

Real Option Pricing with Mean-Reverting Investment and Project Value

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1 Introduction

Quantitative methods to analyze the option to invest in a project enjoy a long and distinguished history. The classical work of McDonald and Siegel (1986) (see also Dixit and Pindyck (1994)) investigates the problem from the point of view of derivative pricing and assigns the project's value as

$$\text{value} = e^{-rT} \mathbb{E}[(V_T - I_T)_+] . \quad (1.1)$$

Here, the expected value is taken under an appropriate risk-adjusted measure. Furthermore, V_T and I_T represent the project's value and amount to be invested, respectively, at time T .

If the project can be started at anytime, then (1.1) is modified to its American counterpart. In this case, the maturity date T is replaced by a stopping time τ ($0 \leq \tau \leq T$) and the investor chooses the stopping time to maximize the option's value. As such, the problem becomes a free boundary problem in which the optimal strategy is computed simultaneously with the option's value.

Traditionally, the project value is assumed to be a geometric Brownian motion (GBM) and the investment amount is constant or deterministic, as in the pioneering work of Tourinho (1979). Stochastic investment amounts have also been investigated: the case of an investment that is driven by a GBM, in the special situation that the opportunity to invest does not expire in time i.e. a perpetual option, is treated in McDonald and Siegel (1986) See also Berk, Green, and Naik (1999). More recently, Elliott, Miao, and Yu (2007) have investigated the case of regime switching investment costs. It should be also pointed out that similar problems arise in swap options, as in Margrabe (1978), and in uncertain payoffs, as in Fischer (1978). **(MAX REFERENCE)**.

However, much of this work—e.g. McDonald and Siegel (1986) and Blenman and Clark (2005)—also assumes that the amount to be invested is also a GBM. A GBM may be a good model for the project value in certain circumstances, since in many cases it represents a net present value. On the other hand, as already noticed in McDonald and Siegel (1986), the investment costs are typically prices of commodities, and thus

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are expected to revert to an equilibrium level. Furthermore, in situations where the cashflows of the project are directed linked to commodities, then the project value is also expected to approach an equilibrium level. **there are many situations in which a GBM does not suffice.** One such situation is the valuation of the option to the invest in an oil field. Like most commodities, oil prices tend to mean-revert and as a direct result the value of investment in an oil field is also mean-reverting. Consequently, it would not be appropriate to use GBM to model the value of such a project. Of course, several authors have noticed this and mean-reverting processes have been considered in the more recent literature. See for example Metcalf and Hasset (1995) and Sarkar (2003). However, combining mean-reverting project value with mean-reverting investment amount has not been considered so far. There are good reasons for the amount to be invested to be mean-reverting. Consider an oil company which is contemplating to invest in a recently found oil field. The oil company's profits and therefore the amount available to invest, will tend to mean-revert.

2 Trigger curves for mean-reversion investments

The difficulty with allowing both project value V_t and investment amount I_t to mean-revert lies in the fact that the problem becomes a two-dimensional one and the optimal policy will depend on both V_t and I_t . In the case when both processes are GBM, the optimal policy depends only on the ratio V_t/I_t and the value of the option becomes homogeneous in I_t (or V_t)—**this has been already observed in McDonald and Siegel (1986) and it seems that this fact has become a paradigm in Real Options pricing.** We therefore seek a new mean-reverting model which produces the qualitative features of mean-reverting V_t and I_t while maintaining the homogeneity of the solution. To this end, our model assumes the following

$$V_t = e^{\theta + X_t}, \tag{2.1}$$

$$dX_t = -\alpha X_t dt + \sigma_X dW_t^X, \tag{2.2}$$

$$I_t = e^{\phi + Y_t}, \tag{2.3}$$

$$dY_t = -((\alpha - \beta)X_t + \beta Y_t) dt + \sigma_Y dW_t^Y, \tag{2.4}$$

Here, W_t^X and W_t^Y are, in general, correlated Brownian motions with correlation ρ . As usual we work on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = \{(\mathcal{F}_t)_{0 \leq t \leq T}\}$ ($\mathcal{F}_t = \sigma((W_s^X, W_s^Y)_{0 \leq s \leq t})$) is the natural filtration generated by the driving Brownian motions and \mathbb{P} is the statistical (historical) probability measure.

In this model, the value V_t of the project mean-reverts to a long-run level θ , while the investment I_t available for the project instantaneously mean-reverts to a stochastic level $\eta_t := \exp\{\phi + \frac{\alpha - \beta}{\beta} X_t\}$. However, the process X_t itself mean-reverts to zero, implying that $\exp\{\phi\}$ is the true long-run level of the investment process. This coupling of investment and value is not entirely artificial. In fact, it is quite reasonable to assume that the amount available for the investment is tied in some way to the value of the project itself. Nonetheless, this coupling of investment can be minimized by appropriate choices of the model parameters. In Figure 2.1, two sample paths for the value and investment are presented. Panel (a) contains no correlation between the increments in the investment level and value; however, since the investment is instantaneously pulled to the stochastic level η_t , there is some feedback effect. In fact, the processes X_t and Y_t are cointegrated.

Under the modeling assumption (2.1)-(2.4), the value V_t of the project and the value to investment ratio V_t/I_t are both mean-reverting processes *and* the dynamics of the ratio depends only on the ratio itself.

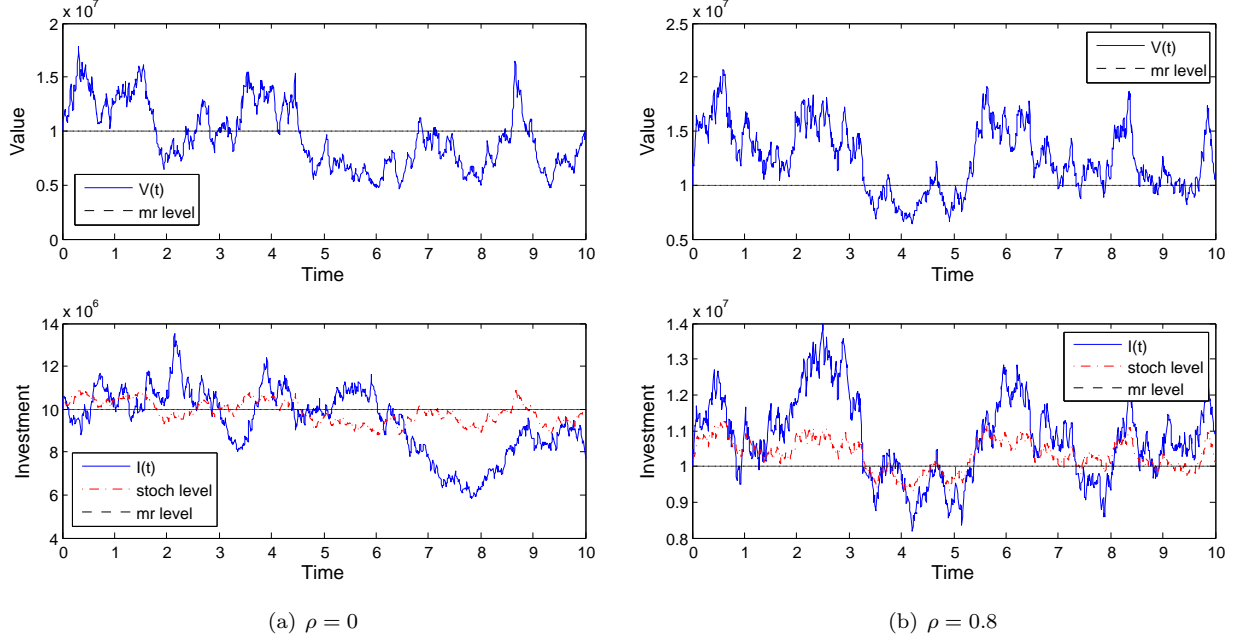


Figure 2.1: Two sample paths with differing levels of correlation. The lines label *mr level* are the long-run mean-reverting levels for the value and investment, while *stoch level* is the instantaneous mean reversion level of the investment.

Specifically, notice that $\frac{V_t}{I_t} = e^{(\theta-\phi)+(X_t-Y_t)}$ and define $Z_t = X_t - Y_t$, then

$$dZ_t = -\beta Z_t dt + \sigma_X dW_t^X - \sigma_Y dW_t^Y . \quad (2.5)$$

This implies that the ratio can be modeled directly as a mean-reverting process with mean-reversion rate β and effective instantaneous variance of $\sigma^2 := \sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y$.

We now investigate the option to invest under the modeling framework (2.1)-(2.4). The Bermudan option to invest, where investment can only be exercised at discrete times $\{t_0, t_1, \dots, t_n\}$ (e.g. quarterly, monthly, or weekly), can be priced recursively on the exercise dates as follows:

$$p_{t_n}(V_{t_n}, I_{t_n}) = (V_{t_n} - I_{t_n})_+ \quad (2.6)$$

$$p_{t_{m-1}} = \max \left\{ e^{-r\Delta t_m} \mathbb{E} \left[p_{t_m}(V_{t_m}, I_{t_m}) \mid \mathcal{F}_{t_{m-1}} \right]; (V_{t_{m-1}} - I_{t_{m-1}})_+ \right\}, \quad (2.7)$$

for $m = \{1, 2, \dots, n\}$. Let us proceed to describe how to value the option. First, we require

$$f_{t_{n-1}} \triangleq \mathbb{E} \left[(V_{t_n} - I_{t_n})_+ \mid \mathcal{F}_{t_{n-1}} \right] \quad (2.8)$$

$$= \mathbb{E} \left[(V_{t_n}/I_{t_n} - 1)_+ I_{t_n} \mid \mathcal{F}_{t_{n-1}} \right] . \quad (2.9)$$

If I_t were a geometric Brownian motion, then it would be straightforward to absorb I_t into a simple measure change – akin to a numeraire change. However, due to the mean-reverting behavior of I_t a more clever measure change is necessary to absorb it. To this end, introduce a new measure \mathbb{P}^T via the Radon-Nikodym derivative process

$$\eta_t^T \triangleq \left(\frac{d\mathbb{P}^T}{d\mathbb{P}} \right)_t = \frac{\mathbb{E}[I_T \mid \mathcal{F}_t]}{\mathbb{E}[I_T \mid \mathcal{F}_0]} . \quad (2.10)$$

Notice that $\eta_T^T = I_T / \mathbb{E}[I_T | \mathcal{F}_0]$ so that,

$$f_{t_{n-1}} = \mathbb{E}[I_{t_n} | \mathcal{F}_{t_{n-1}}] \mathbb{E}^{t_n} [(V_{t_n}/I_{t_n} - 1)_+ | \mathcal{F}_{t_{n-1}}] \quad (2.11)$$

where $\mathbb{E}^{t_n}[\cdot]$ denotes expectation under the new measure \mathbb{P}^{t_n} . Through recursive application of the described measure change, the option can be evaluated through a series of one-dimensional problems because the ratio V_{t_n}/I_{t_n} depends solely on the Z_t process and not X_t and Y_t individually. More specifically, a one-dimensional binomial tree can be developed for the ratio process; however, a new measure must be used between each exercise date. This does not pose any real problems and we are able to compute the optimal exercise policy as a function of V_t/I_t .

3 Beyond mean-reversion

The above procedure is appropriate when the process does not contain any jumps; however, if jumps are present, then alternative methods must be used. Firstly, jumps render a tree approximation inadequate – multinomial trees are possible, but inaccuracies arise quickly. Furthermore, finite-difference schemes require inverting dense matrices resulting in large slowdowns and potential errors due to truncation of large jumps. Secondly, the measure change induced by a jump process is more complicated, and although it is possible to derive the appropriate change, tractability is lost. Instead, we will now describe a variant of the mean-reverting Fourier Space Time-Stepping method of Jaimungal and Surkov (2009) appropriate for this real-options context and which is also easily extensible to incorporate jumps. See also Jackson, Jaimungal, and Surkov (2008) for the FST method without mean-reversion.

Consider the value of the option to invest in between two decision dates, i.e. $t \in (t_{m-1}, t_m)$, without the discount value:

$$\bar{p}_t = \mathbb{E}[p_{t_m}(X_{t_m}, Y_{t_m}) | \mathcal{F}_t] . \quad (3.1)$$

Notice that, without loss of generality, we have chosen to write the option value in terms of the “log” processes X_t and Y_t . When viewed as a process \bar{p}_t is a \mathbb{P} -martingale, consequently it satisfies the PDE

$$\begin{cases} (\partial_t + \mathcal{L}) \bar{p}(t, X, Y) = 0 \\ \bar{p}(t_m, X, Y) = p_{t_m}(X, Y) \end{cases} \quad (3.2)$$

Here, $p_{t_m}(X, Y)$ is already known from the previous step in the iteration and \mathcal{L} is the infinitesimal generator of the process (X_t, Y_t)

$$\mathcal{L} = -\alpha \partial_X + \frac{1}{2} \sigma_X^2 \partial_{XX} - ((\alpha - \beta)X + \beta Y) \partial_Y + \frac{1}{2} \sigma_Y^2 \partial_{YY} + \rho \sigma_X \sigma_Y \partial_{XY} . \quad (3.3)$$

By introducing the 2D-Fourier transform of $\bar{p}(t, X, Y)$ with respect to the X and Y variables, the PDE can be solved explicitly (see Jaimungal and Surkov (2009)) resulting in

$$\bar{p}(t, X, Y) = \mathcal{F}^{-1} \left[\mathcal{F} [\tilde{p}(t_m, X, Y)] (\omega_1, \omega_2) e^{\Psi((t_m-t), \omega_1, \omega_2)} \right] \quad (3.4)$$

Here, $\tilde{p}(t_m, X, Y) = p_{t_m}(X e^{-\alpha(t_m-t)}, (X - Y) e^{-\beta(t_m-t)})$, $\Psi(\cdot, \cdot, \cdot)$ is a particular quadratic form in (ω_1, ω_2) with coefficients that depend on the model parameters, and $\mathcal{F}[\cdot]$ and $\mathcal{F}^{-1}[\cdot]$ represent Fourier and inverse Fourier transforms respectively.

Through the above representation, a recursive formulation for the value at any given time step can be written as

$$p_{t_{m-1}}(X, Y) = \max \left\{ e^{-r\Delta t_m} \mathcal{F}^{-1} \left[\mathcal{F} [\tilde{p}(t_m, X, Y)](\omega_1, \omega_2) e^{\Psi(\Delta t_m, \omega_1, \omega_2)} \right] ; (V_{t_{m-1}} - I_{t_{m-1}})_+ \right\} \quad (3.5)$$

By comparison with the intrinsic value, the optimal strategy can be computed numerically through two fast Fourier transforms which approximately evaluate the Fourier and inverse transforms. This procedure is far more efficient than a tree or finite-difference scheme as it requires $O(N \log N)$ computations per exercise date, while finite difference schemes will require $O(MN)$ where M is the number of steps required between exercise dates. Furthermore, it is straightforward to incorporate jumps into the above representation – it will require a simple modification of the function Ψ – while tree or finite-difference methods will run into stability and computational issues.

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