APERIODIC CLASSES

JIAGANG YANG

Abstract. We show that there exists a C^1 residual subset $R \subset C^1(M)\setminus\overline{HT}$, such that for $f \in R$ and C an aperiodic class of f, C has a non-trivial partial hyperbolic splitting with 1-dimensional central bundle: $T_C M = E^s_{i_0} \oplus E^c_1 \oplus E^u_{i_0+2}$ where $E^s_{i_0}(C), E^u_{i_0+2}(C) \neq \phi$ and C is an index i_0 and i_0+1 fundamental limit. With [6]'s argument, we show C is Hausdorff limit of a family of non-trivial homoclinic classes. As a corollary, we give a new proof for the following two results which have been proved in [37], [38] respectively: suppose C is a non-trivial chain recurrent class of f, if $C \bigcap P_0^* \neq \phi$ or C is Lyapunov stable, C is a homoclinic class.

Contents

1. Introduction

In the middle of last century, with many remarkable work, hyperbolic diffeomorphisms have been understood very well, but soon people discovered that the set of hyperbolic diffeomorphisms are not dense among differential dynamics, such non-hyperbolic example at first was given by Abraham-Smale and

Date: February 21, 2009.

Partially supported by TWAS-CNPq, FAPERJ.

later more examples appeared, until now all the examples about non-hyperbolic systems (persist non-hyperbolic) can be divided to two kinds of cases: one associates with homoclinic tangency and another associates with heterdimensional cycle. In order to describe the non-hyperbolic diffeomorphisms, in 80's Palis gave the following famous conjecture:

Palis Conjecture *Diffeomorphisms of M exhibiting either a homoclinic tangency or heterdimensional cycle are C^r dense in the complement of the C^1 closure of hyperbolic systems.*

Palis conjecture gives the candidates of mechanists for us to understand the robust non-hyperbolic examples. Now understanding the non-hyperbolic systems has become one of the most important aim in modern dynamical system, one way to study it is try to understand every chain recurrent class of a generic subset of diffeomorphisms, and such way has been proved to be very effective and powerful.

But different with hyperbolic case, a non-hyperbolic diffeomorphism can have infinite number of chain recurrent classes, in fact in [4] they gave such an example, they showed that there exists an open set $\mathcal{U} \subset \overline{HT^1}$ and $R \subset \mathcal{U}$ a residual subset such that every $f \in R$ has infinite number of sinks or sources. We call a diffeomorphism wild (tame) if f has infinite (finite) of chain recurrent classes. Since in [4]'s example $\mathcal{U} \subset \overline{HT^1}$ and residual diffeomorphisms in $\mathcal{U}$ are wild, it means that the dynamics in $\overline{HT^1}$ is extremely complicated, so we just consider $(\overline{HT^1})^c$ in this paper, and for well known reason, we just consider C^1 topology here. The following result is the best thing we can hope for:

Tameness conjecture *There exists a generic subset $R \subset (\overline{HT^1})^c$ such that any $f \in R$ is tame.*

But the above conjecture is still far away to be solved, in [38] I gave a weaker conjecture:

Conjecture 5: *There exists a generic subset $R \subset (\overline{HT^1})^c$ such that for any $f \in R$, suppose C is any aperiodic class of f, then C has a partial hyperbolic splitting $T_C M = E^s \oplus E^c \oplus E^u$ where $E^s, E^u \neq \phi$ and $dim(E^c) = 1$.*

The conjecture 5 played an important role in the proof of Palis weak conjecture which claims that for C^1 residual diffeomorphisms either it's Morse-Smale or the diffeomorphism contains non-trivial homoclinic class. [6] proved conjecture 5 in 3-dimensional case and they showed that conjecture 5 implies Palis weak conjecture. But in high dimensional manifold, Palis weak conjecture was proved by Crovisier finally through the studying of minimal non-hyperbolic set with his remarkable central model argument.

In this paper I'll prove the conjecture 5, more precisely statement is following:

Theorem 1: *There exists a generic subset $R \subset (\overline{HT^1})^c$ such that for any $f \in R$, suppose C is an aperiodic class of f, then C has a partial hyperbolic splitting $T_C M = E^s \oplus E^c \oplus E^u$ where $E^s, E^u \neq \phi$ and $dim(E^c) = 1$.*

The following corollary shows the relation between theorem 1 and Palis weakly conjecture:

Corollary 1: *$f \in R$, C is an aperiodic class of f, then C is Hausdorff limit of a family of non-trivial homoclinic classes.*

Proof : We just need the following lemma proved in [6]:

Lemma 1: *For any $f \in R$, suppose C is an aperiodic class of f with partial hyperbolic splitting $E^s \oplus E^c_1 \oplus E^u$ where $dim(E^c_1) = 1$ and E^c_1 is not hyperbolic, then C is Hausdorff limit of a family of non-trivial homoclinic classes.* □

Remark: [8] *showed that for $f \in R$, C is an aperiodic class and $\Lambda \subset C$ is a minimal non-hyperbolic subset, then Λ is the Hausdorff limit of a family of non-trivial locally restricted homoclinic classes.*

With theorem 1, we can easily prove the following two results, anyway they have been proved in [37], [38] already and the statements there are stronger.

Theorem 2: *Suppose $f \in R$, C is a Lyapunov stable chain recurrent class of f, then C is a homoclinic class.*

Theorem 3: *Suppose $f \in R$, C is a chain recurrent class of f satisfying $C \bigcap P^*_0 \neq \phi$, then C is a homoclinic class.*

There is another conjecture given by Bonatti,

Index complement conjecture: (Bonatti) *There exists a residual subset $R \subset C^1(M)$ such that for any $f \in R$ and C is a chain recurrent class of f, let $I = \{i :\ C$ is an index i fundamental limit$\}$, then I is an interval.*

By theorem 1, we can prove index complement conjecture for diffeomorphisms which are far away from tangency and when C is an aperiodic class.

Theorem 4: *There exists a residual subset $R \subset C^1(M) \setminus \overline{HT^1}$ such that for any $f \in R$ and C is an aperiodic class of f, C has a non-trivial partial hyperbolic splitting $E^s_{i_0} \oplus E^c_1 \oplus E^u_{i_0+2}$ and $I(C) = \{i_0, i_1\}$.*

In §3 we'll give two new generic properties which are proved in §6, 7 respectively. Theorem 1, 2 3,4 will be proved in §4, and in §5 we'll introduce some properties for partial hyperbolic splitting set and Crovisier's central model.

After this preprint was written, we learned from S. Crovisier that he has some related results in a preprint that should appear soon.

Acknowledgements: The author would like to thank his advisor Professor Marcelo Viana for the support and enormous encouragements during the preparation of this work. I would like to thank Shaobo Gan for listening the proof. I also thank Jacob Palis, Lan Wen, Enrique R. Pujals, Lorenzo Diaz, Christian

Bonatti for very helpful remarks. Finally I wish to thank my wife, Wenyan Zhong, for her help and encouragement.

2. Definitions and Notations

Let M be a compact boundless Riemannian manifold, since when M is a surface [26] has proved that hyperbolic diffeomorphisms are open and dense in $C^1(M) \setminus \overline{HT}$, we suppose $dim(M) = d > 2$ in this paper.

Let $Per(f)$ denote the set of periodic points of f, for $p \in Per(f)$, $\pi(p)$ means the period of p. If p is a hyperbolic periodic point, the index of p is the dimension of the stable bundle. We denote $Per_i(f)$ the set of the index i periodic periodic points of f, and we call a point x is an index i preperiodic point of f if there exists a family of diffeomorphisms $g_n \xrightarrow{C^1} f$, where g_n has an index i periodic point p_n and $p_n \longrightarrow x$. $P_i^*(f)$ is the set of index i preperiodic points of f.

Remark 2.1. *It's easy to know* $\overline{P_i(f)} \subset P_i^*(f)$.

Let Λ be an invariant compact set of f, we call Λ is an index i fundamental limit if there exists a family of diffeomorphisms g_n C^1 converging to f, p_n is an index i periodic point of g_n and $Orb(p_n)$ converge to Λ in *Hausdorff* topology. So if $\Lambda(f)$ is an index i fundamental limit, we have $\Lambda(f) \subset P_i^*(f)$. Λ is a minimal index i fundamental limit if $\Lambda(f)$ is an index i fundamental limit and any invariant compact subset $\Lambda_0 \subsetneq \Lambda$ is not an index i fundamental limit, we can also define maximal index i fundamental limit. In [37] with Zorn lemma, we have proved the following result:

Lemma 2.2. *Any index i fundamental limit contains a minimal index i fundamental limit.*

In fact, it's easy to show the following similar result is also true.

Lemma 2.3. *Suppose Λ is an invariant compact set of f containing index i fundamental limit, then Λ contains a maximal index i fundamental limit.*

For two points $x, y \in M$ and some $\delta > 0$, we say there exists a δ-pseudo orbit connects x and y means that there exist points $x = x_0, x_1, \cdots, x_n = y$ such that $d(f(x_i), x_{i+1}) < \delta$ for $i = 0, 1, \cdots, n-1$, and we denote it $x \underset{\delta}{\dashv} y$. We say $x \dashv y$ if for any $\delta > 0$ we have $x \underset{\delta}{\dashv} y$ and denote $x \vdash\dashv y$ if $x \dashv y$ and $y \dashv x$. A point x is called a chain recurrent point if $x \vdash\dashv x$. $CR(f)$ denotes the set of chain recurrent points of f, it's easy to know that $\vdash\dashv$ is a closed equivalent relation on $CR(f)$, and every equivalent class of such relation should be compact and is called chain recurrent class.

Let K be a compact invariant set of f, if x, y are two points in K, we'll denote $x \underset{K}{\dashv} y$ if for any $\delta > 0$, we have a δ -pseudo orbit in K connects x and y. If for any two points $x, y \in K$ we have $x \underset{K}{\dashv} y$, we call K a chain recurrent set. Let C be a chain recurrent class of f, we say C is an aperiodic class if C does not contain periodic point.

Let Λ be an invariant compact set of f, for $0 < \lambda < 1$ and $1 \le i < d$, we say Λ has an index $i-(l, \lambda)$ dominated splitting if we have a continuous invariant splitting $T_\Lambda M = E \oplus F$ where $dim(E_x) = i$ and $\| Df^l|_{E(x)} \| \cdot \| Df^{-l}|_{F(f^l x)} \| < \lambda$ for all $x \in \Lambda$. For simplicity, sometimes we just say $\Lambda(f)$ has an index i dominated splitting. A compact invariant set can have many dominated splittings, but for fixed i, the index i dominated splitting is unique.

We say a diffeomorphism f has C^r tangency if $f \in C^r(M)$, f has a hyperbolic periodic point p and there exists a non-transverse intersection between $W^s(p)$ and $W^u(p)$. HT^r denotes the set of the diffeomorphisms which have C^r tangency, usually we just use HT denote HT^1. We call a diffeomorphism f is far away from tangency if $f \in C^1(M) \setminus \overline{HT}$. The following proposition shows the relation between dominated splitting and far away from tangency.

Proposition 2.4. *([29]) f is C^1 far away from tangency if and only if there exists (l, λ) such that $P_i^*(f)$ has index $i - (l, \lambda)$ dominated splitting for $0 < i < d$.*

Usually dominated splitting is not a hyperbolic splitting, Mañé showed that in some special case, one bundle of the dominated splitting is hyperbolic.

Proposition 2.5. *([21]) Suppose $\Lambda(f)$ has an index i dominated splitting $E \oplus F$ ($i \neq 0$), let $j_0 = \min_j\{j : \Lambda$ contains index j fundamental limit$\}$, if $j_0 \geq i$, then E is a contracting bundle.*

3. Generic properties

In this section at first we'll introduce some C^1 generic properties, they are either well known or proved in [37]; and then we'll give two new generic properties lemma 3.5, 3.6 which will be proved in § 6, 7 respectively.

For a topology space X, we call a set $R \subset X$ is a generic subset of X if R is countable intersection of open and dense subsets of X, and we call a property is a generic property of X if there exists some generic subset R of X holds such property. Especially, when $X = C^1(M)$ and R is a generic subset of $C^1(M)$, we just call R is C^1 generic, and we call any generic property of $C^1(M)$ 'a C^1 generic property' or 'the property is C^1 generic'.

At first let's state some well known C^1 generic properties.

Proposition 3.1. *There is a C^1 generic subset R_0' such that for any $f \in R_0'$, one has*

1) *f is Kupka-Smale (every periodic point p in $Per(f)$ is hyperbolic and the invariant manifolds of periodic points are everywhere transverse).*
2) *$CR(f) = \Omega = \overline{Per(f)}$.*
3) *$P_i^*(f) = \overline{P_i(f)}$*
4) *any chain recurrent set is the Hausdorff limit of periodic orbits.*
5) *any index i fundamental limit is the Hausdorff limit of index i periodic orbits of f.*
6) *any chain recurrent class containing a periodic point p is the homoclinic class $H(p, f)$.*
7) *suppose C is a homoclinic class of f, and $i_0 = min\{i : C \bigcap Per_i(f) \neq \phi\}$, $i_1 = max\{i : C \bigcap Per_i(f) \neq \phi\}$, then for any $i_0 \leq i \leq i_1$, we have $C \bigcap Per_i(f) \neq \phi$.*

By proposition 3.1, for any f in R_0', every chain recurrent class C of f is either an aperiodic class or a homoclinic class. If $\#(C) = \infty$, we say C is non-trivial.

The following results are proved in [37]:

Theorem 3.2. *There exists a generic subset $R_0 \subset C^1(M) \setminus \overline{HT}$, such that for any $f \in R_0$ and C is a non-trivial chain recurrent class of f, if $C \bigcap P_0^* \neq \phi$, then C is a homoclinic class containing index 1 periodic points and C is an index 0 fundamental limit.*

Lemma 3.3. *There exists a generic subset $R_0 \subset C^1(M)\backslash\overline{HT}$, such that for $f \in R_0$ and C is a non-trivial chain recurrent class of f, let $j_0 = \min_j\{j : C \bigcap P_j^* \neq \phi\}$ and Λ is a minimal index j_0 fundamental limit in C, then*

- *either Λ is a non-trivial minimal set with partial hyperbolic splitting $E^s_{j_0} \oplus E^c_1 \oplus E^u_{j_0+2}$*
- *or C contains a periodic point with index j_0 or j_0+1 and C is an index j_0 fundamental limit.*

Definition 3.4. *$f \in C^1(M)$, $\{p_n\}_{n=1}^{\infty}$ is a family of index i hyperbolic periodic points and $\lim_{n\to\infty} \pi(p_n) \longrightarrow \infty$, we say $\{p_n\}$ is index stable if for any $\varepsilon > 0$, we have $\#\{n|$ exists diffeomorphism g satisfying $d_{C^1}(g, f) < \varepsilon$ and $Orb_f(p_n)$ is a hyperbolic periodic orbit of g whose index different with $i\} < \infty$.*

Lemma 3.5. *(Shaobo Gan's lemma) There exists a generic subset $R_0 \subset C^1(M) \setminus \overline{HT}$, such that for $f \in R_0$, suppose $\{p_n(f)\}$ is a family of index i_0 ($i_0 \neq 0, d$) periodic points of f which is index stable and satisfies $\lim_{n\to\infty} \pi(p_n) \longrightarrow \infty$, then there exists a subsequence $\{p_{n_i}\}$ such that $W^s_{loc}(Orb(p_{n_i})) \pitchfork W^u_{loc}(Orb(p_{n_j})) \neq \phi$, so especially, if $\lim_{n\to\infty} Orb(p_n) = \Lambda$ and suppose C is the chain recurrent class which contains Λ, then C contains an index i_0 periodic point.*

Now I give two basic generic properties whose proof will be given in §6, 7 respectively.

Lemma 3.6. *For $f \in R_0 \bigcap R_0' \subset C^1(M) \setminus \overline{HT}$, C is a non-trivial chain recurrent class of f, $\Lambda \subsetneq C$ is an invariant compact subset of f, denote $i_0 = \min_i\{i :$ Λ contains an index i fundamental limit$\}$, $i_1 = \max_i\{i :$ Λ contains an index i fundamental limit$\}$. Suppose Λ itself is an index i_0 and index i_1 fundamental limit and $i_1 > i_0 + 1$, then C contains an index i periodic point with $i_0 \leq i \leq i_1$.*

Lemma 3.7. *There exists a generic subset $R \subset R_0 \bigcap R_0'$ such that for $f \in R$, suppose C is a non-trivial chain recurrent class of f, $C_0 \subset C$ is a non-trivial chain recurrent set of f, denote $j_0 = \min_j\{j :$ C_0 contains an index j fundamental limit$\}$ and let $\Lambda \subset C_0$ be a maximal index j_0 fundamental limit of C_0, then if Λ has a partial hyperbolic splitting $E^s_{j_0} \oplus E^c_1 \oplus E^u_{j_0+2}$ where $E^c_1(\Lambda) = 1$ and $E^c_1(\Lambda)$ is not hyperbolic, we have*

- *either $C_0 = \Lambda$,*
- *or C contains index j_0 or $j_0 + 1$ periodic point.*

We'll show that the above residual subset $R \subset (\overline{HT})^c$ satisfies theorem 1, 2 3, 4.

4. Proof of theorem 1, 2, 3, 4

4.1. Proof of theorem 1.

Denote $j_0 = \min_j\{j : C$ contains index j fundamental limit$\}$ and let C_0 be a maximal index j_0 fundamental limit.

Denote $j_{01} = \max_j\{j : C_0$ contains index j fundamental limit$\}$ and let C_{01} be a maximal index j_{01} fundamental limit of C_0.

Denote $j_{010} = \min_j\{j : C_{01}$ contains index j fundamental limit$\}$ and let C_{010} be a maximal index j_{010} fundamental limit of C_{01}.

Define $\alpha_n = (\overbrace{01\cdots01}^{n'01'})$ and $\beta_n = (\overbrace{01\cdots010}^{n-1'01'})$, repeat above induction, we can denote $j_{\beta_n} = \min_j\{j :$ $C_{\alpha_{n-1}}$ contains index j fundamental limit$\}$ and let C_{β_n} be a maximal index j_{β_n} fundamental limit of

$C_{\alpha_{n-1}}$; denote $j_{\alpha_n} = \max_j\{j : C_{\beta_n}$ contains index j fundamental limit$\}$ and let C_{α_n} be a maximal index j_{α_n} fundamental limit of C_{β_n}.

It's easy to know that

- $j_0 \leq j_{010} \leq \cdots \leq j_{\beta_n} \leq j_{\beta_{n+1}} \leq \cdots$ and $j_{01} \geq j_{0101} \geq \cdots \geq j_{\alpha_n} \geq j_{\alpha_{n+1}} \geq \cdots$;
- $j_{\alpha_n} \geq j_{\beta_n} + 1$;
- $C_0 \supset C_{01} \supset \cdots \supset C_{\beta_n} \supset C_{\alpha_n} \supset \cdots$.

Let $C_\infty = \bigcap_n C_{\alpha_n} = \bigcap_n C_{\beta_n}$, denote $i_0 = \lim_{n\to\infty} j_{\beta_n}$ and $i_1 = \lim_{n\to\infty} j_{\alpha_n}$, then by above induction, we can know that C_∞ is an index i_0 and index i_1 fundamental limit and $i_0 = \min_i\{i : C_\infty$ contains index i fundamental limit$\}$, $i_1 = \max_i\{i : C_\infty$ contains index i fundamental limit$\}$.

At first let's note that $i_1 \neq i_0$, since otherwise by proposition 2.4, 2.5, C_∞ is a hyperbolic set, by shadowing lemma, there exists periodic point in the same chain recurrent class with C_∞, that's a contradiction with the fact that C is an aperiodic class. Now we divide the proof into two cases:

A) $i_1 > i_0 + 1$,

B) $i_1 = i_0 + 1$

Case A: Recall $i_0 = \min_i\{i : C_\infty$ contains index i fundamental limit$\}$, $i_1 = \max_i\{i : C_\infty$ contains index i fundamental limit$\}$, lemma 3.6 shows that C contains a periodic point, it's a contradiction since C is an aperiodic class.

Case B: In this case $C_\infty \subset P^*_{i_0} \bigcap P^*_{i_0+1}$, by the fact $i_0 = \min_i\{i : C_\infty$ contains index i fundamental limit$\}$, $i_1 = \max_i\{i : C_\infty$ contains index i fundamental limit$\}$, proposition 2.4, 2.5 show that C_∞ has the following partial hyperbolic splitting $T_{C_\infty}M = E^s_{i_0} \oplus E^c_1 \oplus E^u_{i_0+2}$ where $dim(E^c_1(C_\infty)) = 1$. Then there exists a small neighborhood V of C_∞ such that the maximal invariant set of $\overline{V}$: $\Lambda = \bigcap_i f^i(\overline{V})$ has the same kind of partial hyperbolic splitting. Recall that $C_\infty = \lim_n C_{\alpha_n} = \lim_n C_{\beta_n}$, now we claim that there exists n_0 such that for any $n \geq n_0$ we have $C_\infty = C_{\alpha_n} = C_{\beta_n}$ and $j_{\alpha_n} = i_0 + 1$, $j_{\beta_n} = i_0$.

Proof of the claim: We can choose n_0 big enough such that $C_{\alpha_{n_0}} \subset V$, then $C_{\alpha_{n_0}}$ will has the partial hyperbolic splitting $T_{C_{\alpha_{n_0}}}M = E^s_{i_0} \oplus E^c_1 \oplus E^u_{i_0+2}$. By generic property 4) of proposition 3.1, there is a family of periodic point $\{p_n\}$ satisfying $\lim_{n\to\infty} Orb(p_n) \longrightarrow C_{\alpha_{n_0}}$, and we can suppose the family of periodic points all have index i_0 or $i_0 + 1$, here we just suppose they all have index i_0, then $C_{\alpha_{n_0}}$ is an index i_0 fundamental limit. But from C is an aperiodic class, by Gan's lemma we know that the family of periodic points $\{p_n\}$ is not index stable, it means that for any $\varepsilon > 0$, there exists n big enough and $d_{C^1}(g, f) < \varepsilon$ such that $Orb_f(p_n)$ is an index j periodic point of g where $j \neq i_0$. Since $Orb(p_n)$ stays near $C_{\alpha_{n_0}}$, and $C_{\alpha_{n_0}}$ has the special partial hyperbolic splitting, we can know $j = i_0 + 1$, so $C_{\alpha_{n_0}}$ is also an index $i_0 + 1$ fundamental limit. By the construction of C_∞ we have that

$$C_{\alpha_{n_0}} = C_{\beta_{n_0+1}} = C_{\alpha_{n_0+1}} = \cdots = C_\infty.$$

□

From above claim we know that $i_{\alpha_{n_0}} = \max_i\{i : C_{\beta_{n_0}} = i_0 + 1$ contains index i fundamental limit$\}$ and $C_{\alpha_{n_0}}$ is the maximal index $i_0 + 1$ fundamental limit of $C_{\beta_{n_0}}$, and $C_{\alpha_{n_0}} = C_\infty$ has the partial hyperbolic splitting $T_{C_{\alpha_{n_0}}}M = E^s_{i_0} \oplus E^c_1 \oplus E^u_{i_0+2}$, so by lemma 3.7, $C_{\beta_{n_0}} = C_{\alpha_{n_0}} = C_\infty$.

Repeat the argument, we can know that

$$C_\infty = C_{\alpha_{n_0}} = C_{\beta_{n_0}} = C_{\alpha_{n_0-1}} = \cdots = C_0 = C,$$

so C has the partial hyperbolic splitting $E^s_{i_0} \oplus E^c_1 \oplus E^u_{i_0+2}$. Now we claim that $i_0 \neq 0, d-1$.
Proof of the claim: Here we only need the following result in [6]:

Lemma 3: *$f \in R$, C is a non-trivial chain recurrent class of f with partial hyperbolic splitting $E^c_1 \oplus E^u$ where $dim(E^c = 1)$ and E^c_1 is not hyperbolic, then C is a homoclinic class.* □

4.2. **Proof of theorem 2, 3, 4.**
Proof of theorem 2: It's easy to know that theorem 2 is just a corollary of theorem 1 and the following result has been proved in [6]:

Lemma 2: *$f \in R$, C is a non-trivial chain recurrent class of f with partial hyperbolic splitting $E^s \oplus E^c_1 \oplus E^u$ where $dim(E^c = 1)$ and E^c_1 is not hyperbolic, then C is a homoclinic class.* □

Proof of theorem 3: Now we suppose C is an aperiodic class of f, then by theorem 1, C has partial hyperbolic splitting $E^s \oplus E^c_1 \oplus E^u$ where $dim(E^c = 1)$ and E^c_1 is not hyperbolic, by $C \bigcap P^*_0 \neq \phi$, we know that $E^s(C) = \phi$, so C has partial hyperbolic splitting $E^c_1 \oplus E^u$, that's a contradiction with theorem 1 since $E^s_(C) = \phi$ here. □

Proof of theorem 4: By 4) of proposition 3.1 and $f \in R$, C is the Hausdorff limit of a family of periodic points $p_n(f)$. In theorem 1 we've known that C has partial hyperbolic splitting $E^s_{i_0} \oplus E^c_1 \oplus E^u_{i_0+2}(C)$, so $I(C) \subset i_0, i_0+1$ and we can suppose p_n all have index i_0 (since the other case is similar). By Gan's lemma, $\{p_n(f)\}$ is not index stable (since C doesn't contain periodic point), that means for any $\varepsilon > 0$, there exist n arbitrarily big and a diffeomorphism g satisfying $d_{C^1}(f,g) < \varepsilon$ and $Orb_f(p_n)$ is index i_0+1 periodic point of g, so C is also an index i_0+1 fundamental limit, that means $I(C) = \{i_0, i_0+1\}$ □

5. Partial hyperbolic splitting and Crovisier's central model

In order to do some preparation for the proof of lemma 3.6 given in § 6, in this section we'll introduce some basic facts about partial hyperbolic splitting and Crovisier's central model. The main results are lemma 5.2 and corollary 5.12.

5.1. **Partial hyperbolic splitting.** Suppose $f \in R$, Λ is a compact chain recurrent set of f with a dominated splitting $E^{cs}_{i_0} \oplus E^c_1 \oplus E^{cu}_{i_0+2}$ where $dim(E^c_1(\Lambda)) = 1$, then we can choose a small neighborhood V_0 of Λ such that the maximal invariant set of $\overline{V_0}$: $\Lambda_0 = \bigcap_j f^j(\overline{V_0})$ has the same type of partial hyperbolic splitting $E^{cs}_{i_0} \oplus E^c_1 \oplus E^{cu}_{i_0+2}$ also, in fact, we can extend such splitting to $\overline{V_0}$ (it's not invariant anymore). For every point $x \in \overline{V_0}$, we define some cones on its tangent space $C^i_a(x) = \{v | v \in T_xM$, there exists $v' \in E^i(x)$ such that $d(\frac{v}{|v|}, \frac{v'}{|v'|}) < a\}_{i=cs,c,cu,ccs,ccu}$ where $E^{ccs} = E^{cs}_{i_0} \oplus E^c_1$ and $E^{ccu} = E^{cu}_{i_0} \oplus E^c_1$, and we call it an index i_0-cone C_a^i. When a is small enough, $C_a^i(x)\bigcap C_a^j(x) = \phi$ $_{(i\neq j=cs,c,cu)}$, $C_a^{ccs}(x)\bigcap C_a^{cu}(x) = \phi$, $C_a^{ccu}\bigcap C_a^{cs} = \phi$ and $Df(C_a^i(x)) \subset C_a^i(f(x))$ $_{i=cu,ccu}$, $Df^{-1}(C_a^i(x)) \subset C_a^i(f^{-1}(x))$ $_{i=cs,ccs}$ for $x \in \Lambda_0$.

We say a submanifold D^i $(i = cs, c, cu, ccs, ccu)$ tangents with index i_0 cone C_a^i when $dim(D^i) = dim(E^i)$ and for any $x \in D^i$, $T_xD^i \subset C_a^i(x)$. For simplicity, sometimes we just call it an index i_0 i-disk, especially when $i = c$, we call D^c an index i_0 central curve, and when the index i_0 has been fixed, we just call D^i an i-disk. We say an index i_0 i-disk D^i has center x with size δ if $x \in D^i$, and respecting the Riemannian metric restricting on D^i, the ball centered on x with radius δ is contained in D^i. We say an i-disk D^i has center x with radius δ if $x \in D^i$, and respecting the Riemannian metric restricting on D^i, the distance between any point $y \in D^i$ and x is smaller than δ.

We say an index i_0 central curve γ is an index i_0 **central segment** if $f^i(\gamma) \subset V_0$ and $f^i(\gamma)$ is an index i_0 central curve for any $i \in \mathbb{Z}$, so if γ is a central segment, we have $\gamma \subset \Lambda_0$, and it's easy to know that $T_x\gamma = E_1^c(x)$ for any $x \in \gamma$. We say a index i_0 smooth central curve γ is an index i_0 positive (negative) central segment if $f^i(\gamma) \subset V_0$ and $f^i(\gamma)$ is an index i_0 central curve for any $i \geq (\leq)0$, so if γ is an index i_0 positive (negative) central segment, $\gamma \subset \bigcap\limits_{-\infty}^{0} f^i(\overline{V_0})$ $(\bigcap\limits_{0}^{\infty} f^i(\overline{V_0}))$.

Definition 5.1. *We say $E_1^c(\Lambda)$ has an f-orientation if $E_1^c(\Lambda)$ is orientable and Df preserves its orientation.*

The following result has been stated and proved in [37], [38].

Lemma 5.2. *Suppose Λ has dominated splitting $E_{i_0}^{cs} \oplus E_1^c \oplus E_{i_0+2}^{cu}$, its neighborhood V_0 and the set Λ_0 are given above, then for a small open neighborhood V_1 of Λ satisfying $\overline{V_1} \subset V_0$ and let $\Lambda_1 = \bigcap\limits_{i=-\infty}^{\infty} f^i(\overline{V_1})$, $\Lambda_1^+ = \bigcap\limits_{i=-\infty}^{0} f^i(\overline{V_1})$, $\Lambda_1^- = \bigcap\limits_{i=0}^{\infty} f^i(\overline{V_1})$, there exist $0 < \delta_0 < 1$, $\delta_0/2 > \delta_1 > \delta_2 > 0$ such that they satisfy the following properties:*

a) *if $E_1^c(\Lambda)$ has an f orientation, $E_1^c(\Lambda_1)$ has an f orientation also.*
b) *for any $x \in V_1$, $B_{\delta_0}(x) \subset V_0$ and $E_1^c(B_{\delta_0}(x))$ is orientable, so it gives orientation for any index i_0 central curve in $B_{\delta_0}(x)$, and we choose δ_0 small enough such that any index i_0 central curve in $B_{\delta_0}(x)$ never intersects with itself.*
c) *for any $x \in \Lambda_1$, there exists an index i_0 central curve $l_{\delta_1}(x)$ with center x and radius δ_1, such that there exists a continuous function $\Phi^c : \Lambda_1 \longrightarrow Emb^1(I, M)$ satisfying $\Phi^c(x) = l_{\delta_1}(x)$ where $x \in \Lambda_1$, and if let $l_{\delta_2}(x) \subset l_{\delta_1}(x)$ be the central curve with center x and radius δ_2, then $f(l_{\delta_2}(x)) \subset l_{\delta_1}(f(x))$ and $f^{-1}(l_{\delta_2}(x)) \subset l_{\delta_1}(f^{-1}(x))$.*
d) *for any index i_0 positive central segment γ satisfying length$(f^i(\gamma)) < \delta_1$ for all $i \geq 0$, every $x \in \gamma$ will have uniform size of strong stable manifold: $W_{\delta_1}^{ss}(x)$ where $W_{\delta_1}^{ss}(x)$ is an index i_0-cs disk tangent at x on $E_{i_0}^{cs}(x)$, and $W_{\delta_1}^s(\gamma) = \bigcup\limits_{x\in\gamma} W_{\delta_1}^{ss}(x)$ would be an index i_0 ccs-disk. For any $x \in Int(\gamma)$ and any $\delta > 0$, then there exists $\delta_x > 0$ such that for any $y \in B_{\delta_x}(x)\bigcap\Lambda_1$, for any index i_0-cu disk $D^{cu}(y)$ with center y and size δ, we'll have $D_\delta^{cu}(y) \pitchfork W_{\delta_1}^s(\gamma) \neq \phi$.*

Remark 5.3. *In d) of above lemma, if we have γ belongs to a chain recurrent class C and y is a periodic point with $D^{cu}(y) \subset W_{loc}^u(y)$, then we have $y \dashv x$.*

5.2. **Crovisier's central model.** Suppose Λ has dominated splitting $E^{cs}_{i_0}\oplus E^c_1\oplus E^{cu}_{i_0+2}$, in this subsection, let's fix $V_0, V_1, \Lambda_1, \delta_0/2>\delta_1>\delta_2>0$ given by lemma 5.2 and a small enough, we'll introduce Crovisier's central model. By his work, we can get some dynamical property for the index i_0 central curves of Λ_1. The main result in this subsection is corollary 5.12.

Definition 5.4. *A central model is a pair $(\widetilde{K},\widetilde{f})$ where*

a) *$\widetilde{K}$ is a compact metric space called the base of the central model.*
b) *$\widetilde{f}$ is a continuous map from $\widetilde{K}\times[0,1]$ into $\widetilde{K}\times[0,\infty)$*
c) *$\widetilde{f}(\widetilde{K}\times\{0\})=\widetilde{K}\times\{0\}$*
d) *$\widetilde{f}$ is a local homeomorphism in a neighborhood of $\widetilde{K}\times\{0\}$: there exists a continuous map $g:\ \widetilde{K}\times[0,1]\longrightarrow\widetilde{K}\times[0,\infty)$ such that $\widetilde{f}\circ\widetilde{g}$ and $\widetilde{g}\circ\widetilde{f}$ are identity maps on $\widetilde{g}^{-1}(\widetilde{K}\times[0,1])$ and $\widetilde{f}^{-1}(\widetilde{K}\times[0,1])$ respectively.*
e) *$\widetilde{f}$ is a skew product: there exits two map $\widetilde{f}_1:\ \widetilde{K}\longrightarrow\widetilde{K}$ and $\widetilde{f}_2:\ \widetilde{K}\times[0,1]\longrightarrow[0,\infty)$ respectively such that for any $(x,t)\in\widetilde{K}\times[0,1]$, one has $\widetilde{f}(x,t)=(\widetilde{f}_1(x),\widetilde{f}_2(x,t))$.*

$\widetilde{f}$ general doesn't preserve $\widetilde{K}\times[0,1]$, so the dynamic outside $\widetilde{K}\times\{0\}$ is only partially defined.

The central model $(\widetilde{K},\widetilde{f})$ has a chain recurrent central segment if there is a segment $I=\{x\}\times[0,a]$ contained in a chain recurrent class of $f|_{\widetilde{K}\times[0,1]}$.

A subset $S\subset\widetilde{K}\times[0,1]$ of a product $\widetilde{K}\times[0,\infty)$ is a strip if for any $x\in\widetilde{K}$, the intersection $S\bigcap\{x\}\times[0,\infty)$ is a non-trivial interval.

In his remarkable paper [8], Crovisier got the following important result.

Lemma 5.5. *([8] Proposition 2.5) Let $(\widetilde{K},\widetilde{f})$ be a central model with a chain transitive base, then the two following properties are equivalent:*

a) *there is no chain recurrent central segment;*
b) *there exists some strip S in $\widetilde{K}\times[0,1]$ that is arbitrarily small neighborhood of $\widetilde{K}\times\{0\}$ and it's a trapping region for $\widetilde{f}$ or $\widetilde{f}^{-1}$: either $\widetilde{f}(Cl(S))\subset Int(S)$ or $\widetilde{f}^{-1}(Cl(S))\subset Int(S)$.*

Remark 5.6. *If the central model $(\widetilde{K},\widetilde{f})$ has a chain recurrent central segment and $\widetilde{K}\times\{0\}$ is transitive, from Crovisier's proof, we can know for any small neighborhood V of $\widetilde{K}\times\{0\}$, there exists a segment $x\times[0,a]_{a\neq0}$ contained in the same chain recurrent class of $\widetilde{f}|_V$ with $\widetilde{K}\times\{0\}$.*

An open strip $S\subset\widetilde{f}\times[0,1]$ satisfying $\widetilde{f}(Cl(S))\subset Int(S)$ or $\widetilde{f}^{-1}(Cl(S))\subset Int(S)$ is called a trapping strip, in the first case, we call the trapping strip is 1-step contracting, and the second case is called 1-step expanding.

Definition 5.7. *Let f be a diffeomorphism of a manifold M, Λ is a compact set with dominated splitting $E^{cs}_{i_0}\oplus E^c_1\oplus E^{cu}_{i_0+2}$, $\Lambda_1, V_0, V_1, a, \delta_0/2>\delta_1>\delta_2>0$ are given in §5.1, where Λ_1 also has a dominated splitting $E^{cs}_{i_0}\oplus E^c_1\oplus E^{cu}_{i_0+2}$. A central model $(\widetilde{\Lambda_1},\widetilde{f})$ is an index i_0 central model for (Λ_1,f) if there exists a continuous map $\pi:\ \widetilde{\Lambda}_1\times[0,\infty)\longrightarrow M$ such that:*

a) *π semi-conjugate $\widetilde{f}$ and f : $f\circ\pi=\pi\circ\widetilde{f}$ on $\widetilde{\Lambda}_1\times[0,1]$*
b) *$\pi(\widetilde{\Lambda}_1\times\{0\})=\Lambda_1$*

c) *the collection of map* $t \longrightarrow \pi(\widetilde{x}, t)$ *is a continuous family of* C^1 *embedding of* $[0, \infty)$ *into* M*, parameterized by* $\widetilde{x} \in \widetilde{\Lambda_1}$*;*

d) *for any* $\widetilde{x} \in \widetilde{\Lambda_1}$*, the curve* $\pi(\widetilde{x}, [0, \infty)) \subset U$ *has length bigger than* δ_2 *but smaller than* δ_1*, it's tangent at the point* $x = \pi(\widetilde{x}, 0) \in \Lambda_1$ *to* E_1^c *and it's an index* i_0 *central curve (that means the curve* $\pi(\widetilde{x}, [0, \infty))$ *tangents with the index* i_0 *central cone* $C^c_{a_0}$*).*

Remark 5.8. *From now, if* $(\widetilde{\Lambda}_1, \widetilde{f})$ *is an index* i_0 *central model for* (Λ_1, f) *and* π *is the projection map, we'll denote the central model as* $(\widetilde{\Lambda}_1, \widetilde{f}, \pi)$*. Here I should notice the reader that* π *in this paper has two different meanings, one denote the period of periodic point and another denote the projection map of central model. If there exists any confusion, I'll point out.*

The following lemma shows that central model always exists.

Lemma 5.9. *([8])* $\Lambda, \Lambda_1, V_0, U_1$ *are given in §5.1, then there exists an index* i_0 *central model* $(\widetilde{\Lambda}_1, \widetilde{f}, \pi)$ *for* (Λ_1, f)*. Let's denote* $\widetilde{\Lambda} \subset \widetilde{\Lambda}_1$ *the set satisfying* $\pi^{-1}(\Lambda) \bigcap (\widetilde{\Lambda}_1 \times \{0\}) = \widetilde{\Lambda} \times \{0\}$*, then* $(\widetilde{\Lambda}, \widetilde{f}, \pi)$ *is an index* i_0 *central model for* (Λ, f)*, and if* Λ *is minimal (transitive, chain recurrent set),* $\widetilde{\Lambda} \times \{0\}$ *is also minimal (transitive, chain recurrent set).*

Remark 5.10.

1) *When the cental bundle* $E_1^c(\Lambda_1)$ *has an* f*-orientation (it means that* $E_1^c|_{\Lambda_1}$ *is orientable and* Df *preserves such orientation), we call the orientation 'right', then we can get two index* i_0 *central models* $(\widetilde{\Lambda_1^+}, \widetilde{f}^+, \pi^+)$ *and* $(\widetilde{\Lambda_1^-}, \widetilde{f}^-, \pi^-)$ *for* (Λ_1, f)*, we call them the right model and the left model, where* π^i $_{(i=+,-)}$ *is a bijection between* $\widetilde{\Lambda}_1^i \times \{0\}$ *and* Λ_1*, and for* $\widetilde{x}^i \in \widetilde{\Lambda}_1^i$*,* $\pi(\widetilde{x}^i \times [0, \infty))$ *is a half of index* i_0 *central curve at the right (*$i = +$*) or left (*$i = -$*) of* $x = \pi(\widetilde{x}^i \times \{0\})$*.*
2) *If* f *doesn't preserve any orientation of* $\widetilde{E_1^c}(\Lambda_1)$*, then* $\pi : \widetilde{\Lambda}_1 \longrightarrow \Lambda_1$ *is two-one: any point* $x \in \Lambda_1$ *has two preimages* $\widetilde{x}^-$ *and* $\widetilde{x}^+$ *in* $\widetilde{\Lambda}_1$*, the homeomorphism* σ *of* $\widetilde{\Lambda}_1$ *which exchanges the preimages* $\widetilde{x}^+$ *and* $\widetilde{x}^-$ *of any point* $x \in \Lambda_1$ *commutes with* $\widetilde{f}$*.*

   In § 5.1, we know that any point $x \in \Lambda_1$ *has a local orientation, then* $\pi(\widetilde{x}^+ \times [0, \infty))$ *is an index* i_0 *central curve on the right of* x*,* $\pi(\widetilde{x}^- \times [0, \infty))$ *is on the left of* x*, the union of them is an index* i_0 *central curve with central at* x *and radius* δ_1*.*

The following lemma is proved in [8].

Lemma 5.11. $f \in R$*,* Λ *is a chain recurrent set with a dominated splitting* $E^{cs}_{i_0} \oplus E^c_1 \oplus E^{cu}_{i_0+2}$ *where* $dim(E_1^c(\Lambda)) = 1$ *and* $E_1^c(\Lambda)$ *is not hyperbolic. Let* V, V_1, Λ_1 *be given in §5.1, by lemma 5.9,* (Λ_1, f) *has an index* i_0 *central model* $(\widetilde{\Lambda}_1, \widetilde{f}, \pi)$*, let* $\widetilde{\Lambda} \subset \widetilde{\Lambda}_1$ *be the set satisfying* $\widetilde{\Lambda} \times \{0\} = \pi^{-1}(\Lambda) \bigcap \widetilde{\Lambda}_1 \times \{0\}$*, then* $(\widetilde{\Lambda}, \widetilde{f}, \pi)$ *is a central model for* (Λ, f) *and we have*

a) *either* $(\widetilde{\Lambda}_1, \widetilde{f}, \pi)$ *has a trapping region,*
b) *or* $(\widetilde{\Lambda}, \widetilde{f}, \pi)$ *has a chain recurrent central segment.*

Corollary 5.12. $f \in R$*,* Λ *is a chain recurrent set with a dominated splitting* $E^{cs}_{i_0} \oplus E^c_1 \oplus E^{cu}_{i_0+2}$ *where* $dim(E_1^c(\Lambda)) = 1$ *and* $E_1^c(\Lambda)$ *is not hyperbolic. Let* V, V_1, Λ_1 *be given in §5.1, by lemma 5.9,* (Λ, f) *has an index* i_0 *central model* $(\widetilde{\Lambda}, \widetilde{f}, \pi)$*. Suppose the central model* $(\widetilde{\Lambda}, \widetilde{f}, \pi)$ *has a chain recurrent central segment* $\widetilde{\gamma}_{\widetilde{x}}$ *where* $\widetilde{x} \in \widetilde{\Lambda}$*, denote* $\gamma_x = \pi(\widetilde{\gamma}_{\widetilde{x}})$*, then* $\gamma_x \subset \Lambda_1$ *and it's an index* i_0 *central segment, in fact we have* *that length*$(f^i(\gamma_x)) < \delta_1$ *for any* $i \in \mathbb{Z}$ *and* γ_x *is in the same chain recurrent set with* Λ *respecting the map* $f|_{V_1}$.

6. Proof of lemma 3.6

Proof of lemma 3.6: By Gan's lemma, we know that either Λ is an index i_0+1 fundamental limit or C contains index i_0 periodic point, so we can suppose Λ is always an index i_0+1 fundamental limit. With the same argument, we can suppose Λ is an index i_1-1 fundamental limit also, then by proposition 2.4, 2.5, Λ has the following dominated splitting $T_\Lambda M = E^s_{i_0} \oplus E^{cs}_1 \oplus E^c \oplus E^{cu}_1 \oplus E^u_{i_1+1}$. Now we have two kinds of index different central models: the index i_0 central model and the index i_1-1 central model. We suppose the two central bundles E^{cs}_1 and E^{cu}_1 both have an f-orientation, since the proof for the other case is similar. We give every central bundle an orientation and all it right, then we have two central models (right or left) for every central bundle, at first, let's deal with a simple case.

Claim: If one of the index i_0 central model and one of the index i_1-1 central model both have chain recurrent central segment, then C contains an index i periodic point with $i_0 \le i \le i_1$.

Proof of the claim: By corollary 5.12, suppose $\gamma^{cs}_{x_0}$ is the index i_0 chain recurrent central segment and $\gamma^{cu}_{y_0}$ is the index i_1-1 chain recurrent central segment, then there exists a chain recurrent set $\Lambda^* \subset V_0$ such that $\Lambda \bigcup \gamma^{cs}_{x_0} \bigcup \gamma^{cu}_{y_0} \subset \Lambda^*$. Choose $x \in \gamma^{cs}_{x_0} \setminus x_0$ and $y \in \gamma^{cu}_{y_0} \setminus y_0$, according to 4) of proposition 3.1 there exists a family of periodic orbits $\{Orb(p_n)\}$ in V_0 satisfying $\lim_{n\to\infty} Orb(p_n) \longrightarrow \Lambda^*$, then there exists i_n and j_n such that $f^{i_n}(p_n) \longrightarrow x$ and $f^{j_n}(p_n) \longrightarrow y$. Recall that $\bigcup_n Orb(p_n) \subset \Lambda_0$ and Λ_0 has the following partial hyperbolic splitting $T_\Lambda M = E^s_{i_0} \oplus E^{cs}_1 \oplus E^{cc} \oplus E^{cu}_1 \oplus E^u_{i_1+1}$, we know that every point $q \in \bigcup_n Orb(p_n)$ will have a strong stable manifold $W^{ss}_{loc}(q)$ which is an index i_0 cs disk and have a strong unstable manifold $W^{uu}_{loc}(q)$ which is an index i_1-1 cu disk, by d) of lemma 5.2, $\gamma^{cs}_{x_0}$ has an unstable manifold $W^u_{loc}(\gamma^{cs}_{x_0})$ which is index i_0 ccu disk and $\gamma^{cu}_{y_0}$ has a stable manifold $W^s_{loc}(\gamma^{cu}_{y_0})$ which is index i_1-1 ccs disk, so when n is big enough, we have $W^{ss}_{loc}(f^{i_n}(p_n)) \pitchfork W^u_{loc}(\gamma^{cs}_{x_0}) \neq \phi$ and $W^{uu}_{loc}(f^{j_n}(p_n)) \pitchfork W^s_{loc}(\gamma^{cu}_{y_0}) \neq \phi$, by remark 5.3 we know $p_n \dashv x_0$ and $y_0 \dashv p_n$, by the fact $x_0 \dashv y_0$, p_n is in the same chain recurrent class with x_0, so $Orb(p_n) \subset C$. Recall that $\Lambda^* \subset V_0$ has the special partial hyperbolic splitting and $Orb(p_n)$ stays near Λ^*, we know that Orb(p_n) has index i with $i_0 \le i \le i_1$. □

Now we can suppose that the two index i_0 central models both don't have chain recurrent central segment, then there exist any small trapping regions for these two central models. Now we claim that there always exists $x_0 \in \Lambda$ such that for the $0 < \lambda < 1$ and l given in proposition 2.4 and $1 > \mu > \lambda$, we have

$$\prod_{j=0}^{n-1} \|Df^l|_{E^{cs}_1(f^{jl}(x_0))}\| \le \mu^n \text{ for } n \ge 1. \tag{6.1}$$

Proof of the claim: Here we need the following lemma at first:

Lemma 6.1. *(*[31]*) Assume* $f \in R$*, let* Λ *be an index* i_1 *fundamental limit of* f $(1 \le i_1 \le d-1)$*, and* $E^{cs}_{i_1}(\Lambda) \oplus E^{cu}_{i_1+1}(\Lambda)$ *is an index* $i_1 - (l, \lambda)$ *dominated splitting on* Λ *given by proposition 2.4, then*

1) *either for any* $\mu \in (\lambda, 1)$*, there exists* $c \in \Lambda$ *such that* $\prod_{j=0}^{n-1} \|Df^l|_{E^{cs}_{i_1}(f^{jl}c)}\| \leq \mu^n$ *for* $n \geq 1$*,*

2) *or* $E^{cs}_{i_1}$ *splits into a dominated splitting* $E^{cs}_{i_1-1} \oplus E^c_1$ *with* $dim(E^c_1) = 1$ *such that for any* $\mu \in (\lambda, 1)$*, there is* $c' \in \Lambda$ *such that* $\prod_{j=0}^{n-1} \|Df^l|_{E^{cs}_{i_1-1}(f^{jl}c')}\| \leq \mu^n$ *for all* $n \geq 1$*.*

Lemma 6.2. *Let* Λ *be an invariant compact set of* f*, with two dominated splitting* $E^{cs} \oplus F^{cu}$ *and* $\widetilde{E}^{cs} \oplus \widetilde{F}^{cu}$*, if* $dim(E^{cs}) \leq dim(\widetilde{E}^{cs})$*, then* $E^{cs} \subset \widetilde{E}^{cs}$*.*

Since Λ is an index i_1 fundamental limit, if 1) of lemma 6.1 is true for Λ, then there exists $x \in \Lambda$ such that $\prod_{j=0}^{n-1} \|Df^l|_{E^{ccs}_{i_1}(f^{jl}x)}\| \leq \mu_0^n$ for $n \geq 1$. On Λ we have another dominated splitting $(E^s_{i_0} \oplus E^{cs}_1) \oplus E^{i_0+2}_{cu}$, since $dim(E^s_{j_0} \oplus E^c_1) = i_0 + 1 < i_1 = dim(E^{ccs}_{i_1})$, by lemma 6.2, $E^s_{i_0} \oplus E^c_1|_\Lambda \subset E^{ccs}_{i_1}|_\Lambda$, so we have $\prod_{j=0}^{n-1} \|Df^l|_{E^s_{i_0} \oplus E^{cs}_1(f^{jl}x)}\| \leq \mu_0^n$ for $n \geq 1$.

If 2) of lemma 6.1 is true for Λ, then there exists x' such that $\prod_{j=0}^{n-1} \|Df^l|_{E^{cs}_{i_1-1}(f^{jl}x')}\| \leq \mu_0^n$ for $n \geq 1$, recall that $dim(E^s_{i_0} \oplus E^{cs}_1) = i_0 + 1 \leq i_1 - 1 = dim(E^{cs}_{i_1-1})$, by lemma 6.2, $E^s_{i_0} \oplus E^{cs}_1|_\Lambda \subset E^{cs}_{i_1-1}|_\Lambda$, so we have $\prod_{j=0}^{n-1} \|Df^l|_{E^s_{i_0} \oplus E^{cs}_1(f^{jl}x')}\| \leq \mu_0^n$ for $n \geq 1$. □

Now we claim that for the x_0 above, $length(f^i(\gamma_{x_0})) \longrightarrow 0^+$.

Proof of the claim: Here we just need the following lemma.

Lemma 6.3. *([26]) For any* $0 < \mu < 1$*, there exists* $\varepsilon > 0$ *such that for* $x \in \Lambda_0$ *which satisfies* $\prod_{j=0}^{n-1} \|Df|_{\widetilde{E}(f^j x)}\| \leq \mu^n$ *for all* $n > 0$*, then* $diam(f^n(l^{cs}_\varepsilon(x))) \longrightarrow 0$*, i.e. the central stable manifold of* x *with size* ε *is in fact a stable manifold.*

□

So by the above two claims, we know that the trapping regions for the two index i_0 central models are always 1-step contracting.

Now choose a family of index i_0 periodic point $\{p_n\}$ such that $\lim_{n\to\infty} Orb(p_n) \longrightarrow \Lambda$, and consider the central curves l^{cs}_δ, by trapping region of the two central models (Λ_1, f, π^+) and (Λ_1, f, π^-), there exist $\gamma^{cs,+}_{p_n}$ and $\gamma^{cs,-}_{p_n}$ and $\varepsilon > 0$ such that $f^{\pi(p_n)}(\gamma^{cs,+}_{p_n}) \subset Int(\gamma^{cs,+}_{p_n})$, $f^{\pi(p_n)}(\gamma^{cs,-}_{p_n}) \subset Int(\gamma^{cs,-}_{p_n})$ and $length(\gamma^{cs,+}_{p_n} \setminus f^{\pi(p_n)}(\gamma^{cs,+}_{p_n})) > \varepsilon$, $length(\gamma^{cs,-}_{p_n} \setminus f^{\pi(p_n)}(\gamma^{cs,-}_{p_n})) > \varepsilon$. Now define $\gamma^{cs}_{p_n} = \gamma^{cs,+}_{p_n} \bigcup \gamma^{cs,-}_{p_n}$ and $\Gamma^{cs,i}_{p_n} = \bigcap_j f^{j\pi(p_n)}(\gamma^{cs,i}_{p_n})$ for $(i = +,-)$, $h^{cs,i}_{p_n} = \gamma^{cs,i}_{p_n} \setminus \Gamma^{cs,i}_{p_n}$ for $i = +,-$, and $\Gamma^{cs}_{p_n} = \Gamma^{cs,+}_{p_n} \bigcup \Gamma^{cs,-}_{p_n}$. Denote $q^{cs,+}_{p_n}$ the right extreme point for $\Gamma^{cs}_{p_n}$ and $q^{cs,-}_{p_n}$ the left extreme point for $\Gamma^{cs}_{p_n}$, then by the 1-step contracting property, we know that $h^{cs,i}_{p_n} \subset W^s(q^{cs,i}_{p_n})$ for $i = +,-$.

Now we claim that $length(\Gamma^{cs}_{Orb(p_n)}) \longrightarrow 0$.

Proof of the claim Suppose there exists $q_n \in Orb(p_n)$ and δ such that $length(\Gamma^{cs}_{q_n}) > \delta$ for all n, when n big enough, there exists δ_n and $i_{1,n} < i_{2,n}$ such that $d(f^{i+i_{1,n}}(q_n), f^i(x_0)) < \delta_n$ for $0 \leq i \leq i_{2,n} - i_{1,n}$ where $\delta_n \longrightarrow 0^+$ and $i_{2,n} - i_{1,n} \longrightarrow \infty$. Recall that $length(\Gamma^{cs}_q) < \delta_1$ for any $q \in Orb(p_n)$, by (6.1) we know that $length(\Gamma^{cs}_{f^{i_{2,n}}(q_n)}) \longrightarrow 0^+$ (since from $f^{i_{1,n}}(q_n)$ to $f^{i_{2,n}}(q_n)$, by 6.1, f contracts the central curve with exponential rate). Suppose $q_n \longrightarrow y_0$ and $\Gamma^{cs}_{q_n} \longrightarrow \Gamma^{cs}_{y_0}$, then $\Gamma^{cs}_{y_0} \subset \gamma^{cs}_{y_0}$, $length(\Gamma^{cs}_{y_0}) > \delta$ and it's an index i_0 chain recurrent central segment (since $length(\Gamma^{cs}_{f^{i_{2,n}}(q_n)}) \longrightarrow 0$ and $f^{\pi(p_n)-i_{2,n}}(\Gamma^{cs}_{f^{i_{2,n}}(q_n)}) = \Gamma^{cs}_{q_n}$), that's a contradiction with our assumption. □

By lemma 6.1 and the argument following there, there exists i_n, l, $1 > \mu > \lambda$ and $q_n = f^{i_n}(p_n)$ such that $\prod_{i=0}^{m-1} \|Df^{-l}|_{E^u_{i_0+2}(f^{in-il}(q_n))}\| < \mu^m$ for $n \geq 0$ (since q_n is periodic point with index i_0, here we denote $E^c \oplus E^{cu}_1 \oplus E^u_{i_1+1}|_{Orb(p_n)}$ by $E^u_{i_0+2}(Orb(p_n)))$, then by a similar result with lemma 6.3 for central unstable manifold, q_n has uniform size of strong unstable manifold which is an index i_0 cu disk. In fact, for some $\mu < \mu_0 < 1$, by the property that $\lim_{n\to\infty} length(\Gamma^{cs}_{Orb(p_n)}) \longrightarrow 0$, when n is big enough, every periodic point $q \in \Gamma^{cs}_{q_n}$ satisfies $\prod_{i=0}^{m-1} \|Df^{-l}|_{E^u_{i_0+2}(f^{in-il}(q))}\| < \mu_0^m$ also, so every periodic point in $\Gamma^{cs}_{q_n}$ will have uniform size of strong unstable manifold which is an index i_0 cu disk, then when n and m are big enough, for any periodic point $q \in \Gamma^{cs}_{q_n}$, it has $W^{uu}(q) \pitchfork W^s_{loc}(\Gamma^{cs}_{q_m}) \neq \phi$, (since $W^s_{loc}(\Gamma^{cs}_{q_m})$ contains $W^s_{loc}(\gamma^{cs})$, and by the fact γ^{cs} is a positive central segment, $W^s_{loc}(\gamma^{cs})$ is an index i_0 cu disk with uniform size), so there exists a periodic point $q^* \in \Gamma_{f^{im}(p_m)}$ such that $W^{uu}(q) \pitchfork W^s(q^*) \neq \phi$, and we denote it $q \prec q^*$, so we can define a partial order for the periodic points in $\Gamma^{cs}_{q_n}$ and $\Gamma^{cs}_{q_m}$, it's easy to know that every equivalent class belongs to a non-trivial chain recurrent class. If we suppose $q_n \longrightarrow y_0 \in \Lambda$ and fix n_0 big enough, we know that for any $n > n_0$, there exists a non-trivial homoclinic class containing periodic points in $\Gamma^{cs}_{q_{n_0}}$ and $\Gamma^{cs}_{q_n}$, then it's easy to know that C contains a periodic point of $\Gamma^{cs}_{q_{n_0}}$, and since the periodic orbit stays near Λ, it has index i with $i_0 \leq i \leq i_1$. □

7. Proof of lemma 3.7

The basic idea of proof of lemma 3.7 is that when we suppose $\Lambda \subsetneq C_0$ and C doesn't contain index j_0, $j_0 + 1$ periodic points, we can find a family of periodic points stay a lot of time near Λ and whose Hausdorff limit is contained in C_0 and bigger than Λ, we denote their Hausdorff limit by Λ^*, then by the definition of Λ we know Λ^* is an index j fundamental limit with $j \geq j_0 + 1$. Since the periodic points above stay almost all the time near Λ, we can get a measure μ with $supp(\mu) \subset \Lambda^*$ and μ has index j_0+1 (in fact $supp(\mu)$ is contained in a small neighborhood of Λ), then by C^1 Pesin theory given in [36], C contains index $j_0 + 1$ periodic point and that's a contradiction.

In order to get the above sequence of periodic points, in lemma 7.10 we show that the orbits near Λ have some good position, and then use connecting lemma and generic assumptions, we can get the periodic points we need. A similar argument was used in the proof of "the Technique lemma" ([38]).

In § 7.1 we introduce some new generic properties, in § 7.2 we introduce the connecting lemma because we need a special property which just appears during the proof, in § 7.3 I'll state lemma 7.6 and use it to prove lemma 3.7, the proof of lemma 7.6 is given in § 7.4.

7.1. **Some new generic properties.** Choose $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ a topological basis of M satisfying that for any $\varepsilon > 0$, there exists a subsequence $\{U_{\alpha_i}\}_{i=1}^{\infty}$ such that $diam(U_{\alpha_i}) < \varepsilon$ and $\bigcup_i (U_{\alpha_i})$ is a cover of M. Fix this topological basis, we'll give some new C^1 generic properties.

At first, let's recall some definitions, suppose K is a compact set of M and $f \in C^1(M)$ has been given, $x, y \in K$, $x \underset{K}{\dashv} y$ means that for any $\varepsilon > 0$, there exists an ε-pseudo orbit in K beginning from x and ending at y. If $K = M$, we just denote $x \dashv y$.

The following result has been proved in [7]:

Lemma 7.1. *There exists a generic subset $R^*_{1,0}$ such that any $f \in R^*_{1,0}$ satisfies the following property: suppose K is a compact set, W is any neighborhood of K, $x_0, x_1, \cdots, x_n, \in K$ satisfy $x_0 \underset{K}{\dashv} x_1 \underset{K}{\dashv} \cdots \underset{K}{\dashv} x_n$, $U_0, U_1, \cdots, U_n \subset W$ are neighborhoods of $x_0, x_1, \cdots, x_n$ respectively, then there exists a segment of orbit of f in W beginning from U_0, passing U_i $_{1<i<n}$ and ending in U_n. More precisely, there exists $a \in U_0$ and $j_n > j_i > j_0 = 0$ $_{(0<i<n)}$ such that $f^{j_i}(a) \in U_i$ for $0 \le i \le n$ and $f^j(a) \in W$ for $0 \le j \le j_n$.*

Lemma 7.2. *There exists a generic subset $R^*_{1,1}$ such that any $f \in R^*_{1,1}$ satisfies the following property: for a sequence $\{s_t\}$ where $0 < s_t < 1$ and $s_t \longrightarrow 1^-$, $\{\Phi_i\}_{i=1}^K \subset \{U_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{O_t\}_{j=1}^J \subset \{U_\alpha\}_{\alpha\in\mathcal{A}}$, if for $t_0 \in \mathbb{N}$ there exist $g_n \xrightarrow{C^1} f$ and g_n has periodic point p_n satisfying $\frac{\#\{Orb_{gn}(p_n)\bigcap(\bigcup_{i=1}^K \Phi_i)\}}{\pi_{gn}(p_n)} > s_{t_0}$ and $Orb_{g_n}(p_n)\bigcap O_j \ne \phi$ for $1 \le j \le J$, then f itself has a periodic point p satisfying $\frac{\#\{Orb(p)\bigcap(\bigcup_{i=1}^K \Phi_i\})}{\pi(p)} > s_{t_0}$ and $Orb(p_n)\bigcap O_t \ne \phi$ for $1 \le t \le J$. Especially if there exists $\{U_i\}_{i=1}^k \subset \{U_\alpha\}_{\alpha\in\mathcal{A}}$ such that $Orb_{g_n}(p_n) \subset \bigcup_{i=1}^k U_i$ for all n, then we can let $Orb(p) \subset \bigcup_{i=1}^k U_i$.*

Proof : Here we just proof the first part, let's consider the set $\{(\Phi_{\beta_1}, \cdots, \Phi_{\beta_{N(\beta)}}; O_{\beta_1}, \cdots, O_{\beta_{J(\beta)}})\}_{\beta\in\mathcal{B}_0}$ where $\Phi_{\beta_i}, O_{\beta_j} \in \{U_\alpha\}_{\alpha\in\mathcal{A}}$, it's easy to know $\mathcal{B}_0$ is countable.

For any $\beta \in \mathcal{B}_0$, denote

- $H_{\beta,t} = \{f|\ f \in C(M), f$ has a C^1 neighborhood $\mathcal{U}$ such that for any $g \in \mathcal{U}$, g has a periodic orbit $Orb(p_g)$ satisfying $\frac{\#\{Orb_g(p_g)\bigcap(\bigcup_{i=1}^{N(\beta)} \Phi_{\beta_i})\}}{\pi_g(p_g)} > s_t$ and $Orb_g(p_g)\bigcap O_{\beta_i} \ne \phi$ for $1 \le i \le J(\beta)\}$,
- $N_{\beta,t} = \{f|\ f \in C^1(M), f$ has a C^1 neighborhood $\mathcal{U}$ such that for any $g \in \mathcal{U}$, g has no any periodic orbit p_g satisfying $\frac{\#\{Orb_g(p_g)\bigcap(\bigcup_{i=1}^{N(\beta)} \Phi_{\beta_i})\}}{\pi_g(p_g)} > s_t$ and $Orb_g(p_g)\bigcap O_{\beta_i} \ne \phi$ for $1 \le i \le J(\beta)\}$.

It's easy to know $H_{\beta,t}\bigcup N_{\beta,t}$ is open and dense in $C^1(M)$. Let $R^*_{1,0} = \bigcap_{t\in\mathbb{N}}\bigcap_{\beta\in\mathcal{B}_0}(H_{\beta,t}\bigcup N_{\beta,t})$, we'll show $R^*_{1,0}$ satisfies the property we need.

For any $f \in R^*_{1,0}$ and any $\beta^* \in \mathcal{B}_0$, $t \in \mathbb{N}$, suppose there exists a family of C^1 diffeomorphisms $\{g_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} g_n = f$ and any g_n has a periodic orbit $Orb(p_n)$ satisfying $\frac{\#\{Orb_{gn}(p_n)\bigcap(\bigcup_{i=1}^{N(\beta^*)} \Phi_{\beta^*_i})\}}{\pi_{gn}(p_n)} > s_t$ and $Orb_{g_n}(p_n)\bigcap O_{\beta^*_t} \ne \phi$ for $1 \le t \le J(\beta^*)$, then $f \notin N_{\beta^*,t}$. That means $f \in H_{\beta^*,t}$, so we proved this lemma. □

Now let $R_0 = R_0 \bigcap R^*_{1,0} \bigcap R^*_{1,1}$, in §7.3 we'll show this residual set satisfies lemma 3.7.

7.2. **Introduction of connecting lemma.** Connecting lemma was proved by Hayashi [16] at first, and then was extended to the conservative setting by Xia, Wen [34]. the following statement of connecting lemma was given by Lan Wen as an uniform version of connecting lemma.

Lemma 7.3. *(connecting lemma [30]) For any C^1 neighborhood $\mathcal{U}$ of f, there exist $\rho > 1$, a positive integer L and $\delta_0 > 0$ such that for any z and $\delta < \delta_0$ satisfying $\overline{f^i(B_\delta(z))}\bigcap\overline{f^j(B_\delta(z))} = \phi$ for $0 \le i \ne j \le L$, then for any two points p and q outside the cube $\Delta = \bigcup_{i=1}^L f^i(B_\delta(z))$, if the positive f-orbit of p hits the ball $B_{\delta/\rho}(z)$ after p and if the negative f-orbit of q hits the small ball $B_{\delta/\rho}(z)$, then there is $g \in \mathcal{U}$ such that $g = f$ off Δ and q is on the positive g-orbit of p.*

Remark 7.4. *Suppose we have another point $z_1 \in M$ satisfying $\Delta_1 \bigcap \Delta = \phi$ where $\Delta_1 = \bigcup_{i=1}^{L} f^i(B_\delta(z_1))$, then if we use twice connecting lemma in Δ and Δ_1 respectively, we can still get a diffeomorphism g in $\mathcal{U}$.*

Now we'll show the idea of the proof of connecting lemma, because we need some special property which just appears in the proof.

In the proof, the main idea is Hayashi's 'cutting' tool, by it we can cut some orbits from p's original f-orbit and q's original f-orbit, and then connect the rest part in Δ. More precisely description is following, suppose $f^{s_m}(p) \in B_{\delta/\rho}(z)$ and there exists $0 < s_1 < s_2 < \cdots < s_m$ such that $f^{s_i} \in B_\delta(z)$ for $1 \le i \le m$ and $f^s(p) \notin B_\delta(z)$ for $s \in \{0, 1, \cdots, s_m\} \setminus \{s_1, s_2, \cdots, s_m\}$; for q, there exists $0 < t_1 < t_2 < \cdots < t_n$ such that $f^{-t_i}(q) \in B_\delta(z)$ for $1 \le i \le n$, $f^{-t_n}(q) \in B_{\delta/\rho}(z)$ and $f^{-t}(q) \notin B_\delta(z)$ for $t \in \{0, 1, \cdots, t_n\} \setminus \{t_1, t_2, \cdots, t_n\}$. By some rules, we can cut some f-orbits in p's orbit like $\{f^{s_i+1}(p), f^{s_i+2}(p), \cdots, f^{s_j}(p)\}_{j>i}$ and cut some f-orbits in q's orbit like $\{f^{-t_j+1}(q), f^{-t_j+2}(q), \cdots, f^{-t_i}(q)\}_{j>i}$, then the rest segment looks like:

$$P' = (p, f(p), \cdots, f^{s_{i_1}}(p); f^{s_{i_2}+1}(p), \cdots, f^{s_{i_3}}(p); \cdots; f^{s_{i_{(k(p)-1)}}+1}(p), \cdots, f^{s_{i_{k(p)}}}(p)),$$

$$Q' = (f^{-t_{j_{k(q)}}+1}(q), \cdots, f^{-t_{j_{k(q)-1}}}(q); \cdots; f^{-t_{j_3}+1}, \cdots, f^{-t_{j_2}}(q); f^{-t_{j_1}+1}(q) \cdots, f^{-1}(q), q).$$

Denote $X = P' \bigcup Q'$, and $\pi(X)$ the length of X, it's easy to know X is a 2δ-pseudo orbits. Then we can do several perturbations called 'push' in Δ and get a diffeomorphism g such that q is on the positive g-orbit of p, in fact, we have $g^{\pi(X)}(p) = q$. It's because after the push, we can connect $f^{s_{i_1}}(p)$ and $f^{s_{i_2}+L}(p)$, $\cdots$; $f^{s_{i_{k(p)-2}}}(p)$ and $f^{s_{i_{k(p)-1}}+L}(p)$; $f^{s_{i_{k(p)}}}(p)$ and $f^{-t_{j_{k(q)}}+L}(q)$; $f^{-t_{j_{k(q)-1}}}(q)$ and $f^{-t_{j_{k(q)-2}}+L}(q)$; $\cdots$; $f^{-t_{j_2}}(q)$ and $f^{-t_{j_1}+L}(q)$ every time by L times pushes in Δ, we don't cut orbits this time, and it's important to note that the supports of different pushes don't intersect with each other, so we don't change the length of X, *we just push the points of X in Δ and get a connected orbit.* By the above argument, it's easy to know $g|_{M\setminus\Delta} = f|_{M\setminus\Delta}$ and $g(\Delta) = f(\Delta)$. More details see [16], [30], [34].

Remark 7.5.

a) *In the above argument, suppose there exists an open set V such that $f^i(p) \in V$ for $0 \le i \le s_m$ and $\Delta \subset V$, then after cutting and pushing, we can know $\{p, g(p), \cdots, g^{\pi(P')}(p)\} \subset V$. What's more, we can show that $\#\{\{g^i(p)\}_{i=0}^{\pi(P')+\pi(Q')} \bigcap (V)^c\} < t_n$.*

b) *If there exists an open set V such that $\Delta \subset V$, $f^i(p) \in V$ for $0 \le i \le s_m$ and $f^{-j}(q) \in V$ for $0 \le j \le t_n$, then after cutting and pushing, we can know $g^i(p) \subset V$ for $0 \le i \le \pi(X)$.*

7.3. **Proof of lemma 3.7.** Let's suppose the lemma is false, I'll prove that Λ is included in a bigger index j_0 fundamental limit of C_0.

Now choose $y \in C_0 \setminus \Lambda$ and a small neighborhood V_0 of Λ such that $y \notin \overline{V_0}$ and the maximal invariant subset Λ_0 of $\overline{V_0}$ still has the partial hyperbolic splitting $E^s_{j_0} \oplus E^c_1 \oplus E^u_{j_0+2}$. Choose a family of open sets $\{\Phi_i\}_{i=1}^N \subset \{U_\alpha\}_{\alpha\in\mathcal{A}}$ such that $\Lambda \subset \bigcup_{i=1}^{N} \Phi_i \subset V_0$, choose an open neighborhood V_1 of Λ such that $\overline{V_1} \subset \bigcup_{i=1}^{N} \Phi_i$.

Now we need the following result whose proof will be given in §7.4.

Lemma 7.6. *Under the same assumption with lemma 3.7, suppose $f \in R$, C doesn't contain index j_0 and j_0+1 periodic point and $\Lambda \subsetneq C_0$, then for $s_n \longrightarrow 1^-$ given in lemma 7.2, there exists a family of periodic points $\{p_n(f)\}$ such that $\Lambda \subsetneq \lim_{n\to\infty} Orb(p_n) \subset C_0$ and $\frac{\#\{Orb(p_n)\bigcap \bigcup_1^N \Phi_i\}}{\pi(p_n)} > s_n$.*

In the following proof we'll show that we can always suppose that the above sequence of periodic points all have index j_0, that means $\lim_{n\to\infty} Orb(p_n)$ is an index j_0 fundamental limit of C_0 bigger than Λ, it's a contradiction with the assumption that Λ is the maximal index j_0 fundamental limit of C_0.

Denote $j^* = \min_j\{j : j \geq j_0$ and there exits a family of index j periodic points which satisfies lemma 7.6$\}$, choose $\{p_n\}$ such a family of index j^* periodic points, we claim that $j^* = j_0$.

Proof of the claim: Suppose $j^* \geq j_0+1$, denote $C_1^* = \lim_{n\to\infty} Orb(p_n)$, then $C_1^* \subset P_{j^*}^*$, by $f \in R \subset C^1(M) \setminus \overline{HT}$ and proposition 2.4, C_1^* has the following dominated splitting $E_{j^*}^{cs} \oplus E_{j^*+1}^{cu}$. From the definition of j^*, it's easy to know that $\{Df|_{E_{j^*}^{cs}(Orb(p_n))}\}$ is stable contracting (or by Frank's type of small perturbation, we can change the periodic point's index to j^*-1, with a generic argument like what we do in § 7.1, f itself has a family of index j^*-1 periodic points satisfying lemma 7.6, it's a contradiction with the definition of j^*), then like the argument in [37] (lemma 4.9, 4.10, corollary 4.11), there exist $\mu_0 < 1, l \in \mathbb{N}$ such that for any $\pi(p_n)$ big enough, there exists $c_n' \in Orb(p_n)$ satisfying $\prod_{j=0}^{n-1} \|Df^l|_{E_{j^*}^{cs}(f^{jl}c_n)}\| \leq \mu_0^n$ for $n \geq 1$, since $\lim_{n\to\infty} \pi(p_n) \longrightarrow \infty$, we can suppose all the periodic orbits $Orb(p_n)$ satisfies above property. Then choose $1 > \mu_1 > \mu_0$, By Pliss lemma, there exists a subset $P_n \subset Orb(p_n)$ such that $\frac{\#(P_n)}{\pi(p_n)} > \delta$ and for any $c \in P_n$ we have $\prod_{j=0}^{n-1} \|Df^l|_{E_{j^*}^{cs}(f^{jl}c)}\| \leq \mu_1^n$ for $n \geq 1$. Since $\frac{\#\{Orb(p_n)\bigcap \bigcup_{i=1}^N \Phi_i\}}{\pi(p_n)} > s_n$ and $\lim_{n\to\infty} s_n \longrightarrow 1^-$, so there exist $c_n \in P_n \bigcap \bigcup_{i=1}^N \Phi_i$ and $i_n \longrightarrow \infty$ such that $f^i(c_n) \in \bigcup_{i=1}^N \Phi_i$ for $-i_n \leq i \leq i_n$. Let $c_n \longrightarrow c_0$, then $Orb(c_0) \subset \overline{\bigcup_{i=1}^N \Phi_i}$, and $\prod_{j=0}^{n-1} \|Df^l|_{E_{j^*}^{cs}(f^{jl}c_0)}\| \leq \mu_1^n$ for $n \geq 1$. Denote $C_1 = \overline{Orb(c_0)}$, we have $C_1 \subset \lim_{n\to\infty} Orb(p_n) \subset C_0$, it means C_1 also has index j^* dominated splitting $E_{j^*}^{cs} \oplus F_{j^*+1}^{cu}$. Because C_1 is an invariant compact subset of $\overline{\bigcup_{i=1}^N \Phi_i}$, C_1 has the partial hyperbolic splitting $E_{j_0}^s \oplus E_1^c \oplus E_{j_0+2}^u$. From the assumption we know $j_0+1 \leq j^*$, by lemma 6.2, $E_{j_0}^s \oplus E_1^c(x) \subset E_{j^*}^{cs}(x)$ for $x \in C_1$, so we have $\prod_{j=0}^{n-1} \|Df^l|_{E_1^c(f^{jl}c_0)}\| \leq \mu_1^n$ for $n \geq 1$. It means there is an ergodic measure ν with support in C_1 and the central Lyapunov exponents is negative, so ν is a hyperbolic ergodic measure with index j_0+1.

Definition 7.7. *A hyperbolic ergodic measure ν has index i if the number of negative Lyapunov exponents is i.*

Lemma 7.8. *Suppose $f \in C^1(M) \setminus (\overline{HT})$ and μ is an hyperbolic ergodic measure of f, then there exists a periodic point in the same chain recurrent class with supp(μ) and the periodic point has the same index with the hyperbolic ergodic measure.*

By the above lemma, we can show that there exists a periodic point with index j_0+1 in the same chain recurrent class with C_0. □

Now we know that $\lim_{n\to\infty} Orb(p_n)$ is an index j_0 fundamental limit of C_0 and $\Lambda \subsetneq \lim_{n\to\infty} Orb(p_n)$, it's a contradiction with the fact that Λ is the maximal index j_0 fundamental limit of C_0. □

7.4. **Proof of lemma 7.6.** Choose $x_0 \in \Lambda$, then for any $\delta_n \longrightarrow 0^+$, there exists a δ_n-pseudo orbit in C_0 from y to x_0, denote z_n^+ is the last time the pseudo orbit enters V_1, suppose $\lim_{n\to\infty} z_n^+ = z_0$, then $Orb^+(z_0) \in \overline{V_1}$ and for any δ_n, we have $z_0 \underset{\delta_n}{\dashv} x_0$ and all the pseudo orbits are in $\overline{V_1} \bigcap C_0$, so we get $z_0 \underset{\overline{V_1}\bigcap C_0}{\dashv} x_0$. We can always suppose z_0 is not a periodic point, since if z_0 is a periodic point, by f is a Kupka-Smale diffeomorphism and $Orb^+(z_0) \subset \overline{V_1}$, z_0 should be a hyperbolic periodic point with index i_0 or $i_0 + 1$.

Now for $\{\delta_n\}_{n=1}^{\infty}$ satisfying $\delta_n \longrightarrow 0^+$, for every δ_n, there exists a δ_n-pseudo orbit in C_0 from x_0 to y, denote z_n^- the first time the pseudo orbit leaves V_1, suppose $\lim_{n\to\infty} f^{-1}(z_n^-) = z_1$, then $Orb^-(z_1) \in \overline{V_1}$ and $x_0 \underset{\overline{V_1}\bigcap C_0}{\dashv} z_1$. With the same argument for z_0, we can suppose z_1 is not periodic point.

It's easy to know that lemma 7.6 is equivalent with the following result:

Lemma 7.9. *With the same assumption of lemma 3.7, suppose C doesn't contain index j_0 and j_0+1 periodic point, then for $0 < s < 1$, $\varepsilon, \delta > 0$, and for any $\{U_i\}_{i=1}^k \subset \{U_\alpha\}_{\alpha\in\mathcal{A}}$ an open cover for C_0, denote $U = \bigcup_{i=1}^{k} U_i$ and choose $\{x_0, x_1, \cdots, x_{N_0}\}$ an ε dense subset of Λ, there exists a periodic point $\{p(f)\}$ such that $Orb(p) \subset U$, $\frac{\#\{Orb(p)\bigcap \bigcup_{1}^{N} \Phi_i\}}{\Phi(p)} > s$, $Orb(p) \bigcap B_\delta(x_i) \neq \phi$ for $0 \le i \le N_0$ and $Orb(p) \bigcap B_{\delta_0}(z_0) \neq \phi$.*

Proof : The idea of the proof is following, at first use generic assumption, we can get an orbit in U beginning from a small neighborhood of z_1 and ending in a small neighborhood of z_0, then we'll find another orbit in $U \bigcap \bigcup_{i=1}^{N} \Phi_i$ beginning from a small neighborhood of z_0 passing a very small neighborhood of Λ and ending in the small neighborhood of z_1, more important, the second segment is far more longer than the first segment and its orbit has some kind of good position. Then use twice connecting lemma near z_0 and z_1, we can get a periodic orbit, and from the good position of the second orbit, we can show that the new periodic orbit will preserve a long segment orbit which belongs to the original second segment, so the new periodic orbit satisfies the density assumption, with another generic assumption, f itself will have such kind of periodic orbit.

Now at first let's show that the orbit in V_0 will have some special kind of position, this property is the key for us to get the density control.

Lemma 7.10. *There exists $0 < \delta_0 < \delta$ such that $B_{\delta_0}(z_0), B_{\delta_0}(z_1), B_{\delta_0}(x_0) \subset V_0$, and any segment orbit in $V_0 \bigcap B_{\delta_0}(C_0)$ at the end entering $B_{\delta_0}(x_0)$ never passes $B_{\delta_0}(z_1)$; and any segment of orbit in $V_0 \bigcap B_{\delta_0}(C_0)$ beginning from $B_{\delta_0}(x_0)$ never passes $B_{\delta_0}(z_0)$. More precisely:*

a) *for any a and $i_0 > 0$ satisfying $f^{i_0}(a) \in B_{\delta_0}(x_0)$ and $f^i(a) \in V_0 \bigcap B_{\delta_0}(C_0)$ for $0 \le i \le i_0$, we have $f^i(a) \notin B_{\delta_0}(z_1)$ for $0 \le i \le i_0$,*

b) *for any b and $j_0 > 0$ satisfying $b \in B_{\delta_0}(x_0)$ and $f^j(b) \in V_0 \bigcap B_{\delta_0}(C_0)$ for $0 \le j \le j_0$, we have $f^j(b) \notin B_{\delta_0}(z_0)$ for $0 \le j \le j_0$,*

Proof :We just prove the case a), the proof for the other case is similar.

If a) is false, we'll have $x_0 \underset{\overline{V_0}\bigcap C_0}{\dashv} z_0$, recall that $z_0 \underset{\overline{V_0}\bigcap C_0}{\dashv} x_0$, we know there exists a chain recurrent set $C_1 \subset \overline{V_0} \bigcap C_0$ containing z_0, x_0 and Λ, by 4) or proposition 3.1, there exists a family of periodic orbits $\{Orb(p_n)\}$ satisfying $\lim_{n\to\infty} Orb(p_n) \longrightarrow C_1$. Recall that C_1 has the partial hyperbolic splitting

$T_{C_1}M = E^s_{j_0} \oplus E^c_1 \oplus E^u_{j_0+2}$, so $Orb(p_n)$ has index j_0 or j_0+1. If $\{p_n\}$ all have index j_0, that means C_1 is an index j_0 fundamental limit of C_0, it's a contradiction with the fact that Λ is the maximal index j_0 fundamental limit of C_0. If $\{p_n\}$ all have index j_0+1, then we have

1) either $\{Orb(p_n)\}$ are index stable,
2) or $\{Orb(p_n)\}$ are not index stable.

In the first case, by Gan's lemma, C contains index j_0+1 periodic point. In the second case, for any $\varepsilon > 0$, there exist n big enough and a diffeomorphism g_n such that $Orb_f(p_n)$ is an index j_0 periodic orbit of g_n and $d(f, g_n) < \varepsilon$, it means that C_1 is an index j_0 fundamental limit, it's a contradiction with the fact that Λ is the maximal index j_0 fundamental limit. □

Now we choose another family of open sets $\{U_i^*\}_{i=1}^{k^*} \subset \{U_\alpha\}_{\alpha\in\mathcal{A}}$ which is an open cover of C_0 such that $\bigcup_{i=1}^{k^*} U_i^* \subset B_{\delta_0}(C_0)\bigcap U$, denote $U_0 = \bigcup_{i=1}^{k^*} U_i^*$. From now we just focus the orbits in U_0, all the orbits or pseudo orbits we consider in the following proof will locate in U_0.

Now choose $\varepsilon_n \longrightarrow 0^+$ and after replacing by a subsequence, we can suppose the $\{s_n\}$ fixed in lemma 7.2 satisfies $s < s_n \longrightarrow 1^-$, then by connecting lemma 7.3, the open set $B_{\varepsilon_n}(f) \subset C^1(M)$ gives us a family of parameters: $\delta_n \longrightarrow 0^+, \rho_n \longrightarrow \infty, L_n$.

At first, we choose a sequence $\delta_{0,n}$ such that

A1 $\delta_{0,n} < \delta_0$, $\delta_{0,n+1} < \frac{\delta_{0,n}}{\rho_n}$;
A2 $f^i(B_{\delta_{0,n}}(z_0))\bigcap f^j(B_{\delta_{0,n}}(z_0)) = \phi$ for $0 \le i \ne j \le L_n$ and $f^i(B_{\delta_{0,n}}(z_0)) \subset \bigcup_{i=0}^{N} \Phi_i \bigcap U_0$ for $0 \le i \le L_n$;
A3 define $\Delta_{0,n} = \bigcup_{i=0}^{L_n} f^i(B_{\delta_{0,n}}(z_0))$, we have $\overline{\Delta_{0,n}}\bigcap\Lambda = \phi$.
A4 $\Delta_{0,n} \subset U_0$.

Recall that $z_0 \in C_0 \setminus \Lambda$ is not a periodic point and $Orb^+(z_0) \subset V_1 \subset \bigcup_{1}^{N} \Phi_i$, so we can always choose such sequence, with the same reason, we can also choose a sequence $\delta_{1,n}$ such that

B1 $\delta_{1,n} < \delta_0$, $\delta_{1,n+1} < \frac{\delta_{1,n}}{\rho_n}$;
B2 $f^{-i}(B_{\delta_{1,n}}(z_1))\bigcap f^{-j}(B_{\delta_{1,n}}(z_1)) = \phi$ for $0 \le i \ne j \le L_n$ and $f^{-i}(B_{\delta_{1,n}}(z_1)) \subset \bigcup_{i=0}^{N} \Phi_i$ for $0 \le i \le L_n$;
B3 define $\Delta_{1,n} = \bigcup_{i=0}^{L_n} f^{-i}(B_{\delta_{1,n}}(z_1))$, we have $\overline{\Delta_{1,n}}\bigcap\Lambda = \phi$;
B4 $\Delta_{1,n} \subset U_0$.

Now by lemma 7.1 and $z_0, z_0 \in C_0 \subset U_0$, there exist a family of points a_n and numbers $i_{0,n}$ satisfying the following property:

C1 $a_n \in B_{\delta_{1,n}/\rho_n}(z_1)$ and $f^{i_{0,n}}(a_n) \in B_{\delta_{0,n}/\rho_n}(z_0)$,
C2 $f^i(a_n) \in U_0$ for $0 \le i \le i_{0,n}$.

Then for every n there exists a sequence $\delta_{2,n} \longrightarrow 0^+$ such that:

D1 $\delta_{2,n} < \delta_0$ and $\delta_{2,n+1} < \delta_{2,n}$,
D2 for any $x \in \Lambda$, we have $B_{\delta_{2,n}}(x)\bigcap\Delta_{0,n} = \phi$ and $B_{\delta_{2,n}}(x)\bigcap\Delta_{1,n} = \phi$,
D3 for any $x \in \Lambda$ and any i satisfying $f^i(B_{\delta_{0,n}}(z_0))\bigcap B_{\delta_{2,n}}(x) \ne \phi$, we have $\frac{i_{0,n}}{i-L_n} < 1 - s_n$,

D4 for any $x \in \Lambda$ and any i satisfying $f^{-i}(B_{\delta_{1,n}}(z_1)) \bigcap B_{\delta_{2,n}}(x) \neq \phi$, we have $\frac{i_{0,n}}{i-L_n} < 1 - s_n$.

Since Λ is an invariant compact subset not containing periodic point (that's because C doesn't contain index j_0 and $j_0 + 1$ periodic point), we can always choose such sequence.

By the property $z_0 \underset{\overline{V_1} \bigcap C_0}{\dashv} x_0 \underset{\overline{V_1} \bigcap C_0}{\dashv} x_1 \underset{\overline{V_1} \bigcap C_0}{\dashv} \cdots \underset{\overline{V_1} \bigcap C_0}{\dashv} x_{N_0} \underset{\overline{V_1} \bigcap C_0}{\dashv} z_1$ (the last relation comes from $x_{N_0} \underset{\overline{V_1} \bigcap C_0}{\dashv} x_0 \underset{\overline{V_1} \bigcap C_0}{\dashv} z_1$), there exists a family of points $b_n \in B_{\delta_{0,n}/\rho_n}(z_0)$ and $0 < j_{i,n} < j_{N_0+1,n}$ ($0 \leq i \leq N_0$) such that

E1 $f^{j_{i,n}}(b_n) \in B_{\delta_{2,n}}(x_i)$ for $0 \leq i \leq N_0$ and $f^{j_{N_0+1,n}}(b_n) \in B_{\delta_{1,n}/\rho_n}(z_1)$,

E2 $f^j(b_n) \in \bigcup_{i=1}^{N} \Phi_i \bigcap U_0$ for $0 \leq j \leq j_{N_0+1,n}$.

Denote $J_{0,n} = \min_i\{j_{i,n}, 0 \leq i \leq N_0\}, J_{1,n} = \max_i\{j_{i,n}, 0 \leq i \leq N_0\}$, by the 'good' position given in lemma 7.10, $f^j(b_n) \bigcap B_{\delta_{1,n}}(z_1) = \phi$ for $0 \leq j \leq J_{1,n}$ and $f^j(b_n) \bigcap B_{\delta_{0,n}}(z_0) = \phi$ for $J_{0,n} \leq j \leq j_{N_0+1}$. Suppose $j^*_{0,n}$ the last time the orbit $\{f^j(b_n)\}_{j=0}^{J_{0,n}}$ leaves $B_{\delta_{0,n}}(z_0)$ and $j^*_{1,n}$ the first time the orbit $\{f^j(b_n)\}_{j=J(1,n)}^{j_{N_0+1,n}}$ enters $B_{\delta_{1,n}}(z_1)$. Then by D2, D4 and the 'good' position, we know that $f^j(b_n) \bigcap B_{\delta_{0,n}}(z_0) = \phi$ and $f^j(b_n) \bigcap B_{\delta_{1,n}}(z_1) = \phi$ for $j^*_{0,n} \leq j \leq j^*_{1,n}$, what's more, we have $j^*_{0,n} + L_n < J_{0,n}, j^*_{1,n} - L_n > J_{1,n}$, $\frac{i_{0,n}}{J_{0,n} - j^*_{0,n} - L_n} < 1 - s_n$ and $\frac{i_{0,n}}{j^*_{1,n} - J_{1,n} - L_n} < 1 - s_n$.

In fact, we can split the orbit $\{f^i(b_n)\}_{i=0}^{j_{N_0+1,n}}$ into three sub-segments: segment I, $\{f^i(b_n)\}_{i=0}^{j^*_{0,n}+L_n-1}$; segment II, $\{f^i(b_n)\}_{i=j^*_{0,n}+L_n}^{j^*_{1,n}-L_n}$; segment III, $\{f^i(b_n)\}_{j^*_{,n}-L_n+1}^{j_{N_0+1,n}}$. The segment I doesn't intersect with $\Delta_{1,n} = \bigcup_{j=0}^{L_n-1} f^{-j}(B_{\delta_{1,n}}(z_1))$, segment III doesn't intersect with $\Delta_{0,n} = \bigcup_{j=0}^{L_n-1} f^j(B_{\delta_{0,n}}(z_0))$, segment II doesn't intersect with $\Delta_{0,n}$ and $\Delta_{1,n}$. In the following proof, we'll use connecting lemmas in $\Delta_{1,n}$ and $\Delta_{0,n}$ respectively, then the segment II is unchanged after the perturbation, and in fact we'll get a new periodic orbit which contains segment II.

Since now we'll use twice connecting lemma near z_0 and z_1 and get a periodic orbit.

At first fix an n, we'll do the connecting lemma in a neighborhood of z_1, let's consider the two points $f^{i_{0,n}}(a_n)$ and b_n, we know the positive f-orbit of b_n hits $B_{\delta_{1,n}/\rho_n}(z_1)$ after b_n and the negative f-orbit of $f^{i_{0,n}}(a_n)$ hits $B_{\delta_{1,n}/\rho_n}(z_1)$ also, by connecting lemma and the fact $\Delta_{1,n} \subset \bigcup_{i=1}^{N} \Phi_i \bigcap U_0$, there exists $g^*_n \in B_{\varepsilon_n}(f)$ such that $g^*_n \equiv f$ off $\Delta_{1,n} = \bigcup_{i=0}^{L_n-1} f^{-i}(B_{\delta_{1,n}}(z_1))$ and there exists $0 < j^*_{2,n} < j^*_{3,n}$ such that:

F1 $(g^*_n)^j(b_n) = f^j(b_n)$ for $0 \leq j \leq j^*_{1,n} - L_n$,

F2 $j^*_{2,n} > j^*_{1,n} > J_{1,n}$ and $(g^*_n)^{j^*_{2,n}}(b_n) \in B_{\delta_{1,n}}(z_1)$, $(g^*_n)^{j^*_{3,n}}(b_n) = f^{i_{0,n}}(a_n) \in B_{\delta_{0,n}/\rho_n}(z_0)$,

F3 $(g^*_n)^j(b_n) \in \bigcup_{i=1}^{N} \Phi_i$ for $0 \leq j \leq j^*_{2,n}$ and $j^*_{3,n} - j^*_{2,n} < i_{0,n}$,

F4 $(g^*_n)^j(b_n) \in U_0$ for $0 \leq j \leq j^*_{3,n}$.

Remark 7.11. *Above argument shows that* $\#\{\{(g^*_n)^j(b_n)\}_{j=0}^{j^*_{3,n}} \bigcap (\bigcup_{i=1}^{N} \Phi_i)^c\} < i_{0,n}$*, in the following proof, we'll use connecting lemma again in a neighborhood of* z_0 *and we can get a new diffeomorphism* g *and a periodic point* $p_n(g_n)$ *such that they satisfy the following property:*

- $\#\{\{Orb_{g_n}(p_n) \bigcap (\bigcup_{i=1}^{N} \Phi_i)^c\} < i_{0,n}$
- $\{f^j(b_n)\}_{j=j^*_{0,n}+L_n}^{j^*_{1,n}-L_n} \subset Orb_{g_n}(p_n)$,

- $Orb_{g_n}(p_n) \subset U_0$.

Now we'll use connecting lemma in the neighborhood of z_0, let's consider $f^{j^*_{1,n}-L_n}(b_n)$, since $f^j(b_n) = (g^*_n)^j(b_n)$ for $0 \leq j \leq j^*_{1,n} - L_n$ we know that the negative g^*_n-orbit of $f^{j^*_{1,n}-L_n}(b_n)$ hits $B_{\delta_{0,n}/\rho_n}(z_0)$ after $f^{j^*_{1,n}-L_n}(b_n)$, and by F2, the negative g^*_n-orbit of $f^{j^*_{1,n}-L_n}(b_n)$ hits $B_{\delta_{0,n}/\rho_n}(z_0)$ also. Using connecting lemma, by the fact $\Delta_{0,n} = \bigcup_{j=0}^{L_n} f^j(B_{\delta_{0,n}}(z_0)) \subset \bigcup_{i=1}^{N} \Phi_i \bigcap U_0$, F4 and remark 7.4, there exists $g_n \in B_{\varepsilon_n}(f)$ such that $g_n \equiv g^*_n$ off $\Delta_{0,n}$ and there exists j_0, j_1 such that

G1 $g_n^{j_1}(f^{j^*_{1,n}-L_n}(b_n)) = g_n^{-j_0}(f^{j^*_{1,n}-L_n}(b_n)) \in B_{\delta_{0,n}}(z_0)$,

G2 $f^{(j^*_{1,n}-L_n)-j}(b_n) = (g^*_n)^{-j}(f^{j^*_{1,n}-L_n}(b_n)) = (g_n)^{-j}(f^{j^*_{1,n}-L_n}(b_n))$ for $0 \leq j \leq j^*_{1,n} - j^*_{0,n} - 2L_n$, it means that $\#\{Orb_{g_n}(f^{j_{N_0,n}}(b_n)) \bigcap \bigcup_{i=1}^{N} \Phi_i\} \geq j^*_{1,n} - j^*_{0,n} - 2L_n > j^*_{1,n} - J_{1,n} - L_n$,

G3 $\#\{Orb_{g_n}(f^{j_{N_0,n}}(b_n)) \bigcap (\bigcup_{i=1}^{N} \Phi_i)^c\} \leq j^*_{3,n} - j^*_{2,n} \leq i_{0,n}$,

G4 $Orb_{g_n}(f^{j^*_{1,n}-L_n}(b_n)) \subset U_0$.

We denote the above periodic orbits for g_n by $Orb(p_n)$ where $p_n = g_n^{j_1}(f^{j^*_{1,n}-L_n}(b_n))$, then we know that $\lim_{n\to\infty} p_n \longrightarrow z_0$ and $\frac{\#\{Orb_{gn}(p_n) \bigcap \bigcup_{i=1}^{N} \Phi_i\}}{\pi_{gn}(Orb_{gn}(p_n))} = 1 - \frac{\#\{Orb_{gn}(p_n) \bigcap (\bigcup_{i=1}^{N} \Phi_i)^c\}}{\pi_{gn}(Orb_{gn}(p_n))} \geq 1 - \frac{i_{0,n}}{j^*_{1,n} - J_{0,n} - L_n} > 1 - (1 - s_n) = s_n$. By G2 and $j_{0,n} + L_n < J_{0,n} \leq j_{i,n} \leq J_{1,n} < j^*_{1,n} - L_n$ for $0 \leq i \leq N_0$, $f^{j_{i,n}}(b_n) = f^{j_{i,n}-(j^*_{1,n}-L_n)}(f^{j^*_{1,n}-L_n}(b_n)) = g_n^{j_{i,n}-(j^*_{1,n}-L_n)}(f^{j^*_{1,n}-L_n}(b_n)) \subset Orb_{g_n}(p_n)$, so $Orb_{g_n}(p_n) \bigcap B_{\delta_{2,n}}(x_i) \neq \phi$.

Now we know that there exists a family of diffeomorphisms $\{g_n\}$ such that $g_n \xrightarrow{C^1} f$ and g_n has periodic point p_n such that $Orb_{g_n}(p_n) \subset U_0 \subset U$, $\frac{\#\{Orb_{gn}(p_n) \bigcap \bigcup_{i=1}^{N} \Phi_i\}}{Orb_{gn}(p_n)} > s$, $p_n \longrightarrow z_0$ and $Orb_{g_n}(p_n) \bigcap B_{\delta_{2,n}}(x_i) \neq \phi$. Choose $O_0(x_0), \cdots, O_{N_0}(x_{N_0}), O_{N_0+1}(z_0) \in \{U_\alpha\}_{\alpha\in\mathcal{A}}$ neighborhoods of $x_0, \cdots, x_{N_0}, z_0$ respectively such that $O_{N_0+1}(z_0) \subset B_{\delta_0}(z_0)$, and $O_i(x_i) \subset B_{\delta_0}(x_i)$ for $0 \leq i \leq N_0$, then there exists n_0 such that for $n > n_0$ we have $Orb_{g_n}(p_n) \bigcap O_i \neq \phi$ for $0 \leq i \leq N_0 + 1$, so by generic property lemma 7.2, f itself has periodic point p such that $Orb(p) \subset U$, $\frac{\#\{Orb(p) \bigcap \bigcup_{i=1}^{N} \Phi_i\}}{Orb(p)} > s$ and $Orb(p_n) \bigcap O_i \neq \phi$ for $0 \leq i \leq N_0 + 1$, since $O_i(x_i) \subset B_{\delta_0}(x_i) \subset B_\delta(x_i)$ for $0 \leq i \leq N_0$ and $O_{N_0+1}(z_0) \subset B_\delta(z_0)$, we finish the proof. ☐

References

[1] F. Abdenur, C. Bonatti, S. Crovisier, L.J. Diaz and L. Wen, *Periodic points and homoclinic classes, preprint* (2006).

[2] R. Abraham and S. Smale, Nongenericity of Ω -stability, Global analysis I, *Proc. Symp. Pure Math. AMS* **14** (1970), 5-8.

[3] C. Bonatti and S. Crovisier, Recurrence et genericite(French), *Invent. math.,* **158** (2004), 33-104

[4] C. Bonatti, L. J. Díaz, Connexions hétérocliniques et généricite d'une infinité de puits ou de sources, *Annales Scientifiques de l'cole Normal Suprieure de Paris*, **32 (4)**, (1999) 135-150,

[5] C. Bonatti, L. J. Díaz, and M. Viana, Dyanamics beyond uniform hyperbolic, **Volume 102** of *Encyclopaedia of Mathematical Sciences.* Springer- Verlag, Berlin, 2005. A global geometric and probabilistic perspective, Mathematical Physics, III.

[6] C. Bonatti, S. Gan and L. Wen, On the existence of non-trivial homoclinic class, *preprint* (2005)

[7] S. Crovisier, Periodic orbits and chain transitive sets of C^1 diffeomorphisms, *preprint* (2004).

[8] S. Crovisier, Birth of homoclinic intersections: a model for the central dynamics of partial hyperbolic systems, *preprint* (2006).

[9] J. Franks, Necessary conditions for stability of diffeomorphisms, *Trans. Amer. Math. Soc.*, **158** (1977), 301-308.

[10] S. Gan, Another proof for C^1 stability conjecture for flows, *SCIENCE IN CHINA (Series A)* **41 No. 10** (October 1998) 1076-1082

[11] S. Gan, The Star Systems X^* and a Proof of the $C^1 Omega$-stability Conjecture for Flows, *Journal of Differential Equations*, **163** (2000) 1–17

[12] S. Gan and L. Wen, Heteroclinic cycles and homoclinic closures for generic diffeomorphisms, *Journal of Dynamics and Differential Equations,*, **15** (2003), 451-471.

[13] S. Gan and L. Wen, Nonsingular star flow satisfy Axion M and the nocycle condition, *Ivent. Math.,* **164** (2006), 279-315.

[14] N. Gourmelon, Addapted metrics for dominated splitting, *Ergod. Th. and Dyn. Sys.* **27** (2007), 1839-1849.

[15] N. Gourmelon, Generation of homoclinic tangencies by C^1 perturbations. *Prepublications del Institute Math. de Bourgogne* **502** (2007).

[16] S. Hayashi, Connecting invariant manifolds and the solution of the C^1 stability and Ω-stable conjecture for flows, *Ann. math.*, **145** (1997), 81-137.

[17] M. Hirsch, C. Pugh, and M. Shub, Invariant manifolds, volume 583 of *Lect. Notes in Math.* Springer Verlag, New york, 1977

[18] Shantao Liao, On the stability conjecture, *Chinese Annals of Math.*, **1** (1980), 9-30.(in English)

[19] Shantao Liao, Qualitative Theory of Differentiable Dynamical Systems, *China Science Press*, (1996).(in English)

[20] R. Mañé, Contributions to the stability conjecture, *Topology*, **17** (1978), 383-396.

[21] R. Mañé, An ergodic closing lemma, *Ann. Math.*, **116** (1982), 503-540.

[22] R. Mañé, A proof of the C^1 stability conjecture. *Inst. Hautes Etudes Sci. Publ. Math.* **66** (1988), 161-210.

[23] S. Newhouse, Non-density of Axiom A(a) on S 2. *Proc. A. M. S. Symp pure math*, **14** (1970), 191-202, 335-347.

[24] S. Newhouse, Diffeomorphisms with infinitely many sinks, *Topology*, **13**, 9-18, (1974).

[25] J. Palis and M. Viana, High dimension diffeomorphisms displaying infinitely sinks, *Ann. Math.*, **140** (1994), 1-71.

[26] E. Pujals and M. Sambarino, Homoclinic tangencies and hyperbolicity for surface diffeomorphisms, *Ann. Math.*, **151** (2000), 961-1023.

[27] M. Shub, Topological transitive diffeomorphisms in T^4, Lecture Notes in Math. Vol. **206**, Springer Verlag, 1971.

[28] L. Wen, On the C^1-stability conjecture for flows. *Journal of Differential Equations,* **129**(1995) 334-357.

[29] L. Wen, Homoclinic tangencies and dominated splittings, *Nonlinearity*, **15** (2002), 1445-1469.

[30] L.Wen, A uniform C^1 connecting lemma, *Discrete and continuous dynamical systems*, **8** (2002), 257-265.

[31] L. Wen, Generic diffeomorphisms away from homoclinic tangencies and heterodimensional cycles, *Bull. Braz. Math. Soc. (N.S.)*, **35** (2004), 419- 452.

[32] L. Wen, Selection of Quasi-hyperbolic strings, *preprint* (2006).

[33] L. Wen and S. Gan, Obstruction sets, Obstruction sets, quasihyperbolicity and linear transversality.(Chinese) *Beijing Daxue Xuebao Ziran Kexue Ban*, **42** (2006), 1-10.

[34] Z. Xia and L. Wen, C^1 connecting lemmas, *Trans. Amer. Math. Soc.* **352** (2000), 5213-5230.

[35] D. Yang, S. Gan, L, Wen, Minimal Non-hyperbolicity and Index-Completeness, *preprine* (2007).

[36] J. Yang, Ergodic measure far away from tangencies, *preprint* (2007).

[37] J. Yang, Newhouse phenomenon and homoclinic classes. *preprint* (2007).

[38] J. Yang, Lyapunov stable chain recurrent classes, *preprint* (2007).

IMPA, Est. D. Castorina 110, 22460-320 Rio de Janeiro, Brazil

E-mail address: `yangjg@impa.br`