VANISHING VISCOSITY WITH SHORT WAVE LONG WAVE INTERACTIONS FOR MULTI-D SCALAR CONSERVATION LAWS

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Abstract. We consider a system coupling a multidimensional semilinear Schrödinger equation and a multidimensional nonlinear scalar conservation law with viscosity, which is motivated by a model of short wave-long wave interaction introduced by Benney (1977). We prove the global existence and uniqueness of the solution of the Cauchy problem for this system. We also prove the convergence of the whole sequence of solutions when the viscosity ε and the interaction parameter α approach zero so that $\alpha = o(\varepsilon^{1/2})$. We also indicate how to extend these results to more general systems which couple multidimensional semilinear systems of Schrödinger equations with multidimensional nonlinear systems of scalar conservation laws mildly coupled.

1. Introduction

We consider the Cauchy problem for the multidimensional system

$$iu_t + \Delta u = |u|^\gamma u + \alpha\, g(v)\, h'\left(|u|^2\right) u,$$
$$v_t + \mathbf{a}\cdot\nabla f(v) = \alpha\ \mathbf{a}\cdot\nabla\left(h\left(|u^2|\right) g'(v)\right) + \varepsilon\Delta v,$$

which is motivated by the model of short wave-long wave interaction introduced by Benney [3]. We prove the global existence of a unique solution of this system in $H^1(\mathbb{R}^N)$. We also analize the problem of the convergence of the solutions when $\varepsilon, \alpha \to 0$. We prove the convergence of the whole sequence of solutions when $\varepsilon, \alpha \to 0$ with $\alpha = o(\varepsilon^{1/2})$ to the solution of the corresponding pair of decoupled equations. We also indicate how these results may be extended to systems coupling several semilinear Schrödinger equations and several mildly coupled nonlinear scalar conservation laws.

We recall that in [7] many onedimensional systems coupling a semilinear Schrödinger equation with nonlinear systems of conservation laws, including some of the most representatives, were analyzed. The coupling with a particular type of scalar conservation law was addressed earlier in [6]. Other Benney type models were studied in, e.g., [19], [20], [1] and [2].

The remaining of the paper is organized as follows. In Section 2, we prove the local and global existence of a unique solution of the Cauchy problem. In Section 3, we prove the convergence of the solutions when $\varepsilon, \alpha \to 0$, with $\alpha = o(\varepsilon^{1/2})$. In Section 4, we indicate how these results may be easily extended to more general systems coupling several semilinear Schrödinger equations with several nonlinear mildly coupled scalar conservation laws.

2. Existence and uniqueness of a global solution

We consider the following Cauchy problem

$$iu_t + \Delta u = |u|^\gamma u + \alpha\, g(v)\, h'\left(|u|^2\right) u, \tag{2.1}$$
$$v_t + \mathbf{a}\cdot\nabla f(v) = \alpha\ \mathbf{a}\cdot\nabla\left(h\left(|u^2|\right) g'(v)\right) + \varepsilon\Delta v, \tag{2.2}$$
$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \tag{2.3}$$

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where $u(x,t) \in \mathbb{C}$, $v(x,t) \in \mathbb{R}$, $x \in \mathbb{R}^N$, $N \geq 2$, $t \geq 0$, $0 \leq \gamma < \frac{4}{N-2}$. $\mathbf{a}$ is a velocity field such that

$$\mathbf{a} \in W^{1,\infty}(\mathbb{R}^N)^N$$

and

$$\text{div}\, \mathbf{a} = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{2.4}$$

f, g are C^3 real functions, $\operatorname{supp} g' \subseteq [-M_0, M_0]$, $h : [0, +\infty) \to [0, +\infty)$ is a nonnegative bounded C^3 function with $\operatorname{supp} h' \subseteq [0, M_1]$ for certain positive constants M_0, M_1, and $\alpha, \varepsilon > 0$. For simplicity we also assume $f(0) = g(0) = h(0) = 0$.

We assume

$$u_0 \in H^1(\mathbb{R}^N), \quad v_0 \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \tag{2.5}$$

with $\|v_0\|_\infty \leq M_0$ and M_0 is as above.

Before starting the study of the Cauchy problem (2.1)–(2.3) we recall the Strichartz's estimates which are a powerful tool in the study of multidimensional nonlinear Schrödinger equations. We begin by recalling the definition of an admissible pair.

Definition 2.1. We say that a pair (q, r) is *admissible* if

$$\frac{2}{q} = N(\frac{1}{2} - \frac{1}{r}) \tag{2.6}$$

and

$$2 \leq r \leq \frac{2N}{N-2} \qquad (2 \leq r \leq \infty \text{ if } N = 1,\ 2 \leq r < \infty \text{ if } N = 2). \tag{2.7}$$

We observe that if (q, r) is an admissible pair, then $2 \leq q \leq \infty$. We also note that the pair $(\infty, 2)$ is always admissible and the pair $(2, \frac{2N}{N-2})$ is admissible if $N \geq 3$.

We now state the well known Strichartz's estimates, originally due to Strichartz [16] and generalized and improved by Ginibre and Velo [9], Yajima [17], Cazenave and Weissler [5] and Keel and Tao [11]. We refer to [4] for the proof and comments.

Let $U(t)$ be the unitary group of operators in $L^2(\mathbb{R}^N)$ associated with the Schrödinger operator $i\frac{\partial}{\partial t} + \Delta$, that is, given $\varphi \in L^2$, $U(t)\varphi$ is the solution of $iu_t + \Delta u = 0$, $u(x, 0) = \varphi(x)$.

Theorem 2.1 (Strichartz's estimates)**.** *The following hold:*

(i) *For every $\varphi \in L^2(\mathbb{R}^N)$, the function $t \mapsto U(t)\varphi$ belongs to*

$$L^q(\mathbb{R}, L^r(\mathbb{R}^N)) \cap C(\mathbb{R}, L^2(\mathbb{R}^N))$$

for every admissible pair (q, r). Further, there exists a constant C such that

$$\|U(\cdot)\varphi\|_{L^q(\mathbb{R},L^r)} \leq C\|\varphi\|_{L^2} \qquad \text{for every } \varphi \in L^2(\mathbb{R}^N). \tag{2.8}$$

(ii) *Let I be an interval of $\mathbb{R}$ (bounded or not), $J = \bar{I}$, and $t_0 \in J$. If (κ, ρ) is an admissible pair and $f \in L^{\kappa'}(I, L^{\rho'}(\mathbb{R}^N))$, where $1/\kappa' + 1/\kappa = 1$, $1/\rho' + 1/\rho = 1$, then, for every admissible pair (q, r), the function*

$$t \mapsto \Phi_f(t) = \int_{t_0}^t U(t - s)f(s)\, ds, \qquad \text{for } t \in I,$$

belongs to $L^q(I, L^r(\mathbb{R}^N)) \cap C(J, L^2(\mathbb{R}^N))$. Further, there exists a constant C independent of I such that

$$\|\Phi_f\|_{L^q(I,L^r)} \leq C\|f\|_{L^{\kappa'}(I,L^{\rho'})}, \qquad \text{for every } f \in L^{\kappa'}(I, L^{\rho'}(\mathbb{R}^N)). \tag{2.9}$$

Next, we state what we mean by a solution of the Cauchy problem (2.1)–(2.3).

Definition 2.2. For (u_0, v_0) as above, a pair (u, v) is said to be a solution of (2.1)–(2.3) in $\mathbb{R}^N \times [0, T]$ if

$$(u, v) \in C([0, T], H^1(\mathbb{R}^N)) \cap C^1([0, T], H^{-1}(\mathbb{R}^N)),$$

equations (2.1) and (2.2) are satisfied in $H^{-1}(\mathbb{R}^N)$ and (2.3) holds. We say that a pair (u, v) is a solution of (2.1)–(2.3) in $\mathbb{R}^N \times [0, T)$ (resp., $\mathbb{R}^N \times [0, \infty)$) if (u, v) is a solution of (2.1)–(2.3) in $\mathbb{R}^N \times [0, T_0]$, for all $0 < T_0 < T$ (resp., $0 < T_0 < \infty$).

The following theorem establishes the existence and uniqueness of a local solution to (2.1)–(2.3).

Theorem 2.2. *Let u_0, v_0 satisfy* (2.5)*. Then, there exists $T > 0$ such that the Cauchy problem* (2.1)–(2.3) *has a unique solution $(u, v) \in C([0, T]; H^1(\mathbb{R}^N))$.*

Proof. The following proof is an adaptation of a method due to T. Kato [10] which is based on a fixed point argument using Strichartz's estimates. We closely follow the lines of this method as exposed in [4], section 4.4. For simplicity throughout this proof we make $\varepsilon = \alpha = 1$.

Since

$$|(|u_1|^\gamma u_1) - (|u_2|^\gamma u_2)| \le C(|u_1|^\gamma + |u_2|^\gamma)|u_1 - u_2|,$$

and, by the assumptions on g and h,

$$|g(v_1)\, h'\left(|u_1|^2\right) u_1 - g(v_2)\, h'\left(|u_2|^2\right) u_2| \le C(|u_1 - u_2| + |v_1 - v_2|),$$

by Hölder's inequality, with $r = \gamma + 2$, we deduce that

$$\|(|u_1|^\gamma u_1) - (|u_2|^\gamma u_2)\|_{L^{r'}} \le C(\|u_1\|_{L^r}^\gamma + \|u_2\|_{L^r}^\gamma)\|u_1 - u_2\|_{L^r}, \tag{2.10}$$

where $1/r' + 1/r = 1$, and

$$\|g(v_1)\, h'\left(|u_1|^2\right) u_1 - g(v_2)\, h'\left(|u_2|^2\right) u_2\|_{L^2} \le C(\|u_1 - u_2\|_{L^2} + \|v_1 - v_2\|_{L^2}). \tag{2.11}$$

From (2.10) and (2.11) we deduce

$$\|\nabla(|u|^\gamma u)\|_{L^{r'}} \le C\|u\|_{L^r}^\gamma \|\nabla u\|_{L^r}, \tag{2.12}$$

and

$$\|\nabla\left(g(v)h'(|u|^2)u\right)\|_{L^2} \le C(\|\nabla u\|_{L^2} + \|\nabla v\|_{L^2}). \tag{2.13}$$

Fix $M, T > 0$ to be choosen later and let q be such that (q, r) is an admissible pair. Consider the set

$$E = \Big\{(u, v) \in \left(L^\infty((0, T); H^1(\mathbb{R}^N)) \cap L^q((0, T); W^{1,r}(\mathbb{R}^N))\right) \times \left(L^\infty((0, T); H^1(\mathbb{R}^N)) \cap L^\infty(\mathbb{R}^N \times (0, T))\right) :$$
$$\|u\|_{L^\infty((0,T);H^1)} \le M, \quad \|u\|_{L^q((0,T);W^{1,r})} \le M, \quad \|v\|_{L^\infty((0,T);H^1)} \le M, \quad \|v\|_{L^\infty(\mathbb{R}^N \times (0,T))} \le M\Big\}$$

equipped with the distance

$$d((u_1, v_1), (u_2, v_2)) = \|u_1 - u_2\|_{L^q((0,T);L^r)} + \|u_1 - u_2\|_{L^\infty((0,T);L^2)} + \|v_1 - v_2\|_{L^\infty((0,T);L^2)}. \tag{2.14}$$

We easily see that that (E, d) is a complete metric space.

Let $\Lambda_1(u)$ denote the nonlinear operator $u \mapsto |u|^\gamma u$, which by (2.10) is continuous $L^r(\mathbb{R}^N) \to L^{r'}(\mathbb{R}^N)$, and let $\Lambda_2(u, v)$ denote the nonlinear operator $(u, v) \mapsto g(v)h'(|u|^2)u$, which by (2.11) is continuous $L^2(\mathbb{R}^N)^2 \to L^2(\mathbb{R}^N)$. We have the following: $\Lambda_1(u) \in L^q((0, T), W^{1,r'}(\mathbb{R}^N))$, $\Lambda_2(u, v) \in L^\infty((0, T), H^1(\mathbb{R}^N))$,

$$\|\Lambda_1(u)\|_{L^q((0,T),W^{1,r'})} \le C\|u\|_{L^\infty((0,T),L^r)}^\gamma \|u\|_{L^q((0,T),W^{1,r'})},$$
$$\|\Lambda_2(u, v)\|_{L^\infty((0,T);H^1)} \le C(\|u\|_{L^\infty((0,T),H^1)} + \|v\|_{L^\infty((0,T),H^1)}),$$

and

$$\|\Lambda_1(u_1) - \Lambda_1(u_2)\|_{L^q((0,T),L^{r'})}$$
$$\le C\left(\|u_1\|_{L^\infty((0,T),L^r)}^\gamma + \|u_2\|_{L^\infty((0,T),L^r)}^\gamma\right)\|u_1 - u_2\|_{L^q((0,T),L^r)},$$
$$\|\Lambda_2(u_1, v_1) - \Lambda_2(u_2, v_2)\|_{L^\infty((0,T),L^2)} \le C\left(\|u_1 - u_2\|_{L^\infty((0,T),L^2)} + \|v_1 - v_2\|_{L^\infty((0,T),L^2)}\right).$$

From the embedding $H^1(\mathbb{R}^N) \to L^r(\mathbb{R}^N)$, Hölder's inequality in time and the above estimates we deduce

$$\|\Lambda_1(u)\|_{L^{q'}((0,T),W^{1,r'})} + \|\Lambda_2(u,v)\|_{L^1((0,T),H^1)} \le C(T + T^{\frac{q-q'}{qq'}})(1+M^\gamma)M \tag{2.15}$$

and

$$\begin{aligned}\|\Lambda_1(u_1) - \Lambda_1(u_2)\|_{L^{q'}((0,T),L^{r'})} + \|\Lambda_2(u_1,v_1) - \Lambda_2(u_2,v_2)\|_{L^1((0,T),L^2)} \\ \le C(T + T^{\frac{q-q'}{qq'}})(1+M^\gamma)d((u_1,v_1),(u_2,v_2)).\end{aligned} \tag{2.16}$$

Now, for $(u,v) \in E$, let $\mathcal{H}(u,v)(t) = (\mathcal{H}_1(u,v), \mathcal{H}_2(u,v))$ be defined by

$$\mathcal{H}(u,v)(t) = (\mathcal{H}_1(u,v), \mathcal{H}_2(u,v)) := \Big(U(t)u_0 + i\int_0^t U(t-s)\left(\Lambda_1(u)(s) + \Lambda_2(u,v)(s)\right) ds\,,$$
$$S(t)v_0 - \int_0^t \mathbf{a}\cdot\nabla K(t-s) * (f(v(s)) - (h(|u|^2)g'(v))(s))\, ds \Big)$$

where $U(t)$ is the unitary propagator of the Schrödinger operator as above, $S(t)v_0 := K(t) * v_0$ is the contraction semigroup associated to the heat equation, and $K(x,t) := (4\pi t)^{-N/2}e^{-|x|^2/4t}$ is the well known heat kernell.

It follows from (2.15) and Strichartz's estimates that

$$\mathcal{H}_1(u,v) \in C([0,T]; H^1(\mathbb{R}^N)) \cap L^q((0,T); W^{1,r}(\mathbb{R}^N)), \tag{2.17}$$

and

$$\|\mathcal{H}_1(u,v)\|_{L^\infty((0,T);H^1)} + \|\mathcal{H}_1(u,v)\|_{L^q((0,T);W^{1,r})} \le C\,\|u_0\|_{H^1} + C\,(T + T^{\frac{q-q'}{qq'}})\,(1+M^\gamma)M.$$

Similarly, from (2.16) we deduce that

$$\begin{aligned}\|\mathcal{H}_1(u_1,v_1) - \mathcal{H}_1(u_2,v_2)\|_{L^\infty((0,T);L^2)} + \|\mathcal{H}_1(u_1,v_1) - \mathcal{H}_1(u_2,v_2)\|_{L^q((0,T);L^r)} \\ \le C\,(T + T^{\frac{q-q'}{qq'}})\,(1+M^\gamma)M\,d((u_1,v_1),\,(u_2,v_2)).\end{aligned}$$

We note that

$$\frac{q-q'}{qq'} = 1 - \frac{2}{q} = N\frac{4-(N-2)\gamma}{2N(\gamma+2)} > 0.$$

As to $\mathcal{H}_2(u,v)$, using the well known facts about the heat kernell (cf.,e.g., [15])

$$\|K(t)\|_{L^1(\mathbb{R}^N)} = 1, \qquad \|\nabla K(t)\|_{L^1(\mathbb{R}^N)} \le \frac{C}{\sqrt{t}},$$

and using the boundedness of h in $C^1([0,\infty))$ and the Lipschitz continuity of f and g it is easy to deduce that

$$\mathcal{H}_2(u,v) \in C([0,T], H^1(\mathbb{R}^N)) \cap L^\infty(\mathbb{R}^N \times [0,T]), \tag{2.18}$$

$$\|\mathcal{H}_2(u,v)(t)\|_{H^1} \le \|v_0\|_{H^1} + C\,T^{1/2}M\left(\|u\|_{L^\infty((0,T);H^1)} + \|v\|_{L^\infty((0,T);H^1)}\right)$$

and

$$\|\mathcal{H}_2(u,v)(t)\|_{L^\infty} \le \|v_0\|_{L^\infty} + C\,T^{1/2}M\|v\|_{L^\infty(\mathbb{R}^N\times(0,T))}.$$

Therefore,

$$\begin{aligned}&\|\mathcal{H}_2(u,v)\|_{L^\infty((0,T);H^1)} \\ &\quad \le \|v_0\|_{H^1} + C\,T^{1/2}M\left(\|u\|_{L^\infty((0,T);H^1)} + \|v\|_{L^\infty((0,T);H^1)}\right),\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}&\|\mathcal{H}_2(u,v)\|_{L^\infty(\mathbb{R}^N\times(0,T))}\\ &\qquad\le \|v_0\|_{L^\infty}+C\,T^{1/2}M\|v\|_{L^\infty(\mathbb{R}^N\times(0,T))}.\end{aligned} \tag{2.20}$$

Similarly, we deduce

$$\begin{aligned}\|\mathcal{H}_2(u_1,v_1)(t)-\mathcal{H}_2(u_2,v_2)(t)\|_{L^\infty((0,T);L^2)}&\\ \le CT^{1/2}M\left(\|u_1-u_2\|_{L^\infty((0,T);L^2)}+\|v_1-v_2\|_{L^\infty((0,T);L^2)}\right)&\\ \le CT^{1/2}M\,d((u_1,v_1),(u_2,v_2)).&\end{aligned} \tag{2.21}$$

Now, given $u_0\in H^1(\mathbb{R}^N)$ and $v_0\in H^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, we set

$$M=\max\{2C\|u_0\|_{H^1},2\|v_0\|_{H^1},2\|v_0\|_{L^\infty}\},$$

and choose T small enough so that

$$\max\{C(T+T^{\frac{q-q'}{qq'}})(1+M^\gamma),CT^{1/2}M\}\le\frac{1}{2}.$$

In this way, for $(u,v)\in E$ we obtain

$$\|\mathcal{H}_1(u,v)\|_{L^\infty((0,T),H^1)}+\|\mathcal{H}_1(u,v)\|_{L^q((0,T),W^{1,r})}\le M,$$

and

$$\|\mathcal{H}_2(u,v)\|_{L^\infty((0,T),H^1)}+\|\mathcal{H}_2(u,v)\|_{L^\infty(\mathbb{R}^N\times(0,T))}\le M,$$

so that $\mathcal{H}(u,v)\in E$ and for $(u_1,v_1),(u_2,v_2)\in E$,

$$d(\mathcal{H}(u_1,v_1),\mathcal{H}(u_2,v_2))\le\frac{1}{2}\,d((u_1,v_1),(u_2,v_2)).$$

In particular, $\mathcal{H}$ is a strict contraction and so Banach fixed point theorem implies the existence of a unique fixed point $(u,v)\in E$. Moreover, by (2.17) and (2.18) we deduce that (u,v) is a solution of (2.1)–(2.3) which concludes the proof.

□

To extend the local solution given by Theorem 2.2 to a global one we will make use of the following result, which is the multidimensional analogue of Lemma 2.1 of [7]. The proof is entirely similar to that of the corresponding onedimensional result in [7] and so we omit it.

Lemma 2.1. *Let $M_2>M_0$ so that $\operatorname{supp} g'\subseteq(-M_2,M_2)$. Let (u,v) be a solution of (2.1)–(2.3) in $\mathbb{R}^N\times[0,T)$ and assume that $\|v_0\|_\infty<M_2$. Then*

$$\|v\|_{L^\infty(\mathbb{R}^N\times[0,T))}\le M_2. \tag{2.22}$$

We now establish the existence of a global solution to the Cauchy problem (2.1)–(2.3).

Theorem 2.3. *Let (u,v) be a solution of the Cauchy problem (2.1)–(2.3) in $\mathbb{R}^N\times[0,T)$. Then, there exist functions $b_1,b_2\in C([0,\infty))$, with b_1 independent of $\alpha,\varepsilon\in(0,1)$, such that*

$$\|u(t)\|_{H^1}\le b_1(t),\qquad \|v(t)\|_{H^1}\le b_2(t),\qquad t\in[0,T). \tag{2.23}$$

In particular, (u,v) may be extended to a solution in $\mathbb{R}^N\times[0,\infty)$, which is unique.

Proof. As usual we first obtain some basic integral identities. The first one, the so called conservation of charge, is easily obtained by multiplying (2.1) by $\bar u$, integrating by parts in $\mathbb{R}^N$, and taking the imaginary part, which gives

$$\frac{d}{dt}\int_{\mathbb{R}^N}|u|^2\,dx=0. \tag{2.24}$$

The second one, the conservation of energy for u, is obtained by first multiplying (2.1) by $\bar{u}_t$, integrating in $\mathbb{R}^N$, and taking the real part, which gives

$$\frac{d}{dt}\int_{\mathbb{R}^N}\left\{\frac{1}{2}|\nabla u|^2+\frac{1}{\gamma+2}|u|^{\gamma+2}+\alpha g(v)h(|u|^2)\right\}dx=\int_{\mathbb{R}^N}\alpha h(|u|^2)g(v)_t\,dx.$$

Now, the right-hand side of the above equation is computed by multiplying the equation (2.2) by $\alpha h(|u|^2)g'(v)$, integrating in $\mathbb{R}^N$, using integration by parts, to obtain

$$\begin{aligned}\int_{\mathbb{R}^N}\alpha h(|u|^2)g(v)_t\,dx&=\int_{\mathbb{R}^N}\{\alpha f(v)\mathbf{a}\cdot\nabla(h(|u|^2)g'(v))-\alpha\varepsilon g'(v)h'(|u|^2)\nabla|u|^2\cdot\nabla v-\alpha\varepsilon h(|u|^2)g''(v)|\nabla v|^2\}\,dx\\&=\int_{\mathbb{R}^N}\{f(v)(v_t+\mathbf{a}\cdot\nabla f(v)-\varepsilon\Delta v)-\alpha\,\varepsilon g'(v)h'(|u|^2)\nabla|u|^2\cdot\nabla v-\alpha\,\varepsilon h(|u|^2)g''(v)|\nabla v|^2\}\,dx\\&=\int_{\mathbb{R}^N}\{F(v)_t+\varepsilon f'(v)|\nabla v|^2-\alpha\,\varepsilon g'(v)h'(|u|^2)\nabla|u|^2\cdot\nabla v-\alpha\,\varepsilon h(|u|^2)g''(v)|\nabla v|^2\}\,dx,\end{aligned}$$

which then gives the conservation of energy for u

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^N}&\left\{\frac{1}{2}|\nabla u|^2+\frac{1}{\gamma+2}|u|^{\gamma+2}+\alpha g(v)h(|u|^2)-F(v)\right\}dx\\&=\int_{\mathbb{R}^N}\{\varepsilon f'(v)|\nabla v|^2-\alpha\,\varepsilon g'(v)h'(|u|^2)\nabla|u|^2\cdot\nabla v-\alpha\,\varepsilon h(|u|^2)g''(v)|\nabla v|^2\}\,dx,\end{aligned}\tag{2.25}$$

where $F(v)=\int_0^v f(\sigma)\,d\sigma$. The last integral identity, the conservation of energy for v, is obtained by multiplying (2.2) by v and integrating in $\mathbb{R}^N$, using integration by parts, which then gives

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}v^2\,dx+\varepsilon\int_{\mathbb{R}^N}|\nabla v|^2\,dx=\int_{\mathbb{R}^N}\alpha\,g(v)h'(|u|^2)\mathbf{a}\cdot\nabla|u|^2\,dx.\tag{2.26}$$

Integrating (2.26) in time, recalling that $h'(|u|^2)=0$, for $|u|^2>M_1$, we obtain the estimate

$$\|v(t)\|_2^2+\varepsilon\int_0^t\|\nabla v(s)\|_2^2\,ds\le C+C\alpha\int_0^t\|\nabla u(s)\|_2\,ds,\tag{2.27}$$

where we have used (2.22) and (2.24). Here, $\|\cdot\|_p$ denotes the norm in $L^p(\mathbb{R}^N)$, and, as it will be henceforth, C denotes a positive constant possibly depending only on the initial data and $f,g,h,\mathbf{a},M_0,M_1,M_2$.

Now, we are going to obtain an estimate for $\|\nabla u\|_2^2$ from (2.25) and (2.27). To this, observe that $|F(v(x,t))|\le C|v(x,t)|^2$, because $F(0)=F'(0)=0$ and (2.22), and $|\alpha\,g(v(x,t))h(|u(x,t)|^2)|\le C\,\alpha|u(x,t)|^2$, because $h(0)=0$ and g and h' are bounded. Moreover, $|\alpha\,\varepsilon g'(v)h'(|u|^2)\nabla|u|^2\cdot\nabla v|\le C\alpha\,\varepsilon|\nabla u||\nabla v|\le C\alpha\,\varepsilon(|\nabla u|^2+|\nabla v|^2)$, observing that C may change its value from one occurrence to another. Thus, from (2.25) and (2.27) we get

$$\|\nabla u\|_2^2\le C+C\int_0^t(\|\nabla u(s)\|_2+\|\nabla u(s)\|_2^2)\,ds,\tag{2.28}$$

which from Gronwall's inequality gives

$$\|\nabla u\|_2^2\le e^{C(t+1)},\tag{2.29}$$

where C, as always in this proof, depends only on the initial data and $f,g,h,\mathbf{a},M_0,M_1,M_2$. In particular, (2.29) and (2.23) establish a bound for $\|u\|_{H^1}$ independent of α,ε as claimed.

Now, from (2.2), using a well known fact about the non-homogeneous heat equation, we obtain that, for each $0\le t<T$,

$$\|v_t\|_{L^2([0,t],L^2(\mathbb{R}^N))}\,,\ \varepsilon\|\Delta v\|_{L^2([0,t],L^2(\mathbb{R}^N))}\le C\|\mathbf{a}\cdot\nabla(-f(v)+h(|u|^2)g'(v))\|_{L^2([0,t],L^2(\mathbb{R}^N))},$$

which, from (2.27) and (2.29), gives

$$\|v_t\|_{L^2([0,t],L^2(\mathbb{R}^N))}+\|v\|_{L^2([0,t],H^2(\mathbb{R}^N)}\le b(t),$$

with $b \in C([0,\infty))$ depending on α, ε, which, by interpolation (see, e.g., Theorem 3.1 in [14]), gives

$$\|v(t)\|_{H^1} \le b_2(t),$$

for some $b_2 \in C([0,\infty))$, which concludes the proof of (2.23). The fact that (2.23) allows to extend (u,v) to a solution of (2.1)–(2.3) in $\mathbb{R}^N \times [0,\infty)$ and the uniqueness follow in a standard way.

□

3. Vanishing viscosity and short wave long wave interaction coefficient.

In this section we analyze the problem of the convergence of the solutions of (2.1)–(2.3) when ε and α approach 0.

We recall that a function $v \in L^\infty(\mathbb{R}^N \times [0,\infty))$ is said to be an *entropy solution* of the Cauchy problem for the scalar conservation law

$$v_t + \mathbf{a}\cdot\nabla f(v) = 0, \tag{3.1}$$

$$v(x,0) = v_0(x), \tag{3.2}$$

if for any convex entropy-entropy flux pair, that is, any pair $(\eta,q) \in C^1(\mathbb{R})^2$ satisfying $q'(v) = \eta'(v)f'(v)$ with η convex, we have

$$\int_0^\infty \int_{\mathbb{R}^N} \{\eta(v)\phi_t + q(v)\mathbf{a}\cdot\nabla\phi\}\,dx\,dt + \int_{\mathbb{R}^N} \eta(v_0(x))\phi(x,0)\,dx \ge 0 \tag{3.3}$$

for all nonnegative $\phi \in C_c^\infty(\mathbb{R}^{N+1})$.

We also recall that a complex valued function u is a *weak H^1-solution* of the Cauchy problem for the nonlinear Schrödinger equation

$$iu_t + \Delta u = |u|^\gamma u, \tag{3.4}$$

$$u(x,0) = u_0(x), \tag{3.5}$$

if

$$u \in L^\infty_{\rm loc}([0,\infty), H^1(\mathbb{R}^N)) \cap W^{1,\infty}_{\rm loc}([0,\infty), H^{-1}(\mathbb{R}^N)) \tag{3.6}$$

is such that equation (3.4) is satisfied in $H^{-1}(\mathbb{R}^N)$ for a.e. $t \in [0,\infty)$ and $u(0) = u_0$. The latter makes sense in $L^2(\mathbb{R}^N)$ since (3.6) implies $u \in C([0,\infty), L^2(\mathbb{R}^N))$.

We recall that, by Kruzhkov [12], there is uniqueness of the entropy solution of (3.1)-(3.2) and, by Kato [10] (cf. Proposition 4.2.1 in [4]), there is uniqueness of the weak H^1-solution of the problem (3.4)-(3.5).

We have the following result.

Theorem 3.1. *If $\varepsilon \to 0$ and $\alpha \to 0$ so that $\alpha/\varepsilon^{1/2} \to 0$ also, that is, $\alpha = o(\varepsilon^{1/2})$, then the solutions $(u^\varepsilon, v^\varepsilon)$ of* (2.1)–(2.3) *converge to a pair (u,v) such that u is the weak H^1-solution of* (3.4)-(3.5) *and v is the entropy solution of* (3.1)-(3.2).

Proof. From Theorem 2.3, u^ε is uniformly bounded in $L^\infty_{\rm loc}([0,\infty), H^1(\mathbb{R}^N))$ as $\varepsilon, \alpha \to 0$, with $\alpha = o(\varepsilon^{1/2})$. Also, $|u^\varepsilon|^\gamma u^\varepsilon$ is uniformly bounded in $L_{\rm loc}([0,\infty), H^{-1}(\mathbb{R}^N))$, which follows by Sobolev's embedding $H^1(\mathbb{R}^N) \to L^r(\mathbb{R}^N)$, $r = \gamma+2$, $N \ge 2$, and $\alpha g(v^\varepsilon)h'(|u^\varepsilon|^2 u^\varepsilon \to 0$ in $L^2_{\rm loc}(\mathbb{R}^N \times [0,\infty))$, since h' and g are bounded and by (2.24). From (2.1) we then see that u^ε_t is uniformly bounded in $L^\infty_{\rm loc}([0,\infty), H^{-1}(\mathbb{R}^N))$. From Aubin's lemma (see, e.g., [13]), we deduce that u^ε is precompact in $L^2_{\rm loc}(\mathbb{R}^N \times [0,\infty))$, and so any sequence u^{ε_i} possesses a subsequence converging to a complex valued function $u \in L^\infty_{\rm loc}([0,\infty), H^1(\mathbb{R}^N))$, which clearly satisfies (3.4) in the sense of distributions and $|u|^\gamma u \in L^\infty_{\rm loc}([0,\infty), H^{-1}(\mathbb{R}^N))$. In particular, $\Delta u - |u|^\gamma u \in L^\infty_{\rm loc}([0,\infty), H^{-1}(\mathbb{R}^N))$. Hence, since u satisfies (3.4) in the sense of distributions, we deduce that $u \in W^{1,\infty}_{\rm loc}([0,\infty), H^{-1}(\mathbb{R}^N))$, and so u is the weak H^1-solution of (3.4)-(3.5). This implies that the whole sequence u^ε converges to u.

To prove the convergence of v^ε to the entropy solution v of (3.1)-(3.2) we are going to apply DiPerna's theorem on the uniqueness of admissible measure-valued solutions of (3.1)-(3.2) (cf. [8]). We recall that an admissible measure-valued solution of (3.1)-(3.2) is a measurable map $(x,t)\mapsto \nu_{x,t}$, from $\mathbb{R}^N\times[0,\infty)$ into the space of probability measures on a compact $K\subseteq\mathbb{R}$, which satisfies

$$\int_0^\infty\int_{\mathbb{R}^N}\{\langle\nu_{x,t},\eta(\lambda)\rangle\phi_t+\langle\nu_{x,t},q(\lambda)\rangle\mathbf{a}\cdot\nabla\phi\}\,dx\,dt\geq 0, \tag{3.7}$$

for all convex entropy-entropy flux pairs (η,q) for (3.1) and all $\phi\in C_c^\infty(\mathbb{R}^N\times(0,\infty))$, and such that

$$\lim_{T\to 0}\frac{1}{T}\int_0^T\int_{|x|<R}\langle\nu_{x,t},|\lambda-v_0(x)|\rangle\,dx\,dt=0 \qquad \text{for all } R>0. \tag{3.8}$$

DiPerna's theorem asserts that an admissible measure-valued solution of (3.1)-(3.2) must coincide a.e. $(x,t)\in\mathbb{R}^N\times(0,\infty)$ with $\delta_{v(x,t)}$, the Dirac measure concentrated at $v(x,t)$, where $v(x,t)$ is the entropy solution of (3.1)-(3.2). We recall that when $\nu_{x,t}$ is a Dirac measure almost everywhere in $\mathbb{R}^N\times(0,\infty)$, the associated subsequence converges in $L^1_{\text{loc}}(\mathbb{R}^N\times(0,\infty))$ (cf. [18]).

Now, we have that v^ε is uniformly bounded in $L^\infty(\mathbb{R}^N\times[0,\infty))$ because of Lemma 2.1. Therefore, we can apply Tartar's theorem on the existence of Young measures associated to subsequences of a sequaence of uniformly bounded functions in L^∞ (cf. [18]). So, let $\{\nu_{x,t}\,:\,(x,t)\in\mathbb{R}^N\times(0,\infty)\}$ be the parametrized family of Young measures associated to a subsequence v^{ε_i} of v^ε, which we will keep denoting v^ε. Clearly, $\operatorname{supp}\nu_{x,t}\subseteq[-M_2,M_2]$, by Lemma 2.1.

By (2.26) and the fact that $\alpha=o(\varepsilon^{1/2})$, we deduce that

$$\alpha\mathbf{a}\cdot\nabla(g'(v^\varepsilon)h(|u^\varepsilon|^2))\to 0 \qquad \text{in } L^2_{\text{loc}}(\mathbb{R}^N\times[0,\infty)). \tag{3.9}$$

Now, given any convex entropy-entropy flux pair for (3.1), (η,q), and any nonnegative $\phi\in C_c^\infty(\mathbb{R}^{N+1})$, multiply equation (2.2), with $u^\varepsilon,v^\varepsilon$ replacing u,v, by $\eta'(v^\varepsilon)\phi$, integrate in $\mathbb{R}^N\times(0,\infty)$, using integration by parts, and make $\varepsilon\to 0$ to obtain

$$\int_0^\infty\int_{\mathbb{R}^N}\{\langle\nu_{x,t},\eta(\lambda)\rangle\phi_t+\langle\nu_{x,t},q(\lambda)\rangle\mathbf{a}\cdot\nabla\phi\}\,dx\,dt+\int_{\mathbb{R}^N}\eta(v_0(x))\phi(x,0)\,dx\geq 0 \tag{3.10}$$

In particular, $\nu_{x,t}$ satisfies (3.8) for nonnegative $\phi\in C_c^\infty(\mathbb{R}^N\times(0,\infty))$.

Let us take in (3.10) $\phi(x,t)=\psi(x)\chi_k(t)$ with $0\leq\psi\in C_c^\infty(\mathbb{R}^N)$ and $\chi_k(t)=1$, for $t\in(-T,T)$, $\chi_k(t)=\max\{1-k|T-t|,0\}$, for $|t|\geq T$, where T is a Lebesgue point of $\int_{\mathbb{R}^N}\langle\nu_{x,t},\eta(\lambda)\rangle\psi\,dx$, which we can obviously do by a standard approximation argument. Then, letting $k\to\infty$ we get from (3.10)

$$-\int_{\mathbb{R}^N}\langle\nu_{x,T},\eta(\lambda)\rangle\psi(x)\,dx+\int_0^T\int_{\mathbb{R}^N}\langle\nu_{x,t},q(\lambda)\rangle\mathbf{a}\cdot\nabla\psi(x)\,dx\,dt+\int_{\mathbb{R}^N}\eta(v_0(x))\psi(x)\,dx\geq 0, \tag{3.11}$$

and so, making $T\to 0$, we obtain

$$\limsup_{T\to 0}\int_{\mathbb{R}^N}\langle\nu_{x,T},\eta(\lambda)\rangle\psi(x)\,dx\leq\int_{\mathbb{R}^N}\eta(v_0(x))\psi(x)\,dx \tag{3.12}$$

The above inequality is easily extended to $0\leq\psi\in L^1(\mathbb{R}^N)$ and $\eta(v)=|v-\xi|$, $\xi\in\mathbb{R}$. Therefore, approaching $v_0(x)$ in $L^1_{\text{loc}}(\mathbb{R}^N)$ by linear combinations of characteristic functions $\varphi^\nu:=\sum_{j=1}^N\xi_j\chi_{E_j}$, using (3.12) with $\eta(v)=|v-\xi_j|$ and $\psi=\chi_{E_j}$, adding up for $j=1,\dots,N$, and passing to the limit when $\varphi^\nu\to v_0$ in $L^1_{\text{loc}}(\mathbb{R}^N)$ we get

$$\lim_{T\to 0}\int_{|x|<R}\langle\nu_{x,T},|\lambda-v_0(x)|\rangle\,dx=0, \qquad \text{for all } R>0, \tag{3.13}$$

which implies (3.8). Hence, $\nu_{x,t}$ is an admissible measure-valued solution of (3.1)-(3.2) and by DiPerna's theorem we conclude that the subsequence v^ε converges in $L^1_{\text{loc}}(\mathbb{R}^N\times(0,\infty))$ to the entropy solution $v(x,t)$

of (3.1)-(3.2). By the uniqueness of the limit, we finally conclude that the whole sequence v^ε converges to v, which finishes the proof.

□

4. Final Remarks

The results of the previous sections may be easily extended to more general systems of the form

$$i\partial_t u_j + \Delta u_j = u_j|u|^{\gamma_k} + \alpha_k g_k(v_k)h_k'(|u|^2)u_j, \tag{4.1}$$

$$\partial_t v_k + \mathbf{a}_k \cdot \nabla f_k(v_k) = \alpha_k \mathbf{a}_k \cdot \nabla(g_k'(v_k)h_k(|u|^2)) + B_k(v)v_k + \varepsilon_k \Delta v_k, \qquad j = 1,\dots,r,\ k = 1,\dots,s, \tag{4.2}$$

with $u_j, v_k, \alpha_k, \varepsilon_k\gamma_k, f_k, g_k, h_k$ satisfying the same hypotheses as $u, v, \alpha, \varepsilon, \gamma, f, g, h$ in (2.1)-(2.2), respectively, $B_k : \mathbb{R}^s \to \mathbb{R}$ satisfying $B_k \in C^2(\mathbb{R}^s) \cap L^\infty(\mathbb{R}^s)$, $|u|^2 = |u_1|^2 + \cdots + |u_r|^2$, $v = (v_1, \dots, v_s)$.

The local existence follows as in the proof of Theorem 2.2, since it is based on Strichartz estimates for the Schrödinger operator, and standard estimates for the heat operator, which may be easily extended to system (4.1)-(4.2).

Concerning the global existence, it also follows as in the proof of Theorem 2.3. Clearly, now, instead of (2.22), we have

$$\|v_k(t)\|_\infty \le M_2 e^{C_k t}, \qquad t \in [0,T),\ k = 1,\dots,s, \tag{4.3}$$

where $C_k = \|B_k\|_\infty$. The analogues of (2.24), (2.25) and (2.26) are obtained in similar way and from the corresponding identities one easily proves the necessary energy estimates which allow to prolong the local solution to $[0,\infty)$.

Convergence of solutions when $\varepsilon_k \to 0$ and $\alpha_k = o(\varepsilon_k^{1/2})$, $k = 1,\dots,s$, is also proved by arguments completely analogous to those in the proof of Theorem 3.1.

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