MULTIPLICATIVE FORMS AT THE INFINITESIMAL LEVEL

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ABSTRACT. We describe arbitrary multiplicative differential forms on Lie groupoids infinitesimally, i.e., in terms of Lie algebroid data. This description is based on the study of linear differential forms on Lie algebroids and encompasses many known integration results related to Poisson geometry. We also revisit multiplicative multivector fields and their infinitesimal counterparts, drawing a parallel between the two theories.

CONTENTS

1. INTRODUCTION

This paper is devoted to the study of multiplicative differential forms on Lie groupoids, with focus on their infinitesimal counterparts. Given a Lie groupoid $\mathcal{G}$ over a manifold M, recall that a k-form $\omega \in \Omega^k(\mathcal{G})$ is called *multiplicative* if $m^*\omega = \mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega$, where $m : \mathcal{G}^{(2)} = \mathcal{G} \times_M \mathcal{G} \to \mathcal{G}$ is the groupoid multiplication, and $\mathrm{pr}_i : \mathcal{G}^{(2)} \to \mathcal{G}$, $i = 1, 2$, are the natural projections. Our goal is to characterize multiplicative forms on $\mathcal{G}$ solely in terms of information from its Lie algebroid.

Much of the motivation for this work comes from symplectic geometry and its generalizations, including Poisson and Dirac structures [10, 26], as well as generalized complex structures [16, 17]; it is known (e.g. from [5, 9, 11]) that all these geometrical structures encode infinitesimal data relative to multiplicative 2-forms and, as a result, Lie groupoids equipped with multiplicative 2-forms provide global models for these geometries. There are further connections between multiplicative 2-forms and the theory of moment maps, found e.g. in [4, 5, 23, 27].

Multiplicative differential forms also arise as constituents of the Bott-Schulman double complex of Lie groupoids [2] (see also [1] and references therein), which computes the cohomology of their classifying spaces. So the problem of understanding multiplicative forms infinitesimally may be seen as part of the problem of finding infinitesimal models for the cohomology of classifying spaces. This broader perspective is explored in the recent work [1], leading to results closely related to ours; a comparison between our viewpoint and the one in [1] is also discussed in this paper.

Our approach to describe multiplicative forms infinitesimally starts with the study of *linear* differential forms on vector bundles $A \to M$. We observe (Theorem 2.5) that any linear k-form on A is equivalent to a pair (μ, ν) of vector-bundle maps $\mu : A \to \wedge^{k-1}T^*M$, $\nu : A \to \wedge^k T^*M$, covering the identity on M. If A carries a Lie algebroid structure, with bracket $[\cdot,\cdot]$ and anchor ρ, we say that the pair (μ, ν) is an *IM k-form* (*IM* standing for *infinitesimally multiplicative*) if the following compatibility conditions are satisfied: for all $u, v \in \Gamma(A)$,

(1) $i_{\rho(u)}\mu(v) = -i_{\rho(v)}\mu(u)$,
(2) $\mu([u,v]) = \mathcal{L}_{\rho(u)}\mu(v) - i_{\rho(v)}\mathrm{d}\mu(u) - i_{\rho(v)}\nu(u)$,
(3) $\nu([u,v]) = \mathcal{L}_{\rho(u)}\nu(v) - i_{\rho(v)}\mathrm{d}\nu(u)$.

We prove in Theorem 4.6 that multiplicative k-forms on a source-simply-connected Lie groupoid $\mathcal{G}$ over M are in one-to-one correspondence with IM k-forms on its Lie algebroid $A \to M$. Concretely, the IM k-form (μ, ν) associated with a multiplicative k-form $\omega \in \Omega^k(\mathcal{G})$ is defined by

$$\langle \mu(u), X_1 \wedge \ldots \wedge X_{k-1} \rangle = \omega(u, X_1, \ldots, X_{k-1}),$$
$$\langle \nu(u), X_1 \wedge \ldots \wedge X_k \rangle = \mathrm{d}\omega(u, X_1, \ldots, X_k),$$

where $X_i \in TM$, $i = 1, \ldots, k$, and we view $M \subseteq \mathcal{G}$ and $A \subseteq T\mathcal{G}|_M$.

A special class of IM-forms is obtained as follows. Any closed form $\phi \in \Omega^{k+1}(M)$ determines a map $\nu : A \to \wedge^k T^*M$, $\nu(u) = -i_{\rho(u)}\phi$, satisfying condition (3) above. The IM k-forms (μ, ν) with ν of this type are referred to as *IM k-forms relative to* ϕ; they are the infinitesimal versions of multiplicative k-forms satisfying

$$\mathrm{d}\omega = \mathsf{s}^*\phi - \mathsf{t}^*\phi,$$

where s and t denote the groupoid source and target maps. For $k = 2$, IM forms relative to ϕ include ϕ-twisted Poisson and Dirac structures [26], and our Theorem 4.6 recovers their known integrations [5, 8]. For arbitrary k, IM forms relative to ϕ were studied in [1] in connection with the Weil algebra of a Lie algebroid. These and other examples are discussed in this paper.

The method we use to integrate IM forms on Lie algebroids to multiplicative forms on Lie groupoids relies entirely on the known correspondence between Lie-algebroid and Lie-groupoid morphisms (Lie's second theorem for Lie algebroids); in particular, we do not resort to the path spaces of [7, 12, 25], hence avoiding infinite dimensional constructions. Although our method is inspired by [3, 21, 22], it brings a technical difference in that we represent differential k-forms on a manifold N by *functions* $\oplus^k TN \to \mathbb{R}$ (as opposed to maps $\oplus^{k-1}TN \to T^*N$); this small variation greatly simplifies computations, so even when restricted to known situations, our general proof seems more direct than existing ones. The integration of IM forms is carried out in two steps: first, we show that an IM k-form on a Lie algebroid $A \to M$ defines an element in $\Omega^k(A)$ whose associated function $\oplus^k TA \to \mathbb{R}$ is a Lie-algebroid morphism; second, upon integration, one obtains a groupoid morphism $\oplus^k T\mathcal{G} \to \mathbb{R}$ which defines a multiplicative k-form.

In the last part of the paper, we revisit multiplicative multivector fields on Lie groupoids, as in [18]. We show how the very same techniques used to study multiplicative forms apply to the dual situation of multivector fields, leading to an alternative proof of the universal lifting theorem of [18] (not involving path spaces) and drawing a clear parallel between the two theories.

As a final remark, we note that the results in this paper admit a natural formulation in terms of graded geometry. Multiplicative forms and multivector fields on a given Lie groupoid $\mathcal{G}$ may be seen as multiplicative *functions* on the associated *graded* Lie groupoids $T[1]\mathcal{G}$ and $T^*[1]\mathcal{G}$, respectively. On ordinary Lie groupoids, the infinitesimal counterpart of a multiplicative function is a Lie-algebroid cocycle. The same holds at the graded level and, from this perspective, our results consist in using the geometry of $T[1]\mathcal{G}$ and $T^*[1]\mathcal{G}$ to obtain concrete descriptions of their Lie-algebroid cocycles. For example, using the the natural multiplicative vector field on $T[1]\mathcal{G}$ (the de Rham differential on $\mathcal{G}$), one identifies its Lie-algebroid cocycles with IM forms (see Theorem 3.1); for an analogous description of the Lie-algebroid cocycles of the graded groupoid $T^*[1]\mathcal{G}$ (see Theorem 6.1), ones uses its canonical multiplicative symplectic structure (defined by the Schouten bracket on $\mathcal{G}$). We will not elaborate on the supergeometric viewpoint in this paper, though it makes our results more intuitive.

The paper is organized as follows. In Section 2, we consider linear differential forms on vector bundles $A \to M$ and establish their correspondence with pairs of vector-bundle maps (μ, ν), where $\mu : A \to \wedge^{k-1}T^*M$ and $\nu : A \to \wedge^k T^*M$. In Section 3, we define IM k-forms on Lie algebroids and prove a compatibility result with tangent Lie algebroid structures (Theorem 3.1). Section 4 is devoted to Theorem 4.6, which is the correspondence between IM forms on Lie algebroids and multiplicative forms on Lie groupoids; we also discuss several special cases of this result. Section 5 explains the relationship between Theorem 4.6 and the Van Est isomorphism of [1]. In Section 6, we revisit the theory of multivector fields from [18].

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Notation, conventions and identities. For vector bundles $A \to M$ and $B \to M$ over the *same base* M, a vector-bundle map $\Psi : A \to B$ is always assumed to cover the identity map on M, unless stated otherwise. We denote its transpose, or dual, by $\Psi^t : B^* \to A^*$. We denote the k-fold direct sum of a vector bundle $q_A : A \to M$ by $\oplus^k_M A$, or simply $\oplus^k A$ if there is no risk of confusion. We may also use the notation

$\prod_{q_A}^k A$ if we want to be explicit about the projection map q_A (this is relevant when dealing with double vector bundles).

For a Lie groupoid $\mathcal{G}$ over M, we usually denote its source and target maps by s and t. The set $\mathcal{G}^{(2)} \subset \mathcal{G} \times \mathcal{G}$ of composable pairs is defined by the condition $\mathsf{s}(g) = \mathsf{t}(h)$, and the multiplication is denoted by $m : \mathcal{G}^{(2)} \to \mathcal{G}$, $m(g,h) = gh$. The unit map $\epsilon : M \to \mathcal{G}$ is often used to identify M with its image in $\mathcal{G}$. The Lie algebroid of $\mathcal{G}$ is $A\mathcal{G} = \ker(T\mathsf{s})|_M$, with anchor $T\mathsf{t}|_A : A \to M$ and bracket induced by right-invariant vector fields. For a Lie algebroid $A \to M$, we denote its anchor by ρ_A and bracket by $[\cdot,\cdot]_A$ (or simply ρ and $[\cdot,\cdot]$, if there is no risk of confusion).

We introduce some notation and collect some identities that will be useful for later computations. If $U_1, \dots, U_m$ are vector fields on a manifold M, we set

$$I^U_{m,r} := i_{U_m} \dots i_{U_r}, \quad r \leq m, \tag{1.1}$$

where i_U is the usual contraction. An inductive application of Cartan's formula gives

$$I^U_{m,1}\mathrm{d} = \sum_{l=1}^{m} (-1)^{l+1} I^U_{m,l+1} \mathcal{L}_{U_l} I^U_{l-1,1} + (-1)^m \mathrm{d} I^U_{m,1}, \tag{1.2}$$

where d denotes the de Rham differential and $\mathcal{L}_U$ is the Lie derivative. Given another vector field X and recalling the commutator formula $i_{[X,U_l]} = \mathcal{L}_X i_{U_l} - i_{U_l}\mathcal{L}_X$, we obtain

$$\mathcal{L}_X I^U_{m,1} = \sum_{l=1}^{m} I^U_{m,l+1} i_{[X,U_l]} I^U_{l-1,1} + I^U_{m,1}\mathcal{L}_X. \tag{1.3}$$

Given a differential form α, we also have

$$I^U_{m,1}(\mathrm{d}f \wedge \alpha) = \sum_{l=1}^{m} (-1)^{l+1} \mathrm{d}f(U_l) I^U_{m,l+1} I^U_{l-1,1}\alpha + (-1)^m \mathrm{d}f \wedge I^U_{m,1}\alpha. \tag{1.4}$$

We often use Einstein's summation convention when there is no risk of confusion.

2. Linear forms on vector bundles

In order to define linear forms, we recall a few facts about tangent and cotangent bundles of vector bundles.

2.1. **Tangent and cotangent bundles of vector bundles.** Let $q_A : A \longrightarrow M$ be a vector bundle, and let TA be the tangent bundle of the total space A. Besides its natural vector bundle structure over A, with projection map denoted by $p_A : TA \longrightarrow A$, it is also a vector bundle over TM, with respect to the map $Tq_A : TA \longrightarrow TM$.

It is useful to consider a coordinate description of these bundles. Let (x^j) be coordinates on M, $j = 1, \dots, \dim(M)$, and let $\{e_d\}$ be a basis of local sections of A, $d = 1, \dots, \mathrm{rank}(A)$. The corresponding coordinates on A are denoted by (x^j, u^d), and tangent coordinates on TA by $(x^j, u^d, \dot{x}^j, \dot{u}^d)$. In this notation, given $x = (x^j)$, the coordinates (u^d) specify a point in A_x, $(\dot{x}^j)$ a point in T_xM, whereas $(\dot{u}^d)$ determines a point on a second copy of A_x, tangent to the fibres of $A \longrightarrow M$. Note that $p_A(x^j, u^d, \dot{x}^j, \dot{u}^d) = (x^j, u^d)$, and $Tq_A(x^j, u^d, \dot{x}^j, \dot{u}^d) = (x^j, \dot{x}^j)$.

Similarly, consider the cotangent bundle T^*A, with local coordinates (x^j, u^d, p_j, ξ_d), where (p_j) determines a point in T^*_xM, and (ξ_d) a point in A^*_x, dual to the direction tangent to the fibres of $A \longrightarrow M$. In this case, besides the natural vector bundle structure $c_A : T^*A \longrightarrow A$, $c_A(x^j, u^d, p_j, \xi_d) = (x^j, u^d)$, T^*A is also a vector bundle over A^* [21], with respect to the projection map given in coordinates by

$$r : T^*A \longrightarrow A^*, \ \ r(x^j, u^d, p_j, \xi_d) = (x^j, \xi_d). \tag{2.1}$$

The total spaces TA and T^*A are examples of *double vector bundles*, see [20, 24]. They fit into the following commutative diagrams:

$$\begin{array}{ccc} TA & \xrightarrow{Tq_A} & TM \\ {\scriptstyle p_A}\downarrow & & \downarrow{\scriptstyle p_M} \\ A & \xrightarrow[q_A]{} & M \end{array} \qquad\qquad \begin{array}{ccc} T^*A & \xrightarrow{r} & A^* \\ {\scriptstyle c_A}\downarrow & & \downarrow{\scriptstyle q_{A^*}} \\ A & \xrightarrow[q_A]{} & M \end{array}$$

where

$$p_M : TM \longrightarrow M, \ p_M(x^j, \dot{x}^j) = (x^j), \quad q_{A^*} : A^* \longrightarrow M, \ q_{A^*}(x^j, \xi_d) = (x^j), \tag{2.2}$$

are the natural projections. Recall, see e.g. [20], that the intersection of the kernels of the top and left arrows on each diagram defines a vector bundle over M, known as the *core*. In the case of TA, the core is identified with $A \longrightarrow M$, with coordinates $(x^j, \dot{u}^d)$; for T^*A, the core is T^*M, with coordinates (x^j, p_j).

2.2. **The structure of linear forms on vector bundles.** Let $A \longrightarrow M$ be a vector bundle, with local coordinates (x^j, u^d), and let us consider the k-fold direct sum of TA over A,

$$\oplus_A^k TA := TA \times_A \ldots \times_A TA,$$

locally described by coordinates $(x^j, u^d, \dot{x}_1^j, \ldots, \dot{x}_k^j, \dot{u}_1^d, \ldots, \dot{u}_k^d)$. It is a vector bundle over A, with projection map

$$(x^j, u^d, \dot{x}_1^j, \ldots, \dot{x}_k^j, \dot{u}_1^d, \ldots, \dot{u}_k^d) \mapsto (x^j, u^d),$$

and also a vector bundle over $\oplus^k TM = TM \times_M \ldots \times_M TM$, with projection map

$$(x^j, u^d, \dot{x}_1^j, \ldots, \dot{x}_k^j, \dot{u}_1^d, \ldots, \dot{u}_k^d) \mapsto (x^j, \dot{x}_1^j, \ldots, \dot{x}_k^j).$$

Given a k-form $\Lambda \in \Omega^k(A)$ on the total space of $A \longrightarrow M$, let us consider the induced maps

$$\Lambda^\sharp : \oplus_A^{k-1} TA \longrightarrow T^*A, \quad \Lambda^\sharp(U_1, \ldots, U_{k-1}) = i_{U_{k-1}} \ldots i_{U_1} \Lambda, \tag{2.3}$$

$$\overline{\Lambda} : \oplus_A^k TA \longrightarrow \mathbb{R}, \qquad \overline{\Lambda}(U_1, \ldots, U_k) = i_{U_k} \ldots i_{U_1} \Lambda, \tag{2.4}$$

which are alternating and linear in each of their entries[1].

Definition 2.1. *A k-form Λ is called* ***linear*** *if the induced map $\Lambda^\sharp$ (2.3) is a morphism of vector bundles with respect to the vector bundle structures $\oplus_A^{k-1} TA \longrightarrow \oplus^{k-1} TM$ and $T^*A \longrightarrow A^*$. The space of linear k-forms on A is denoted by $\Omega^k_{\mathrm{lin}}(A)$.*

[1]Notice that, since $\Lambda^\sharp$ (resp. $\overline{\Lambda}$) is *multilinear* in its entries, it is *not* a vector-bundle morphism from the direct sum $\oplus^{k-1} TA \to A$ (resp. $\oplus^k TA \to A$) to $T^*A \to A^*$, unless $k = 2$ (resp. $k = 1$).

In particular, $\Lambda^\sharp$ covers a base map $\lambda : \oplus^{k-1}TM \longrightarrow A^*$,

$$\begin{array}{ccc} \oplus_A^{k-1}TA & \xrightarrow{\Lambda^\sharp} & T^*A \\ \downarrow & & \downarrow r \\ \oplus^{k-1}TM & \xrightarrow[\lambda]{} & A^*. \end{array} \tag{2.5}$$

The map λ is skew symmetric on its entries, so it can be viewed as a vector-bundle map $\wedge^{k-1}TM \longrightarrow A^*$. Its transpose is the vector-bundle map

$$\lambda^t : A \longrightarrow \wedge^{k-1}T^*M. \tag{2.6}$$

A simple computation in coordinates shows the following.

Lemma 2.2. *Given a k-form $\Lambda \in \Omega^k(A)$, the following are equivalent:*

(1) *Λ is linear.*

(2) *In local coordinates (x^j, u^d) on A, Λ has the form*

$$\Lambda = \frac{1}{k!}\Lambda_{i_1\dots i_k,d}(x)u^d \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_k} + \frac{1}{(k-1)!}\lambda_{i_1\dots i_{k-1}d}(x)\mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_{k-1}} \wedge \mathrm{d}u^d, \tag{2.7}$$

where $\lambda_{i_1\dots i_{k-1}d} = \langle \lambda(\partial_{x^{i_1}}, \dots, \partial_{x^{i_{k-1}}}), e_d\rangle$, and $\partial_{x^j} = \frac{\partial}{\partial x^j}$.

(3) *The map $\overline{\Lambda} : \oplus_A^k TA \longrightarrow \mathbb{R}$ defines a vector-bundle map*

$$\begin{array}{ccc} \oplus_A^{k}TA & \xrightarrow{\overline{\Lambda}} & \mathbb{R} \\ \downarrow & & \downarrow \\ \oplus^{k}TM & \longrightarrow & \{*\}. \end{array} \tag{2.8}$$

Given a vector-bundle map $\mu : A \longrightarrow \wedge^k T^*M$, let us consider the linear k-form Λ_μ on A given at a point $u \in A$ by

$$(\Lambda_\mu)_u := Tq_A|_u^t \mu(u). \tag{2.9}$$

In local coordinates (x^i, u^d) on A, Λ_μ is written as

$$(\Lambda_\mu)_u = \frac{1}{k!}\mu_{i_1\dots i_k,d}(x)u^d \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_k}, \tag{2.10}$$

where $\mu_{i_1\dots i_k,d}$ is defined by

$$\mu_{i_1\dots i_k,d} = \left\langle \mu(e_d), \frac{\partial}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{i_k}} \right\rangle.$$

Example 2.3. *When $k = 1$, a direct computation in coordinates shows that the linear 1-form Λ_μ, defined by the vector-bundle map $\mu : A \longrightarrow T^*M$, satisfies*

$$\Lambda_\mu = \mu^*\theta_{can}, \tag{2.11}$$

where $\theta_{can} = p_i \mathrm{d}x^i$ is the canonical 1-form on T^*M. (When $A = M \longrightarrow M$ is the vector bundle with zero fibres, (2.11) *recovers the well-known "tautological" property $\mu^*\theta_{can} = \mu$.)*

Lemma 2.4. *A linear k-form Λ covers the fibrewise zero map in* (2.5) *if and only if it is of the form Λ_μ (as in* (2.9)*) for a vector bundle map $\mu : A \longrightarrow \wedge^k T^*M$.*

Proof. We can use Lemma 2.2, or argue more globally as follows. Consider the projection $r : T^*A \longrightarrow A^*$, as in (2.1). One can directly check that

$$\ker(r)_u = (\ker(Tq_A|_u))^\circ = \mathrm{im}(Tq_A|_u^t),$$

where $^\circ$ stands for the annihilator. It follows from (2.9) that $r \circ \Lambda_\mu^\sharp = 0$, which means that Λ_μ covers the fibrewise zero map in (2.5). Conversely, if Λ covers the fibrewise zero map, then $r \circ \Lambda^\sharp = 0$; so, given $U_1, \ldots, U_k \in T_uA$, $\Lambda^\sharp(U_1, \ldots, U_{k-1}) = Tq_A|_u^*\alpha$ for some $\alpha \in T^*_{q_A(u)}M$. Since Λ is skew symmetric, we conclude that $\Lambda(U_1, \ldots, U_k)$ only depends on $Tq_A|_u(U_j)$, $j = 1, \ldots, k$. Hence, for each $u \in A$, there exists $\mu(u) \in \wedge^l T^*_{q_A(u)}M$ such that $(\Lambda)_u = Tq_A|_u^*\mu(u)$. The linear dependence of $\mu(u)$ on u follows from the linear dependence of $(\Lambda)_u$ on u, see (2.7); the resulting vector bundle map $\mu : A \longrightarrow \wedge^k T^*M$ is smooth by the local expression (2.10). □

Proposition 2.5. *There is a one-to-one correspondence between linear k-forms Λ on A, covering a map $\lambda : \oplus^{k-1}TM \longrightarrow A^*$, and pairs (μ, ν), where $\mu : A \longrightarrow \wedge^{k-1}T^*M$ and $\nu : A \longrightarrow \wedge^k T^*M$ are vector bundle morphisms. The correspondence is given by*

$$\Lambda = \mathrm{d}\Lambda_\mu + \Lambda_\nu, \tag{2.12}$$

where $\mu = (-1)^{k-1}\lambda^t$.

Proof. Let Λ be a linear k-form on A, and set $\mu = (-1)^{k-1}\lambda^t$. A direct computation using the local expression (2.10) and Lemma 2.2 shows that the k-form $d\Lambda_\mu$ is linear and covers the same map λ, hence the linear k-form $\Lambda - \mathrm{d}\Lambda_\mu$ covers the fibrewise zero map. By Lemma 2.4, there is a unique $\nu : A \longrightarrow \wedge^k T^*M$ such that $\Lambda - \mathrm{d}\Lambda_\mu = \Lambda_\nu$. □

A direct consequence of (2.12) is that if Λ is a linear form, then so is $\mathrm{d}\Lambda$.

Example 2.6. *Let $\Lambda \in \Omega^2(A)$ be a linear 2-form with $\mathrm{d}\Lambda = 0$. According to the previous proposition, we can write it as $\Lambda = \mathrm{d}\Lambda_\mu + \Lambda_\nu$, and Λ being closed amounts to $\mathrm{d}\Lambda_\nu = 0$; this condition immediately implies that $\nu = 0$, so $\Lambda = d\Lambda_\mu$. Using* (2.11)*, it follows that*

$$\Lambda = (\lambda^t)^*\omega_{can},$$

where $\omega_{can} = -\mathrm{d}\theta_{can} = \mathrm{d}x^i \wedge \mathrm{d}p_i$ the canonical symplectic form on T^*M (see [19, Sec. 7.3]*, and also* [3, Prop. 4.3]*).*

2.3. **Tangent lifts.** We now briefly discuss linear forms obtained via the *tangent lift* operation [14, 28] (see also [3]), that assigns to any k-form on a manifold M a linear k-form on the total space of its tangent bundle $p_M : TM \to M$.

Let us consider the operation

$$\tau : \Omega^l(M) \longrightarrow \Omega^{l-1}(TM), \quad \tau(\beta)|_X := (Tp_M|_X)^t(i_X\beta), \tag{2.13}$$

where $X \in TM$ and $l \geq 1$; i.e., for $U_1, \ldots, U_{l-1} \in T_X(TM)$,

$$i_{U_{l-1}} \ldots i_{U_1}\tau(\beta)|_X = \beta(X, Tp_M(U_1), \ldots, Tp_M(U_{l-1})).$$

In the notation of Section 2.2, $\tau(\beta)$ is a linear $(l-1)$-form on the vector bundle $A = TM$ of type Λ_ν, where

$$\nu : TM \longrightarrow \wedge^{l-1}T^*M, \quad \nu(X) = i_X\beta.$$

It directly follows from Example 2.3 that, if $\omega \in \Omega^2(M)$, then $\tau(\omega) = (\omega^\sharp)^*\theta_{can}$, where $\theta_{can} = p_i \mathrm{d}x^i$ is the canonical 1-form on T^*M.

The **tangent lift** operation,

$$\Omega^k(M) \longrightarrow \Omega^k(TM), \ \ \alpha \mapsto \alpha_T, \tag{2.14}$$

assigns to $\alpha \in \Omega^k(M)$ the form $\alpha_T \in \Omega^k(TM)$ defined by the Cartan-like formula

$$\alpha_T = \mathrm{d}\tau(\alpha) + \tau(\mathrm{d}\alpha). \tag{2.15}$$

It follows directly from (2.15) that α_T is linear and that the operation (2.14) is compatible with exterior derivatives, in the sense that $(\mathrm{d}\alpha)_T = \mathrm{d}\alpha_T$.

We will also need an equivalent characterization of the tangent lift, see e.g. [14]. Given $\alpha \in \Omega^k(M)$, consider the associated map

$$\overline{\alpha} : \oplus^k TM \longrightarrow \mathbb{R}, \ \ (X_1, \dots, X_k) \mapsto \alpha(X_1, \dots, X_k).$$

Let $\prod_{Tp_M}^k T(TM)$ denote the fibred product with respect to the vector bundle

$$Tp_M : T(TM) \longrightarrow TM, \ \ Tp_M(x^j, \dot{x}^j, \delta x^j, \delta \dot{x}^j) = (x^j, \delta x^j),$$

where $(x^j, \dot{x}^j, \delta x^j, \delta \dot{x}^j)$ are the local coordinates on $T(TM)$ induced by the tangent coordinates $(x^j, \dot{x}^j)$ on TM. We have a natural identification $T(\oplus^k TM) \cong \prod_{Tp_M}^k T(TM)$, so we can view the differential of the function $\overline{\alpha}$ in $C^\infty(\oplus^k TM)$ as a map

$$\mathrm{d}\overline{\alpha} : \prod_{Tp_M}^k T(TM) \longrightarrow \mathbb{R}.$$

Note that the canonical involution

$$J_M : T(TM) \longrightarrow T(TM), \quad J_M(x^j, \dot{x}^j, \delta x^j, \delta \dot{x}^j) = (x^j, \delta x^j, \dot{x}^j, \delta \dot{x}^j), \tag{2.16}$$

induces an identification

$$J_M^{(k)} : \prod_{p_{TM}}^k T(TM) \longrightarrow \prod_{Tp_M}^k T(TM).$$

One can prove (see e.g. [14]) that, given $\alpha \in \Omega^k(M)$, its tangent lift $\alpha_T \in \Omega^k(TM)$ is uniquely determined by the condition

$$\overline{\alpha_T} = \mathrm{d}\overline{\alpha} \circ J_M^{(k)} : \prod_{p_{TM}}^k T(TM) \longrightarrow \mathbb{R}. \tag{2.17}$$

3. Linear forms on Lie algebroids

3.1. **Core and linear sections.** Let us consider local coordinates (x^j) on M, a basis of local sections $\{e_d\}$ of A, and dual basis $\{e^d\}$ of A^*. As in Section 2, we denote the corresponding coordinates on A by (x^j, u^d), and on A^* by (x^j, ξ_d), while coordinates on TA are denoted by $(x^j, u^d, \dot{x}^j, \dot{u}^d)$, and on T^*A by (x^j, u^d, p_j, ξ_d).

Each local section e_a of A defines two local sections of $TA \longrightarrow TM$ by

$$\widehat{e}_a(x^j, \dot{x}^j) = (x^j, 0, \dot{x}^j, \delta_a^d), \quad Te_a(x^j, \dot{x}^j) = (x^j, \delta_a^d, \dot{x}^j, 0), \tag{3.1}$$

where δ_a^d is the d-th component of e_a, i.e., 1 if $d = a$ or zero otherwise.

More generally, e_a defines two types of local sections of $\oplus^k_A TA \longrightarrow \oplus^k TM$ as follows: the first type is given, for each $n \in \{1, \ldots, k\}$, by

$$\widehat{e}_{a,n}(\dot{x}_1 \oplus \ldots \oplus \dot{x}_k) := \widehat{0}(\dot{x}_1) \oplus \ldots \oplus \widehat{0}(\dot{x}_{n-1}) \oplus \widehat{e}_a(\dot{x}_n) \oplus \widehat{0}(\dot{x}_{n+1}) \oplus \ldots \oplus \widehat{0}(\dot{x}_k), \tag{3.2}$$

where $\dot{x}_l = (x^j, \dot{x}^j_l)$ belongs to the l-th component of $\oplus^k TM$ and $\widehat{0}(\dot{x}_l) = (x^j, 0, \dot{x}^j_l, 0)$; the second type is

$$(Te_a)^k(\dot{x}_1 \oplus \ldots \oplus \dot{x}_k) := Te_a(\dot{x}_1) \oplus \ldots \oplus Te_a(\dot{x}_k). \tag{3.3}$$

The sections $\widehat{e}_{a,n}$ and $(Te_a)^k$ are examples of *core* and *linear* sections, respectively, of double vector bundles (see e.g. [13, 20]). A key property is that they generate the module of local sections of $\oplus^k_A TA \longrightarrow \oplus^k TM$. Note also that, under the natural projection $\oplus^k_A TA \longrightarrow A$, core sections $\widehat{e}_{a,n}$ are sent to the zero section of $A \longrightarrow M$, while linear sections $(Te_a)^k$ map to the section e_a.

3.2. **Tangent Lie algebroids.** Suppose that $A \longrightarrow M$ carries a Lie algebroid structure (see e.g. [6, 20]), with Lie bracket $[\cdot, \cdot]_A$ on $\Gamma(A)$ and anchor map $\rho_A : A \longrightarrow TM$. Then the vector bundle $TA \longrightarrow TM$ inherits a natural Lie algebroid structure, known as the *tangent Lie algebroid*, see e.g. [21]. We will need local expressions for the tangent Lie algebroid in terms of the coordinates introduced in Section 3.1.

The Lie algebroid $A \longrightarrow M$ is locally determined by structure functions ρ^j_a and C^c_{ab} defined by

$$\rho_A(e_a) = \rho^j_a \frac{\partial}{\partial x^j}, \quad [e_a, e_b]_A = C^c_{ab} e_c. \tag{3.4}$$

The tangent Lie algebroid structure on $TA \longrightarrow TM$ is defined in terms of core and linear sections (3.1) by

$$[\widehat{e}_a, \widehat{e}_b]_{TA} = 0, \;\; [Te_a, \widehat{e}_b]_{TA} = C^c_{ab}\widehat{e}_c, \;\; [Te_a, Te_b]_{TA} = C^c_{ab} Te_c + \dot{x}^i \frac{\partial C^c_{ab}}{\partial x^i} \widehat{e}_c, \tag{3.5}$$

$$\rho_{TA}(Te_a) = \rho^j_a \frac{\partial}{\partial x^j} + \dot{x}^i \frac{\partial \rho^j_a}{\partial x^i} \frac{\partial}{\partial \dot{x}^j}, \quad \rho_{TA}(\widehat{e}_a) = \rho^j_a \frac{\partial}{\partial \dot{x}^j}. \tag{3.6}$$

In (3.6), we have identified points in $T(TM)$, written in coordinates as $(x^j, \dot{x}^j, \delta x^j, \delta \dot{x}^j)$, with tangent vectors

$$\delta x^j \frac{\partial}{\partial x^j} + \delta \dot{x}^j \frac{\partial}{\partial \dot{x}^j}\Big|_{(x^j, \dot{x}^j)}.$$

We notice that the tangent Lie algebroid induces a Lie algebroid structure on the direct sum $\oplus^k_A TA \longrightarrow \oplus^k TM$. This is a general property of $\mathcal{VB}$*-algebroids*, which we directly verify in this example. A simple consequence of (3.5) and (3.6) is that if U and V are local sections of $TA \longrightarrow TM$, each of type $\widehat{e}_a$ or Te_a, then

$$Tp_M(\rho_{TA}(U)) = \rho_A(p_A(U)), \quad p_A([U, V]_{TA}) = [p_A(U), p_A(V)]_A, \tag{3.7}$$

where $p_M : TM \longrightarrow M$ and $p_A : TA \longrightarrow A$ are the natural projections. It follows from the first equation in (3.7) that if $U_1 \oplus \ldots \oplus U_k \in \oplus^k_A TA$ is of type (3.2) or (3.3), then

$$Tp_M(\rho_{TA}(U_l)) = Tp_M(\rho_{TA}(U_m)), \;\; \forall l, m \in \{1, \ldots, k\}.$$

As a result, $(\rho_{TA}(U_1), \ldots, \rho_{TA}(U_k))$ defines an element in $\prod^k_{Tp_M} T(TM)$. Using the natural identification $\prod^k_{Tp_M} T(TM) = T(\oplus^k TM)$, we obtain a vector bundle map

$\rho_k : \oplus^k_A TA \longrightarrow T(\oplus^k TM)$,

$$\rho_k(U_1 \oplus \ldots \oplus U_k) := \rho_{TA}(U_1) \oplus \ldots \oplus \rho_{TA}(U_k). \tag{3.8}$$

Writing $\oplus^k TM$ in local coordinates $(x^j, \dot{x}^j_1, \ldots, \dot{x}^j_k)$, we have the following explicit formulas:

$$\rho_k(\widehat{e}_{a,n}) = \rho^j_a \frac{\partial}{\partial \dot{x}^j_n}, \tag{3.9}$$

$$\rho_k((Te_a)^k) = \rho^j_a \frac{\partial}{\partial x^j} + \sum_{n=1}^{k} W^j_{a,n} \frac{\partial}{\partial \dot{x}^j_n}, \tag{3.10}$$

where $W^j_{a,n} = \dot{x}^i_n \frac{\partial \rho^j_a}{\partial x^i} \in C^\infty(\oplus^k TM)$.

The second equation in (3.7) implies that if $U_1 \oplus \ldots \oplus U_k$ and $V_1 \oplus \ldots \oplus V_k$ are local sections of $\oplus^k_A TA \longrightarrow \oplus^k TM$ of type (3.2) or (3.3), then

$$[U_1 \oplus \ldots \oplus U_k, V_1 \oplus \ldots \oplus V_k]_k := [U_1, V_1]_{TA} \oplus \ldots \oplus [U_k, V_k]_{TA} \tag{3.11}$$

is a well-defined local section of $\oplus^k_A TA \longrightarrow \oplus^k TM$. Explicitly, we have:

$$[\widehat{e}_{a,n}, \widehat{e}_{b,m}]_k = 0, \tag{3.12}$$

$$[(Te_a)^k, \widehat{e}_{b,m}]_k = C^d_{ab} \widehat{e}_{d,m} \tag{3.13}$$

$$[(Te_a)^k, (Te_b)^k]_k = C^d_{ab}(Te_d)^k + \sum_{n=1}^{k} \dot{x}^i_n \frac{\partial C^d_{ab}}{\partial x^i} \widehat{e}_{d,n}. \tag{3.14}$$

The induced Lie algebroid structure on $\oplus^k_A TA \longrightarrow \oplus^k TM$ is defined by ρ_k and the extension of $[\cdot,\cdot]_k$ to all sections via the Leibniz rule[2].

3.3. **IM-forms.** Let $\Lambda \in \Omega^k(A)$ be a linear k-form on a Lie algebroid $A \longrightarrow M$, $k \geq 1$. Following Prop. 2.5, let $\mu : A \longrightarrow \wedge^{k-1} T^*M$ and $\nu : A \longrightarrow \wedge^k T^*M$ be the vector-bundle maps such that $\Lambda = \mathrm{d}\Lambda_\mu + \Lambda_\nu$. Let us consider the bundle map

$$\begin{array}{ccc} \oplus^k_A TA & \xrightarrow{\overline{\Lambda}} & \mathbb{R} \\ \downarrow & & \downarrow \\ \oplus^k TM & \longrightarrow & \{*\}. \end{array} \tag{3.15}$$

The following is the main result of this section.

Theorem 3.1. *The map* (3.15) *is a Lie algebroid morphism if and only if the following holds for all* $u, v \in \Gamma(A)$*:*

$$i_{\rho(u)}\mu(v) = -i_{\rho(v)}\mu(u) \tag{3.16}$$

$$\mu([u,v]) = \mathcal{L}_{\rho(u)}\mu(v) - i_{\rho(v)}\mathrm{d}\mu(u) - i_{\rho(v)}\nu(u) \tag{3.17}$$

$$\nu([u,v]) = \mathcal{L}_{\rho(u)}\nu(v) - i_{\rho(v)}\mathrm{d}\nu(u). \tag{3.18}$$

---

[2] We adopt the simplified notation ρ_k, $[\cdot,\cdot]_k$, instead of $\rho_{\oplus^k_A TA}$ and $[\cdot,\cdot]_{\oplus^k_A TA}$; in particular, $\rho_1 = \rho_{TA}$ and $[\cdot,\cdot]_1 = [\cdot,\cdot]_{TA}$.

For a Lie algebroid $A \to M$ and vector-bundle maps

$$\mu : A \longrightarrow \wedge^{k-1}T^*M, \quad \nu : A \longrightarrow \wedge^k T^*M, \quad k \geq 1,$$

we say that the pair (μ, ν) is an **IM k-form** on A if conditions (3.16), (3.17) and (3.18) are satisfied. The terminology IM stands for *infinitesimally multiplicative*, and it will be clarified in Section 4. The space of IM k-forms on A is denoted by $\Omega_{\mathrm{IM}}(A)$.

We note that Theorem 3.1 can be alternatively phrased in terms of the map $\Lambda^\sharp$ (2.5), as this map is a Lie algebroid morphism if and only if so is $\overline{\Lambda}$.

Remark 3.2. *Given an IM-form (μ, ν), it follows from* (3.17)*, using the skew-symmetry and Jacobi identity for the Lie algebroid bracket $[\cdot, \cdot]$ on $\Gamma(A)$, that ν automatically satisfies*

$$i_{\rho(u)}\nu(v) = -i_{\rho(v)}\nu(u), \tag{3.19}$$

$$i_{\rho(w)}(\mathcal{L}_{\rho(v)}\nu(u) - \mathcal{L}_{\rho(u)}\nu(v)) + c.p. = 0 \tag{3.20}$$

for all $u, v, w \in \Gamma(A)$, where c.p. stands for cyclic permutations in u, v, w.

Example 3.3. *Consider a Lie algebroid $A \to M$ and a k-form $\eta \in \Omega^k(M)$. Then the pair (μ, ν) of vector-bundle maps*

$$\mu : A \to \wedge^{k-1}T^*M, \;\; \mu(u) = -i_{\rho(u)}\eta, \quad \textit{and} \quad \nu : A \to \wedge^k T^*M, \;\; \nu(u) = -i_{\rho(u)}\mathrm{d}\eta,$$

defines an IM k-form on A.

Example 3.4. *Let $A \to M$ be a Lie algebroid, and let $\phi \in \Omega^{k+1}(M)$ be such that $i_{\rho(u)}\mathrm{d}\phi = 0$, $\forall u \in \Gamma(A)$. One directly checks that the vector-bundle map $\nu : A \longrightarrow \wedge^k T^*M$ given by*

$$\nu(u) := -i_{\rho(u)}\phi \tag{3.21}$$

verifies (3.18)*. The particular IM k-forms (μ, ν) on A for which ν is given as in* (3.21) *for a closed form $\phi \in \Omega^{k+1}(M)$ are called **IM k-forms relative to ϕ**. These special types of IM forms have first appeared in* [5] *(for $k = 2$), and more recently in* [1] *(for arbitrary k), in the study of multiplicative forms (see Section 4).*

Remark 3.5. *Let $\iota_{\mathcal{C}} : \mathcal{C} \hookrightarrow M$ be an orbit of the Lie algebroid $A \to M$, i.e., an integral leaf of the distribution $\rho(A) \subset TM$. If (μ, ν) is an IM k-form on A, then we have induced forms $\mu_{\mathcal{C}} \in \Omega^k(\mathcal{C})$ and $\nu_{\mathcal{C}} \in \Omega^{k+1}(\mathcal{C})$ defined by*

$$i_{\rho(u)}\mu_{\mathcal{C}} = \iota_{\mathcal{C}}^*\mu(u), \quad i_{\rho(u)}\nu_{\mathcal{C}} = \iota_{\mathcal{C}}^*\nu(u).$$

It follows from (3.16) *and* (3.19) *that the formulas above do define differential forms on $\mathcal{C}$; moreover,* (3.17) *implies that $\mathrm{d}\mu_{\mathcal{C}} = \nu_{\mathcal{C}}$. In particular, we see that any IM k-form on a transitive Lie algebroid is like the one in Example 3.3.*

In order to prove Thm. 3.1, we need some lemmas. We work in local coordinates (x^j, u^d) on A, induced by coordinates (x^j) on M and the choice of a basis of local sections $\{e_d\}$ of A (see Section 3.1).

Lemma 3.6. *Let $\dot{x} = (\dot{x}_1, \ldots, \dot{x}_k) \in \oplus^k TM$, where $\dot{x}_l = (x^j, \dot{x}_l^j)$ belongs to the l-th copy of TM. Then:*

$$\overline{\Lambda}(\widehat{e}_{a,n}(\dot{x}_1, \ldots, \dot{x}_k)) = (-1)^{n-1} I^{\dot{x}}_{k,n+1} I^{\dot{x}}_{n-1,1}\mu(e_a), \tag{3.22}$$

$$\overline{\Lambda}((Te_a)^k(\dot{x}_1, \ldots, \dot{x}_k)) = I^{\dot{x}}_{k,1}(\mathrm{d}\mu(e_a) + \nu(e_a)), \tag{3.23}$$

seen as functions in $C^\infty(\oplus^k TM)$ (see (1.1) *for notation).*

Proof. Writing $\Lambda = \mathrm{d}\Lambda_\mu + \Lambda_\nu$ and recalling the local expressions of Λ_μ and Λ_ν (see (2.10)), we have

$$\begin{aligned}(3.24)\qquad \Lambda|_{(x^j,u^d)} =& \frac{1}{(k-1)!}u^d \mathrm{d}\mu_{i_1\ldots i_{k-1},d}(x)\wedge \mathrm{d}x^{i_1}\wedge\ldots\wedge \mathrm{d}x^{i_{k-1}}+\\ & \frac{1}{(k-1)!}\mu_{i_1\ldots i_{k-1},d}(x)\mathrm{d}u^d\wedge \mathrm{d}x^{i_1}\wedge\ldots\wedge \mathrm{d}x^{i_{k-1}}+\\ & \frac{1}{k}\nu_{i_1\ldots i_k,d}(x)u^d \mathrm{d}x^{i_1}\wedge\ldots\wedge \mathrm{d}x^{i_k}.\end{aligned}$$

We write points in TA with coordinates $(x^j, u^d, \dot{x}^j, \dot{u}^d)$ in terms of horizontal tangent vectors $\frac{\partial}{\partial x^j}$ and vertical tangent vectors $\frac{\partial}{\partial u^d}$ as

$$\dot{x}^j\frac{\partial}{\partial x^j} + \dot{u}^d\frac{\partial}{\partial u^d}\Big|_{(x^j,u^d)}.$$

In particular, recalling the local sections $\widehat{e}_a, \widehat{0}$ and Te_a of $TA\to TM$ from Section 3.1, we have

$$\widehat{0}(\dot{x}) = \dot{x}^j\frac{\partial}{\partial x^j}\Big|_{(x^j,0)},\quad \widehat{e}_a(\dot{x}) = \dot{x}^j\frac{\partial}{\partial x^j}+\frac{\partial}{\partial u^a}\Big|_{(x^j,0)},\quad Te_a(\dot{x}) = \dot{x}^j\frac{\partial}{\partial x^j}\Big|_{(x^j,\delta^d_a)},$$

where $\dot{x} = (x^j, \dot{x}^j)\in TM$. Using (3.2) and (3.3), formulas (3.22) and (3.23) follow from a direct calculation. □

Let $(x^j, \dot{x}^j_1,\ldots,\dot{x}^j_k)$ be local coordinates on $\oplus^k TM$, and fix $n\in\{1,\ldots,k\}$.

Lemma 3.7. *Let $\alpha\in\Omega^l(\oplus^k TM)$ be such that $\mathcal{L}_{\frac{\partial}{\partial \dot{x}^j_n}}\alpha = 0$ $\forall j$, and consider on $\oplus^k TM$ the local vector fields $\dot{x}_n = \dot{x}^j_n\frac{\partial}{\partial x^j}$, $V^v = v^j(x)\frac{\partial}{\partial \dot{x}^j_n}$, and $V^h = v^j(x)\frac{\partial}{\partial x^j}$. Then $\mathcal{L}_{V^v} i_{\dot{x}_n}\alpha = i_{V^h}\alpha$.*

Proof. The proof follows from the identity $i_{[X,Y]} = \mathcal{L}_X i_Y - i_Y\mathcal{L}_X$ and the fact that $[v^i(x)\frac{\partial}{\partial \dot{x}^i_n}, \dot{x}^j_n\frac{\partial}{\partial x^j}] = v^j\frac{\partial}{\partial x^j} - \dot{x}^j_n\frac{\partial v^i}{\partial x^j}\frac{\partial}{\partial \dot{x}^i_n}$. □

We now proceed to the proof of the main result.

Proof. (of Theorem 3.1)

To show that the map $\overline{\Lambda}$ in (3.15) is a Lie algebroid morphism (see e.g. [20]), the only condition to be verified is

$$(3.25)\qquad \overline{\Lambda}([U,V]_k) = \mathcal{L}_{\rho_k(U)}\overline{\Lambda}(V) - \mathcal{L}_{\rho_k(V)}\overline{\Lambda}(U)$$

for all U, V sections of $\oplus^k_A TA\longrightarrow\oplus^k TM$. Since sections of type $\widehat{e}_{a,n}$ (core) and $(Te_b)^k$ (linear) locally generate the space of sections of $\oplus^k_A TA\longrightarrow\oplus^k TM$, it suffices to verify (3.25) taking U and V to be of these types.

Core-Core: Let us consider two core sections $\widehat{e}_{a,n}$ and $\widehat{e}_{b,m}$. Since $[\widehat{e}_{a,n}, \widehat{e}_{b,m}]_k = 0$ (3.12), condition (3.25) in this case becomes

$$(3.26)\qquad \mathcal{L}_{\rho_k(\widehat{e}_{a,n})}\overline{\Lambda}(\widehat{e}_{b,m}) - \mathcal{L}_{\rho_k(\widehat{e}_{b,m})}\overline{\Lambda}(\widehat{e}_{a,n}) = 0.$$

Using (3.9) and (3.22), we see that

$$\mathcal{L}_{\rho_k(\widehat{e}_{a,n})}\overline{\Lambda}(\widehat{e}_{b,m}) = (-1)^{n-1}\mathcal{L}_{\rho^i_a\frac{\partial}{\partial \dot{x}^i_n}} I^{\dot{x}}_{k,m+1} I^{\dot{x}}_{m-1,1}\mu(e_b).$$

This condition is trivially satisfied when $n = m$, so we may assume that $n > m$ (the case $n < m$ leads to the same). Using Lemma 3.7, we see that the right-hand side of the last equation agrees with

$$(-1)^{n-1} I^{\dot{x}}_{k,n+1} i_{\rho(e_a)} I^{\dot{x}}_{n-1,m+1} I^{\dot{x}}_{m-1,1} \mu(e_b) = \\ (-1)^{n-1}(-1)^{n-2} I^{\dot{x}}_{k,n+1} I^{\dot{x}}_{n-1,m+1} I^{\dot{x}}_{m-1,1} i_{\rho(e_a)} \mu(e_b).$$

Hence we obtain

$$\mathcal{L}_{\rho_k(\widehat{e}_{a,n})} \overline{\Lambda}(\widehat{e}_{b,m}) = -I^{\dot{x}}_{k,n+1} I^{\dot{x}}_{n-1,m+1} I^{\dot{x}}_{m-1,1} i_{\rho(e_a)} \mu(e_b).$$

An analogous computation leads to

$$\mathcal{L}_{\rho_k(\widehat{e}_{b,m})} \overline{\Lambda}(\widehat{e}_{a,n}) = I^{\dot{x}}_{k,n+1} I^{\dot{x}}_{n-1,m+1} I^{\dot{x}}_{m-1,1} i_{\rho(e_b)} \mu(e_a).$$

It follows that (3.26) is equivalent to

$$i_{\rho(e_a)} \mu(e_b) = -i_{\rho(e_b)} \mu(e_a).$$

Core-Linear: We now consider sections $\widehat{e}_{b,m}$ and $(Te_a)^k$, so that (3.25) reads

$$\overline{\Lambda}([(Te_a)^k, \widehat{e}_{b,m}]_k) = \mathcal{L}_{\rho_k((Te_a)^k)} \overline{\Lambda}(\widehat{e}_{b,m}) - \mathcal{L}_{\rho_k(\widehat{e}_{b,m})} \overline{\Lambda}((Te_a)^k). \tag{3.27}$$

Using the linearity of Λ, (3.13) and (3.22), we have

$$\begin{aligned} \overline{\Lambda}([(Te_a)^k, \widehat{e}_{b,m}]_k) = \overline{\Lambda}(C^d_{ab} \widehat{e}_{d,m}) &= C^d_{ab} (-1)^{m-1} I^{\dot{x}}_{k,m+1} I^{\dot{x}}_{m-1,1} \mu(e_d) \\ &= (-1)^{m-1} I^{\dot{x}}_{k,m+1} I^{\dot{x}}_{m-1,1} \mu([e_a, e_b]). \end{aligned} \tag{3.28}$$

For each fixed n, consider the functions $W^j_{a,n} = \frac{\partial \rho^j_a}{\partial x^i} \dot{x}^i_n$ defined in (3.10), noticing the following identity (of local vector fields on $\oplus^k TM$):

$$W^j_{a,n} \frac{\partial}{\partial x^j} = -[\rho(e_a), \dot{x}_n], \tag{3.29}$$

where $\dot{x}_n = \dot{x}^i_n \frac{\partial}{\partial x^i}$. Using (3.29) and Lemma 3.7, we see that

$$\begin{aligned} \mathcal{L}_{\rho_k((Te_a)^k)} \overline{\Lambda}(\widehat{e}_{b,m}) &= \left(\mathcal{L}_{\rho(e_a)} + \sum_{l=1}^{k} \mathcal{L}_{W^i_{a,l} \frac{\partial}{\partial \dot{x}^i_l}} \right) (-1)^{m-1} I^{\dot{x}}_{k,m+1} I^{\dot{x}}_{m-1,1} \mu(e_b) \\ &= (-1)^{m-1} \left(\mathcal{L}_{\rho(e_a)} I^U_{k,1} \mu(e_b) - \sum_{l=1}^{k} I^U_{k,l+1} i_{[\rho(e_a), U_l]} I^U_{l-1,1} \mu(e_b) \right) \end{aligned}$$

where $U = (U_1, \dots, U_{k-1}) = (\dot{x}_1, \dots, \dot{x}_{m-1}, \dot{x}_{m+1}, \dots, \dot{x}_k)$. It follows from (1.3) that

$$\mathcal{L}_{\rho_k((Te_a)^k)} \overline{\Lambda}(\widehat{e}_{b,m}) = (-1)^{m-1} I^{\dot{x}}_{k,m+1} I^{\dot{x}}_{m-1,1} \mathcal{L}_{\rho(e_a)} \mu(e_b). \tag{3.30}$$

Using (3.23) and Lemma 3.7, we obtain

$$\begin{aligned} \mathcal{L}_{\rho_k(\widehat{e}_{b,m})} \overline{\Lambda}((Te_a)^k) = \mathcal{L}_{\rho^i_a \frac{\partial}{\partial \dot{x}^i_m}} I^{\dot{x}}_{k,1} (\mathrm{d}\mu(e_a) + \nu(e_a)) &= I^{\dot{x}}_{k,m+1} i_{\rho(e_a)} I^{\dot{x}}_{m-1,1} (\mathrm{d}\mu(e_a) + \nu(e_a)) \\ &= (-1)^{m-1} I^{\dot{x}}_{k,m+1} I^{\dot{x}}_{m-1,1} i_{\rho(e_a)} (\mathrm{d}\mu(e_a) + \nu(e_a)). \end{aligned}$$

Combining this last equation with (3.28) and (3.30), we see that (3.27) is equivalent to

$$\mu([e_a, e_b]) = \mathcal{L}_{\rho(e_a)} \mu(e_b) - i_{\rho(e_b)} \mathrm{d}\mu(e_a) - i_{\rho(e_b)} \nu(e_a). \tag{3.31}$$

Linear-Linear: We finally consider condition (3.25) for two linear sections:

$$\overline{\Lambda}([(Te_a)^k,(Te_b)^k]_k) = \mathcal{L}_{\rho_k((Te_a)^k)}\overline{\Lambda}((Te_b)^k) - \mathcal{L}_{\rho_k((Te_b)^k)}\overline{\Lambda}((Te_a)^k). \tag{3.32}$$

Using (3.14) and the linearity of Λ, we have

$$\begin{aligned}\overline{\Lambda}([(Te_a)^k,(Te_b)^k]_k) &= C^d_{ab}\overline{\Lambda}((Te_d)^k) + \sum_{n=1}^{k} \mathrm{d}C^d_{ab}(\dot{x}_n)\overline{\Lambda}(\widehat{e}_{d,n}) \\ &= C^d_{ab}I^{\dot{x}}_{k,1}(\mathrm{d}\mu(e_d)+\nu(e_d)) + \sum_{n=1}^{k}(-1)^{n-1}\mathrm{d}C^d_{ab}(\dot{x}_n)I^{\dot{x}}_{k,n+1}I^{\dot{x}}_{n-1,1}\mu(e_d). \end{aligned}\tag{3.33}$$

It follows from (1.4) (also using that $I^{\dot{x}}_{k,1}\mu(e_d) = 0$, since $\mu(e_d)$ is a $(k-1)$-form) that

$$\begin{aligned} I^{\dot{x}}_{k,1}C^d_{ab}\mathrm{d}\mu(e_d) &= I^{\dot{x}}_{k,1}\mathrm{d}(C^d_{ab}\mu(e_d)) - I^{\dot{x}}_{k,1}(\mathrm{d}C^d_{ab}\wedge\mu(e_d)) \\ &= I^{\dot{x}}_{k,1}\mathrm{d}(C^d_{ab}\mu(e_d)) - \sum_{n=1}^{k}(-1)^{n+1}\mathrm{d}C^d_{ab}(\dot{x}_n)I^{\dot{x}}_{k,n+1}I^{\dot{x}}_{n-1,1}\mu(e_d). \end{aligned}$$

Comparing with (3.33), we conclude that

$$\begin{aligned}\overline{\Lambda}([(Te_a)^k,(Te_b)^k]_k) &= I^{\dot{x}}_{k,1}(\mathrm{d}\mu(C^d_{ab}e_d)+\nu(C^d_{ab}e_d)) \\ &= I^{\dot{x}}_{k,1}(\mathrm{d}\mu([e_a,e_b])+\nu([e_a,e_b])).\end{aligned}\tag{3.34}$$

Using Lemma 3.7, (3.29) and (1.3), we directly obtain

$$\begin{aligned}\mathcal{L}_{\rho((Te_a)^k)}\overline{\Lambda}((Te_b)^k) &= \left(\mathcal{L}_{\rho(e_a)} + \sum_{n=1}^{k}\mathcal{L}_{W^i_{a,n}\frac{\partial}{\partial \dot{x}^i_n}}\right)I^{\dot{x}}_{k,1}(\mathrm{d}\mu(e_b)+\nu(e_b)) \\ &= I^{\dot{x}}_{k,1}\mathcal{L}_{\rho(e_a)}(\mathrm{d}\mu(e_b)+\nu(e_b)).\end{aligned}\tag{3.35}$$

Similarly

$$\mathcal{L}_{\rho((Te_b)^k)}\overline{\Lambda}((Te_a)^k) = I^{\dot{x}}_{k,1}\mathcal{L}_{\rho(e_b)}(\mathrm{d}\mu(e_a)+\nu(e_a)). \tag{3.36}$$

Combining (3.34), (3.35) and (3.36), we see that (3.32) is equivalent to

$$\mathrm{d}\mu([e_a,e_b]) + \nu([e_a,e_b]) = \mathcal{L}_{\rho(e_a)}(\mathrm{d}\mu(e_b)+\nu(e_b)) - \mathcal{L}_{\rho(e_b)}(\mathrm{d}\mu(e_a)+\nu(e_a)).$$

We may assume that (3.31) holds, in which case one can directly check that the last equation is equivalent to

$$\nu([e_a,e_b]) = \mathcal{L}_{\rho(e_a)}\nu(e_b) - i_{\rho(e_b)}\mathrm{d}\nu(e_a).$$

□

4. Infinitesimal description of multiplicative forms

In this section, we relate IM-forms on Lie algebroids with multiplicative forms on Lie groupoids. Let $\mathcal{G}$ be a Lie groupoid over M, with source and target maps denoted by $\mathsf{s},\mathsf{t} : \mathcal{G} \longrightarrow M$, respectively, multiplication $m : \mathcal{G}^{(2)} \longrightarrow \mathcal{G}$, and unit map $\epsilon : M \longrightarrow \mathcal{G}$ (that we often use to view M as a submanifold of $\mathcal{G}$). The Lie algebroid of $\mathcal{G}$ is denoted by $A(\mathcal{G})$, or simply A if there is no risk of confusion; see Section 1.

A k-form $\alpha \in \Omega^k(\mathcal{G})$ is called **multiplicative** if

$$m^*\alpha = pr_1^*\alpha + pr_2^*\alpha, \tag{4.1}$$

where $pr_1, pr_2 : \mathcal{G}^{(2)} \longrightarrow \mathcal{G}$ are the natural projections. Alternatively, one may define multiplicative forms in terms of a natural groupoid structure on $T\mathcal{G}$ over TM, known as the *tangent groupoid*, see e.g. [20]; it has source (resp. target) map $T\mathsf{s} : T\mathcal{G} \longrightarrow TM$ (resp. $T\mathsf{t} : T\mathcal{G} \longrightarrow TM$), multiplication $Tm : (T\mathcal{G})^{(2)} = T\mathcal{G}^{(2)} \longrightarrow T\mathcal{G}$, and unit map $T\epsilon : TM \longrightarrow T\mathcal{G}$. This groupoid structure can be naturally extended to the direct sum $\oplus^k_{\mathcal{G}} T\mathcal{G}$, $k \geq 1$, making it a Lie groupoid over $\oplus^k TM$, with source (resp. target) map $\oplus^k T\mathsf{s}$ (resp. $\oplus^k T\mathsf{t}$), multiplication map $\oplus^k Tm$, etc.

Let $\alpha \in \Omega^k(\mathcal{G})$, and let us consider the associated map

$$\overline{\alpha} : \oplus^k_{\mathcal{G}} T\mathcal{G} \longrightarrow \mathbb{R}, \quad \overline{\alpha}(U_1, \ldots, U_k) = i_{U_k} \ldots i_{U_1} \alpha. \tag{4.2}$$

The following observation is immediate from (4.1).

Lemma 4.1. *α is multiplicative if and only if $\overline{\alpha}$ is a groupoid morphism. (Here $\mathbb{R}$ is viewed as an additive group.)*

We denote the space of multiplicative k-forms on $\mathcal{G}$ by $\Omega^k_{\mathrm{mult}}(\mathcal{G})$.

4.1. **From multiplicative to IM forms.** Let $\mathcal{G}$ be a Lie groupoid over M, and consider the tangent lift operation $\Omega^k(\mathcal{G}) \to \Omega^k(T\mathcal{G})$, $\alpha \mapsto \alpha_T$, recalled in Section 2.3. Using the natural inclusion $\iota_A : A = A\mathcal{G} \hookrightarrow T\mathcal{G}$, we define a map

$$\mathrm{Lie} : \Omega^k(\mathcal{G}) \longrightarrow \Omega^k(A), \quad \alpha \mapsto \mathrm{Lie}(\alpha) = \iota_A^* \alpha_T. \tag{4.3}$$

Given $\alpha \in \Omega^k(\mathcal{G})$, let us consider the associated bundle maps $\mu : A \longrightarrow \wedge^{k-1} T^*M$ and $\nu : A \longrightarrow \wedge^k T^*M$,

$$\langle \mu(u), X_1 \wedge \ldots \wedge X_{k-1} \rangle = \alpha(u, X_1, \ldots, X_{k-1}), \tag{4.4}$$

$$\langle \nu(u), X_1 \wedge \ldots \wedge X_k \rangle = \mathrm{d}\alpha(u, X_1, \ldots, X_k), \tag{4.5}$$

for $X_1, \ldots, X_k \in TM$ and $u \in A$ (here we use the natural inclusions $TM \hookrightarrow T\mathcal{G}|_M$ and $A \hookrightarrow T\mathcal{G}|_M$).

Lemma 4.2. *The k-form $\mathrm{Lie}(\alpha) \in \Omega^k(A)$ is linear and satisfies*

$$\mathrm{Lie}(\alpha) = \mathrm{d}\Lambda_\mu + \Lambda_\nu.$$

Proof. Let $\beta \in \Omega^l(\mathcal{G})$ be any l-form on $\mathcal{G}$, and let us consider the $l-1$-form on A given by $\iota_A^* \tau(\beta)$ (see (2.13)), i.e.,

$$\begin{aligned} \iota_A^* \tau(\beta)|_u = (T\iota_A|_u)^t \tau(\beta)|_{\iota_A(u)} &= (T\iota_A|_u)^t (Tp_{\mathcal{G}}|_{\iota_A(u)})^t i_{\iota_A(u)} \beta \\ &= (T(p_{\mathcal{G}} \circ \iota_A)|_u)^t i_{\iota_A(u)} \beta. \end{aligned}$$

From the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & T\mathcal{G} \\ {\scriptstyle q_A}\downarrow & & \downarrow{\scriptstyle p_{\mathcal{G}}} \\ M & \xrightarrow[\epsilon]{} & \mathcal{G}, \end{array}$$

we see that

$$\iota_A^* \tau(\beta)|_u = (Tq_A|_u)^t (T\epsilon|_{q_A(u)})^t i_{\iota_A(u)} \beta.$$

It immediately follows (see (2.9)) that

$$\iota_A^* \tau(\alpha) = \Lambda_\mu, \quad \iota_A^* \tau(\mathrm{d}\alpha) = \Lambda_\nu.$$

Using (2.15), we see that

$$\Lambda = \iota_A^*(\mathrm{d}\tau(\alpha) + \tau(\mathrm{d}\alpha)) = \mathrm{d}\iota_A^*\tau(\alpha) + \iota_A^*\tau(\mathrm{d}\alpha) = \mathrm{d}\Lambda_\mu + \Lambda_\nu.$$

□

Recall that any groupoid morphism $\psi : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ defines a Lie algebroid morphism $\mathrm{Lie}(\psi) : A\mathcal{G}_1 \longrightarrow A\mathcal{G}_2$ that fits into the diagram

(4.6)
$$\begin{array}{ccc} T\mathcal{G}_1 & \xrightarrow{T\psi} & T\mathcal{G}_2 \\ {\scriptstyle \iota_{A_1}}\uparrow & & \uparrow{\scriptstyle \iota_{A_2}} \\ A\mathcal{G}_1 & \xrightarrow[\mathrm{Lie}(\psi)]{} & A\mathcal{G}_2 \end{array}$$

When $\alpha \in \Omega^k(\mathcal{G})$ is multiplicative, we saw in Lemma 4.1 that $\overline{\alpha} : \oplus_{\mathcal{G}}^k T\mathcal{G} \longrightarrow \mathbb{R}$ is a groupoid morphism; we consider its infinitesimal counterpart,

$$\mathrm{Lie}(\overline{\alpha}) : A(\oplus_{\mathcal{G}}^k T\mathcal{G}) \longrightarrow \mathbb{R},$$

where now $\mathbb{R}$ is viewed as the trivial Lie algebroid over a point. The natural projection $p_{\mathcal{G}} : T\mathcal{G} \longrightarrow \mathcal{G}$ is a groupoid morphism, and there is a canonical identification of Lie algebroids[3]

$$A(\oplus_{\mathcal{G}}^k T\mathcal{G}) = \prod_{\mathrm{Lie}(p_{\mathcal{G}})}^{k} A(T\mathcal{G}).$$

Our next goal is to compare the following two maps:

$$\overline{\mathrm{Lie}(\alpha)} : \oplus_A^k T(A\mathcal{G}) \longrightarrow \mathbb{R} \quad \text{and} \quad \mathrm{Lie}(\overline{\alpha}) : \prod_{\mathrm{Lie}(p_{\mathcal{G}})}^{k} A(T\mathcal{G}) \longrightarrow \mathbb{R}.$$

The involution $J_{\mathcal{G}} : T(T\mathcal{G}) \longrightarrow T(T\mathcal{G})$ (see (2.16)) defines an identification of Lie algebroids $j_{\mathcal{G}} : T(A\mathcal{G}) \longrightarrow A(T\mathcal{G})$ via the diagram

(4.7)
$$\begin{array}{ccc} T(A\mathcal{G}) & \xrightarrow{j_{\mathcal{G}}} & A(T\mathcal{G}) \\ {\scriptstyle T\iota_{A\mathcal{G}}}\downarrow & & \downarrow{\scriptstyle \iota_{A(T\mathcal{G})}} \\ T(T\mathcal{G}) & \xrightarrow[J_{\mathcal{G}}]{} & T(T\mathcal{G}) \end{array}$$

Note that the property $Tp_{\mathcal{G}} \circ J_{\mathcal{G}} = p_{T\mathcal{G}}$ implies that

$$\mathrm{Lie}(p_{\mathcal{G}}) \circ j_{\mathcal{G}} = p_A.$$

As a result, we have a natural identification of Lie algebroids,

(4.8)
$$j_{\mathcal{G}}^{(k)} : \oplus_A^k TA = \prod_{p_A}^{k} T(A\mathcal{G}) \xrightarrow{\sim} \prod_{\mathrm{Lie}(p_{\mathcal{G}})}^{k} A(T\mathcal{G}),$$

[3]The Lie algebroid $A(T\mathcal{G}) \to TM$ is a $\mathcal{VB}$-algebroid with respect to the vector bundle structure $\mathrm{Lie}(p_{\mathcal{G}}) : A(T\mathcal{G}) \to A$; the algebroid structure on $A(T\mathcal{G})$ can be extended to $\prod_{\mathrm{Lie}(p_{\mathcal{G}})}^k A(T\mathcal{G})$ in terms of core and linear sections, just as described in Section 3.2.

fitting into the diagram

$$
\begin{array}{ccc}
\prod^k_{p_A} T(A\mathcal{G}) & \xrightarrow{j^{(k)}_{\mathcal{G}}} & \prod^k_{\mathrm{Lie}(p_{\mathcal{G}})} A(T\mathcal{G}) \\
{\scriptstyle (T\iota_{A\mathcal{G}})^k}\downarrow & & \downarrow{\scriptstyle (\iota_{A(T\mathcal{G})})^k} \\
\prod^k_{p_{T\mathcal{G}}} T(T\mathcal{G}) & \xrightarrow[J^{(k)}_{\mathcal{G}}]{} & \prod^k_{Tp_{\mathcal{G}}} T(T\mathcal{G}).
\end{array} \tag{4.9}
$$

Lemma 4.3. *Let $\alpha \in \Omega^k(\mathcal{G})$ be multiplicative. Then*

$$\mathrm{Lie}(\overline{\alpha}) \circ j^{(k)}_{\mathcal{G}} = \overline{\mathrm{Lie}(\alpha)}. \tag{4.10}$$

In particular, $\overline{\mathrm{Lie}(\alpha)} : \oplus^k_A T(A\mathcal{G}) \longrightarrow \mathbb{R}$ is a Lie algebroid morphism.

Proof. By definition, $\mathrm{Lie}(\overline{\alpha}) = \mathrm{d}\overline{\alpha} \circ (\iota_{A(T\mathcal{G})})^k$, and using (4.9) and (2.17) we obtain

$$
\begin{aligned}
\mathrm{Lie}(\overline{\alpha}) \circ j^{(k)}_{\mathcal{G}} = \mathrm{d}\overline{\alpha} \circ (\iota_{A(T\mathcal{G})})^k \circ j^{(k)}_{\mathcal{G}} &= \mathrm{d}\overline{\alpha} \circ J^{(k)}_{\mathcal{G}} \circ (T\iota_{A\mathcal{G}})^k \\
&= \overline{\alpha_T} \circ (T\iota_{A\mathcal{G}})^k = \overline{\iota^*_A \alpha_T}.
\end{aligned}
$$

□

Proposition 4.4. *Let $\alpha \in \Omega^k(\mathcal{G})$ be multiplicative, and let μ and ν be defined as in* (4.4) *and* (4.5)*. Then (μ, ν) is an IM k-form on $A\mathcal{G}$.*

Proof. The result is a direct consequence of Lemmas 4.2, 4.3, and Theorem 3.1. □

4.2. **Integration of IM forms.** Let $\mathcal{G}$ be a Lie groupoid over M, with Lie algebroid $A = A\mathcal{G}$. Assume that $\mathcal{G}$ is *source-simply-connected* (i.e., the s-fibres are connected with trivial fundamental group), so that $\oplus^k_{\mathcal{G}} T\mathcal{G}$ is also a source-simply-connected groupoid[4]. Let $\Lambda \in \Omega^k(A)$ be a k-form on A for which $\overline{\Lambda} : \oplus^k_A TA \longrightarrow \mathbb{R}$ is a Lie algebroid morphism.

Lemma 4.5. *There is a unique multiplicative k-form $\alpha \in \Omega^k(\mathcal{G})$ such that* $\mathrm{Lie}(\alpha) = \Lambda$ *(see* (4.3)*).*

Proof. Since $\overline{\Lambda}$ is a morphism of Lie algebroids, the identification (4.8) also leads to a Lie algebroid morphism

$$\overline{\Lambda} \circ (j^{(k)}_{\mathcal{G}})^{-1} = \prod^k_{\mathrm{Lie}(p_{\mathcal{G}})} A(T\mathcal{G}) \cong A(\oplus^k T\mathcal{G}) \longrightarrow \mathbb{R}. \tag{4.11}$$

As $\oplus^k_{\mathcal{G}} T\mathcal{G}$ is a source-simply-connected groupoid, we can use Lie's second theorem (see e.g. [20]) to obtain a unique groupoid morphism

$$I_\Lambda : \oplus^k_{\mathcal{G}} T\mathcal{G} \longrightarrow \mathbb{R} \tag{4.12}$$

[4] Given any $X = (X_1, \ldots, X_k) \in \oplus^k T_x M$, the projection $(p_{\mathcal{G}})^k : \oplus^k_{\mathcal{G}} T\mathcal{G} \longrightarrow \mathcal{G}$ makes the source fibre $((T\mathsf{s})^k)^{-1}(X) \subseteq T\mathcal{G}$ into an affine bundle over the source fibre $\mathsf{s}^{-1}(x) \subseteq \mathcal{G}$

integrating the morphism (4.11), i.e., such that $\mathrm{Lie}(I_\Lambda) = \overline{\Lambda} \circ (j_{\mathcal{G}}^{(k)})^{-1}$. To check that $I_\Lambda = \overline{\alpha}$, for $\alpha \in \Omega^k(\mathcal{G})$, it suffices to verify that the following conditions hold:

$$I_\Lambda(U_1, \ldots, U_i, \ldots, U_j, \ldots, U_k) = - I_\Lambda(U_1, \ldots, U_j, \ldots, U_i, \ldots, U_k), \tag{4.13}$$

$$I_\Lambda(U_1, \ldots, U_{i-1}, cU_i, U_{i+1}, \ldots, U_k) = cI_\Lambda(U_1, \ldots, U_k), \tag{4.14}$$

$$\begin{aligned} I_\Lambda(U_1, \ldots, U_{i-1}, U_i + U_i', U_{i+1}, \ldots, U_k) =& I_\Lambda(U_1, \ldots, U_i, \ldots, U_k) + \\ & I_\Lambda(U_1, \ldots, U_i', \ldots, U_k), \end{aligned} \tag{4.15}$$

for all $U_i, U_i' \in T_g\mathcal{G}$, $g \in \mathcal{G}$, $c \in \mathbb{R}$, where $1 \leq i < j \leq k$. As we now show, all conditions can be verified with the same type arguments (cf. [21]).

To prove that (4.13) holds, one directly checks that the map $I_\Lambda^{(ij)} : \oplus_{\mathcal{G}}^k T\mathcal{G} \longrightarrow \mathbb{R}$,

$$I_\Lambda^{(ij)}(U_1, \ldots, U_i, \ldots, U_j, \ldots, U_k) := -I_\Lambda(U_1, \ldots, U_j, \ldots, U_i, \ldots, U_k),$$

is a groupoid morphism, and $\mathrm{Lie}(I_\Lambda^{(ij)}) : \prod_{\mathrm{Lie}(p_{\mathcal{G}})}^k A(T\mathcal{G}) \longrightarrow \mathbb{R}$ satisfies

$$\begin{aligned} \mathrm{Lie}(I_\Lambda^{(ij)})(V_1, \ldots, V_i, \ldots, V_j, \ldots, V_k) &= -\mathrm{Lie}(I_\Lambda)(V_1, \ldots, V_j, \ldots, V_i, \ldots, V_k) \\ &= -\Lambda \circ (j_{\mathcal{G}}^{(k)})^{-1}(V_1, \ldots, V_j, \ldots, V_i, \ldots, V_k) \\ &= \mathrm{Lie}(I_\Lambda)(V_1, \ldots, V_i, \ldots, V_j, \ldots, V_k), \end{aligned}$$

since Λ is skew-symmetric. So $\mathrm{Lie}(I_\Lambda^{(ij)}) = \mathrm{Lie}(I_\Lambda)$, and the uniqueness of integration in Lie's second theorem implies that $I_\Lambda^{(ij)} = I_\Lambda$, which is (4.13).

Similarly, for a fixed $c \in \mathbb{R}$, one can directly show that both the left and right-hand sides of (4.14) define groupoid morphisms $\oplus_{\mathcal{G}}^k T\mathcal{G} \longrightarrow \mathbb{R}$, whose infinitesimal counterparts agree at the level of Lie algebroids due to the multilinearity of Λ. Then (4.14) follows again by the uniqueness part of Lie's second theorem.

The last condition (4.15) can be treated in a completely analogous way, by first noticing that both sides of (4.15) define groupoid morphisms $\oplus_{\mathcal{G}}^{k+1} T\mathcal{G} \longrightarrow \mathbb{R}$, where now we need an extra copy of $T\mathcal{G}$ for U_i'. Again, these morphisms agree at the infinitesimal level due to the multilinearity of Λ, and hence agree globally.

The fact that α is multiplicative follows from Lemma 4.1, and the equality $\Lambda = \mathrm{Lie}(\alpha)$ is a consequence of Lemma 4.3. ☐

A direct consequence of Lemmas 4.3 and 4.5 is that the map

$$\Omega_{\mathrm{mult}}^k(\mathcal{G}) \to \Omega^k(A), \quad \alpha \mapsto \mathrm{Lie}(\alpha),$$

is a bijection onto the subspace of k-forms $\Lambda \in \Omega^k(A)$ such that $\overline{\Lambda} : \oplus_A^k TA \to \mathbb{R}$ is a morphism of Lie algebroids. By the correspondence in Theorem 3.1, this bijection can be alternatively phrased in terms of IM-forms on A:

Theorem 4.6. *Let $\mathcal{G}$ be a source-simply-connected Lie groupoid over M with Lie algebroid $A \longrightarrow M$. For each positive integer k, there is a 1-1 correspondence*

$$\Omega_{\mathrm{mult}}^k(\mathcal{G}) \longrightarrow \Omega_{\mathrm{IM}}^k(A), \;\; \alpha \mapsto (\mu, \nu), \tag{4.16}$$

where μ, ν are given by

$$\langle \mu(u), X_1 \wedge \ldots \wedge X_{k-1} \rangle = \alpha(u, X_1, \ldots, X_{k-1}), \tag{4.17}$$

$$\langle \nu(u), X_1 \wedge \ldots \wedge X_k \rangle = \mathrm{d}\alpha(u, X_1, \ldots, X_k). \tag{4.18}$$

Proof. The result follows from Lemma 4.2 and Theorem 3.1. ☐

The following is a simple example of correspondence in Theorem 4.6.

Example 4.7. *Let us equip $A = T^*M \to M$ with the trivial Lie algebroid structure (both anchor and bracket are identically zero), so we may identify $\mathcal{G} = T^*M$ (with groupoid multiplication given by fibrewise addition). Fixing $\mu = \mathrm{Id} : T^*M \to T^*M$, then any vector-bundle map $\nu : T^*M \to \wedge^2 T^*M$ defines an IM 2-form (μ, ν). When $\nu = 0$, then (μ, ν) corresponds under (4.16) to the canonical symplectic form ω_{can} on $\mathcal{G} = T^*M$; for an arbitrary ν, the corresponding multiplicative 2-form is given, at each $g = (q^j, p_j) \in T^*M$, by*

$$\omega|_g = \omega_{can}|_g + c_M^*\nu(g)$$

where $c_M : T^*M \to M$ is the natural projection.

Let us list some immediate consequences of Theorem 4.6, illustrating how the correspondence (4.16) restricts to subclasses of multiplicative and IM forms:

(a) Let $\eta \in \Omega^k(M)$. Following Example 3.3, we know that (μ, ν), where $\mu(u) = -i_{\rho(u)}\eta$ and $\nu(u) = -i_{\rho(u)}\mathrm{d}\eta$, defines an IM k-form. One directly verifies that the corresponding multiplicative k-form is $\alpha = \mathsf{s}^*\eta - \mathsf{t}^*\eta$.

(b) Let $\phi \in \Omega^{k+1}(M)$ be a closed $k+1$-form. Then Theorem 4.6 gives a bijective correspondence between IM k-forms on A relative to ϕ (i.e., $\nu(u) = -i_{\rho(u)}\phi$, see Example 3.4) and multiplicative k-forms α satisfying $\mathrm{d}\alpha = \mathsf{s}^*\phi - \mathsf{t}^*\phi$. To verify this fact, just notice that $\mathrm{d}\alpha$ is a multiplicative $(k+1)$-form corresponding to an IM $(k+1)$-form of the type discussed in item (a). This recovers [5, Thm. 2.5] when $k = 2$ (cf. [3]), as well as [1, Thm. 2] when k is an arbitrary positive integer.

(c) Let $\alpha \in \Omega^k_{\mathrm{mult}}(\mathcal{G})$ be a given closed multiplicative k-form, with associated IM k-form (μ_α, ν_α) (note that $\nu_\alpha = 0$, necessarily). It follows from Theorem 4.6 that there is a 1-1 correspondence between multiplicative $(k-1)$-forms θ with $\mathrm{d}\theta = \alpha$ and vector-bundle maps $\mu : A \to \wedge^{k-2}T^*M$ satisfying, for all $u, v \in \Gamma(A)$,

$$i_{\rho(u)}\mu(v) = -i_{\rho(v)}\mu(u), \quad \mu([u,v]) = \mathcal{L}_{\rho(u)}\mu(v) - i_{\rho(v)}d\mu(u) - i_{\rho(v)}\mu_\alpha(u).$$

The reason is that (μ, μ_α) is the IM $(k-1)$-form associated with θ (note that μ_α satisfies (3.18) as a result of (μ_α, ν_α) being an IM k-form for α and ν_α being zero). This correspondence is the content of [1, Thm. 3].

For $k = 2$, one has further refinements of Theorem 4.6 based on the general study of multiplicative 2-forms carried out in [5, Section 4], leading to natural generalizations of twisted Poisson and Dirac structures (in the sense of [26]). On the vector bundle $TM \oplus T^*M$, consider the pairing $\langle (X,\alpha), (Y,\beta) \rangle = \beta(X) + \alpha(Y)$, and the natural projections $pr_T : TM \oplus T^*M \to TM$ and $pr_{T^*} : TM \oplus T^*M \to T^*M$. As in Theorem 4.6, we denote by $\mathcal{G}$ a source-simply-connected groupoid with Lie algebroid $A \to M$.

(d) Given an IM 2-form (μ, ν) on A, we consider the vector-bundle map $(\rho, \mu) : A \to TM \oplus T^*M$ and its image

$$L = \{(\rho(u), \mu(u)) \mid u \in A\} \subset TM \oplus T^*M, \tag{4.19}$$

which is a subbundle whenever it has constant rank. By (3.16), over each point of M, L is isotropic with respect to the pairing $\langle \cdot, \cdot \rangle$. It follows from

[5, Cor. 4.8] that, under the assumption that $\dim(\mathcal{G}) = 2\dim(M)$, the correspondence (4.16) restricts to a bijection between multiplicative 2-forms ω such that

$$\ker(T\mathsf{s})_x \cap \ker(\omega)_x \cap \ker(T\mathsf{t})_x = \{0\}, \quad \forall x \in M, \tag{4.20}$$

and IM 2-forms (μ, ν) for which $L = L^\perp$ (i.e., L is *lagrangian* with respect to $\langle\cdot,\cdot\rangle$; in particular, it is a subbundle with $\mathrm{rank}(L) = \dim(M)$) and

$$(\rho, \mu) : A \longrightarrow L \subset TM \oplus T^*M \tag{4.21}$$

is an isomorphism of vector bundles. Moreover, $\mathsf{t} : \mathcal{G} \to M$ relates ω and L as a *forward Dirac map* (see, e.g., [5, Sec. 2.1]). If we define $\nu_L : L \to \wedge^2 T^*M$ by $\nu_L((\rho(u), \mu(u))) = \nu(u)$, then the identification (4.21) induces a Lie algebroid structure on L with anchor $pr_T|_L$ and bracket on $\Gamma(L)$ given by

$$[(X,\alpha),(Y,\beta)]_L := ([X,Y], \mathcal{L}_X\beta - i_Y d\alpha - i_Y\nu_L(X,\alpha)). \tag{4.22}$$

Conversely, if a lagrangian subbundle $L \subset TM \oplus T^*M$ is equipped with $\nu_L : L \to \wedge^2 T^*M$ for which (4.22) is a Lie bracket on $\Gamma(L)$, then $A = L$ is a Lie algebroid with anchor $pr_T|_L$; if ν_L also satisfies (3.18), then $(pr_{T^*}|_L, \nu_L)$ is an IM 2-form. Then, under the bijection (4.16), $(pr_{T^*}|_L, \nu_L)$ corresponds to a multiplicative 2-form ω on $\mathcal{G}$ satisfying (4.20). Taking ν_L to be of type $\nu_L(X,\alpha) = -i_X\phi$ for a closed 3-form $\phi \in \Omega^3(M)$, we recover the integration of twisted Dirac structures by twisted presymplectic groupoids of [5, Sec. 2].

(e) When a multiplicative 2-form is nondegenerate, then $\dim(\mathcal{G}) = 2\dim(M)$ automatically (see, e.g., [5, Lem. 3.3]), and (4.20) trivially holds. Under (4.16), this situation corresponds to the case where the IM 2-form (μ,ν) is such that $\mu : A \to T^*M$ is an isomorphism; in other words, L is the graph of a bivector field π on M: $L = \{(i_\alpha\pi, \alpha) \mid \alpha \in T^*M\}$. Following (d) above, the fact that $\Gamma(L)$ is closed under the bracket (4.22) is expressed by the compatibility condition

$$\frac{1}{2}[\pi,\pi](\alpha,\beta,\gamma) = \nu_L((i_\alpha\pi,\alpha))(i_\beta\pi, i_\gamma\pi)$$

for all $\alpha,\beta,\gamma \in \Omega^1(M)$. When ν_L is of type $\nu_L(X,\alpha) = -i_X\phi$ for a closed 3-form $\phi \in \Omega^3(M)$, one recovers twisted Poisson structures, and (4.16) gives their integration to twisted symplectic groupoids (cf. [8]).

5. Relation with the Weil algebra and Van Est isomorphism

This section clarifies how linear and IM-forms on Lie algebroids fit into the Weil algebra of [1, Sec. 3], and how the infinitesimal description of multiplicative forms relates to the general Van Est isomorphism of [1, Sec. 4].

Let A be a Lie algebroid over M. We consider the associated *Weil algebra* $W(A)$ as in [1, Sec. 3], which is a bi-graded differential algebra. The space of elements of degree (p,k) is denoted by $W^{p,k}(A)$, and the differential on $W(A)$ is a sum of differentials $d^h + d^v$, where

$$d^h : W^{p,k}(A) \longrightarrow W^{p+1,k}(A), \quad d^v : W^{p,k}(A) \longrightarrow W^{p,k+1}(A).$$

We will be mostly concerned with $W^{p,k}(A)$ for $p = 0,1,2$.

For $p = 0$, we have $W^{0,k}(A) = \Omega^k(M)$. An element $\Lambda_W \in W^{1,k}(A)$ is given by a pair $((\Lambda_W)_0, (\Lambda_W)_1)$, where

$$(\Lambda_W)_0 : \Gamma(A) \longrightarrow \Omega^k(M), \quad (\Lambda_W)_1 \in \Omega^{k-1}(M, A^*), \tag{5.1}$$

subject to the compatibility condition

$$(\Lambda_W)_0(fu) = f(\Lambda_W)_0(u) - df \wedge (\Lambda_W)_1(u), \tag{5.2}$$

for $f \in C^\infty(M)$, $u \in \Gamma(A)$, and $(\Lambda_W)_1$ viewed as a $C^\infty(M)$-linear map

$$(\Lambda_W)_1 : \Gamma(A) \longrightarrow \Omega^{k-1}(M). \tag{5.3}$$

An element $c_W \in W^{2,k}(A)$ is a triple $((c_W)_0, (c_W)_1, (c_W)_2)$, where

$$\begin{aligned} &(c_W)_0 : \Gamma(A) \times \Gamma(A) \longrightarrow \Omega^k(M), \ \ (c_W)_1 : \Gamma(A) \longrightarrow \Omega^{k-1}(M, A^*), \\ &(c_W)_2 \in \Omega^{k-1}(M, S^2(A^*)), \end{aligned} \tag{5.4}$$

and such that $(c_W)_0$ is skewsymmetric and $\mathbb{R}$-bilinear, subject to suitable compatibility conditions (extending (5.2)) that we will not need explicitly.

We need to recall the expression for d^h restricted to $W^{1,k}(A)$. By definition (see [1, Sec. 3.1]), for $\Lambda_W = ((\Lambda_W)_0, (\Lambda_W)_1) \in W^{1,k}(A)$, $d^h\Lambda_W \in W^{2,k}(A)$ is given by (cf. (5.4))

$$(d^h\Lambda_W)_0(u, v) = -(\Lambda_W)_0([u, v]) + \mathcal{L}_{\rho(u)}((\Lambda_W)_0(v)) - \mathcal{L}_{\rho(v)}((\Lambda_W)_0(u)), \tag{5.5}$$

$$(d^h\Lambda_W)_1(u)(v) = \mathcal{L}_{\rho(u)}((\Lambda_W)_1(v)) - (\Lambda_W)_1([u, v]) + i_{\rho(v)}((\Lambda_W)_0(u)), \tag{5.6}$$

$$(d^h\Lambda_W)_2(u) = -i_{\rho(u)}((\Lambda_W)_1(u)), \tag{5.7}$$

for $u, v \in \Gamma(A)$.

We also need the expression for d^v in the following particular situation. Any bundle map $\mu : A \to \wedge^k T^*M$ (equivalently seen as a $C^\infty(M)$-linear map $\mu : \Gamma(A) \longrightarrow \Omega^k(M)$) defines an element $\mu_W \in W^{1,k}(A)$ by

$$(\mu_W)_0 = \mu, \quad (\mu_W)_1 = 0. \tag{5.8}$$

In this case, $d^v(\mu_W) \in W^{1,k+1}(A)$ is defined by (see [1, Sec. 3.1])

$$(d^v\mu_W)_0(u) = -\mathrm{d}\mu(u), \quad (d^v\mu_W)_1 = \mu. \tag{5.9}$$

Proposition 5.1. *Consider the map* $\psi : \Omega^k_{\mathrm{lin}}(A) \longrightarrow W^{1,k}(A)$, $k = 1, 2, \ldots$,

$$\Lambda = \mathrm{d}\Lambda_\mu + \Lambda_\nu \ \mapsto \ \Lambda_W := -d^v\mu_W + \nu_W.$$

The following holds:

(1) ψ *induces a* $C^\infty(M)$*-linear isomorphism* $\Omega^\bullet_{\mathrm{lin}}(A) \xrightarrow{\sim} W^{1,\bullet}(A)$.
(2) $\psi \circ \mathrm{d} = -d^v \circ \psi$.
(3) ψ *restricts to a linear isomorphism* $\Omega^k_{\mathrm{IM}}(A) \xrightarrow{\sim} \ker(d^h|_{W^{1,k}(A)})$.

Proof. It is clear from (5.8) and (5.9) that the map ψ is injective. Let us check that any $\Lambda_W \in W^{1,k}(A)$ can be written in the form $-d^v\mu_W + \nu_W$ for $C^\infty(M)$-linear maps $\mu : \Gamma(A) \longrightarrow \Omega^{k-1}(M)$, $\nu : \Gamma(A) \longrightarrow \Omega^k(M)$. Let us write $\Lambda_W = ((\Lambda_W)_0, (\Lambda_W)_1)$, and set $\mu = -(\Lambda_W)_1$. Then $(d^v\mu_W)_1 = -(\Lambda_W)_1$, so the element $c = d^v\mu_W + \Lambda_W \in W^{1,k}(A)$ is such that $c_1 = 0$, which implies that $c = \nu_W$ for a bundle map $\nu : A \longrightarrow \wedge^k T^*M$. The $C^\infty(M)$-linearity of ψ results from the following properties: $f\mathrm{d}\Lambda_\mu = \mathrm{d}\Lambda_{f\mu} - \Lambda_{\mathrm{d}f\wedge\mu}$ and $d^v(f\mu)_W = fd^v\mu_W - (\mathrm{d}f \wedge \mu)_W$. Hence (1) is proven.

To prove (2), writing $\Lambda = \mathrm{d}\Lambda_\mu + \Lambda_\nu$, we have $\mathrm{d}\Lambda = \mathrm{d}\Lambda_\nu$. By definition of ψ, it follows that $\psi(\mathrm{d}\Lambda) = -d^v\nu_W$. On the other hand, $-d^v(\psi(\Lambda)) = -d^v(-d^v\mu_W+\nu_W) = -d^v\nu_W$, hence (2) holds.

For (3), we must consider the condition $d^h\Lambda_W = 0$. Written in terms of its components (5.5), (5.6), and (5.7), we obtain three equations involving $(\Lambda_W)_0$ and $(\Lambda_W)_1$, which must be shown to agree with conditions (3.16), (3.17) and (3.18) in Thm. 3.1. Using (5.8), (5.9), we see that

$$(\Lambda_W)_0(u) = \mathrm{d}\mu(u) + \nu(u), \qquad (\Lambda_W)_1(u) = -\mu(u) \tag{5.10}$$

for all $u \in \Gamma(A)$, and it is clear that (5.7) and (5.6) coincide with conditions (3.16) and (3.17), respectively.

For the degree-0 condition (5.5), using (5.10) and (3.17), we find

$$\nu([u,v]) = \mathrm{d}i_{\rho(v)}d\mu(u) + \mathrm{d}i_{\rho(v)}\nu(u) + \mathcal{L}_{\rho(u)}\nu(v) - \mathcal{L}_{\rho(v)}\mathrm{d}\mu(u) - \mathcal{L}_{\rho(v)}\nu(u).$$

Using Cartan's formula $\mathcal{L}_X = i_X\mathrm{d} + \mathrm{d}i_X$, one directly verifies that the last equation agrees with (3.18). □

Let $\mathcal{G}$ be a source-simply-connected Lie groupoid over M, with Lie algebroid $A \longrightarrow M$. There is a double complex $\Omega^k(\mathcal{G}^{(p)})$ associated to $\mathcal{G}$, known as the *Bott-Shulman complex*, see [2]. It is equipped with a differential $\partial : \Omega^k(\mathcal{G}^{(p)}) \longrightarrow \Omega^k(\mathcal{G}^{(p+1)})$, as well as the de Rham differential $\mathrm{d} : \Omega^k(\mathcal{G}^{(p)}) \longrightarrow \Omega^{k+1}(\mathcal{G}^{(p)})$. The *Van Est isomorphism* constructed in [1] relates the cohomologies of $\Omega^k(\mathcal{G}^{(p)})$ and $W^{p,k}(A)$. We will only need a few results of the theory, for $p = 0, 1$.

For $p = 0$, $\Omega^k(\mathcal{G}^{(0)}) = \Omega^k(M) = W^{0,k}(A)$, and

$$\partial : \Omega^k(M) \longrightarrow \Omega^k(\mathcal{G}), \qquad \partial(\eta) = \mathsf{t}^*\eta - \mathsf{s}^*\eta.$$

For $p = 1$, the differential $\partial : \Omega^k(\mathcal{G}) \longrightarrow \Omega^k(\mathcal{G}^{(2)})$ is

$$\partial(\alpha) = pr_1^*\alpha - m^*\alpha + pr_2^*\alpha,$$

and the Van Est map of [1, Sec. 4] restricts to a map

$$\mathbb{V} : \Omega^k_{\mathrm{mult}}(\mathcal{G}) = \ker(\partial|_{\Omega^k(\mathcal{G})}) \longrightarrow \ker(d^h|_{W^{1,k}(A)}) \subset W^{1,k}(A), \tag{5.11}$$

given by

$$\mathbb{V}(\alpha)_0(u) = \epsilon^*(\mathrm{d}i_u\alpha + i_u\mathrm{d}\alpha), \qquad \mathbb{V}(\alpha)_1(u) = -\epsilon^*(i_u\alpha) \tag{5.12}$$

where $\alpha \in \Omega^k_{\mathrm{mult}}(\mathcal{G})$, $u \in \Gamma(A)$ (we view A as a subbundle of $T\mathcal{G}|_M$) and $\epsilon : M \to \mathcal{G}$ is the unit map of $\mathcal{G}$. The map $\mathbb{V}$ satisfies

$$\mathbb{V} \circ \mathrm{d} = -d^v \circ \mathbb{V}, \quad \mathbb{V}(\partial(\eta)) = d^h\eta, \tag{5.13}$$

for $\eta \in \Omega^k(M)$. The general Van Est isomorphism of [1] implies that the induced map

$$\frac{\Omega^k_{\mathrm{mult}}(\mathcal{G})}{\mathrm{Im}(\partial|_{\Omega^k(M)})} \longrightarrow \frac{\ker(d^h|_{W^{1,k}(A)})}{\mathrm{Im}(d^h|_{\Omega^k(M)})} \tag{5.14}$$

is a bijection. In this specific situation, a stronger fact holds.

Proposition 5.2. *The map* $\mathbb{V} : \Omega^k_{\mathrm{mult}}(\mathcal{G}) \longrightarrow \ker(d^h|_{W^{1,k}(A)})$ *is a bijection.*

The proof of the proposition uses the following observation (cf. [1, Sec. 6]).

Lemma 5.3. *Let* $\sigma \in W^{1,k}(A)$*,* $\omega \in \Omega^{k+1}_{\mathrm{mult}}(\mathcal{G})$ *be such that* $d^h\sigma = 0$ *and* $\mathbb{V}(\omega) = -d^v\sigma$*. Then there exists a unique* $\beta \in \Omega^k_{\mathrm{mult}}(\mathcal{G})$ *such that*

$$\mathbb{V}(\beta) = \sigma, \quad \mathrm{d}\beta = \omega.$$

Proof. The key fact to prove the lemma is shown in [1, Lem. 6.3]: for a closed k-form $\alpha \in \Omega^k_{\mathrm{mult}}(\mathcal{G})$, $\mathbb{V}(\alpha) = 0$ if and only if $\alpha = 0$. As an application, we see that ω is necessarily closed, since $\mathbb{V}(\mathrm{d}\omega) = d^v(d^v\sigma) = 0$.

Since $d^h\sigma = 0$, the isomorphism (5.14) implies that there exists $\widetilde{\beta} \in \Omega^k_{\mathrm{mult}}(\mathcal{G})$ such that $\mathbb{V}(\widetilde{\beta}) = \sigma + d^h\eta$. If $\beta = \widetilde{\beta} - \partial\eta$, then by (5.13) we have

$$\mathbb{V}(\beta) = \mathbb{V}(\widetilde{\beta}) - \mathbb{V}(\partial\eta) = \sigma.$$

To conclude that $\mathrm{d}\beta = \omega$, note that $\mathrm{d}\beta - \omega$ is multiplicative, closed, and $\mathbb{V}(\mathrm{d}\beta - \omega) = -d^v\sigma + d^v\sigma = 0$. ☐

We can now prove the proposition.

Proof. (of Prop. 5.2) Let us fix $\xi \in W^{1,k}(A)$, $d^h\xi = 0$. Let $\sigma = -d^v\xi$. Then $d^v\sigma = 0$, $d^h\sigma = 0$, and Lemma 5.3 implies that there exists a unique $\beta \in \Omega^{k+1}_{\mathrm{mult}}(\mathcal{G})$ such that

$$\mathbb{V}(\beta) = \sigma, \quad \mathrm{d}\beta = 0. \tag{5.15}$$

Since $\mathbb{V}(\beta) = -d^v\xi$, and, by assumption, $d^h\xi = 0$, we can apply Lemma 5.3 to conclude that there exists a unique $\theta \in \Omega^k_{\mathrm{mult}}(\mathcal{G})$ such that $\mathbb{V}(\theta) = \xi$ and $\mathrm{d}\theta = \beta$. But notice that the condition $\mathrm{d}\theta = \beta$ is automatically satisfied if $\mathbb{V}(\theta) = \xi$, since $\mathbb{V}(\mathrm{d}\theta) = \sigma$ and the conditions in (5.15) determine β uniquely. ☐

Composing the bijection (5.11) with the identification $\Omega^k_{\mathrm{IM}}(A) \cong \ker(d^h|_{W^{1,k}}(A))$ of (3) in Prop. 5.1, we obtain a bijection

$$\Omega^k_{\mathrm{mult}}(\mathcal{G}) \xrightarrow{\sim} \Omega^k_{\mathrm{IM}}(A).$$

Using (5.10) and (5.12), we see that this bijection is explicitly given by $\alpha \mapsto (\mu, \nu)$, where μ, ν are defined as in (4.4), (4.5), hence agreeing with Theorem 4.6.

6. The dual picture: multiplicative multivector fields

In this section, we illustrate how the techniques used in the paper to study infinitesimal versions of multiplicative forms can be equally applied to multiplicative multivector fields.

We keep the notation introduced in Section 2.1. We focus on the cotangent bundle $c_A : T^*A \to A$ of a vector bundle $A \to M$, described in local coordinates (x^j, u^d, p_j, ξ_d), where (x^j, u^d) are relative to a basis of local sections $\{e_d\}$ of A. The local coordinates on A^* relative to the dual basis $\{e^d\}$ are denoted by (x^j, ξ_d); recall from (2.1) that we have a vector-bundle structure $r : T^*A \to A^*$, $(x^j, u^d, p_j, \xi_d) \mapsto (x^j, \xi_d)$. As in Section 2.2, we also consider the k-fold direct sum $\oplus^k_A T^*A$, described by local coordinates $(x^j, u^d, p^1_j, \dots, p^k_j, \xi^1_d, \dots, \xi^k_d)$, as a vector bundle over $\oplus^k A^*$, with projection map $(x^j, u^d, p^1_j, \dots, p^k_j, \xi^1_d, \dots, \xi^k_d) \mapsto (x^j, \xi^1_d, \dots, \xi^k_d)$.

As in Section 3.1, we will need special sections of the bundle $\oplus^k_A T^*A \to \oplus^k A^*$. For the bundle $T^*A \longrightarrow A^*$, we consider local sections

$$\mathrm{d}\widehat{x}^i(x^j, \xi_d) = (x^j, 0, \delta^i_j, \xi_d), \quad e^L_a(x^j, \xi_d) = (x^j, \delta^d_a, 0, \xi_d), \tag{6.1}$$

which are core and linear sections, respectively; these sections generate the module of local sections of $T^*A \longrightarrow A^*$, and the projection $T^*A \longrightarrow A$ maps core sections to the zero section of $A \longrightarrow M$ and linear sections e_a^L to the section e_a. More generally, local sections of $\oplus_A^k T^*A \to \oplus^k A^*$ are generated by sections of types

$$\mathrm{d}\widehat{x}^{i,n}(\xi^1 \oplus \ldots \oplus \xi^k) = 0(\xi^1) \oplus \ldots 0(\xi^{n-1}) \oplus \mathrm{d}\widehat{x}^i(\xi^n) \oplus 0(\xi^{n+1}) \ldots \oplus 0(\xi^k), \tag{6.2}$$

$$(e_a^L)^k(\xi^1 \oplus \ldots \oplus \xi^k) = e_a^L(\xi^1) \oplus \ldots \oplus e_a^L(\xi^k), \tag{6.3}$$

where $\xi^1 \oplus \ldots \oplus \xi^k \in \oplus^k A^*$ and $0 : A^* \to T^*A$, $0(x^j, \xi_d) = (x^j, 0, 0, \xi_d)$, is the zero section. For each k, we will use these sections to express the natural Lie algebroid structure on $\oplus_A^k T^*A \to \oplus^k A^*$, similarly to Section 3.2.

Using the notation in (3.4), the defining relations for the cotangent Lie algebroid structure on $T^*A \to A^*$ are

$$[\mathrm{d}\widehat{x}^i, \mathrm{d}\widehat{x}^j]_{T^*A} = 0, \tag{6.4}$$

$$[e_a^L, \mathrm{d}\widehat{x}^j]_{T^*A} = \frac{\partial \rho_a^j}{\partial x^i}\mathrm{d}\widehat{x}^i, \quad [e_a^L, e_b^L]_{T^*A} = -\frac{\partial C_{ab}^c}{\partial x^i}\xi_c \mathrm{d}\widehat{x}^i + C_{ab}^c e_c^L, \tag{6.5}$$

$$\rho_{T^*A}(\mathrm{d}\widehat{x}^i) = \rho_d^i \frac{\partial}{\partial \xi_d}, \quad \rho_{T^*A}(e_a^L) = \rho_a^j \frac{\partial}{\partial x^j} + C_{ab}^c \xi_c \frac{\partial}{\partial \xi_b}. \tag{6.6}$$

This Lie algebroid structure is extended to direct sums $\oplus_A^k T^*A \to \oplus^k A^*$ in total analogy to what was done for the tangent Lie algebroid in Section 3.2; we adopt the simplified notation $\rho_k = \rho_{\oplus_A^k T^*A}$ and $[\cdot,\cdot]_k = [\cdot,\cdot]_{\oplus_A^k T^*A}$ for the resulting anchor and bracket[5]. Explicitly, the anchor is given by

$$\rho_k(\mathrm{d}\widehat{x}^{i,n}) = \rho_d^i \frac{\partial}{\partial \xi_d^n}, \qquad \rho_k((e_a^L)^k) = \rho_a^j \frac{\partial}{\partial x^j} + C_{ab}^c \xi_c^n \frac{\partial}{\partial \xi_b^n}, \tag{6.7}$$

whereas for the bracket we have

$$[\mathrm{d}\widehat{x}^{i,n}, \mathrm{d}\widehat{x}^{j,m}]_k = 0, \quad [(e_a^L)^k, \mathrm{d}\widehat{x}^{j,m}]_k = \frac{\partial \rho_a^j}{\partial x^i}\mathrm{d}\widehat{x}^{i,m} \tag{6.8}$$

$$[(e_a^L)^k, (e_b^L)^k]_k = C_{ab}^d (e_d^L)^k - \frac{\partial C_{ab}^c}{\partial x^i}\xi_c^n \mathrm{d}\widehat{x}^{i,n}. \tag{6.9}$$

6.1. **Linear multivector fields and derivations.** Let $\pi \in \mathcal{X}^k(A) = \Gamma(\wedge^k TA)$ be a k-vector field on the total space of a vector bundle $q_A : A \to M$. Let us consider the function (cf. (2.4))

$$\overline{\pi} : \oplus_A^k T^*A \to \mathbb{R}, \quad \overline{\pi}(\Upsilon_1, \ldots, \Upsilon_k) = i_{\Upsilon_k} \ldots i_{\Upsilon_1}\pi.$$

We say that $\pi \in \mathcal{X}^k(A)$ is **linear** if $\overline{\pi}$ defines a vector-bundle map

$$\begin{array}{ccc} \oplus_A^k T^*A & \xrightarrow{\overline{\pi}} & \mathbb{R} \\ \downarrow & & \downarrow \\ \oplus^k A^* & \longrightarrow & \{*\}, \end{array} \tag{6.10}$$

similarly to (2.8). One can directly verify that the notion of linear multivector field agrees with the one considered in [18, Section 3.2]. The space of linear k-vector

[5] Since tangent Lie algebroids are not used in this section, this notation should not cause any confusion with the one in Section 3.2.

fields is denoted by $\mathcal{X}^k_{\text{lin}}(A)$. As in Lemma 2.2 (cf. (2.7)), π is expressed in local coordinates (x^j, u^d) of A as

$$\pi = \frac{1}{k!}\pi_d^{b_1\ldots b_k}(x)u^d\frac{\partial}{\partial u^{b_1}}\wedge\ldots\wedge\frac{\partial}{\partial u^{b_k}} + \frac{1}{(k-1)!}\pi^{b_1\ldots b_{k-1}j}(x)\frac{\partial}{\partial u^{b_1}}\wedge\ldots\wedge\frac{\partial}{\partial u^{b_{k-1}}}\wedge\frac{\partial}{\partial x^j}. \tag{6.11}$$

We have the following analog of Proposition 2.5 for linear multivector fields, proven in [18, Prop. 3.7]: there is a 1-1 correspondence between elements in $\mathcal{X}^k_{\text{lin}}(M)$ and pairs (δ_0, δ_1), where $\delta_0 : C^\infty(M) \to \Gamma(\wedge^{k-1}A)$ and $\delta_1 : \Gamma(A) \to \Gamma(\wedge^k A)$ are linear maps satisfying

$$\delta_0(fg) = g\delta_0 f + f\delta_0 g, \quad \delta_1(fu) = (\delta_0 f)\wedge u + f\delta_1 u, \tag{6.12}$$

for all $f, g \in C^\infty(M)$ and $u \in \Gamma(A)$. Equivalently, one may view such pairs (δ_0, δ_1) as restrictions of linear maps $\delta : \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^{\bullet+k-1}A)$ satisfying the property

$$\delta(u\wedge v) = (\delta u)\wedge v + (-1)^{p(k-1)}u\wedge(\delta v) \tag{6.13}$$

for $u \in \Gamma(\wedge^p A)$ and $v \in \Gamma(\wedge^q A)$; i.e., δ is a degree-$(k-1)$ derivation of the exterior algebra $\Gamma(\wedge^\bullet A)$. For this reason, we denote both maps δ_0 and δ_1 by δ. The explicit correspondence between π and δ is given by[6] (see [18, Section 3.2])

$$\pi(\mathrm{d}l_{\xi^1}, \ldots, \mathrm{d}l_{\xi^{k-1}}, \mathrm{d}q_A^* f) = q_A^*\Big\langle \delta f, \xi^1\wedge\ldots\wedge\xi^{k-1}\Big\rangle, \tag{6.14}$$

$$\pi(\mathrm{d}l_{\xi^1}, \ldots, \mathrm{d}l_{\xi^k})(u) = \sum_{i=1}^{k}(-1)^{i+k}\pi(\mathrm{d}l_{\xi^1}, \ldots, \widehat{\mathrm{d}l_{\xi^i}}, \ldots, \mathrm{d}l_{\xi^k}, \mathrm{d}q_A^*\langle\xi^i, u\rangle) - \Big\langle \delta u, \xi^1\wedge\ldots\wedge\xi^k\Big\rangle, \tag{6.15}$$

where $f \in C^\infty(M)$, $u \in \Gamma(A)$, $\xi^1, \ldots, \xi^k \in \Gamma(A^*)$, and $l_{\xi^i} \in C^\infty(A)$ is the linear function $l_{\xi^i}(u) = \langle\xi^i, u\rangle$. In coordinates, we have

$$\delta x^i = \frac{1}{(k-1)!}\pi^{b_1\ldots b_{k-1}i}(x)e_{b_1}\wedge\ldots\wedge e_{b_{k-1}}, \quad \delta e_a = -\frac{1}{k!}\pi_a^{b_1\ldots b_k}(x)e_{b_1}\wedge\ldots\wedge e_{b_k}, \tag{6.16}$$

where $\{e_d\}$ is a basis of local sections of A.

Let $A \to M$ be a Lie algebroid. The Lie bracket $[\cdot,\cdot]$ on $\Gamma(A)$ has a natural extension (still denoted by $[\cdot,\cdot]$) to the exterior algebra $\Gamma(\wedge^\bullet A)$,

$$[\cdot,\cdot] : \Gamma(\wedge^p A)\times\Gamma(\wedge^q A) \to \Gamma(\wedge^{p+q-1}A),$$

making it into a *Gerstenhaber algebra* (see e.g. [6]): for $u \in \Gamma(\wedge^p A)$, $v \in \Gamma(\wedge^q A)$, and $w \in \Gamma(\wedge^r A)$, we have

$$[u, v] = -(-1)^{(p-1)(q-1)}[v, u], \tag{6.17}$$

$$[u, v\wedge w] = [u, v]\wedge w + (-1)^{(p-1)q}v\wedge[u, w]. \tag{6.18}$$

The next result is the analog of Theorem 3.1 for linear multivector fields.

[6] To see that (6.14) and (6.15) determine the linear k-vector π, note that fibres of $T^*A \to A$ are generated by elements of types $\mathrm{d}l_\xi$ and $\mathrm{d}q_A^* f$, and by linearity $i_{\mathrm{d}q_A^* f_2} i_{\mathrm{d}q_A^* f_1}\pi = 0$.

Theorem 6.1. *Let $\pi \in \mathcal{X}^k_{\text{lin}}(A)$ be a linear k-vector field on a Lie algebroid A, and let $\delta : \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^{\bullet+k-1} A)$ be the associated derivation (as in (6.14) and (6.15)). Then the map $\overline{\pi}$ (6.10) is a Lie algebroid morphism if and only if*

$$\delta[u, v] = [\delta u, v] + (-1)^{(p-1)(k-1)}[u, \delta v], \tag{6.19}$$

for all $u \in \Gamma(\wedge^p A)$, $v \in \Gamma(\wedge^q A)$ (i.e., δ is a $(k-1)$-derivation of the Gerstenhaber bracket).

To draw a clear parallel with Theorem 3.1, we denote by $\mathcal{X}^k_{\text{IM}}(A)$ the space of degree $(k-1)$ derivations $\delta : \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^{\bullet+k-1} A)$ of the Gerstenhaber structure (i.e., (6.13) and (6.19) hold), in analogy with IM k-forms.

Proof. We work locally, so the condition that $\overline{\pi}$ is a Lie algebroid morphism is

$$\overline{\pi}([\Upsilon_1, \Upsilon_2]_k) = \mathcal{L}_{\rho_k(\Upsilon_1)}\overline{\pi}(\Upsilon_2) - \mathcal{L}_{\rho_k(\Upsilon_2)}\overline{\pi}(\Upsilon_1), \tag{6.20}$$

where Υ_1, Υ_2 are local sections of $\oplus^k_A T^*A \to \oplus^k A^*$ of types (6.2) or (6.3) (cf. (3.25)); hence, just as in the proof of Theorem 3.1, there are 3 cases to be analyzed. The assertion of Theorem 6.1 is a direct consequence of the following claims:

(c1) Let $\Upsilon_1 = d\widehat{x}^{i,l}$ and $\Upsilon_2 = d\widehat{x}^{j,m}$. If $l = m$, then (6.20) is automatically satisfied; if $l \neq m$, then (6.20) is equivalent to

$$\delta[x^i, x^j] = [\delta x^i, x^j] + (-1)^{k-1}[x^i, \delta x^j] = 0. \tag{6.21}$$

(c2) Let $\Upsilon_1 = d\widehat{x}^{i,l}$ and $\Upsilon_2 = (e^L_b)^k$. Then (6.20) is equivalent to

$$\delta[x^i, e_b] = [\delta x^i, e_b] + (-1)^{k-1}[x^i, \delta e_b]. \tag{6.22}$$

(c3) Let $\Upsilon_1 = (e^L_a)^k$ and $\Upsilon_2 = (e^L_b)^k$. Then (6.20) is equivalent to

$$\delta[e_a, e_b] = [\delta e_a, e_b] + [e_a, \delta e_b]. \tag{6.23}$$

In order to prove claims (c1), (c2) and (c3), we need some general observations. For any function $F : \oplus^k A^* \to \mathbb{R}$ which is k-linear over $C^\infty(M)$ and skew symmetric, let $\Phi_F \in \Gamma(\wedge^k A)$ be the unique element such that

$$F(\xi^1, \dots, \xi^k) = \left\langle \Phi_F, \xi^1 \wedge \dots \wedge \xi^k \right\rangle.$$

E.g., for $F(\xi^1, \dots, \xi^k) = F^{b_1 \dots b_k} \xi^1_{b_1} \dots \xi^k_{b_k}$ with $F^{b_1 \dots b_k}$ totally antisymetric in its indices, we have $\Phi_F = \frac{1}{k!} F^{b_1 \dots b_k} e_{b_1} \wedge \dots \wedge e_{b_k}$. We will consider the cases where $F = \overline{\pi}(d\widehat{x}^{j,m})$ and $F = \overline{\pi}((e^L_a)^k)$. Using the local expressions (6.2), (6.3) as well as (6.11) and (6.16), one may directly verify the following identities:

$$\overline{\pi}(d\widehat{x}^{j,m}(\xi^1, \dots, \xi^k)) = (-1)^{k-m}\left\langle \delta x^j, \xi^1 \wedge \dots \wedge \widehat{\xi^m} \wedge \dots \wedge \xi^k \right\rangle, \tag{6.24}$$

$$\overline{\pi}((e^L_a)^k(\xi^1, \dots, \xi^k)) = -\left\langle \delta e_a, \xi^1 \wedge \dots \wedge \xi^k \right\rangle, \tag{6.25}$$

where the notation $\xi^1 \wedge \dots \wedge \widehat{\xi^m} \wedge \dots \wedge \xi^k$ means that ξ^m is omitted.

Let us now consider $F(\xi^1, \dots, \xi^k) = F^{b_1 \dots b_k} \xi^1_{b_1} \dots \xi^k_{b_k}$ and the vector fields $\rho_k(d\widehat{x}^{i,l})$ and $\rho_k((e^L_a)^k)$ on $\oplus^k A^*$, see (6.7). Then a direct computation shows the following identities:

$$\mathcal{L}_{\rho_k(\mathrm{d}\widehat{x}^{i,l})}(F(\xi^1,\dots,\xi^k)) = (-1)^l\Big\langle [x^i,\Phi_F],\xi^1\wedge\ldots\wedge\widehat{\xi^l}\wedge\ldots\wedge\xi^k\Big\rangle, \tag{6.26}$$

$$\mathcal{L}_{\rho_k((e_a^L)^k)}(F(\xi^1,\dots,\xi^k)) = \Big\langle [e_a,\Phi_F],\xi^1\wedge\ldots\wedge\ldots\wedge\xi^k\Big\rangle. \tag{6.27}$$

From (6.24) and (6.26), we directly see, assuming that $l<m$, that

$$\begin{aligned}&\mathcal{L}_{\rho_k(\mathrm{d}\widehat{x}^{i,l})}(\overline{\pi}(\mathrm{d}\widehat{x}^{j,m})(\xi^1,\dots,\xi^k)) =\\ &\qquad (-1)^{l+k-m}\Big\langle [x^i,\delta x^j],\xi^1\wedge\ldots\wedge\widehat{\xi^l}\wedge\ldots\wedge\widehat{\xi^m}\wedge\ldots,\xi^k\Big\rangle,\\ &\mathcal{L}_{\rho_k(\mathrm{d}\widehat{x}^{j,m})}(\overline{\pi}(\mathrm{d}\widehat{x}^{i,l})(\xi^1,\dots,\xi^k)) =\\ &\qquad (-1)^{m-1+k-l}\Big\langle [x^j,\delta x^i],\xi^1\wedge\ldots\wedge\widehat{\xi^l}\wedge\ldots\wedge\widehat{\xi^m}\wedge\ldots\wedge\xi^k\Big\rangle.\end{aligned}$$

Combining these two equations with (6.17), we conclude that claim (c1) holds.

To prove the other two claims, note first that the derivation property for functions in (6.12) implies that

$$\delta f = \frac{\partial f}{\partial x^j}\delta x^j,\quad f\in C^\infty(M). \tag{6.28}$$

As a result, since $[e_b,x^i]=\mathcal{L}_{\rho(e_b)}x^i=\rho_b^i$, we have that

$$\delta[x^i,e_b] = -\delta[e_b,x^i] = -\frac{\partial\rho_b^i}{\partial x^j}\delta x^j. \tag{6.29}$$

Using the second formula in (6.8) together with (6.24) and (6.29), we obtain

$$\begin{aligned}\overline{\pi}([\mathrm{d}\widehat{x}^{i,l},(e_b^L)^k]_k(\xi^1,\dots,\xi^k)) &= -(-1)^{k-l}\Big\langle \frac{\partial\rho_b^i}{\partial x^j}\delta x^j,\xi^1\wedge\ldots\wedge\widehat{\xi^l}\wedge\ldots\wedge\xi^k\Big\rangle\\ &= (-1)^{k-l}\Big\langle \delta[x^i,e_b],\xi^1\wedge\ldots\wedge\widehat{\xi^l}\wedge\ldots\wedge\xi^k\Big\rangle.\end{aligned} \tag{6.30}$$

Combining (6.24) and (6.27), as well as (6.25) and (6.26), we immediately get

$$\mathcal{L}_{\rho_k((e_b^L)^k)}(\overline{\pi}(\mathrm{d}\widehat{x}^{i,l}(\xi^1,\dots,\xi^k))) = (-1)^{k-l}\Big\langle [e_b,\delta x^i],\xi^1\wedge\ldots\wedge\widehat{\xi^l}\wedge\ldots\wedge\xi^k\Big\rangle, \tag{6.31}$$

$$\mathcal{L}_{\rho_k(\mathrm{d}\widehat{x}^{i,l})}(\overline{\pi}((e_b^L)^k(\xi^1,\dots,\xi^k))) = -(-1)^{l}\Big\langle [x^i,\delta e_b],\xi^1\wedge\ldots\wedge\widehat{\xi^l}\wedge\ldots\wedge\xi^k\Big\rangle. \tag{6.32}$$

Now claim (c2) is a direct consequence of (6.30), (6.31) and (6.32).

Finally, to prove (c3), we observe a few facts. From (6.28), we see that

$$\delta[e_a,e_b] = \delta(C_{ab}^c e_c) = \frac{\partial C_{ab}^c}{\partial x^j}\delta x^j\wedge e_c + C_{ab}^c\delta e_c. \tag{6.33}$$

The usual formula for the wedge product gives us the identity

$$\frac{\partial C_{ab}^c}{\partial x^j}\Big\langle \delta x^j\wedge e_c,\xi^1\wedge\ldots\wedge\xi^k\Big\rangle = \sum_{n=1}^k(-1)^{k-n}\frac{\partial C_{ab}^c}{\partial x^j}\xi_c^n\Big\langle \delta x^j,\xi^1\wedge\ldots\wedge\widehat{\xi^n}\wedge\ldots\wedge\xi^k\Big\rangle;$$

using it, we immediately obtain from (6.9), (6.24) and (6.25) that

$$\overline{\pi}([(e_a^L)^k,(e_b^L)^k]_k(\xi^1,\dots,\xi^k)) = -\Big\langle \delta[e_a,e_b],\xi^1\wedge\ldots\wedge\xi^k\Big\rangle. \tag{6.34}$$

On the other hand, from (6.25) and (6.27) we have that

$$\mathcal{L}_{\rho_k((e_a^L)^k)}(\overline{\pi}((e_b^L)^k(\xi^1,\dots,\xi^k))) = -\Big\langle [e_a,\delta e_b],\xi^1\wedge\ldots\wedge\xi^k\Big\rangle. \tag{6.35}$$

Using (6.34) and (6.35), we can immediately verify that claim (c3) holds. □

6.2. **Infinitesimal description of multiplicative multivector fields.** We now discuss the analogs of the results in Section 4 for multiplicative multivector fields.

Let $\mathcal{G}$ be a Lie groupoid over M. Its cotangent bundle $T^*\mathcal{G}$ has a natural Lie groupoid structure over A^*, known as the **cotangent groupoid** of $\mathcal{G}$, see [9] and [20] for a full description. For us, it will suffice to recall that the unit map $\widetilde{\epsilon}: A^* \to T^*\mathcal{G}|_M$ identifies A^* with the annihilator of $TM \subset T\mathcal{G}$, and that the source map $\widetilde{\mathsf{s}}: T^*\mathcal{G} \to A^*$ is defined by

$$\langle \widetilde{\mathsf{s}}(\alpha_g), u\rangle = \langle \alpha_g, Tl_g(u - T\mathsf{t}(u))\rangle, \qquad \alpha_g \in T^*_g\mathcal{G},\ u \in A_{\mathsf{s}(g)}, \tag{6.36}$$

where l_g denotes left translation in $\mathcal{G}$. Note that $\widetilde{\mathsf{s}}$ is a vector-bundle map covering $\mathsf{s}: \mathcal{G} \to M$; using coordinates (z^l) on $\mathcal{G}$, it has the form

$$\widetilde{\mathsf{s}}(z^l, \alpha_l) = (\mathsf{s}(z)^j, C^l_d(z)\alpha_l) \in A^*|_{\mathsf{s}(z)}. \tag{6.37}$$

We will not need the explicit expression for $C^l_d(z)$, just to note that $\widetilde{\mathsf{s}}(\mathrm{d}\mathsf{t}^* f) = 0$ for all $f \in C^\infty(M)$ (by (6.36)), which implies that

$$C^l_d(z)\frac{\partial(\mathsf{t}^* f)}{\partial z^l} = 0, \quad \forall f \in C^\infty(M). \tag{6.38}$$

Similarly to what happens for the tangent groupoid, the cotangent groupoid structure extends to direct sums $\oplus^k_{\mathcal{G}} T^*\mathcal{G}$ over $\oplus^k A^*$. A multivector field $\Pi \in \mathcal{X}^k(\mathcal{G})$ is called **multiplicative** if the associated map

$$\overline{\Pi}: \oplus^k_{\mathcal{G}} T^*\mathcal{G} \to \mathbb{R}, \quad \overline{\Pi}(\zeta_1, \ldots, \zeta_k) = i_{\zeta_k}\ldots i_{\zeta_1}\Pi \tag{6.39}$$

is a groupoid morphism (cf. Lemma 4.1). We denote the space of multiplicative k-vector fields on $\mathcal{G}$ by $\mathcal{X}^k_{\mathrm{mult}}(\mathcal{G})$.

Remark 6.2. *We may equivalently consider the map*

$$\Pi^\sharp: \oplus^{k-1}_{\mathcal{G}} T^*\mathcal{G} \to T\mathcal{G}, \ \ \Pi^\sharp(\zeta_1, \ldots, \zeta_{k-1}) = i_{\zeta_{k-1}}\ldots i_{\zeta_1}\Pi, \tag{6.40}$$

and verify that Π is multiplicative if and only if $\Pi^\sharp$ is a groupoid morphism.

Let us recall, see e.g. [21], the identification of Lie algebroids

$$\theta_{\mathcal{G}}: A(T^*\mathcal{G}) \xrightarrow{\sim} T^*(A\mathcal{G}), \tag{6.41}$$

which extends to an identification $\theta^k_{\mathcal{G}}: A(\oplus^k_{\mathcal{G}} T^*\mathcal{G}) = \prod_{\mathrm{Lie}(c_{\mathcal{G}})} A(T^*\mathcal{G}) \to \oplus^k_A T^*A$ ($c_{\mathcal{G}}: T^*\mathcal{G} \to \mathcal{G}$ is a groupoid morphism). Given $\Pi \in \mathcal{X}^k_{\mathrm{mult}}(\mathcal{G})$, we consider the infinitesimal map $\mathrm{Lie}(\overline{\Pi}): A(\oplus^k_{\mathcal{G}} T^*\mathcal{G}) \to \mathbb{R}$ (see (4.6)), as well as the composition

$$\mathrm{Lie}(\overline{\Pi}) \circ (\theta^k_{\mathcal{G}})^{-1}: \oplus^k_A T^*A \to \mathbb{R}. \tag{6.42}$$

The exact same arguments as in Lemma 4.5 directly show that there is a unique k-vector field $\mathrm{Lie}(\Pi) \in \mathcal{X}^k(A)$ satisfying

$$\overline{\mathrm{Lie}(\Pi)} = \mathrm{Lie}(\overline{\Pi}) \circ (\theta^k_{\mathcal{G}})^{-1}; \tag{6.43}$$

moreover, the map

$$\mathcal{X}^k_{\mathrm{mult}}(\mathcal{G}) \longrightarrow \mathcal{X}^k(A), \quad \Pi \mapsto \mathrm{Lie}(\Pi),$$

is a bijection onto the subspace of k-vector fields $\pi \in \mathcal{X}^k_{\mathrm{lin}}(A)$ for which $\overline{\pi}: \oplus^k_A T^*A \to \mathbb{R}$ is a morphism of Lie algebroids. An immediate consequence of Theorem 6.1 is

Corollary 6.3. *There is a bijective correspondence*

$$\mathcal{X}^k_{\mathrm{mult}}(\mathcal{G}) \longrightarrow \mathcal{X}^k_{\mathrm{IM}}(A), \quad \Pi \mapsto \delta, \tag{6.44}$$

where δ is the derivation associated with $\pi = \mathrm{Lie}(\Pi) \in \mathcal{X}^k_{\mathrm{lin}}(A)$ (via (6.14) *and* (6.15)*).*

This result is parallel to Theorem 4.6, except that it provides no explicit way of computing δ directly out of Π (analogous to (4.17) and (4.18)). This missing aspect will be clarified in the next section.

6.3. **The universal lifiting theorem revisited.** For $u \in \Gamma(\wedge^p A)$, let us denote by u^r the corresponding right-invariant p-vector field on $\mathcal{G}$. As observed in [18, Section 2], given $\Pi \in \mathcal{X}^k_{\mathrm{mult}}(\mathcal{G})$, then $[\Pi, u^r]$ is again right invariant, which means that there exists $\delta_\Pi u \in \Gamma(\wedge^{p+k-1}A)$ such that $(\delta_\Pi u)^r = [\Pi, u^r]$. One can check that the map $\delta_\Pi : \Gamma(\wedge^p A) \to \Gamma(\wedge^{p+k-1}A)$ is a derivation of the Gerstenhaber structure, i.e., $\delta_\Pi \in \mathcal{X}^k_{\mathrm{IM}}(M)$.

Proposition 6.4. *The map $\mathcal{X}^k_{\mathrm{mult}}(\mathcal{G}) \longrightarrow \mathcal{X}^k_{\mathrm{IM}}(A)$, $\Pi \mapsto \delta_\Pi$, where δ_Π is defined by*

$$(\delta_\Pi f)^r = [\Pi, \mathsf{t}^* f], \quad (\delta_\Pi u)^r = [\Pi, u^r], \tag{6.45}$$

for $f \in C^\infty(M)$ and $u \in \Gamma(A)$, coincides with the map (6.44)*; in particular, it is a bijection.*

The fact that the correspondence in Proposition 6.4 is a bijection is the *universal lifting theorem* of [18] (see Theorem 2.34 therein), which we recover here as a consequence of Corollary 6.3. We need to collect some observations before getting into the proof of Proposition 6.4.

Let us consider the isomorphism

$$\Theta : T(T^*\mathcal{G}) \longrightarrow T^*(T\mathcal{G}), \ \ (z^j, \alpha_j, \dot{z}^j, \dot{\alpha}_j) \mapsto (z^j, \dot{z}^j, \dot{\alpha}_j, \alpha_j), \tag{6.46}$$

which is related to the identification $\theta_\mathcal{G}$ in (6.41) via

$$\theta_\mathcal{G} = (T\iota_A)^t \circ \Theta \circ \iota_{A(T^*\mathcal{G})}, \tag{6.47}$$

where $(T\iota_A)^t$ is the fibrewise dual to the vector-bundle map $T\iota_A : TA \to \iota_A^* T(T\mathcal{G})$ (the composition in (6.47) is well defined since $\Theta \circ \iota_{A(T^*\mathcal{G})}(A(T^*\mathcal{G})) \subset \iota_A^* T^*(T\mathcal{G})$; this can be derived directly from (6.37)). For a k-vector field $\Pi \in \mathcal{X}^k(\mathcal{G})$, its **tangent lift** is the k-vector field $\Pi_T \in \mathcal{X}^k(T\mathcal{G})$ defined by the condition (cf. (2.17))

$$\overline{\Pi_T} = \mathrm{d}\overline{\Pi} \circ (\Theta^{-1})^k, \tag{6.48}$$

where $\mathrm{d}\overline{\Pi} : T(\oplus^k_\mathcal{G} T^*\mathcal{G}) = \prod^k_{T c_\mathcal{G}} T(T^*\mathcal{G}) \to \mathbb{R}$ is the differential of the function $\overline{\Pi}$ in $C^\infty(\oplus^k_\mathcal{G} T^*\mathcal{G})$ defined by (6.39).

Remark 6.5. *As observed in* [14]*, one may alternatively define the tangent lift Π_T in terms of $\Pi^\sharp$* (6.40)*:*

$$\Pi_T^\sharp : \oplus^{k-1}_{T\mathcal{G}} T^*(T\mathcal{G}) \to T(T\mathcal{G}), \ \ \Pi_T^\sharp = (J_\mathcal{G})^{-1} \circ T\Pi^\sharp \circ (\Theta^{-1})^{(k-1)}, \tag{6.49}$$

where $J_\mathcal{G} : T(T\mathcal{G}) \to T(T\mathcal{G})$ is the involution (2.16)*.*

When Π is multiplicative, it follows from (6.43) that

$$\overline{\pi} = \overline{\mathrm{Lie}(\Pi)} = \overline{\Pi}_T \circ (\Theta \circ \iota_{AT^*\mathcal{G}} \circ \theta_{\mathcal{G}}^{-1})^k. \tag{6.50}$$

We will need this characterization of π in the proof of Proposition 6.4.

For local computations, it will be convenient to consider adapted local coordinates

$$(x^j, y^d) \ \text{ on } \mathcal{G} \text{ around } M \subset \mathcal{G}, \tag{6.51}$$

where y^d are coordinates along the s-fibres. We will also use the induced coordinates $((x^j, y^d), (\dot{x}^j, \dot{y}^d))$ on $T\mathcal{G}$, and similarly for $T^*\mathcal{G}$, $T(T^*\mathcal{G})$ and $T^*(T\mathcal{G})$. In these coordinates, $\iota_A : A \to T\mathcal{G}|_M$, $\iota_A(x^j, u^d) = ((x^j, 0), (0, u^d))$, and $T\iota_A : TA \to \iota_A^* T(T\mathcal{G})$ is given by

$$T\iota_A \left(\dot{x}^j \frac{\partial}{\partial x^j} + \dot{u}^d \frac{\partial}{\partial u^d} \right) \Big|_u = \dot{x}^j \frac{\partial}{\partial x^j} + \dot{u}^d \frac{\partial}{\partial \dot{y}^d} \Big|_{\iota_A(u)}, \quad u \in A,$$

whereas for $(T\iota_A)^t : \iota_A^* T^*(T\mathcal{G}) \to T^*A$ we have

$$(T\iota_A)^t (p_j dx^j + \gamma_a dy^a + \overline{p}_j d\dot{x}^j + \overline{\gamma}_a d\dot{y}^a)|_{\iota_A(u)} = (p_j dx^j + \overline{\gamma}_a du^a)|_u, \ \ u \in A.$$

Since the unit map $\widetilde{\epsilon} : A^* \to T^*\mathcal{G}|_M$ identifies A^* with the annihilator of $TM \subset T\mathcal{G}$, given $\xi \in \Gamma(A^*)$, locally written as (x^j, ξ_d), the local 1-form on $\mathcal{G}$ given by

$$\widetilde{\xi}(x^j, y^d) = \xi_d(x) dy^d \tag{6.52}$$

extends $\widetilde{\epsilon}(\xi(x))$ to a neighborhood of M in $\mathcal{G}$. We denote by $l_{\widetilde{\xi}} \in C^\infty(T\mathcal{G})$ the linear function determined by $\widetilde{\xi}$. The following lemma is key to compare the map in Proposition 6.4 with the map (6.44).

Lemma 6.6. *Let $\mathcal{J} = \Theta \circ \iota_{AT^*\mathcal{G}} \circ \theta_{\mathcal{G}}^{-1} : T^*A \to \iota_A^* T^*(T\mathcal{G})$, and let $u_0 \in A$. Then, for any $f \in C^\infty(M)$ and $\xi \in \Gamma(A^*)$, we have*

$$\mathcal{J}(\mathrm{d}q_A^* f|_{u_0}) = \mathrm{d}(\mathsf{t}^* f)^\vee|_{\iota_A(u_0)}, \tag{6.53}$$

$$\mathcal{J}(\mathrm{d}l_\xi|_{u_0}) = (\mathrm{d}l_{\widetilde{\xi}} + \mathrm{d}h^\vee)|_{\iota_A(u_0)}, \tag{6.54}$$

where $^\vee$ means the pull-back of functions on $\mathcal{G}$ by $p_{\mathcal{G}} : T\mathcal{G} \to \mathcal{G}$, and $h \in C^\infty(\mathcal{G})$ is a function that vanishes on $M \subset \mathcal{G}$.

Proof. The proof follows from some observations, all of which can be checked through computations in adapted local coordinates (x^j, y^d) as in (6.51).

The first observation one can directly verify is that

$$(T\iota_A)^t (\mathrm{d}(\mathsf{t}^* f)^\vee|_{\iota_A(u)}) = \mathrm{d}q_A^* f|_u, \ \ u \in A. \tag{6.55}$$

Using the local expression (6.37) for the source map $\widetilde{\mathsf{s}}$, the property (6.38), and the definition of Θ, a direct computation shows that

$$T\widetilde{\mathsf{s}}(\Theta^{-1}(\mathrm{d}(\mathsf{t}^* f)^\vee|_{\iota_A(u)})) = 0 \in TA^*|_{q_A(u)}, \quad q_A(u) \in M \subset A^*. \tag{6.56}$$

It follows that $\Theta^{-1}(\mathrm{d}(\mathsf{t}^* f)^\vee|_{\iota_A(u)})$ is in the image of $\iota_{A(T^*\mathcal{G})}$, hence there is a unique $\Upsilon \in T^*A|_{q_A(u)}$ such that

$$\Theta^{-1}(\mathrm{d}(\mathsf{t}^* f)^\vee|_{\iota_A(u)}) = \iota_{A(T^*\mathcal{G})}(\theta_{\mathcal{G}}^{-1}(\Upsilon)), \ \text{ i.e., } \ \mathcal{J}(\Upsilon) = \mathrm{d}(\mathsf{t}^* f)^\vee|_{\iota_A(u)}.$$

Using (6.55) and (6.47). we conclude that $\Upsilon = \mathrm{d}q_A^* f|_u$, which proves (6.53).

With respect to the coordinates $((x^j, y^d), (\dot{x}^j, \dot{y}^d))$ on $T\mathcal{G}$, one can write

$$\mathrm{d}l_{\widetilde{\xi}}|_{\iota_A(u)} = \frac{\partial \xi_a}{\partial x^i} u^a \mathrm{d}x^i + \xi_a \mathrm{d}\dot{y}^a \in T^*(T\mathcal{G})|_{\iota_A(u)}, \tag{6.57}$$

from where we conclude that

$$(T\iota_A)^t(\mathrm{d}l_{\widetilde{\xi}}|_{\iota_A(u)}) = \left(\frac{\partial \xi_a}{\partial x^j} u^a \mathrm{d}x^j + \xi_a \mathrm{d}u^a\right)\Big|_u = q_A^* \mathrm{d}l_\xi|_u \ \in T^*A|_u. \tag{6.58}$$

Let us now consider $\mathcal{J}(q_A^*\mathrm{d}l_\xi) \in \iota_A^* T^*(T\mathcal{G})$. Since $\Theta^{-1}(\mathcal{J}(q_A^*\mathrm{d}l_\xi))$ lies in $T(T^*\mathcal{G})|_{A^*}$, one can directly verify that $\mathcal{J}(q_A^*\mathrm{d}l_\xi)$ can be written as

$$p_j \mathrm{d}x^j + \gamma_a \mathrm{d}y^a + \overline{\gamma}_a \mathrm{d}\dot{y}^a,$$

i.e., its components relative to $\mathrm{d}\dot{x}^j$ vanish. By (6.47), $(T\iota_A)^t(\mathcal{J}(q_A^*\mathrm{d}l_\xi)|_u) = q_A^*\mathrm{d}l_\xi|_u$, so from the second equality in (6.58) we conclude that $p_j = \frac{\partial \xi_a}{\partial x^j} u^a$ and $\overline{\gamma}_a = \xi_a$, i.e.,

$$\mathcal{J}(q_A^*\mathrm{d}l_\xi|_u) = \left(\frac{\partial \xi_a}{\partial x^j} u^a \mathrm{d}x^j + \overline{\gamma}_a \mathrm{d}y^a + \xi_a \mathrm{d}\dot{y}^a\right)\Big|_{\iota_A(u)}.$$

For each given $u_0 \in A$, one can find $h \in C^\infty(\mathcal{G})$ vanishing on $M \subset \mathcal{G}$ and such that $\mathrm{d}h^\vee|_{\iota_A(u_0)} = \overline{\gamma}_a(\iota_A(u_0))\mathrm{d}y^a$, and (6.54) follows by a direct comparison with (6.57). □

We will need the following immediate observations about linear functions on vector bundles.

Lemma 6.7. *Let $q_B : B \to N$ be a vector bundle, with coordinates (x^j, b^d) relative to a basis of local sections $\{e_d\}$, and consider $b = b^d e_d \in \Gamma(B)$, $b^\vee = b^d \frac{\partial}{\partial b^d} \in \mathcal{X}(B)$, and $\beta = \beta_d e^d \in \Gamma(B^*)$. Let $l_\beta \in C^\infty(B)$ be the linear function defined by β, and fix $b_0 = b(x_0) \in B$, for a given $x_0 \in N$. Then $\mathcal{L}_{b^\vee} l_\beta = q_B^*\langle \beta, b\rangle$ and*

$$l_\beta(b_0) = (\mathcal{L}_{b^\vee} l_\beta)(b_0). \tag{6.59}$$

We now prove Proposition 6.4.

Proof. (of Proposition 6.4)
Let $\pi = \mathrm{Lie}(\Pi)$, and consider $\xi^1, \dots, \xi^{k-1} \in \Gamma(A^*)$ and $f \in C^\infty(M)$. Let us fix $u_0 \in A$, $x_0 = q_A(u_0) \in M$. By (6.50) and Lemma 6.6, we have

$$\begin{aligned}\pi(\mathrm{d}l_{\xi^1}, \dots, \mathrm{d}l_{\xi^{k-1}}, dq_A^* f)|_{u_0} &= \Pi_T(\mathcal{J}\mathrm{d}l_{\xi^1}, \dots, \mathcal{J}\mathrm{d}l_{\xi^{k-1}}, \mathcal{J}\mathrm{d}q_A^* f)|_{\iota_A(u_0)} \\ &= \Pi_T(\mathrm{d}l_{\widetilde{\xi}^1} + \mathrm{d}h_1^\vee, \dots, \mathrm{d}l_{\widetilde{\xi}^{k-1}} + \mathrm{d}h_{k-1}^\vee, \mathrm{d}(\mathsf{t}^* f)^\vee)|_{\iota_A(u_0)},\end{aligned} \tag{6.60}$$

with $h_i \in C^\infty(\mathcal{G})$, $h_i|_M = 0$, and $\widetilde{\xi^i}$ as in (6.52) . We directly check from the definition of Π_T that it is a linear multivector, $\Pi_T \in \mathcal{X}^k_{\mathrm{lin}}(T\mathcal{G})$, so (see footnote 6)

$$i_{\mathrm{d}f_1^\vee} i_{\mathrm{d}f_2^\vee} \Pi_T = 0, \quad \forall f_1, f_2 \in C^\infty(\mathcal{G}). \tag{6.61}$$

Hence the expression in (6.60) agrees with

$$\Pi_T(\mathrm{d}l_{\widetilde{\xi}^1}, \dots, \mathrm{d}l_{\widetilde{\xi}^{k-1}}, \mathrm{d}(\mathsf{t}^* f)^\vee)|_{\iota_A(u_0)} = [\Pi_T, (\mathsf{t}^* f)^\vee](\mathrm{d}l_{\widetilde{\xi}^1}, \dots, \mathrm{d}l_{\widetilde{\xi}^{k-1}})|_{\iota_A(u_0)}, \tag{6.62}$$

where $[\cdot, \cdot]$ is the Schouten bracket on $\mathcal{X}^\bullet(T\mathcal{G})$.

Let us consider the *vertical lift* operation $\mathcal{X}^\bullet(\mathcal{G}) \to \mathcal{X}^\bullet(T\mathcal{G})$, $\Pi \mapsto \Pi^\vee$: in coordinates (z^l) on $\mathcal{G}$, it sends the vector field $Y = Y^l \frac{\partial}{\partial z^l}$ to $Y^\vee = Y^l \frac{\partial}{\partial \dot{z}^l}$, and this is extended to a graded algebra homomorphism of multivector fields. From the Schouten bracket relations for vertical and tangent lifts, see e.g. [15], we obtain

$$[\Pi_T, (\mathsf{t}^* f)^\vee] = [\Pi, \mathsf{t}^* f]^\vee = ((\delta_\Pi f)^r)^\vee.$$

Letting $x_0 = q_A(u_0) \in M$, a direct computation in coordinates (6.51) shows that

$$\begin{aligned} ((\delta_\Pi f)^r)^\vee(\mathrm{d}l_{\widetilde{\xi}^1}, \ldots, \mathrm{d}l_{\widetilde{\xi}^{k-1}})|_{\iota_A(u_0)} &= \Big\langle (\delta_\Pi f)^r, \widetilde{\xi}^1 \wedge \ldots \wedge \widetilde{\xi}^{k-1} \Big\rangle\Big|_{\epsilon(x_0)}, \\ &= \Big\langle \delta_\Pi f, \xi^1 \wedge \ldots \wedge \xi^{k-1} \Big\rangle\Big|_{x_0}, \end{aligned}$$

from where it follows that

$$\pi(\mathrm{d}l_{\xi^1}, \ldots, \mathrm{d}l_{\xi^{k-1}}, \mathrm{d}q_A^* f)|_{u_0} = \Big\langle \delta_\Pi f, \xi^1 \wedge \ldots \wedge \xi^{k-1} \Big\rangle\Big|_{x_0} = q_A^* \Big\langle \delta_\Pi f, \xi^1 \wedge \ldots \wedge \xi^{k-1} \Big\rangle\Big|_{u_0}.$$

Comparing with (6.14), we conclude that δ (see (6.44)) and δ_Π agree on $C^\infty(M)$. It remains to check that they agree on $\Gamma(A)$.

We now consider $\xi^1, \ldots, \xi^k \in \Gamma(A^*)$ and describe $\pi(\mathrm{d}l_{\xi^1}, \ldots, \mathrm{d}l_{\xi^k})|_{u_0}$ in terms of δ_Π. By (6.50) and (6.54), we have (keeping the notation of Lemma 6.6)

$$\begin{aligned} \pi(\mathrm{d}l_{\xi^1}, \ldots, \mathrm{d}l_{\xi^k})|_{u_0} &= \Pi_T(\mathcal{J}\mathrm{d}l_{\xi^1}, \ldots, \mathcal{J}\mathrm{d}l_{\xi^k})|_{u_0} \\ &= \Pi_T(\mathrm{d}l_{\widetilde{\xi}^1} + \mathrm{d}h_1^\vee, \ldots, \mathrm{d}l_{\widetilde{\xi}^k} + \mathrm{d}h_k^\vee)|_{\iota_A(u_0)}. \end{aligned} \tag{6.63}$$

From (6.61), we see that the expression $\Pi_T(\mathrm{d}l_{\widetilde{\xi}^1} + \mathrm{d}h_1^\vee, \ldots, \mathrm{d}l_{\widetilde{\xi}^k} + \mathrm{d}h_k^\vee)$ can be re-written as

$$\Pi_T(\mathrm{d}l_{\widetilde{\xi}^1}, \ldots, \mathrm{d}l_{\widetilde{\xi}^k}) + \sum_{j=1}^{k} \Pi_T(\mathcal{J}\mathrm{d}l_{\xi^1}, \ldots, \mathcal{J}\mathrm{d}l_{\xi^{j-1}}, dh_j^\vee, \mathcal{J}\mathrm{d}l_{\xi^{j+1}}, \ldots, \mathcal{J}\mathrm{d}l_{\xi^k}). \tag{6.64}$$

We claim that, for all $j = 1, \ldots, k$, we have

$$\Pi_T(\mathcal{J}\mathrm{d}l_{\xi^1}, \ldots, \mathcal{J}\mathrm{d}l_{\xi^{j-1}}, \mathrm{d}h_j^\vee, \mathcal{J}\mathrm{d}l_{\xi^{j+1}}, \ldots, \mathcal{J}\mathrm{d}l_{\xi^k}) = 0. \tag{6.65}$$

To see that, recall from Remark 6.5 that Π_T satisfies $\Pi_T^\sharp \circ \Theta^{(k-1)} = (J_\mathcal{G})^{-1} \circ T\Pi^\sharp$, and, since $\Pi^\sharp : \oplus_\mathcal{G}^{k-1} T^*\mathcal{G} \to T\mathcal{G}$ is a groupoid morphism (see Remark 6.2),

$$T\Pi^\sharp \circ (\iota_{A(T^*\mathcal{G})})^{(k-1)} \subseteq A(T\mathcal{G}).$$

It follows from (4.7) and the definition of $\mathcal{J}$ that $\Pi_T^\sharp \circ \mathcal{J}^{(k-1)} \subseteq T\iota_A(TA) \subset \iota_A^* T(T\mathcal{G})$. Relative to the adapted coordinates (x^j, y^d) in (6.51), elements in $T\iota_A(TA) \subset \iota_A^* T(T\mathcal{G})$ are combinations of $\frac{\partial}{\partial x^j}$ and $\frac{\partial}{\partial \dot{y}^d}$, whereas $\mathrm{d}h_j^\vee|_{\iota_A(u)}$ is in the span of $\mathrm{d}y^d$. So (6.65) follows, and we conclude that

$$\pi(\mathrm{d}l_{\xi^1}, \ldots, \mathrm{d}l_{\xi^k})|_{u_0} = \Pi_T(\mathrm{d}l_{\widetilde{\xi}^1}, \ldots, \mathrm{d}l_{\widetilde{\xi}^k})|_{\iota_A(u_0)}. \tag{6.66}$$

To proceed, we observe that $\Pi_T(\mathrm{d}l_{\widetilde{\xi}^1}, \ldots, \mathrm{d}l_{\widetilde{\xi}^k})$ defines a linear function on $T\mathcal{G}$, and, using (6.59) in Lemma 6.7 (with $B = T\mathcal{G}$), we write

$$\Pi_T(\mathrm{d}l_{\widetilde{\xi}^1}, \ldots, \mathrm{d}l_{\widetilde{\xi}^k})|_{\iota_A(u_0)} = \mathcal{L}_{(u^r)^\vee}(\Pi_T(\mathrm{d}l_{\widetilde{\xi}^1}, \ldots, \mathrm{d}l_{\widetilde{\xi}^k}))|_{\iota_A(u_0)}, \tag{6.67}$$

where $u \in \Gamma(A)$ is such that $u(x_0) = u_0$. But

$$\begin{aligned}\mathcal{L}_{(u^r)^\vee}(\Pi_T(\mathrm{d}l_{\widetilde{\xi}^1}, \ldots, \mathrm{d}l_{\widetilde{\xi}^k})) =& (\mathcal{L}_{(u^r)^\vee}\Pi_T)(\mathrm{d}l_{\widetilde{\xi}^1}, \ldots, \mathrm{d}l_{\widetilde{\xi}^k}) \\ &+ \sum_{j=1}^{k} \Pi_T(\mathrm{d}l_{\widetilde{\xi}^1}, \mathrm{d}l_{\widetilde{\xi}^{j-1}}, \mathcal{L}_{(u^r)^\vee}(\mathrm{d}l_{\widetilde{\xi}^j}), \mathrm{d}l_{\widetilde{\xi}^{j+1}}, \ldots, \mathrm{d}l_{\widetilde{\xi}^k})),\end{aligned}$$

and note that

$$\mathcal{L}_{(u^r)^\vee}(\mathrm{d}l_{\widetilde{\xi}^l}) = \mathrm{d}(\widetilde{\xi}^l(u^r))^\vee, \quad \mathcal{L}_{(u^r)^\vee}\Pi_T = [u^r, \Pi]^\vee = -((\delta_\Pi u)^r)^\vee,$$

where we used the Schouten-bracket relations for tangent and vertical lifts in the second equation. One can directly check that

$$((\delta_\Pi u)^r)^\vee(\mathrm{d}l_{\widetilde{\xi}^1}, \ldots, \mathrm{d}l_{\widetilde{\xi}^k})|_{\iota_A(u_0)} = \Big\langle (\delta_\Pi u)^r, \widetilde{\xi}^1 \wedge \ldots \wedge \widetilde{\xi}^k \Big\rangle\Big|_{\epsilon(x_0)} = \Big\langle \delta_\Pi u, \xi^1 \wedge \ldots \wedge \xi^k \Big\rangle\Big|_{x_0}.$$

Thus $\mathcal{L}_{(u^r)^\vee}(\Pi_T(\mathrm{d}l_{\widetilde{\xi}^1}, \ldots, \mathrm{d}l_{\widetilde{\xi}^k}))|_{\iota_A(u_0)}$ equals

$$-\Big\langle \delta_\Pi u, \xi^1 \wedge \ldots \wedge \xi^k \Big\rangle\Big|_{x_0} + \sum_{j=1}^{k} \Pi_T(\mathrm{d}l_{\widetilde{\xi}^1}, \ldots, \mathrm{d}l_{\widetilde{\xi}^{j-1}}, \mathrm{d}(\widetilde{\xi}^j(u^r))^\vee, \mathrm{d}l_{\widetilde{\xi}^{j+1}}, \ldots, \mathrm{d}l_{\widetilde{\xi}^k})|_{\iota_A(u_0)}.$$

Using local coordinates (x^j, y^d) as in (6.51), one can check the identity

$$\mathrm{d}(\widetilde{\xi}^j(u^r))^\vee|_{\iota_A(u_0)} = \mathrm{d}(\mathsf{t}^*\langle \xi^j, u\rangle)^\vee|_{\iota_A(u_0)} + \mathrm{d}h_j^\vee|_{\iota_A(u_0)},$$

where $h_j \in C^\infty(\mathcal{G})$ vanishes on $M \subset \mathcal{G}$. It follows that

$$\begin{aligned}\Pi_T(\mathrm{d}l_{\widetilde{\xi}^1}, \ldots, &\mathrm{d}l_{\widetilde{\xi}^{j-1}}, \mathrm{d}(\widetilde{\xi}^j(u^r))^\vee, \mathrm{d}l_{\widetilde{\xi}^{j+1}}, \ldots, \mathrm{d}l_{\widetilde{\xi}^k})|_{\iota_A(u_0)} = \\ &\Pi_T(\mathrm{d}l_{\widetilde{\xi}^1}, \ldots, \mathrm{d}l_{\widetilde{\xi}^{j-1}}, \mathrm{d}(\mathsf{t}^*\langle \xi^j, u\rangle)^\vee, \mathrm{d}l_{\widetilde{\xi}^{j+1}}, \ldots, \mathrm{d}l_{\widetilde{\xi}^k})|_{\iota_A(u_0)} + \\ &\Pi_T(\mathrm{d}l_{\widetilde{\xi}^1}, \ldots, \mathrm{d}l_{\widetilde{\xi}^{j-1}}, \mathrm{d}h_j^\vee, \mathrm{d}l_{\widetilde{\xi}^{j+1}}, \ldots, \mathrm{d}l_{\widetilde{\xi}^k})|_{\iota_A(u_0)}.\end{aligned} \tag{6.68}$$

Using the linearity of Π_T (see footnote 6) and (6.65), we see that

$$\begin{aligned}\Pi_T(\mathrm{d}l_{\widetilde{\xi}^1}, \ldots, &\mathrm{d}l_{\widetilde{\xi}^{j-1}}, \mathrm{d}h_j^\vee, \mathrm{d}l_{\widetilde{\xi}^{j+1}}, \ldots, \mathrm{d}l_{\widetilde{\xi}^k}) \\ &= \Pi_T(\mathcal{J}\mathrm{d}l_{\widetilde{\xi}^1}, \ldots, \mathcal{J}\mathrm{d}l_{\widetilde{\xi}^{j-1}}, \mathrm{d}h_j^\vee, \mathcal{J}\mathrm{d}l_{\widetilde{\xi}^{j+1}}, \ldots, \mathcal{J}\mathrm{d}l_{\widetilde{\xi}^k}) = 0.\end{aligned}$$

A direct comparison with (6.60), (6.62) gives that

$$\begin{aligned}\Pi_T(\mathrm{d}l_{\widetilde{\xi}^1}, \ldots, \mathrm{d}l_{\widetilde{\xi}^{j-1}}, \mathrm{d}(\mathsf{t}^*\langle \xi^j, u\rangle)^\vee, &\mathrm{d}l_{\widetilde{\xi}^{j+1}}, \ldots, \mathrm{d}l_{\widetilde{\xi}^k})|_{\iota_A(u_0)} = \\ &(-1)^{k-j}\pi(\mathrm{d}l_{\xi^1}, \ldots, \widehat{\mathrm{d}l_{\xi^j}}, \ldots, \mathrm{d}l_{\xi^k}, \mathrm{d}q_A^*\langle \xi^j, u\rangle)|_{u_0}\end{aligned}$$

Going back to (6.66), we finally conclude that

$$\begin{aligned}\pi(\mathrm{d}l_{\xi^1}, \ldots, \mathrm{d}l_{\xi^k})|_{u_0} =& -\Big\langle \delta_\Pi u, \xi^1 \wedge \ldots \wedge \xi^k \Big\rangle\Big|_{x_0} \\ &+ \sum_{j=1}^{k}(-1)^{k-j}\pi(\mathrm{d}l_{\xi^1}, \ldots, \widehat{\mathrm{d}l_{\xi^j}}, \ldots, \mathrm{d}l_{\xi^k}, \mathrm{d}q_A^*\langle \xi^j, u\rangle)|_{u_0}.\end{aligned}$$

Comparing with (6.15), we conclude that $\delta = \delta_\Pi$. □

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