equation. According to this, introduce the velocity potencial $\phi = \sqrt{\beta}U_0L\tilde{\phi}$ and the tilded dimensionless variables for the lower region

$$x = L\tilde{x}, \qquad z = L\tilde{z}, \qquad t = \frac{L}{U_0}\tilde{t}, \qquad \eta = h_1\tilde{\eta},$$
$$p_2 = (\rho_1U_0^2)\tilde{p}_2, \qquad u_2 = \sqrt{\beta}U_0\tilde{u}_2, \qquad w_2 = \sqrt{\beta}U_0\tilde{w}_2.$$

Note that the definition for $\tilde{z}$ is different from the one for the upper region, since the relation $O(\tilde{z}) = 1$ is needed.

In these dimensionless variables, the Bernoulli law for the interface reads

$$\sqrt{\beta}\,\phi_t + \frac{\beta}{2}(\phi_x^2 + \phi_z^2) + \eta + \frac{\rho_1}{\rho_2}P = C(t),$$

where the tilde has been ignored. $C(t)$ is an arbitrary function of time. Then, up to order β, the pressure derivative P_x is

$$P_x = -\frac{\rho_2}{\rho_1}\Big(\eta_x + \sqrt{\beta}\,\phi_{xt}(x,0,t)\Big) + O(\beta), \tag{11}$$

because

$$\left[\sqrt{\beta}\phi_t\left(x,\ \sqrt{\beta}\eta,t\right)\right]_x = \sqrt{\beta}\phi_{tx}\left(x,\ \sqrt{\beta}\eta,t\right)+\beta\phi_{tz}\left(x,\ \sqrt{\beta}\eta,t\right)\eta_x$$
$$= \sqrt{\beta}\phi_{tx}\left(x,\ \sqrt{\beta}\eta,t\right)+O(\beta)$$
$$= \sqrt{\beta}\phi_{tx}(x,0,t)+O(\beta).$$

From Eq. (11) it is clear that it is sufficient to find the linear solution for the horizontal velocity ϕ_x at $z = 0$ in order to obtain P_x at the interface. Therefore, problem

$$\begin{cases} \phi_{xx}+\phi_{zz}=0, & \text{on } -\dfrac{h_2}{L}+\dfrac{h(Lx)}{L}\le z\le \sqrt{\beta}\,\eta(x,t), \\ \phi_z=\eta_t+\sqrt{\beta}\,\eta_x\phi_x, & \text{at } z=\sqrt{\beta}\,\eta(x,t), \\ -h'(Lx)\phi_x+\phi_z=0, & \text{at } z=-\dfrac{-h_2}{L}+\dfrac{h(Lx)}{L}, \end{cases}$$

is linearized around $z = 0$ to give

$$\begin{cases} \phi_{xx}+\phi_{zz}=0, & \text{on } -\dfrac{h_2}{L}+\dfrac{h(Lx)}{L}\le z\le 0, \\ \phi_z=\eta_t, & \text{at } z=0, \\ -h'(Lx)\phi_x+\phi_z=0, & \text{at } z=-\dfrac{-h_2}{L}+\dfrac{h(Lx)}{L}. \end{cases} \tag{12}$$

In this systematic reduction we use

$$\phi_x(x,\ \sqrt{\beta}\,\eta,t)=\phi_x(x,0,t)+O\left(\sqrt{\beta}\right),$$
$$\phi(x,\ \sqrt{\beta}\,\eta,t)=\phi(x,0,t)+O\left(\sqrt{\beta}\right),$$

and so on.

To find the horizontal velocity $\phi_x(x,0,t)$ in problem (12), a conformal mapping between the lower unperturbed layer and the flat strip $\xi\in\left[0,-\frac{h_2}{L}\right]$ is performed. See Fig. 2, where $H(x)=\frac{h(Lx)}{L}$.

The problem in conformal coordinates is

$$\begin{cases} \phi_{\xi\xi}+\phi_{\zeta\zeta}=0, & \text{on } -\dfrac{h_2}{L}\le\zeta\le 0, \\ \phi_\zeta(\xi,0,t)=M(\xi)\,\eta_t(\mathrm{x}(\xi,0),t), & \text{at } \zeta=0, \\ \phi_\zeta=0, & \text{at } \zeta=-\dfrac{h_2}{L}, \end{cases} \tag{13}$$

where $M(\xi)$ is the variable free surface coefficient $\mathrm{z}_\zeta(\xi,0)$, as in Nachbin (2003).

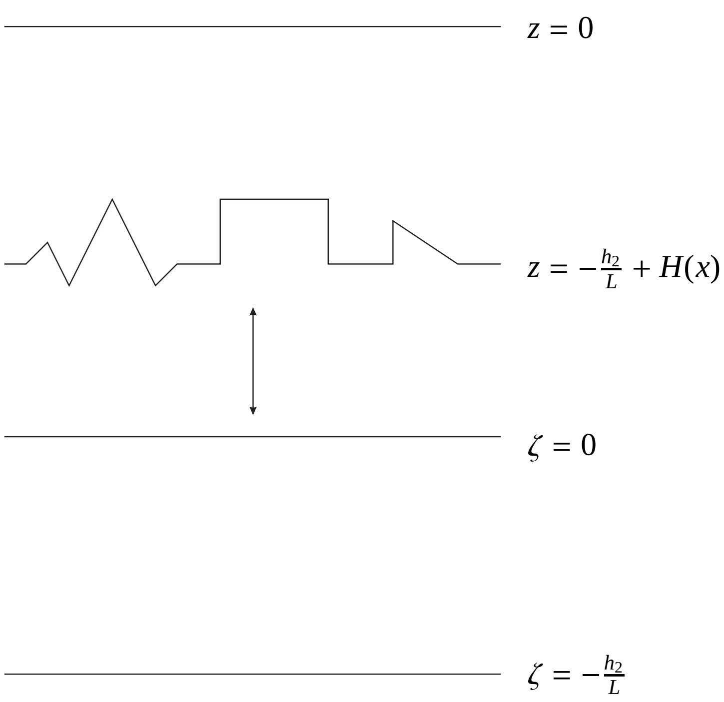

Figure 2: Conformal mapping, $(x, z) = (\mathrm{x}(\xi, \zeta), \mathrm{z}(\xi, \zeta))$.

To obtain the Neumann condition at the unperturbed interface in problem (13), consider

$$\phi_\zeta = \phi_x \mathrm{x}_\zeta + \phi_z \mathrm{z}_\zeta$$

evaluated at $z = 0$ (equivalently $\zeta = 0$) and

$$\phi_\zeta(\xi, 0, t) = \phi_x(x, 0, t)\,\mathrm{x}_\zeta(\xi, 0) + \phi_z(x, 0, t)\,\mathrm{z}_\zeta(\xi, 0).$$

Cauchy-Riemann relations and the fact that $\mathrm{z}(\xi, 0) = 0$ and $\mathrm{z}_\xi(\xi, 0) = 0$ imply that $\mathrm{x}_\zeta(\xi, 0) = 0$, which leads to the Neumann condition employed.

Since a conformal mapping was used in the coordinate transformation, $\mathrm{x}_\xi(\xi, 0) \neq 0$. From $\phi_\xi(\xi, 0, t) = \phi_x(x, 0, t)\,\mathrm{x}_\xi(\xi, 0)$, velocity $\phi_x(x, 0, t)$ is recovered as

$$\phi_x(x, 0, t) = \frac{\phi_\xi(\xi, 0, t)}{M(\xi)}.$$

Notice that $\phi_\xi(\xi, 0, t)$ is a tangential derivative on the boundary for problem (13). Hence it can be obtained as the Hilbert transform on the strip applied to the Neumann data. Namely

$$\phi_\xi(\xi, 0, t) = \mathcal{T}\left[M(\cdot)\eta_t(\mathrm{x}(\cdot, 0), t)\right](\xi),$$

where

$$\mathcal{T}[f](\xi) = \frac{L}{2h_2} \fint f(\tilde{\xi}) \coth\left(\frac{\pi L}{2h_2}(\tilde{\xi} - \xi)\right)\, d\tilde{\xi} \qquad (14)$$

is the Hilbert transform on the strip of height $\frac{h_2}{L}$. The integral must be interpreted as a principal value. The effect of the two-dimensional layer below the interface is being collapsed onto a one-dimensional integral. This integral is easily computed by FFTs leading to efficient computational schemes such as in Artiles and Nachbin (2004).

Now, $\phi_x(x,0,t)$ is also a tangential derivative on the flat upper boundary for problem (12), whose domain is a corrugated strip. Hence, it is also expressed as a Hilbert transform acting on Neumann data. Since

$$\phi_x(x,0,t) = \frac{L}{2h_2} \fint \frac{M(\tilde{\xi})}{M(\xi(x,0))}\, \eta_t(\mathrm{x}(\tilde{\xi},0),t) \coth\left(\frac{\pi L}{2h_2}(\tilde{\xi} - \xi(x,0))\right)\, d\tilde{\xi},$$

a Hilbert-like transform on the corrugated strip has been defined:

$$\mathcal{T}_c[f](x) = \frac{L}{2h_2} \fint \frac{M(\tilde{\xi})}{M(\xi(x,0))}\, f(\mathrm{x}(\tilde{\xi},0)) \coth\left(\frac{\pi L}{2h_2}(\tilde{\xi} - \xi(x,0))\right)\, d\tilde{\xi},$$

which unlike Eq. (14) is not a convolution operator.

Finally, substituting the expression for P_x in the upper layer averaged equations:

$$\begin{cases} \eta_t - [(1-\eta)\overline{u_1}]_x = 0, \\ \overline{u_1}_t + \overline{u_1}\,\overline{u_1}_x + \left(1 - \frac{\rho_2}{\rho_1}\right)\eta_x = \\ \qquad \sqrt{\beta}\frac{L}{2h_2}\frac{\rho_2}{\rho_1}\frac{1}{M(\xi(x,0))} \fint M(\tilde{\xi})\eta_{tt}(\mathrm{x}(\tilde{\xi},0),t) \coth\left(\frac{\pi L}{2h_2}(\tilde{\xi} - \xi(x,0))\right)\, d\tilde{\xi} + O(\beta). \end{cases}$$

In a compact notation this becomes

$$\begin{cases} \eta_t - [(1-\eta)\overline{u_1}]_x = 0, \\ \overline{u_1}_t + \overline{u_1}\,\overline{u_1}_x + \left(1 - \frac{\rho_2}{\rho_1}\right)\eta_x = \sqrt{\beta}\frac{\rho_2}{\rho_1}\mathcal{T}\left[\frac{M(\cdot)}{M(\xi)}\eta_{tt}(\mathrm{x}(\cdot,0),t)\right] + O(\beta). \end{cases}$$

Note that the first equation is exact. According to it $\eta_{tt} = ((1-\eta)\overline{u_1})_{xt}$ so only the first time derivative of $\overline{u_1}$ needs to enter the right-hand side of the second equation.

In conclusion, the reduced one-dimensional model (shallow water for the upper layer and intermediate depth for the lower one) is:

$$\begin{cases} \eta_t - [(1-\eta)\overline{u_1}]_x = 0, \\ \overline{u_1}_t + \overline{u_1}\,\overline{u_1}_x + \left(1 - \frac{\rho_2}{\rho_1}\right)\eta_x = \sqrt{\beta}\frac{\rho_2}{\rho_1}\mathcal{T}\left[\frac{M(\cdot)}{M(\xi)}((1-\eta)\overline{u_1})_{xt}(\mathrm{x}(\cdot,0),t)\right]. \end{cases} \tag{15}$$

This is a Boussinesq-type system for the perturbation of the interface η and the mean-layer horizontal upper velocity $\overline{u_1}$. It involves a Hilbert transform on the strip characterizing the presence of harmonic functions (or the potential flow) below the interface. Efficient computational methods can be produced for this accurate reduced model which governs, to leading order, a complex two-dimensional problem.

There is an interesting limit for this model when the lower depth tends to infinity ($h_2 \to \infty$). In this case the bottom is not seen anymore ($M(\xi) \to 1$ and $\mathrm{x}(\tilde{\xi}, 0) \to \tilde{\xi}$). Therefore

$$\phi_{xt}(x, 0, t) \to \frac{1}{\pi} ⨍ \frac{((1-\eta)\overline{u_1})_{xt}(\tilde{x}, t)}{\tilde{x} - x}\, d\tilde{x} = \mathcal{H}\Big[((1-\eta)\overline{u_1})_{xt}(\cdot, t)\Big],$$

where $\mathcal{H}$ is the usual Hilbert transform defined as

$$\mathcal{H}[f](x) = \frac{1}{\pi} ⨍ \frac{f(x')}{x' - x}\, dx'.$$

In this (shallow upper layer) infinite lower layer regime, system (15) becomes

$$\begin{cases} \eta_t - [(1-\eta)\overline{u_1}]_x = 0, \\ \overline{u_1}_t + \overline{u_1}\,\overline{u_1}_x + \left(1 - \dfrac{\rho_2}{\rho_1}\right)\eta_x = \sqrt{\beta}\dfrac{\rho_2}{\rho_1}\mathcal{H}\Big[((1-\eta)\overline{u_1})_{xt}(\cdot, t)\Big]. \end{cases} \tag{16}$$

The FFT of a Hilbert transform is easily computed. We now make a comment regarding using FFTs in numerical schemes. Both operators $\mathcal{T}[f]$ and $\mathcal{H}[f]$ have Fourier transforms of the form

$$\widehat{\mathcal{T}[f]} = i \coth\left(\frac{kh_2}{L}\right)\hat{f},$$
$$\widehat{\mathcal{H}[f]} = i\,\mathrm{sgn}(k)\hat{f},$$

where the operator's symbol multiplies the transform of f, namely $\hat{f}$. Therefore in Eqs. (15) and (16) a pseudospectral scheme would Fourier transform (namely through a FFT) the terms inside the square brackets.

Details of the computational methods available will be published elsewhere in the near future.

3. DISPERSION RELATIONS FOR THE LINEARIZED MODELS

Consider the flat bottom case in system (15), that is, $M \equiv 1$:

$$\begin{cases} \eta_t - [(1-\eta)\overline{u_1}]_x = 0, \\ \overline{u_1}_t + \overline{u_1}\,\overline{u_1}_x + \left(1 - \dfrac{\rho_2}{\rho_1}\right)\eta_x = \dfrac{\rho_2}{\rho_1}\sqrt{\beta}\mathcal{T}\Big[((1-\eta)\overline{u_1})_{xt}\Big] + O(\beta). \end{cases}$$

Its linearization around the steady state $\eta = 0$, $\overline{u_1} = 0$ gives:

$$\begin{cases} \eta_t - \overline{u_1}_x = 0, \\ \overline{u_1}_t + \left(1 - \dfrac{\rho_2}{\rho_1}\right)\eta_x = \dfrac{\rho_2}{\rho_1}\sqrt{\beta}\mathcal{T}\left[\overline{u_1}_{xt}\right]. \end{cases}$$

By differentiating once in t, η can be eliminated from the second equation:

$$\overline{u_1}_{tt} + \left(1 - \frac{\rho_2}{\rho_1}\right)\overline{u_1}_{xx} = \frac{\rho_2}{\rho_1}\sqrt{\beta}\mathcal{T}\left[\overline{u_1}_{xtt}\right].$$

Let $\overline{u_1} = Ae^{i(kx-\omega t)}$ and substituting above

$$e^{i(kx-\omega t)}\left(-\omega^2 - \left(1-\frac{\rho_2}{\rho_1}\right)k^2\right) = \frac{\rho_2}{\rho_1}\sqrt{\beta}\,k\omega^2\,e^{-i\omega t}\,\mathcal{T}\left[-ie^{ikx}\right].$$

Since $\mathcal{T}[e^{ikx}] = i\coth\left(\frac{kh_2}{L}\right)e^{ikx}$,

$$\omega^2 = \frac{\left(\frac{\rho_2}{\rho_1}-1\right)k^2}{1+\frac{\rho_2}{\rho_1}k\sqrt{\beta}\coth\left(\frac{kh_2}{L}\right)}$$

which is the correct limit for the full dispersion relation when $kh_1 \to 0$. Observe that as $\frac{\omega^2}{k^2} \to 0$, bounded phase velocities are obtained as k becomes large.

In the limit $h_2 \to \infty$ the operator $\mathcal{T}$ becomes $\mathcal{H}$. Since $\mathcal{H}[e^{ikx}] = i\,\mathrm{sgn}(k)e^{ikx}$, the dispersion relation for system (16) is

$$\omega^2 = \frac{k^2\left(\frac{\rho_2}{\rho_1}-1\right)}{1+\frac{\rho_2}{\rho_1}\sqrt{\beta}\,|k|}$$

and $\frac{\omega^2}{k^2} \to 0$ as $k \to \infty$.

The computational methods will capture these dispersion relations since the spatial discretizations are highly accurate through the use of FFTs.

4. UNIDIRECTIONAL WAVE REGIME

For weakly nonlinear unidirectional waves and slowly varying topography, our model reduces to the ILW equation and the BO equation.

Consider again system (15). Set $\eta = \alpha\eta^*$, $\overline{u_1} = \alpha c_0\overline{u_1}^*$, $t = \frac{t^*}{c_0}$ where $c_0^2 = \left(\frac{\rho_2}{\rho_1}-1\right)$ and $\alpha = O\left(\sqrt{\beta}\right)$. Depending on the root c_0 chosen, there will be a right- or left-travelling wave.

Then

$$\begin{cases} \eta_t - [(1-\alpha\eta)\overline{u_1}]_x = 0, \\ \overline{u_1}_t + \alpha\overline{u_1}\,\overline{u_1}_x - \eta_x = \sqrt{\beta}\frac{\rho_2}{\rho_1}\mathcal{T}\left[\frac{M(\cdot)}{M(\xi)}[(1-\alpha\eta)\overline{u_1}]_{xt}\right] + O(\beta). \end{cases} \tag{17}$$

Note that

$$\eta_t = \overline{u_1}_x + O\left(\alpha,\ \sqrt{\beta}\right); \qquad \eta_x = \overline{u_1}_t + O\left(\alpha,\ \sqrt{\beta}\right).$$

Look for a solution, corrected to first order in α and $\sqrt{\beta}$, in the form

$$\eta = A_1\overline{u_1} + \alpha A_2\overline{u_1}^2 + \sqrt{\beta}A_3\mathcal{T}\left[\frac{M(\cdot)}{M(\xi)}\overline{u_1}_t\right].$$

Substituting in the system of Eqs. (17) up to order α, $\sqrt{\beta}$, two equations for $\overline{u_1}$ are obtained:

$$0 = A_1\overline{u_1}_t + 2\alpha A_2\overline{u_1}\,\overline{u_1}_t + 2\alpha A_1\overline{u_1}\,\overline{u_1}_x - \overline{u_1}_x + \sqrt{\beta}A_3\mathcal{T}\left[\frac{M(\cdot)}{M(\xi)}\overline{u_1}_{tt}\right]$$

and

$$0 = \overline{u_1}_t + \alpha \overline{u_1}\,\overline{u_1}_x - \left[A_1 \overline{u_1}_x + 2\alpha A_2 \overline{u_1}\,\overline{u_1}_x + \sqrt{\beta} A_3 \mathcal{T}\left[\frac{M(\cdot)}{M(\xi)} \overline{u_1}_{xt} \right] \right] - \\ - \sqrt{\beta}\frac{\rho_2}{\rho_1} \mathcal{T}\left[\frac{M(\cdot)}{M(\xi)} \overline{u_1}_{xt} \right] + O\left(\alpha^2, \beta, \alpha\sqrt{\beta}\right).$$

These two equations are consistent if $A_1 = 1$, $A_2 = -\frac{1}{4}$ and $A_3 = -\frac{\rho_2}{2\rho_1}$. Therefore, the evolution equation for $\overline{u_1}$ is

$$\overline{u_1}_t + \frac{3}{2}\alpha \overline{u_1}\,\overline{u_1}_x - \overline{u_1}_x - \frac{\rho_2}{\rho_1}\frac{\sqrt{\beta}}{2} \mathcal{T}\left[\frac{M(\cdot)}{M(\xi)} \overline{u_1}_{xt} \right] = O\left(\beta, \alpha^2, \alpha\sqrt{\beta}\right).$$

For the elevation of the interface η a similar equation can be obtained through asymptotic relations which permit (to leading order) to exchange derivatives in η for derivatives in $\overline{u_1}$, as well as time derivatives for spatial derivatives. The evolution equation for the elevation of the interface is

$$\eta_t - \eta_x + \frac{3}{2}\alpha\eta\eta_x - \frac{\rho_2}{\rho_1}\frac{\sqrt{\beta}}{2} \mathcal{T}\left[\frac{M(\cdot)}{M(\xi)} \eta_{xt} \right] = O\left(\beta, \alpha^2, \alpha\sqrt{\beta}\right).$$

This is a BO-type equation called ILW equation. Instead of the usual Hilbert transform on the half-space, a Hilbert transform on the strip appears. The dispersion relation is

$$\omega = \frac{k}{1 - \frac{\rho_2}{\rho_1}\frac{\sqrt{\beta}}{2} k \coth\left(\frac{kh_2}{L}\right)}.$$

Mathematical details will be provided elsewhere to explain why the x-derivatives can be exchanged with the operator $\mathcal{T}_c$. We only remark here that there is no need to specify the slow parameter ε in the equations above if we establish a relation between α, β and ε of type $\varepsilon^2 = O\left(\sqrt{\beta}\right)$.

For system (16) a similar reduction is obtained

$$\eta_t - \eta_x + \frac{3}{2}\alpha\eta\eta_x - \frac{\rho_2}{\rho_1}\frac{\sqrt{\beta}}{2} \mathcal{H}[\eta_{xt}] = 0,$$

which is a regularized BO equation.

CONCLUSIONS

A one-dimensional Boussinesq-type model for the evolution of internal waves in a two-layer system is derived. The regime considered is a shallow water configuration for the upper layer and an intermediate or infinite depth for the lower layer. The bottom has an arbitrary, not necessarily smooth, profile generalizing the flat bottom model derived by Choi and Camassa (1999). This arbitrary topography is dealt with by performing a conformal mapping as in Nachbin (2003), which leads to a Hilbert-like transform on the corrugated strip. In the unidirectional propagation regime the model reduces to a generalized BO equation when a slow varying topography is assumed.

We intend to use this model to study the interaction of internal waves with different types of bottom profiles such as submerged structures and multiscale topography profiles. Stability analysis for the hyperbolic and weakly dispersive regimes is to be performed.

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REFERENCES

Artiles, W. & Nachbin, A., 2004. Nonlinear evolution of surface gravity waves over highly variable depth. *Physical Review Letters*, vol. 93, n. 23, pp. 234501–1–234501–4.

Benjamin, T. B., 1967. Internal waves of permanent form of great depth. *Journal of Fluid Mechanics*, vol. 29, pp. 559–592.

Choi, W. & Camassa, R., 1996a. Long internal waves of finite amplitude. *Physical Review Letters*, vol. 77, pp. 1759–1762.

Choi, W. & Camassa, R., 1996b. Weakly nonlinear internal waves in a two-fluid system. *Journal of Fluid Mechanics*, vol. 313, pp. 83–103.

Choi, W. & Camassa, R., 1999. Fully nonlinear internal waves in a two-fluid system. *Journal of Fluid Mechanics*, vol. 396, pp. 1–36.

Davis, R. E. & Acrivos, A., 1967. Solitary internal waves in deep water. *Journal of Fluid Mechanics*, vol. 29, pp. 593–607.

Joseph, R. I., 1977. Solitary waves in finite depth fluid. *Journal of Physics* A, vol. 10, pp. L225–L227.

Kubota, T., Ko, D., & Dobbs, L., 1978. Propagation of weakly non linear internal waves in a stratified fluid of finite depth. *AIAA Journal Hydrodynamics*, vol. 11, pp. 157–165.

Matsuno, Y., 1993. A unified theory of nonlinear wave propagation in two-fluid systems. *Journal of the Physical Society of Japan*, vol. 62, pp. 1902–1916.

Nachbin, A., 2003. A terrain-following Boussinesq system. *SIAM Journal on Applied Mathematics*, vol. 63, pp. 905–922.

Ono, H., 1975. Algebraic solitary waves in stratified fluids. *Journal of the Physical Society of Japan*, vol. 39, pp. 1082–1091.