NEUMANN-NEUMANN METHODS FOR A DG DISCRETIZATION OF ELLIPTIC PROBLEMS WITH DISCONTINUOUS COEFFICIENTS ON GEOMETRICALLY NONCONFORMING SUBSTRUCTURES

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Abstract. A discontinuous Galerkin discretization for second order elliptic equations with *discontinuous coefficients* in 2-D is considered. The domain of interest Ω is assumed to be a union of polygonal substructures Ω_i of size $O(H_i)$. We allow this substructure decomposition to be geometrically nonconforming. Inside each substructure Ω_i, a conforming finite element space associated to a triangulation $\mathcal{T}_{h_i}(\Omega_i)$ is introduced. To handle the nonmatching meshes across $\partial\Omega_i$, a discontinuous Galerkin discretization is considered. In this paper additive and hybrid Neumann-Neumann Schwarz methods are designed and analyzed. Under natural assumptions on the coefficients and on the mesh sizes across $\partial\Omega_i$, a condition number estimate $C(1 + \max_i \log \frac{H_i}{h_i})^2$ is established with C independent of h_i, H_i, h_i/h_j, and the jumps of the coefficients. The method is well suited for parallel computations and can be straightforwardly extended to three dimensional problems. Numerical results are included.

1. Introduction

In this paper a *discontinuous Galerkin* (DG) approximation of elliptic problems with *discontinuous coefficients* is considered [8]. See [4] and references therein for an overview on local DG discretizations. The problem is considered in a polygonal region Ω which is a *geometrically nonconforming* union of disjoint polygonal substructures Ω_i, $i = 1, \dots, N$. The discontinuities of the coefficients are assumed to occur only across the interfaces of the substructures $\partial\Omega_i$. Inside each substructure Ω_i, a conforming finite element method is introduced to discretize the local problem, and is allowed *nonmatching triangulations* to occur across the $\partial\Omega_i$. This kind of composite discretization is motivated by the location of the discontinuities of the coefficients and by the regularity of the solution of the problem. The discrete problem is formulated using a symmetric DG method with interior penalty (IPDG) terms on $\partial\Omega_i$. To deal with the discontinuities of the coefficients across the substructure interfaces, *harmonic averages* of the coefficients are considered on

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these interfaces; see [8]. The consistency of this discretization is given in [9] while an optimal a priori error estimate is established in [8]; see also Lemma 2.2 below. IPDG methods based on harmonic averages of the coefficients were also considered for advection-diffusion-reaction problems [6] to obtain stable discretizations.

The main goal of this paper is to design and analyze additive and hybrid Neumann - Neumann algorithms for the resulting DG-discrete problem. This type of algorithms is well established for standard conforming and nonconforming discretizations [14, 26, 27, 30, 29, 20], however, no enough attention were payed to DG discretization. We note that other types of preconditioners were considered for solving discrete IPDG problems. In connection with two-level domain decomposition preconditioners, we mention [15, 16, 23, 5, 1, 2, 7, 25], where small and generous overlapping Schwarz methods were considered for DG discretizations. In connection with multilevel preconditioners for DG problems, we mention [17, 19, 24, 22, 21]. These papers focus on the scalability of the preconditioners with respect to mesh parameters, however, only few discussions were considered on the robustness with respect to jumps of the coefficients across the substructuring interfaces. For classical conforming and nonconforming discretizations, it is known that, in two dimensions, domain decomposition and multilevel methods may lead to robust preconditioners with respect to jumps of the coefficients; see [30]. In three dimensions, however, the robustness of these methods can be achieved only in special circumstances such as when every subdomain touches part of the Dirichlet boundary or when only few cross points do not satisfy the quasi-monotonicity condition on the jumps of the coefficients; see [12, 31, 18]. For more general discontinuous coefficients, the robustness of these methods can be achieved when coarse problems based on discrete harmonic extensions are introduced; see [14, 12, 14, 27, 29, 13, 28]. The same robustness issues also occur for DG discretizations, hence, the notion of discrete harmonic extension in the DG sense was also introduced in the Technical Report [11] in order to design robust N-N algorithms; see also [10] for numerical experiments. We point out that only the geometrically conforming case was treated in these works. Here in this paper we extend these results to the *geometrically non-conforming case* and introduce new N-N coarse spaces and solvers. We note that, using the techniques developed in this paper, we can extend the Balancing Domain Decomposition by Constraints (BDDC) methods for DG discretizations [9] to the geometrically nonconforming case.

The problem is reduced to the Schur complement form with respect to unknowns on $\partial\Omega_i$, for $i = 1, \ldots, N$. Discrete harmonic functions defined in a special way are used in this step. The methods are designed and analyzed for the Schur complement problem using the general theory of N-N methods; see [30]. The local problems are defined on Ω_i and faces or part of faces of $\partial\Omega_j$ which are common to Ω_i. The coarse space is defined using a special partitioning of unity with respect to the subdomains Ω_i and introducing master and slave sides of local interfaces between substructures. Recall that we work with a geometrically nonconforming partition of Ω into substructures Ω_i, $i = 1, \ldots, N$. A (part of) face $F_{ij} = \partial\Omega_i \cap \partial\Omega_j$ is a master when $\rho_i \geq C\rho_j$, otherwise it is a slave, so if $F_{ij} \subset \partial\Omega_i$ is a master then $F_{ji} \subset \partial\Omega_j$, $F_{ij} = F_{ji}$, is a slave. The h_i-triangulation on F_{ij} and h_j-triangulation on F_{ji} are built in a way that $h_i \geq Ch_j$ if $\rho_i \geq C\rho_j$. Here h_i and h_j are the parameters of the triangulation in Ω_i and Ω_j, respectively, and C is a generic constant of $O(1)$. We prove that the algorithms are almost optimal and their rate of convergence is independent of the mesh parameters, the number of subdomains Ω_i and the jumps of coefficients. The algorithms are well suited for parallel computations and they can be straightforwardly extended to three-dimensional problems.

The paper is organized as follows. In Section 2 the differential problem and its DG discretization are formulated. In Section 2.3 the Schur complement problem is derived using discrete harmonic functions in an special way. Some technical tools are introduced in Section 3. Section 4 is dedicated to introduce important notations and the *interface condition* on the coefficients and the parameters steps, see Assumption 4.1. Two additive Neumann-Neumann Schwarz preconditioners, one based on a small coarse space and another based on a larger coarse space, are defined and analyzed in Section 5. In Section 6 we present the Balancing Domain Decomposition versions. Finally, in Section 7 some numerical experiments are presented which confirm the theoretical results. The numerical results show that the introduced Assumption 4.1 is necessary and sufficient.

2. Differential and discrete problems

In this section we study in detail properties of the discrete problem.

2.1. **Differential problem.** Consider the following problem: Find $u^* \in H_0^1(\Omega)$ such that

$$a(u^*, v) = f(v) \quad \text{for all } v \in H_0^1(\Omega) \tag{2.1}$$

where

$$a(u,v) := \sum_{i=1}^{N} \int_{\Omega_i} \rho_i \nabla u \cdot \nabla v dx \text{ and } f(v) := \int_{\Omega} f v dx.$$

Here, $\overline{\Omega} = \cup_{i=1}^N \overline{\Omega}_i$ where the substructures Ω_i are disjoint regular polygonal subregions of diameter $O(H_i)$. We assume the substructures Ω_i form a geometrically nonconforming partition of Ω, therefore, for all $i \neq j$ the intersection $\partial\Omega_i \cap \partial\Omega_j$ is empty, a vertex of Ω_i and/or Ω_j, or a common face or part of a face of $\partial\Omega_i$ and $\partial\Omega_j$. In case the intersection is empty or a common vertex Ω_i and Ω_j, or a common face of Ω_i and Ω_j, we say that a partition is geometrically conforming. For simplicity of presentation we assume that the right-hand side $f \in L^2(\Omega)$ and the coefficients ρ_i are all positive constants.

2.2. **Discrete problem.** In each Ω_i we introduce a shape regular triangulation $\mathcal{T}_i(\Omega_i)$ in each Ω_i with triangular elements and mesh parameter h_i. The resulting triangulation on Ω is in general nonmatching across $\partial\Omega_i$. We introduce $X_i(\Omega_i)$ to be the regular finite element (FE) space of piecewise linear and continuous functions in $\mathcal{T}_i(\Omega_i)$. We do not assume that functions in $X_i(\Omega_i)$ vanish on $\partial\Omega_i \cap \partial\Omega$. We define

$$X_h(\Omega) := X_1(\Omega_1) \times \cdots \times X_N(\Omega_N)$$

and represent functions v of $X_h(\Omega)$ as $v = \{v_i\}_{i=1}^N$ with $v_i \in X_i(\Omega_i)$.

The discrete problem obtained by the DG method, see [4, 8], is of the form: Find $u_h^* \in X_h(\Omega)$ such that

$$a_h(u_h^*, v_h) = f(v_h) \quad \text{for all } v_h \in X_h(\Omega) \tag{2.2}$$

where

$$a_h(u,v) = \sum_{i=1}^{N} \hat{a}_i(u,v) \text{ and } f(v) = \sum_{i=1}^{N} \int_{\Omega_i} f v_i dx. \tag{2.3}$$

Each bilinear form $\hat{a}_i$ is given as a sum of three bilinear forms:

$$\hat{a}_i(u,v) := a_i(u,v) + s_i(u,v) + p_i(u,v), \tag{2.4}$$

where

$$a_i(u,v) := \int_{\Omega_i} \rho_i \nabla u_i \nabla v_i dx, \tag{2.5}$$

$$s_i(u,v) := \sum_{F_{ij} \subset \partial\Omega_i} \int_{F_{ij}} \frac{\rho_{ij}}{l_{ij}} \left(\frac{\partial u_i}{\partial n_i}(v_j - v_i) + \frac{\partial v_i}{\partial n_i}(u_j - u_i) \right) ds$$

and

$$p_i(u,v) := \sum_{F_{ij} \subset \partial\Omega_i} \int_{F_{ij}} \frac{\rho_{ij}}{l_{ij}} \frac{\delta}{h_{ij}} (u_j - u_i)(v_j - v_i) ds. \tag{2.6}$$

Here, the bilinear form p_i is called the penalty term with a positive penalty parameter δ. In the above equations, we set $l_{ij} = 2$ when $F_{ij} = \partial\Omega_i \cap \partial\Omega_j$ is a common face (or part of a face) of $\partial\Omega_i$ and $\partial\Omega_j$, and define $\rho_{ij} := 2\rho_i\rho_j/(\rho_i + \rho_j)$ as the harmonic average of ρ_i and ρ_j, and $h_{ij} := 2h_ih_j/(h_i + h_j)$. In order to simplify notations we include the index $j = \partial$ when $F_{i\partial} := \partial\Omega_i \cap \partial\Omega$ is a face of $\partial\Omega_i$ and set $l_{i\partial} := 1$ and let $v_\partial = 0$ for all $v \in X_h(\Omega)$, and define $\rho_{i\partial} := \rho_i$ and $h_{i\partial} := h_i$. The outward normal derivative on $\partial\Omega_i$ is denoted by $\frac{\partial}{\partial n_i}$. We note that when ρ_{ij} is given by the harmonic average then $\min\{\rho_i, \rho_j\} \le \rho_{ij} \le 2\min\{\rho_i, \rho_j\}$.

We also define the positive bilinear forms d_i as

$$d_i(u,v) := a_i(u,v) + p_i(u,v), \tag{2.7}$$

and the broken bilinear form d_h for $X_h(\Omega)$ with weights given by ρ_i and $\frac{\delta}{l_{ij}}\frac{\rho_{ij}}{h_{ij}}$ by

$$d_h(u,v) := \sum_{i=1}^{N} d_i(u,v). \tag{2.8}$$

For $u = \{u_i\}_{i=1}^N \in X_h(\Omega)$ the associated broken norm is then defined by

$$\|u\|_h^2 := d_h(u,u) = \sum_{i=1}^{N} \{\rho_i \parallel \nabla u_i \parallel_{L^2(\Omega_i)}^2 + \sum_{F_{ij} \subset \partial\Omega_i} \frac{\delta}{l_{ij}} \frac{\rho_{ij}}{h_{ij}} \int_{F_{ij}} (u_i - u_j)^2 ds\}. \tag{2.9}$$

It is known that there exist constants $\delta_0 = O(1) > 0$ and $0 < c < 1$ such that for $\delta \ge \delta_0$, we have $|s_i(u,u)| < c d_i(u,u)$ and $\sum_i s_i(u,u) < c d_h(u,u)$, and the lemma follows:

Lemma 2.1. *There exists $\delta_0 > 0$ such that for $\delta \geq \delta_0$ and for all $u \in X_h(\Omega)$ the following inequalities hold:*

$$\gamma_0 d_i(u,u) \leq \hat{a}_i(u,u) \leq \gamma_1 d_i(u,u), \; i = 1, \ldots, N, \tag{2.10}$$

and

$$\gamma_0 d_h(u,u) \leq a_h(u,u) \leq \gamma_1 d_h(u,u) \tag{2.11}$$

where γ_0 and γ_1 are positive constants independent of the ρ_i, h_i and H_i.

For the proof we refer to [8] or [9]. This result implies that the problem (2.2) is elliptic and has a unique solution.

A priori error estimates for the method are optimal for constant coefficients, and also for the case where h_i and h_j are of the same order; see [3, 4]. For discontinuous coefficients ρ_i and/or for h_i and h_j are not on the same order, we have the following Lemma 2.2. For the proof, see Theorem 4.2 of [8] and Lemma 2.2 of [9].

Lemma 2.2. *Let u^* and u_h^* be the solutions of (2.1) and (2.2). For $u^* \in H_0^1(\Omega)$ and $u^*|_{\Omega_i} \in H^2(\Omega_i)$, $i = 1, \ldots, N$, we have*

$$\|u^* - u_h^*\|_h^2 \leq C \sum_{i=1}^{N} \left(h_i^2 + \sum_{F_{ij} \subset \partial\Omega_i} \frac{h_j^3}{h_i} \right) \rho_i |u^*|_{H^2(\Omega_i)}^2$$

where C is independent of h_i, H_i and ρ_i.

2.3. **Schur complement problem.** In this subsection we derive the Schur complement bilinear form for the problem (2.2). We first introduce auxiliary notations.

Define $X_i^\circ(\Omega_i)$ as the subspace of $X_i(\Omega_i)$ of functions that vanish on $\partial\Omega_i$. A function $u_i \in X_i(\Omega)$ can be represented as

$$u_i = \mathcal{H}_i u_i + \mathcal{P}_i u_i \tag{2.12}$$

where $\mathcal{H}_i u_i$ is the discrete harmonic part of u_i in the sense of $a_i(.,.)$, see (2.5), i.e.,

$$\begin{cases} a_i(\mathcal{H}_i u_i, v_i) = & 0 \quad \text{for all } v_i \in X_i^\circ(\Omega_i) \\ \mathcal{H}_i u_i = & u_i \quad \text{on} \quad \partial\Omega_i, \end{cases} \tag{2.13}$$

while $\mathcal{P}_i u_i$ is the projection of u_i into $X_i^\circ(\Omega_i)$ in the sense of $a_i(.,.)$, i.e.,

$$a_i(\mathcal{P}_i u_i, v_i) = a_i(u_i, v_i) \quad \text{for all } v_i \in X_i^\circ(\Omega_i). \tag{2.14}$$

Note that $\mathcal{H}_i u_i$ is the classical discrete harmonic part of u_i. Let us denote by $X_h^\circ(\Omega)$ the subspace of $X_h(\Omega)$ defined by $X_h^\circ(\Omega) := X_1^\circ(\Omega_1) \times \cdots \times X_N^\circ(\Omega_N)$ and consider the global projections $\mathcal{H}u := \{\mathcal{H}_i u_i\}_{i=1}^N$ and $\mathcal{P}u := \{\mathcal{P}_i u_i\}_{i=1}^N : X_h(\Omega) \rightarrow X_h^\circ(\Omega)$ in the sense of $\sum_{i=1}^N a_i(\cdot,\cdot)$. Hence, a function $u \in X_h(\Omega)$ can then be decomposed as

$$u = \mathcal{H}u + \mathcal{P}u. \tag{2.15}$$

Alternatively to (2.15), a function $u \in X_h(\Omega)$ can be represented as

$$u = \hat{\mathcal{H}}u + \hat{\mathcal{P}}u, \tag{2.16}$$

where $\hat{\mathcal{P}}u = \{\hat{\mathcal{P}}_i u_i\}_{i=1}^N : X_h(\Omega) \rightarrow X_h^\circ(\Omega)$ is the projection in the sense of the original bilinear for $a_h(\cdot,\cdot)$, see (2.3), while $\hat{\mathcal{H}}u = \{\hat{\mathcal{H}}_i u\}_{i=1}^N \in X_h(\Omega)$ where $\hat{\mathcal{H}}_i u$ is the discrete harmonic part of u in the sense of $\hat{a}_i(.,.)$ defined in (2.4), i.e., $\hat{\mathcal{H}}_i u \in X_i(\Omega_i)$ is the solution of

$$\left\{\begin{array}{lll} \hat{a}_i(\hat{\mathcal{H}}_i u, v_i) & = 0 & \text{for all } v_i \in X_i^\circ(\Omega_i), \\ \hat{\mathcal{H}}_i u & = u_i & \text{on } \partial\Omega_i \\ \hat{\mathcal{H}}_i u & = u_j & \text{on every (part of) face } F_{ji} \subset \partial\Omega_j. \end{array}\right. \tag{2.17}$$

Here the index j in the last equation of (2.17) runs over all Ω_j and $j = \partial$ such that $\overline{\Omega}_i \cap \overline{\Omega}_j$ and $\overline{\Omega}_i \cap \overline{\Omega}$ has nonzero measure, respectively. In the latter case, recall that $u_\partial = 0$.

Observe that since $\hat{\mathcal{P}}_i u_i \in X_i^\circ(\Omega_i)$ we have that for all $v_i \in X_i^\circ(\Omega_i)$,

$$a_i(\hat{\mathcal{P}}_i u, v_i) = a_h(u, E_i v_i),$$

where E_i is the standard discrete zero extension operator, i.e., $E_i v_i := \{v_j\}_{j=1}^N$, where v_j vanishes for $j \neq i$; see also Section 4 for the definition of others zero extension operators I_i and $\tilde{I}_i$.

The discrete solution of (2.2) can be decomposed as $u_h^* = \hat{\mathcal{H}} u_h^* + \hat{\mathcal{P}} u_h^*$. To compute the projection $\hat{\mathcal{P}} u_h^*$ we need to solve the following set of standard discrete Dirichlet problems:

$$a_i(\hat{\mathcal{P}}_i u_h^*, v_i) = f(E_i v_i) \qquad \text{for all } v_i \in X_i^\circ(\Omega_i). \tag{2.18}$$

Note that these problems, for $i = 1, \ldots N$, are local and independent, and so, they can be solved in parallel. This is a precomputational step.

We next formulate the problem for $\hat{\mathcal{H}} u_h^*$. We first point out that for $v_i \in X_i^\circ(\Omega_i)$ we have

$$\hat{a}_i(u_i, v_i) = (\rho_i \nabla u_i, \nabla v_i)_{L^2(\Omega_i)} + \sum_{F_{ij} \subset \partial\Omega_i} \frac{\rho_{ij}}{l_{ij}} (\frac{\partial v_i}{\partial n}, u_j - u_i)_{L^2(F_{ij})}. \tag{2.19}$$

Note that (2.17) has a unique solution. To see this, let us rewrite (2.17) in the form

$$\rho_i (\nabla \hat{\mathcal{H}}_i u, \nabla \varphi_k^i)_{L^2(\Omega_i)} = - \sum_{F_{ij} \subset \partial\Omega_i} \frac{\rho_{ij}}{l_{ij}} (\frac{\partial \varphi_k^i}{\partial n}, u_j - u_i)_{L^2(F_{ij})} \tag{2.20}$$

where φ_k^i is the nodal basis function of $X_i^\circ(\Omega_i)$ associated with any interior nodal point x_k of the h_i-triangulation of Ω_i. The normal derivative $\frac{\partial \varphi_k^i}{\partial n}$ does not vanish on $\partial\Omega_i$ when x_k is a node of an element of the triangulation $\mathcal{T}_i(\Omega_i)$ touching $\partial\Omega_i$. We see that $\hat{\mathcal{H}}_i u$ is a special extension into Ω_i where u is given on $\partial\Omega_i$ and on all (part of) faces F_{ji}. Therefore, $\hat{\mathcal{H}}_i u$ depends not only on the values of u_i on $\partial\Omega_i$ but also on the values of u_j given on $F_{ji} = \partial\Omega_i \cap \partial\Omega_j$ and on $F_{\partial i}$ (we already have assumed $u_\partial = 0$). Note that $\hat{\mathcal{H}}_i u$ is discrete harmonic except at nodal points close to $\partial\Omega_i$. We will sometimes call $\hat{\mathcal{H}}_i u$ as the discrete harmonic in a special sense, i.e., in the sense of $\hat{a}_i(\cdot,\cdot)$.

Observe that (2.17) for $u \in X_h(\Omega)$ is obtained from

$$a_h(\hat{\mathcal{H}} u, v) = 0 \tag{2.21}$$

when taking $v = \{v_i\}_{i=1}^N \in X_h^\circ(\Omega)$. It is easy to see that $\hat{\mathcal{H}}u = \{\hat{\mathcal{H}}_i u\}_{i=1}^N$ and $\hat{\mathcal{P}}u = \{\hat{\mathcal{P}}_i u_i\}_{i=1}^N$ are orthogonal in the sense of $a_h(.,.)$, i.e.,

$$a_h(\hat{\mathcal{H}}u, \hat{\mathcal{P}}v) = 0, \qquad u, v \in X^h(\Omega). \tag{2.22}$$

In addition,

$$\mathcal{H}\hat{\mathcal{H}}u = \mathcal{H}u \ \text{ and } \ \hat{\mathcal{H}}\mathcal{H}u = \hat{\mathcal{H}}u \tag{2.23}$$

since neither $\hat{\mathcal{H}}u$ nor $\mathcal{H}u$ changes the values of u at the nodes on the boundaries of the subdomains Ω_i; see (2.13) and (2.17).

Define

$$\Gamma_h := (\cup_i \partial\Omega_{ih_i}), \tag{2.24}$$

where $\partial\Omega_{ih_i}$ is the set of nodal points of $\partial\Omega_i$. We note that the definition of Γ_h includes the nodes on both triangulations of $\cup_i \partial\Omega_i$.

We are now in a position to derive the Schur complement problem for (2.2). Applying the decomposition (2.16) in (2.2) we obtain

$$a_h(\hat{\mathcal{H}}u_h^* + \hat{\mathcal{P}}u_h^*, \hat{\mathcal{H}}v_h + \hat{\mathcal{P}}v_h) = f(\hat{\mathcal{H}}v_h + \hat{\mathcal{P}}v_h)$$

or

$$a_h(\hat{\mathcal{H}}u_h^*, \hat{\mathcal{H}}v_h) + 2a_h(\hat{\mathcal{H}}u_h^*, \hat{\mathcal{P}}v_h) + a_h(\hat{\mathcal{P}}u_h^*, \hat{\mathcal{P}}v_h) = f(\hat{\mathcal{H}}v_h) + f(\hat{\mathcal{P}}v_h).$$

Using (2.18) and (2.21) we have

$$a_h(\hat{\mathcal{H}}u_h^*, \hat{\mathcal{H}}v_h) = f(\hat{\mathcal{H}}v_h) \qquad \text{for all } v_h \in X_h(\Omega). \tag{2.25}$$

This is the Schur complement problem for (2.2). We denote by V the set of all functions v_h in $X_h(\Omega)$ such that $v_h \equiv \hat{\mathcal{H}}v_h$, i.e., the space of discrete harmonic functions in the sense of the $\hat{\mathcal{H}}$. We rewrite the Schur complement problem as follows: Find $u_h^* \in V$ such that

$$\mathcal{S}(u_h^*, v_h) = g(v_h) \qquad \text{for all } v_h \in V \tag{2.26}$$

where, here and below, $u_h^* \equiv \hat{\mathcal{H}}u_h^*$ and

$$\mathcal{S}(u_h, v_h) := a_h(\hat{\mathcal{H}}u_h, \hat{\mathcal{H}}v_h) \ \text{ and } \ g(v_h) := f(\hat{\mathcal{H}}v_h). \tag{2.27}$$

The Schur complement problem (2.26) has a unique solution.

3. Technical tools

Our main goal is to design and analyze Neumann-Neumann (N-N) methods for solving (2.26). This will be done in the next sections. We now introduce some notations and facts to be used later. Let $u = \{u_i\}_{i=1}^N \in X_h(\Omega)$ and consider $d_i(\cdot,\cdot)$ and $d_h(\cdot,\cdot)$, the bilinear forms defined in (2.7) and (2.8), respectively. First note that for $u \in X_h(\Omega)$, Lemma 2.1 states that

$$\gamma_0 d_h(u,u) \le a_h(u,u) \le \gamma_1 d_h(u,u), \tag{3.1}$$

where γ_0 and γ_1 are positive constants independent of h_i, H_i and ρ_i. Additionally, the following lemma shows the equivalence between the discrete harmonic functions in the sense of $\mathcal{H}$ and in the sense of $\hat{\mathcal{H}}$. For the proof of the following lemma we refer to [9].

Lemma 3.1. *For $u \in X_h(\Omega)$ we have*

$$d_i(\mathcal{H}_i u, \mathcal{H}_i u) \leq d_i(\hat{\mathcal{H}}_i u, \hat{\mathcal{H}}_i u) \leq C d_i(\mathcal{H}_i u, \mathcal{H}_i u), \; i = 1, \dots, N, \tag{3.2}$$

and

$$d_h(\mathcal{H} u, \mathcal{H} u) \leq d_h(\hat{\mathcal{H}} u, \hat{\mathcal{H}} u) \leq C d_h(\mathcal{H} u, \mathcal{H} u) \tag{3.3}$$

where $\mathcal{H}u = \{\mathcal{H}_i u_i\}_{i=1}^N$ and $\hat{\mathcal{H}}u = \{\hat{\mathcal{H}}_i u\}_{i=1}^N$ are defined by (2.13) and (2.17) respectively, and C is a positive constant independent of h_i, u, ρ_i and H_i.

From (3.1) and (3.3) we have

$$\gamma_0 d_h(\mathcal{H} u, \mathcal{H} u) \leq a_h(\hat{\mathcal{H}} u, \hat{\mathcal{H}} u) \leq C \gamma_1 d_h(\mathcal{H} u, \mathcal{H} u) \tag{3.4}$$

and therefore, we can take advantages of all the discrete Sobolev norm results known for $\mathcal{H}$ discrete harmonic extensions and for the norm d_h.

4. Notations and the interface condition

In this section we introduce local and global subspaces and bilinear forms on the interface Γ_h; see (2.24). We also introduce a sufficient condition (Assumption 4.1) for designing robust preconditioners and for deriving quasi-optimal bounds for the condition number of the preconditioners. In Section 7 we show numerically that Assumption 4.1 is indeed necessary for robustness.

First we classify substructures according to their position with respect to the boundary $\partial\Omega$. We say that a substructure Ω_i is an *interior substructure* or *floating substructures* if Ω_i does not share a face with the boundary of Ω, i.e., when the size (the Lebesque measure) of $\partial\Omega_i \cap \partial\Omega$ vanishes. Otherwise, we say it is a *boundary substructure* or *nonfloating substructure*. We denote by $\mathcal{N}_I$ and $\mathcal{N}_B$ the sets of indices of interior and boundary substructures, respectively.

Recall that a common (part of) face of $\partial\Omega_i$ to $\partial\Omega_j$ has two sides, the side contained in $\partial\Omega_i$, denoted by F_{ij}, and the side contained in $\partial\Omega_j$, denoted by F_{ji}. Note also that geometrically $F_{ji} = F_{ij}$. For convenience of notation we also introduce fictitious faces $F_{\partial i} = F_{i\partial}$ where $F_{\partial i} = \partial\Omega_i \cap \partial\Omega$; see Subsection 2.2. Since in this paper we consider only the zero Dirichlet boundary condition, functions defined on $F_{\partial i}$ must vanish, while functions on $F_{i\partial}$ are free to take any value. Throughout this paper, F_{ij} stands for a face (or part of a face) of positive Lebesque measure.

Let $\overset{\circ}{\Omega}_{ih_i}$ and $\partial\Omega_{ih_i}$ be the interior and boundary nodes of $\mathcal{T}_{h_i}(\overline{\Omega}_i)$ in Ω_i and on $\partial\Omega_i$, respectively. Define F_{ijh_i} as the set of nodes of $\partial\Omega_{ih_i}$ that are on F_{ij}. Recall that F_{ij} is a *closed* interval. We also define ∂F_{ijh_i} as the set of nodes on F_{ijh_i} that are closest to the boundary ∂F_{ij}. Let $\overset{\circ}{F}_{ijh_i} := F_{ijh_i} \setminus \partial F_{ijh_i}$ be the set of interior nodes in F_{ij}. Additionally, we define the extended boundary nodes $\partial^e F_{ijh_i}$ as the union of ∂F_{ijh_i} and the nodal points $y \in \partial\Omega_i \setminus F_{ij}$ closest to $x \in \partial F_{ij}$ when x is not a nodal point. Note that when F_{ij} is a full face of $\partial\Omega_i$, then $\partial^e F_{ijh_i} = \partial F_{ij}$. Let $\overline{F}_{ijh_i} := \overset{\circ}{F}_{ijh_i} \cup \partial^e F_{ijh_i}$. See Figure 1 for an example. We define

$$\Gamma_i := \partial\Omega_{ih_i} \cup \bigcup_{F_{ij} \subset \partial\Omega_i} \overline{F}_{jih_j}. \tag{4.1}$$

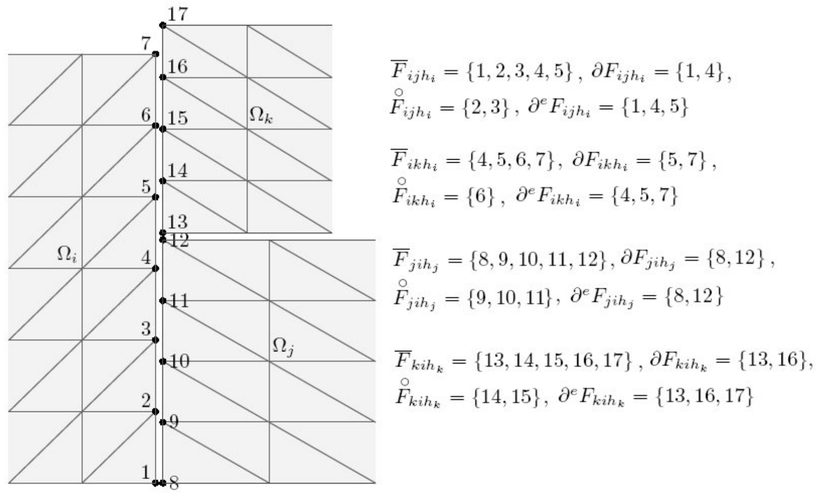

FIGURE 1. An example of nodes classification on an interface.

Note that Γ_i is defined to include the nodes on Γ_h necessary for computing $\hat{\mathcal{H}}_i$; see (2.17). Define W_i as the space of piecewise linear functions defined by the nodal values on Γ_i extended via $\hat{\mathcal{H}}_i$ (defined in (2.17)) inside Ω_i, i.e.,

$$W_i := \left\{ v : \text{ nodal values of } v \text{ defined on } \overset{\circ}{\Omega}_{ih_i} \cup \Gamma_i \text{ and } v \equiv \hat{\mathcal{H}}_i v \text{ in } \Omega_i \right\}. \tag{4.2}$$

Observe that a function $u^{(i)} \in W_i$ can be represented as

$$u^{(i)} = \{u_l^{(i)}\}_{l \in \#(i)} \quad \text{where} \quad \#(i) = \{i\} \cup \{j : F_{ij} \in \partial\Omega_i\}.$$

Here $u_i^{(i)}$ and $u_j^{(i)}$ stand for the nodal values of $u^{(i)}$ on $\overline{\Omega}_i$ and on $\overline{F}_{jih_j}$, respectively. Recall also that sometimes we write $u = \{u_i\}_{i=1}^N \in V$ to refer to a function defined on all of Γ_h with each u_i defined (only) on $\partial\Omega_i$; see Subsection 2.2. We point out that F_{ij} and F_{ji} are geometrically the same even though the mesh on the side F_{ij} is inherited from the Ω_i triangulation while the mesh on the side F_{ji} corresponds from the Ω_j triangulation. Note also that, according to our conventions, if $i \in \mathcal{N}_B$ and $u^{(i)} \in W_i$ then $u_\partial^{(i)} = 0$ on the fictitious face $F_{\partial i}$.

Define the extension operator $\tilde{I}_i : W_i \to V$ as follows: Given $u^{(i)} \in W_i$, let $\tilde{I}_i u^{(i)}$ be equal to $u^{(i)}$ at nodes of Γ_i and $\overset{\circ}{\Omega}_{ih_i}$ and zero on $\Gamma_h \backslash \Gamma_i$, and extended by $\hat{\mathcal{H}} \tilde{I}_i u^{(i)}$ elsewhere and denoted also by $\tilde{I}_i$, i.e.,

$$\tilde{I}_i u(x) = \begin{cases} u(x) & \text{if } x \in \Gamma_i \cup \overset{\circ}{\Omega}_{ih_i} \\ 0 & \text{if } x \in \Gamma_h \backslash \Gamma_i \\ \hat{\mathcal{H}} \tilde{I}_i u \equiv \tilde{I}_i u & \text{elsewhere.} \end{cases} \tag{4.3}$$

In addition, we assign to each pair $\{F_{ij}, F_{ji}\}$ a master and a slave side. If F_{ij} is a slave side then F_{ji} is a master side and vice versa. If F_{ij} is a slave side we will use the notation δ_{ij} (instead of F_{ij}) to emphasis this fact, while if F_{ij} is a master side we will use the notation γ_{ij}. Note that since we are working with a geometrically nonconforming decomposition of Ω, a part of a face can be labeled as master while other part of the same face can be marked as slave. We will use the notation $\gamma_{ijh_i} := F_{ijh_i}$, $\overset{\circ}{\gamma}_{ijh_i} := \overset{\circ}{F}_{ijh_i}$, $\overline{\gamma}_{ijh_i} := \overline{F}_{ijh_i}$ $\partial\gamma_{ijh_i} := \partial F_{ijh_i}$, $\partial^e\gamma_{ijh_i} := \partial^e F_{ijh_i}$ when F_{ij} is a master side. Analogous notation will be used also for a slave side δ_{ij}. The choice of slave-master sides are such that the *interface condition*, stated next in Assumption 4.1, can be satisfied. Under this assumption, Theorems 5.2, 5.5 and Theorem 6.1 below hold with constants C independent of the ρ_i, h_i and H_i. This assumption says basically that the coarser meshes h_i should be chosen where the coefficient ρ_i are larger, and additionally, the master side should be chosen on the side where the coefficient is larger.

Assumption 4.1 (The interface condition)**.** *We say that the coefficients $\{\rho_i\}$ and the local mesh sizes $\{h_i\}$ satisfy the interface condition if there exist constants β_1 and β_2, of the order $O(1)$, such that for any (part of) face F_{ij}, one of the following inequalities hold:*

$$\begin{cases} h_i \le \beta_1 h_j \text{ and } \rho_i \le \beta_2 \rho_j & \text{if } F_{ij} \text{ is a slave side, or} \\ h_j \le \beta_1 h_i \text{ and } \rho_j \le \beta_2 \rho_i & \text{if } F_{ij} \text{ is a master side.} \end{cases} \tag{4.4}$$

We associate to each $\Omega_i, i = 1, \cdots, N$ a weighting diagonal matrices $D^{(i)} = \{D_l^{(i)}\}_{l\in\#(i)}$ on $\Gamma_i \cup \overset{\circ}{\Omega}_{ih_i}$. Let x be a node of $\Gamma_i \cup \overset{\circ}{\Omega}_{ih_i}$. Then, the diagonal element of $D^{(i)}$ associated to x is defined by:

- On $\overset{\circ}{\Omega}_{ih_i} \cup \partial\Omega_{i,h_i}$ $(l = i)$

$$D_i^{(i)}(x) = \begin{cases} 0 & \text{if } x \in \overset{\circ}{F}_{ijh_i} \text{ and } F_{ij} \text{ is a slave side} \\ 1 & \text{otherwise,} \end{cases} \tag{4.5}$$

- On $\overline{F}_{jih_j}$ $(l = j)$

$$D_j^{(i)}(x) = \begin{cases} 0 & \text{if } x \in \partial^e F_{jih_j}, \\ 1 & \text{if } x \in \overset{\circ}{F}_{jih_j} \text{ and } F_{ij} \text{ is a master side} \\ 0 & \text{if } x \in \overset{\circ}{F}_{jih_j} \text{ and } F_{ij} \text{ is a slave side,} \end{cases} \tag{4.6}$$

- For $x \in \overline{F}_{i\partial h_i}$ we set $D_i^{(i)}(x) = 1$.

Remark 4.2. We can define any value for $D^{(i)}$ on $\overset{\circ}{\Omega}_{ih_i}$ since, as we will see below, the operator of interest is $I_i := \tilde{I}_i D^{(i)}$ and $\tilde{I}_i u^{(i)}$ does not depend on the values of $u^{(i)}$ on $\overset{\circ}{\Omega}_{ih_i}$.

There are two alternative ways of defining the diagonal matrices $D^{(i)}$ on Γ_i and still ensuring Theorems 5.2, 5.5 and 6.1 below to hold: 1) On (part of) faces F_{ij}, where h_i and h_j are of the same order, the values of (4.5) and (4.6) at nodal points x of $\overset{\circ}{F}_{jih_j}$ can be replaced by $\frac{\rho_i^\beta}{\rho_i^\beta + \rho_j^\beta}, \beta \ge 1/2$ (see [29]); 2) Similarly, on (part of) faces F_{ij}, where ρ_i and ρ_j are of the same order, we can replace (4.5) and (4.6) at nodal nodes x of $\overset{\circ}{F}_{ijh_i}$ and $\overset{\circ}{F}_{jih_j}$ by $\frac{h_i}{h_i+h_j}$.

The prolongation operators $I_i : W_i \to V$, $i = 1, \ldots, N$, are defined as

$$I_i = \tilde{I}_i D^{(i)}. \tag{4.7}$$

It is easy to see that the image of I_i forms a decomposition (a direct sum) for V since

$$\sum_{i=1}^{N} I_i \tilde{I}_i^T u = u, \tag{4.8}$$

where the $\tilde{I}_i^T$ stand for the restriction of V to W_i.

5. Additive preconditioners

To design and analyze additive N-N type methods for solving (2.26) we use the general framework of ASM; see Lemma 5.1 below and [30]. In the Section 5.1 we consider an additive Schwarz method based on the coarse space $V_{0,I}$, i.e., a coarse space with one degree of freedom per interior substructure and no degree of freedom per boundary substructure; see (5.5). Then we consider several variants of this method.

5.1. Additive Schwarz method with the $V_{0,I}$ coarse space.

We now introduce the local and coarse problems to define the additive Schwarz method $T_{as,I}$.

5.1.1. *Local problems.* Recall the definition of Γ_i in (4.1) and the space W_i in (4.2). Define

$$\begin{cases} V_i = V_i(\Gamma_i) := \left\{ u^{(i)} \in W_i : \int_{\partial\Omega_i} u_i^{(i)} = 0 \right\}, & \text{if } i \in \mathcal{N}_I \\ V_i = V_i(\Gamma_i) := W_i, & \text{if } i \in \mathcal{N}_B \end{cases} \tag{5.1}$$

i.e., for interior substructures Ω_i, V_i is the subspace of W_i consisting of functions with zero average value on $\partial\Omega_i$, while for boundary substructures, V_i is the whole space W_i. We recall that a function $v^{(i)} \in W_i$ (or V_i) then $v^{(i)} \equiv \hat{\mathcal{H}}_i v^{(i)}$ and $v \in V$ then $v \equiv \hat{\mathcal{H}} v$.

For $u^{(i)}, v^{(i)} \in V_i$, $i = 1, \ldots, N$, we define the local bilinear form b_i as

$$b_i(u^{(i)}, v^{(i)}) := \hat{a}_i(u^{(i)}, v^{(i)}), \tag{5.2}$$

where the bilinear form $\hat{a}_i$ is defined in (2.4). We define the operators $T_i : V \to V$, $i = 1, \ldots, N$, by defining $\tilde{T}_i : V \to V_i$ as

$$b_i(\tilde{T}_i u, v^{(i)}) = a_h(u, I_i v^{(i)}) \text{ for all } v^{(i)} \in V_i, \tag{5.3}$$

and then set $T_i = I_i \tilde{T}_i$. It is easy to see, from Lemma (2.1), that these problems are well posed.

5.1.2. *Coarse problems.* Let $e^{(i)} \in W_i$ be the vector with value one at the nodes of Γ_i and on $\overset{\circ}{\Omega}_{ih_i}$. Recall that the prolongation operators $\tilde{I}_i$ and I_i are defined in (4.3) and (4.7), respectively. Define $\Theta_i \in V$, for $i = 1, \ldots, N$, as $\Theta_i := \tilde{I}_i \Theta^{(i)}$ where $\Theta^{(i)} = D^{(i)} e^{(i)}$. Hence, $\Theta_i = I_i e^{(i)}$ and $\Theta_i \equiv \hat{\mathcal{H}} \Theta_i$. Note from (4.5) and (4.6) we have that

$$\sum_{i=1}^{N} \Theta_i = 1 \text{ on } \Gamma_h. \tag{5.4}$$

We consider the following coarse space:

$$V_{0,I} = \text{Span}\,\{\Theta_i\}_{i\in\mathcal{N}_I} \subset V. \tag{5.5}$$

The coarse bilinear form is defined according to

$$b_0(u,v) = \left(1+\log\frac{H}{h}\right)^{-2} a_h(u,v), \quad u,v \in V_{0,I}. \tag{5.6}$$

Next we define the projection-like operator $T_0 : V \to V_{0,I}$ as

$$b_0(T_0 u, v^{(0)}) = a_h(u, v^{(0)}) \text{ for all } v^{(0)} \in V_{0,I}. \tag{5.7}$$

Let us denote below $V_0 = V_{0,I}$ and I_0 by the identity operator defined on functions $V_0 \subset V$.

The additive preconditioner is defined by

$$T_{as,I} = \sum_{i=0}^{N} T_i, \tag{5.8}$$

Note that $T_{as,I}$ is symmetric and from the abstract theory of ASM we have the following:

Lemma 5.1 (See Theorem 2.7 in [30]). *Suppose that the following three assumptions hold:*
Assumption i) There exists a constant C_0 such that for all $u \in V$ there exists a decomposition $u = \sum_{i=0}^{N} I_i u^{(i)}$ with $u^{(i)} \in V_i$, $i = 0,1,\ldots,N$, such that

$$b_0(u^{(0)}, u^{(0)}) + \sum_{i=1}^{N} b_i(u^{(i)}, u^{(i)}) \le C_0^2 a_h(u,u). \tag{5.9}$$

Assumption ii) There exist constants ϵ_{ij}, $i,j = 1,\ldots,N$, such that for all $u^{(i)} \in V_i$, $u^{(j)} \in V_j$ we have

$$a_h(I_i u^{(i)}, I_j u^{(j)}) \le \epsilon_{ij} a_h(I_i u^{(i)}, I_i u^{(i)})^{1/2} a_h(I_j u^{(j)}, I_j u^{(j)})^{1/2}.$$

Assumption iii) There exists a constant ω such that

$$a_h(I_i u^{(i)}, I_i u^{(i)}) \le \omega b_i(u^{(i)}, u^{(i)}) \text{ for all } u^{(i)} \in V_i,\ i = 0,1,\ldots,N.$$

Then, $T_{as,I}$ is invertible and

$$C_0^{-2} a_h(u,u) \le a_h(T_{as,I} u, u) \le (\rho(\epsilon)+1)\omega a_h(u,u) \text{ for all } u \in V.$$

Here, $\rho(\epsilon)$ is the spectral radius of the matrix $\epsilon = \{\epsilon_{ij}\}_{i,j=1}^{N}$.

5.1.3. *Condition number estimation for $T_{as,I}$.* In this section we state and prove the main result concerning the preconditioner defined in (5.8) with $V_0 = V_{0,I}$.

To avoid the proliferation of constants, we will use sometimes the notation $A \preceq B$ to represent the inequality $A \leq (constant)B$, and $A \asymp B$ if $A \preceq B$ and $B \preceq A$, where the $(constant)$ does not depend on H_i, h_i and ρ_i.

Theorem 5.2. *Let the Assumption 4.1 be satisfied. In addition, assume that for $i \in \mathcal{N}_B$, the size of $\partial\Omega_i \cap \partial\Omega$ is of the same order as the diameter of Ω_i. Then there exist positive constants C_1 and C_2 independent of h_i, H_i and the jumps of ρ_i such that*

$$C_1 a_h(u,u) \leq a_h(T_{as,I}u,u) \leq C_2 \left(1 + \log\frac{H}{h}\right)^2 a_h(u,u) \quad \textit{for all } u \in V. \tag{5.10}$$

Here $\log(H/h) = \max_i \ \log(H_i/h_i)$.

Proof. By the general theory of ASMs we need to check the three key assumptions of Lemma 5.1.

<u>*Assumption*</u> *i)* In order to verify (5.9) it is enough to prove (see Lemma 2.1) that for every $u = \{u_i\}_{i=1}^N \in V$, there exist $u^{(i)} \in V_i$, $i = 0, \ldots, N$, such that $u = u^{(0)} + \sum_{i=1}^N I_i u^{(i)}$ and

$$b_0(u^{(0)}, u^{(0)}) + \sum_{i=1}^{N} b_i(u^{(i)}, u^{(i)}) \leq C_0^2 d_h(u,u) \tag{5.11}$$

where C_0 does not independ on h_i, H_i and ρ_i.

Recall that $\Theta_i = I_i e^{(i)}$ where $e^{(i)}$ has value one at the nodes of Γ_i and $\overset{\circ}{\Omega}_{ih_i}$. See also (4.5), (4.6) and (4.7). Let $u = \{u_i\}_{i=1}^N \in V$ and define

$$u^{(0)} = \sum_{i \in \mathcal{N}_I} \overline{u}_i \Theta_i = \sum_{i \in \mathcal{N}_I} I_i \overline{u}_i e^{(i)}, \tag{5.12}$$

with

$$\overline{u}_i = \frac{1}{|\partial\Omega_i|} \int_{\partial\Omega_i} u_i dx, \ i = 1, \ldots N. \tag{5.13}$$

Since the operators I_i defined in (4.7) form a partition of unity on Γ_h, see (4.8), we can write

$$u - u^{(0)} = \sum_{i \in \mathcal{N}_I}^{N} I_i(\tilde{I}_i^T u - \overline{u}_i e^{(i)}) + \sum_{i \in \mathcal{N}_B} I_i(\tilde{I}_i^T u) = \sum_{i=1}^{N} I_i u^{(i)}, \tag{5.14}$$

where $u^{(i)} := \tilde{I}_i^T u - \overline{u}_i e^{(i)}$ if $i \in \mathcal{N}_I$, and $u^{(i)} := \tilde{I}_i^T u$ if $i \in \mathcal{N}_B$. Note that $u^{(i)} \in V_i, i = 1, \cdots, N$.

Note that $u^{(i)}$ can be represented as $u^{(i)} = \{u_l^{(i)}\}_{l \in \#(i)} \in V_i$, for $i = 1, \cdots, N$. For $i \in \mathcal{N}_I$ we have

$$\begin{cases} u_i^{(i)} = u_i - \overline{u}_i e_i^{(i)} = u_i - \overline{u}_i & \text{on } \overline{\Omega}_i, \\ u_j^{(i)} = u_j - \bar{u}_i e_j^{(i)} = u_j - \bar{u}_i & \text{on } F_{ji}, \text{ for all } F_{ij} \subset \partial\Omega_i, \end{cases} \tag{5.15}$$

while for $i \in \mathcal{N}_B$ we have

$$\left\{ \begin{array}{ll} u_i^{(i)} = u_i & \text{on } \partial\Omega_i, \\ u_j^{(i)} = u_j & \text{on } F_{ji}, \text{ for all } F_{ij} \subset \partial\Omega_i. \end{array} \right. \tag{5.16}$$

Using Lemma 2.1 we have that for $i = 1, \cdots, N$,

$$\begin{aligned} b_i(u^{(i)}, u^{(i)}) &\preceq \rho_i \parallel \nabla \mathcal{H}_i u_i^{(i)} \parallel^2_{L^2(\Omega_i)} + \sum_{F_{ij} \subset \partial\Omega_i} \delta \frac{\rho_{ij}}{h_{ij}} \parallel u_i^{(i)} - u_j^{(i)} \parallel^2_{L^2(F_{ij})} \\ &= \rho_i \parallel \nabla \mathcal{H}_i u_i \parallel^2_{L^2(\Omega_i)} + \sum_{F_{ij} \subset \partial\Omega_i} \delta \frac{\rho_{ij}}{h_{ij}} \parallel u_i - u_j \parallel^2_{L^2(F_{ij})} . \end{aligned} \tag{5.17}$$

It remains to estimate $b_0(u^{(0)}, u^{(0)})$. In Lemma 5.3, see below, we will prove that

$$d_h(u^{(0)}, u^{(0)}) \leq C \left(1 + \log \frac{H}{h}\right)^2 d_h(u, u), \tag{5.18}$$

and therefore, together with Lemma 2.1 and the definition of b_0 in (5.6), we have that

$$b_0(u^{(0)}, u^{(0)}) \leq C a_h(u, u) \tag{5.19}$$

where C does not independ on h_i, H_i and ρ_i.

Assumption ii) We need to prove that

$$a_h(I_i u^{(i)}, I_j u^{(j)}) \leq \varepsilon_{ij} a_h^{1/2}(I_i u^{(i)}, I_i u^{(i)})\, a_h^{1/2}(I_j u^{(j)}, I_j u^{(j)}) \tag{5.20}$$

for $u^{(i)} \in V_i$ and $u^{(j)} \in V_j$, $i, j = 1, \cdots, N$, and that the spectral radius of $\varepsilon = \{\varepsilon_{ij}\}_{i,j=1}^N$, $\varrho(\varepsilon)$, is bounded. In our case $\varrho(\varepsilon) \leq C$ with constant independent of h_i, H_i and ρ_i, $i = 1, \ldots, N$. This follows from the fact that ε_{ij} vanishes when Γ_i and Γ_j do not touch each other.

Assumption iii). We need to prove that for $i = 0, 1, \cdots, N$,

$$a_h(I_i u^{(i)}, I_i u^{(i)}) \leq \omega b_i(u^{(i)}, u^{(i)}) \quad \text{for all } u^{(i)} \in V_i \tag{5.21}$$

with $\omega \leq C\,(1 + \log(H/h))^2$ where C is a positive constant independent of h_i, H_i and the jumps of ρ_i. The proof of (5.21) for $i = 0$ with $\omega = C\,(1 + \log(H/h))^2$ follows from the definition of $b_0(\cdot,\cdot)$, while for $i = 1, \ldots, N$, the proof will be presented separately in Lemma 5.4 below. □

We now complete the proof of Theorem 5.2 by proving auxiliary results associated with (5.18) and (5.21). See Lemmas 5.3 and 5.4 below.

Lemma 5.3. *Let the Assumption 4.1 be satisfied. Then for any $u \in V$ and $u^{(0)}$ defined by (5.12), the following inequality holds*

$$d_h(u^{(0)}, u^{(0)}) \leq C \left(1 + \log \frac{H}{h}\right)^2 d_h(u, u) \tag{5.22}$$

where the constant C does not independ of h_i, H_i and the jumps of ρ_i.

Proof. Let us denote $u^{(0)} = \{u_i^{(0)}\}_{i=1}^N$. By Lemma 3.1 it is enough to prove the estimate (5.22) for $\mathcal{H}u^{(0)} = \{\mathcal{H}_i u_i^{(0)}\}_{i=1}^N$. Let us denote $\mathcal{H}u^{(0)}$ by $u^{(0)}$. We have (5.23)

$$d_h(u^{(0)},u^{(0)}) = \sum_{i=1}^N \left\{ \rho_i \parallel \nabla u_i^{(0)} \parallel_{L^2(\Omega_i)}^2 + \sum_{F_{ij}\subset\partial\Omega_i} \frac{\delta}{l_{ij}} \frac{\rho_{ij}}{h_{ij}} \parallel u_i^{(0)} - u_j^{(0)} \parallel_{L^2(F_{ij})}^2 \right\}.$$

We now estimate the first term in (5.23). Let us consider first the case where $i \in \mathcal{N}_I$. From the definition of $u^{(0)}$ in (5.12) we see that on $\partial\Omega_i$

$$u_i^{(0)} = \overline{u}_i \Theta_i^{(i)} + \sum_{\delta_{ij}\subset\partial\Omega_i} \overline{u}_j \Theta_i^{(j)} - \sum_{\delta_{ij}\subset\partial\Omega_i,\ j\in\mathcal{N}_B} \overline{u}_j \Theta_i^{(j)}. \tag{5.24}$$

It is easy to see from (4.6) that when $\delta_{ij} = F_{ij}$ is a slave side and $\overset{\circ}{F}_{ijh_i}$ is empty then $\Theta_i^{(j)}$ vanishes. Hence, we consider only the cases in (5.24) which $\overset{\circ}{F}_{ijh_i}$ is not empty, and hence from the definition of $\overset{\circ}{F}_{ijh_i}$ we have $h_i \preceq |F_{ij}|$, where $|F_{ij}|$ denotes the size (the Lebesque measure) of F_{ij}. In general, $|F_{ij}|$ can be very tiny due to the geometrically nonconformity of the Ω_i partition, however, this is not the case when $\overset{\circ}{F}_{ijh_i}$ is not empty. Additionally, because F_{ij} is a slave side and the Assumption 4.1 hypothesis holds, we have $h_i \asymp h_{ij} \preceq h_j$. From (5.4) we have

$$u_i^{(0)} - \overline{u}_i = \sum_{\delta_{ij}\subset\partial\Omega_i} (\overline{u}_j - \overline{u}_i)\Theta_i^{(j)} - \sum_{\delta_{ij}\subset\partial\Omega_i,\ j\in\mathcal{N}_B} \overline{u}_j \Theta_i^{(j)} \quad \text{on } \partial\Omega_i. \tag{5.25}$$

Using (5.25) we obtain

$$\begin{aligned} \parallel \nabla u_i^{(0)} \parallel_{L^2(\Omega_i)}^2 &= \parallel \nabla (u_i^{(0)} - \overline{u}_i) \parallel_{L^2(\Omega_i)}^2 \\ &\preceq \left(1 + \log \frac{H_i}{h_i}\right) \left(\sum_{\delta_{ij}\subset\partial\Omega_i} (\overline{u}_i - \overline{u}_j)^2 + \sum_{\delta_{ij}\subset\partial\Omega_i,\ j\in\mathcal{N}_B} \overline{u}_j^2 \right) \end{aligned} \tag{5.26}$$

where we have used the following extension theorem

$$\parallel \nabla \Theta_i^{(j)} \parallel_{L^2(\Omega_i)}^2 \preceq \parallel \Theta_i^{(j)} \parallel_{H_{00}^{1/2}(\delta_{ij})}^2$$

and the discrete inequality (see [30])

$$\parallel \Theta_i^{(j)} \parallel_{H_{00}^{1/2}(\delta_{ij})}^2 \preceq \left(1 + \log \frac{H_i}{h_i}\right).$$

Now we estimate the term $(\overline{u}_i - \overline{u}_j)^2$ in (5.26). Denote

$$\overline{u}_{ij} = \frac{1}{|F_{ij}|} \int_{F_{ij}} u_i ds \quad \text{and} \quad \overline{u}_{ji} = \frac{1}{|F_{ji}|} \int_{F_{ji}} u_j ds. \tag{5.27}$$

Note that $h_{ij} \asymp h_i \preceq |F_{ij}|$ and so

$$(\overline{u}_{ij} - \overline{u}_{ji})^2 = \frac{1}{|F_{ij}|^2} (u_i - u_j, 1)_{L^2(F_{ij})}^2 \preceq \frac{1}{h_i} \|u_i - u_j\|_{L^2(F_{ij})}^2.$$

By the discrete and Poincaré inequalities, and using again that $h_i \preceq |F_{ij}|$ we obtain

$$(\overline{u}_i - \overline{u}_{ij})^2 \preceq \left(1 + \log \frac{H_i}{h_i}\right) \parallel \nabla u_i \parallel_{L^2(\Omega_i)}^2 . \tag{5.28}$$

Using the above estimates we obtain

$$\begin{aligned}(\overline{u}_i - \overline{u}_j)^2 &\preceq (\overline{u}_i - \overline{u}_{ij})^2 + (\overline{u}_{ij} - \overline{u}_{ji})^2 + (\overline{u}_{ji} - \overline{u}_j)^2 \\ &\preceq \left(1 + \log \frac{H_i}{h_i}\right) \| \nabla u_i \|^2_{L^2(\Omega_i)} + \frac{1}{h_{ij}} \| u_i - u_j \|^2_{L^2(F_{ij})} \\ &+ \left(1 + \log \frac{H_j}{h_j}\right) \| \nabla u_j \|^2_{L^2(\Omega_j)} . \end{aligned} \tag{5.29}$$

We point out that the log factor above in (5.28) can be dropped if $|F_{ij}| \asymp H_i$. See Remark 5.6 below.

Now we estimate the term $\overline{u}_j^2$ in (5.26) for $j \in \mathcal{N}_B$. Recall that $u_\partial = 0$, hence, $\overline{u}_{\partial j} = 0$. Then, using the notation (5.27) we obtain

$$\begin{aligned}(\overline{u}_j)^2 &= (\overline{u}_j - \overline{u}_{j\partial} + \overline{u}_{j\partial} - \overline{u}_{\partial j})^2 \\ &\preceq \left(1 + \log \frac{H_j}{h_j}\right) \|\nabla u_j\|^2_{L^2(\Omega_j)} + \frac{1}{h_{j\partial}} \|u_j - u_\partial\|^2_{L^2(F_{j\partial})}. \end{aligned} \tag{5.30}$$

We now estimate the first term in (5.23) for $i \in \mathcal{N}_B$, see (5.24). We obtain

$$\begin{aligned}\| \nabla u_i^{(0)} \|^2_{L^2(\Omega_i)} &\preceq \sum_{\delta_{ij} \subset \partial\Omega_i,\ j \in \mathcal{N}_I} (\overline{u}_j)^2 \| \Theta_i^{(j)} \|^2_{H_{00}^{1/2}(\delta_{ij})} \\ &\preceq \left(1 + \log \frac{H_i}{h_i}\right) \sum_{\delta_{ij} \subset \partial\Omega_i,\ j \in \mathcal{N}_I} \overline{u}_j^2. \end{aligned} \tag{5.31}$$

Here again, the log factor above in (5.31) can be dropped if $|F_{j\partial}| \asymp H_j$. See also Remark 5.6.

To estimate the term $\overline{u}_j^2$ with $j \in \mathcal{N}_I$ we use

$$(\overline{u}_j)^2 \preceq \left\{ (\overline{u}_j - \overline{u}_{ji})^2 + (\overline{u}_{ji} - \overline{u}_{ij})^2 + (\overline{u}_{ij} - \overline{u}_i)^2 + (\overline{u}_i - \overline{u}_{i\partial} + \overline{u}_{i\partial} - \overline{u}_{\partial i})^2 \right\}$$

and then apply the same arguments given above. Substituting (5.29) and (5.30) into (5.26) and recalling that $\rho_i \asymp \rho_{ij} \preceq \rho_j$ and $h_i \asymp h_{ij} \preceq h_j$ on every slave side δ_{ij}, we obtain

$$\begin{aligned}\rho_i \| \nabla u^{(0)} \|^2_{L^2(\Omega_i)} &\preceq \left(1 + \log \frac{H}{h}\right)^2 \Big\{ \rho_i \| \nabla u_i \|^2_{L^2(\Omega_i)} \\ + \sum_{\delta_{ij} \subset \Omega_i} \rho_j \| \nabla u_j \|^2_{L^2(\Omega_j)} &+ \frac{\rho_{ij}}{h_{ij}} \| u_i - u_j \|^2_{L^2(F_{ij})} + \frac{\rho_{j\partial}}{h_{j\partial}} \| u_j - u_\partial \|^2_{L^2(F_{j\partial})} \Big\}. \end{aligned} \tag{5.32}$$

It remains to estimate the second term of (5.23). Observe that the estimate is obvious for $F_{i\partial}$ since $u^{(0)} = 0$ on $F_{\partial i}$ and $F_{i\partial}$. when $i \in \mathcal{N}_B$. Assume now that $i \in \mathcal{N}_I$ and $j \in \mathcal{N}_I$. We consider separately the cases when F_{ij} is a master and a slave side. Suppose that $F_{ij} = \gamma_{ij}$ is a master side. We have on F_{ij}

$$\begin{aligned} u_i^{(0)} - u_j^{(0)} &= \overline{u}_i \Theta_i^{(i)} - \left(\overline{u}_j \Theta_j^{(j)} + \overline{u}_i \Theta_j^{(i)} \right) \\ &= (\overline{u}_i - \overline{u}_j) \Theta_j^{(j)}. \end{aligned} \tag{5.33}$$

Hence,

$$\frac{1}{h_{ij}} \parallel u_i^{(0)} - u_j^{(0)} \parallel^2_{L^2(F_{ij})} = \frac{1}{h_{ij}} (\overline{u}_i - \overline{u}_j)^2 \parallel \Theta_j^{(j)} \parallel^2_{L^2(F_{ij})} \preceq (\overline{u}_i - \overline{u}_j)^2$$

where we have used that $h_j \asymp h_{ij} \preceq h_i$ and

$$\parallel \Theta_j^{(j)} \parallel^2_{L^2(F_{ij})} \preceq h_j \tag{5.34}$$

since $\Theta_j^{(j)}$ vanishes on $\overset{\circ}{F}_{jih_j}$. Using (5.29) and $\rho_j \asymp \rho_{ij} \preceq \rho_i$ we obtain

$$\begin{aligned} \frac{\rho_{ij}}{h_{ij}} \parallel u_i^{(0)} - u_j^{(0)} \parallel^2_{L^2(F_{ij})} &\preceq \left(1 + \log \frac{H_i}{h_i}\right) \rho_i \parallel \nabla u_i \parallel^2_{L^2(\Omega_i)} \\ &+ \frac{\rho_{ij}}{h_{ij}} \parallel u_i - u_j \parallel^2_{L^2(F_{ij})} \\ &+ \left(1 + \log \frac{H_j}{h_j}\right) \rho_j \parallel \nabla u_j \parallel^2_{L^2(\Omega_j)} . \end{aligned} \tag{5.35}$$

Now assume that $F_{ij} = \delta_{ij}$ is a slave side. In this case on F_{ij} we have, see (5.33),

$$u_i^{(0)} - u_j^{(0)} = \bar{u}_i \Theta_i^{(i)} + \bar{u}_j \Theta_i^{(j)} - \bar{u}_j \Theta_j^{(j)} = (\overline{u}_i - \overline{u}_j)\Theta_i^{(i)},$$

therefore, we get

$$\begin{aligned} \frac{\rho_{ij}}{h_{ij}} \parallel u_i^{(0)} - u_j^{0)} \parallel^2_{L^2(F_{ij})} &= \frac{\rho_{ij}}{h_{ij}} (\overline{u}_i - \overline{u}_j)^2 \parallel \Theta_i^{(i)} \parallel^2_{L^2(F_{ij})} \\ \preceq \rho_{ij}(\overline{u}_i - \overline{u}_j)^2 &\preceq \left(1 + \log \frac{H_i}{h_i}\right) \rho_i \parallel \nabla u_i \parallel^2_{L^2(\Omega_i)} \\ + \frac{\rho_{ij}}{h_{ij}} \parallel u_i - u_j \parallel^2_{L^2(F_{ij})} &+ \left(1 + \log \frac{H_j}{h_j}\right) \rho_j \parallel \nabla u_j \parallel^2_{L^2(\Omega_j)} \end{aligned} \tag{5.36}$$

in view of (5.34) for $\delta_{ij} \subset \partial\Omega_i$ and (5.29).

Assume now $i \in \mathcal{N}_I$ and $j \in \mathcal{N}_B$. Since $i \in \mathcal{N}_I$ then $u_j^{(0)}$ vanishes on F_{ji}. If $F_{ij} = \delta_{ij}$ is a slave side or $F_{ij} = \gamma_{ij}$ is a master side then

$$\frac{\rho_{ij}}{h_{ij}} \parallel u_i^{(0)} - u_j^{(0)} \parallel^2_{L^2(F_{ij})} \preceq \rho_{ij} (\overline{u}_i)^2$$

and using that

$$\overline{u}_i^2 \preceq (\overline{u}_i - \overline{u}_{ij})^2 + (\overline{u}_{ij} - \overline{u}_{ji})^2 + (\overline{u}_{ji} - \overline{u}_j)^2 + (\overline{u}_j - \overline{u}_{j\partial} + \overline{u}_{j\partial} - \overline{u}_{\partial j})^2$$

and the same arguments given before, the estimate follows. The case $i \in \mathcal{N}_B$ and $j \in \mathcal{N}_I$ follows from the previous case.

Substituting (5.32), (5.35) and (5.36) into (5.23) we get (5.22). □

In order to complete the proof of Theorem 5.2 in the next lemma we prove the inequality (5.21).

Lemma 5.4. *Let the Assumption 4.1 be satisfied. In addition, assume that for $i \in \mathcal{N}_B$ the size of $\partial\Omega_i \cap \partial\Omega$ is of the same order as the diameter of Ω_i. Then for $u^{(i)} \in V_i$, $i = 1, \ldots, N$, we have*

$$a_h(I_i u^{(i)}, I_i u^{(i)}) \leq C \left(1 + \log \frac{H}{h}\right)^2 b_i(u^{(i)}, u^{(i)}), \tag{5.37}$$

where C does not independ on h_i, H_i and the jumps of ρ_i.

Proof. To prove (5.37) we can replace the terms $a_h(I_iu^{(i)}, I_iu^{(i)})$ and $b_i(u^{(i)}, u^{(i)})$ by $d_h(\mathcal{H}_iI_iu^{(i)}, \mathcal{H}I_iu^{(i)})$ and $d_i(\mathcal{H}_iu^{(i)}, \mathcal{H}_iu^{(i)})$, respectively; see Lemma 2.1 and Lemma 3.1.

In order to simplify notations, all the functions are considered as harmonic extensions in the $\mathcal{H}$ sense. Hence, we denote $\mathcal{H}I_iu^{(i)}$ by $D^{(i)}u^{(i)}$ and $\mathcal{H}u^{(i)}$ by $u^{(i)}$ and let $u^{(i)} = \{u_l^{(i)}\}_{l\in\#(i)} \in V_i$. Using (2.7), (2.8) and (4.7) we obtain

$$d_h(D^{(i)}u^{(i)}, D^{(i)}u^{(i)}) = d_i(D^{(i)}u^{(i)}, D^{(i)}u^{(i)}) + \sum_j d_j(D^{(i)}u^{(i)}, D^{(i)}u^{(i)}) \tag{5.38}$$

where the sum is taken over Ω_j with common faces or part of faces to Ω_i. The first term of the right-hand side of (5.38) can be estimated as follows. From the definition of d_i in (2.7) we write

$$\begin{aligned} &d_i(D^{(i)}u^{(i)}, D^{(i)}u^{(i)}) \\ &= \rho_i\|\nabla D_i^{(i)}u_i^{(i)}\|^2_{L^2(\Omega_i)} + \sum_{F_{ij}\subset\partial\Omega_i} \frac{\delta}{l_{ij}}\frac{\rho_{ij}}{h_{ij}} \parallel D_i^{(i)}u_i^{(i)} - D_j^{(i)}u_j^{(i)} \parallel^2_{L^2(F_{ij})} . \end{aligned} \tag{5.39}$$

We now bound the first term of (5.39). We note that

$$\rho_i \parallel \nabla D_i^{(i)}u_i^{(i)} \parallel^2_{L^2(\Omega_i)} \le 2\rho_i \left\{ \parallel \nabla(D_i^{(i)}u_i^{(i)} - u_i^{(i)}) \parallel^2_{L^2(\Omega_i)} + \parallel \nabla u_i^{(i)} \parallel^2_{L^2(\Omega_i)} \right\} \tag{5.40}$$

and observe that from the definition of $D_i^{(i)}$ in (4.5) and (4.6) we have

$$\rho_i \parallel \nabla(D_i^{(i)}u_i^{(i)} - u_i^{(i)}) \parallel^2_{L^2(\Omega_i)} \le C \sum_{\delta_{ij}\subset\partial\Omega_i} \rho_i \parallel \tilde{u}_i^{(i)} \parallel^2_{H_{00}^{1/2}(\delta_{ij})} . \tag{5.41}$$

Here, $\tilde{u}_i^{(i)} = u_i^{(i)}$ at the nodal points in $\overset{\circ}{\delta}_{ijh_i}$, and $\tilde{u}_i^{(i)} = 0$ at $\partial^e\delta_{ijh_i}$ and at the remaining nodes of $\partial\Omega_{ih_i}$. Note that the support of $\tilde{u}_i^{(i)}$ is contained in δ_{ij}. Also recall that δ_{ij} denotes F_{ij} when F_{ij} is a slave side. For $i \in \mathcal{N}_I$, the local function $u_i^{(i)}$ has zero average value on $\partial\Omega_i$, hence, we can bound the $H_{00}^{1/2}-$norm of $\tilde{u}_i^{(i)}$ by (see for example [30])

$$\rho_i \parallel \tilde{u}_i^{(i)} \parallel^2_{H_{00}^{1/2}(\delta_{ij})} \preceq \left(1 + \log\frac{H_i}{h_i}\right)^2 \rho_i|u_i^{(i)}|^2_{H^1(\Omega_i)}.$$

For $i \in \mathcal{N}_B$ we have

$$\begin{aligned} \rho_i\|\tilde{u}^{(i)}\|^2_{H_{00}^{1/2}(\delta_{ij})} &\preceq \left(1 + \log\frac{H_i}{h_i}\right)^2 \rho_i \left\{ |u_i^{(i)}|^2_{H^1(\Omega_i)} + \frac{1}{H_i^2}\|u_i^{(i)}\|^2_{L^2(\Omega_i)} \right\} \\ &\preceq \left(1 + \log\frac{H_i}{h_i}\right)^2 \rho_i \left\{ |u_i^{(i)}|^2_{H^1(\Omega_i)} + \frac{1}{h_i}\|u_i^{(i)}\|^2_{L^2(F_{i\partial})} \right\} \end{aligned} \tag{5.42}$$

where $F_{i\partial} \subset \partial\Omega$. To get the inequality in (5.42) we have used the following estimate

$$\begin{aligned} \frac{1}{H_i^2}\|u_i^{(i)}\|^2_{L^2(\Omega_i)} &\le \frac{2}{H_i^2}\left\{ \|u_i^{(i)} - \bar{u}_{i\partial}^{(i)}\|^2_{L^2(\Omega_i)} + \|\bar{u}_{i\partial}^{(i)}\|^2_{L^2(\Omega_i)} \right\} \\ &\preceq |u_i^{(i)}|^2_{H^1(\Omega_i)} + \frac{1}{h_i}\|u_i^{(i)}\|^2_{L^2(F_{i\partial})} \end{aligned} \tag{5.43}$$

where $\bar{u}_{i\partial}^{(i)} = \int_{F_{i\partial}} u_i^{(i)}ds/|F_{i\partial}|$. Note that we have used the assumption that the size $|F_{i\partial}| \asymp H_i$ in order to avoid an extra log factor in the inequality (5.43). In the case this assumption is not satisfied, the coarse basis function Θ_i must be added to the coarse problem to obtain the estimate with $(1+\log(H_i/h_i))^2$ factor. We point out that the coarse space $V_{0,I\cup B}$ defined later in (5.59) includes automatically such functions; see that Theorem 5.5 below does not assume that the size of $\partial\Omega_i \cap \partial\Omega$ is of the same order as the diameter of Ω_i. Using the estimates (5.41) and (5.42) in (5.40) we get

$$\begin{aligned}
&\rho_i \|\nabla D_i^{(i)} u_i^{(i)}\|^2_{L^2(\Omega_i)} \\
\preceq\; &\left(1+\log\frac{H_i}{h_i}\right)^2 \rho_i \Big\{ \|\nabla u_i^{(i)}\|^2_{L^2(\Omega_i)} + \frac{1}{h_i}\|u_i^{(i)}\|_{L^2(F_{i\partial})} \Big\}.
\end{aligned} \tag{5.44}$$

We now estimate the second terms of (5.39) with $F_{ij} \subset \partial\Omega_i$. First note that the estimate is straightforward for boundary faces $F_{i\partial}$ since by definition $u_\partial^{(i)} = 0$ and $D_i^{(i)} = 1$ on $F_{i\partial}$. We now estimate the terms of (5.39) when the $\delta_{ij} = F_{ij}$ is a slave side. From (4.5) and (4.6) we have

$$\| D_i^{(i)} u_i^{(i)} - D_j^{(i)} u_j^{(i)} \|^2_{L^2(\delta_{ij})} \preceq h_i \max_{\delta_{ij}} |u_i^{(i)}|^2,$$

and recalling that $\rho_i \asymp \rho_{ij} \preceq \rho_j$ and $h_i \asymp h_{ij} \preceq h_j$ we obtain

$$\begin{aligned}
&\frac{\rho_{ij}}{h_{ij}} \| D_i^{(i)} u_i^{(i)} - D_j^{(i)} u_j^{(i)} \|^2_{L^2(\delta_{ij})} \preceq \rho_i \max_{\delta_{ij}} |u_i^{(i)}|^2 \\
\preceq\; &\left(1+\log\frac{H_i}{h_i}\right) \rho_i \left\{ |u_i^{(i)}|^2_{H^1(\Omega_i)} + \frac{1}{H_i^2}\|u_i^{(i)}\|^2_{L^2(\Omega_i)} \right\}.
\end{aligned} \tag{5.45}$$

To estimate of the second term of the right-hand side in (5.45) we use a Poincaré inequality (recall $u_i^{(i)}$ has zero average value on $\partial\Omega_i$) when $i \in \mathcal{N}_I$, and we use the inequality (5.43) when $i \in \mathcal{N}_B$. Thus

$$\begin{aligned}
&\frac{\rho_{ij}}{h_{ij}} \|D_i^{(i)} u_i^{(i)} - D_j^{(i)} u_j^{(i)}\|^2_{L^2(\delta_{ij})} \\
\preceq\; &(1+\log\frac{H_i}{h_i})\; \rho_i \left\{ |u_i^{(i)}|^2_{H^1(\Omega_i)} + \frac{1}{h_i}\|u_i^{(i)}\|^2_{L^2(F_{i\partial})} \right\}.
\end{aligned} \tag{5.46}$$

Now consider the case where $\gamma_{ij} = F_{ij}$ is a master side. Remember that on a master side, $h_j \asymp h_{ij} \preceq h_i$ and $\rho_j \asymp \rho_{ij} \preceq \rho_i$. We have

$$\| D_i^{(i)} u_i^{(i)} - D_j^{(i)} u_j^{(i)} \|_{L^2(\gamma_{ij})} \leq \| u_i^{(i)} - u_j^{(i)} \|_{L^2(\gamma_{ij})} + \| z_j^{(i)} \|_{L^2(F_{ji})}, \tag{5.47}$$

where

$$z_j^{(i)} = \sum_{x_k^j \in \partial^e F_{jih_j}} u_j^{(i)}(x_k^j)\varphi_k^j.$$

Here, φ_k^j are the nodal basis functions on F_{ji,h_j} corresponding to the nodes x_k^j on $\partial^e F_{ji,h_j}$. Let us denote the support of $z_j^{(i)}$ by $S_{z_j^{(i)}}$ on F_{ji} and see that $|S_{z_j^{(i)}}| \preceq h_j$. We have

$$\| z_j^{(i)} \|_{L^2(S_{z_j^{(i)}})} \preceq \| u_j^{(i)} \|_{L^2(S_{z_j^{(i)}})} \preceq \|u_j^{(i)} - u_i^{(i)}\|^2_{L^2(\gamma_{ij})} + \|u_i^{(i)}\|^2_{L^2_{S(z_j^{(i)})}}. \tag{5.48}$$

The second term of the right-hand side of (5.48) can be estimated by

$$\|u_i^{(i)}\|^2_{L^2(S_{z_j^{(i)}})} \preceq Ch_j \max_{F_{ij}} |u_i^{(i)}|^2$$

$$\preceq h_i \left(1 + \log \frac{H_i}{h_i}\right) \left\{ |u_i^{(i)}|^2_{H^1(\Omega_i)} + \frac{1}{h_i} \|u_i^{(i)}\|^2_{L^2(F_{i\partial})} \right\}, \tag{5.49}$$

where we have used a Poincaré inequality for $i \in \mathcal{N}_I$ and the estimate (5.43) for $i \in \mathcal{N}_B$. Using (5.48) and (5.49) in (5.47) we get

$$\frac{\rho_{ij}}{h_{ij}} \|D_i^{(i)} u_i^{(i)} - D_j^{(i)} u_j^{(i)}||^2_{L^2(F_{ij})} \preceq \tag{5.50}$$

$$\frac{\rho_{ij}}{h_{ij}} \|u_j^{(i)} - u_i^{(i)}\|^2_{L^2(F_{ij})} + \left(1 + \log \frac{H_i}{h_i}\right) \rho_i \left\{ \|\nabla u_i^{(i)}\|^2_{L^2(\Omega_i)} + \frac{1}{h_i} \|u_i^{(i)}\|^2_{L^2(F_{i\partial})} \right\}.$$

We now use the estimates (5.44), (5.45) and (5.50) in (5.39) and Lemma 2.1 to obtain

$$d_i(D^{(i)} u^{(i)}, D^{(i)} u^{(i)}) \preceq \left(1 + \log \frac{H_i}{h_i}\right)^2 b_i(u^{(i)}, u^{(i)}). \tag{5.51}$$

We now estimate the second terms of (5.38) by bounding $d_j(D^{(i)} u^{(i)}, D^{(i)} u^{(i)})$ by $b_i(u^{(i)}, u^{(i)})$. For $u = \{u_j^{(i)}\} \in V_i$ we have

$$d_j(\tilde{I}_i D^{(i)} u^{(i)}, \tilde{I}_i D^{(i)} u^{(i)})$$

$$= \rho_j \parallel \nabla D_j^{(i)} u_j^{(i)} \parallel^2_{L^2(\Omega_j)} + \frac{\delta}{l_{ij}} \frac{\rho_{ij}}{h_{ij}} \int_{F_{ij}} (D_i^{(i)} u_i^{(i)} - D_j^{(i)} u_j^{(i)})^2 dx. \tag{5.52}$$

We only need to estimate the first term of (5.52) since the second term has been already estimated; see (5.45) and (5.50). If F_{ij} is a slave side of $\partial\Omega_i$ then $D_j^{(i)}$ vanishes, and so vanishes $\parallel \nabla D_j^{(i)} u_j^{(i)} \parallel^2_{L^2(\Omega_j)}$. We now estimate the case where γ_{ij} is a master side of $F_{ij} = \partial\Omega_i \cap \partial\Omega_j$. On F_{ji} we decompose $u_j^{(i)} = w_j^{(i)} + \sum_{x_k^j \in \partial^e F_{jih_j}} u_j^{(i)}(x_k^j) \varphi_k^j$, where $w_j^{(i)} = D_j^{(i)} u_j^{(i)}$, i.e., $w_j^{(i)}$ equals $u_j^{(i)}$ at the nodes in $\overset{\circ}{F}_{jih_j}$ and zero at the nodes in $\partial^e F_{jih_j}$. Note that the support of $w_j^{(i)}$ belongs to F_{ji}. We have

$$\parallel \nabla w_j^{(i)} \parallel^2_{L^2(\Omega_j)} \preceq \parallel w_j^{(i)} \parallel^2_{H^{1/2}_{00}(F_{ji})}$$

$$= \{ |w_j^{(i)}|^2_{H^{1/2}(F_{ji})} + \int_{F_{ji}} \frac{(w_j^{(i)})^2}{\text{dist}(s, \partial F_{ji})} ds \}. \tag{5.53}$$

We now estimate the first term of (5.53). Let Q_j be the L_2- projection on the h_j-triangulation of F_{ji}. Then

$$|w_j^{(i)}|^2_{H^{1/2}(F_{ji})} \leq 2\{ |w_j^{(i)} - Q_j u_i^{(i)}|^2_{H^{1/2}(F_{ji})} + |Q_j u_i^{(i)}|^2_{H^{1/2}(F_{ji})} \} \tag{5.54}$$

$$\preceq \frac{1}{h_j} \parallel w_j^{(i)} - u_i^{(i)} \parallel^2_{L^2(F_{ji})} + \parallel \nabla u_i^{(i)} \parallel^2_{L^2(\Omega_i)}$$

and

$$\begin{aligned}&\| \ w_j^{(i)} - u_i^{(i)} \ \|^2_{L^2(F_{ji})} \\ (5.55)\qquad &\leq 2\{\| \ u_j^{(i)} - u_i^{(i)} \ \|^2_{L^2(F_{ji})} + \| \sum_{x_k^j \in \partial^e F_{ji,h_j}} u_j^{(i)}(x_k^j)\varphi_v^j \ \|^2_{L^2(F_{ji})}\}\end{aligned}$$

where the second term of (5.55) can be bounded as before, see (5.47)-(5.49).

It remains to estimate the second term of (5.53). In order to simplify the arguments, we take F_{ij} as the interval $[0, H]$. Note that

$$(5.56)\qquad \int_{F_{ji}} \frac{(w_j^{(i)})^2}{\mathrm{dist}(s, \partial F_{ji})} ds \preceq \int_0^{H/2} \frac{(w_j^{(i)})^2}{s} ds + \int_{H/2}^{H} \frac{(w_j^{(i)})^2}{(H-s)}.$$

Let us estimate the first term in the right-hand side of (5.56). Let A be the most left node of $\overset{\circ}{F}_{jih_j}$ in $[0, H/2]$ and note the size of the interval of $[0, A]$ is $O(h_j)$. We have

$$\begin{aligned}\int_0^{H/2} \frac{(w_j^{(i)})^2}{s} ds \;&=\; \int_0^{A} \frac{(w_j^{(i)})^2}{s} ds + \int_A^{H/2} \frac{(u_j^{(i)})^2}{s} ds \\ &\preceq\; (u_j^{(i)}(A))^2 + \int_A^{H/2} \frac{(u_i^{(i)} - u_j^{(i)})^2}{s} ds + \int_A^{H/2} \frac{(u_i^{(i)})^2}{s} ds \\ &\preceq\; (u_j^{(i)}(A))^2 + \frac{1}{h_j} \| \ u_i^{(i)} - u_j^{(i)} \ \|^2_{L^2(F_{ji})} \\ &\quad + \left(1 + \log \frac{H_j}{h_j}\right) \max_{F_{ij}} |u_i^{(i)}|^2 \\ &\preceq\; \frac{1}{h_j} \| \ u_i^{(i)} - u_j^{(i)} \ \|^2_{L^2(F_{ij})} \\ &\quad + \left(1 + \log \frac{H_i}{h_i}\right)\left(1 + \log \frac{H_j}{h_j}\right) \left\{|u_i^{(i)}|^2_{H^1(\Omega_i)} + \frac{1}{h_i}\|u_i^{(i)}\|^2_{L^2(F_{i\partial})}\right\}\end{aligned}$$

where $(u_j^{(i)}(A))^2$ has been estimated using (5.48) and (5.49). The second term of (5.56) is estimated similarly. Substituting these estimates in (5.56) we get

$$\begin{aligned}(5.57)\qquad &\int_{F_{ji}} \frac{(w_j^{(i)})^2}{\mathrm{dist}(s, \delta F_{ji})} ds \\ &\preceq \left(1 + \log \frac{H}{h}\right)^2 \{\| \ \nabla u_i^{(i)} \ \|^2_{L^2(\Omega_i)} \ + \ \frac{1}{h_j} \| \ u_i^{(i)} - u_j^{(i)} \ \|^2_{L^2(F_{ij})} + \frac{1}{h_i}\|u_i^{(i)}\|^2_{L^2(F_{i\partial})}\}.\end{aligned}$$

Substituting (5.54) and (5.57) and together with (5.55) into (5.53), and then substituting this resulting estimate with (5.45) and (5.50) into (5.52), and using Lemma 2.1, we get

$$(5.58)\qquad d_j(D^{(i)}u^{(i)}, D^{(i)}u^{(i)}) \preceq \left(1 + \log \frac{H}{h}\right)^2 b_i(u^{(i)}, u^{(i)}).$$

Using (5.51) and (5.58) in (5.38), we get

$$d_h(D^{(i)}u^{(i)}, D^{(i)}u^{(i)}) \preceq \left(1 + \log \frac{H}{h}\right)^2 b_i(u^{(i)}, u^{(i)}).$$

□

5.2. **Additive Schwarz method with the $V_{0,I\cup B}$ coarse space.** We recall that the upper bound $C(1+\log H/h)^2$ in Theorem 5.2 requires the condition that $|\partial\Omega_i \cap \partial\Omega| \asymp H_i$ for all $i \in \mathcal{N}_B$. Without this condition we obtain an upper bound $C(1+\log H/h)^3$; see the discussion below (5.43). To obtain an upper bound $C(1+\log H/h)^2$ without this condition, we enhance the coarse space $V_{0,I}$, see (5.5), by adding boundary coarse basis functions, i.e.,

$$V_{0,I\cup B} = \text{Span}\,\{\Theta_i\}_{i\in\mathcal{N}_I\cup\mathcal{N}_I}\,. \tag{5.59}$$

The additive preconditioner is then defined by

$$T_{as,I\cup B} = \sum_{i=0}^{N} T_i, \tag{5.60}$$

where the T_0 is defined as in (5.7) except that now we replace $V_{0,I}$ by $V_{0,I\cup B}$. We then obtain:

Theorem 5.5. *Let the Assumption 4.1 be satisfied. Then there exist positive constants C_1 and C_2 independent of h_i, H_i and the jumps of ρ_i such that*

$$C_1 a_h(u,u) \le a_h(T_{as,I\cup B}u,u) \le C_2\left(1+\log\frac{H}{h}\right)^2 a_h(u,u) \quad \textit{for all } u\in V. \tag{5.61}$$

Here $\log(H/h) = \max_i\ \log(H_i/h_i)$.

Proof. Use that $V_{0,I} \subset V_{0,I\cup B} \subset V$ and repeat the proof of Theorem 5.2 with the discussion below (5.43). □

Remark 5.6. There are cases where we can use the following coarse bilinear form,

$$\tilde{b}_0(u,v) = \left(1+\log\frac{H}{h}\right)^{-1} a_h(u,v), \quad u,v\in V_0 \tag{5.62}$$

and still keeping the two logs result (5.10) and (5.61) of Theorem 5.2 and Theorem 5.5, respectively. The cases are when the size of any face or part of a face F_{ij} and $F_{i\partial}$ are of the same order as H_i. In such cases, it is easy to see that (5.22) in Lemma 5.3 will hold with only one log; see the discussions in (5.29) and (5.31).

Remark 5.7. Finally we point ourt that all the bilinear forms $b_i, i=0,\cdots,N$ considered until now were based on exact solvers, i.e., based on the bilinear forms $a_h(\hat{\mathcal{H}}u^{(0)},\hat{\mathcal{H}}u^{(0)})$ and $\hat{a}_i(\hat{\mathcal{H}}_i u^{(i)},\hat{\mathcal{H}}_i u^{(i)})$. We note that, due to Lemma 2.1 and Lemma 3.1, all the results will still hold if we replace those bilinear forms by $d_h(\mathcal{H}u^{(0)},\mathcal{H}u^{(0)})$, and $d_i(\mathcal{H}_i u^{(i)},\mathcal{H}_i u^{(i)})$, respectively.

6. Hybrid preconditioners

In this section we design and analyze an hybrid type (BDD) method for solving (2.26); see [26, 30]. We consider the hybrid version of $T_{as,I}$, see (5.8). The hybrid version of $T_{as,I\cup B}$, see (5.60), can be treated similarly.

6.1. **The method.** Recall the definition of the Γ_i in (4.1), the spaces W_i in (4.2), the local subspaces V_i in (5.1) and the coarse subspace $V_0 = V_{0,I}$ in (5.5). Consider the bilinear forms $b_i, i = 1, \cdots, N$, defined in (5.2).

Now define bilinear form a_0 as the exact bilinear form a_h, i.e.,

$$a_0(u,v) = a_h(u,v), \quad u,v \in V_0, \tag{6.1}$$

with $\hat{\mathcal{H}}$ defined in (2.17). Introduce the coarse projection $P_0 : V \to V_0$ defined by

$$a_0(P_0 u, v) = a_h(u,v) \quad \text{for all } v \in V_0. \tag{6.2}$$

The hybrid method is defined as (see [30])

$$T_{hyb,I} = P_0 + (I - P_0)\left(\sum_{i=1}^{N} T_i\right)(I - P_0), \tag{6.3}$$

where the operators T_i were defined as $T_i = I_i\tilde{T}_i$ with $\tilde{T}_i$ defined by (5.3), $i = 1, \ldots, N$.

Let the subspace $V_0^{\perp} \subset V$ consists of functions $w \in V$ such that $a_h(w, v_0) = 0$, for all $v_0 \in V_0$. It is easy to check that if $w \in V_0^{\perp}$ then $T_{hyb,I}w \in V_0^{\perp}$. The PCG algorithm for solving $T_{hyb,I}v = w$, $w \in V_0^{\perp}$, searches for the best approximation to the solution in the Krylov subspace generated by powers of $T_{hyb,I}$ applied to w. Assume the goal is to solve $Su = g$, where $u = u_h^*$, see (2.26). We replace this equation by $T_{hyb,I}u = \tilde{g}$ where $\tilde{g} = T_{hyb,I}u$, and compute $u_0 = P_0u$. The computations of $\tilde{g}$ and u_0 can be obtained directly from g without the knowledge of u by (5.3) and (5.7), respectively; see also [30]. Note that in our case $u = v + u_0$ and $w = T_{hyb,I}u - P_0u$ belongs to $V_0^{\perp}$. Then we can solve $T_{hyb,I}v = w$ using the PCG algorithm operated on the subspace $V_0^{\perp}$.

6.2. **Condition number estimate for $T_{hyb,I}$.** ¿From the analysis of the additive method $T_{as,I}$ developed in Theorem 5.2 we can derive an analysis for the hybrid method $T_{hyb,I}$. Observe that in both methods we have considered the same local and coarse spaces. Note also that in the design of the hybrid method $T_{hyb,I}$ we have considered the bilinear form $a_0(\cdot,\cdot)$ defined in (6.1) rather than the bilinear form $b_0(\cdot,\cdot)$ defined in (5.6). These two bilinear forms differ only from each other by a scaling factor. For both methods we have considered the same local bilinear forms $b_i(\cdot,\cdot)$ defined in (5.2).

Theorem 6.1. *Let the Assumption 4.1 be satisfied. In addition, assume that for $i = 1, \cdots, N$, the size of $\partial\Omega_i \cap \partial\Omega$ is of the same order as the diameter of Ω_i. Then there exists a positive constant C independent of h_i, H_i and the jumps of ρ_i such that*

$$a_h(u,u) \le a_h(T_{hyb,I}u,u) \le C\left(1 + \log\frac{H}{h}\right)^2 a_h(u,u) \text{ for all } u \in V_0^{\perp}. \tag{6.4}$$

Here $\log(H/h) = \max_i \log(H_i/h_i)$.

Proof. Upper Bound: Using Rayleigh quotient arguments and properties of the orthogonal projection P_0, i.e., that $(I - P_0)P_0 = 0$, we obtain

$$\lambda_{\max}(T_{hyb,I|_{V_0^{\perp}}}) = \max_{u \in V_0^{\perp}\setminus\{0\}} \frac{a(T_{hyb,I}u,u)}{a(u,u)}$$

$$= \max_{u \in V_0^\perp \setminus \{0\}} \frac{a_h(\sum_{i=1}^N T_i u, u)}{a_h(u,u)} = \max_{u \in V_0^\perp \setminus \{0\}} \frac{a_h([\gamma P_0 + \sum_{i=1}^N T_i] u, u)}{a_h(u,u)}$$

$$\leq \max_{u \in V \setminus \{0\}} \frac{a_h([\gamma P_0 + \sum_{i=1}^N T_i] u, u)}{a_h(u,u)} = \lambda_{\max}(T_{as,I}),$$

where $\gamma = \left(1 + \log \frac{H}{h}\right)^{-2}$ and $T_{as,I}$ defined in (5.8). Hence, the upper bound follows from the upper bound of Theorem 5.2.

Lower Bound: We obtain

$$\lambda_{\min}(T_{hyb,I|V_0^\perp}) = \min_{u \in V_0^\perp \setminus \{0\}} \frac{a(T_{hyb,I} u, u)}{a(u,u)}$$

$$= \min_{u \in V_0^\perp \setminus \{0\}} \frac{a_h(\sum_{i=1}^N T_i u, u)}{a_h(u,u)} = \sup_{\epsilon > 0} \min_{u \in V_0^\perp \setminus \{0\}} \frac{a_h([\epsilon P_0 + \sum_{i=1}^N T_i] u, u)}{a_h(u,u)}$$

$$\geq \sup_{\epsilon > 0} \min_{u \in V \setminus \{0\}} \frac{a_h([\epsilon P_0 + \sum_{i=1}^N T_i] u, u)}{a_h(u,u)}.$$

It remains to show

$$\sup_{\epsilon > 0} \min_{u \in V \setminus \{0\}} \frac{a_h([\epsilon P_0 + \sum_{i=1}^N T_i] u, u)}{a_h(u,u)} \geq 1. \tag{6.5}$$

Let $\epsilon > 0$ be fixed. A lower bound estimation for $\epsilon P_0 + \sum_{i=1}^N T_i$ can be obtained from the general theory of ASMs where we need to check the Assumption i) of Lemma 5.1. To check this assumption, let $u \in V$ and consider the same decomposition $\sum_{i=1}^N I_i u^{(i)} = u$ described in the proof of Theorem 5.2, i.e., $u^{(0)}$ defined in (5.12) and the $u^{(i)}, i = 1, \cdots, N$ defined in (5.14). Using the same steps of the proof of Theorem 5.2 we obtain

$$\epsilon a_h(u_0, u_0) \leq C \epsilon (1 + \log \frac{H}{h})^2 a_h(u,u),$$

and

$$\sum_{i=1}^N b_i(u^{(i)}, u^{(i)}) = a_h(u,u).$$

Note that to obtain this equality we do not use d_i as in (5.17). Instead, we work with $\hat{a}_i$ and we get an equality in (5.17) with right-hand side equals to $\hat{a}_i$. Summing these equalities we get the above estimates. Hence, we obtain

$$\epsilon a_h(u_0, u_0) + \sum_{i=1}^N b_i(u^{(i)}, u^{(i)}) \leq \left(1 + C\epsilon (1 + \log \frac{H}{h})^2\right) a_h(u,u),$$

and therefore

$$\sup_{\epsilon > 0} \min_{u \in V \setminus \{0\}} \frac{a_h([\epsilon P_0 + \sum_{i=1}^N T_i] u, u)}{a_h(u,u)} \geq \sup_{\epsilon > 0} \left(1 + C\epsilon (1 + \log \frac{H}{h})^2\right)^{-1} = 1.$$

□

7. Numerical experiments

In this section, we present numerical results for the preconditioners introduced in (5.8), (5.60) and (6.3), for the geometrically conforming and nonconforming cases. We also show that the bounds of Theorems 5.2, 5.5 and 6.1 are reflected in the numerical tests. In particular we show that the interface condition (Assumption 4.1) is necessary and sufficient.

7.1. **Geometrically conforming case.** Let choose the domain $\Omega = (0,1)^2$ and divide it into $N = M \times M$ equally spaced squares subdomains Ω_i. Inside each subdomain Ω_i we generate a structured triangulation with n_i subintervals in each coordinate direction and apply the discretization presented in Section 2 with $\delta = 4$. In the numerical experiments we use a red and black checkerboard type of subdomain partition. On the black subdomains we let $n_b = 2 * 2^{L_b}$ and on the red subdomains we let $n_r = 3{*}2^{L_r}$, where L_b and L_r are integers denoting the number of refinements inside each subdomain Ω_i. Hence, the mesh sizes are $h_b = \frac{2^{-L_b}}{2M}$ and $h_r = \frac{2^{-L_r}}{3M}$, respectively. We solve the second order elliptic problem $-\mathrm{div}(\rho(x)\nabla u^*(x)) = 1$ in Ω with homogeneous Dirichlet boundary conditions. We consider $\rho(x)$ to be piecewise constant with different positive constants in substructures. In the numerical experiments, we run PCG until the l_2 initial residual is reduced by a factor of 10^6.

7.1.1. *Hybrid preconditioner.* We first test the hybrid preconditioner (6.3). In the first test we consider the constant coefficient case $\rho_r = \rho_b = 1$. We consider different values of $M \times M$ coarse partitions and different values of local refinements $L_b = L_r$, therefore, keeping constant the mesh ratio $h_b/h_r = 3/2$. We place the master on the black subdomains. Table 1 lists the number of PCG iterations and in parenthesis the condition number of the preconditioned system. We note that the interface condition (Assumption 4.1) is satisfied. As expected from the analysis, the condition numbers appear to be independent of the number of subdomains and grow by a polylogarithmical factor when the size of the local problems increases. Note that in the case of continuous coefficients, as expected, the Theorem 6.1 is valid without any assumption on h_b and h_r since the master sides are chosen on the larger meshes.

Table 1. Geometrically conforming case: $T_{hyb,I}$ iterations count and condition numbers for different sizes of coarse and local problems and with constant coefficients $\rho_b = \rho_r = 1$ and $L_b = L_r$.

$L_r \downarrow M \rightarrow$	2	4	8	16
0	13 (6.86)	18 (8.39)	20 (8.89)	19 (9.02)
1	17 (8.97)	22 (11.30)	24 (11.57)	24 (11.63)
2	18 (12.12)	26 (14.74)	28 (14.82)	27 (14.83)
3	19 (16.82)	30 (19.98)	32 (20.03)	32 (20.05)
4	21 (22.23)	33 (26.64)	37 (26.64)	37 (26.67)
5	22 (28.25)	36 (34.19)	42 (34.04)	42 (34.06)

We now consider the discontinuous coefficient case where we set $\rho_b = 1$ on the black subdomains and $\rho_r = \mu$ on the red subdomains. The subdomain partition is

TABLE 2. Geometrically conforming case: $T_{hyb,I}$ iterations count and condition numbers for different values of the coefficients $\rho_r = \mu$ and local meshes with L_r refinements on the red subdomains. On black subdomains the coefficients $\rho_r = 1$ and refinement $L_b = 0$ are kept fixed. The subdomain partition is also kept fixed to 4×4.

$L_r \downarrow \mu \rightarrow$	1000	10	0.1	0.001
0	90 (2556)	33 (29.16)	17 (8.28)	18 (8.83)
1	133 (3744)	40 (42.31)	19 (8.70)	18 (8.95)
2	184 (5362)	47 (58.20)	19 (9.21)	18 (9.46)
3	237 (7178)	52 (75.55)	19 (9.50)	18 (9.83)
4	303 (9102)	57 (94.59)	19 (9.65)	18 (10.08)

kept fixed to 4×4. Table 2 lists the results on runs for different values of μ and for different levels of refinements L_r on the red subdomains. On the black subdomains $n_b = 2$ is kept fixed. The masters are placed on the black subdomains. It is easy to see that the interface condition (Assumption 4.1) is a sufficient and necessary condition for the robustness of the solver.

7.1.2. *Additive preconditioner.* We repeat the tests above however now for the additive preconditioner (5.8) and with the coarse bilinear form defined in (5.6). Numerically the coarse bilinear form defined in (5.6) showed slightly better results (not presented in this paper) than the coarse bilinear form defined according to (5.62). Tables 3 and 4 show the results. The results, as expected, are similar to the hybrid preconditioner and are consistent with Theorem 5.2. Even though the number of iterations for the additive Schwarz method is slightly larger than for the hybrid Schwarz method, we point out that the additive version has the advantage that it requires only one residual calculation per each PCG iteration while the hybrid version requires two residual calculations. In addition, the additive version, unlike the hybrid version, allows the use of some inexact local and global solvers; see Remark 5.7.

TABLE 3. Geometrically conforming case: $T_{as,I}$ iterations count, the condition numbers and minimal eigenvalue for different sizes of coarse and local problems and constant coefficients $\rho_b = \rho_r = 1$ and $L_b = L_r$. Here we use the coarse bilinear form b_0 defined in (5.6).

$L_r \downarrow M \rightarrow$	2	4	8	16
0	14 (8.10,1.00)	25 (30.26,0.44)	34 (38.89,0.34)	39 (40.46,0.33)
1	16 (10.50,1.00)	27 (28.23,0.59)	35 (35.82,0.48)	39 (37.58,0.47)
2	19 (14.23,1.01)	29 (29.94,0.70)	37 (39.86,0.59)	42 (41.80,0.58)
3	20 (18.40,1.03)	32 (36.06,0.78)	42 (46.88,0.68)	46 (49.02,0.66)
4	20 (23.47,1.03)	34 (44.14,0.83)	47 (55.90,0.75)	52 (58.23,0.73)

In the case of discontinuous coefficient we set as before $\rho_b = 1$ on the black subdomains and $\rho_r = \mu$ on the red subdomains. The subdomain partition is kept fixed to 4×4. Table 4 lists the results. On the black subdomains $n_b = 2$ is kept fixed

TABLE 4. Geometrically conforming case: $T_{as,I}$ iterations count and condition numbers of the additive preconditioner for different values of coefficients $\rho_r = \mu$ and the local mesh refinements L_r on the red subdomains only. The coefficients and the local mesh sizes on the black subdomains are kept fixed to $\rho_b = 1$ and $L_b = 0$. The subdomain partition is also kept fixed to 4×4. Here we used the coarse bilinear form b_0 defined in (5.6).

$L_r \downarrow \mu \rightarrow$	1000	10	1	0.1	0.001
0	179 (465239.38)	46 (184.52)	25 (30.26)	21 (18.13)	19 (13.79)
1	291 (638410.53)	61 (293.31)	26 (34.61)	21 (16.05)	18 (12.24)
2	400 (710636.95)	71 (429.62)	30 (43.08)	22 (15.45)	18 (11.55)
3	518 (666219.34)	78 (593.21)	32 (54.57)	22 (15.43)	18 (11.21)
4	660 (559822.47)	82 (788.51)	34 (68.52)	22 (15.71)	18 (11.04)
5	776 (449924.63)	89 (1019.62)	36 (84.68)	22 (16.14)	18 (10.97)

($L_b = 0$) and masters are placed on the black subdomains. We can see in Table 4 that the interface condition (Assumption 4.1) holds if and only if the preconditioner is robust.

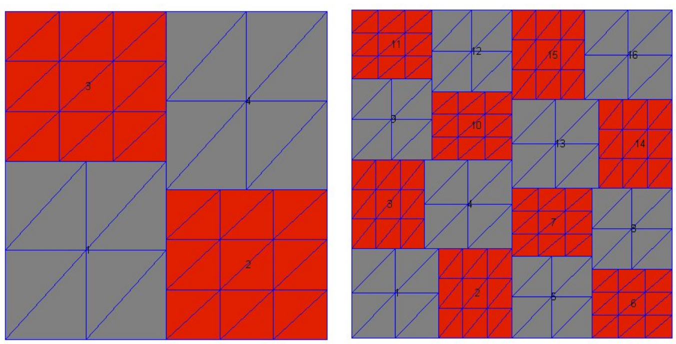

FIGURE 2. Geometrically nonconforming partition.

7.2. **Geometrically nonconforming case.** We consider the domain $\Omega = (0,1)^2$ and divide it into $N = M \times M$ rectangular geometrically nonconforming subdomains Ω_i as in Figure 2. In each subdomain, the next level of refinement is obtained from a regular conforming 2×2 rectangular refinement by enlarging (or decreasing) the width or high of some rectangles by a factor $fac = 1 + 1/23$ (or $1 - 1/23$); see Figure 2.

Note that $(fac)^{\log_2(M)} = O(1)$, therefore, $H \simeq 1/M$. Inside each subdomain Ω_i we generate a structured triangulation with n_i subintervals in each coordinate direction and apply the discretization presented in Section 2 with $\delta = 4$. In the numerical experiments we use a red and black checkerboard type of subdomain partition. On the black subdomains we let $n_b = 2 * 2^{L_b}$ and on the red subdomains we let $n_r = 3 * 2^{L_r}$, where L_b and L_r are integers denoting the number of refinements inside each subdomain Ω_i. Hence, since the size of each subdomain is $O(1/M)$ then the mesh sizes are $h_b \simeq \frac{1}{Mn_b}$ and $h_r \simeq \frac{1}{Mn_r}$, respectively. We solve the second order elliptic problem $-\mathrm{div}(\rho(x)\nabla u^*(x)) = 1$ in Ω with homogeneous Dirichlet boundary conditions. We repeat the experiment of Section 7.1.

7.2.1. *Hybrid preconditioner.* We first test the case $\rho_r = \rho_b = 1$. We consider also different values of $M \times M$ coarse partitions and different values of local refinements $L_b = L_r$. Here and on the tests below, we place the master on the black subdomain in the case that (a part of) a face $F_{ij} = \partial\Omega_i \cap \partial\Omega_j$ shares two different colors subdomains, and place on the most north-east subdomain otherwise. Table 5 lists the number of PCG iterations and in parenthesis the condition number estimate of the preconditioned system. We note that the interface condition (Assumption 4.1) is satisfied. As expected from the analysis, the condition numbers appear to be independent of the number of subdomains and grow by a polylogarithmical factor when the size of the local problems increases. Note that in the case of continuous coefficients Theorem 6.1 is valid without any assumption on h_b and h_r since the master sides are chosen on the larger meshes.

TABLE 5. Geometrically nonconforming case: $T_{hyb,I}$ iterations count, condition number and smallest eigenvalue of the hybrid preconditioner for different sizes of coarse and local problems and with constant coefficients $\rho_b = \rho_r = 1$.

$L_r \downarrow M \rightarrow$	2	4	8	16
0	12(6.22,1.00)	18(7.83,1.03)	22(10.79,1.02)	23 (12.23,1.01)
1	16(7.95,1.00)	24(11.74,1.02)	26(13.98,1.01)	27 (14.76,1.01)
2	20(16.03,1.05)	29(18.87,1.01)	32(18.77,1.01)	32 (18.96,1.00)
3	22(20.24,1.06)	32(24.13,1.00)	36(23.87,1.00)	36 (24.51,1.00)
4	23(29.05,1.02)	37(33.34,1.00)	42(33.33,1.00)	43 (34.60,1.00)

TABLE 6. Geometrically nonconforming case: $T_{hyb,I}$ iterations count and condition numbers for different values of the coefficients $\rho_r = \mu$ and local meshes with L_r refinements) on the red subdomains. On black subdomains the coefficients $\rho_r = 1$ and refinement $L_b = 0$ are kept fixed. The subdomain partition is also kept fixed to 4×4.

$L_r \downarrow \mu \rightarrow$	1000	10	1	0.1	0.001
0	93(3069.53)	34(34.84)	18(7.83)	18(8.93)	18(9.73)
1	120(4530.84)	43(50.36)	21(10.35)	19(9.60)	19(10.45)
2	175(4990.32)	48(54.73)	23(14.81)	20(15.60)	19(16.24)
3	235(6496.58)	53(69.84)	25(17.54)	20(17.41)	19(18.12)
4	336(7542.38)	57(79.24)	26(20.02)	21(20.05)	19(20.98)

We now consider the discontinuous coefficient case where we set $\rho_b = 1$ on the black subdomains and $\rho_r = \mu$ on the red subdomains. The substructures partition is kept fixed to 4×4. Table 6 lists the results on runs for different values of μ and for different levels of refinements L_r on the red subdomains. On the black subdomains $n_b = 2$ is kept fixed. It is easy to see in Table 6 that the interface condition (Assumption 4.1) holds if and only if the preconditioner is robust.

7.2.2. *Additive preconditioner.* We repeat the experiments done for the hybrid preconditioner in the geometrically nonconforming case. As before we consider the constant coefficient case $\rho_r = \rho_r = 1$, the mesh ratio $h_b/h_r = 3/2$. Table 7 shows that the condition numbers appear to be independent of the number of subdomains and grow by a polylogarithmical factor when the size of the local problems increases. As expected by Theorem 5.5, Table 8 shows that condition numbers do not change much when we replace $T_{as,I}$ to $T_{as,I\cup B}$.

TABLE 7. Geometrically nonconforming case: $T_{as,I}$ iterations count and condition numbers for different sizes of coarse and local problems and constant coefficients $\rho_b = \rho_r = 1$ and $L_b = L_r$.

$L_r \downarrow M \rightarrow$	2	4	8	16
0	11 (7.32,1.00)	24 (36.03,0.40)	42 (50.00,0.31)	51 (55.82,0.29)
1	17 (15.01,1.00)	30 (40.09,0.53)	39 (51.82,0.42)	45 (56.56,0.40)
2	22 (20.19,1.03)	32 (47.28,0.63)	42 (59.81,0.52)	46 (62.27,0.50)
3	23 (23.76,1.05)	35 (48.76,0.71)	43 (62.95,0.60)	47 (67.01,0.58)

TABLE 8. Geometrically nonconforming case: $T_{as,I\cup B}$ iterations count and condition numbers for different sizes of coarse and local problems and constant coefficients $\rho_b = \rho_r = 1$ and $L_b = L_r$.

$L_r \downarrow M \rightarrow$	2	4	8	16
0	13 (8.20,1.00)	23 (36.99,0.44)	40 (49.11,0.35)	44 (53.93,0.33)
1	18 (17.25,1.00)	31 (44.82,0.55)	40 (56.05,0.44)	46 (60.45,0.43)
2	23 (22.83,1.05)	35 (53.68,0.63)	43 (62.59,0.53)	48 (67.40,0.51)
3	25 (27.10,1.06)	37 (55.76,0.71)	44 (65.16,0.61)	48 (69.74,0.59)

TABLE 9. Geometrically nonconforming case: $T_{hyb,I}$ iterations count and condition numbers for different values of coefficients and the local mesh sizes on the red subdomains only. The coefficients and the local mesh sizes on the black subdomains are kept fixed. The number of subdomains are also kept fixed to 4×4 and $L_b = 0$.

$L_r \downarrow \mu \rightarrow$	1000	10	1	0.1	0.001
0	137 (225164.57)	47 (205.41)	24 (36.03)	22 (23.75)	22 (24.41)
1	218 (294622.11)	60 (333.17)	26 (42.50)	22 (23.05)	23 (22.86)
2	291 (291958.12)	67 (462.11)	28 (54.06)	22 (36.76)	23 (35.45)
3	395 (289006.68)	72 (621.10)	30 (65.35)	23 (39.06)	22 (36.61)
4	529 (273591.72)	77 (790.20)	32 (78.79)	24 (44.24)	23 (40.29)

Now the case of discontinuous coefficient $\rho_b = 1$ on the black subdomains and $\rho_r = \mu$ on the red subdomains. The subdomain partition is kept fixed to 4×4. Table 9 lists the results on runs for different values of μ and for different levels of refinements L_r on the red subdomains. On the black subdomains $n_b = 2$ is kept fixed, i.e., $L_b = 2$. It is easy to see in Table 9 that the interface condition (Assumption 4.1) holds if and only if the preconditioner is robust.

8. Conclusions

In this paper we consider a discontinuous Galerkin discretization of second order elliptic equations with discontinuous coefficients and nonmatching meshes on geometrically nonconforming substructures. We designed and analyzed Neumann-Neumann methods of additive and additive-multiplicative. We prove that the method is almost optimal and very well suited for parallel computations. The coarse space is constructed using a special partition of unity. The rate of convergence of both methods are polylogarithmically with repect to the local mesh size, and does not depend on the number of substructures and on the jumps of coefficients. The numerical tests confirm the theoretical results. The methods can be straightforwardly extended to 3-D cases.

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