ERGODIC MEASURES FAR AWAY FROM TANGENCIES

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Abstract. We show that for C^1 diffeomorphisms far away from homoclinic tangencies, every ergodic invariant measure has at most one zero Lyapunov exponent, and the Oseledets splitting corresponding to positive, zero, and negative exponents is dominated. When the invariant ergodic measure is hyperbolic (all exponents non-zero), then almost every point has a local stable manifold and a local unstable manifold both of which are differentiable embedded disks. Moreover, a version of the classical shadowing lemma holds, so that the hyperbolic measure is the weak limit of a sequence of atomic measures supported on periodic orbits belonging to the same homoclinic class.

Together with a recent result of [7], this allows us to prove that there exists a residual subset R of C^1 diffeomorphisms far away from tangencies such that for any $f \in R$, either it's Axiom A, or it has a non-hyperbolic ergodic measure.

1. Introduction

In his famous paper [15], the first time Oseledets gave the definition and existence of Lyapunov exponents for any invariant measure: for an ergodic measure μ of a diffeomorphism f, there exist $k \in \mathbb{N}$, real numbers $\lambda_1 > \cdots > \lambda_k$, and for $\mu-$almost all $x \in M$, there exists a splitting $T_xM = E^1_x \oplus \cdots \oplus E^k_x$ of the tangent space, such that the splitting is invariant under Df, and

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|Df^n(x)v_i\| = \lambda_i, \quad v_i \in E^i_x \setminus \{0\}.$$

We call λ_i the Lyapunov exponent of μ and the splitting $E^1 \oplus \cdots \oplus E^k$ Oseledets splitting. Usually the splitting is just defined on a full measure subset, not continuous just measurable changed with the points. In fact, for any measurable bundle on M, Oseledets proved the existence of Lyapunov exponents for any invariant measure.

Since then, Lyapunov exponents have played a key role in studying the ergodic behavior of a dynamical system, understanding the Lyapunov exponents also become to one of the classical problems of the theory of differential dynamical systems. Especially when all the Lyapunov exponents are not vanishing, such kind of ergodic measure is called hyperbolic measure and which attracts a lot of attention.

Here we prove that if the diffeomorphism is far away from homoclinic tangencies, the Lyapunov exponents of its ergodic measures can be given a good description. Here a diffeomorphism is far away from homoclinic tangencies means that no diffeomorphism in a neighborhood exhibits a non-transverse intersection between the stable manifold and the unstable manifold of some periodic point.

Theorem 1: *Suppose f is far away from tangencies and μ is an ergodic measure of f, then*

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- *either μ is hyperbolic with index i and the index i Oseledets splitting is a dominated splitting,*
- *or μ has just one zero Lyapunov exponent, and the Oseledets splitting corresponding to negative, zero and positive Lyapunov exponents is a dominated splitting.*

Remark: *The definition of dominated splitting is given in section 1, since dominated splitting is always continuous, the above special kind of Oseledets splitting is always continuous.*

By the definition, the tangent space of almost every point of a hyperbolic measure is splitted as the sum of two subspaces which are exponentially contracted or expanded by all large enough iterated of the derivative, it means the hyperbolic measure has some 'weak' hyperbolic property on the tangent space. Pesin showed that with some additional regularity assumption on the diffeomorphism (C^2 or $C^{1+Holder}$), the hyperbolic ergodic measure shares many properties with hyperbolic set, for example, there exists a family of local stable manifolds on a positive subset which is continuous and with uniform size, such property is called the stable manifold theorem; Katok gave also a shadowing lemma, with it he proved that the hyperbolic ergodic measure is the weak limit of a sequence of atomic invariant measures supported on periodic orbits belonging to the same homoclinic class, such property is called Katok's closing lemma. Along this direction several deeper results have been proved, such as entropy formula, dimension theory etc, all these results are called Pesin theory.

Now usually we call a hyperbolic measure together with the diffeomorphism a non-uniform hyperbolic system, Pesin theory has been proved a very important and powerful tool to understand the non-uniform hyperbolic system. But there is a restriction because the Pesin theory always needs the diffeomorphism be $C^{1+Holder}$, for C^1 diffeomorphism the arguments fail to work (see [19]).

In [1], they begin to consider C^1 Pesin theory, they proved that with a dominated assumption on the tangent space, the stable manifold theorem is still true, and if the diffeomorphism is 'tame', then there exist a lot of hyperbolic ergodic measures.

In this paper we treat Pesin theory as a theory derives topological information from hyperbolic measure, it means that we just consider the stable manifold theorem and Katok's closing lemma. With such understanding, we show that when the diffeomorphism is C^1 far from tangencies, C^1 Pesin theory is still true. The precisely statement is following:

Theorem 2: *Suppose $f \in Diff^1(M) \setminus \overline{HT}$ and μ is a hyperbolic ergodic measure of f with i negative Lyapunov exponents, then C^1 Pesin theory is true:*

a) *almost every point has a local stable manifold and a local unstable manifold both of which are differentiable embedded disks, and there exists a compact positive measure subset Λ^s (resp. Λ^u) which has continuous and uniform size of stable (resp. unstable) manifolds.*
b) *μ is the weak limit of a sequence of invariant measures μ_n supported by periodic orbits p_n with index i, and the periodic orbits are homoclinic related with each other.*

Remark:

- *The stable manifolds we get usually is not absolutely continuous, that's because the absolutely continuous property heavily depends on distortion which just holds under $C^{1+\alpha}$ assumption.*
- *In fact, from the proof, it's easy to see that the above theorem can be stated in the following classical way of Pesin theory in the $C^{1+\alpha}$ case:*

  Suppose f is far away from tangencies and μ is a hyperbolic ergodic measure of f with i negative Lyapunov exponents, then there exists a family of compact set $\Lambda_0 \subset \Lambda_1 \subset \cdots$ with positive measure such that $f^{+(-)}(\Lambda_i) \subset \Lambda_{i+1}$, $\mu(\bigcup_i \Lambda_i) = 1$ and they satisfy the following properties:
  - *for every Λ_i, there exist local continuous stable and unstable manifolds on it with uniform size;*
  - *for every Λ_i, there exist $\varepsilon_i > 0$, $L_i > 0$ and $N_i \in \mathbb{N}$, such that if there exist $x \in \Lambda_i$ and $m > N_i$ satisfying $f^m(x) \in \Lambda_i$ and $d(x, f^m(x)) < \varepsilon_i$, then there exists periodic point p with period m and $d(f^j(x), f^j(p)) < L_i \cdot d(x, f^m(x))$ for $0 \leq j < m$. Moreover, some point in the periodic orbit has uniform size of local stable and local unstable manifolds and the size just depends on Λ_i.*

In [7], Díaz and Gorodetski started to consider the generic existence of non-hyperbolic ergodic measure and gave the following conjecture:

Conjecture 1: *There exists a generic subset R in $Diff^1(M)$ such that for any $f \in R$, either f is Axiom A or it has a non-hyperbolic ergodic measure μ.*

With a result of [7], we prove conjecture 1 for diffeomorphisms far from tangencies:

Theorem 3: *There exists a residual subset R in $Diff^1(M) \setminus \overline{HT}$ such that for any $f \in R$*

- *either f is hyperbolic,*
- *or f has a non-hyperbolic ergodic measure μ.*

The structure of this paper is following: in § 2 we give some definitions and notations, theorem 1 is proved in § 3, in § 4, we give the proof of theorem 2, in § 5 we give some basic C^1 generic properties and theorem 3 is proved in § 6.

After this preprint was written, we learned from S. Crovisier that he got some similar results in [6].
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2. Definitions and Notations

Let M be a compact boundlessness Riemannian manifold with $dim(M) = d \geq 2$. Denote by $Per(f)$ the set of periodic points of f and $\Omega(f)$ the non-wondering set of f, for $p \in Per(f)$, $\pi(p)$ is the period of p. When p is a hyperbolic periodic point, the index of p is the dimension of its stable bundle, $Per_i(f)$ is the set of index i periodic points of f, and we call $x \in M$ an index i preperiodic point of f if there exists a family of diffeomorphisms $g_n \xrightarrow{C^1} f$, where g_n has an index i periodic point p_n and $p_n \longrightarrow x$. Denote by $P_i^*(f)$ the set of index i preperiodic point of f, it's easy to see $\overline{P_i(f)} \subset P_i^*(f)$.

An invariant compact set Λ is called index i fundamental limit if there exists a family of diffeomorphisms g_n C^1 converging to f, p_n is an index i periodic point of g_n and $Orb(p_n)$ converge to Λ in *Hausdorff* topology. By the definition, if Λ is an index i fundamental limit of f, then $\Lambda \subset P_i^*(f)$.

For two points $x, y \in M$ and $\delta > 0$, if there exist points $x = x_0, x_1, \cdots, x_n = y$ such that $d(f(x_i), x_{i+1}) < \delta$ for $i = 0, 1, \cdots, n-1$, we say there is a δ-pseudo orbit connecting x and y and denote by $x \underset{\delta}{\dashv} y$. If $x \underset{\delta}{\dashv} y$ is true for any $\delta > 0$, we denote $x \dashv y$, and $x \vdash\!\dashv y$ if $x \dashv y$ and $y \dashv x$. When $x \vdash\!\dashv x$, it's called chain recurrent point, $CR(f)$ is the set of chain recurrent points of f, it's easy to see that $\vdash\!\dashv$ is an closed equivalent relation on $CR(f)$, every equivalent class of such relation should be compact and is called chain recurrent class. Let K be a compact invariant set of f, for $x, y \in K$, we denote $x \underset{K}{\dashv} y$ if for any $\delta > 0$, one has a δ -pseudo orbit in K connecting x and y. If $x \underset{K}{\dashv} y$ is true for any $x, y \in K$, we call K a chain recurrent set. Let C be a chain recurrent class of f, it's called aperiodic class if C does not contain periodic point.

Let Λ be an invariant compact set of f, for $0 < \lambda < 1$ and $1 \leq i < d$, we say Λ admits an index $i-(l, \lambda)$ dominated splitting if there is a continuous invariant splitting $T_\Lambda M = E \oplus F$ where $dim(E_x) = i$ for any $x \in \Lambda$ and $\| Df^l|_E(x) \| \cdot \| Df^{-l}|_F(f^l x) \| < \lambda$ for all $x \in \Lambda$. It's well known that when Λ admits a dominated splitting $E \oplus F$, then the two bundles E, F are continuous bundles. A compact invariant set can have many dominated splittings, but for fixed i, index i dominated splitting is unique. We also use dominated splitting for several bundles, an invariant splitting $E_1 \oplus E_2 \oplus \cdots \oplus E_k$ on the tangent space of Λ is called dominated splitting if there is l, λ such that for any $x \in \Lambda$ and $\| Df^l|_{E_j(x)} \| \cdot \| Df^{-l}|_{E_{j+1}(f^l x)} \| < \lambda$ for all $x \in \Lambda$ and $1 \leq j < k$.

Remark 2.1. *Suppose μ is an ergodic measure of diffeomorphism f, and $supp(\mu)$ admits an index i dominated splitting $E \oplus F$, because the bundles E, F are continuous, consider the Lyapunov exponents for μ on bundle E and F respectively, denote $\lambda_1 \leq \cdots \lambda_i \leq \lambda_{i+1} \leq \cdots \leq \lambda_d$ the exponents of μ, by the definition of dominated splitting, vectors of F expand faster than of E, so the exponents for μ on bundle E are smaller than the exponents for μ on bundle F, it implies $\lambda_1, \cdots, \lambda_i$ are the exponents for μ on bundle E and $\lambda_{i+1}, \cdots, \lambda_d$ are the exponents for μ on bundle F.*

We say an ergodic invariant measure μ of diffeomorphism f has type (i, k) if the number of negative and vanishing Lyapunov exponents of μ is i and k respectively. And when $k = 0$, we say μ has index i.

For $f \in Diff^r(M)$, if there is a non-transverse intersection between the stable and unstable manifolds of some hyperbolic periodic point p, we say f has C^r tangency. Denote HT^r the set of diffeomorphisms which have C^r tangency, usually we just use HT denote HT^1. And a diffeomorphism f is far away from tangency if $f \in Diff^1(M) \setminus \overline{HT}$. The following proposition shows the relation between dominated splitting and far away from tangencies.

Proposition 2.2. *([20]) f is C^1 far away from tangencies if and only if there exists (l,λ) such that $P_i^*(f)$ has index $i-(l,\lambda)$ dominated splitting for $0<i<d$.*

Mañé showed that in some special case, one bundle of the dominated splitting is hyperbolic.

Proposition 2.3. *([14]) Suppose $\Lambda(f)$ has an index i dominated splitting $E\oplus F$ $(i\neq 0)$, if $\Lambda(f)\bigcap P_j^*(f)=\phi$ for $0\leq j<i$, then E is a contracting bundle.*

3. **Proof of theorem 1**

Here we need the following special statement of ergodic closing lemma which is a little stronger than the original statement given in [14] and whose proof will be given in Appendix.

Lemma 3.1. *(New statement of Ergodic closing lemma) Suppose μ is a type (i,k) ergodic measure of f, then for any $i\leq j\leq i+k$, $supp(\mu)$ is an index j fundamental limit. And there exists a family of diffeomorphisms g_n, such that:*

1) *: $g_n \xrightarrow{C^1} f$,*
2) *: g_n has periodic point p_n with index j, denote by μ_n the invariant atom measure on $Orb_{g_n}(p_n)$, then $\mu_n \xrightarrow{*-weak} \mu$.*

Proof of theorem 1: We divide the proof into two cases:

a) μ is hyperbolic with index i;
b) μ has type (i,k) where $k\neq 0$.

In case a), by lemma 3.1 and proposition 2.2, $supp(\mu)\subset P_i^*$ and $supp(\mu)$ has index i dominated splitting $E_i^{cs}\oplus E_{i+1}^{cu}$. By the definition of dominated splitting and remark 2.1, the Lyapunov exponents for μ on bundle E_i^{cs} are strictly smaller than the Lyapunov exponents for μ on bundle E_{i+1}^{cu}, since the number of negative exponents of μ is i, the Lyapunov exponents for μ on bundle E_i^{cs} are negative and the dominated splitting $E_i^{cs}\oplus E_{i+1}^{cu}$ is the Oseledets splitting corresponding to the positive and negative exponents.

In case b), at first we show $k=1$.

If $k>1$, by lemma 3.1, $supp(\mu)\subset P_i^*\bigcap P_{i+1}^*\bigcap\cdots\bigcap P_{i+k}^*$, then by proposition 2.2 and $f\in(\overline{HT})^c$, $supp(\mu)$ admits index i dominated splitting $E_i^{cs}\oplus E_{i+1}^{cu}$; index $i+1$ dominated splitting $E_{i+1}^{cs}\oplus E_{i+2}^{cu}$; $\cdots$, and index $i+k$ dominated splitting $E_{i+k}^{cs}\oplus E_{i+k+1}^{cu}$. Let

$$E_{i+1,1}^{c}=E_{i+1}^{cs}\bigcap E_i^{cu},\ E_{i+2}^{c}=E_{i+2}^{cs}\bigcap E_{i+1}^{cu},\ \cdots,\ E_{i+k}^{c}=E_{i+k}^{cs}\bigcap E_{i+k-1}^{cu},$$

then $supp(\mu)$ admits a new dominated splitting

$$T_{supp(\mu)}M=E_i^{cs}\oplus E_{i+1,1}^{c}\oplus E_{i+2,1}^{c}\oplus\cdots\oplus E_{i+k,1}^{c}\oplus E_{i+k+1}^{cu}.$$

We denote $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_d$ the Lyapunov exponents of μ, since μ has type (i,k), $\lambda_{i+1} = \cdots = \lambda_{i+k} = 0$, then for almost every point x and $v_x \in E^c_{j,1}(x) \setminus \{0\}$ $_{(1\leq j\leq k)}$, one has $\lim_{n\to\pm\infty} \frac{1}{n}\log\|Df^n(x)v_x\| = \lambda_j = 0$. Since $E^c_{i+1,1} \oplus E^c_{i+2,1}$ is a dominated splitting, there exists $l \in \mathbb{N}$ and $\lambda < 1$ such that $\|Df^l(v_y)\|/\|v_y\| < \lambda\|D^l f(w_y)\|/\|w_y\|$ for any $y \in supp(\mu)$ and $v_y \in E^c_{i+1,1}(y), w_y \in E^c_{i+2,1}(y)$, then $\|Df^{nl}(v_x)\| < \lambda^n\|v_y\|\|D^{nl}f(w_y)\|/\|w_y\|$

$$0 = \lambda_{i+1} = \lim_{n\to\pm\infty}\frac{1}{nl}\log\|Df^{nl}(x)v_x\| < \lim_{n\to\pm\infty}\frac{1}{nl}(n\log\lambda + \log\|Df^{nl}(x)w_x\|) = \log\lambda + \lambda_{i+2} < 0,$$

that's a contradiction.

So μ has type $(i,1)$, by lemma 3.1, and above argument, $supp(\mu)$ admits the following dominated splitting $T_{supp(\mu)}M = E^{cs}_i \oplus E^c_{i+1,1} \oplus E^{cu}_{i+2}$, using remark 2.1, with the same argument in case (a), the Lyapunov exponents for μ on bundle E^{cs}_i are smaller than the Lyapunov exponent for μ on bundle $E^c_{i+1,1}$, and the Lyapunov exponent for μ on bundle $E^c_{i+1,1}$ is smaller than the Lyapunov exponents for μ on bundle E^{cu}_{i+2}. Because μ has type $(i,1)$, the dominated splitting $E^{cs}_i \oplus E^c_{i+1,1} \oplus E^{cu}_{i+2}$ is also the Oseledets splitting corresponding to the negative, zero, positive exponents. □

4. Proof of theorem 2

Before we give the proof, we need the following lemma whose proof is given in § 4.1, which claims that with a dominated assumption, the C^1 Pesin theory stated in theorem 2 is true. Such idea was at first given in [1]. Here we cite one of their result (the stable manifold theorem) and add another new property (similar with Katok's closing lemma in C^2 case).

Lemma 4.1. *Suppose $f \in Diff^1(M)$ and μ a hyperbolic ergodic measure of f with index i, if $supp(\mu)$ admits an i-dominated splitting, then the following properties are true:*

a) *there exists a compact positive measure subset Λ^s (resp. Λ^u) such that every point in Λ^s (resp. Λ^u) has continuous and uniform size of stable (resp. unstable) manifolds,*
b) *μ is the weak limit of a sequence of invariant measures μ_n supported by periodic orbits p_n with index i, and $supp(\mu)$ is contained in the homoclinic class $H(p_n, f)$.*

a) was given in [1] at first, we state it here just in order to make the statement complete. b) generalizes [13]'s result to C^1, in the proof of b) we use Liao-Gan shadowing lemma (see [10]) which is similar with the shadowing lemma in $C^{1+\alpha}$ Pesin theory.

Proof of theorem 2: By lemma 3.1 and proposition 2.2, we know that $supp(\mu) \subset P^*_i(f)$, and $supp(\mu)$ admits index i dominated splitting, recall that μ has index i, theorem 1 is a simple corollary of lemma 4.1. □

It's easy to see that the following corollary is true:

Corollary 4.2. *Suppose f is a diffeomorphism far from tangencies, and C is an aperiodic class of f, then any ergodic measure with support on C is not hyperbolic.*

4.1. C^1 **Pesin theory.** In this subsection we'll give the proof of lemma 4.1. a) of theorem 1 was given in [1], but for completeness, we still give a proof here.

Proof of a): We just prove the stable manifold theorem, the proof for unstable manifold theorem is similar. Denote $E^{cs}\oplus E^{cu}$ the index i dominated splitting on $supp(\mu)$, it's easy to know that the Lyapunov exponents on bundle E^{cs} (resp. E^{cu}) are negative (resp. positive), by sub-ergodic theorem, there exists $\lambda>0$ such that:

$$\lim_{n\to\infty}\int\frac{1}{n}\ln\|Df^n|_{E^{cs}(x)}\|d\mu(x)<-\lambda<0, \tag{4.1}$$

choose N big enough such that $\int\ln\|Df^N|_{E^{cs}(x)}\|d\mu(x)<-\lambda<0$, from Birkhoff ergodic theorem, there exists a subset A^s with μ full measure such that for any $x\in A^s$ one has:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\ln\|Df^N|_{E^{cs}(f^{jN}(x))}\|<-\lambda. \tag{4.2}$$

Choose $0<\lambda_0<\lambda$, denote Λ^s the set of point $y\in\Lambda^s$ satisfying $\frac{1}{n}\sum\limits_{j=0}^{n-1}\ln\|Df^N|_{E^{cs}(f^{jN}(y))}\|<-\lambda_0$ for all $n>0$. It's easy to see that Λ^s is a closed set.

Now we need the following well known Pliss lemma:

Lemma 4.3. *(Pliss lemma) For $K>0$ and $\lambda<\lambda_1$, there exists $\delta>0$ such that for any sequence $\{a_n\}$ satisfying $\|a_n\|<K$ and $\limsup\limits_{n\to\infty}\frac{1}{n}\sum\limits_{j=1}^{n}a_j<\lambda$, there exist $\{N_t\}$ and a subsequence $\{a_{n_i}\}$ such that $\frac{1}{m}\sum\limits_{j=1}^{m}a_{n_i+j}<\lambda_1$ for any $m\in\mathbb{N}$ and $\liminf\limits_{t\to\infty}\frac{\#\{a_{n_i};1<n_i\le N_t\}}{N_t}>\delta$.*

In lemma 4.3 (Pliss lemma), let $K=\max\{\|Df_x^N\|,x\in M\}$, and consider $-\lambda<-\lambda_0<0$, we get $\delta>0$ and we'll show that $\mu(\Lambda^s)>\delta/N$. The following result is well known.

Lemma 4.4. *There exists a μ full measure subset A_0^s such that for any $x\in A_0^s$, we have*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(x)}\overset{weaktoplogy}{\longrightarrow}\mu.$$

We can suppose $A_0^s\subset A^s$ always, for $x\in A_0^s$, consider $\{a_j=\|Df^N|_{E^{cs}(f^{jN}(x))}\|\}_{j=1}^{\infty}$ and $-\lambda<-\lambda_0<0$, by lemma 4.4,there is a subsequence $\{N_t\}_{t=1}^{\infty}$ such that $\lim\limits_{t\to\infty}\frac{1}{N_t}\sum\limits_{j=0}^{N_t-1}\delta_{f^{jN}(x)}(\Lambda^s)>\delta$, since Λ^s is compact, $\mu(\Lambda^s)\ge\lim\limits_{t\to\infty}\frac{1}{N_tN}\sum\limits_{j=0}^{N_t-1}\delta_{f^{jN}(x)}(\Lambda^s)>\delta/N>0$.

Now we'll show that Λ^s has continuous and uniform size of stable manifolds.

Let $I_1^{s,(u)}=(-1,1)^{i,(n-i)}$ and $I_\varepsilon^{s,(u)}=(-\varepsilon,\ \varepsilon)^{i,(n-i)}$, denote by $Emb^1(I^{s(u)},M)$ the set of C^1-embedding of $I_1^{s(u)}$ on M, recall by [12] that $\widetilde{\Lambda}$ has index i dominated splitting $\widetilde{E}\oplus\widetilde{F}$ implies the following.

Lemma 4.5. *There exist two continuous function $\Phi^{cs}:\ \widetilde{\Lambda}\longrightarrow Emb^1(I^s,M)$ and $\Phi^{cu}:\ \widetilde{\Lambda}\longrightarrow Emb^1(I^u,M)$ such that, with $W_\varepsilon^{cs}(x)=\Phi^{cs}(x)I_\varepsilon^s$ and $W_\varepsilon^{cu}(x)=\Phi^{cu}(x)I_\varepsilon^u$, the following properties hold:*

a) $T_xW^{cs}_\varepsilon = \widetilde{E}(x)$ *and* $T_xW^{cu}_\varepsilon = \widetilde{F}(x)$,

b) *For all* $0 < \varepsilon_1 < 1$, *there exists* ε_2 *such that* $f(W^{cs}_{\varepsilon_2}(x)) \subset W^{cs}_{\varepsilon_1}(f(x))$ *and* $f^{-1}(W^{cu}_{\varepsilon_2}(x)) \subset W^{cu}_{\varepsilon_1}(f^{-1}(x))$.

c) *For all* $0 < \varepsilon < 1$, *there exists* $\delta > 0$ *such that if* $y_1,\ y_2 \in \widetilde{\Lambda}$ *and* $d(y_1,\ y_2) < \delta$, *then* $W^{cs}_\varepsilon(y_1) \pitchfork W^{cu}_\varepsilon(y_2) \neq \phi$.

The following lemma given by [18] shows that there really exists stable manifolds on Λ^s which are continuous and with uniform size.

Lemma 4.6. *([18]) For any* $0 < \lambda < 1$, *there exists* $\varepsilon > 0$ *such that for* $x \in \widetilde{\Lambda}$ *which satisfies* $\prod_{j=0}^{n-1} \|Df^{N_1}|_{\widetilde{E}(f^{jN_1}x)}\| \leq \lambda^n$ *for all* $n > 0$, *then* $diam(f^n(W^{cs}_\varepsilon)) \longrightarrow 0$, *i.e. the central stable manifold of* x *with size* ε *is in fact a stable manifold.*

Proof of b): Here we should use a special shadowing lemma given by [10]:

Lemma 4.7. *([10], theorem 1.1): Let* $f \in Diff^1(M)$, *assume that* Λ *is a closed invariant set of* f *and there is a continuous invariant splitting* $T_\Lambda M = E \oplus F$ *on* Λ, *i.e.* $Df(E_x) = E_{f(x)}$ *and* $Df(F_x) = F_{f(x)}$ *for* $x \in \Lambda$. *For any* $\lambda_1 < 1$ *there exist* $L > 0$, $\delta_0 > 0$, N_1 *such that for any* $\delta < \delta_0$ *if we have an orbit segment* $(x, f(x), \cdots, f^{nN_1}(x))$ *satisfies the following properties:*

$$\begin{aligned}
\prod_{i=0}^{s-1} \|Df^{N_1}|_{E(f^{jN_1}(x))}\| &\leq (\lambda_1)^s \textit{ for } 0 \leq s \leq n-1,\\
\prod_{i=0}^{s-1} \|Df^{-N_1}|_{E(f^{(n-j)N_1}(x))}\| &\leq (\lambda_1)^s \textit{ for } 0 \leq s \leq n-1,\\
d(x, f^{nN_1}(x)) &< \delta,
\end{aligned}$$

then there exists a periodic point p *with period* nN_1 *and* $L\delta$*-shadows* $(x, f(x), \cdots, f^{nN_1}(x))$.

Now from the proof of a), there also exists a positive measure subset Λ^u such that for any $x \in \Lambda^u$, $\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \ln \|Df^{-N}|_{E^{cu}(f^{-jN}(x))}\| < -\lambda_0$. Since μ is ergodic, there exists n_0 such that $\Lambda^{su} = f^{n_0}(\Lambda^s) \bigcap \Lambda^u$ has positive measure, now from the proof of a), for any $x \in \Lambda^{su}$,

$$\frac{1}{n}\sum_{j=0}^{n-1} \ln \|Df^{-N}|_{E^{cu}(f^{-jN}(x))}\| < -\lambda_0;\ \ \frac{1}{n}\sum_{j=0}^{n-1} \ln \|Df^{N}|_{E^{cs}(f^{jN-n_0}(x))}\| < -\lambda_0.$$

Choose n_1 big enough and $0 < \lambda_1 < \lambda_0$, for $N_1 = n_1 \cdot N$ and any $x \in \Lambda^{su}$, we have

$$\frac{1}{n}\sum_{j=0}^{n-1} \ln \|Df^{N_1}|_{E^{cs}(f^{jN_1}(x))}\| < -\lambda_1;\ \ \frac{1}{n}\sum_{j=0}^{n-1} \ln \|Df^{-N_1}|_{E^{cu}(f^{-jN_1}(x))}\| < -\lambda_1. \tag{4.3}$$

Now we need the following result:

Lemma 4.8. *There exists a subset* $\Lambda_0 \subset \Lambda^{su}$, *such that* $\mu(\Lambda_0) = \mu(\Lambda^{su})$ *and for any* $x \in \Lambda_0$

(A) x *is a recurrent point, i.e. there exists* $0 < i_1 < i_2 < \cdots i_n < \cdots$ *such that* $f^{i_nN_1}(x) \in \Lambda^{su}$ *and* $\lim_{n\to\infty} d(x, f^{i_nN_1}(x)) \longrightarrow 0$.

(B) $\lim\limits_{n\to\infty} \frac{1}{i_n N_1} \sum\limits_{i=0}^{i_n N_1 - 1} \delta_{f^i(x)} \longrightarrow \mu$.

Remark 4.9. *Above lemma can be proved by Poincaré recurrence theorem and Birkhoff ergodic theorem, it's also easy to see that for any $x \in \Lambda_0$, $supp(\mu) \subset \overline{Orb^+(x)}$.*

Fix a point $x \in \Lambda_0$, by (A) of lemma 4.8, we can choose i_n such that $d(x, f^{i_n N_1}(x)) \ll \delta_1$, by lemma 4.7 and (4.3), there is a periodic point p_n with period $i_n N_1$ which $L \cdot d(x, f^{i_n N_1}(x))$-shadows $(x, f(x), \cdots, f^{i_n N_1}(x))$, by (B) of lemma 4.8, $\lim\limits_{n\to\infty} \frac{1}{i_n N_1} \sum\limits_{i=0}^{i_n N_1 - 1} \delta_{f^i(p_n)} \longrightarrow \mu$.

Now we claim that the above family of periodic points $\{p_n\}$ have uniform size of stable and unstable manifolds.

Proof of the claim: Because $supp(\mu)$ admits index i dominated splitting, there is a small neighborhood U of $supp(\mu)$ such that the maximal invariant set $\widetilde{\Lambda}$ of U admits index i dominated splitting also. So when n big enough, $Orb(p_n) \in U$ admits index i dominated splitting.

Choose $1 > \lambda_2 > \lambda_1$, then there exists a δ_0 such that for any two points $y_1, y_2 \in \widetilde{\Lambda}$ and $d(y_1, y_2) < \delta_0$, one has $\ln \|Df^{N_1}|_{E^{cs}(y_1)}\| - \ln \|Df^{N_1}|_{E^{cs}(y_2)}\| < \lambda_2 - \lambda_1$. Because $Orb(p_n)$ $L \cdot d(x, f^{i_n N_1}(x))$-shadows $(x, f(x), \cdots, f^{i_n N_1}(x))$, by $\lim\limits_{n\to\infty} d(x, f^{i_n N_1}(x)) \longrightarrow 0$ and (4.3), for n big enough, we have $\prod\limits_{j=0}^{m-1} \|Df^{N_1}|_{\widetilde{E}(f^{jN_1} p_n)}\| \le \lambda_2^m$ and $\prod\limits_{j=0}^{m-1} \|Df^{-N_1}|_{\widetilde{E}(f^{-jN_1} p_n)}\| \le \lambda_2^m$ for $0 \le m \le i_n N_1$, since p_n is periodic point with period $i_n N_1$, we have

$$\prod_{j=0}^{m-1} \|Df^{N_1}|_{\widetilde{E}(f^{jN_1} p_n)}\| \le \lambda_2^m; \prod_{j=0}^{m-1} \|Df^{-N_1}|_{\widetilde{E}(f^{-jN_1} p_n)}\| \le \lambda_2^m \text{for } \ m \ge 0 \tag{4.4}$$

Now by lemma 4.5 and 4.6, p_n has uniform size of stable manifold and unstable manifold. □

Since $p_n \longrightarrow x$, and p_n has uniform size of stable and unstable manifold, by (3) of lemma 4.5, when n, m big enough, $W^s_{loc}(p_n) \pitchfork W^u_{loc}(p_m) \neq \phi$ and $W^s_{loc}(p_n) \pitchfork W^u_{loc}(p_m) \neq \phi$, so p_n and p_m are homoclinic related. Replace by a subsequence, we can suppose $\{p_n\}$ are all homoclinic related with each other, so $\{p_n\}$ and x all belong to the same homoclinic class, by remark 4.9, $supp(\mu) \subset \overline{Orb^+(x)}$, so we get that $supp(\mu)$ and $\{p_n\}$ all belong to the same homoclinic class. □

5. C^1 Generic Properties

At first let's state several well known C^1 generic properties.

Lemma 5.1. *There exists a C^1 residual subset R such that for any $f \in R$, the following properties are right:*

1) *(Kupka-Smale) all the periodic points are hyperbolic and the intersection between stable manifold and unstable manifold of periodic points are always transverse,*
2) *([5]) suppose C is a chain recurrent class of f, and it contains a periodic point p, then $C = H(p, f)$,*
3) *([5]) suppose Λ is an index i fundamental limit of f, then there exists a family of index i periodic points $\{p_n\}$ such that $\lim\limits_{n\to\infty} Orb(p_n) \overset{Hausdorff}{\longrightarrow} \Lambda$.*

4) *([7]) if C is a homoclinic class contains periodic points with different indexes, then there exists a non-trivial non-hyperbolic ergodic measure with support in C.*

The following result is given by Shaobo Gan, a proof can be found in [22].

Lemma 5.2. *$f \in Diff^1(M)$ and $\{p_n\}$ is a family of index i periodic points of f satisfying $\lim_{n\to\infty} \pi(p_n) \longrightarrow \infty$, if $\{p_n\}$ is index stable, then there exists a subsequence $\{p_{i_n}\}$ such that p_{i_m} and p_{i_n} are homoclinic related for $n \neq m$.*

Corollary 5.3. *Suppose $f \in R$, C is a chain recurrent class of f and $\Lambda \subset C$ is an index i fundamental limit, if C doesn't contain index i periodic point, then Λ is index $i-1$ or $i+1$ fundamental limit.*

Proof : Suppose C doesn't contain index i periodic point, then Λ is not an orbit of index i periodic orbit. By 3) of 5.1, there exists a family of index i periodic points $\{p_n\}$ satisfying $\lim_{n\to\infty} Orb(p_n) = \Lambda$ and $\lim_{n\to\infty} \pi(p_n) = \infty$.

By lemma 5.2, the family of periodic points are not index stable, with the argument in [4], there exists a subsequence of periodic orbits $\{Orb(p_{n_j})\}$ and a family of diffeomorphisms $g_{n_j} \xrightarrow{C^1} f$ such that $Orb(p_{n_j})$ is index $i+1$ or $i-1$ periodic points of g_{n_j}. So Λ is also an index $i+1$ or $i-1$ fundamental limit. □

Lemma 5.4. *For $f \in R$, it satisfies Axiom A if and only if every chain recurrent class of f is hyperbolic.*

Proof :

□

6. Proof of theorem 3:

At first we need the following lemma whose proof is given in § 6.1:

Lemma 6.1. *There exists a generic subset R in $Diff^1(M) \setminus \overline{HT}$ such that for any $f \in R$ and $C(f)$ is a homoclinic class whose periodic points are all hyperbolic and have an unique index i, then*

- *either C is a hyperbolic set,*
- *or there exists a non-hyperbolic ergodic measure μ with $supp(\mu) \subset C$.*

Proof of theorem 3: Suppose $f \in R$ and C is a chain recurrent class of f, we can always suppose C is not trivial ($\#(C) = \infty$) since if $\#(C)$ is finite, by 1) of lemma 5.1, it's a hyperbolic periodic orbit, and there is unique invariant measure with support on C and the measure is hyperbolic.

We divide the proof into three cases:

1) C is an aperiodic class;
2) C contains periodic points and all the periodic points in C have the same index;
3) C contains index different periodic point.

In case 1), Corollary 4.2 shows any ergodic measure μ with support on C is not hyperbolic and has just 1 zero Lyapunov exponent.

In the case 2), lemma 6.1 shows that either C is hyperbolic or there exists a non-hyperbolic ergodic measure μ with $supp(\mu) \subset C$.

In the case 3), we need the generic property 4) of lemma 5.1 which was proved in [7] shows that there always exists a non-hyperbolic ergodic measure μ with $supp(\mu) \subset C$. $\square$

6.1. **Proof of lemma 6.1.**

Proof : Here we suppose that C is not hyperbolic and all the ergodic measures with support on C are hyperbolic, we'll show the contradiction.

Suppose C contains index i ($i \neq 0, d$) periodic point p, then $C \subset H(p, f) \subset P_i^*$, by proposition 2.2, C has an index i dominated splitting $E_i^{cs} \oplus E_{i+1}^{cu}$. Since C is not hyperbolic, the splitting is not hyperbolic splitting, we can suppose the bundle E_i^{cs} is not hyperbolic, by proposition 2.3, there exists $j < i$ such that $C \bigcap P_j^* \neq \phi$, it means there exist $g_n \xrightarrow{C^1} f$ and p_n index j periodic points of g_n such that $p_n \xrightarrow{C^1} x \in C$, from the definition of chain recurrent class, it's easy to know that $\limsup\limits_{n\to\infty} Orb(p_n) \subset C$ and the set is an index j fundamental limit, denote $i_0 = \min\{j : C \text{ contains index } j \text{ fundamental limit}\}$, then we have $C \bigcap P_j^* = \phi$ for $j < i_0$.

Choose $\Lambda_0 \subset C$ an index i_0 fundamental limit, by proposition 2.2, Λ_0 has an index i_0 dominated splitting $E_{i_0}^{cs} \oplus E_{i_0+1}^{cu}$, by proposition 2.3 and the definition of i_0, the bundle $E_{i_0}^{cs}$ is contracting, we denote $E_{i_0}^{cs}$ by $E_{i_0}^{s}$ since now. By generic properties in lemma 5.1, there exists a family of index i_0 periodic points $\{p_n\}$ such that $\lim\limits_{n\to\infty} Orb(p_n) \longrightarrow \Lambda_0$. By lemma 5.3, $\{p_n\}$ cannot be index stable and Λ_0 is an index $i_0 + 1$ fundamental limit also, so $\Lambda \subset P_{i_0+1}^*$. By proposition 2.2 again, Λ_0 has index $i_0 + 1$ dominated splitting $E_{i_0+1}^{cs} \oplus E_{i_0+2}^{cu}$, denote $E_{i_0+1,1}^{c} = E_{i_0+1}^{cs} \bigcap E_{i_0+1}^{cu}$, then Λ_0 has the following dominated splitting $E_{i_0}^{s} \oplus E_{i_0+1,1}^{c} \oplus E_{i_0+2}^{cu}$.

Since Λ_0 is an index i_0 fundamental limit, that means the bundle $E_{i_0+1,1}^{c}$ is not contracting, now we need the following lemma whose proof is easy and we just omit.

Lemma 6.2. *suppose Λ is a compact invariant subset of f with dominated splitting $E \oplus F$ and the bundle $E(\Lambda)$ is not contracting, then there exists a point $x \in \Lambda$ such that $\|Df^n|_{E(x)}\| \geq 1$ for $n \geq 0$.*

By the above lemma there exists $x \in \Lambda_0$ such that $\prod\limits_{i=0}^{n-1} \|Df|_{E_{i+1,1}^c(f^i(x))}\| \geq 1$ for $n \geq 0$ (since $dim(E_{i+1,1}^c) = 1$), choose a converge subsequence from $\{\sum\limits_{j=0}^{n-1} \delta_{f^j(x)}\}_{n=1}^{\infty}$ and suppose $\lim\limits_{j\to\infty} \sum\limits_{j=0}^{n-1} \delta_{f^j(x)} \longrightarrow \nu_0$, then ν_0 is an invariant measure with $supp(\nu_0) \subset \omega(x) \subset \Lambda_0$ such that $\int_{\Lambda_0} \|Df|_{E_{i+1,1}^c}\| d\nu_0 \geq 0$. By ergodic decomposition theorem on Λ_0, we can suppose there exists an ergodic measure ν with support on Λ_0 satisfying $\int_{\Lambda_0} \|Df|_{E_{i_0+1,1}^c}\| d\nu \geq 0$. Denote $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_d$ the Lyapunov exponents of ν, then $\lambda_{i_0+1} = \int_{\Lambda_0} \|Df|_{E_{i_0+1,1}^c}\| d\nu \geq 0$. Recall that we have supposed ν is hyperbolic, so $\lambda_{i_0+1} > 0$, that means ν has index smaller than $i_0 + 1$, by $f \in R \subset Diff^1(M) \setminus \overline{HT}$ and theorem 1, C contains periodic points with index smaller than $i_0 + 1$. Recall that $i_0 < i$, C contains index different periodic points, that's a contradiction. $\square$

Appendix

Proof : Suppose the theorem is false, then there is a non-trivial measure and j with $i \leq j \leq i + k$ which don't satisfy the theorem.

In the following we'll get the contradiction by showing that $\#\{$negative Lyapunov exponents of $\mu\} > j \geq i$ or $\#\{$positive Lyapunov exponents of $\mu\} > d - j \geq d - (i + k)$. In order to prove this, we need show that there is a positive measure subset such that for every point in this subset, on its tangent space, the tangent map Df is exponentially contracting on a subspace with dimension larger than j or exponentially expanding on a subspace with dimension larger than $d - j$.

Lemma 6.3. *(Ergodic closing lemma) Suppose μ is an ergodic measure of f, then there exists a family of diffeomorphisms g_n, such that:*

1) *: $g_n \xrightarrow{C^1} f$,*
2) *: g_n has periodic point p_n, let μ_n denote the invariant atom measure on $Orb_{g_n}(p_n)$, we have $\mu_n \xrightarrow{*-weak} \mu$.*

From Mañé's ergodic closing lemma, there always exists a family of diffeomorphisms $g_n \xrightarrow{C^1} f$ where g_n has an invariant measure μ_n supported on periodic orbit $p_n(g_n)$ and $\mu_n \xrightarrow{*-weak} \mu$, suppose the periodic points' indices are all the same and strictly bigger than j.

Denote $j_0 = \min_{t \geq j}\{t :$ exists a family of diffeomorphisms $g_n \xrightarrow{C^1} f$ where g_n has an invariant measure μ_n supported on index t periodic orbit $p_n(g_n)$ and $\mu_n \xrightarrow{*-weak} \mu$ $\}$, then by our assumption, $j_0 > j$. Choose such a family of diffeomorphisms $\{g_n\}$ which has periodic point $\{p_n(g_n)\}$ with index j_0 and $Orb_{g_n}(p_n)$ supports an invariant measure μ_n for g_n satisfying $\mu_n \xrightarrow{*-weak} \mu$, since μ is not trivial, $\lim_{n\to\infty} \pi_{g_n}(p_n(g_n)) \longrightarrow \infty$. Denote $E^s_{j_0,n}(Orb(p_n))$ the contracting subspace on $Orb(p_n)$ with dimension j_0, the family of periodic linear maps $\{Dg_n|_{E^s_{j_0,n}(Orb(p_n))}\}$ over $\mathbb{R}^{j_0}$ is called uniformly periodic contracting if there exists $\varepsilon > 0$ such that for any n large enough and any periodic linear map $\{A_1, \cdots, A_{\pi_{pn}(g_n(p_n))}\}$ over $\mathbb{R}^{j_0}$ satisfying $\|A_j - Dg_n|_{E^s_{j_0,n}(g_n^{j-1}(p_n))}\| < \varepsilon$, we have all the eigenvalues of $\prod_{j=1}^{\pi_{pn}} A_j < 1$.

Now we'll show that the above sequence of periodic linear maps $\{Dg_n|_{E^s_{j_0,n}(Orb(p_n))}\}$ we've got is uniformly periodic contracting. At first, we need the well known Franks lemma:

Lemma 6.4. *$g_n \xrightarrow{C^1} f$, suppose p_n is a periodic point of g_n, $A|_{Orb(p_n)}$ is an ε-perturbation of $\{Dg_n|_{Orb(p_n)}\}$, then for any neighborhood U of $Orb(p_n)$, there exists g'_n such that $g'_n \equiv g_n$ on $(M \setminus U)\bigcup Orb(p_n)$, $d_{C^1}(g_n, g'_n) < \varepsilon$ and $\{Dg'_n|_{orb(p_n)}\} = \{A|_{Orb(p_n)}\}$.*

As a corollary of Franks lemma, we can show that the family of periodic linear maps is uniformly periodic contracting:

Corollary 6.5. *There exists $\varepsilon > 0$ such that for any periodic linear map $\{A_1, \cdots, A_{\pi_{gn}(p_n)}\}$ over $\mathbb{R}^{j_0}$ satisfying $\|A_j - Dg_n|_{E^s_{j_0,n}(g_n^{j-1}(p_n))}\| < \varepsilon$, we have all the eigenvalues of $\prod_{j=1}^{\pi_{pn}} A_j < 1$.*

Proof : If the sequence is not uniformly periodic contracting, there exists $\varepsilon_{n_j} \longrightarrow 0$ and a sequence of periodic linear maps $\{(A_{n_j,1}, \cdots, A_{n_j,\pi_{gn_j}(g_{n_j}(p_{n_j}))})\}_j$ over $\mathbb{R}^{j_0}$ such that $\|A_{n_j,k} - Dg_{n_j}|_{E^s_{j_0,n_j}(g_{n_j}^{k-1}(p_{n_j}))}\| < \varepsilon_{n_j}$ and one eigenvalue of $\prod_{k=1}^{\pi_{gn_j}(p_{n_j})} A_{n_j,k} > 1$.

Now we claim that replace by another sequence of periodic linear maps over $\mathbb{R}^{j_0}$, we can always suppose $\prod_{k=1}^{\pi_{g_{n_j}}(p_{n_j})} A_{n_j,k}$ has index j_0-1.

Proof of the claim: We can choose a new sequence periodic linear map $\{(B_{n_j,1},\cdots,B_{n_j,\pi_{g_{n_j}}(g_{n_j}(p_{n_j}))})\}_j$ over $\mathbb{R}^{j_0}$ such that $\|B_{n_j,k}-Dg_{n_j}|_{E^s_{j_0,n_j}(g^{k-1}_{n_j}(p_{n_j}))}\|<\varepsilon_{n_j}$, all the eigenvalues of $\prod_{k=1}^{\pi_{g_{n_j}}(p_{n_j})} B_{n_j,k}\leq 1$ except one real or a couple of complex eigenvalues with norm 1.

If $\prod_{k=1}^{\pi_{g_{n_j}}(p_{n_j})} B_{n_j,k}$ has only one real eigenvalue with norm 1, after small perturbation, we get a new periodic linear map $\{(\widetilde{A}_{n_j,1},\cdots,\widetilde{A}_{n_j,\pi_{g_{n_j}}(p_{n_j})})\}_j$ over $\mathbb{R}^d$ such that $\prod_{k=1}^{\pi_{g_{n_j}}(p_{n_j})} \widetilde{A}_{n_j,k}$ has index j_0-1.

If $\prod_{k=1}^{\pi_{g_{n_j}}(p_{n_j})} B_{n_j,k}$ has a couple of complex eigenvalues with norm 1, lemma 3.7 of [4] shows after small perturbation, we can let the two complex eigenvalues to be real with norm 1, then by another perturbation, we get a new periodic linear map $\{(\widetilde{A}_{n_j,1},\cdots,\widetilde{A}_{n_j,\pi_{g_{n_j}}(p_{n_j})})\}_j$ over $\mathbb{R}^d$ such that $\prod_{k=1}^{\pi_{g_{n_j}}(p_{n_j})} \widetilde{A}_{n_j,k}$ has index j_0-1.

By above arguments, we can always get a new sequence of periodic linear maps $\{(\widetilde{A}_{n_j,1},\cdots,\widetilde{A}_{n_j,\pi_{g_{n_j}}(p_{n_j})})\}_j$ over $\mathbb{R}^{j_0}$ which satisfying $\|\widetilde{A}_{n_j,k}-Dg_{n_j}(g^{k-1}_{n_j}(p_{n_j}))\|<2\varepsilon_{n_j}$ and $\prod_{k=1}^{\pi_{g_{n_j}}(p_{n_j})} \widetilde{A}_{n_j,k}$ has index j_0-1.

Replace the sequence of periodic linear maps $\{(A_{n_j,1},\cdots,A_{n_j,\pi_{g_{n_j}}(g_{n_j}(p_{n_j}))})\}_j$ over $\mathbb{R}^{j_0}$ by the sequence of periodic linear maps $\{(\widetilde{A}_{n_j,1},\cdots,\widetilde{A}_{n_j,\pi_{g_{n_j}}(p_{n_j})})\}_j$ over $\mathbb{R}^{j_0}$ and finished the proof of the claim. □

Now use Franks lemma, by ε_{n_j} perturbation, we can get a new diffeomorphism g'_{n_j} such that $Orb_{g_{n_j}}(p_{n_j}(g_{n_j}))$ is an index j_0-1 periodic orbit of g'_{n_j}, that's a contradiction with the definition of j_0. □

For such kind of uniformly contracting periodic linear maps, [14] gave the following lemma:

Lemma 6.6. *([14] Lemma II.4): $g_n\xrightarrow{C_1} f$, suppose p_n is index j_0 periodic point of g_n and $\lim_{n\to\infty}\pi_{g_n}(p_n)\longrightarrow\infty$. If the sequence of periodic linear maps $\{Dg_n|_{E^s_{j_0,n}(Orb(p_n))}\}$ is uniformly periodic contracting, then there exist $l>0$, $N_0>0$ and $\lambda<1$ such that for any periodic orbit p_n with period $\pi(p_n)>N_0$, we have*

$$\prod_{i=0}^{[\frac{\pi(pn)}{l}]}\|Dg^l|_{E^s_{j_0,n}(g^{il}_n(p_n))}\|<\lambda^{[\frac{\pi(pn)}{l}]}. \tag{6.1}$$

Remark 6.7. *Under the same assumption with lemma 6.6, and $l>0$, $N_0>0$, $\lambda<1$ given there, for any periodic orbit p_n with period $\pi(p_n)>N_0$ and any $k>0$, we have*

$$\prod_{i=0}^{k[\frac{\pi(pn)}{l}]}\|Dg^l|_{E^s_{j_0,n}(g^{il}_n(p_n))}\|<\lambda^{k[\frac{\pi(pn)}{l}]}. \tag{6.2}$$

That's because we can consider the new sequence of periodic linear maps

$$\{(Dg_n|_{E^s_{j_0,n}(Orb(p_n))});\ (Dg_n|_{E^s_{j_0,n}(Orb(g_n(p_n)))});\ \cdots;\ (Dg_n|_{E^s_{j_0,n}(Orb(g_n^{\pi(pn)-1}(p_n)))})\}.$$

Then (6.1) *is true for* $g_n^{k\cdot l\cdot[\frac{\pi(pn)}{l}](p_n)}$ *where* $k > 0$.

For the uniformly contracting periodic linear maps $\{Dg_n|_{E^s_{j_0,n}(Orb(p_n))}\}$, remark 6.6 gives parameters $l > 0$, $N_0 > 0$ and $\lambda < 1$, choose $\lambda < \lambda_0 < 1$, by (6.2) and lemma 4.3 (Pliss lemma), with the fact $\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \delta_{g_n^j(p_n)} \longrightarrow \mu_n$ where μ_n is the ergodic measure on $Orb_{g_n}(p_n)$, if denote $\Lambda_n = \{y \in Orb(p_n) : \prod_{i=0}^{m} \|Dg_n^l|_{E^s_{n,j_0}(g_n^{il}(y))}\| < \lambda_0^m\}$, then for $\pi(p_n)$ big enough, there exists a uniformly number $\delta > 0$, such that $\mu_n(\Lambda_n) > \delta$.

Proposition 6.8. *Suppose* X *is a compact metric space, denote* $CX = \{K : K$ *is compact subset of* $X\}$, *the space* CX *with Hausdorff topology is still a compact space.*

Since Λ_n is compact, with proposition 6.8, there is a compact set Λ such that $\lim_{n\to\infty} \Lambda_n \longrightarrow \Lambda$. It's easy to know that $\mu(\Lambda) > \delta$ and for every point $y \in \Lambda$, there exists a j_0 dimension space $E_{j_0}(y)$ in it's tangent space such that $\prod_{i=0}^{m} \|Dg_n^l|_{E_{j_0}(f^{il}(y))}\| < \lambda_0^m$ for $m \geq 0$, so every point in Λ has at least j_0 number of negative Lyapunov exponents. Since the measure μ is ergodic and Λ has positive measure, μ has at least j_0 number of negative Lyapunov exponents. That's a contradiction, since μ has just i number of negative Lyapunov exponents and $i \leq j < j_0$. □

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