HOMOGENIZATION OF NONLINEAR PDE'S IN THE FOURIER-STIELTJES ALGEBRAS

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Abstract. We introduce the Fourier-Stieltjes algebra in $\mathbb{R}^n$ which we denote by $\mathrm{FS}(\mathbb{R}^n)$. It is a subalgebra of the algebra of bounded uniformly continuous functions in $\mathbb{R}^n$, $\mathrm{BUC}(\mathbb{R}^n)$, strictly containing the almost periodic functions, whose elements are invariant by translations and possess a mean-value. Thus, it is a so called algebra with mean value, a concept introduced by Zhikov and Krivenko (1986). Namely, $\mathrm{FS}(\mathbb{R}^n)$ is the closure in $\mathrm{BUC}(\mathbb{R}^n)$, with the sup norm, of the real valued functions which may be represented by a Fourier-Stieltjes integral of a complex valued measure with finite total variation. We prove that it is an ergodic algebra and that it shares many interesting properties with the almost periodic functions. In particular, we prove its invariance under the flow of Lipschitz Fourier-Stieltjes fields. We analyse the homogenization problem for nonlinear transport equations with oscillatory velocity field in $\mathrm{FS}(\mathbb{R}^n)$. We also consider the corresponding problem for porous medium type equations on bounded domains with oscillatory external source belonging to $\mathrm{FS}(\mathbb{R}^n)$. We further address a similar problem for a system of two such equations coupled by a nonlinear zero order term. Motivated by the application to nonlinear transport equations, we also prove basic results on flows generated by Lipschitz vector fields in $\mathrm{FS}(\mathbb{R}^n)$ which are of interest in their own.

1. Introduction

The purpose of this paper is to introduce a large algebra with mean value (w.m.v.), strictly containing the almost periodic functions and to consider the homogenization problem for some nonlinear partial differential equations with oscillatory behavior governed by functions belonging to that algebra w.m.v.. Namely, denoting by $\mathrm{BUC}(\mathbb{R}^n)$ the space of bounded uniformly continuous functions, we are going to deal with the algebra $\mathrm{FS}(\mathbb{R}^n)$ defined as the closure in the sup norm of the functions in $\mathrm{BUC}(\mathbb{R}^n)$ whose Fourier transform is a complex-valued measure with finite total variation. We show that this algebra shares many important properties with the almost periodic functions. In particular, it is an ergodic algebra, which also contains the perturbations of almost periodic functions by continuous functions vanishing at infinity.

We then consider the homogenization problem for certain nonlinear PDE's. More specifically, we begin by analysing the homogenization of nonlinear transport equations where the associated vector field belongs to the algebra $\mathrm{FS}(\mathbb{R}^n)$. This discussion extends and improves the one corresponding to the same problem in [2], in the context of almost periodic functions, as well as the pioneering one provided by W. E in [18] in the context of periodic functions.

Next, we consider the homogenization problem for a porous medium type equation on a bounded domain with a stiff oscillatory external source in $\mathrm{FS}(\mathbb{R}^n)$. The latter was addressed in [3] for the Cauchy problem in $\mathbb{R}^n$ with oscillatory external source belonging to a general ergodic algebra and "well-behaved" initial data, i.e., initial data which are solutions of the associated steady equation in the fast variable. Here we restrict the discussion to $\mathrm{FS}(\mathbb{R}^n)$ which allows us to consider more general initial data not necessarily "well-behaved". Finally, we also address the homogenization problem for a system of two such porous medium type equations coupled by a nonlinear zero-order term.

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Both applications given here, namely, to nonlinear transport equations and to porous medium type equations in bounded domains with more general initial data, make essential use of nice properties of the functions in $\mathrm{FS}(\mathbb{R}^n)$ established in this paper, whose extension to general ergodic algebras is not known as yet. More specifically, in the first mentioned application, we make use of the fact that $\mathrm{FS}(\mathbb{R}^n)$ is invariant under the flows generated by Lipschitz vector fields whose components belong to $\mathrm{FS}(\mathbb{R}^n)$. This property was proved for the class of almost periodic functions in [2] and it is extended here to the functions in the Fourier-Stieltjes algebra (see Theorem 4.1 below). In the second mentioned application, we make use of the property that if $f \in \mathrm{FS}(\mathbb{R}^n)$ has mean value zero, then, for any $\varepsilon > 0$, there is $u \in \mathrm{FS}(\mathbb{R}^n)$ satisfying $f - \varepsilon \le \Delta u \le f + \varepsilon$ in $\mathbb{R}^n$. As it is shown in Lemma 3.3 below, such property follows easily from the definition of $\mathrm{FS}(\mathbb{R}^n)$. In both cases, the possibility of extending the corresponding property to general ergodic algebras is an open question.

The ergodic algebra $\mathrm{FS}(\mathbb{R}^n)$ contains the class $\mathrm{PAP}(\mathbb{R}^n)$ of functions $f \in \mathrm{BUC}(\mathbb{R}^n)$ such $f = g + \psi$ with $g \in \mathrm{AP}(\mathbb{R}^n)$ and $\psi \in C_0(\mathbb{R}^n)$, where $\mathrm{AP}(\mathbb{R}^n)$ is the space of almost periodic functions and $C_0(\mathbb{R}^n)$ is the space of continuous functions vanishing at infinity. In particular, this ergodic algebra not only strictly contains the almost periodic functions, but it is also stable under perturbations given by continuous functions vanishing at infinity. Moreover, it contains also the Fourier transforms of Cantor measures, which do not belong to $\mathrm{PAP}(\mathbb{R}^n)$ and so it is strictly larger than $\mathrm{PAP}(\mathbb{R}^n)$ (see Proposition 3.1). In this way, $\mathrm{FS}(\mathbb{R}^n)$ is capable of describing much more general oscillatory profiles than $\mathrm{AP}(\mathbb{R}^n)$. In fact, it is also a challenging open problem whether there exist or not ergodic algebras that are not subalgebras of $\mathrm{FS}(\mathbb{R}^n)$.

We use the existence of multiscale Young measures in the setting of vector valued algebras with mean value proved in [3]. As in the case of almost periodic functions, we make essential use of the fact that associated with any algebra w.m.v. $\mathcal{A}$ there exists a compact space $\mathcal{K}$ such that any $f \in \mathcal{A}$ may be viewed as an element of $C(\mathcal{K})$. Such compact space associated with the algebra w.m.v. provides the additional parameter of the multiscale (two-scale) Young measures. The latter are useful tools for the search of corrector functions in nonlinear homogenization problems.

Multiscale Young measures have been introduced in periodic problems by W. E [18] as a broader tool extending the previous concept of multiscale convergence introduced by Nguetseng [28] and further developed by Allaire [1]. It refines to multiple scale analysis the classical concept of Young measures introduced in [35], so fundamentally useful, especially after its striking applications in connection with problems concerning compactness of solution operators for nonlinear partial differential equations by Tartar [34], Murat [26], DiPerna [15, 16, 17], etc.. We recall that in [3] it was established a link between multiscale Young measures and the more general setting of homogenization of random stationary ergodic processes (see, e.g., [30], [23], [22], [13], [32], [24], [9]).

The extension of the multiscale Young measures from the periodic setting to the almost periodic one was carried out in [2] where applications to nonlinear transport equations, scalar conservation laws with oscillatory external sources, Hamilton-Jacobi equations and fully nonlinear elliptic equations are provided. In this connection, we recall that the two-scale convergence has been extended to the context of almost periodic homogenization and, more generally, to generalized Besicovitch spaces in [10] (see also, e.g., [28, 29]). We also recall that the method of two-scale convergence was extended to the context of stochastic homogenization, under separability assumption, in [6]. The applications in the cited references [10, 28, 29, 6] are basically to linear or monotone operators.

This paper is organized as follows. In Section 2, we recall the basic facts about algebras w.m.v. introduced in [36] and, in particular, ergodic algebras. In Section 3, we introduce the algebra $\mathrm{FS}(\mathbb{R}^n)$ and establish a number of its important properties. In Section 4, we analyse flows of Lipschitz vector fields in $\mathrm{FS}(\mathbb{R}^n)$ establishing some basic results which are of interest in their own and will be needed in our study of the homogenization of nonlinear transport equations. In Section 5, we briefly recall the concept of vector-valued algebras w.m.v. and the theorem on the existence of multiscale Young measures from homogenization in algebras w.m.v. established in [3]. In Section 6, we consider the application to nonlinear transport equations improving and extending to the context $\mathrm{FS}(\mathbb{R}^n)$ a previous result of [2], in the almost periodic setting, and the pioneering one by W. E [18] in the periodic setting. Finally, in Sections 7 and 8 we give applications to porous medium type equations on bounded domains, with oscillatory external source and initial data in $\mathrm{FS}(\mathbb{R}^n)$ with respect to the oscillatory variable, and to a system of such equations coupled by a nonlinear zero-order term, respectively.

2. Ergodic Algebras

In this section we recall the basic facts concerning algebras with mean values and, in particular, ergodic algebras. To begin, we recall the notion of mean value for functions defined in $\mathbb{R}^n$.

Definition 2.1. Let $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. A number $M(g)$ is called the *mean value of g* if

$$\lim_{\varepsilon\to 0}\int_A g(\varepsilon^{-1}x)\,dx = |A|M(g) \tag{2.1}$$

for any Lebesgue measurable bounded set $A \subseteq \mathbb{R}^n$, where $|A|$ stands for the Lebesgue measure of A. This is equivalent to saying that $g(\varepsilon^{-1}x)$ converges, in the duality with L^∞ and compactly supported functions, to the constant $M(g)$. Also, if $A_t := \{x \in \mathbb{R}^n : t^{-1}x \in A\}$ for $t > 0$ and $|A| \neq 0$, (2.1) may be written as

$$\lim_{t\to\infty}\frac{1}{t^n|A|}\int_{A_t} g(x)\,dx = M(g). \tag{2.2}$$

We will also use the notation $⨍_{\mathbb{R}^n} g\,dx$ for $M(g)$.

Notation: As usual, we denote by $\mathrm{BUC}(\mathbb{R}^n)$ the space of the bounded uniformly continuous real-valued functions in $\mathbb{R}^n$.

We recall now the definition of algebras with mean value introduced in [36].

Definition 2.2. Let $\mathcal{A}$ be a linear subspace of $\mathrm{BUC}(\mathbb{R}^n)$. We say that $\mathcal{A}$ is an *algebra with mean value* (or *algebra w.m.v.*, in short), if the following conditions are satisfied:

(A) If f and g belong to $\mathcal{A}$, then the product fg belongs to $\mathcal{A}$.
(B) $\mathcal{A}$ is invariant with respect to translations τ_y in $\mathbb{R}^n$.
(C) Any $f \in \mathcal{A}$ possesses a mean value.
(D) $\mathcal{A}$ is closed in $\mathrm{BUC}(\mathbb{R}^n)$ and contains the unity, i.e., the function $e(x) := 1$ for $x \in \mathbb{R}^n$.

Remark 2.1. Condition (D) was not originally in [36] but we include it here since any algebra satisfying (A), (B) and (C) can be extended to an algebra satisfying (A)–(D) in a unique way modulo isomorphisms.

For the development of the homogenization theory in algebras with mean value, as is done in [36, 22] (see also [10]), in similarity with the case of almost periodic functions, one introduces, for $1 \le p < \infty$, the space $\mathcal{B}^p$ as the abstract completion of the algebra $\mathcal{A}$ with respect to the Besicovitch seminorm

$$|f|_p^p := \limsup_{L\to\infty}\frac{1}{(2L)^n}\int_{[-L,L]^n}|f|^p\,dx.$$

Both the action of translations and the mean value extend by continuity to $\mathcal{B}^p$, and we will keep using the notation $\tau_y f$ and $M(f)$ even when $f \in \mathcal{B}^p$ and $y \in \mathbb{R}^n$. Furthermore, for $p > 1$ the product in the algebra extends to a bilinear operator from $\mathcal{B}^p \times \mathcal{B}^q$ into $\mathcal{B}^1$, with q equal to the dual exponent of p, satisfying

$$|fg|_1 \le |f|_p|g|_q.$$

In particular, the operator $M(fg)$ provides a nonnegative definite bilinear form on $\mathcal{B}^2$.

Since there is an obvious inclusion between members of this family of spaces, we may define the space $\mathcal{B}^\infty$ as follows:

$$\mathcal{B}^\infty = \{f \in \bigcap_{1\le p<\infty}\mathcal{B}^p \;:\; \sup_{1\le p<\infty}|f|_p < \infty\}.$$

We endow $\mathcal{B}^\infty$ with the (semi)norm

$$|f|_\infty := \sup_{1\le p<\infty} |f|_p.$$

Obviously the corresponding quotient spaces for all these spaces (with respect to the null space of the seminorms) are Banach spaces, and we get a Hilbert space in the case $p = 2$. We denote by $\overset{\mathcal{B}^p}{=}$, the equivalence relation given by the equality in the sense of the $\mathcal{B}^p$ semi-norm.

Remark 2.2. A classical argument going back to Besicovitch [4] (see also [22], p.239) shows that the elements of $\mathcal{B}^p$ can be represented by functions in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $1 \le p < \infty$.

We next recall a result established in [3] which provides a connection between algebras with mean value and compactifications of $\mathbb{R}^n$ endowed with a group of "translations" and an invariant probability measure.

Theorem 2.1 (cf. [3])**.** *For an algebra $\mathcal{A}$, we have:*

(i) *There exist a compact space $\mathcal{K}$ and an isometric isomorphism i identifying $\mathcal{A}$ with the algebra $C(\mathcal{K})$ of continuous functions on $\mathcal{K}$. Moreover, if $\mathcal{A}$ separates points of $\mathbb{R}^n$, then $\mathcal{K}$ is a compactification of $\mathbb{R}^n$.*
(ii) *The translations $T(y) : \mathbb{R}^n \to \mathbb{R}^n$, $T(y)x = x + y$, extend to a group of homeomorphisms $T(y) : \mathcal{K} \to \mathcal{K}$, $y \in \mathbb{R}^n$.*
(iii) *There exists a Radon probability measure $\mathfrak{m}$ on $\mathcal{K}$ which is invariant under the group of transformations $T(y)$, $y \in \mathbb{R}^n$, such that*

$$⨍_{\mathbb{R}^n} f\,dx = \int_{\mathcal{K}} i(f)\,d\mathfrak{m}$$

(iv) *The family $T(y)$, $y \in \mathbb{R}^n$, is a continuous n-dimensional dynamical system on $\mathcal{K}$.*
(v) *For $1 \le p \le \infty$, the Besicovitch space $\mathcal{B}^p/\overset{\mathcal{B}^p}{=}$ is isometrically isomorphic to $L^p(\mathcal{K}, \mathfrak{m})$.*

Remark 2.3. When $p = 2$, we denote $L^2(\mathcal{K}, \mathfrak{m})$ simply by $L^2(\mathcal{K})$.

A group of unitary operators $T(y) : \mathcal{B}^2 \to \mathcal{B}^2$ is then defined by setting $[T(y)f] = \tau_y f$, where τ_y denotes the map induced by the translation $x \mapsto x + y$. Since the elements of $\mathcal{A}$ are uniformly continuous in $\mathbb{R}^n$, the group $\{T(y)\}$ is strongly continuous, i.e. $T(y)f \to f$ in $\mathcal{B}^2$ as $y \to 0$ for all $f \in \mathcal{B}^2$. The notion of invariant function is then introduced by simply saying that a function in $\mathcal{B}^2$ is *invariant* if $T(y)f \overset{\mathcal{B}^2}{=} f$, for all $y \in \mathbb{R}^n$. More clearly, $f \in \mathcal{B}^2$ is invariant if

$$M\big(|T(y)f - f|^2\big) = 0, \qquad \forall y \in \mathbb{R}^n. \tag{2.3}$$

The concept of ergodic algebra is then introduced as follows.

Definition 2.3. An algebra $\mathcal{A}$ w.m.v. is called *ergodic* if any invariant function f belonging to the corresponding space $\mathcal{B}^2$ is equivalent (in $\mathcal{B}^2$) to a constant.

In [22] an alternative definition of ergodic algebra is also given which is shown therein to be equivalent to Definition 2.3, by using von Neumann's Mean Ergodic Theorem. We state that as the following lemma, whose detailed proof may be found in [22], p.247.

Lemma 2.1. *Let $\mathcal{A} \subseteq \mathrm{BUC}(\mathbb{R}^n)$ be an algebra with mean value. Then $\mathcal{A}$ is ergodic if and only if*

$$\lim_{t\to\infty} M_y \left(\Big| \frac{1}{|B(0;t)|} \int_{B(0;t)} f(x+y)\,dx - M(f) \Big|^2 \right) = 0 \qquad \forall f \in \mathcal{A}. \tag{2.4}$$

3. The algebra $\mathrm{FS}(\mathbb{R}^n)$

For any $f \in L^\infty(\mathbb{R}^n)$, let us denote by $\hat{f}$ the Fourier transform of f defined as the following distribution

$$\langle \hat{f}, \phi \rangle := \int f(x)\check{\phi}(x)\,dx, \qquad \text{for all } \phi \in C_c^\infty(\mathbb{R}^n),$$

where $\check{\phi}$ denotes the usual inverse Fourier transform of ϕ defined by

$$\check{\phi}(x) = \frac{1}{(2\pi)^{\frac{n}{2}})}\int \phi(y)e^{iy\cdot x}\,dy.$$

Next we introduce an important algebra w.m.v. for the purposes of this paper.

Definition 3.1. We denote by $\mathrm{FS}(\mathbb{R}^n)$ the closure in $\mathrm{BUC}(\mathbb{R}^n)$ of the space $\mathrm{FS}_*(\mathbb{R}^n)$, defined by

$$\mathrm{FS}_*(\mathbb{R}^n) := \left\{ f : \mathbb{R}^n \to \mathbb{R} \,:\, f(x) = \int_{\mathbb{R}^n} e^{ix\cdot y}\,d\nu(y) \text{ for some } \nu \in \mathcal{M}_*(\mathbb{R}^n) \right\}, \tag{3.1}$$

where by $\mathcal{M}_*(\mathbb{R}^n)$ we denote the space of complex-valued measures μ with finite total variation, i.e., $|\mu|(\mathbb{R}^n) < \infty$. We call $\mathrm{FS}(\mathbb{R}^n)$ the *Fourier-Stieltjes algebra*.

Recall that a subalgebra $\mathcal{B}$ of an algebra $\mathcal{A}$ is called an ideal of $\mathcal{A}$ if for any $f \in \mathcal{A}$ and $g \in \mathcal{B}$ we have $fg \in \mathcal{B}$. Let $C_0(\mathbb{R}^n)$ denote the closure of $C_c^\infty(\mathbb{R}^n)$ with respect to the sup norm, and let $\mathrm{AP}(\mathbb{R}^n)$ denote the algebra of almost periodic functions in $\mathbb{R}^n$. We denote by $\mathrm{PAP}(\mathbb{R}^n)$ the space of perturbed almost periodic functions in $\mathbb{R}^n$ defined by

$$\mathrm{PAP}(\mathbb{R}^n) := \{ f \in \mathrm{BUC}(\mathbb{R}^n) \,:\, f = g + \psi,\ g \in \mathrm{AP}(\mathbb{R}^n),\ \psi \in C_0(\mathbb{R}^n)\}.$$

We have the following result.

Proposition 3.1. $\mathrm{FS}(\mathbb{R}^n) \subseteq \mathrm{BUC}(\mathbb{R}^n)$ *is an algebra w.m.v. containing* $C_0(\mathbb{R}^n)$ *as an ideal. Moreover,* $\mathrm{FS}(\mathbb{R}^n)$ *is an ergodic algebra and* $\mathrm{PAP}(\mathbb{R}^n)$ *is a closed strict subalgebra of* $\mathrm{FS}(\mathbb{R}^n)$*.*

Proof. 1. Obviously all functions in $\mathrm{FS}_*(\mathbb{R}^n)$ are bounded and uniformly continuous, and the measure ν in (3.1) is the Fourier transform of f. That $\mathrm{FS}_*(\mathbb{R}^n)$, and therefore $\mathrm{FS}(\mathbb{R}^n)$, is an algebra follows from the fact that the Fourier transform of the product is the convolution of the Fourier transforms of the factors, and $\mathcal{M}_*(\mathbb{R}^n)$ is stable under convolution. The invariance by translations, follows from the fact that the Fourier transform of $f(\cdot + t)$ equals $e^{it\cdot y}\hat{f}(y)$. Finally, the mean value property follows from the fact that if $f \in \mathrm{FS}_*(\mathbb{R}^n)$, then its mean value exists and it is equal to $\hat{f}(\{0\}) := \nu(\{0\})$. The latter follows easily from the fact that $\hat{f}$ is a complex-valued measure with finite total variation and the mean value of $\int_{y\neq 0} e^{ix\cdot y}d\hat{f}(y)$ is equal to zero by Fubini and dominated convergence theorems.

2. The Fourier transform maps the Schwartz space of smooth fast decaying functions $\mathcal{S}(\mathbb{R}^n)$ into itself and therefore $C_c^\infty(\mathbb{R}^n) \subseteq \mathrm{FS}_*(\mathbb{R}^n)$. It follows that its closure, namely $C_0(\mathbb{R}^n)$, is an ideal of $\mathrm{FS}(\mathbb{R}^n)$. That $\mathrm{AP}(\mathbb{R}^n)$ is a subalgebra of $\mathrm{FS}(\mathbb{R}^n)$ follows easily from the fact that the Fourier transform of $e^{i\lambda\cdot x}$ is δ_λ, where δ_λ denotes the Dirac measure concentrated at λ, and by the fact that the vector space spanned by these functions is dense in $\mathrm{AP}(\mathbb{R}^n)$ with respect to the sup norm (Bohr theorem). It follows that $\mathrm{PAP}(\mathbb{R}^n)$ is a subalgebra of $\mathrm{FS}(\mathbb{R}^n)$.

3. The fact that $\mathrm{PAP}(\mathbb{R}^n)$ is closed can be seen as follows. Given $\varepsilon > 0$, $g_1, g_2 \in \mathrm{AP}(\mathbb{R}^n)$ and $\psi_1, \psi_2 \in C_0(\mathbb{R}^n)$ such that $\|f_1 - f_2\|_\infty < \varepsilon$, with $f_i = g_i + \psi_i$, $i = 1, 2$, then, for a suitable compact K, we have $|g_1(x) - g_2(x)| < 2\varepsilon$, for $x \notin K$. The almost periodicity then implies that $\|g_1 - g_2\|_\infty < 3\varepsilon$ and so $\|\psi_1 - \psi_2\|_\infty < 4\varepsilon$. Hence, if $(f_k = g_k + \psi_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{PAP}(\mathbb{R}^n)$, then (g_k) and (ψ_k) are Cauchy sequences in $\mathrm{AP}(\mathbb{R}^n)$ and $C_0(\mathbb{R}^n)$, respectively. Since these spaces are closed in $\mathrm{BUC}(\mathbb{R}^n)$, we conclude that the limit of (f_k) is in $\mathrm{PAP}(\mathbb{R}^n)$. Therefore, $\mathrm{PAP}(\mathbb{R}^n)$ is closed.

4. The fact that $\mathrm{PAP}(\mathbb{R}^n)$ is strictly contained in $\mathrm{FS}(\mathbb{R}^n)$ is seen as follows. If $n > 1$, say $n = 2$, then for $f(x_1, x_2) = \varphi(x_1)$, with $\varphi \in \mathcal{S}(\mathbb{R})$, we have that $\hat{f}$ is the product measure $\hat{\varphi} \times \delta_0$, where $\hat{\varphi} \in \mathcal{S}(\mathbb{R})$ is the

one-dimensional Fourier transform of φ and δ_0 is the Dirac measure on $\mathbb{R}$ concentrated at 0. Hence, such f belongs to $\mathrm{FS}(\mathbb{R}^2)$. On the other hand, $f \notin \mathrm{PAP}(\mathbb{R}^2)$ since it is obviously impossible to have $f = g + \psi$, with $g \in \mathrm{AP}(\mathbb{R}^2)$ and $\psi \in C_0(\mathbb{R}^2)$.

5. It remains then to prove that $\mathrm{PAP}(\mathbb{R}) \subsetneq \mathrm{FS}(\mathbb{R})$. To see that, consider the Cantor set $\mathcal{C}$ contained in $[-\frac{1}{2}, \frac{1}{2}]$ obtained by excluding the middle third $(-\frac{1}{6}, \frac{1}{6})$, then the middle thirds $(-\frac{4}{9}, -\frac{2}{9})$ and $(\frac{2}{9}, \frac{4}{9})$, and so on. Consider the Cantor measure $\mu_{\mathcal{C}}$ characterized by the fact that $f_{\mathcal{C}}(x) := \mu_C((-\infty, x))$ is the probability that an element of $\mathcal{C}$ taken at random belongs to $(-\infty, x)$. It is easy to see that $f_{\mathcal{C}}(x)$ is the distribution of $\sum_{n=1}^{\infty} X_n/3^n$, where X_n are independent random variables assuming values -1 and 1 with probability 1/2 each. Therefore, the inverse Fourier transform of $\mu_{\mathcal{C}}$, $F_{\mathcal{C}} := \check{\mu}_{\mathcal{C}}$, can be expressed as the infinite product (see, e.g., problem 31.15 in [5], [33] and [21])

$$F_{\mathcal{C}}(x) = \prod_{k=1}^{\infty} \cos\left(\frac{\pi x}{3^k}\right).$$

In particular,

$$F_{\mathcal{C}}(x) = \cos\left(\frac{\pi x}{3}\right) F_{\mathcal{C}}\left(\frac{x}{3}\right).$$

Using this recursion formula it is possible to find a sequence x_k, with $x_k \to \infty$, for which $|F_{\mathcal{C}}(x_k)| > \delta_0$ for all $k \in \mathbb{N}$, for some $\delta_0 > 0$, that is, $F_{\mathcal{C}} \notin C_0(\mathbb{R})$ (see, e.g., the graph of $F_{\mathcal{C}}$ in Figure 2 in [21]). More specifically, if $x_k = 2 \cdot 3^k$, then $F_{\mathcal{C}}(x_{k+1}) = F_{\mathcal{C}}(x_k)$. It is possible to show directly, analysing the series $\sum_{n=1}^{\infty} \log\left|\cos\left(\frac{2\pi}{3^{k-1}}\right)\right|$, that $|F_{\mathcal{C}}(6)| > 0$, so $|F_{\mathcal{C}}(x_k)| = |F_{\mathcal{C}}(6)| > 0$, for all $k \in \mathbb{N}$. Now, if $F_{\mathcal{C}} = g + \psi$, with $g \in \mathrm{AP}(\mathbb{R})$ and $\psi \in C_0(\mathbb{R})$, then $\hat{\psi}$ is a complex valued measure with finite total variation, which must attribute value zero to any point so that we can have $|\psi(x)| \to 0$ as $|x| \to \infty$. Hence, since μ_C also attributes value zero to any point, we conclude that $g = 0$ and so $F_{\mathcal{C}} \in C_0(\mathbb{R})$ which is not true. Hence, $F_{\mathcal{C}} \notin \mathrm{PAP}(\mathbb{R})$ although, obviously, $F_{\mathcal{C}} \in \mathrm{FS}(\mathbb{R})$, which concludes the proof that $\mathrm{PAP}(\mathbb{R}^n) \subsetneq \mathrm{FS}(\mathbb{R}^n)$ for all $n \in \mathbb{N}$.

6. The fact that $\mathrm{FS}(\mathbb{R}^n)$ is an ergodic algebra is proved as follows. First, we claim that any function in $\mathrm{FS}(\mathbb{R}^n)$ such that $M(f) = 0$ may be uniformly approximated by functions $\phi \in \mathrm{FS}_*(\mathbb{R}^n)$ with the property that the support of $\hat{\phi}$ is a compact at positive distance from the origin. Indeed, given $f \in \mathrm{FS}(\mathbb{R}^n)$ with $M(f) = 0$, there exists $(\varphi_k)_{k\in\mathbb{N}} \subseteq \mathrm{FS}_*(\mathbb{R}^n)$ with $M(\varphi_k) = 0$ and $\varphi_k \to f$ uniformly. Setting $\mu_k = \hat{\varphi}_k$, since $\mu_k(\{0\}) = 0$ and $|\mu_k|(\mathbb{R}^n) < \infty$, we can find $0 < r_k < R_k$ such that

$$|\mu_k|\left(\mathbb{R}^n \setminus \{x \,:\, |x| < r_k \text{ or } |x| > R_k\}\right) < \frac{1}{k}.$$

Defining, $\nu_k := \mu_k \lfloor \{x \,:\, r_k \le |x| \le R_k\}$ and $\phi_k := \check{\nu}_k$, we obtain that the sequence $(\phi_k)_{k\in\mathbb{N}} \subseteq \mathrm{FS}_*(\mathbb{R}^n)$ has the required properties.

7. To complete the proof that $\mathrm{FS}(\mathbb{R}^n)$ is an ergodic algebra we are going to use Lemma 2.1. Since $f - M(f)$ has mean value zero, it suffices to verify (2.4) assuming $M(f) = 0$. For $\phi \in \mathrm{FS}_*(\mathbb{R}^n)$ satisfying the property described in the preceding step, it is possible to prove that $M_x(\phi(x + y)) = 0$ uniformly with respect to $y \in \mathbb{R}^n$ (see [22], p.246). Hence, taking such ϕ so that $\|f - \phi\|_\infty < \sqrt{\varepsilon}/2$ and taking $t_0 > 0$ large enough so that

$$\frac{1}{|B(0;t)|}\left|\int_{B(0;t)} \phi(x+y)\,dx\right| < \frac{\sqrt{\varepsilon}}{2}, \qquad \text{for } t > t_0,$$

uniformly with respect to $y \in \mathbb{R}^n$, we arrive at

$$M_y\left(\Big|\frac{1}{|B(0;t)|}\int_{B(0;t)} f(x+y)\,dx\Big|^2\right) < \varepsilon, \qquad \text{for } t > t_0,$$

which proves the ergodicity of $\mathrm{FS}(\mathbb{R}^n)$. $\square$

Remark 3.1. We recall that an example presented in [22] allows to construct an algebra w.m.v. which is not ergodic. Namely, we take the closed algebra with unity in $\mathrm{BUC}(\mathbb{R})$ generated by the function $\cos\sqrt[3]{x}$ and its translates $\cos\sqrt[3]{x+t}$, $t\in\mathbb{R}$. Indeed, $\cos\sqrt[3]{x}-\cos\sqrt[3]{x+t}\to 0$ as $|x|\to\infty$ for all $t\in\mathbb{R}$, and each of the functions $\cos^k\sqrt[3]{x}$, $k\in\mathbb{N}$, possesses a mean value M_k. Clearly, any product of translates $\cos\sqrt[3]{x+t_1}\cdots\cos\sqrt[3]{x+t_k}$ is $\mathcal{B}^2$-equivalent to $\cos^k\sqrt[3]{x}$, $k\in\mathbb{N}$. Moreover, $M_1=0$ and $M_2=\frac{1}{2}$. Hence, this algebra is an algebra w.m.v. which is not ergodic since the function $\cos\sqrt[3]{x}$ is invariant and is not $\mathcal{B}^2$-equivalent to a constant. The question remains whether or not there exist ergodic algebras which are not subalgebras of $\mathrm{FS}(\mathbb{R}^n)$.

Remark 3.2. It follows from Proposition 3.1, in particular, that we can apply Theorem 2.1 to the algebra w.m.v. $\mathrm{FS}(\mathbb{R}^n)$. Henceforth, $\mathcal{K}$ and $\mathcal{B}^2$ are, respectively, the compactification of $\mathbb{R}^n$ and the Besicovitch space associated with $\mathrm{FS}(\mathbb{R}^n)$, and $\mathfrak{m}$ is the corresponding Randon probability measure on $\mathcal{K}$.

The following lemma concerning functions in $\mathrm{FS}_*(\mathbb{R}^n)$ will be used in the next section.

Lemma 3.1. *Let $f\in\mathrm{FS}_*(\mathbb{R}^n)$ and $F\subseteq\mathbb{R}^n$ such that $|\mu|(F)=0$, where $\mu:=\hat{f}$. Then, there exists $(f_j)_{j\ge1}\subseteq\mathrm{FS}_*(\mathbb{R}^n)$ such that $f_j\to f$ uniformly and each $\hat{f}_j$ has compact support such that* $\operatorname{supp}\hat{f}_j\cap F=\emptyset$.

Proof. Given $\epsilon>0$, there is an open set $V\supset F$ and $R>0$ such that

$$|\mu|(\{x\,:\,|x|>R\})<\frac{\epsilon}{2}\qquad\text{and}\qquad|\mu|(V)<\frac{\epsilon}{2}.$$

Define $\mu_1:=\mu\lfloor\{x\,:\,|x|>R\}$, $\mu_2:=\mu\lfloor\{V\cap\{x\,:\,|x|<R\}\}$ and $\mu_3:=\mu-\mu_1-\mu_2$. Let $f_\epsilon:=\check{\mu_3}$ and observe that

$$f-f_\epsilon=\check{\mu_1}+\check{\mu_2}=\int_{\mathbb{R}^n}e^{-ix\cdot y}\,d\mu_1(y)+\int_{\mathbb{R}^n}e^{-ix\cdot y}\,d\mu_2(y)$$

and so

$$\|f-f_\epsilon\|_\infty\le|\mu_1|(\mathbb{R}^n)+|\mu_2|(\mathbb{R}^n)<\varepsilon.$$

Further, we have $\operatorname{supp}\hat{f}_\epsilon\subseteq\{x\,:\,|x|\le R\}-\{V\cap\{x\,:\,|x|<R\}\}$ and it is separated from F. ☐

We will also make use of the following property of $\mathrm{FS}(\mathbb{R}^n)$ in the next section.

Lemma 3.2. *Let*

$$E_1:=\big\{\varphi\in\mathrm{FS}_*(\mathbb{R}^n)\,:\,\operatorname{supp}\hat{\varphi}\subseteq\{x\,:\,x_n=0\}\big\},$$
$$E_2:=\big\{\varphi\in\mathrm{FS}_*(\mathbb{R}^n)\,:\,\operatorname{supp}\hat{\varphi}\subseteq\{x\,:\,x_n\neq0\}\,\text{and it is compact}\big\}.$$

Then, we have the following orthogonal decomposition for $\mathcal{B}^2$:

$$\mathcal{B}^2=\overline{E_1}\oplus\overline{E_2},\tag{3.2}$$

where $\overline{}$ means the closure in $\mathcal{B}^2$.

Proof. 1. Given $\psi\in\mathrm{FS}_*(\mathbb{R}^n)$, there exist $\psi_1,\psi_2\in\mathrm{FS}_*(\mathbb{R}^n)$ such that $\operatorname{supp}\hat{\psi_1}\subseteq\{x;\,x_n=0\}$, $\operatorname{supp}\hat{\psi_2}\subseteq\{x;\,x_n\neq0\}$ and $\psi=\psi_1+\psi_2$. To see this, let $\mu:=\hat{\psi}$ and note that $\mu=\mu\lfloor\{x\,:\,x_n=0\}+\mu\lfloor\{x\,:\,x_n\neq0\}=\nu_1+\nu_2$. Define $\psi_1:=\check{\nu_1}$ and $\psi_2:=\check{\nu_2}$.

2. Besides, by Lemma 3.1, ψ_2 may be uniformly approximated by functions $\{\psi_2^{(j)}\}_{j\ge1}\subseteq\mathrm{FS}_*(\mathbb{R}^n)$ such that $\operatorname{supp}\widehat{\psi_2^{(j)}}$ is compact and $\operatorname{supp}\widehat{\psi_2^{(j)}}\cap\{x;\,x_n=0\}=\emptyset$. The functions ψ_1 and ψ_2 are orthogonal as elements of $\mathcal{B}^2$. Indeed, setting $\nu_2^j=\widehat{\psi_2^{(j)}}$, we have $\langle\psi_1,\psi_2^j\rangle=\widehat{\psi_1\psi_2^j}\{0\}=\nu_1*\nu_2^j\{0\}=0$, since the supports of ν_1 and ν_2^j are disjoint. Letting $j\to\infty$ we obtain $\langle\psi_1,\psi_2\rangle=0$.

3. Now, given any $v\in\mathcal{B}^2$, there exists a sequence $(\psi^j)_{j\in\mathbb{N}}\subseteq\mathrm{FS}_*(\mathbb{R}^n)$ such that $\psi^j\to v$ in $\mathcal{B}^2$. For each ψ^j we have a decomposition $\psi^j=\psi_1^j+\psi_2^j$, with $\psi_1^j\in E_1$ and $\psi_2^j\in\overline{E_2}$. By the orthogonality between ψ_1^j and ψ_2^j and the boundedness of ψ^j in $\mathcal{B}^2$, we deduce that the functions ψ_1^j and ψ_2^j are uniformly bounded in

$\mathcal{B}^2$. Hence, by passing to a subsequence if necessary, there exist $v_1, v_2 \in \mathcal{B}^2$ such that $\psi_1^j \rightharpoonup v_1$ and $\psi_2^j \rightharpoonup v_2$, where $\rightharpoonup$ means weak convergence in $\mathcal{B}^2$. Since E_1 and E_2 are convex, we have that $v_1 \in \overline{E_1}$ and $v_2 \in \overline{E_2}$. It is immediate to see that v_1 is orthogonal to $\overline{E_2}$ and the same for v_2 and $\overline{E_1}$. Hence, v_1 and v_2 are orthogonal. By orthogonality, we also deduce that the decomposition $v = v_1 + v_2$ with $v_1 \in \overline{E_1}$ and $v_2 \in \overline{E_2}$ is unique and this concludes the proof of the asserted orthogonal decomposition for $\mathcal{B}^2$.

□

The following fact concerning functions in $\mathrm{FS}(\mathbb{R}^n)$ will be used in our application to homogenization of porous medium type equations in the final part of this paper.

Lemma 3.3. *If $f \in \mathrm{FS}(\mathbb{R}^n)$ and $M(f) = 0$, then for any $\varepsilon > 0$ there exists a bounded smooth function u_ε satisfying the inequalities*

$$f - \varepsilon \le \Delta u_\varepsilon \le f + \varepsilon. \tag{3.3}$$

Proof. Clearly, the stated property is stable under uniform approximations. Hence, we may assume further that $f \in \mathrm{FS}_*(\mathbb{R}^n)$. Let $\mu = \hat{f}$. In this case, the assumption that $M(f) = 0$ is equivalent to $\mu(\{0\}) = 0$ as we have seen in the proof of Proposition 3.1.

Now, given any $\varepsilon > 0$, for $R > 0$ sufficiently large and for $r > 0$ sufficiently small we have

$$|\mu|(\{x : |x| > R\}) < \frac{\varepsilon}{2} \qquad \text{and} \qquad |\mu|(\{x : |x| < r\}) < \frac{\varepsilon}{2},$$

the latter because $\mu(\{0\}) = 0$. Let $\nu_1 := \mu\lfloor\{x : |x| > R\}$ and $\nu_2 := \mu\lfloor\{x : |x| < r\}$. We easily verify that $\|\check{\nu_1}\|_\infty < \varepsilon/2$ and $\|\check{\nu_2}\|_\infty < \varepsilon/2$.
Now, set $\nu := \mu - \nu_1 - \nu_2$ and let $g := \check{\nu}$. We claim that g and all its derivatives belong to $\mathrm{BUC}(\mathbb{R}^n)$ and there is a bounded and smooth solution for the equation $\Delta u = g$ in $\mathbb{R}^n$. Indeed, this follows immediately from the fact that $\hat{g}$ has compact support separated from zero. In fact, we have $g = g * \phi$ where $\phi \in \mathcal{S}(\mathbb{R}^n)$ satisfies $\hat{\phi} \in C_c^\infty(\mathbb{R}^n)$, $\hat{\phi} = 1$ on $\operatorname{supp} \nu$ and $\hat{\phi} = 0$ in a neighborhood of the origin. Further, define h by $\hat{h}(\xi) = -\hat{\phi}(\xi)/|\xi|^2$ and $u = g * h$. Then $\Delta(g * h) = g * \Delta h = g * \phi = g$ (*cf.* [22], p.246). This proves the claim and concludes the proof. □

4. Flows Generated By Lipschitz Vector Fields in $\mathrm{FS}(\mathbb{R}^n)$

In this section we investigate the flows generated by Lipschitz fields in $\mathrm{FS}(\mathbb{R}^n)$.

Let $a \in \mathrm{FS} \cap W^{1,\infty}(\mathbb{R}^n; \mathbb{R}^n)$, and let us assume that a is incompressible, i.e.

$$\nabla_z \cdot a = 0. \tag{4.1}$$

Let us consider the Cauchy problem

$$\begin{cases} \dfrac{dX}{dt}(z,t) = a(X(z,t)), \\ X(z,0) = z. \end{cases} \tag{4.2}$$

In some occasions we will denote the map $t \mapsto X(z,t)$ by $X_t(z)$.

We are interested in the properties of the map $\boldsymbol{X}_t : \mathrm{BUC}(\mathbb{R}^n) \to \mathrm{BUC}(\mathbb{R}^n)$ defined by $g \mapsto g \circ X_t$.

Theorem 4.1. *$\varphi \circ X_t \in \mathrm{FS}(\mathbb{R}^n)$ for any $\varphi \in \mathrm{FS}(\mathbb{R}^n)$ and*

$$⨍_{\mathbb{R}^n} |\varphi(X(z,t))|^2 \, dz = ⨍_{\mathbb{R}^n} |\varphi(z)|^2 \, dz. \tag{4.3}$$

Therefore $\boldsymbol{X}_t$ can be extended to an operator in $\mathcal{B}^2$ satisfying

$$⨍_{\mathbb{R}^n} |\boldsymbol{X}_t(\varphi)|^2 \, dz = ⨍_{\mathbb{R}^n} |\varphi(z)|^2 \, dz \qquad \forall \varphi \in \mathcal{B}^2. \tag{4.4}$$

Proof. 1. Assume that $b \in \mathrm{FS}_*(\mathbb{R}^n;\mathbb{R}^n)$ and that $\varphi \in \mathrm{FS}_*(\mathbb{R}^n)$ is such that the support of $\mu := \hat{\varphi}$ is compact. Define

$$\gamma(x) := \varphi(x + b(x)) = \int_{\mathbb{R}^n} e^{ix\cdot y} e^{ib(x)\cdot y}\, d\mu(y).$$

We have

$$\begin{aligned}\langle \hat{\gamma}, \psi \rangle &= \langle \gamma, \check{\psi} \rangle = \int_{\mathbb{R}^n} \gamma(x)\check{\psi}(x)\, dx = \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} e^{ix\cdot y} e^{ib(x)\cdot y}\, d\mu(y) \right\} \check{\psi}(x)\, dx \\ &= \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} e^{ix\cdot y} e^{ib(x)\cdot y} \check{\psi}(x)\, dx \right\} d\mu(y) = \int_{\mathbb{R}^n} \langle e_y \cdot e_y \circ b, \check{\psi} \rangle\, d\mu(y) \\ &= \int_{\mathbb{R}^n} \langle \delta_y * \widehat{e_y \circ b}, \psi \rangle\, d\mu(y)\end{aligned}$$

where $e_y(x) := e^{ix\cdot y}$. Moreover, $\widehat{e_y \circ b}$ is a complex-valued measure with total variation $|\widehat{e_y \circ b}|(\mathbb{R}^n) \leq e^{|y||\hat{b}|(\mathbb{R}^n)}$. Since the support of μ is compact, then $\int_{\mathbb{R}^n} \delta_y * \widehat{e_y \circ b}\, d\mu(y)$ is a well defined complex-valued measure. Thus, $\gamma \in \mathrm{FS}_*(\mathbb{R}^n)$.

2. Now, take $\varphi \in \mathrm{FS}(\mathbb{R}^n)$ and observe that $\exists \{\varphi_j\}_{j\geq 1} \subseteq \mathrm{FS}_*(\mathbb{R}^n)$ such that the support of $\hat{\varphi}_j$ is compact and $\varphi_j \to \varphi$ uniformly. Thus, $\varphi_j(\cdot + b(\cdot)) \to \varphi(\cdot + b(\cdot))$ uniformly and, by the step 1, $\varphi_j(\cdot + b(\cdot)) \in \mathrm{FS}_*(\mathbb{R}^n)$ for all j, which implies that $\varphi(\cdot + b(\cdot)) \in \mathrm{FS}(\mathbb{R}^n)$.

3. Let $b \in \mathrm{FS}(\mathbb{R}^n;\mathbb{R}^n)$. Recalling the definition of $\mathrm{FS}(\mathbb{R}^n)$, we have that there exists $(b_j)_{j\in\mathbb{N}} \subseteq \mathrm{FS}_*(\mathbb{R}^n;\mathbb{R}^n)$ such that $b_j \to b$ uniformly. Hence, $\varphi(\cdot + b_j(\cdot)) \to \varphi(\cdot + b(\cdot))$ uniformly. By the step 2, $\varphi(\cdot + b_j(\cdot)) \in \mathrm{FS}(\mathbb{R}^n)$ for any j. Thus, we have proved that $\varphi(\cdot + b(\cdot)) \in \mathrm{FS}(\mathbb{R}^n)$ for any $\varphi \in \mathrm{FS}(\mathbb{R}^n)$ and $b \in \mathrm{FS}(\mathbb{R}^n;\mathbb{R}^n)$.

4. Define $Y := \{ f \in C(\mathbb{R}^n;\mathbb{R}^n);\ f(x) - x \in L^\infty(\mathbb{R}^n;\mathbb{R}^n) \}$. Note that if $f, g \in Y$, then, $f - g \in L^\infty(\mathbb{R}^n;\mathbb{R}^n)$. We define in Y the metric $d_Y(f,g) := \|f - g\|_\infty$ and observe that (Y, d_Y) is a complete metric space. Fix $T > 0$ and let $X := C([-T,T];Y)$ with the metric $d(\varphi_1,\varphi_2) := \sup_{t\in[-T,T]} d_Y(\varphi_1(t),\varphi_2(t))$. The space (X,d) is a complete metric space and define $F : X \to X$ by:

$$F(\varphi)(t,z) := z + \int_0^t a(\varphi(s,z))ds.$$

Using a standard argument, we see that there is a single fixed point $X_t(z)$ of F which is the unique solution of (4.2). Moreover, $\forall\, \Phi \in X$, we have that $F^{(j)}(\Phi) \to X_t(z)$ in X. Therefore, for each t fix, $F^{(j)}(\Phi)(t) \to X_t$ uniformly in z. Now, we take $\Phi(t,z) = z$ and $X^{(j)} := F^{(j)}(\Phi) = F(F^{(j-1)}(\Phi)) = F(X^{(j-1)})$ and note that $X^{(1)}(t,z) = z + ta(z)$ and it is uniformly continuous in $[-T,T]\times\mathbb{R}^n$. By step 3, $\varphi \circ X^{(1)}(t,\cdot) = \varphi(\cdot + ta(\cdot)) \in \mathrm{FS}(\mathbb{R}^n)$ for any $\varphi \in \mathrm{FS}(\mathbb{R}^n)$.

5. Suppose that for all $\varphi \in \mathrm{FS}(\mathbb{R}^n)$, we have $\varphi(X^{(j-1)}(t,\cdot)) \in \mathrm{FS}(\mathbb{R}^n)$ for each fixed $t \in [-T,T]$ and $X^{(j-1)}$ is uniformly continuous in $[-T,T]\times\mathbb{R}^n$. Observe that,

$$\varphi(X^{(j)}(z,t)) = \varphi(F(X^{(j-1)})(z,t)) = \varphi\biggl(z + \int_0^t a(X^{(j-1)}(z,s))\, ds \biggr).$$

Since $X^{(j-1)}$ is uniformly continuous, then the Riemann sums of $\int_0^t a(X^{(j-1)}(z,s))\, ds$ uniformly converge in z. Therefore, $\int_0^t a(X^{(j-1)}(z,s))\, ds \in \mathrm{FS}(\mathbb{R}^n;\mathbb{R}^n)$ and by step 3, $\varphi(X^{(j)}(t,\cdot)) \in \mathrm{FS}(\mathbb{R}^n)$. Moreover, it is easy to see that $X^{(j)}$ is uniformly continuous in $[-T,T]\times\mathbb{R}^n$. Thus, we have proved, by induction, that $\varphi(X^{(j)}(t,\cdot)) \in \mathrm{FS}(\mathbb{R}^n)$ for all j. Therefore, the uniform convergence of $X^{(j)}(t,\cdot)$ to $X_t(\cdot)$ provides the proof of the first part of the theorem.

6. Now we prove (4.3). The incompressibility assumption (4.1) implies that the Jacobian determinant of X_t is a.e. equal to 1, and we have

$$\begin{aligned}
&\frac{1}{L^n}\int_{[0,L]^n}|\varphi(X(z,t))|^2\,dz=\frac{1}{L^n}\int_{X_t([0,L]^n)}|\varphi(w)|^2\,dw\\
&=\frac{1}{L^n}\int_{[0,L]^n}|\varphi(w)|^2\,dw-\frac{1}{L^n}\int_{[0,L]^n\setminus X_t([0,L]^N)}|\varphi(w)|^2\,dw\\
&+\frac{1}{L^n}\int_{X_t([0,L]^n)\setminus[0,L]^n}|\varphi(w)|^2\,dw.
\end{aligned}$$

Take the limit as $L\to\infty$ observing that the two last terms on the right-hand side of the last equality above go to 0 as $L\to\infty$ since

$$[\|a\|_\infty t, L-\|a\|_\infty t]^n\subseteq X_t([0,L]^n)\subseteq[-\|a\|_\infty t, L+\|a\|_\infty t]^n.$$

We then obtain (4.3). Relation (4.3) immediately implies that $\boldsymbol{X}_t$ can be extended to an operator in $\mathcal{B}^2$, and that $\boldsymbol{X}_t$ fulfills (4.4). □

We will make use of the following lemma which is a generalization of a lemma of [2], whose simple proof remains essentially the same and for which, therefore, we refer to [2].

Lemma 4.1. *Let X_1, X_2 be compact spaces, R_1 a dense subset of X_1 and $W:R_1\to X_2$. Suppose that for all $g\in C(X_2)$ the function $g\circ W$ is the restriction to R_1 of some (unique) $g_1\in C(X_1)$. Then W can be extended to a continuous mapping $\underline{W}:X_1\to X_2$.*
Further, suppose in addition that R_2 is a dense set of X_2, W is a bijection from R_1 onto R_2 and for all $f\in C(X_1)$, $f\circ W^{-1}$ is the restriction to R_2 of some (unique) $f_2\in C(X_2)$. Then W can be extended to a homeomorphism $\underline{W}:X_1\to X_2$.

Corollary 4.1. *For any $t\in\mathbb{R}$ the flow map X_t can be uniquely extended to a homeomorphism $\underline{X}_t$ of $\mathcal{K}$ and $\boldsymbol{X}_t(\varphi)=\varphi(\underline{X}_t)$ for any $\varphi\in L^2(\mathcal{K})$.*

Proof. Since $C(\mathcal{K})$ is isomorphic to $\mathrm{FS}(\mathbb{R}^n)$, it is a direct consequence of the invariance of $\mathrm{FS}(\mathbb{R}^n)$ under $\boldsymbol{X}_t$ and of Lemma 4.1, with $R_1=R_2=\mathbb{R}^n$, $X_1=X_2=\mathcal{K}$ and $W=X_t$. □

Let $\mathcal{S}$ be the closed subspace of $\mathcal{B}^2$ defined as follows. Let us consider the equation

$$\nabla\cdot(av)=0. \tag{4.5}$$

We define a class of asymptotic solutions of (4.5) as follows. Let us define the space of test functions

$$\mathcal{T}:=\{v\in\mathrm{FS}(\mathbb{R}^n)\cap W^{1,\infty}(\mathbb{R}^n)\,:\,\nabla_a v:=a\cdot\nabla v\in\mathrm{FS}(\mathbb{R}^n)\}. \tag{4.6}$$

We then define

$$\mathcal{S}:=\left\{v\in\mathcal{B}^2\ :\ \fint_{\mathbb{R}^n}v(z)\nabla_a\varphi(z)\,dz=0,\ \text{ for all }\varphi\in\mathcal{T}\right\}. \tag{4.7}$$

We also consider the following subspaces of $\mathcal{S}$:

$$\mathcal{S}^*:=\left\{v\in\mathrm{FS}(\mathbb{R}^n)\cap W^{1,\infty}(\mathbb{R}^n):\ \nabla_a v=0\quad\text{a.e.}\right\}, \tag{4.8}$$

$$\mathcal{S}^\dagger:=\left\{v\in\mathcal{S}\,:\,\exists\,(v_k)_{k\in\mathbb{N}}\subseteq\mathcal{T}, v_k\overset{\mathcal{B}^2\cap L^2_{\rm loc}}{\longrightarrow}v\text{ and }\nabla_a v_k\overset{\mathcal{B}^2\cap L^2_{\rm loc}}{\longrightarrow}0\right\}. \tag{4.9}$$

and

$$\mathcal{S}^\flat:=\left\{v\in\mathcal{S}\,:\,\exists\,(v_k)_{k\in\mathbb{N}}\subseteq\mathcal{T}, v_k\overset{\mathcal{B}^2}{\longrightarrow}v\text{ and }\nabla_a v_k\overset{\mathcal{B}^2}{\longrightarrow}0\right\}. \tag{4.10}$$

Clearly, we have $\mathcal{S}^* \subseteq \mathcal{S}^\dagger \subseteq \mathcal{S}^\flat$. We will also be concerned in this section with the question whether $\mathcal{S}^\dagger$ is dense in $\mathcal{S}$ because of our application to nonlinear transport equations. In the case of periodic functions, the analogues of $\mathcal{S}^\dagger$ and $\mathcal{S}^\flat$ coincide since convergences in $L^2_{\rm loc}$ and $\mathcal{B}^2$ are equivalent in that case. We will see in the next proposition that in fact $\mathcal{S}^\flat = \mathcal{S}$. The analogous result in the periodic case implies $\mathcal{S}^\dagger = \mathcal{S}$. The latter was given another proof in [12] by a standard argument using convolutions with approximations of the Dirac measure. Nevertheless, this argument is heavily supported on the equivalence between $L^2_{\rm loc}$ and $\mathcal{B}^2$ convergences in the periodic case and so cannot be extended even to the case of almost periodic functions. Therefore, here, as in [2], instead of considering the question of the density of $\mathcal{S}^\dagger$ in $\mathcal{S}$, we will address the question concerning the stronger property of the density of $\mathcal{S}^*$ in $\mathcal{S}$.

Let us consider for a moment the one-parameter group of unitary operators $\boldsymbol{X}_t : \mathcal{B}^2 \to \mathcal{B}^2$ defined by $\boldsymbol{X}_t v = v \circ X_t$, where X_t is the flow generated by the vector field a. Let us denote also by $\mathcal{B}^2$ its standard complex extension and consider the natural extension of $\boldsymbol{X}_t$ to the complexification of $\mathcal{B}^2$. By Stone's Theorem (see, e.g., [31], p. 266) there is a self-adjoint operator A on $\mathcal{B}^2$ so that $\boldsymbol{X}_t = e^{itA}$. We have the following fact.

Proposition 4.1. *The self-adjoint operator A such that $\boldsymbol{X}_t = e^{itA}$ is essentially self-adjoint on $\mathcal{T}$ and $A|\mathcal{T} = \frac{1}{i}\nabla_a$. Moreover, $\mathcal{S}$ is the invariant space under $\boldsymbol{X}_t$ and $\mathcal{S}^\flat = \mathcal{S}$.*

Proof. 1. The proof follows ideas in the proof of Stone's Theorem. We denote by $\langle\cdot,\cdot\rangle$ the inner product in $\mathcal{B}^2$:

$$\langle u, v\rangle := \fint_{\mathbb{R}^n} u(z)\bar{v}(z)\,dz, \qquad \text{for } u, v \in \mathcal{B}^2.$$

Let $\phi \in C_c^\infty(\mathbb{R})$ and for each $v \in \mathcal{T}$ define

$$v_\phi := \int_{-\infty}^{\infty} \phi(t)\boldsymbol{X}_t v\,dt = \int_{-\infty}^{\infty} \phi(t) v \circ X_t\,dt.$$

Let $\mathcal{T}^*$ be the set of finite linear combinations of all such v_ϕ for $v \in \mathcal{T}$ and $\phi \in C_c^\infty(\mathbb{R})$. We claim that $\mathcal{T}^* \subseteq \mathcal{T}$ and that $\mathcal{T}^*$ is dense in $\mathcal{T}$ in the uniform topology and hence it is also dense in $\mathcal{B}^2$.

2. First we see that $v_\phi \in \mathrm{FS} \cap W^{1,\infty}(\mathbb{R}^n)$. Indeed, that $v_\phi \in \mathrm{FS}(\mathbb{R}^n)$ follows from the uniform convergence of the Riemann sums and the invariance of FS by the flow X_t given by Theorem 4.1. The fact that $v_\phi \in W^{1,\infty}(\mathbb{R}^n)$ follows from the fact that $v \in W^{1,\infty}(\mathbb{R}^n)$ and the Lipschitz continuity of the X_t with respect to the initial data, which follows from the Lipschitz continuity of the vector field a through Duhamel's formula and Grönwall's inequality. Finally, the fact that $\nabla_a v_\phi \in \mathrm{FS}(\mathbb{R}^n)$ is seen as follows. By the Lipschitz continuity of v_ϕ we deduce that

$$\nabla_a v_\phi(x) = \lim_{h\to 0} \frac{v_\phi \circ X_h(x) - v_\phi(x)}{h},$$

where the limit exists for a.e. $x \in \mathbb{R}^n$. On the other hand we have

$$\begin{aligned}\left(\frac{\boldsymbol{X}_h - I}{h}\right) v_\phi &= \int_{-\infty}^{\infty} \phi(t)\left(\frac{\boldsymbol{X}_{t+h} - \boldsymbol{X}_t}{h}\right) v\,dt \\ &= \int_{-\infty}^{\infty} \frac{\phi(t-h) - \phi(t)}{h}\boldsymbol{X}_t v\,dt \\ &\xrightarrow{\mathcal{B}^2} -\int \phi'(t)\boldsymbol{X}_t v\,dt \\ &= v_{-\phi'}.\end{aligned}$$

Hence, $\nabla_a v_\phi = v_{-\phi'}$ and so $\nabla_a v_\phi \in \mathcal{T}^*$, which gives in particular that $\mathcal{T}^* \subseteq \mathcal{T}$. The density of $\mathcal{T}^*$ in $\mathcal{T}$ follows by taking an approximate identity sequence $\varphi_\varepsilon(t) = \varepsilon^{-1}\varphi(\varepsilon^{-1}t)$, with $0 \le \varphi \in C_c^\infty(\mathbb{R})$, $\int \varphi\,dt = 1$, and noticing that v_{φ_ε} converges uniformly to v for any $v \in \mathcal{T}$. Since $\mathcal{T}$ is dense in $\mathcal{B}^2$, we conclude that $\mathcal{T}^*$ is dense in $\mathcal{B}^2$.

3. For $v_\phi \in \mathcal{T}^*$ we define $Bv_\phi := i^{-1}\nabla_a v_\phi = i^{-1}v_{-\phi'}$. Notice that $\boldsymbol{X}_t : \mathcal{T}^* \to \mathcal{T}^*$, $B : \mathcal{T}^* \to \mathcal{T}^*$ and $\boldsymbol{X}_t B v_\phi = B\boldsymbol{X}_t v_\phi$, for $v_\phi \in \mathcal{T}^*$. Furthermore, given $v_1, v_2 \in \mathcal{T}^*$ we clearly have

$$\begin{aligned}\langle Bv_1, v_2\rangle &= \lim_{s\to 0}\left\langle \left(\frac{\boldsymbol{X}_s - I}{is}\right) v_1, v_2\right\rangle \\ &= \lim_{s\to 0}\left\langle v_1, \left(\frac{I-\boldsymbol{X}_{-s}}{is}\right) v_2\right\rangle \\ &= \langle v_1, Bv_2\rangle,\end{aligned}$$

and so B is symmetric.

4. The proof that B is essentially self-adjoint uses the criterion that this fact is equivalent to $\mathrm{Ker}(B^* \pm i) = \{0\}$ (see [31], p. 257) and follows the lines of the argument for a similar assertion in the proof Stone's Theorem. The argument is as follows. Suppose that there is a $u \in D(B^*)$ so that $B^*u = iu$. Then for each $v \in D(B) = \mathcal{T}^*$,

$$\begin{aligned}\frac{d}{dt}\langle \boldsymbol{X}_t v, u\rangle &= \langle iB\boldsymbol{X}_t v, u\rangle = -i\langle \boldsymbol{X}_t v, B^*u\rangle = -i\langle \boldsymbol{X}_t v, iu\rangle \\ &= \langle \boldsymbol{X}_t v, u\rangle.\end{aligned}$$

Thus, the complex-valued function $f(t) = \langle \boldsymbol{X}_t v, u\rangle$ satisfies the ordinary differential equation $f' = f$ which means $f(t) = f(0)e^t$, and so $f(0) = \langle v, u\rangle = 0$, because $f(t)$ is bounded. Since $\mathcal{T}^*$ is dense, $u = 0$. A similar argument shows that $\mathrm{Ker}(B^* + i) = \{0\}$, which completes the proof that B is essentially self-adjoint. In particular, $A = \bar{B}$.

5. Now, we observe that $\mathcal{T} \subseteq D(A)$. Indeed, this follows if we can show that $\lim_{t\to 0}\dfrac{\boldsymbol{X}_t v - v}{t}$ exists in $\mathcal{B}^2$ for all $v \in \mathcal{T}$ (see, e.g., Theorem VIII.7 in [31]). But from the definition of $\mathcal{T}$ we immediately have that this limit exists in the sense of uniform convergence and so also in $\mathcal{B}^2$, and it is equal to $\nabla_a v$. In particular, $A|\mathcal{T} = i^{-1}\nabla_a$.

6. We show now that $\mathcal{S} \subseteq D(A)$ and that $\mathcal{S}$ is the invariant space under $\boldsymbol{X}_t$. The fact that $\mathcal{S} \subseteq D(A)$ follows from the definition of $\mathcal{T}$ since it implies trivially that $\mathcal{S} \subseteq D(B^*) = D(A)$. Now, for each $u \in \mathcal{S}$, given any $v \in \mathcal{T}$, the function $g(t) := \langle \boldsymbol{X}_t u, v\rangle = \langle u, \boldsymbol{X}_{-t} v\rangle$ satisfies $g'(t) = 0$ and so $g(t) = g(0)$. Hence, $\boldsymbol{X}_t u = u$ for all $u \in \mathcal{S}$. On the other hand, if $u \in \mathcal{B}^2$ is invariant by $\boldsymbol{X}_t$ we have

$$0 = \langle (\boldsymbol{X}_t - I)u, v\rangle = \langle u, (\boldsymbol{X}_{-t} - I)v\rangle.$$

Dividing by t and making $t \to 0$ we obtain that $u \in \mathcal{S}$.

7. Finally, the fact that $\mathcal{S}^\flat = \mathcal{S}$ is a consequence of the fact that A is essentially self-adjoint when restricted to $\mathcal{T}^*$ and $\mathcal{S} \subseteq D(A)$. Indeed, given $u \in \mathcal{S}$, since $(u, 0) \in \mathrm{graph}(A)$, we have that there exists a sequence $v_\alpha \in \mathcal{T}^*$ such that $v_\alpha \to u$ in $\mathcal{B}^2$ and $\nabla_a v_\alpha \to 0$ in $\mathcal{B}^2$. This means that $u \in \mathcal{S}^\flat$ and so $\mathcal{S}^\flat = \mathcal{S}$.

□

Recalling the canonical isomorphism between $\mathcal{B}^2$ and $L^2(\mathcal{K})$, when v is viewed as a function in $L^2(\mathcal{K})$, we can say that $v \in \mathcal{S}$ if

$$\int_{\mathcal{K}} v(z)\nabla_a\varphi(z)\,d\mathfrak{m}(z) = 0 \qquad \text{for all } \varphi \in \mathcal{T}, \tag{4.11}$$

where, for simplicity, we use the same notation for a function g in $\mathrm{FS}(\mathbb{R}^n)$ or $\mathcal{B}^2$ and its extension $\underline{g}$ to $C(\mathcal{K})$ or $L^2(\mathcal{K})$.

Given $g \in \mathcal{B}^2$, we denote by $\tilde{g} \in \mathcal{S}$ its orthogonal projection on $\mathcal{S}$. Accordingly, we denote by $\tilde{a}$ the vector field whose components $\tilde{a}_i$ are the projections on $\mathcal{S}$ of a_i.

By the properties of orthogonal projections, $\tilde{g}$ is characterized by

$$\int_{\mathcal{K}} gh\,d\mathfrak{m} = \int_{\mathcal{K}} \tilde{g}h\,d\mathfrak{m}, \qquad g \in L^2(\mathcal{K}),\ h \in \mathcal{S}. \tag{4.12}$$

The next result improves a similar proposition in [2] for the case of almost periodic functions. The proof is similar to the corresponding one in [2] but now we can dispense with the density of $\mathcal{S}^*$ in $\mathcal{S}$ by using the equality $\mathcal{S}^\flat = \mathcal{S}$ which holds in general by Proposition 4.1.

Proposition 4.2. *$\mathcal{S} \cap L^\infty(\mathcal{K})$ is an algebra and*

$$\widetilde{gr} = g\tilde{r} \qquad \forall g \in \mathcal{S} \cap L^\infty(\mathcal{K}),\ r \in L^2(\mathcal{K}). \tag{4.13}$$

Proof. Given $g \in \mathcal{S} \cap L^\infty(\mathcal{K})$, let $(g_k)_{k\in\mathbb{N}} \subseteq \mathcal{T}$ be such that $g_k \to g$ and $\nabla_a g_k \to 0$ in $L^2(\mathcal{K})$, which exists since $\mathcal{S} = \mathcal{S}^\flat$. As $g \in L^\infty(\mathcal{K})$, we may choose (g_k) to be uniformly bounded, by replacing g_k by $\rho \circ g_k$ where $\rho \in C_c^\infty(\mathbb{R})$ is such that $\rho(s) = s$ for $s \in [-\|g\|_\infty, \|g\|_\infty]$. Now, given $r \in \mathcal{S}$, since $g_k \in \mathcal{T}$, we have the identity

$$rg_k\nabla_a\varphi + r\varphi\nabla_a g_k = r\nabla_a(\varphi g_k),$$

for all $\varphi \in \mathcal{T}$. Integrating, we obtain

$$\langle rg_k, \nabla_a\varphi\rangle = -\langle r\varphi, \nabla_a g_k\rangle,$$

and letting $k \to \infty$ we find $\langle rg, \nabla_a\varphi\rangle = 0$, for all $\varphi \in \mathcal{T}$, which means that $rg \in \mathcal{S}$.

Finally, let g, r be as in (4.13). For any $h \in \mathcal{S}$ we have

$$\int_{\mathcal{K}} h\widetilde{gr}\,d\mathfrak{m} = \int_{\mathcal{K}} h(gr)\,d\mathfrak{m} = \int_{\mathcal{K}} hg\tilde{r}\,d\mathfrak{m}$$

because $hg \in \mathcal{S}$. Since $g\tilde{r} \in \mathcal{S}$ and $h \in \mathcal{S}$ is arbitrary this proves that $\widetilde{gr} = g\tilde{r}$. $\square$

We remark that the Mean Ergodic Theorem (see [14], Theorem VIII.7.1), which is applicable due to Theorem 4.1 and to the fact that $\mathcal{S}$ is the invariant space of $\boldsymbol{X}_t$ (see Proposition 4.1 above) implies that for $g \in \mathcal{B}^2$ we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \boldsymbol{X}_s g(z)\,ds = \tilde{g}(z) \qquad \forall g \in \mathcal{B}^2, \tag{4.14}$$

in the sense of convergence in $\mathcal{B}^2$, and one can use this formula to link in a more explicit way $\tilde{g}$ to g.

For the remaining of this section we will be concerned with the question of the density of $\mathcal{S}^*$ in $\mathcal{S}$ which obviously implies the density of $\mathcal{S}^\dagger$ in $\mathcal{S}$ and is motivated by the application to nonlinear transport equations to be considered later on. More specifically, we will be interested in establishing conditions on the vector field a such that the density of $\mathcal{S}^*$ in $\mathcal{S}$ holds.

For the sake of reference, we state the following elementary lemma whose proof is a simple calculus exercise left to the reader.

Lemma 4.2. *Let $W : \mathbb{R}^n \to \mathbb{R}^n$ be a bi-Lipschitz map, $\Phi := W^{-1}$ and b a vector field. Then, the following are equivalent:*

(i) $b \cdot \nabla(\varphi \circ W) = (D_n\varphi) \circ W$ *for all* $\varphi \in C^1(\mathbb{R}^n)$.

(ii) $(D_n\Phi) \circ W = b$.

where D_n is the partial derivative with respect to the n-th coordinate. Thus, Φ satisfies the ordinary differential equation

$$\frac{d\Phi}{dy_n} = b(\Phi).$$

The first result that we establish giving sufficient conditions for the density of $\mathcal{S}^*$ in $\mathcal{S}$ is the analogue of a lemma established in [2] for the case of almost periodic functions. The proof follows the same lines as the proof of the analogous result in [2] with the exception that here we have to replace the use made therein of the orthogonal family $\{\cos\lambda\cdot x,\ \sin\lambda\cdot x\ :\ \lambda \in \mathbb{R}^n\}$ spanning $\mathrm{AP}(\mathbb{R}^n)$ by the decomposition given by Lemma 3.2. We omit the proof since it may be easily achieved from what we have said and also because it is similar to the proof of the Lemma 4.4 stated below, whose details will be provided.

Lemma 4.3. *Suppose that $W:\mathbb{R}^n_z\to\mathbb{R}^n_w$ is a bi-Lipschitz map satisfying: for all $g\in\mathrm{FS}(\mathbb{R}^n)$, $g\circ W\in\mathrm{FS}(\mathbb{R}^n)$ and $g\circ W^{-1}\in\mathrm{FS}(\mathbb{R}^n)$. Let $J=|\det\frac{\partial W}{\partial z}|$ and assume that $J\in\mathrm{FS}(\mathbb{R}^n)$ and $\kappa_1\le J\le\kappa_2$ for certain constants $0<\kappa_1\le\kappa_2$. Assume that the vector field $a(z)$ satisfies*

$$\frac{1}{J(z)}a(z)\cdot\nabla_z(\varphi\circ W)=\left(\frac{\partial\varphi}{\partial w_n}\right)\circ W \qquad \forall\varphi\in C^1(\mathbb{R}^n_w).$$

Then $\mathcal{S}^*$ is dense in $\mathcal{S}$.

The following lemma is a more efficient tool in providing concrete examples where $\mathcal{S}^*$ is dense in $\mathcal{S}$.

Lemma 4.4. *If $a=(a_1,\cdots,a_n)$ is a vector field and $x=(x_1,\cdots,x_n)$, let us use the notation $\bar a=(a_1,\cdots,a_{n-1})$ and $\bar x=(x_1,\cdots,x_{n-1})$. Suppose the following assumptions hold:*

(A1) *$a\in\mathrm{FS}\cap W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^n)$, $\operatorname{div}a=0$ and $|a_n|\ge\delta>0$;*

(A2) *The function defined by $\Phi(x):=(\Phi_{x_n}(\bar x),x_n)$, where $\Phi_{x_n}(\bar x)$ is the flow associated with the Cauchy problem*

$$\begin{cases}\dfrac{dX}{dx_n}(\bar x,x_n)=\dfrac{\bar a(X(\bar x,x_n),x_n)}{a_n(X(\bar x,x_n),x_n)},\\ X(\bar x,0)=\bar x,\end{cases}\tag{4.15}$$

is such that $\Phi:\mathbb{R}^n\mapsto\mathbb{R}^n$ is a bi-Lipschitz map and $g\circ\Phi$, $g\circ\Phi^{-1}\in\mathrm{FS}(\mathbb{R}^n)$ for any $g\in\mathrm{FS}(\mathbb{R}^n)$.

Then, $\mathcal{S}^*$ is dense in $\mathcal{S}$.

Proof. 1. Its easy to check that

$$\operatorname{div}_{\bar x}\left(\frac{a}{a_n}\right)=-\frac{1}{{a_n}^2}\,a\cdot\nabla a_n$$

By Lemma 4.2, $a\cdot\nabla a_n=a_nD_n(a_n\circ\Phi)\circ\Phi^{-1}$. This gives

$$\operatorname{div}_{\bar x}\left(\frac{a}{a_n}\right)(\Phi)=\frac{-1}{a_n\circ\Phi}\,D_n(a_n\circ\Phi)=\frac{D_n((a_n\circ\Phi)^{-1})}{(a_n\circ\Phi)^{-1}}.$$

By the well known Euler's formula for Jacobians (see, e.g., [19]), we have

$$D_nJ=\operatorname{div}_{\bar x}\left(\frac{a}{a_n}\right)(\Phi)J$$

where $J:=|\det(\partial\Phi/\partial z)|$. Hence, combining these last two equalities, we conclude that

$$J(x)=\frac{a_n(\bar x,0)}{a_n(\Phi(x))}.$$

In particular, $J\in\mathrm{FS}(\mathbb{R}^n)$.

2. Let E_1 and E_2 be as in Lemma 3.2. Given $\psi\in E_2$, there is $\varphi\in\mathrm{FS}_*(\mathbb{R}^n)$ such that $D_n\varphi=\frac{\psi}{a_n(\bar x,0)}$. Indeed, let $f\in C_c^\infty(\mathbb{R}^n)$ be such that $\hat f=1$ on $\operatorname{supp}\hat\psi$ and $\hat f=0$ on a neighborhood of $\{x:x_n=0\}$. Set $\hat h:=\frac{\hat f}{ix_n}$ and define $\varphi:=\frac{1}{a_n(\bar x,0)}\psi*h$. Therefore, it is sufficient to observe that $D_n(\psi*h)=\psi*D_nh=\psi*f=\psi$.

3. Let $W:=\Phi^{-1}$. Using the fact that

$$[-L/C,L/C]^n\subseteq W([-L,L]^n)\subseteq[-LC,LC]^n$$

for a suitable constant C, it is easy to see that $g\circ W\in\mathcal{B}^2$ if and only if $g\in\mathcal{B}^2$. Now, if f, $g\in\mathrm{FS}_*(\mathbb{R}^n)$, then the following important relation holds:

$$\lim_{L\to+\infty}\frac{1}{(2L)^n}\int_{W([-L,L]^n)}f(x)g(x)\,dx=M(fg)M(J^{-1}).\tag{4.16}$$

Indeed, let $\mu = \widehat{fg}$. Hence, we have

$$\begin{aligned}
\frac{1}{(2L)^n}\int_{W([-L,L]^n)} f(x)g(x)\,dx &= \frac{1}{(2L)^n}\int_{[-L,L]^n} f\circ W g\circ W J^{-1}(x)\,dx \\
&= \frac{1}{(2L)^n}\int_{[-L,L]^n}\int_{\mathbb{R}^n} e^{iW(x)\cdot y}\,d\mu(y)J^{-1}(x)\,dx \\
&= \mu(\{0\})\frac{1}{(2L)^n}\int_{[-L,L]^n} J^{-1}(x)\,dx \\
&+ \frac{1}{(2L)^n}\int_{[-L,L]^n}\int_{\mathbb{R}^n\setminus\{0\}} e^{iW(x)\cdot y}\,d\mu(y)J^{-1}(x)\,dx \\
&= M(fg)\frac{1}{(2L)^n}\int_{[-L,L]^n} J^{-1}(x)\,dx + \int_{\mathbb{R}^n\setminus\{0\}}\frac{1}{(2L)^n}\int_{W([-L,L]^n)} e^{ix\cdot y}\,dx\,d\mu(y) \\
&\to M(fg)M(J^{-1}) \qquad \text{as } L\to\infty,
\end{aligned}$$

where the last limit is due to the fact that for $y \neq 0$ the inner integral is $O(L^{n-1})$. Clearly (4.16) may be extended to hold for $f, g \in \mathcal{B}^2$.

4. Given $v \in \mathcal{S}$, we have by the definition of $\mathcal{S}$

$$0 = \fint_{\mathbb{R}^n} v(z)a(z)\cdot\nabla\varphi(z)\,dz \qquad \text{for all } \varphi\in\mathcal{T}.$$

Changing φ by $\varphi\circ W$ in the equality above and taking into account Lemma 4.2 which yields

$$a\cdot\nabla(\varphi\circ W) = (a_n(\bar{x},0)\circ W)\,(D_n\varphi)\circ W\,J^{-1},$$

we get

$$\begin{aligned}
0 &= \fint_{\mathbb{R}^n} v(x)a(x)\cdot\nabla(\varphi\circ W)(x)\,dx = \lim_{L\to+\infty}\frac{1}{(2L)^n}\int_{[-L,L]^n} v(x)(a_n(\bar{x},0)\circ W)\,(D_n\varphi)\circ W\,J^{-1}\,dx \\
&= \lim_{L\to+\infty}\frac{1}{(2L)^n}\int_{W([-L,L]^n)} v\circ W^{-1}a_n(\bar{x},0)D_n\varphi\,dx = \big\langle v\circ W^{-1}, a_n(\bar{x},0)D_n\varphi\big\rangle M(J^{-1}),
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the usual inner product of $\mathcal{B}^2$. Hence, by the step 2,

$$\langle v\circ W^{-1},\psi\rangle = 0, \qquad \forall\psi\in\overline{E_2}.$$

Since $\mathcal{B}^2 = \overline{E_1}\oplus\overline{E_2}$, we have that $v\circ W^{-1}\in\overline{E_1}$ and from this it follows that there exists $(\varphi_j)_{j\geq1}\subseteq E_1$ such that $\varphi_j\to v\circ W^{-1}$ in the sense of $\mathcal{B}^2$. Using (4.16) and the fact that $0<k_1\leq J^{-1}\leq k_2$ for suitable positive constants k_1 and k_2, we see that

$$\|\varphi_j\circ W - v\|^2 \leq \frac{k_2}{k_1}\,\|\varphi_j - v\circ W^{-1}\|^2,$$

where $\|\cdot\|$ is the norm induced by the standard inner product of $\mathcal{B}^2$. Therefore, $\varphi_j\circ W\to v$ in $\mathcal{B}^2$ and, since $(\varphi_j\circ W)_{j\geq1}\subseteq\mathcal{S}^*$, the result follows. □

We conclude this section with some examples of vector fields satisfying the assumptions (A1) and (A2) of Lemma 4.4. For this purpose the next lemma will be important.

Lemma 4.5. *Let us consider the Cauchy problem*

$$\begin{cases} \frac{dX}{dt}(z,t) = \bar{\alpha}(X(z,t))\theta(t), \\ X(z,0) = z. \end{cases} \tag{4.17}$$

where $\bar\alpha \in \mathrm{FS}\cap W^{1,\infty}(\mathbb{R}^{n-1};\mathbb{R}^{n-1})$ *and* $\theta \in \mathrm{FS}(\mathbb{R})$ *is such that* $\varrho(t) := \int_0^t \theta(s)\,ds \in \mathrm{FS}(\mathbb{R})$. *Let* $X_t(x)$ *be the flow generated by the vector field* $\bar\alpha(x)$. *Then,* $\Phi(x,t) := (X_{\varrho(t)}(x), t)$ *satisfies* $\varphi\circ\Phi$, $\varphi\circ\Phi^{-1} \in \mathrm{FS}(\mathbb{R}^n)$ *for any* $\varphi \in \mathrm{FS}(\mathbb{R}^n)$.

Proof. The proof follows the same method as the one of Theorem 4.1 and so we only describe the main steps.

1. As in the proof of the referred theorem, we first show that $\varphi(x+\beta(x,t),t) \in \mathrm{FS}(\mathbb{R}^n)$ for any $\varphi \in \mathrm{FS}(\mathbb{R}^n)$ and $\beta \in \mathrm{FS}(\mathbb{R}^n;\mathbb{R}^{n-1})$.

2. Using the same notation in the proof of Theorem 4.1 with $T := \|\varrho\|_\infty$, we then prove that $X^{(j)}(\cdot,\varrho(\cdot)) \to X_{\varrho(\cdot)}(\cdot)$ uniformly with respect to x, t. Moreover, $\varphi(X^{(1)}(\cdot,\varrho(\cdot)),\cdot) \in \mathrm{FS}(\mathbb{R}^n)$ for any $\varphi \in \mathrm{FS}(\mathbb{R}^n)$.

3. Finally, we prove that if $X^{(j-1)}$ is uniformly continuous in $\mathbb{R}^{n-1}\times[-T,T]$ and $\varphi(X^{(j-1)}(\cdot,\varrho(\cdot)),\cdot) \in \mathrm{FS}(\mathbb{R}^n)$ for all $\varphi \in \mathrm{FS}(\mathbb{R}^n)$, then the same is true of $X^{(j)}$. To this we observe that

$$\varphi(X^{(j)}(x,\varrho(t)),t) = \varphi\Big(x + \int_0^{\varrho(t)} \bar\alpha(X^{(j-1)}(x,s))\,ds\,,\,t\Big)$$

and, by the uniform continuity of $\bar\alpha(X^{(j-1)}(x,s))$ and $\varrho(t)$, the Riemann sums corresponding to the integral $\int_0^{\varrho(t)} \bar\alpha(X^{(j-1)}(x,s))\,ds$ uniformly converge in x,t. The conclusion of the proof is as in the one of Theorem 4.1. □

Corollary 4.2. *Let* a *be a vector field of the form* $a(\bar x, x_n) := \Big(\alpha_1(\bar x)\theta(x_n),\cdots,\alpha_{n-1}(\bar x)\theta(x_n),\alpha_n(\bar x)\Big)$, *where* $\alpha_i \in \mathrm{FS}\cap W^{1,\infty}(\mathbb{R}^{n-1})$, $i=1,\dots,n$, $|\alpha_n| \ge \delta > 0$, $\theta \in \mathrm{FS}\cap W^{1,\infty}(\mathbb{R})$, $\int_0^{x_n}\theta(y)\,dy \in \mathrm{FS}(\mathbb{R})$ *and* $\operatorname{div}_{\bar x}\bar\alpha = 0$. *Then,* $\mathcal{S}^*$ *is dense in* $\mathcal{S}$.

Proof. It suffices to observe that a satisfies the assumptions (A1) and (A2) of Lemma 4.4, which is immediate from Lemma 4.5. □

Remark 4.1. We observe that if $\theta \in \mathrm{FS}_*(\mathbb{R})$ is such that $\frac{1}{y} \in L^1_{|\hat\theta|}(\mathbb{R})$, then the indefinite integral of θ belongs to $\mathrm{FS}(\mathbb{R})$.

5. Two-scale Young Measures

In this section we recall the theorem giving the existence of two-scale Young measures established in [3]. We begin by recalling the concept of vector-valued algebra with mean value.

Given a Banach space E and an algebra w.m.v. $\mathcal{A}$, we denote by $\mathcal{A}(\mathbb{R}^n;E)$ the space of functions $f \in \mathrm{BUC}(\mathbb{R}^n;E)$ such that $L_f := \langle L, f\rangle$ belongs to $\mathcal{A}$ for all $L \in E^*$ and the family $\{L_f\,:\, L\in E^*,\ \|L\|\le 1\}$ is relatively compact in $\mathcal{A}$.

For bounded Borel sets $Q \subseteq \mathbb{R}^n$ and $f \in \mathrm{BUC}(\mathbb{R}^n;E)$, it is easily checked by an approximation with Riemann sums that $L \mapsto \int_Q \langle L, f\rangle\,dx$ defines a linear functional on E^*, continuous for the weak topology $\sigma(E^*,E)$; as a consequence, there exists a unique element of E, that we shall denote by $\int_Q f\,dx$, satisfying

$$\langle L, \int_Q f\,dx\rangle = \int_Q \langle L, f\rangle\,dx \qquad \forall L \in E^*.$$

For similar reasons, if $f \in \mathcal{A}(\mathbb{R}^n;E)$ the integrals $⨍_{Q_t} f\,dx$ weakly converge in E, as $t\to+\infty$, to a vector, that we shall denote by $⨍_{\mathbb{R}^n} f\,dx$, characterized by

$$\langle L,\ ⨍_{\mathbb{R}^n} f\,dx\rangle = ⨍_{\mathbb{R}^n}\langle L, f\rangle\,dx \qquad \forall L \in E^*.$$

Theorem 5.1 (cf. [3])**.** *Let E be a Banach space, $\mathcal{A}$ an algebra and $\mathcal{K}$ be the compact associated with $\mathcal{A}$. There is an isometric isomorphism between $\mathcal{A}(\mathbb{R}^n; E)$ and $C(\mathcal{K}; E)$. Denoting by $g \mapsto \underline{g}$ the canonical map from $\mathcal{A}$ to $C(\mathcal{K})$, the isomorphism associates to $f \in \mathcal{A}(\mathbb{R}^n; E)$ the map $\underline{f} \in C(\mathcal{K}; E)$ satisfying*

$$\underline{\langle L, f\rangle} = \langle L, \underline{f}\rangle \in C(\mathcal{K}) \qquad \forall L \in E^*. \tag{5.1}$$

In particular, for each $f \in \mathcal{A}(\mathbb{R}^n; E)$, $\|f\|_E \in \mathcal{A}$.

We define the space $L^p(\mathcal{K}; E)$ as the completion of $C(\mathcal{K}; E)$ with respect to the norm $\|\cdot\|_p$, defined as usual:

$$\|f\|_p := \left(\int_{\mathcal{K}} \|f\|_E^p \, d\mathfrak{m} \right)^{1/p}.$$

As usual, we identify functions in L^p that coincide $\mathfrak{m}$-a.e. in $\mathcal{K}$.

The next theorem gives the existence of two-scale Young measures associated with an algebra $\mathcal{A}$. For the proof, we again refer to [3].

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and $\{u_\varepsilon(x)\}_{\varepsilon>0}$ be a family of functions in $L^\infty(\Omega; K)$, for some compact metric space K.

Theorem 5.2. *Given any infinitesimal sequence $\{\varepsilon_i\}_{i\in\mathbb{N}}$ there exist a subnet $\{u_{\varepsilon_{i(d)}}\}_{d\in D}$, indexed by a certain directed set D, and a family of probability measures on K, $\{\nu_{z,x}\}_{z\in\mathcal{K}, x\in\Omega}$, weakly measurable with respect to the product of the Borel σ-algebras in $\mathcal{K}$ and $\mathbb{R}^n$, such that*

$$\lim_D \int_\Omega \Phi\Big(\frac{x}{\varepsilon_{i(d)}}, x, u_{\varepsilon_{i(d)}}(x)\Big)\, dx = \int_\Omega \int_{\mathcal{K}} \langle \nu_{z,x}, \underline{\Phi}(z, x, \cdot)\rangle \, d\mathfrak{m}(z)\, dx \qquad \forall \Phi \in \mathcal{A}\left(\mathbb{R}^n; C_0(\Omega \times K)\right). \tag{5.2}$$

Here $\underline{\Phi} \in C\left(\mathcal{K}; C_0(\Omega \times K)\right)$ denotes the unique extension of Φ. Moreover, equality (5.2) *still holds for functions Φ in the following function spaces:*

1. *$\mathcal{B}^1(\mathbb{R}^n; C_0(\Omega \times K))$;*
2. *$\mathcal{B}^p(\mathbb{R}^n; C(\bar{\Omega} \times K))$ with $p > 1$;*
3. *$L^1(\Omega; \mathcal{A}(\mathbb{R}^n; C(K)))$.*

As in the classical theory of Young measures we have the following consequence of Theorem 5.2.

Theorem 5.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, let $\{u_\varepsilon\} \subseteq L^\infty(\Omega; \mathbb{R}^m)$ be uniformly bounded and let $\nu_{z,x}$ be a two-scale Young measure generated by a subnet $\{u_{\varepsilon(d)}\}_{d\in D}$, according to Theorem 5.2. Assume that U belongs either to $L^1(\Omega; \mathcal{A}(\mathbb{R}^n; \mathbb{R}^m)))$ or to $\mathcal{B}^p(\mathbb{R}^n; C(\bar{\Omega}; \mathbb{R}^m))$ for some $p > 1$. Then*

$$\nu_{z,x} = \delta_{\underline{U}(z,x)} \quad \textit{if and only if} \quad \lim_D \|u_{\varepsilon(d)}(x) - U(\frac{x}{\varepsilon(d)}, x)\|_{L^1(\Omega)} = 0. \tag{5.3}$$

6. Application to Nonlinear Transport equations

In this section we study the homogenization problem for a nonlinear transport equation with an incompressible and autonomous velocity field. The main result here improves and extends the one corresponding to the same problem in [2] in the context of homogenization in $\mathrm{AP}(\mathbb{R}^n)$.

Let $a \in \mathrm{FS} \cap W^{1,\infty}(\mathbb{R}^n; \mathbb{R}^n)$, and let us assume that a is incompressible, i.e.

$$\nabla_z \cdot a(z) = 0. \tag{6.1}$$

We consider the equation

$$\partial_t u_\varepsilon + \nabla_x \cdot (a(\frac{x}{\varepsilon}) f(u_\varepsilon)) = 0, \qquad t > 0,\ x \in \mathbb{R}^n, \tag{6.2}$$

with $f \in C^1(\mathbb{R})$, and the initial data given by

$$u_\varepsilon(x, 0) = U_0(\frac{x}{\varepsilon}, x), \tag{6.3}$$

where $U_0(z,x) \in L^1_{loc}(\mathbb{R}^n; \mathrm{FS}(\mathbb{R}^n))$. For each $z \in \mathcal{K}$, we also consider the auxiliary initial value problem given by

$$U_t + \nabla_x \cdot (\tilde{a}(z) f(U)) = 0, \qquad t > 0,\ x \in \mathbb{R}^n, \tag{6.4}$$

and the initial data

$$U(z,x,0) = U_0(z,x), \qquad x \in \mathbb{R}^n. \tag{6.5}$$

In (6.4), $\tilde{a}$ is the vector field whose components are the images of the corresponding components of the vector field a by the projection of $L^2(\mathcal{K})$ onto $\mathbb{S}$ as defined by (4.12), in accordance with the notation introduced in Section 4.

The stability properties of entropy solutions to scalar conservation laws show that, possibly modifying $\tilde{a}$ in a negligible set, U may be viewed as a Borel map from $\mathcal{K}$ into $L^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$, where $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,+\infty))$. Using this fact, one can for instance find a Borel function $\bar{U}$, setting

$$\bar{U}(z,x,t) := \liminf_{\varepsilon \to 0} U(z,\cdot,t) * \rho_\varepsilon(x).$$

Hence, in the following we can assume with no loss of generality that U is a Borel map.

We will need the following theorem (see [2] for the proof) which provides a comparison principle between two parametrized families of measures satisfying a first-order differential inequality in conservation form, which extends a theorem of DiPerna [17].

Theorem 6.1. *Let $\{\mu^i_{x,t}\}$, $(x,t) \in \mathbb{R}^{n+1}_+$, $i = 1,2$, be two weakly measurable parametrized families of probability measures over a compact separable metric space K. Let $\{\mu^i_{x,0}\}_{x \in \mathbb{R}^n}$, $i = 1,2$, be two parametrized families of probability measures over K satisfying*

$$\begin{aligned} &\lim_{t \to 0} \frac{1}{t} \int_0^t \int_{\mathbb{R}^n} \langle \mu^1_{x,s}, g\rangle \phi(x)\, dx\, ds = \int_{\mathbb{R}^n} \langle \mu^1_{x,0}, g\rangle \phi(x)\, dx, \\ &\lim_{t \to 0} \int_{\{|x|<R\}} |\langle \mu^2_{x,s}, g\rangle - \langle \mu^2_{x,0}, g\rangle|\, dx = 0, \end{aligned} \tag{6.6}$$

for all $g \in C(K)$, $\phi \in C_c(\mathbb{R}^n)$ and $R > 0$. Let $I : K \times K \to \mathbb{R}$, $G : K \times K \to \mathbb{R}^n$ be continuous functions with $I \ge 0$ and $|G(\rho,\lambda)| \le C\, I(\rho,\lambda)$, for some $C > 0$. Assume

$$\begin{aligned} &\partial_t \langle \mu^1_{x,t}, I(\cdot,\lambda)\rangle + \nabla_x \cdot \langle \mu^1_{x,t}, G(\cdot,\lambda)\rangle \le 0, \qquad \text{for all } \lambda \in K, \\ &\partial_t \langle \mu^2_{x,t}, I(\rho,\cdot)\rangle + \nabla_x \cdot \langle \mu^2_{x,t}, G(\rho,\cdot)\rangle \le 0, \qquad \text{for all } \rho \in K, \end{aligned} \tag{6.7}$$

in the sense of the distributions in $\mathbb{R}^{n+1}_+$. Then, for a.e. $t > 0$, we have

$$\int_{\{|x|<R\}} \langle \mu^1_{x,t} \otimes \mu^2_{x,t}, I(\cdot,\cdot)\rangle\, dx \le \int_{\{|x|<R+Ct\}} \langle \mu^1_{x,0} \otimes \mu^2_{x,0}, I(\cdot,\cdot)\rangle\, dx. \tag{6.8}$$

In the following theorem we extend to the context of the algebra FS a result of W. E (*cf.* [18]), relative to the periodic case. We relax the restriction $f' > 0$ imposed on [18], asking only that the set of zeros of f' is nowhere dense. We characterize the weak limit of u_ε and, under suitable additional regularity assumptions on U, we prove a strong correctors formula. By the latter we mean an oscillatory profile $U(\frac{x}{\varepsilon}, x, t)$ which corrects the weak convergence of u_ε to a strong one in L^1_{loc}.

We remark that in the periodic setting A.-L. Dalibard [12] has recently obtained a characterization of the weak limit of u_ε, with no strong correctors formula, without restrictions on f, as a consequence of a more general rather technical analysis using the kinetic formulation.

Theorem 6.2. *Let $a \in W^{1,\infty} \cap \mathrm{FS}(\mathbb{R}^n;\mathbb{R}^n)$ and $U_0 \in L^1_{loc}(\mathbb{R}^n; \mathrm{FS}(\mathbb{R}^n))$. Let $\{u_\varepsilon\}_{\varepsilon>0}$ be the sequence of entropy solutions of* (6.2), (6.3)*. Assume that the set $E = \{u \in \mathbb{R} : f'(u) = 0\}$ has one-dimensional Lebesgue measure zero, that U_0 is bounded and satisfies*

$$U_0(\cdot,x) \in \mathcal{S} \text{ for a.e. } x \in \mathbb{R}^n, \text{ with } \mathcal{S} \text{ defined in } (4.7) \tag{6.9}$$

and finally that the set $\mathcal{S}^\dagger$ defined in (4.9) *is dense in* $\mathcal{S}$.
Then u_ε weakly star converge in $L^\infty(\mathbb{R}^{n+1}_+)$ to

$$u(x,t) := \int_{\mathcal{K}} U(z,x,t)\, d\mathfrak{m}(z), \tag{6.10}$$

where U is the solution of (6.4), (6.5). *Suppose further that*

$$\text{either } U \in L^1_{loc}(\mathbb{R}^n \times [0,T]; C(\mathcal{K})) \text{ or } U \in \bigcap_{R>0} L^2(\mathcal{K}; C(\overline{B}_R(0)\times[0,T])) \tag{6.11}$$

for some $T>0$. Then

$$\lim_{\varepsilon\to 0}\Big(u_\varepsilon(x,t) - U(\frac{x}{\varepsilon},x,t)\Big) = 0 \quad \text{in } L^1_{loc}(\mathbb{R}^n\times[0,T]). \tag{6.12}$$

Proof. 1. We first observe that the entropy solutions u_ε of (6.2), (6.3) are uniformly bounded in $L^\infty(\mathbb{R}^{n+1}_+)$. Hence, taking into account Theorem 5.3, it suffices to show that any two-scale Young measure $\nu_{z,x,t}$ generated by a subnet of $\{u_\varepsilon\}_{\varepsilon>0}$ satisfies

$$\nu_{z,x,t} = \delta_{U(z,x,t)}, \tag{6.13}$$

for a.e. $(z,x,t)\in\mathcal{K}\times\mathbb{R}^{n+1}_+$. Let then $\nu_{z,x,t}$ be a two-scale Young measure generated by a subnet of $\{u_\varepsilon\}_{\varepsilon>0}$ which, for notational simplicity, we still denote by $\{u_\varepsilon\}$. For any nonnegative $\psi\in L^1(\mathbb{R}^{n+1}_+)$ we set also

$$\sigma_z^\psi := \int_{\mathbb{R}^{n+1}_+} \psi(x,t)\nu_{z,x,t}\, dx\, dt. \tag{6.14}$$

2. We use Theorem 6.1 to prove (6.13). So, let us consider the family of Kruzkhov's entropies

$$\eta(\lambda,k) = |\lambda - k|, \qquad q(\lambda,k) = \operatorname{sgn}(\lambda-k)(f(\lambda)-f(k)), \tag{6.15}$$

so that the entropy solution of (6.2) satisfies

$$\partial_t\eta(u_\varepsilon,k) + \nabla_x\cdot(a(\frac{x}{\varepsilon})q(u_\varepsilon,k)) \le 0 \qquad \forall k\in\mathbb{R}, \tag{6.16}$$

in the sense of distributions: it means that for all $0\le\phi\in C_c^\infty(\mathbb{R}^{n+1})$ we have

$$\int_{\mathbb{R}^{n+1}_+}\{\eta(u_\varepsilon,k)\phi_t + q(u_\varepsilon,k)(a(\frac{x}{\varepsilon})\cdot\nabla_x\phi)\}\,dx\,dt + \int_{\mathbb{R}^n}\eta(U_0(\frac{x}{\varepsilon},x),k)\phi(x,0)\,dx \ge 0. \tag{6.17}$$

In (6.17) we take $\phi(x,t) = \varepsilon\varphi(\frac{x}{\varepsilon})\psi(x,t)$, where $0\le\varphi\in\mathcal{T}$, with $\mathcal{T}$ defined in (4.6), and $0\le\psi\in C_c^\infty(\mathbb{R}^{n+1})$, and let $\varepsilon\to0$ to get

$$\int_{\mathcal{K}}\langle\sigma_z^\psi, q(\cdot,k)\rangle\nabla_a\varphi\, d\mathfrak{m}(z) \ge 0. \tag{6.18}$$

By applying this inequality with $C\pm\varphi$, with $C=\|\varphi\|_\infty$, and using the arbitrariness of φ we get (recalling (4.11))

$$z\mapsto\langle\sigma_z^\psi, q(\cdot,k)\rangle\in\mathcal{S}. \tag{6.19}$$

3. Relation (6.19) is true also for the entropy fluxes $s_+(u,k)$, $s_-(u,k)$ associated with the convex entropies $r_+(u,k)=\max\{0,u-k\}$, $r_-(u,k)=\max\{0,k-u\}$. By linearity, we deduce that (6.19) holds for any entropy flux q associated with a Lipschitz entropy η satisfying $\eta'=\chi_I$, where χ_I is the characteristic function of any interval $I\subseteq\mathbb{R}$. Since any Lipschitz function may be locally uniformly approximated by finite linear combinations of Lipschitz entropies of that form, we deduce that (6.19) holds for all entropy fluxes associated with any Lipschitz entropy (not necessarily convex!). Now, if $\bar I\subseteq\mathbb{R}\setminus E$, then the entropy flux associated with the Lipschitz entropy η_I satisfying $\eta_I' = \chi_I/f'$ is q_I defined modulo constants by $q_I'=\chi_I$. Hence, $z\mapsto\langle\sigma_z^\psi,q_I\rangle\in\mathcal{S}$, for any interval I with $\bar I\subseteq\mathbb{R}\setminus E$. By linearity and the fact that E has measure zero,

we deduce that the latter holds for any interval I. Again using the fact that any Lipschitz function may be locally uniformly approximated by finite linear combinations of such functions q_I, we conclude that

$$z \mapsto \langle \sigma_z^{\psi}, g\rangle \in \mathcal{S} \qquad \text{for any Lipschitz function } g:\mathbb{R}\to\mathbb{R}. \tag{6.20}$$

By approximation we deduce that (6.20) holds for any $\psi \in L^1(\mathbb{R}_+^{n+1})$.

4. Let $\phi(x,t)=\varphi(\frac{x}{\varepsilon})\psi(x,t)$, where $0\le\varphi\in\mathcal{S}^\dagger$ and $0\le\psi\in C_c^\infty(\mathbb{R}^{n+1})$. By the definition of $\mathcal{S}^\dagger$, there exists $(\varphi_k)_{k\in\mathbb{N}}\subseteq\mathcal{T}$, where $\mathcal{T}$ is defined in (4.6), such that $\varphi_k\to\varphi$ and $\nabla_a\varphi_k\to 0$, both in $\mathcal{B}^2$ and in L^2_{loc}. We first consider (6.17) with $\phi(x,t)$ replaced by $\phi_k(x,t)=\varphi_k(\frac{x}{\varepsilon})\psi(x,t)$ and make $k\to\infty$. Then we pass to the limit as $\varepsilon\to 0$ to get

$$\begin{aligned}\int_{\mathbb{R}_+^{n+1}}\int_{\mathcal{K}}\{\langle\nu_{z,x,t},\eta(\cdot,k)\rangle\underline{\varphi}\psi_t+\langle\nu_{z,x,t},q(\cdot,k)\rangle\underline{\varphi}(\underline{a}\cdot\nabla_x\psi)\}\,d\mathfrak{m}(z)\,dx\,dt&\\ +\int_{\mathbb{R}^n}\int_{\mathcal{K}}\eta(U_0(z,x),k)\underline{\varphi}(z)\psi(x,0)\,d\mathfrak{m}(z)\,dx\ge 0.&\end{aligned}\tag{6.21}$$

5. By Proposition 4.2 and (6.19) the maps $z\mapsto\underline{\varphi}(z)\langle\sigma_z^{\partial_i\psi},q(\cdot,k)\rangle$ belong to $\mathcal{S}$. Therefore, taking (4.12) into account, we can rewrite (6.21) as

$$\begin{aligned}\int_{\mathbb{R}_+^{n+1}}\int_{\mathcal{K}}\{\langle\nu_{z,x,t},\eta(\cdot,k)\rangle\underline{\varphi}\psi_t+\langle\nu_{z,x,t},q(\cdot,k)\rangle\underline{\varphi}(\tilde a\cdot\nabla_x\psi)\}\,d\mathfrak{m}(z)\,dx\,dt&\\ +\int_{\mathbb{R}^n}\int_{\mathcal{K}}\eta(U_0(z,x),k)\underline{\varphi}(z)\psi(x,0)\,d\mathfrak{m}(z)\,dx\ge 0,&\end{aligned}\tag{6.22}$$

for all $0\le\varphi\in\mathcal{S}^\dagger$ and all $0\le\psi\in C_c^\infty(\mathbb{R}^{n+1})$. But then, using also the fact that $z\mapsto\langle\sigma_z^{\psi_t},\eta(\cdot,k)\rangle$ (see (6.20)) and $\tilde a_i\langle\sigma_z^{\partial_i\psi},q(\cdot,k)\rangle$ belong to $\mathcal{S}$ and assumption (6.9) on U_0, we obtain that (6.22) holds for all $0\le\varphi\in L^2(\mathcal{K})$ (here we use the density of $\mathcal{S}^\dagger$ in $\mathcal{S}$). In particular, for each fixed $0\le\psi\in C_c^\infty(\mathbb{R}^{n+1})$, inequality (6.22) can be strengthened to an inequality a.e. on $z\in\mathcal{K}$. A density argument on the class of test functions ψ then gives that for a.e. $z\in\mathcal{K}$ the following property is fulfilled:

$$\begin{aligned}\int_{\mathbb{R}_+^{n+1}}\{\langle\nu_{z,x,t},\eta(\cdot,k)\rangle\psi_t+\langle\nu_{z,x,t},q(\cdot,k)\rangle(\tilde a(z)\cdot\nabla_x\psi)\}\,dx\,dt&\\ +\int_{\mathbb{R}^n}\eta(U_0(z,x),k)\psi(x,0)\,dx\ge 0,&\end{aligned}\tag{6.23}$$

for all $0\le\psi\in C_c^\infty(\mathbb{R}^{n+1})$.

6. We are going to apply Theorem 6.1 to show that $\nu_{z,x,t}$ is a Dirac measure for almost every $(z,x,t)\in\mathcal{K}\times\mathbb{R}_+^{n+1}$. To do this, first we observe that (6.23) implies

$$\lim_{t\to 0}\frac{1}{t}\int_0^t\int_{\mathbb{R}^n}\langle\nu_{z,x,\tau},g\rangle\phi(x)\,dx\,d\tau=\int_{\mathbb{R}^n}\langle\delta_{U_0(z,x)},g\rangle\phi(x)\,dx,\tag{6.24}$$

for all $g\in C(\mathbb{R})$ and $\phi\in C_c(\mathbb{R}^n)$. Indeed, choosing $\psi(x,t)=\delta_h(t)\phi(x)$, with $\delta_h(t)=\max\{h^{-1}(h-t),0\}$, for $t\ge 0$, $h>0$, $\phi\in C_c^\infty(\mathbb{R}^n)$, $\phi\ge 0$, in (6.23), we obtain

$$\lim_{h\to 0}\frac{1}{h}\int_0^h\int_{\mathbb{R}^n}\langle\nu_{z,x,t},|\cdot-k|\rangle\phi(x)\,dx\,dt\le\int_{\mathbb{R}^n}|U_0(z,x)-k|\phi(x)\,dx,\tag{6.25}$$

for all $\phi\in C_c^\infty(\mathbb{R}^n)$, $\phi\ge 0$, and a fortiori also for all $0\le\phi\in L^1(\mathbb{R}^n)$. Taking advantage of the flexibility given by the presence of $\phi\in L^1(\mathbb{R}^n)$ in (6.25), we may replace k by any function $k(x)$ in $L^\infty(\mathbb{R}^n)$, in particular, $k(x)=U_0(z,x)$. This proves (6.24).

7. Now, let $U(z,x,t)$ be the solution of (6.4), (6.5). The entropy condition states that

$$\partial_t\eta(\lambda,U)+\nabla_x\cdot(\tilde a(z)q(\lambda,U))\le 0\qquad\text{for all }\lambda\in\mathbb{R},\ z\in\mathcal{K}.\tag{6.26}$$

and

$$\lim_{t\to 0}\int_{\{|x|<R\}} |U(z,x,t)-U_0(z,x)|\,dx = 0, \qquad \text{for all } R>0. \tag{6.27}$$

Therefore, we can apply Theorem 6.1 with $\mu^1_{x,t}=\nu_{z,x,t}$, $\mu^2_{x,t}=\delta_{U(z,x,t)}$, $I=\eta$ and $G=\tilde a(z)q$, for a.e. $z\in\mathcal{K}$. From this we easily deduce that $\nu_{z,x,t}=\delta_{U(z,x,t)}$, for a.e. $(z,x,t)\in\mathcal{K}\times\mathbb{R}^{n+1}_+$.

8. To prove the weak convergence $u_\varepsilon \rightharpoonup u$, with $u(x,t)$ given by (6.10), we argue as follows. Let $U^\delta\in C(\mathcal{K}\times\mathbb{R}^{n+1}_+)$ be bounded. Using (5.2) with test function

$$\Phi(\lambda,z,x,t) := |\lambda-U^\delta(z,x,t)|\psi(x,t)$$

with $0\le\psi\in C_c(\mathbb{R}^{n+1}_+)$, we obtain

$$\limsup_{\varepsilon\to 0}\int_{\mathbb{R}^{n+1}_+}\psi(x,t)|u_\varepsilon(x)-U^\delta(\frac{x}{\varepsilon},x,t)|\,dx\,dt = \int_{\mathbb{R}^{n+1}_+}\int_{\mathcal{K}}|U\psi-U^\delta\psi|\,d\mathfrak{m}(z)\,dx\,dt.$$

On the other hand, the continuity of U_δ gives

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^{n+1}_+}U^\delta(\frac{x}{\varepsilon},x,t)\psi(x,t)\,dx\,dt = \int_{\mathbb{R}^{n+1}_+}\int_{\mathcal{K}}U^\delta(z,x,t)\,d\mathfrak{m}(z)\psi(x,t)\,dx\,dt.$$

Hence, combining the previous two formulas, we get

$$\limsup_{\varepsilon\to 0}\left|\int_{\mathbb{R}^{n+1}_+}u_\varepsilon(x)\psi(x,t)-\bar U^\delta(x,t)\psi(x,t)\,dx\,dt\right| \le \|U^\delta\psi-U\psi\|_{L^1},$$

with $\bar U^\delta(x,t) := \int_{\mathcal{K}}U^\delta(z,x,t)\,d\mathfrak{m}(z)$. By a density argument we obtain the weak star convergence of u_ε to $\lim_\delta \bar U^\delta$, i.e. $\int_{\mathcal{K}}U(z,x,t)\,d\mathfrak{m}(z)$. Finally the fact that $u_\varepsilon(x,t)-U(\frac{x}{\varepsilon},x,t)\to 0$ in $L^1_{\rm loc}(\mathbb{R}^n\times[0,T])$ as $\varepsilon\to 0$, under assumption (6.11), follows directly by Lemma 5.3. □

Concerning (6.11), ensuring the existence of strong correctors, we observe that the first alternative is trivially satisfied if U_0 and $\tilde a$ are independent of z, in which case we may take any $T>0$. A simple example is provided, for $N=2$, by the incompressible vector field $a(z)=(g(z_2),\beta)$ with $g\in \mathrm{FS}(\mathbb{R})\cap W^{1,\infty}(\mathbb{R})$ and $\beta\neq 0$. In this case $\tilde a(z)=(\fint g,\beta)$, which follows easily from (4.14). The following lemma gives sufficient conditions for the verification of the second alternative in (6.11).

Lemma 6.1. *If the range of a is contained in a closed convex set $\mathcal{P}$, then $U\in L^2(\mathcal{K};C(\overline{B}(0,R)\times[0,T]))$ for any $R>0$, for any $T>0$ such that the entropy solutions V_b of*

$$\partial_t V_b+\nabla_x\cdot(bf(V_b))=0, \qquad t>0,\ x\in\mathbb{R}^n, \tag{6.28}$$

$$V_b(x,0)=U_0(z,x), \qquad x\in\mathbb{R}^n, \tag{6.29}$$

have locally uniformly bounded Lipschitz constant in $\mathbb{R}^n\times[0,T]$, with respect to $b\in\mathcal{P}$ and $z\in\mathbb{R}^n$.

Proof. By applying (4.14) we obtain that also the range of $\tilde a$ is contained in $\mathcal{P}$. We will prove that $U(z,x,t)\in L^2(\mathcal{K};C(\overline{B}_R(0)\times[0,T]))$ for any $R>0$. Since U is bounded we need only to check its measurability. This follows by the fact that for any $\delta>0$ it is possible to find a compact $K_\delta\subseteq\mathcal{K}$ such that $U(z,x,t)\in C(K_\delta;C(\mathbb{R}^n\times[0,T]))$. Indeed, given $\delta>0$ we may find K_δ such that the restriction of $\tilde a$ to $\mathcal{K}_\delta$ is continuous. Now, the stability properties of entropy solutions tell us that $z\mapsto U(z,\cdot,\cdot)$ is continuous from $K_\delta\subseteq\mathcal{K}$ into $L^1_{loc}(\mathbb{R}^{n+1}_+)$. The local uniform Lipschitz bound then gives continuity with respect to the stronger topology. □

An example where Lemma 6.1 applies is provided by the case in which all the components of a are nonnegative, $f''(u) \geq 0$ for all $u \in \mathbb{R}$ and $\frac{\partial U_0}{\partial x_i}(z,x) \geq 0$ for all $(z,x) \in \mathbb{R}^n \times \mathbb{R}^n$, $i = 1, \dots, n$. In this case, if $b \in \mathcal{P} = [0,M]^n$ for some $M > 0$, then it is well known that the entropy solution U_b of (6.28), (6.29) can be constructed by the method of characteristics in such a way that $U_b \in W^{1,\infty}_{loc}(\mathbb{R}^{n+1}_+))$ if the initial datum is a Lipschitz function. We recall that, in general, entropy solutions are discontinuous.

7. Porous medium type equations with oscillatory external sources on bounded domains

In this section we consider a homogenization problem for a porous medium type equation with oscillatory external force, similar to the one considered in [3]. The differences with respect to the earlier problem lie in the fact that we now consider the equation on a bounded domain and we allow more general initial data. The price we pay for these extensions is that we need to impose some restrictions on the ergodic algebra, namely requiring the external source as a function of the oscillatory variable to belong to $\mathrm{FS}(\mathbb{R}^n)$. Also, for our result on the existence of oscillatory profiles enhancing the weak convergence to a strong convergence we ask the "pressure" function to be convex, which was not necessary for the corresponding result in [3]. To simplify our boundary conditions we consider external sources depending explicitly also on the non-oscillatory variable.

So, let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary. We consider the initial-boundary value problems

$$\partial_t u = \Delta f(u) - \Delta G(x, \frac{x}{\varepsilon}) \qquad (x,t) \in \Omega \times (0,\infty), \tag{7.1}$$

$$u|\partial\Omega = p_0, \tag{7.2}$$

$$u(x,0) = u_0(x, \frac{x}{\varepsilon}) := \varphi_0(x, \frac{x}{\varepsilon}) + p_0. \tag{7.3}$$

The main goal here will be to apply the theory developed in earlier sections and not to solve the problem (7.1)-(7.3) in its greatest generality. Therefore in what follows we will be quite generous in our regularity assumptions, without paying attention to possible extensions under weaker regularity hypotheses. To avoid confusion we explicitly state that $\Delta G(x, \frac{x}{\varepsilon}) := \sum_{i=1}^n \left(G_{x_i x_i}(x,y) + \frac{2}{\varepsilon} G_{x_i y_i}(x,y) + \frac{1}{\varepsilon^2} G_{y_i y_i}(x,y) \right)$ with $y = \frac{x}{\varepsilon}$.

Notation: We shall denote by $C^\gamma(\bar{\Omega})$ the Hölder continuous functions in $\bar{\Omega}$ with Hölder exponent $\gamma \in (0,1)$. By $C_0^\gamma(\Omega)$ we shall denote the functions in $C^\gamma(\bar{\Omega})$ which vanish at $\partial\Omega$. Also, we shall denote by $C^{2+\gamma}(\bar{\Omega})$ the functions in $C^2(\bar{\Omega})$ whose second derivatives are in $C^\gamma(\bar{\Omega})$ and by $C_0^{2+\gamma}(\Omega)$ the functions in $C^{2+\gamma}(\bar{\Omega}) \cap C_0^\gamma(\Omega)$ whose derivatives up to second order are in $C_0^\gamma(\Omega)$.

In the case of the porous medium equation, typical examples of the function $f(u)$ are given by $f(u) = u^\gamma$, with $\gamma > 0$. With this application in mind, we will not assume in principle that f is defined in the whole line, which causes the need of more technical specifications in the assumptions to follow.

We make the following assumptions on f, G, p_0 and φ_0:

- (A3) The function f is defined and smooth on an interval $(a,b) \subseteq \mathbb{R}$ on which it satisfies $f' > 0$.
- (A4) The function $G(x,y)$ satisfies $G \in C^{2+\gamma}(\bar{\Omega} \times \mathbb{R}^n) \cap \mathrm{FS}(\mathbb{R}^n; C_0^{2+\gamma}(\Omega))$, for some $0 < \gamma < 1$;
- (A5) $\varphi_0 \in C^{2+\gamma}(\bar{\Omega} \times \mathbb{R}^n) \cap \mathrm{FS}(\mathbb{R}^n; C_0^{2+\gamma}(\Omega))$;
- (A6) $p_0 \in (a,b)$ and there exist $q_1, q_2 \in f((a,b))$ such that $q_1 < f(p_0) < q_2$ and

$$\alpha < q_1 + G(x,y) < f(u_0(x,y)) < q_2 + G(x,y) < \beta, \qquad \text{for all } (x,y) \in \Omega \times \mathbb{R}^n,$$

with $[\alpha, \beta] \subseteq f((a,b))$. In particular, we have

$$a < \Phi_{q_1}(x, \frac{x}{\varepsilon}) < u_0(x, \frac{x}{\varepsilon}) < \Phi_{q_2}(x, \frac{x}{\varepsilon}) < b, \tag{7.4}$$

where, for $q \in f((a,b))$,

$$\Phi_q(x,y) := f^{-1}\left(G(x,y)+q\right). \tag{7.5}$$

For $q \in [q_1, q_2]$ define

$$\bar{g}(x,q) := M_y\left(f^{-1}\big(G(x,y)+q\big)\right).$$

For each $x \in \Omega$, $p \in [\bar{g}(x,q_1), \bar{g}(x,q_2)]$, we define $\bar{f}(x,p)$ implicitly by the formula

$$p = M_y\left(f^{-1}\big(G(x,y)+\bar{f}(x,p)\big)\right). \tag{7.6}$$

Clearly, we have $\bar{g}(x,\bar{f}(x,p)) = p$ and $\bar{f}(x,\bar{g}(x,q)) = q$.

We notice that for all $q \in \mathbb{R}$ such that $G(x,y)+q \in f((a,b))$, for $(x,y) \in \Omega \times \mathbb{R}^n$, the function $\Phi_q(x,y)$ defined in (7.5) is such that $\Phi_q(x, \frac{x}{\varepsilon})$ is a stationary solution of (7.1).

Under the assumptions (A3)-(A6) we have the following result.

Theorem 7.1. *Let u_ε be the (classical) solution of problem* (7.1)-(7.3)*. Then u_ε is uniformly bounded in $L^\infty(\Omega \times (0,\infty))$ and $u_\varepsilon \overset{*}{\rightharpoonup} \bar{u}$ in $L^\infty(\Omega \times (0,\infty))$ where $\bar{u} \in C^{2,1}(\bar{\Omega} \times [0,\infty))$ is the (classical) solution of the initial-value problem*

$$\partial_t \bar{u} = \Delta \bar{f}(x,\bar{u}), \tag{7.7}$$

$$\bar{u}|\partial\Omega = p_0, \tag{7.8}$$

$$\bar{u}(x,0) = p_0 + M_y\left(\varphi_0(x,y)\right), \tag{7.9}$$

where for each $x \in \Omega$, $\bar{f}(x,\cdot)$ is implicitly defined on $[\bar{g}(x,q_1), \bar{g}(x,q_2)]$ by the relation (7.6)*. Moreover, if $f''(u) \neq 0$ for all $a < u < b$ then*

$$u_\varepsilon(x,t) - \Phi_{\bar{f}(x,\bar{u}(x,t))}(x, \frac{x}{\varepsilon}) \to 0 \qquad \text{in } L^1_{loc}(\Omega \times (0,\infty)). \tag{7.10}$$

Proof. 1. Existence, uniqueness and smoothness of solutions of (7.1)-(7.3) follow from the classical theory developed in [25]. The fact that the solutions of (7.1)-(7.3) form a uniformly bounded sequence in $L^\infty(\Omega \times (0,\infty))$ follows from the inequalities (see, e.g., [7])

$$\int_0^\infty \int_\Omega \{(u-v)_\pm \phi_t \mp H(\pm(u-v))\nabla(f(u)-f(v)) \cdot \nabla\phi\}\, dxdt \geq 0 \tag{7.11}$$

for all $0 \leq \phi \in C_0^\infty(\mathbb{R}^n \times (0,\infty))$, which hold for smooth solutions of (7.1)-(7.3), satisfying $(u-v)_\pm = 0$ on $\partial\Omega \times (0,\infty)$, where by $(s)_\pm$ we denote the function $\max\{\pm s, 0\}$ and by $H(s)$ we denote the Heaviside function $H(s) = 1$ for $s > 0$ and $H(s) = 0$ for $s < 0$. Indeed, we may apply (7.11) for $(u-v)_-$ and for $(u-v)_+$ with $v = v_1(x, \frac{x}{\varepsilon}) := \Phi_{q_1}(x, \frac{x}{\varepsilon})$ and $v = v_2(x, \frac{x}{\varepsilon}) := \Phi_{q_2}(x, \frac{x}{\varepsilon})$, respectively, for q_1 and q_2 as in (7.4). Therefore, using a suitable sequence of test functions $\phi(x,t)$ we arrive at the inequalities

$$a < v_1(x, \frac{x}{\varepsilon}) \leq u_\varepsilon(x,t) \leq v_2(x, \frac{x}{\varepsilon}) < b. \tag{7.12}$$

2. We now define $U_\varepsilon(x,t)$ in $\Omega \times [0,\infty)$ as the smooth bounded solution of

$$\Delta U = u_\varepsilon(x,t), \tag{7.13}$$

$$U|\partial\Omega = 0, \tag{7.14}$$

We then notice that U_ε is the (viscosity) solution of

$$\partial_t U - f(\Delta U) = -G(x, \frac{x}{\varepsilon}) - f(p_0), \tag{7.15}$$

$$U|\partial\Omega = 0, \tag{7.16}$$

$$U(x,0) = U_{0,\varepsilon}(x), \tag{7.17}$$

where $U_{0,\varepsilon}$ is the smooth bounded solution of (7.13)-(7.14) for $t=0$. We refer to [11] for a self-contained exposition of the theory of viscosity solutions for fully nonlinear elliptic equations, and to [8] for the corresponding regularity theory. Next we shall study the homogenization of (7.15)-(7.17) using a method motivated by [20].

3. Since $u_\varepsilon(x,t)$ is uniformly bounded in $L^\infty(\Omega\times[0,\infty))$, we easily see that $U_\varepsilon(x,t)$ form a uniformly bounded sequence in $L^\infty([0,\infty);W^{2,p}(\Omega))$ for all $p\in(1,\infty)$. On the other hand, from (7.15) we easily deduce that $|U_\varepsilon(x,t)-U_\varepsilon(x,s)|\le C|t-s|$ for all $x\in\Omega$ for some constant $C>0$, independent of ε. Hence, we see that U_ε is uniformly bounded in $W^{1,\infty}(\bar\Omega\times[0,\infty))$. In particular, there is a subsequence U_{ε_i} of U_ε converging locally uniformly in $\bar\Omega\times[0,\infty)$ to a function $\bar U\in W^{1,\infty}(\bar\Omega\times[0,\infty))$.

4. We claim that $\bar U(x,t)$ is the viscosity solution of the initial-boundary value problem

$$U_t-\bar f(x,\Delta U)=-f(p_0), \tag{7.18}$$

$$U|\partial\Omega=0, \tag{7.19}$$

$$U(x,0)=\bar U_0(x), \tag{7.20}$$

where $\bar U_0$ is the solution of

$$\Delta U=p_0+M_y\left(\varphi_0(x,y)\right), \tag{7.21}$$

$$U|\partial\Omega=0, \tag{7.22}$$

5. Indeed, let $(\hat x,\hat t)\in\Omega\times(0,\infty)$ and let $v_\delta\in\mathrm{FS}(\mathbb{R}^n)$ be smooth bounded and such that

$$\Delta_y v_\delta\le f^{-1}\left(G(\hat x,y)+\bar f(\hat x,p)\right)-p+\delta, \tag{7.23}$$

$$\Delta_y v_\delta\ge f^{-1}\left(G(\hat x,y)+\bar f(\hat x,p)\right)-p-\delta, \tag{7.24}$$

with $p=\Delta\bar U(\hat x,\hat t)$, whose existence is given by Lemma 3.3. In particular, given any $\delta'>0$ we can find $\delta>0$ sufficiently small such that

$$f(\Delta\bar U(\hat x,\hat t)+\Delta v_\delta(y))\le G(\hat x,y)+\bar f(\hat x,\Delta\bar U(\hat x,\hat t))+\delta',$$
$$f(\Delta\bar U(\hat x,\hat t)+\Delta v_\delta(y))\ge G(\hat x,y)+\bar f(\hat x,\Delta\bar U(\hat x,\hat t))-\delta'.$$

Take $\rho>0$, and let $x_j\in\Omega$ be a point of maximum of

$$U_j(x,\hat t)-\bar U(x,\hat t)-\varepsilon_j^2 v_\delta(\frac{x}{\varepsilon_j})-\rho|x-\hat x|^2+\rho,$$

which exists since v_δ is bounded. Here we denote $U_j=U_{\varepsilon_j}$. We clearly have $x_j\to\hat x$ as $j\to\infty$. We have

$$U_{jt}(x_j,\hat t)-f\left(\Delta\bar U(x_j,\hat t)+\Delta v_\delta(\frac{x_j}{\varepsilon_j})+\rho\right)\le -G(x_j,\frac{x_j}{\varepsilon_j})-f(p_0),$$

and

$$f\left(\Delta\bar U(\hat x,\hat t)+\Delta v_\delta(\frac{x_j}{\varepsilon_j})\right)\le G(\hat x,\frac{x_j}{\varepsilon_j})+\bar f(\hat x,\Delta\bar U(\hat x,\hat t))+\delta',$$

which, after addition, gives

$$U_{jt}(x_j,\hat t)-\bar f(\hat x,\Delta\bar U(\hat x,\hat t))\le -f(p_0)+O(|x_j-\hat x|)+O(\rho)+\delta'.$$

Hence, letting $j\to\infty$ first, and then letting $\rho,\delta'\to0$, we obtain

$$\bar U_t(\hat x,\hat t)-\bar f(\hat x,\Delta\bar U(\hat x,\hat t))\le -f(p_0).$$

The opposite inequality follows in a similar way and hence we have proved the claim.

6. By the uniqueness of the viscosity solution of (7.18)–(7.20), we conclude that the whole sequence $U_\varepsilon(x,t)$ converges uniformly to $\bar U(x,t)$. Consequently, $u_\varepsilon(x,t)$ converges in the weak-$*$ topology of $L^\infty(\Omega\times(0,\infty))$ to $\bar u=\Delta\bar U(x,t)$, which is the classical solution of (7.7)–(7.9), and this concludes the proof of the first part of the theorem.

7. We are now going to prove (7.10) under the additional assumption that $f''(u) \neq 0$ for all $u \in (a, b)$. We write the identity

$$\partial_t U_\varepsilon - f(\Delta U_\varepsilon) = -G(x, \frac{x}{\varepsilon}) - f(p_0).$$

Multiplying by $\phi(x,t)\varphi(\frac{x}{\varepsilon})$ with $\phi \in C_0^\infty(\Omega \times (0,\infty))$ and $\varphi \in \mathrm{FS}(\mathbb{R}^n)$, then integrating in $\Omega \times (0,\infty)$ and next taking the limit along a suitable subnet $\varepsilon(d)$, $d \in D$, we obtain by Theorem 5.2

$$\int_0^\infty \int_\Omega \int_{\mathcal{K}} \{\langle \nu_{x,t,z}, f(\cdot)\rangle - G(x,z) - \bar{f}(x, \Delta\bar{U})\}\phi(x,t)\varphi(z)\, d\mathfrak{m}(z)\, dx\, dt = 0,$$

where $\mathcal{K}$ is the compactification of $\mathbb{R}^n$ associated with $\mathrm{FS}(\mathbb{R}^n)$. Since ϕ and φ are arbitrary, we have

$$\langle \nu_{x,t,z}, f(\cdot)\rangle = G(x,z) + \bar{f}(x, \Delta\bar{U}) = f\left(f^{-1}\big(G(x,z) + \bar{f}(x,\Delta\bar{U})\big)\right), \quad \text{for a.e. } (x,t,z) \in \Omega \times (0,\infty) \times \mathcal{K}.$$

Since $f'' \neq 0$ we conclude that

$$\nu_{x,t,z} = \delta_{f^{-1}\left(G(x,z)+\bar{f}(x,\Delta\bar{U})\right)}, \quad \text{for a.e. } (x,t,z) \in \Omega \times (0,\infty) \times \mathcal{K}.$$

and this implies that (7.10) holds. □

8. A system of porous medium type equations with oscillatory external sources

Again, we let Ω be an open bounded subset of $\mathbb{R}^n$ with smooth boundary. We consider the following system of porous media equations

$$\begin{cases} u_t = \Delta f_1(u) + h(v) - \Delta G_1(x, \frac{x}{\varepsilon}), \\ v_t = \Delta f_2(v) - \Delta G_2(x, \frac{x}{\varepsilon}). \end{cases} \tag{8.1}$$

As we observe, the equation for u has an additional source term $h(v)$ which couples it with the equation for v. We then prescribe the following initial and boundary data

$$u|\partial\Omega = p_{01}, \qquad v|\partial\Omega = p_{02}, \tag{8.2}$$

$$u(x,0) = u_0(x, \frac{x}{\varepsilon}) := \varphi_{01}(x, \frac{x}{\varepsilon}) + p_{01}, \qquad v(x,0) = v_0(x, \frac{x}{\varepsilon}) := \varphi_{02}(x, \frac{x}{\varepsilon}) + p_{02}. \tag{8.3}$$

We suppose that assumptions (A3)–(A6) are satisfied with f, G and φ_0 replaced by f_1, f_2, G_1, G_2 and $\varphi_{01}, \varphi_{02}$, respectively. For simplicity we assume now that $(a,b) = (0, +\infty)$, as is the case in the model of porous media where u, v represent densities. As for h, we assume the following.

(A7) The function h is defined and smooth in the interval $(0, +\infty)$ and $h(v) \geq 0$ for all $v \in (0, \infty)$.

We further assume:

(A8) The function f_2 also satisfies $f_2''(v) \neq 0$ for all $v \in (0, \infty)$.

Concerning problem (8.1)–(8.3), under the assumptions (A3)–(A8), we have the following result.

Theorem 8.1. *The (classical) solutions $(u_\varepsilon, v_\varepsilon)$ of the problem (8.1)-(8.3) form a uniformly bounded sequence in $L^\infty(\Omega \times (0,T))^2$ and $(u_\varepsilon, v_\varepsilon) \overset{*}{\rightharpoonup} (\bar{u}, \bar{v})$ in $L^\infty(\Omega \times (0,T))$ for all $T > 0$ where $(\bar{u}, \bar{v})$ is the (classical) solution of the initial-boundary value problem*

$$\begin{cases} \partial_t u = \Delta \bar{f}_1(x,u) + \bar{h}(x,v), \\ \partial_t v = \Delta \bar{f}_2(x,v), \end{cases} \tag{8.4}$$

$$(u,v)|\partial\Omega = (p_{01}, p_{02}), \tag{8.5}$$

$$(u(x,0), v(x,0)) = \left(p_{01} + M_y\big(\varphi_{01}(x,y)\big), p_{02} + M_y\big(\varphi_{02}(x,y)\big)\right), \tag{8.6}$$

where for each $x \in \Omega$, $\bar f_1(x,\cdot), \bar f_2(x,\cdot)$ are implicitly defined on $[\bar g_1(x,q_{11}), \bar g_1(x,q_{12})]$ and $[\bar g_2(x,q_{21}), \bar g_2(x,q_{22})]$, respectively, by the relation (7.6) with f replaced by f_1, f_2, where $\bar g_1$ and $\bar g_2$ are the corresponding inverses. The function $\bar h(x,v)$ is defined by

$$\bar h(x,v) = M_y\left(h\big(f_2^{-1}(G_2(x,y) + \bar f_2(x,v))\big)\right).$$

Moreover,

$$v_\varepsilon(x,t) - \Phi^2_{\bar f_2(x,\bar v(x,t))}(x,\frac{x}{\varepsilon}) \to 0 \qquad \text{in } L^1_{loc}(\Omega\times(0,\infty)), \tag{8.7}$$

where

$$\Phi^2_q(x,y) = f_2^{-1}(G_2(x,y) + q).$$

Further, if $f_1''(u) \neq 0$ we also have

$$u_\varepsilon(x,t) - \Phi^1_{\bar f_1(x,\bar u(x,t))}(x,\frac{x}{\varepsilon}) \to 0 \qquad \text{in } L^1_{loc}(\Omega\times(0,\infty)), \tag{8.8}$$

where

$$\Phi^1_q(x,y) = f_1^{-1}(G_1(x,y) + q).$$

Proof. 1. By Theorem 7.1, we immediately have the assertions concerning v_ε.

2. The fact that u_ε are uniformly bounded in $L^\infty(\Omega\times(0,T))$ for all T is proved as follows. Since $h \geq 0$, we have that u_ε satisfies

$$\int_0^\infty \int_\Omega \{(u - \Phi^1_{q_{01}})_-\phi_t + H(\Phi^1_{q_{01}} - u)\nabla(f_1(u) - f_1(\Phi^1_{q_{01}}))\cdot\nabla\phi\}\,dx\,dt \geq 0,$$

for all $0 \leq \phi \in C_0^\infty(\Omega\times(0,\infty))$, where $q_{01} = f(p_{01})$. Then, using a suitable sequence of test functions ϕ we arrive at the inequality

$$\int_\Omega (u(x,t) - \Phi^1_{q_{01}}(x,\frac{x}{\varepsilon}))_-\,dx \leq \int_\Omega (u_0(x,\frac{x}{\varepsilon}) - \Phi^1_{q_{01}}(x,\frac{x}{\varepsilon}))_-\,dx,$$

and so we get $u_\varepsilon(x,t) \geq \Phi^1_{q_{01}}(x,\frac{x}{\varepsilon})$ for all $(x,t) \in \Omega\times(0,\infty)$.

3. To get a uniform bound from above for u_ε on $\Omega\times(0,T)$ we proceed as follows. Let $M > 0$ be such that $h(v_\varepsilon(x,t)) < M$ for $(x,t) \in \Omega\times(0,\infty)$. We easily verify the validity of the inequality

$$\int_0^\infty \int_\Omega \{(u - \Phi^1_{q_{02}} - Mt)_+\phi_t - H(u - \Phi^1_{q_{02}} - Mt)\nabla(f_1(u) - f_1(\Phi^1_{q_{02}}))\cdot\nabla\phi\}\,dx\,dt \geq 0,$$

for all $0 \leq \phi \in C_0^\infty(\Omega\times(0,\infty))$, where $q_{02} = f(p_{02})$, from which it follows, by taking a suitable sequence of test functions ϕ,

$$\int_\Omega (u(x,t) - \Phi^1_{q_{02}}(x,\frac{x}{\varepsilon}) - Mt)_+\,dx \leq \int_\Omega (u_0(x,\frac{x}{\varepsilon}) - \Phi^1_{q_{01}}(x,\frac{x}{\varepsilon}))_+\,dx.$$

Therefore, we get $u_\varepsilon(x,t) \leq \Phi^1_{q_{02}}(x,\frac{x}{\varepsilon}) + Mt$ for all $(x,t) \in \Omega\times(0,\infty)$.

4. Now, let us denote by $\Delta^{-1}g$ the solution of the boundary-value problem

$$\Delta w = g, \qquad x \in \Omega, \tag{8.9}$$

$$w|\partial\Omega = 0. \tag{8.10}$$

As $h(v_\varepsilon(x,t)) - h\big(\Phi^2_{\bar f_2(x,\bar v(x,t))}(x,\frac{x}{\varepsilon})\big) \to 0$ in $L^1_{\rm loc}(\Omega\times(0,\infty))$, we obtain that

$$\Delta^{-1}\left(h(v_\varepsilon(x,t)) - h\big(\Phi^2_{\bar f_2(x,\bar v(x,t))}(x,\frac{x}{\varepsilon})\big)\right) \to 0, \quad \text{uniformly in } \Omega \text{ for a.e. } t\in(0,T),$$

for all $T > 0$. Also, since $h\big(\Phi^2_{\bar f_2(x,\bar v(x,t))}(x,\frac{x}{\varepsilon})\big)$ converges weakly to $M_y\left(h\big(\Phi^2_{\bar f_2(x,\bar v(x,t))}(x,y)\big)\right)$ we obtain that

$$\Delta^{-1}h\big(\Phi^2_{\bar f_2(x,\bar v(x,t))}(x,\frac{x}{\varepsilon})\big) \to \Delta^{-1}M_y\left(h\big(\Phi^2_{\bar f_2(x,\bar v(x,t))}(x,y)\big)\right), \quad \text{uniformly in } \Omega \text{ for a.e. } t\in(0,T),$$

for all $T > 0$. Hence we conclude that

$$\Delta^{-1}h(v_\varepsilon(x,t)) \to \Delta^{-1}M_y\left(h\big(\Phi^2_{\bar f_2(x,\bar v(x,t))}(x,y)\big)\right), \quad \text{uniformly in } \Omega \text{ for a.e. } t\in(0,T),$$

for all $T > 0$. Let us denote

$$\psi(x,t) := \Delta^{-1}M_y\left(h\big(\Phi^2_{\bar f_2(x,\bar v(x,t))}(x,y)\big)\right) = \Delta^{-1}\bar h(x,\bar v(x,t)).$$

5. Again, let $U^\varepsilon := \Delta^{-1}u_\varepsilon$. Hence, U^ε is a viscosity solution to

$$U_t - f_1(\Delta U) = -G_1(x,\frac{x}{\varepsilon}) + \psi(x,t) + f(p_{01}) + O_t(\varepsilon), \tag{8.11}$$

$$U|\partial\Omega = 0, \tag{8.12}$$

$$U(x,0) = \Delta^{-1}u_0(x,\frac{x}{\varepsilon}), \tag{8.13}$$

where

$$O_t(\varepsilon) := \Delta^{-1}h(v_\varepsilon(x,t)) - \psi(x,t) \to 0, \quad \text{uniformly in } \Omega \text{ for a.e. } t\in(0,\infty).$$

6. We claim that $U^\varepsilon(x,t)$ converges uniformly in Ω for a.e. $t\in(0,\infty)$ to $\bar U(x,t)$, where $\bar U(x,t)$ is the viscosity solution of

$$U_t - \bar f_1(x,\Delta U) = \psi(x,t) + f_1(p_{01}), \tag{8.14}$$

$$U|\partial\Omega = 0, \tag{8.15}$$

$$U(x,0) = \Delta^{-1}M_y(u_0(x,y)). \tag{8.16}$$

7. Indeed, let $(\hat x,\hat t)\in\Omega\times(0,\infty)$, where $\hat t$ is such that $O_{\hat t}(\varepsilon)\to 0$ uniformly in Ω, and v_δ satisfy (7.23) and (7.24), with $f,\bar f,G$ replaced by $f_1,\bar f_1,G_1$. Proceeding exactly as in the proof of Theorem 7.1 we prove the claim, observing that the presence now of the term $O_{\hat t}(\varepsilon)$ does not affect the validity of the arguments.

8. Since $u_\varepsilon = \Delta U^\varepsilon$ is a uniformly bounded sequence in $L^\infty(\Omega\times(0,T))$, we then conclude that the whole sequence u_ε converges in the weak-$*$ topology of $L^\infty(\Omega\times(0,T))$ to the (classical) solution $\bar u(x,t)$ of

$$u_t - \Delta\bar f_1(x,u) = \bar h(x,\bar v(x,t)), \tag{8.17}$$

$$u|\partial\Omega = p_0, \tag{8.18}$$

$$u(x,0) = M_y(u_0(x,y)). \tag{8.19}$$

9. The final assertion of the theorem is proved exactly as the analogous one in Theorem 7.1.

□

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