Subdivision curves on surfaces with arc-length control

J. Estrada-Sarlabous[1], V. Hernández-Mederos[1], D. Martínez-Morera[2], L. Velho[3]

[1]ICIMAF, Cuba
[2]UFAL, Brazil
[3]IMPA, Brazil

Abstract

Dans ce travail, nous présentons un nouveau schéma de subdivision de courbes non stationnaire et interpolatoire, adapté à la conception des courbes sur les surfaces.
Nous montrons que le schéma converge et que la courbe de subdivision est continue. De plus, en commençant avec une certaine paramétrisation naturelle du polygone initial on obtient une courbe de subdivision paramétréé par un multiple de la longueur d'arc.

In this paper we present a new non-stationary, interpolatory, curve subdivision scheme, suitable for designing curves on surfaces.
We show that the scheme converges and the subdivision curve is continuous. Moreover, starting with a certain natural parametrization of the initial polygon, we obtain a subdivision curve parametrized by a multiple of the arc-length.

1. Introduction

Subdivision curves are very important for several science and engineering applications. In the Euclidean space R^n, they are easy to define by just selecting a small set of control points. A drawing algorithm typically performs some subdivision steps, thus approximating the limit curve by a polygonal line in a matter of milliseconds. In this setting, there are many references on the convergence of these schemes.

Given their good properties and the advantages of using these curves, it is natural to extend them to non-euclidean geometries, such as Riemannian manifolds, Lie groups or triangulations. In [PR95, RLJ05, PH05] one may find some of these extensions.

In this paper we propose a simple method to define a subdivision scheme on a two dimensional manifold S that is easy to be implemented on triangle meshes. Under mild conditions, the limit curve of the proposed scheme is a continuous curve on S. Higher order continuity has not been explored in this work, since it depends also on the smoothness of S.

The classical 4-point scheme [Dub86, DLG87] is one of the earliest and most popular interpolatory curve subdivision schemes. It is a member of the Dubuc-Deslauriers family of subdivision schemes [DD89], where the new points lie on a polynomial interpolating consecutive vertices of the control polygon. More precisely, starting from an initial polygon $P^0 = \{P_i^0, i \in Z\}$ the 4-point scheme is defined by the equations

$$P_{2i}^{j+1} = P_i^j, \qquad P_{2i+1}^{j+1} = g_i^j(t_{2i+1}^{j+1}) \tag{1}$$

where $g_i^j(t)$ is the cubic polynomial interpolating the points P_k^j at uniform parameter values $t_k^j = k/2^j$ for $k = i-1, i, i+1, i+2$, and $t_{2i+1}^{j+1} = (2i+1)/2^{j+1}$.

Several authors [KS98, DFH09] have noticed that the limit curves of the 4-point scheme fit tightly to the long edges of the initial control polygon and loosely to the short edges, see Figure 1, left. This is a result of the uniform parametrization $t_i^0 = i$ for all i: the same time is used to travel between two consecutive points P_i^0, P_{i+1}^0 of the initial polygon, regardless of their distance. In other words, the limit curve of the uniform 4-point subdivision scheme is far away from being arc-length parametrized.

One way to address this problem is to use a non-uniform

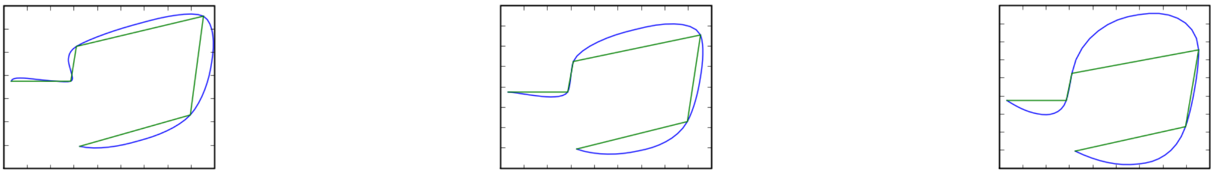

Figure 1: *The limit curve of the 4-point subdivision scheme (1) with the initial parametrization (2) with* $\beta = 0$ *(left),* $\beta = 0.5$ *(middle) and* $\beta = 1$ *(right)*

parameterization for P^0

$$t^0_{i+1} = t^0_i + \|P^0_{i+1} - P^0_i\|^\beta \tag{2}$$

Then, the parameter values at the step $j+1$, are computed from the parameters of the previous step as

$$t^{j+1}_{2i} = t^j_i, \qquad t^{j+1}_{2i+1} = \frac{t^j_i + t^j_{i+1}}{2} \tag{3}$$

Taking this idea one step further, in [DFH09], a reparametrization is introduced in each step, defined by the equation

$$t^j_0 = 0, \qquad t^j_{i+1} = t^j_i + \|P^j_{i+1} - P^j_i\|^\beta \tag{4}$$

Points in the step $j+1$ are given by $P^{j+1}_{2i} = P^j_i, P^{j+1}_{2i+1} = f^j_i((t^j_i + t^j_{i+1})/2)$, where $f^j_i(t)$ is the cubic polynomial interpolating (t^j_k, P^j_k) for $k = i-1, i, i+1, i+2$. The limit curve of this nonlinear scheme is smooth and when the centripetal parametrization is used, it is relatively close to the initial polygon and its shape is pleasing.

In Figure 1 we also show the limit curves of the 4-point subdivision scheme corresponding to different values of β. Notice that the change in the parameterization of the initial polygon affects the shape of the limit curve.

Motivated by the idea of keeping some control on the geometry of the limit curves by means of a simple nonlinear subdivision scheme, in [HEIM09] is presented a nonstationary, interpolatory, plane curve subdivision scheme, whose limit curve is continuous and parametrized by a multiple of the arc-length. The itermediate polygons P^j are provided with non-uniform parametrizations reflecting the relationship between the length of a side $P^j_i P^j_{i+1}$ of P^j and the length of the subpolygon of P^{j+1} obtained applying the subdivision scheme to $P^j_i P^j_{i+1}$. Based on this result, it becomes a natural idea to try to extend this scheme to a curved surface S in the most intrinsic way, i.e., using the geometry of the surface S.
The problem of designing curves on smooth manifolds has been addressed in several works [PR95, RLJ05, PH05], and also on triangulations [MCV07]. Some general frameworks for linear subdivision on smooth and discrete manifolds have been defined [WP06, Wal06, WD05, MVC05] in the last years.

In this work, we take another way. We translate the geometric ideas from [HEIM09] to the geometry of the manifold. Doing so, we are able to prove some properties of the new curves, which are defined on the manifold, in a way similar to [HEIM09]. In the interpolatory scheme proposed in this paper, the new points do not necessarily lie on the cubic polynomial (1). Instead, starting from a parametrization t^0 of the original polygon P^0 computed from the arc-lengths ρ^0_i of the *geodesic curves* on S joining two consecutive vertices P^0_i and P^0_{i+1}, we control the length of the j-th subdivision polygons P^j in such a way that, after k subdivision steps applied to the side $P^j_i P^j_{i+1}$ of polygon P^j, the length of the obtained polygon tends to be proportional, with *the same* $proportionality\ factor$ for all i, to the length of the parameter interval $t^j_{i+1} - t^j_i$ corresponding to the parametrization t^j assigned to P^j. Furthermore, the limit curve is continuous and it is parametrized by a multiple of the arc-length.

A bound for the Hausdorff distance between the limit curve and the initial polygon is also obtained.

The proposed scheme is nonlinear, hence we cannot study its properties through the Laurent polynomials formalism [Dyn92]. Instead, we rely on analytical and geometric arguments that are particular to this type of schemes.

2. The subdivision scheme

2.1. General definitions

Let $P^0 = \{P^0_i, i \in Z\}$ be an initial polygon with vertices on a given surface S, where three consecutive vertices are always noncollinear. The equations giving the polygon at step $j+1$ can be written as

$$P^{j+1}_{2i} = P^j_i, \qquad P^{j+1}_{2i+1} = g^j_i(P^j_{i+1}, P^j_i, \alpha^j) \tag{5}$$

where P^{j+1}_{2i+1} is a point on S and the control parameters $\alpha^j > 1$ satisfy the condition

$$\alpha := \prod_j \alpha^j \ < \ \infty \tag{6}$$

For any pair of points $Q_1, Q_2 \in S$ denote by $d_g(Q_1, Q_2)$ the arc-length of the *geodesic* curve on S (we use the term *geodesic* meaning the *locally shortest curve* or *shortest geodesic*, if S is a triangulation, see [MMP87]) with initial point

Q_1 and final point Q_2. In particular, for two consecutive vertices of polygon P^k, $P^k_{r+1}, P^k_r \in S$, we denote $d_g(P^k_r, P^k_{r+1})$ by ρ^k_r. The new vertex $P^{j+1}_{2i+1} = g^j_i(P^j_{i+1}, P^j_i, \alpha^j)$ is computed in such a way, that for a given parameter $\alpha^j > 1$

$$\rho^{j+1}_{2i} + \rho^{j+1}_{2i+1} = \alpha^j \, \rho^j_i \tag{7}$$

Condition (7) means that the new point P^{j+1}_{2i+1} is in the set

$$E_g := \{Q \in S \,/\, d_g(Q, P^j_i) + d_g(Q, P^j_{i+1}) = \alpha^j \, \rho^j_i\} \tag{8}$$

due its similarity with the classical definition of an ellipse, we will call this set the *geodesic ellipse* on S with foci P^j_i, P^j_{i+1} and eccentricity $1/\alpha^j$.

2.2. Convergence

To study the convergence of the subdivision scheme, we first define the parametric values corresponding to each point on the subdivision polygon.
For the initial a non-uniform parametrization t^0 of polygon P^0, we set:

$$t^0_0 = 0 \,,\; t^0_{i+1} = t^0_i + \rho^0_i \tag{9}$$

For the parametrization t^j of polygon P^j, we keep the parameters of the even indices at level $j+1$ the same as at level j, and set the new parameter t^{j+1}_{2i+1} in the interval $[t^j_i, t^j_{i+1}]$ in such a way that

$$\frac{\rho^{j+1}_{2i}}{t^{j+1}_{2i+1} - t^j_i} = \frac{\rho^{j+1}_{2i+1}}{t^j_{i+1} - t^{j+1}_{2i+1}}$$

That is,

$$t^{j+1}_{2i} = t^j_i, \quad t^{j+1}_{2i+1} = \delta^j_i t^j_i + (1-\delta^j_i) t^j_{i+1} \tag{10}$$

with $\delta^j_i = \frac{\rho^{j+1}_{2i+1}}{\rho^{j+1}_{2i} + \rho^{j+1}_{2i+1}}$.

Theorem 1 Consider the subdivision scheme (5)-(7) using the parametrization (9)-(10). If the new points P^{j+1}_{2i+1} are selected in such a way that, for all i, j

$$\rho^{j+1}_k \leq \Gamma \, \rho^j_i \quad \text{for} \quad k = 2i, 2i+1 \quad \text{with} \quad \Gamma < 1 \tag{11}$$

then the subdivision scheme *converges* and the limit curve $c(t)$ is *continuous*.

Proof: Let $P^j(t)$ be the piecewise linear function interpolating (t^j_i, P^j_i). We will show that $\|P^j(t) - P^{j+1}(t)\|_\infty$ tends uniformly to 0 when $j \to \infty$. We have

$$\begin{aligned} \|P^j(t) - P^{j+1}(t)\|_\infty &= \\ \max_i \max_{t^j_i \leq t t^j_{i+1}} \|P^j(t) - P^{j+1}(t)\| &= \\ \max_i \|P^j(t^{j+1}_{2i+1}) - P^{j+1}(t^{j+1}_{2i+1})\| & \end{aligned} \tag{12}$$

Since $P^j(t)$ is linear in $[t^j_i, t^j_{i+1}]$ and t^{j+1}_{2i+1} is given by (10) we obtain,

$$P^j(t^{j+1}_{2i+1}) = \delta^j_i P^j_i + (1-\delta^j_i) P^j_{i+1} \tag{13}$$

Substituting (13) in (12), using the value of δ^j_i in (10), and recalling that $P^{j+1}(t^{j+1}_{2i+1}) = P^{j+1}_{2i+1}$, we obtain

$$\begin{aligned} \|P^j - P^{j+1}\|_\infty = \max_i \|P^{j+1}_{2i+1} - (\delta^j_i P^j_i + (1-\delta^j_i) P^j_{i+1})\| &= \\ \max_i \|\delta^j_i (P^{j+1}_{2i+1} - P^j_i) + (1-\delta^j_i)(P^{j+1}_{2i+1} - P^j_{i+1})\| &\leq \\ \max_i \{\delta^j_i \|P^{j+1}_{2i+1} - P^j_i\| + (1-\delta^j_i)\|P^{j+1}_{2i+1} - P^j_{i+1}\|\} &\leq \\ \max_i \{\delta^j_i \rho^{j+1}_{2i} + (1-\delta^j_i)\rho^{j+1}_{2i+1}\} &\leq \\ 2\max_i \{\frac{\rho^{j+1}_{2i+1}\rho^{j+1}_{2i}}{\alpha^j \, \rho^j_i}\} & \end{aligned} \tag{14}$$

since $P^k_r = P^{k+1}_{2r}$ and , $\|P^k_{r+1} - P^k_r\|$, the euclidean distance from P^k_r to P^k_{r+1}, is smaller than the geodesic distance (on S) ρ^k_r from P^k_r to P^k_{r+1}. Using (7) and the arithmetic-geometric mean inequality, we get

$$2\frac{\rho^{j+1}_{2i+1}\rho^{j+1}_{2i}}{\alpha^j \rho^j_i} = 2\frac{\rho^{j+1}_{2i+1}\rho^{j+1}_{2i}}{\rho^{j+1}_{2i+1} + \rho^{j+1}_{2i}} \leq \frac{\rho^{j+1}_{2i+1} + \rho^{j+1}_{2i}}{2} = \frac{\alpha^j \rho^j_i}{2}$$

Therefore, from (14) we obtain,

$$\|P^j(t) - P^{j+1}(t)\|_\infty \leq \frac{\alpha^j}{2} \max_i \{\, \rho^j_i\}$$

Assuming (11), we get

$$\|P^j(t) - P^{j+1}(t)\|_\infty \leq \Gamma(\frac{\alpha^j}{2} \max_i \{\rho^{j-1}_i\}) \leq \Gamma^2(\frac{\alpha^j}{2} \max_i \{\rho^{j-2}_i\})$$

$$\leq \cdots \leq \Gamma^j(\frac{\alpha^j}{2} \max_i \{\rho^0_i\})$$

Observe that $\Gamma < 1$ and that (11) implies $\alpha^j \leq 2$, therefore passing to the limit we obtain, $\lim_{j\to\infty} \|P^j - P^{j+1}\|_\infty = 0$. The last expression means that the sequence $\{P^j(t)\}$ is a Cauchy sequence in the sup norm and in consequence it *converges*. Since we have proved that $\{P^j(t)\}$ converges uniformly, the limit function $c(t)$ has to be *continuous*. □

Remark 1 Notice that (11) is sufficient but not necessary condition. In particular, if the hypothesis holds only after a certain step j_0, the scheme still converges to a continuous curve as we can see by applying the same proof to the polygon P^{j_0}.

3. Properties of the limit curve

3.1. Distance from the subdivision curve to the polygon

In this section a bound for the Hausdorff distance between the limit curve and the initial polygon P^0 is obtained.

Lemma 1 Any vertex P_k^j of the j-th subdivision of the edge $P_i^0 P_{i+1}^0$ belongs to the set

$$P_k^j \in \{Q \in S \;/\; d_g(Q,P_i^0) + d_g(Q,P_{i+1}^0) \leq (\alpha^0\alpha^1\cdots\alpha^{j-1})\,\rho_i^0\} \tag{15}$$

i.e. , P_k^j is inside the *geodesic ellipse* with foci P_i^0, P_{i+1}^0 and eccentricity $(\alpha^0\alpha^1\cdots\alpha^{j-1})^{-1}$.

Proof: We have $P_i^0 = P_{2^j i}^j$ and $P_{i+1}^0 = P_{2^j(i+1)}^j$. The vertices in j-th step corresponding to the edge $P_i^0 P_{i+1}^0$ are P_k^j, for $k = 2^j i, \ldots, 2^j(i+1)$ and

$$\begin{aligned} d_g(P_i^0,P_k^j) + d_g(P_k^j,P_{i+1}^0) &= \\ d_g(P_{2^j i}^j,P_k^j) + d_g(P_k^j,P_{2^j(i+1)}^j) &\leq \\ \sum_{l=2^j i}^{k-1}\rho_l^j + \sum_{l=k}^{2^j(i+1)-1}\rho_l^j &= \\ \sum_{l=2^j i}^{2^j(i+1)-1}\rho_l^j &= \\ \alpha^{j-1}\sum_{l=2^{j-1}i}^{2^{j-1}(i+1)-1}\rho_l^{j-1} &= \\ \cdots & \\ = \alpha^{j-1}\alpha^{j-2}\cdots\alpha^0\rho_i^0 & \end{aligned}$$

Hence, the sum of the geodesic distances from P_k^j to $P_l^0, l = i, i+1$ is smaller or equal to $\alpha^{j-1}\alpha^{j-2}\cdots\alpha^0$ times the geodesic distance from P_i^0 to P_{i+1}^0, ρ_i^0. □

Using Lemma 1, we obtain an upper bound of the Hausdorff distance d_H between the segment of the limit curve $\{c(t), t \in [t_i^j, t_{i+1}^j]\}$ and the edge $P_i^0 P_{i+1}^0$.

Theorem 2 Let $c(t)$ be the limit curve of the subdivision scheme (5). Assume that $\alpha = \prod_{j=0}^{\infty}\alpha^j$ is finite. Then

$$d_H(\{c(t), t \in [t_i^0, t_{i+1}^0]\}, P_i^0 P_{i+1}^0) \leq \frac{\|P_{i+1}^0 - P_i^0\|\sqrt{\omega^2 - 1}}{2} \tag{16}$$

with $\omega := \frac{\alpha\rho_i^0}{\|P_{i+1}^0 - P_i^0\|}$.

Proof: From Lemma 1, we know that all points P_k^j obtained at the j-th subdivision of the edge $P_i^0 P_{i+1}^0$ are contained in the *geodesic ellipse* with foci P_i^0, P_{i+1}^0 and eccentricity $(\alpha^0\alpha^1\cdots\alpha^{j-1})^{-1}$, given by

$$\{Q \in S \;/\; d_g(Q,P_i^0) + d_g(Q,P_{i+1}^0) = (\alpha^0\alpha^1\cdots\alpha^{j-1})\rho_i^0\}$$

Let be Q a point on the curve segment $c[t_i^0, t_{i+1}^0]$, and denote by Π_Q the plane spanned by Q, P_i^0 and P_{i+1}^0. Since the euclidean distance is smaller than the geodesic distance, then Q is in the interior of the *euclidean* ellipse on Π_Q with foci P_i^0, P_{i+1}^0 and eccentricity $1/\omega$, defined by

$$E_Q := \{R \in \Pi_Q \;/\; \|R - P_i^0\| + \|R - P_{i+1}^0\| = \alpha\rho_i^0\}$$

with $\omega := \frac{\alpha\rho_i^0}{\|P_{i+1}^0 - P_i^0\|}$.
Observe that the length of the semiminor axis of the *euclidean* ellipse E_Q is $\frac{\|P_{i+1}^0 - P_i^0\|\sqrt{\omega^2-1}}{2}$, while the euclidean distance from each focus to the closest intersection point between the semimajor axis of E_Q and the euclidean ellipse E_Q is $\frac{\|P_{i+1}^0 - P_i^0\|(\omega-1)}{2}$. Since $\omega \geq 1$, the first one is bigger than the second one. Therefore, the Hausdorff distance from the section of the limit curve corresponding to the parameter interval $[t_i^0, t_{i+1}^0]$ to the edge $P_i^0 P_{i+1}^0$ is bounded above by $\frac{\|P_{i+1}^0 - P_i^0\|\sqrt{\omega^2-1}}{2}$. □

3.2. Parametrization

In this section we prove properties of the subdivision scheme, when t^0 is the parametrization defined in (9).

Theorem 3 Consider the subdivision scheme (5)-(7) using the parametrization (9)-(10) and assume that conditions (6) and (11) hold. Denote by $l^j(P_0^j, P^j(t))$ the length of the subdivision curve P^j in the step j between points P_0^j and $P^j(t)$. Then, for any $\varepsilon > 0$ exists j_0, such that

$$\left| \, l^{j_0+k}(P_0^{j_0+k}, P^{j_0+k}(t)) - \frac{R(j_0)}{R(j_0+k)}\, t \, \right| < \varepsilon$$

holds, for all $k > 0$, where $R(k) := \prod_{j \geq k}\alpha^j$.

Proof: Assume that $t_i^{j_0} < t \leq t_{i+1}^{j_0}$. Then, it is not difficult to show that using the recurrence (7) we obtain

$$l^{j_0+k}(P_0^{j_0+k}, P^{j_0+k}(t_i^{j_0})) = \frac{R(j_0)}{R(j_0+k)}\, t_i^{j_0}$$

Therefore, the equality

$$\left| \, l^{j_0+k}(P_0^{j_0+k}, P^{j_0+k}(t)) - \frac{R(j_0)}{R(j_0+k)}\, t \, \right| =$$

$$\left| l^{j_0+k}(P^{j_0+k}(t_i^{j_0}), P^{j_0+k}(t)) - \frac{R(j_0)}{R(j_0+k)}\,(t - t_i^{j_0}) \right| \tag{17}$$

holds for $t_i^{j_0} < t \leq t_{i+1}^{j_0}$.
Let be $s_i^{j_0+k}$ the sum of the geodesic distances between two consecutive vertices of the subpolygon of P^{j_0+k} that is obtained after k subdivision steps of the side $P_i^{j_0} P_{i+1}^{j_0}$ of P^{j_0}. Clearly, holds

$$l^{j_0+k}(P^{j_0+k}(t_i^{j_0}), P^{j_0+k}(t)) \leq s_i^{j_0+k} \tag{18}$$

On the other hand, if we take P^{j_0} as initial polygon and t^{j_0} as initial parametrization, we get the same limit curve $c(t)$ as when we take P^0 as initial polygon and t^0 as initial parametrization, thus after a similar argument to the one used in Lemma 1, we get that $s_i^{j_0+k}$ has the upper bound

$$s_i^{j_0+k} \leq \frac{R(j_0)}{R(j_0+k)}\,\rho_i^{j_0} \tag{19}$$

Therefore, sustituting (18) and (19) in (17), we get the following inequalities,

$$| \, l^{j_0+k}(P_0^{j_0+k}, P^{j_0+k}(t)) - \frac{R(j_0)}{R(j_0+k)} \, t \, | \; =$$

$$|l^{j_0+k}(P^{j_0+k}(t_i^{j_0}), P^{j_0+k}(t)) - \frac{R(j_0)}{R(j_0+k)} \, (t - t_i^{j_0})| \; \leq$$

$$\frac{R(j_0)}{R(j_0+k)} \, (\rho_i^{j_0} - (t - t_i^{j_0})) \; <$$

$$\frac{R(j_0)}{R(j_0+k)} \, \rho_i^{j_0}$$

Recall that (6) implies that $R(j_0+k) \rightarrow_{k\rightarrow\infty} 1$. On the other hand, after Theorem 1, the subdivision scheme (5)-(7) using the parametrization (9)-(10) is convergent. Hence, exists j_0, such that

$$\frac{R(j_0)}{R(j_0+k)} < 2 \text{ and } \max_i \{\rho_i^{j_0}\} < \varepsilon/2$$

hold, and substituting these inequalities in the last expression above we get the desired result. □

Remark 2 The above result means that for sufficiently large j, the piecewise linear function $P^j(t)$ interpolating (t_i^j, P_i^j) is approximately parametrized by a multiple of the arc-length. Indeed, let $c(t)$ be the limit curve and assume that conditions (6) and (11) hold. Defining $L(0,t) := \lim_{k\rightarrow\infty} l^{j_0+k}(P_0^0, P^{j_0+k}(t))$ as the arc-length of the section of $c(t)$ between points $c(0) = P_0^0$ and $c(t)$, we get that $L(0,t) \simeq R(j_0)\, t$ holds. Hence, $c(t)$ is parametrized approximately by a multiple of the arc-length.

4. Conclusions

We described a subdivision scheme with control over the length of the limit curve, suitable for designing curves on surfaces.

At each subdivision step, similarly to the classic 4-point scheme, the existing vertices are retained, making the scheme interpolatory.

Despite the parametrization is not uniform, it is possible to compute a sequence of m points approximately on the subdivision curve, with approximately uniform arc-length distribution. Recall that even when these points are not exactly on the subdivision curve, a bound for the Hausdorff distance between any point of the polygon in the last step and the subdivision curve can be computed.

The formulation of the scheme is presented for general two dimensional manifolds. In the particular case of triangulated surfaces, we believe that the results obtained in [MCV07, MVC08] will make it possible to obtain efficient implementations. That should be discussed in detail in a future work.

Acknowledgments

The first two authors has been supported by CITMA/Cuba under grant PNCB0409. J. Estrada-Sarlabous and V. Hernández-Mederos acknowledge also the support of TWAS-UNESCO-CNPq and Visgraf-IMPA Brazil in the frame of the TWAS-UNESCO / CNPq-Brazil Associateship Ref. 3240173676 and Ref. 3240173677 respectively.

References

[DD89] DESLAURIERS G., DUBUC S.: Symmetric iterative interpolation processes. *Constructive Approximation 5*, 1 (1989), 49–68.

[DFH09] DYN N., FLOATER M., HORMANN K.: Four point curve subdivision based on iterated chordal and centripetal parameterizations. *Computer Aided Geometric Design 26* (2009), 279–286.

[DLG87] DYN N., LEVIN D., GREGORY J. A.: A four-point interpolatory subdivision scheme for curve design. *Computer Aided Geometric Design 4* (1987), 257–268.

[Dub86] DUBUC S.: Interpolation through an iterative scheme. *Journal of Mathematical Analysis and Applications 114* (1986), 185–204.

[Dyn92] DYN N.: Subdivision schemes in computer-aided geometric design. In *Advances in numerical analysis* (1992), Light W., (Ed.), vol. 2, Clarendon Press, pp. 36–104.

[HEIM09] HERNÁNDEZ V., ESTRADA J., IVRISSIMTZIS I., MORALES S.: Curve subdivision with arc-length control. *To appear in Computing* (2009).

[KS98] KOBBELT L., SCHRÖDER P.: A multiresolution framework for variational subdivision. *ACM Trans. on Graph. 17*, 4 (1998), 209–237.

[MCV07] MARTÍNEZ D., CARVALHO P. C., VELHO L.: Geodesic Bézier curves: A tool for modeling on triangulations. In *Proceedings of SIBGRAPI 2007. XX Brazilian Symposium on Computer Graphics and Image Processing* (2007), IEEE Computer Society, pp. 71–78.

[MMP87] MITCHELL J. S. B., MOUNT D. M., PAPADIMITRIOU C. H.: The discrete geodesic problem. *SIAM Journal on Computing 16*, 4 (1987), 647–668.

[MVC05] MARTÍNEZ D., VELHO L., CARVALHO P. C.: Computing geodesics on triangular meshes. *Computer and Graphics 29*, 5 (October 2005), 667–675.

[MVC08] MARTÍNEZ D., VELHO L., CARVALHO P.: Subdivision curves on surfaces and applications. In *CIARP* (2008), pp. 405–412.

[PH05] POTTMANN H., HOFER M.: A variational aproach to spline curves on surfaces. *Computer Aided Geometric Design 22*, 7 (October 2005), 693–709.

[PR95] PARK F. C., RAVANI B.: Bezier curves on Riemannian manifolds and Lie groups with kinematic applications. *ASME Journal of Mechanical Design 117* (1995), 36–40.

[RLJ05] RODRIGUEZ R. C., LEITE F. S., JACUBIAK J.: A new geometric algorithm to generate smooth interpolating curves on riemannian manifolds. *LMS Journal of Computation and Mathematics 8* (2005), 251–266.

[Wal06] WALLNER J.: Smoothness analysis of subdivision schemes by proximity. *Constr. Approx. 24*, 3 (2006), 289–318.

[WD05] WALLNER J., DYN N.: Convergence and C^1 analysis of subdivision schemes on manifolds by proximity. *Comput. Aided Geom. Design 22*, 7 (2005), 593–622.

[WP06] WALLNER J., POTTMANN. H.: Intrinsic subdivision with smooth limits for graphics and animation. *ACM Trans. Graphics 25*, 2 (2006), 356–374.