# Morita equivalence and characteristic classes of star products

H. Bursztyn, V. Dolgushev and S. Waldmann

*Belatedly to Giovanni Felder and Boris Tsygan on the occasion of their 50th birthday.*

**Abstract**

This paper deals with two aspects of the theory of characteristic classes of star products: first, on an arbitrary Poisson manifold, we describe Morita equivalent star products in terms of their Kontsevich classes; second, on symplectic manifolds, we describe the relationship between Kontsevich's and Fedosov's characteristic classes of star products.

## Contents

# 1 Introduction

Given a smooth real manifold $M$, consider the set $\mathrm{FPoiss}(M)$ of equivalence classes of formal Poisson structures $\pi = \hbar\pi_1 + \hbar^2\pi_2 + \ldots \in \Gamma(\wedge^2 TM)[[\hbar]]$ on $M$, and let $\mathrm{Def}(M)$ denote the set of equivalence classes of star products $*$ on $M$. The celebrated Kontsevich's formality theorem [46, 47] provides a bijective correspondence

$$\mathcal{K}_* : \mathrm{FPoiss}(M) \longrightarrow \mathrm{Def}(M), \tag{1.1}$$

in such a way that if $\pi = \hbar\pi_1 + \ldots$ and $*$ are related by $\mathcal{K}_*$, then $*$ is a deformation quantization, in the sense of [1], of the ordinary Poisson structure $\pi_1$. In particular, any star product on $M$ can be assigned to an equivalence class of a formal Poisson structure via (1.1), called the *Kontsevich characteristic class* or simply the *Kontsevich class* of the star product.

For a given star product, finding the associated Kontsevich class is often a hard problem, to which no effective solution is currently available. There are two main approaches to tackle this problem: the first one makes use of algebraic index theorems, see e.g. [11, 12, 21, 28, 29, 50], while the second is based on homological algebra arguments, such as the duality between Hochschild cohomology and homology [10, 17, 26, 30, 37, 58].

The first goal of the present paper is to describe the Kontsevich classes of Morita equivalent star-product algebras on a smooth real manifold $M$ (in this paper, star products are defined on the algebra of *complex-valued* smooth functions on $M$). As shown in [4, 6], two star products on $M$ are Morita equivalent if and only if they lie in the same orbit of a canonical action of the group $\mathrm{Diff}(M) \ltimes \mathrm{Pic}(M)$ on the moduli space $\mathrm{Def}(M)$ of equivalence classes of star products; here $\mathrm{Diff}(M)$ denotes the group of diffeomorphisms of $M$, and $\mathrm{Pic}(M) \cong H^2(M, \mathbb{Z})$ is the *Picard group*, i.e., the group of isomorphism classes of complex line bundles over $M$. The action of $\mathrm{Diff}(M)$ on star products is the natural one by pull-back, while the action of $\mathrm{Pic}(M)$ on $\mathrm{Def}(M)$ is defined in a less obvious way [4]. Hence the problem of expressing Morita equivalent star products in terms of their Kontsevich classes amounts to describing the action of $\mathrm{Diff}(M) \ltimes \mathrm{Pic}(M)$ on the moduli space of formal Poisson structures $\mathrm{FPoiss}(M)$ making the map $\mathcal{K}_*$ in (1.1) equivariant.

The group $\mathrm{Diff}(M)$ naturally acts on formal Poisson structures, and it follows from [20, 47] that the map (1.1) is $\mathrm{Diff}(M)$-equivariant; so in order to describe Morita equivalent star products one only needs to focus on the action of the Picard group $\mathrm{Pic}(M)$. A key observation is that the set of formal Poisson structures on $M$ carries a natural action of the abelian group of closed ($\mathbb{C}[[\hbar]]$-valued) 2-forms, defined by a formal version of the *gauge transformations* of [55, Sec. 3] (also known as *B-field transforms* in the context of generalized complex geometry [39, 42]); moreover, we prove that this action naturally descends to an action of the abelian group $H^2(M, \mathbb{C})[[\hbar]]$ on the moduli space $\mathrm{FPoiss}(M)$. Our first main result is Theorem 3.11, which asserts that two star products are related by the action of a line bundle $L$, representing an element in $\mathrm{Pic}(M)$, if and only if their classes in $\mathrm{FPoiss}(M)$ are connected by the action of the element $2\pi i c_1(L)$, where $c_1(L)$ is the Chern class of $L$. For a further discussion relating this result to Morita equivalence of Poisson manifolds, we refer to [8].

Morita equivalent star products have been also considered in the physics literature in the context of noncommutative gauge theory [43, 44, 54]. The essence of the statement of our Theorem 3.11 may be found in these works, as well as ideas concerning its proof; here we provide a complete proof of this result based on the explicit globalization of Kontsevich's formality quasi-isomorphism constructed in [18, 20, 47] and some general facts about formal differential equations.

On a symplectic manifold $(M, \omega)$, equivalence classes of star products quantizing the associated non-degenerate Poisson bracket are classified by their *Fedosov classes*, which are elements in

$$\frac{1}{\hbar}[\omega] + H^2(M, \mathbb{C})[[\hbar]], \tag{1.2}$$

see, e.g., [3, 16, 15, 27, 28, 50, 51]. Hence to each star product $*$ on $(M, \omega)$ one may assign either its Kontsevich class, defined by the class of a formal Poisson structure $\pi = \hbar\pi_1 + \dots$, where $\pi_1$ is the Poisson bivector field defined by the symplectic form $\omega$, or its Fedosov class in (1.2). The nondegeneracy of $\pi_1$ implies that the series $\pi = \hbar\pi_1 + \dots$ can be formally inverted to a series of closed 2-forms, defining an element in (1.2), and the fact that this element agrees with the Fedosov class of the star product $*$ has been conjectured by A. Chervov and L. Rybnikov in [13, Conjecture 4]. In this paper, we prove this conjecture, which allows us to recover the description of Morita equivalent star products on symplectic manifolds of [7] as a particular case of our Theorem 3.11. The proof of this conjecture about the relationship between Fedosov's and Kontsevich's classes in Theorem 4.1 also partially closes the project mentioned in item 1) of [46, Section 0.2].

We remark that the construction of the map $\mathcal{K}_*$ in (1.1) involves choices. We prove in Theorem 2.6 that the definition of the Kontsevich classes of star products does not depend on the choices made in the globalization procedure of [20]. This definition may depend, however, on the specific choice of formality quasi-isomorphism between polyvector fields and polydifferential operators on $\mathbb{R}^d$. In this paper, we tacitly assume that the formality quasi-isomorphism on $\mathbb{R}^d$ is the one constructed by M. Kontsevich in [46] with the angle function defined via hyperbolic geometry of the Lobachevsky plane.

Finally, we point out that there is an alternative construction of global star products in [9], which we believe can also be used to study Morita equivalence as well as to relate Fedosov's and Kontsevich's classes of star products on symplectic manifolds.

Let us briefly describe the organization of the paper.

In Section 2, we recall Fedosov's resolutions and the globalization of Kontsevich's formality quasi-isomorphism [20, 18], which we use to define the Kontsevich classes of star products. We verify in Theorem 2.6 (whose proof is deferred to Appendix C) that the definition of the Kontsevich classes is independent of the choices made in the globalization procedure.

Section 3 is devoted to the description of the Kontsevich classes of Morita equivalent star products. The key step consists in verifying that the map $\mathcal{K}_*$ in (1.1) satisfies an equivariance property with respect to appropriate actions of the Picard group, as explained in Theorem 3.11.

In Section 4 we describe the relationship between Fedosov's and Kontsevich's classes of star products on symplectic manifolds. The main result of this section is formulated in Theorem 4.1, and its proof is divided into several parts. First, we introduce a modification of Fedosov's construction [27]. Second, we describe a version of the Emmrich-Weinstein connection [25], which is then used to show that Kontsevich star products are equivalent to the star products constructed using the modified Fedosov construction. Finally, we show that the original Fedosov star products coincide with the star products obtained using the modified Fedosov construction.

In the end of the paper, Appendix A collects the necessary facts about formal differential equations, Appendix B recalls some key facts about DGLAs, Maurer-Cartan elements and $L_\infty$-morphisms. Finally, Appendix C contains the somewhat technical proof of Theorem 2.6.

**Acknowledgements:** H.B. thanks CNPq and Faperj for financial support. The work of V.D. was partially supported by the NSF grant DMS 0856196, Regent's Faculty Fellowship, and the Grant for Support of Scientific Schools NSh-8065.2006.2. V.D.'s part of the work was done when his mother-in-law Anna Vainer was baby sitting little Nathan. V.D. would like to thank his mother-in-law for her help. The authors thank P. Etingof, M. Gualtieri and V. Rubtsov for useful conversations, and G. Dito and P. Schapira for the invitation to the meeting on Algebraic Analysis and Deformation Quantization in Scalea, which facilitated our collaboration. We also thank the institutions that hosted us at various stages of this project: Freiburg University (H.B.), U. C. Riverside (H.B.), Northwestern University (V.D.), the University of Chicago (V.D.), and IMPA (S.W.).

## 1.1 Notation and conventions

Throughout this paper $M$ is a smooth real manifold, $\mathcal{O}_M$ is the sheaf of smooth *complex-valued* functions on $M$, and $\mathcal{O}(M)$ is the algebra of its global sections. The algebra of smooth complex-valued polyvector fields is denoted by $\mathcal{X}^\bullet(M)$; it is equipped with the Schouten-Nijenhuis bracket $[\,,\,]_{SN}$, which is a degree 0 Lie bracket for the shifted grading $\mathcal{X}^{\bullet+1}(M)$. The space of complex-valued differentiable forms is denoted by $\Omega^\bullet(M)$. For a sheaf of $\mathcal{O}_M$-modules $\mathcal{G}$ and an open subset $U \subset M$, we denote by $\Gamma(U, \mathcal{G})$ the vector space of global sections of $\mathcal{G}$ and by $\mathcal{O}(U)$ the algebra of smooth complex-valued functions on $U$. Furthermore, we denote by $\Omega^\bullet(U, \mathcal{G})$ the graded vector space of exterior forms on $U$ with values in $\mathcal{G}$.

For a vector $v$ in a graded vector space or a cochain complex $V$, its degree is denoted by $|v|$. By the *suspension* $\mathbf{s}V$ of a graded vector space (or a cochain complex) $V$ we mean $\varepsilon \otimes V$, where $\varepsilon$ is a one-dimensional vector space placed in degree $+1$. The *desuspension* $\mathbf{s}^{-1}V$ is the inverse operation. Throughout this paper we use the Koszul rule of signs. If $V$ is a graded vector space, we denote its symmetric algebra by $S(V)$, whereas $S^k(V)$ is the $k$-th component of this algebra.

For a unital associative algebra $\mathcal{A}$, we denote by $C^\bullet(\mathcal{A})$ the (normalized) Hochschild cochain complex of $\mathcal{A}$ with coefficients in $\mathcal{A}$,

$$C^\bullet(\mathcal{A}) = \mathrm{Hom}\big((\mathcal{A}/\mathbb{C}\,1)^\bullet, \mathcal{A}\big). \tag{1.3}$$

The coboundary operator $\partial^{\mathrm{Hoch}}$ on (1.3) is given by

$$\begin{aligned}(\partial^{\mathrm{Hoch}}P)(a_0, a_1, \dots, a_k) = & a_0 P(a_1, \dots, a_k) - P(a_0 a_1, \dots, a_k) + \\ & P(a_0, a_1 a_2, a_3, \dots, a_k) - \dots + (-1)^k P(a_0, \dots, a_{k-2}, a_{k-1}a_k) + \\ & (-1)^{k+1} P(a_0, \dots, a_{k-2}, a_{k-1}) a_k,\end{aligned} \tag{1.4}$$

where $P \in C^k(\mathcal{A})$ and $a_i \in \mathcal{A}$. In particular, for a degree-zero cochain $P \in C^0(\mathcal{A}) = \mathcal{A}$, we have

$$(\partial^{\mathrm{Hoch}}P)(a_0) = a_0\, P - P\, a_0\,. \tag{1.5}$$

The Hochschild cochain complex with the shifted grading $C^{\bullet+1}(\mathcal{A})$ carries the structure of a differential graded Lie algebra (or *DGLA* for short). The differential is exactly the Hochschild coboundary operator $\partial^{\mathrm{Hoch}}$ (1.4) and the Lie bracket is the well-known Gerstenhaber bracket[1] [33],

$$[Q_1, Q_2]_G = \sum_{i=0}^{k_1} (-1)^{(i+k_1)k_2} Q_1(a_0, \dots, Q_2(a_i, \dots, a_{i+k_2}), \dots, a_{k_1+k_2}) - (-1)^{k_1 k_2}(1 \leftrightarrow 2)\,, \tag{1.6}$$

where $Q_i \in C^{k_i+1}(\mathcal{A})$, and $a_j \in \mathcal{A}$.

As usual in this subject, we use adapted versions of Hochschild (co)chains for the algebra $\mathcal{O}(M)$; we denote by $C^\bullet(\mathcal{O}_M)$ the proper subcomplex of polydifferential operators in the full Hochschild cochain complex of $\mathcal{O}(M)$.

In this paper every DGLA $(\mathcal{L}, d_{\mathcal{L}}, [\,,\,]_{\mathcal{L}})$ is equipped with a complete descending filtration

$$\dots \supset \mathcal{F}^{-2}\mathcal{L} \supset \mathcal{F}^{-1}\mathcal{L} \supset \mathcal{F}^{0}\mathcal{L} \supset \mathcal{F}^{1}\mathcal{L} \supset \dots\,, \qquad \mathcal{L} = \lim_n \mathcal{L}/\mathcal{F}^n\mathcal{L}\,. \tag{1.7}$$

In most cases this filtration will be bounded from the left. We will often use a formal deformation parameter $\hbar$ to obtain a complete descending filtration on $\mathcal{L}$. For example, extending the field of scalars $\mathbb{C}$ to the ring $\mathbb{C}[[\hbar]]$ of formal power series, we obtain from the DGLA $C^{\bullet+1}(\mathcal{A})$ (resp. the

[1]Note that our sign convention for the Gerstenhaber bracket differs from the standard one.

graded Lie algebra $\mathcal{X}^{\bullet+1}(M)$) the DGLA $C^{\bullet+1}(\mathcal{A})[[\hbar]]$ (resp. the graded Lie algebra $\mathcal{X}^{\bullet+1}(M)[[\hbar]]$); the descending filtrations are

$$\mathcal{F}^k C^{\bullet+1}(\mathcal{A})[[\hbar]] = \hbar^k C^{\bullet+1}(\mathcal{A})[[\hbar]] , \qquad \mathcal{F}^k \mathcal{X}^{\bullet+1}(M)[[\hbar]] = \hbar^k \mathcal{X}^{\bullet+1}(M)[[\hbar]].$$

We assume that every morphism $\kappa : \mathcal{L} \to \widetilde{\mathcal{L}}$ of two such DGLAs is compatible with the filtrations. In addition, every quasi-isomorphism $\kappa : \mathcal{L} \stackrel{\sim}{\to} \widetilde{\mathcal{L}}$ is assumed to satisfy the following:

**Condition 1.1** *The restriction of a quasi-isomorphism $\kappa$ to each filtration subcomplex $\mathcal{F}^m\mathcal{L}^\bullet$*

$$\kappa\Big|_{\mathcal{F}^m\mathcal{L}^\bullet} : \mathcal{F}^m\mathcal{L}^\bullet \to \mathcal{F}^m\widetilde{\mathcal{L}}^\bullet$$

*is a quasi-isomorphism.*

We will also need $L_\infty$ morphisms and $L_\infty$ quasi-isomorphisms of DGLAs. We recall them in Appendix B; see Definitions B.2 and B.3. For $L_\infty$ morphisms or $L_\infty$ quasi-isomorphisms between DGLAs, we reserve the arrow $\succ\!\to$.

# 2 Global formality and star products

## 2.1 Fedosov's resolutions

We now briefly recall Fedosov's resolutions (see [18, Chapter 4]) of polyvector fields, and Hochschild cochains of $\mathcal{O}(M)$. This construction has various incarnations, and it is referred to as the Gelfand-Fuchs trick [31], or formal geometry [32] in the sense of Gelfand and Kazhdan, or mixed resolutions [61] of Yekutieli.

We denote by $x^i$ local coordinates on $M$ and by $y^i$ fiber coordinates in the tangent bundle $TM$. We denote by $\mathcal{S}M$ the formally completed symmetric algebra of the cotangent bundle $T^*M$. We regard $\mathcal{S}M$ as a sheaf of algebras over $\mathcal{O}_M$, whose sections can be viewed as formal power series in tangent coordinates $y^i$. In particular, $C^\bullet(\mathcal{S}M)$ is the *sheaf* of normalized Hochschild cochains of $\mathcal{S}M$ over $\mathcal{O}_M$. Namely, sections of $C^k(\mathcal{S}M)$ over an open subset $U \subset M$ are $\mathcal{O}(U)$-linear maps[2]

$$P : \Gamma(U, \mathcal{S}M)^{\otimes k} \to \Gamma(U, \mathcal{S}M), \tag{2.1}$$

which are continuous in the $y$-adic topology on $\Gamma(U, \mathcal{S}M)$ and satisfy the normalization condition

$$P(\dots, 1, \dots) = 0\,. \tag{2.2}$$

We let $\mathcal{T}^\bullet_{poly}$ be the sheaf of fiberwise polyvector fields, which is the cohomology of the complex of sheaves $C^\bullet(\mathcal{S}M)$ (see [18, page 60]). The grading convention for $\mathcal{T}^\bullet_{poly}$ coincides with the one for $\mathcal{X}^\bullet(M)$.

It is shown in [18, Theorem 4] that the algebra $\Omega^\bullet(M, \mathcal{S}M)$ can be equipped with a differential of the form

$$D = \nabla - \delta + A, \tag{2.3}$$

where

$$\nabla = dx^i \frac{\partial}{\partial x^i} - dx^i \Gamma^k_{ij}(x) y^j \frac{\partial}{\partial y^k} \tag{2.4}$$

[2]The sheaf $C^\bullet(\mathcal{S}M)$ is the $y$-adic completion of the sheaf $\mathcal{D}^\bullet_{poly}$ of fiberwise polydifferential operators. It is the latter sheaf that was used in [18] (see Definition 12 on page 60), and it is not hard to see that the sheaf $\mathcal{D}^\bullet_{poly}$ can be replaced by its completion $C^\bullet(\mathcal{S}M)$ in all the constructions of [18].

is a torsion free connection with Christoffel symbols $\Gamma^k_{ij}(x)$,

$$\delta = dx^i \frac{\partial}{\partial y^i}\,, \tag{2.5}$$

and

$$A = \sum_{p=2}^{\infty} dx^k A^j_{ki_1\dots i_p}(x) y^{i_1}\dots y^{i_p}\frac{\partial}{\partial y^j} \in \Omega^1(\mathcal{T}^1_{poly})\,. \tag{2.6}$$

**Remark 2.1** Although the construction of the differential (2.3) is very similar to the construction of the Fedosov differential in [27], they are not be confused; we refer to the differential (2.3) as a *geometric Fedosov differential* while the original Fedosov differential is referred to as a *quantum Fedosov differential*.

Note that $\delta$ in (2.5) is also a differential on $\Omega^\bullet(M,\mathcal{S}M)$, and (2.3) can be viewed as a deformation of $\delta$ via the connection $\nabla$. Let us recall from [18] the following operator on $\Omega^\bullet(M,\mathcal{S}M)$:

$$\delta^{-1}(a) = \begin{cases} y^k \dfrac{\vec{\partial}}{\partial(dx^k)} \displaystyle\int_0^1 a(x,ty,tdx)\frac{dt}{t}, & \text{if } a\in\Omega^{>0}(M,\mathcal{S}M)\,, \\ 0, & \text{otherwise}\,. \end{cases} \tag{2.7}$$

The arrow over $\partial$ in (2.7) means that we use the left derivative with respect to the “anti-commuting” variable $dx^k$. The operator (2.7) satisfies the following properties:

$$\delta^{-1}\circ\delta^{-1} = 0, \tag{2.8}$$

$$a = \sigma(a) + \delta\delta^{-1}a + \delta^{-1}\delta a, \qquad \forall\, a\in\Omega^\bullet(M,\mathcal{S}M), \tag{2.9}$$

where

$$\sigma(a) = a\Big|_{y^i=dx^i=0}. \tag{2.10}$$

**Remark 2.2** Following [18, Chapter 4], we extend the operators $\delta$, $\delta^{-1}$ and $\sigma$ to $\Omega^\bullet(M,\mathcal{T}^\bullet_{poly})$ and $\Omega^\bullet(M,C^\bullet(\mathcal{S}M))$ so that (2.9) holds also for all $a\in\Omega^\bullet(M,\mathcal{T}^\bullet_{poly})$, and for all $a\in\Omega^\bullet(M,C^\bullet(\mathcal{S}M))$.

According to [18, Prop. 10] the sheaves $\mathcal{T}^\bullet_{poly}$ and $C^\bullet(\mathcal{S}M)$ are equipped with a canonical action of the sheaf of Lie algebras $\mathcal{T}^1_{poly}$, and this action is compatible with the corresponding DGLA structures. Using this action in [18, Chp. 4], the Fedosov differential (2.3) is extended to differentials on $\Omega^\bullet(M,\mathcal{T}^\bullet_{poly})$ and $\Omega^\bullet(M,C^\bullet(\mathcal{S}M))$. Propositions 13 and 14 in [18] provide us with a quasi-isomorphism from the graded Lie algebra $\mathcal{X}^{\bullet+1}(M)$ to the DGLA

$$(\Omega^\bullet(M,\mathcal{T}^{\bullet+1}_{poly}), D, [\,,\,]_{SN}), \tag{2.11}$$

and a quasi-isomorphism from the DGLA $C^{\bullet+1}(\mathcal{O}_M)$ to the DGLA

$$(\Omega^\bullet(M,C^{\bullet+1}(\mathcal{S}M)), D+\partial^{\text{Hoch}}, [\,,\,]_G)\,. \tag{2.12}$$

To construct these quasi-isomorphisms we recall that the restriction of the map $\sigma$ (2.10) to the subspace of $D$-flat sections $\Gamma(M,\mathcal{S}M)\cap\ker D$ gives a bijection

$$\sigma : \Gamma(M,\mathcal{S}M)\cap\ker D \to \mathcal{O}(M) \tag{2.13}$$

onto the algebra of functions $\mathcal{O}(M)$. The inverse map

$$\tau : \mathcal{O}(M) \to \Gamma(M, \mathcal{S}M) \cap \ker D \tag{2.14}$$

is defined by iterating the equation

$$\tau(f) = f + \delta^{-1}(\nabla \tau(f) + A \cdot \tau(f)), \qquad f \in \mathcal{O}(M), \tag{2.15}$$

in degrees in the fiber coordinates $y$'s.

One may verify that the map $\tau$ satisfies

$$\frac{\partial}{\partial y^{i_1}} \dots \frac{\partial}{\partial y^{i_p}} \tau(f)\Big|_{y=0} = \partial_{x^{i_1}} \dots \partial_{x^{i_p}} f(x) + \text{lower order derivatives of } f. \tag{2.16}$$

To proceed further, we need the subspace

$$\Gamma_\delta(M, C^{\bullet+1}(\mathcal{S}M)) = \Gamma(M, C^{\bullet+1}(\mathcal{S}M)) \cap \ker \delta \tag{2.17}$$

of $\delta$-flat cochains of the sheaf $\mathcal{S}M$ . This subspace consists of $\mathcal{O}_M$-linear polydifferential operators on $\mathcal{S}M$ whose coefficients do not depend on the fiber coordinates $y$'s. The graded vector space $\Gamma_\delta(M, C^{\bullet+1}(\mathcal{S}M))$ is, in fact, isomorphic to the subspace

$$\Gamma(M, C^{\bullet+1}(\mathcal{S}M)) \cap \ker D$$

of $D$-flat sections of $C^{\bullet+1}(\mathcal{S}M)$. The corresponding isomorphism,

$$\varrho : \Gamma_\delta(M, C^{\bullet+1}(\mathcal{S}M)) \stackrel{\cong}{\longrightarrow} \Gamma(M, C^{\bullet+1}(\mathcal{S}M)) \cap \ker D, \tag{2.18}$$

is defined by iterating the equation

$$\varrho(P) = P + \delta^{-1}(\nabla \varrho(P) + [A, \varrho(P)]_G), \qquad P \in \Gamma_\delta(M, C^{\bullet+1}(\mathcal{S}M)) \tag{2.19}$$

in degrees in the fiber coordinates $y$'s.

On the other hand, using $\tau$ (2.14) we construct the map

$$\begin{aligned} &\nu : \Gamma_\delta(M, C^{\bullet+1}(\mathcal{S}M)) \longrightarrow C^{\bullet+1}(\mathcal{O}_M), \\ &\nu(P)(a_0, a_1, \dots, a_k) = P(\tau(a_0), \tau(a_1), \dots, \tau(a_k))\Big|_{y=0}, \end{aligned} \tag{2.20}$$

for $a_i \in \mathcal{O}(M)$ and $P \in \Gamma_\delta(M, C^{k+1}(\mathcal{S}M))$. Due to property (2.16), the map $\nu$ is also an isomorphism of graded vector spaces.

Composing $\varrho$ with $\nu^{-1}$, we get the isomorphism

$$\tau_{\text{ext}} = \varrho \circ \nu^{-1} : C^{\bullet+1}(\mathcal{O}_M) \stackrel{\cong}{\longrightarrow} \Gamma(M, C^{\bullet+1}(\mathcal{S}M)) \cap \ker D, \tag{2.21}$$

as well as the following embedding (for which we keep the same notation):

$$\tau_{\text{ext}} = \tau \circ \nu^{-1} : C^{\bullet+1}(\mathcal{O}_M) \hookrightarrow \Omega^\bullet(M, C^{\bullet+1}(\mathcal{S}M)). \tag{2.22}$$

Restricting (2.21) and (2.22) to the graded Lie algebra $\mathcal{X}^{\bullet+1}(M)$ of polyvector fields, we get the isomorphism

$$\tau_{\text{ext}} : \mathcal{X}^{\bullet+1}(M) \stackrel{\cong}{\longrightarrow} \Gamma(M, \mathcal{T}^{\bullet+1}_{poly}) \cap \ker D \tag{2.23}$$

and the embedding (for the both maps we keep the same notation $\tau_{\text{ext}}$)

$$\tau_{\text{ext}} : \mathcal{X}^{\bullet+1}(M) \hookrightarrow \Omega^\bullet(M, \mathcal{T}^{\bullet+1}_{poly}), \tag{2.24}$$

respectively. According to [18, Chapter 4] both maps (2.22) and (2.24) are compatible with the corresponding DGLA structures. Furthermore, the acyclicity of $D$ in positive exterior degrees implies that the maps (2.22) and (2.24) are quasi-isomorphisms.

To simplify our notation, we shall denote all three maps (2.14), (2.22), and (2.24) simply by $\tau$, avoiding the notation $\tau_{\rm ext}$ henceforth. This simplification does not lead to confusion because the restriction of $\tau_{\rm ext}$ (2.22) to

$$\mathcal{O}(M) = C^0(\mathcal{O}_M) = \mathcal{X}^0(M)$$

coincides with $\tau$.

## 2.2 A sequence of quasi-isomorphisms between $\mathcal{X}^{\bullet+1}(M)$ and $C^{\bullet+1}(\mathcal{O}_M)$

We now outline the construction of a sequence of quasi-isomorphisms between the DGLAs $\mathcal{X}^{\bullet+1}(M)$ and $C^{\bullet+1}(\mathcal{O}_M)$. The construction makes use of Kontsevich's $L_\infty$-quasi-isomorphism [46] $K$ from the DGLA $\mathcal{X}^{\bullet+1}(\mathbb{R}^d)$ of polyvector fields on $\mathbb{R}^d$ to the DGLA $C^{\bullet+1}(\mathcal{O}_{\mathbb{R}^d})$ of Hochschild cochains of $\mathcal{O}_{\mathbb{R}^d}$ (see Remark 2.7). Let us list some key properties of the structure maps

$$K_n : \Big(\mathcal{X}^{\bullet+1}(\mathbb{R}^d)\Big)^{\otimes n} \to C^{\bullet+1}(\mathcal{O}_{\mathbb{R}^d})[1-n]\,, \qquad n \ge 1 \tag{2.25}$$

of this $L_\infty$-quasi-isomorphism $K$:

**P 1** $K_n(\dots,\gamma_1,\gamma_2,\dots) = -(-1)^{|\gamma_1||\gamma_2|}K_n(\dots,\gamma_2,\gamma_1,\dots)$, where $|\gamma_i|$ is the degree of $\gamma_i$ in the vector space $\mathcal{X}^{\bullet+1}(\mathbb{R}^d)$ with the shifted grading.

**P 2** The map $K_1 : \mathcal{X}^{\bullet+1}(\mathbb{R}^d) \to C^{\bullet+1}(\mathcal{O}_{\mathbb{R}^d})$ coincides with the canonical embedding of the space of polyvectors into the space of polydifferential operators.

**P 3** The maps $K_n$ are $\mathfrak{gl}_d$ equivariant.

**P 4** $K_n(v,\dots) = 0$ if $n \ge 2$ and $v$ is a vector field which depends linearly on the coordinates of $\mathbb{R}^d$.

**P 5** If $v_1, v_2, \dots, v_n$ are vector fields and $n \ge 2$ then $K_n(v_1,\dots,v_n) = 0$.

**P 6** For every $n \ge 2$ we have $K_n(\dots,c) = 0$ if $c$ is a constant viewed as a degree-zero polyvector field $c \in \mathcal{X}^0(\mathbb{R}^d) = \mathcal{O}(\mathbb{R}^d)$.

Properties P 1 – P 5 allow us to construct an $L_\infty$-quasi-isomorphism

$$K^{tw} : (\Omega^\bullet(\mathcal{T}^{\bullet+1}_{poly}), D, [\,,\,]_{SN}) \succ\!\!\to (\Omega^\bullet(C^{\bullet+1}(\mathcal{S}M)), D + \partial^{\rm Hoch}, [\,,\,]_G) \tag{2.26}$$

as follows. For every coordinate neighborhood $U$ the part

$$\mu^D_U = -dx^i\Gamma^k_{ij}(x)y^j\frac{\partial}{\partial y^k} - \delta + A \tag{2.27}$$

of the geometric Fedosov differential (2.3) may be viewed as a Maurer-Cartan element of the DGLA

$$(\Omega^\bullet(U, \mathcal{T}^{\bullet+1}_{poly}), d, [\,,\,]_{SN}),$$

where $d$ is the de Rham differential. Kontsevich's $L_\infty$ quasi-isomorphism [46] for $\mathbb{R}^d$ may be viewed as an $L_\infty$ quasi-isomorphism

$$K : (\Omega^\bullet(U, \mathcal{T}^{\bullet+1}_{poly}), d, [\,,\,]_{SN}) \succ\!\!\to (\Omega^\bullet(U, C^{\bullet+1}(\mathcal{S}M)), d + \partial^{\rm Hoch}, [\,,\,]_G)\,. \tag{2.28}$$

Using this $L_\infty$ quasi-isomorphism and equation (B.17) from Appendix B, we can send the Maurer-Cartan element $\mu_U^D$ to the Maurer-Cartan element

$$\sum_{n=1}^{\infty} \frac{1}{n!} K_n(\mu_U^D, \mu_U^D, \dots, \mu_U^D)$$

of the DGLA $(\Omega^\bullet(U, C^{\bullet+1}(\mathcal{S}M)), d + \partial^{\mathrm{Hoch}}, [\,,\,]_G)$. Properties P 2 and P 5 imply that

$$\sum_{n=1}^{\infty} \frac{1}{n!} K_n(\mu_U^D, \mu_U^D, \dots, \mu_U^D) = \mu_U^D\,.$$

Therefore, twisting $K$ by the Maurer-Cartan element $\mu_U^D$ (see Appendix B) we get an $L_\infty$ quasi-isomorphism

$$K^{\mu_U^D} : (\Omega^\bullet(U, \mathcal{T}_{poly}^{\bullet+1}), D, [\,,\,]_{SN}) \succ\!\!\rightarrow (\Omega^\bullet(U, C^{\bullet+1}(\mathcal{S}M)), D + \partial^{\mathrm{Hoch}}, [\,,\,]_G)\,. \tag{2.29}$$

Properties P 3 and P 4 imply that $K^{\mu_U^D}$ does not depend on the choice of trivialization of the tangent bundle $TM$ over $U$. Hence we get an $L_\infty$ quasi-isomorphism (2.26) by setting

$$K_n^{tw}(\gamma_1, \gamma_2, \dots, \gamma_n)\Big|_U = K_n^{\mu_U^D}(\gamma_1, \gamma_2, \dots, \gamma_n)\,. \tag{2.30}$$

Combining $K^{tw}$ with the maps (2.22) and (2.24), we obtain a sequence of quasi-isomorphisms

$$\mathcal{X}^{\bullet+1}(M) \stackrel{\tau}{\longrightarrow} \Omega^\bullet(M, \mathcal{T}_{poly}^{\bullet+1}) \stackrel{K^{tw}}{\succ\!\!\rightarrow} \Omega^\bullet(M, C^{\bullet+1}(\mathcal{S}M)) \stackrel{\tau}{\longleftarrow} C^{\bullet+1}(\mathcal{O}_M). \tag{2.31}$$

Using [19, Lemma 1], we can reduce the sequence (2.31) to a single $L_\infty$ quasi-isomorphism

$$\mathcal{K} : \mathcal{X}^{\bullet+1}(M) \succ\!\!\rightarrow C^{\bullet+1}(\mathcal{O}_M) \tag{2.32}$$

between the DGLAs $\mathcal{X}^{\bullet+1}(M)$ and $C^{\bullet+1}(\mathcal{O}_M)$. More precisely, in [18, Chapter 4, Eq. (4.36)] a chain homotopy is constructed which contracts the complex $(\Omega^\bullet(M, C^{\bullet+1}(\mathcal{S}M)), D + \partial^{\mathrm{Hoch}})$ to its sub-DGLA

$$(\Gamma(M, C^{\bullet+1}(\mathcal{S}M)) \cap \ker D, \partial^{\mathrm{Hoch}}, [\,,\,]_G)\,. \tag{2.33}$$

Using this chain homotopy and [19, Lemma 1], one constructs an $L_\infty$ quasi-isomorphism

$$\widetilde{\mathcal{K}} : \mathcal{X}^{\bullet+1}(M) \succ\!\!\rightarrow (\Omega^\bullet(M, C^{\bullet+1}(\mathcal{S}M)), D + \partial^{\mathrm{Hoch}}, [\,,\,]_G) \tag{2.34}$$

satisfying two properties: first, all the structure maps $\widetilde{\mathcal{K}}_n$ of $\widetilde{\mathcal{K}}$ take values in the sub-DGLA (2.33); second, $\widetilde{\mathcal{K}}$ is homotopy equivalent to the composition

$$K^{tw} \circ \tau : \mathcal{X}^{\bullet+1}(M) \succ\!\!\rightarrow (\Omega^\bullet(M, C^{\bullet+1}(\mathcal{S}M)), D + \partial^{\mathrm{Hoch}}, [\,,\,]_G)\,.$$

Composing $\widetilde{\mathcal{K}}$ with the inverse of the isomorphism (2.21) we get the desired $L_\infty$ quasi-isomorphism $\mathcal{K}$ (2.32). Going through details of this construction and using the fact that the structure maps (2.25) land in normalized Hochschild cochains, one verifies the following:

**Proposition 2.3** *For every constant $c$, viewed as a degree zero polyvector field $c \in \mathcal{X}^0(M) = \mathcal{O}(M)$, we have*

$$\mathcal{K}_1(c) = c, \quad \textit{and} \quad \mathcal{K}_n(\dots, c) = 0, \quad \forall\ n \geq 2.$$

**Remark 2.4** From now on we extend the field of scalars $\mathbb{C}$ to the ring $\mathbb{C}[[\hbar]]$ in all our constructions. In other words, we replace the DGLAs $\mathcal{X}^{\bullet+1}(M)$, $\Omega^{\bullet}(M, \mathcal{T}^{\bullet+1}_{poly})$, $\Omega^{\bullet}(M, C^{\bullet+1}(\mathcal{S}M))$, and $C^{\bullet+1}(\mathcal{O}_M)$ in (2.31) with

$$\mathcal{X}^{\bullet+1}(M)[[\hbar]], \quad \Omega^{\bullet}(M, \mathcal{T}^{\bullet+1}_{poly})[[\hbar]], \quad \Omega^{\bullet}(M, C^{\bullet+1}(\mathcal{S}M))[[\hbar]], \qquad C^{\bullet+1}(\mathcal{O}_M)[[\hbar]], \tag{2.35}$$

respectively. We also replace the connection form $\Gamma$ in (2.4) by a general formal Taylor power series in $\hbar$:

$$\Gamma_{\hbar} = \Gamma_0 + \hbar\Gamma_1 + \hbar^2\Gamma_2 + \dots ,$$

where $\Gamma_0$ is an ordinary torsion free connection form and $\Gamma_1$, $\Gamma_2$, ... are global sections of $TM \otimes S^2(T^*M)$. Finally, we allow $A$ (2.6) to have the more general form:

$$A = \sum_{p=2, r=0}^{\infty} dx^k \hbar^r A^j_{r;ki_1\dots i_p}(x) y^{i_1} \dots y^{i_p} \frac{\partial}{\partial y^j} \in \Omega^1(M, \mathcal{T}^1_{poly})[[\hbar]] \,. \tag{2.36}$$

It is not hard to see that the constructions described in Subsection 2.1 and in this subsection can be generalized to this setting. Furthermore, the resulting $L_\infty$ quasi-isomorphisms (2.31) connecting the DGLAs in (2.35) agree with the $\hbar$-adic filtration and satisfy Condition 1.1.

**Remark 2.5** We will need the following complete descending filtration on the sheaf of algebras $\mathcal{S}M[[\hbar]]$ (see Appendix C):

$$\mathcal{S}M[[\hbar]] = \mathcal{F}^0\mathcal{S}M[[\hbar]] \supset \mathcal{F}^1\mathcal{S}M[[\hbar]] \supset \mathcal{F}^2\mathcal{S}M[[\hbar]] \supset \dots , \tag{2.37}$$

where local sections of $\mathcal{F}^m\mathcal{S}M[[\hbar]]$ are the series

$$a = \sum_{2k+l \ge m} \hbar^k \, a_{k;i_1,\dots,i_l}(x) \, y^{i_1} y^{i_2} \dots y^{i_l} \,.$$

## 2.3 Star products and their equivalence classes

A *star product* [1, 2] on a manifold $M$ is a $\mathbb{C}[[\hbar]]$-linear associative product on $\mathcal{O}(M)[[\hbar]]$ of the form

$$f * g = fg + \sum_{k=1}^{\infty} \hbar^k \Pi_k(f, g), \tag{2.38}$$

where $f, g \in \mathcal{O}(M)[[\hbar]]$ and $\Pi_k \in C^2(\mathcal{O}_M)$, i.e., $\Pi_k$ are normalized bidifferential operators. Because we deal with normalized Hochschild cochains (1.3), star products satisfy

$$f * 1 = 1 * f = f \,. \tag{2.39}$$

Since $f * g = fg \mod \hbar$, star products should be viewed as an associative (but not necessarily commutative) formal deformation of the ordinary product of functions on $M$.

Two star products $*$ and $*'$ are *equivalent* if there exist (normalized) differential operators $T_i : \mathcal{O}(M) \to \mathcal{O}(M)$, $i = 1, 2, \dots$, so that the formal series $T = \mathrm{id} + \hbar T_1 + \hbar^2 T_2 + \dots$ intertwines the star products,

$$T(f * g) = T(f) *' T(g). \tag{2.40}$$

We denote the set of equivalence classes of star products on $M$ by $\mathrm{Def}(M)$.

The associativity property of a star product (2.38) can be equivalently expressed as the Maurer-Cartan equation

$$\partial^{\mathrm{Hoch}}\Pi + \frac{1}{2}[\Pi, \Pi]_G = 0$$

for the formal series of bidifferential operators $\Pi := \sum_{k=1}^{\infty} \hbar^k \Pi_k \in C^2(\mathcal{O}_M)[[\hbar]]$. Thus Maurer-Cartan elements of the DGLA $C^{\bullet+1}(\mathcal{O}_M)[[\hbar]]$ (see Appendix B) are exactly the star products on $M$. Furthermore, one can verify that equivalent Maurer-Cartan elements of $C^{\bullet+1}(\mathcal{O}_M)[[\hbar]]$ (in the sense of (B.4)) correspond to equivalent star products.

Maurer-Cartan elements in $\mathcal{X}^{\bullet+1}(M)[[\hbar]]$ are formal series of bivector fields in $\hbar$,

$$\pi = \hbar\pi_1 + \hbar^2\pi_2 + \cdots \in \hbar\,\mathcal{X}^2(M)[[\hbar]] \tag{2.41}$$

satisfying the equation

$$[\pi, \pi]_{SN} = 0\,. \tag{2.42}$$

We refer to Maurer-Cartan elements in $\mathcal{X}^{\bullet+1}(M)[[\hbar]]$ as *formal Poisson structures*, by analogy with the usual definition of Poisson structures in geometry (cf. [23, 48, 60]); further properties of formal Poisson structures are discussed in Section 3.2.

As recalled in Appendix B, the prounipotent group of formal diffeomorphisms,

$$\mathfrak{G}(\mathcal{X}^{\bullet+1}(M)[[\hbar]]) = \exp\left(\hbar\mathcal{X}^1(M)[[\hbar]]\right), \tag{2.43}$$

acts on Maurer-Cartan elements of $\mathcal{X}^{\bullet+1}(M)[[\hbar]]$ according to

$$\pi^{\exp(X)} = \exp([\cdot, X]_{SN})\pi, \tag{2.44}$$

where $X \in \hbar\mathcal{X}^1(M)[[\hbar]]$. Two formal Poisson structures $\pi$ and $\tilde{\pi}$ are said to be *equivalent* if they lie in the same orbit of this action. We denote the set of equivalence classes of formal Poisson structures by[3] FPoiss($M$). The equivalence class of a star product $*$ and of a formal Poisson structure $\pi$ will be denoted by

$$[*] \in \mathrm{Def}(M) \quad \text{and} \quad [\pi] \in \mathrm{FPoiss}(M),$$

respectively.

The $L_\infty$ quasi-isomorphism $\mathcal{K}$ in (2.32) establishes, according to (B.17), a correspondence between formal Poisson structures and star products on $M$,

$$\pi \mapsto *_K, \quad \text{where} \quad f *_K g = fg + \sum_{n=1}^{\infty} \mathcal{K}_n(\pi, \pi, \ldots, \pi)(f, g), \tag{2.46}$$

for $f, g \in \mathcal{O}(M)[[\hbar]]$. We refer to $*_K$ as the *Kontsevich star product* associated with $\pi$. By Proposition B.4, the correspondence (2.46) induces a bijection

$$\mathcal{K}_* : \mathrm{FPoiss} \to \mathrm{Def}(M), \tag{2.47}$$

associating to each equivalence class of formal Poisson structures an equivalence class of star products. We call the class $[\pi] = \mathcal{K}_*^{-1}([*])$ in FPoiss *Kontsevich's class* of the star product $*$.

Regarding the choices involved in the definition of Kontsevich's classes, we first observe that the bijection $\mathcal{K}_*$ agrees with the bijection induced by the sequence of $L_\infty$ quasi-isomorphisms (2.31); indeed, the fact that shortening the sequence (2.31) using [19, Lemma 1] does not change the correspondence between equivalence classes of Maurer-Cartan elements follows from Lemma B.5 in Appendix B. On the other hand, as discussed in Section 2.2, the middle $L_\infty$ quasi-isomorphism in the sequence (2.31) requires the choice of a Fedosov differential (2.3). As shown by the next result, this choice does not affect the Kontsevich class of a star product.

[3] In the notation of Appendix B,

$$\mathrm{Def}(M) = \pi_0(\mathrm{MC}(C^{\bullet+1}(\mathcal{O}_M)[[\hbar]])), \quad \mathrm{FPoiss}(M) = \pi_0(\mathrm{MC}(\mathcal{X}^{\bullet+1}(M)[[\hbar]])). \tag{2.45}$$

**Theorem 2.6** *The map $\mathcal{K}_*$ in* (2.47) *does not depend on the choice of the Fedosov differential.*

The proof of Theorem 2.6 is in Appendix C.

**Remark 2.7** We note that the constructions in Section 2.2 (and hence the notion of Kontsevich's class) are based on Kontsevich's $L_\infty$ quasi-isomorphism $K$ [46] from $\mathcal{X}^{\bullet+1}(\mathbb{R}^d)$ to $C^{\bullet+1}(\mathcal{O}_{\mathbb{R}^d})$, which is fixed throughout this paper. It is known [45] that there are other $L_\infty$ quasi-isomorphisms from $\mathcal{X}^{\bullet+1}(\mathbb{R}^d)$ to $C^{\bullet+1}(\mathcal{O}_{\mathbb{R}^d})$ which are not homotopy equivalent to $K$; In fact, Tamarkin's proof [22, 41, 57] of Kontsevich's formality theorem indicates that homotopy equivalence classes of such $L_\infty$ quasi-isomorphisms are acted upon by the Grothendieck-Teichmüller group introduced by Drinfeld in [24].

# 3 The characteristic classes of Morita equivalent star products

Two unital rings are called *Morita equivalent* if they have equivalent categories of modules [49]. We view star-product algebras on a manifold $M$ as unital algebras over the ground ring $\mathbb{C}[[\hbar]]$ and consider the problem of describing their Morita equivalence classes. Following [4, 6], this classification is given by the orbits of a canonical action of $\mathrm{Diff}(M) \ltimes \mathrm{Pic}(M)$ on the space $\mathrm{Def}(M)$. Here $\mathrm{Diff}(M)$ is the group of diffeomorphisms of $M$, and $\mathrm{Pic}(M) \cong H^2(M, \mathbb{Z})$ is its *Picard group*, i.e., the group of isomorphism classes of complex line bundles over $M$; $\mathrm{Diff}(M) \ltimes \mathrm{Pic}(M)$ is the semi-direct product group with respect to the action of $\mathrm{Diff}(M)$ on $\mathrm{Pic}(M)$ by pull-back. We will briefly recall how this action is defined, and then give its explicit description in terms of Kontsevich's classes.

## 3.1 An action of the Picard group on star products

Given a diffeomorphism $\varphi : M \to M$ and a star product $*$, we obtain a new star product $*_\varphi$,

$$f *_\varphi g = (\varphi^{-1})^*(\varphi^* f * \varphi^* g),$$

where $f, g \in \mathcal{O}(M)$, and this induces an action

$$\mathrm{Diff}(M) \times \mathrm{Def}(M) \to \mathrm{Def}(M), \quad (\varphi, [*]) \mapsto [*_\varphi]. \tag{3.1}$$

It is known [40] that every isomorphism between two star-product algebras on $M$ is a composition of an equivalence (2.40) with an element in $\mathrm{Diff}(M)$ (viewed as an automorphism of $\mathcal{O}(M)$ via pull-back). This gives a simple interpretation of the action (3.1): the classes of $*$ and $*'$ in $\mathrm{Def}(M)$ are in the same $\mathrm{Diff}(M)$-orbit if and only if the two star-product algebras are isomorphic.

The space $\mathrm{Def}(M)$ also carries a natural action of $\mathrm{Pic}(M)$ [4]. Given a complex line bundle $L \to M$, we view $\Gamma(M, L)$ as a right module over $\mathcal{O}(M)$. We denote by $\mathrm{End}(\Gamma(M, L)) = \Gamma(M, \mathrm{End}(L))$ the algebra of endomorphisms of this module, noticing that there is a canonical identification

$$\mathrm{End}(\Gamma(M, L)) \cong \mathcal{O}(M). \tag{3.2}$$

As shown in [5], for a given star product $*$ on $M$, there is a unique way (up to equivalence) of deforming this module structure to make $\Gamma(M, L)[[\hbar]]$ a right module over the star-product algebra $(\mathcal{O}(M)[[\hbar]], *)$. For $s \in \Gamma(M, L)$, $f \in \mathcal{O}(M)$ we denote the deformed module structure by

$$s \bullet f = sf \mod \hbar,$$

and write $\mathrm{End}(\Gamma(M, L)[[\hbar]], \bullet)$ for the algebra of endomorphisms of this module. One can always find an identification of $\mathcal{O}(M)[[\hbar]]$ with $\mathrm{End}(\Gamma(M, L)[[\hbar]], \bullet)$ as $\mathbb{C}[[h]]$-modules, in such a way that for $\hbar = 0$ one recovers the natural identification (3.2). As a consequence, we obtain a star product $*'$ on $M$ for which

$$(\mathcal{O}(M)[[\hbar]], *') \cong \mathrm{End}(\Gamma(M, L)[[\hbar]], \bullet) \tag{3.3}$$

is an isomorphism of $\mathbb{C}[[\hbar]]$-algebras deforming (3.2). This construction defines an action

$$\Phi : \mathrm{Pic}(M) \times \mathrm{Def}(M) \to \mathrm{Def}(M), \quad (L, [*]) \mapsto \Phi_L([*]) = [*'], \tag{3.4}$$

where $*'$ is characterized, up to equivalence, by (3.3).

The actions (3.1) and (3.4) provide a convenient characterization of Morita equivalence for star products [4]: Two star products $*$ and $*'$ on $M$ are Morita equivalent if and only if their classes in $\mathrm{Def}(M)$ are related by the actions (3.1) and (3.4), i.e., $[*'] = \Phi_L([*_\varphi])$ for some line bundle $L$ and diffeomorphism $\varphi$.

It will be useful to describe the action (3.4) in terms of transition functions. Let us consider a star product $*$, a line bundle $L \to M$, and a deformed right-module structure $\bullet$ on $\Gamma(M, L)[[\hbar]]$ over $(\mathcal{O}(M)[[\hbar]], *)$. Let $\{U_\alpha\}$ be a cover of $M$ by contractible open subsets. We can define local $\mathcal{O}(U_\alpha)$-linear trivialization isomorphisms $\psi_\alpha : \Gamma(U_\alpha, L) \to \mathcal{O}(U_\alpha)$ and transition functions $g_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$ such that $\psi_\alpha(\psi_\beta^{-1})(f)(x) = g_{\alpha\beta}(x)f(x)$, which satisfy $g_{\alpha\beta}^{-1} = g_{\beta\alpha}$ and, on triple intersections, the cocycle condition

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1.$$

As shown in [7], one can always find $\mathbb{C}[[\hbar]]$-linear deformed trivialization isomorphisms $\Psi_\alpha = \psi_\alpha \mod \hbar : \Gamma(U_\alpha, L)[[\hbar]] \to \mathcal{O}(U_\alpha)[[\hbar]]$ satisfying

$$\Psi_\alpha(s \bullet f) = \Psi_\alpha(s) * f,$$

and define deformed transition functions $G_{\alpha\beta} = g_{\alpha\beta} + \mod \hbar \in \mathcal{O}(U_\alpha \cap U_\beta)[[\hbar]]$ such that

$$\Psi_\alpha(s) = G_{\alpha\beta} * \Psi_\beta(s).$$

Since 1 is the unit for $*$ (see (2.39)), it is clear that

$$G_{\alpha\alpha} = 1, \quad G_{\alpha\beta} * G_{\beta\alpha} = 1, \quad G_{\alpha\beta} * G_{\beta\gamma} * G_{\gamma\alpha} = 1. \tag{3.5}$$

Let us consider a $\mathbb{C}[[\hbar]]$-algebra isomorphism

$$T : (\mathcal{O}(M)[[\hbar]], *') \to \mathrm{End}(\Gamma(M, L)[[\hbar]], \bullet),$$

coinciding with (3.2) at the classical limit $\hbar = 0$. The isomorphism $T$ is totally determined by a collection of local equivalences $T_\alpha : (\mathcal{O}(U_\alpha)[[\hbar]], *') \to (\mathcal{O}(U_\alpha)[[\hbar]], *)$ satisfying

$$T_\alpha \circ T_\beta^{-1}(f) = G_{\alpha\beta} * f * G_{\beta\alpha}, \tag{3.6}$$

for $f \in \mathcal{O}(U_\alpha \cap U_\beta)$. One recovers $T$ from the collection $\{T_\alpha\}$ by

$$T(f)(s) = \Psi_\alpha^{-1}(T_\alpha(f) * \Psi_\alpha(s)), \quad s \in \Gamma(U_\alpha, L). \tag{3.7}$$

**Proposition 3.1** *Let $*$ and $*'$ be star products on $M$. The following are equivalent:*

*(i) The star products $*$ and $*'$ are related by (3.4), i.e., there exists a line bundle $L \to M$ for which $\Phi_L([*]) = [*']$;*

(ii) *There exists an open cover* $\{U_\alpha\}$ *of* $M$*, with trivialization maps* $\psi_\alpha$ *and transition functions* $g_{\alpha\beta}$ *for* $L \to M$*, as well as deformed transition functions* $G_{\alpha\beta} = g_{\alpha\beta}$ *mod* $\hbar \in \mathcal{O}(U_\alpha \cap U_\beta)[[\hbar]]$ *satisfying the cocycle conditions* (3.5) *and a collection of equivalences* $T_\alpha : (\mathcal{O}(U_\alpha)[[\hbar]], *') \to (\mathcal{O}(U_\alpha)[[\hbar]], *)$ *for which the compatibility* (3.6) *holds.*

Proof. The implication $(i) \implies (ii)$ was already discussed. We explain how $(ii)$ implies $(i)$. We first show that we can find local deformed trivializations $\Psi_\alpha = \psi_\alpha$ mod $\hbar$ such that

$$\Psi_\alpha \Psi_\beta^{-1}(f) = G_{\alpha\beta} * f \tag{3.8}$$

for all $f \in \mathcal{O}(U_\alpha \cap U_\beta)[[\hbar]]$. We know from [7] that we can find deformed trivializations $\Psi'_\alpha = \psi_\alpha$ mod $\hbar$; let us define $G'_{\alpha\beta}$ by $G'_{\alpha\beta} * f = \Psi'_\alpha(\Psi'_\beta)^{-1}(f)$. We now modify $\Psi'_\alpha$ to obtain $\Psi_\alpha$ satisfying (3.8). Let $\{\chi_\alpha\}$ be a partition of unity on $M$ subordinated to $\{U_\alpha\}$. Consider

$$S_\alpha = \sum_\gamma G'_{\alpha\gamma} * \chi_\gamma * G_{\gamma\alpha},$$

viewed as an element in $\mathcal{O}(U_\alpha)[[\hbar]]$ (note that each summand has a natural extension from $\mathcal{O}(U_\alpha \cap U_\beta)[[\hbar]]$ to $\mathcal{O}(U_\alpha)[[\hbar]]$). Note also that $S_\alpha$ is invertible with respect to $*$, since $S_\alpha = 1$ mod $\hbar$. Finally note that, using the cocycle conditions for $G'_{\alpha\beta}$ and $G_{\alpha\beta}$, we have

$$S_\alpha * G_{\alpha\beta} = \sum_\gamma G'_{\alpha\gamma} * \chi_\gamma * G_{\gamma\alpha} * G_{\alpha\beta} = \sum_\gamma G'_{\alpha\beta} * G'_{\beta\gamma} * \chi_\gamma * G_{\gamma\beta} = G'_{\alpha\beta} * S_\beta.$$

In other words, $G_{\alpha\beta} = S_\alpha^{-1} * G'_{\alpha\beta} * S_\beta$. Let us now define $\Psi_\alpha$ by $\Psi_\alpha(s) = S_\alpha^{-1} * \Psi'_\alpha(s)$. Then

$$\Psi_\alpha \Psi_\beta^{-1}(f) = S_\alpha^{-1} * G'_{\alpha\beta} * S_\beta * f = G_{\alpha\beta} * f,$$

as desired. We now use the local equivalences $T_\alpha$ and $\Psi_\alpha$ to define an isomorphism $T : (\mathcal{O}(M), *') \to \mathrm{End}(\Gamma(M, L)[[\hbar]], \bullet)$ via (3.7). □

## 3.2 An action of closed 2-forms on formal Poisson structures

The description of how formal Poisson structures are acted upon by closed 2-forms is a simple adaptation of the discussion of *gauge transformations* in [55, Sec. 3]; in the context of generalized complex geometry, the same operation appears under the name of *B-field transform*, see e.g. [39, Sec. 3]. We start by recalling standard facts and alternative views of formal Poisson structures.

Given a formal bivector field $\pi = \sum_{k=1}^\infty \hbar^k \pi_k \in \hbar\mathcal{X}^2(M)[[\hbar]]$, we consider the $\mathbb{C}$-bilinear brackets

$$\{\cdot,\cdot\}_k : \mathcal{O}(M) \times \mathcal{O}(M) \to \mathcal{O}(M), \;\; \{f, g\}_k = \pi_k(df, dg),$$

and the induced $\mathbb{C}[[\hbar]]$-bilinear bracket $\{\cdot,\cdot\}_\pi : \mathcal{O}(M)[[\hbar]] \times \mathcal{O}(M)[[\hbar]] \to \mathcal{O}(M)[[\hbar]]$, uniquely determined by

$$\{f, g\}_\pi = \pi(df, dg) = \sum_{k=1}^\infty \hbar^k \{f, g\}_k, \qquad f, g \in \mathcal{O}(M). \tag{3.9}$$

Let $\mathrm{Jac}_\pi : \mathcal{O}(M)[[\hbar]] \times \mathcal{O}(M)[[\hbar]] \times \mathcal{O}(M)[[\hbar]] \to \mathcal{O}(M)[[\hbar]]$ be given by

$$\mathrm{Jac}_\pi(f, g, h) = \{f, \{g, h\}_\pi\}_\pi + \{h, \{f, g\}_\pi\}_\pi + \{g, \{h, f\}_\pi\}_\pi. \tag{3.10}$$

We also consider the $\mathcal{O}(M)[[\hbar]]$-linear map

$$\pi^\sharp : \Omega^1(M)[[\hbar]] \to \hbar\mathcal{X}^1(M)[[\hbar]], \quad \pi^\sharp(\xi) = \sum_{k=1}^\infty \hbar^k \pi_k^\sharp(\xi), \tag{3.11}$$

where $\pi_k^\sharp(\xi) = i_\xi \pi_k$ for $\xi \in \Omega^1(M)$, and its unique extension (as an algebra homomorphism)

$$\pi^\sharp : \Omega^\bullet(M)[[\hbar]] \to \hbar\mathcal{X}^\bullet(M)[[\hbar]]. \tag{3.12}$$

**Proposition 3.2** *Let $\pi \in \hbar\mathcal{X}^2(M)[[\hbar]]$ and $\partial_\pi = [\pi, \cdot]_{SN}$. Then*

$$\frac{1}{2}[\pi,\pi]_{SN}(df, dg, dh) = \mathrm{Jac}_\pi(f,g,h) = \partial_\pi^2(f)(dg, dh), \tag{3.13}$$

*for $f, g, h \in \mathcal{O}(M)[[\hbar]]$.*

Proof. It suffices to verify (3.13) for $f, g, h \in \mathcal{O}(M)$. Given bivector fields $\pi_k, \pi_l \in \mathcal{X}^2(M)$, let $\mathrm{Jac}_{k,l} : \mathcal{O}(M) \times \mathcal{O}(M) \times \mathcal{O}(M) \to \mathcal{O}(M)$ be defined by

$$\mathrm{Jac}_{k,l}(f,g,h) = \{f, \{g,h\}_k\}_l + \{h, \{f,g\}_k\}_l + \{g, \{h,f\}_k\}_l.$$

The Schouten bracket satisfies (see e.g. [23])

$$[\pi_k, \pi_l]_{SN}(df, dg, dh) = \mathrm{Jac}_{k,l}(f,g,h) + \mathrm{Jac}_{l,k}(f,g,h).$$

As a result, the $n^{th}$-order term in $\hbar$ of $\frac{1}{2}[\pi,\pi](df, dg, dh)$ is

$$\sum_{i=1}^{n-1} \mathrm{Jac}_{i,n-i}(f,g,h), \tag{3.14}$$

which agrees with the $n^{th}$-order term in $\hbar$ of $\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\}$, proving the first equality in (3.13). For the second equality, recall that the Schouten bracket satisfies

$$[\pi_l, f]_{SN} = -\pi_l^\sharp(df), \quad [\pi_l, X]_{SN} = -\mathcal{L}_X \pi_l, \tag{3.15}$$

for $f \in \mathcal{O}(M)$, $X \in \mathcal{X}^1(M)$. A direct computation shows that

$$[\pi_k, [\pi_l, f]_{SN}]_{SN}(dg, dh) = \{f, \{g,h\}_k\}_l + \{h, \{f,g\}_l\}_k + \{g, \{h,f\}_l\}_k,$$

and, as a consequence, the $n^{th}$-order term in $\hbar$ of $\partial_\pi^2(f)(dg, dh)$ coincides with (3.13). □

**Corollary 3.3** *If $[\pi,\pi]_{SN} = 0$, then the map $\pi^\sharp$ in* (3.12) *satisfies $\pi^\sharp \circ d = -\partial_\pi \circ \pi^\sharp$.*

Proof. It suffices to verify that $\pi^\sharp \circ d = -\partial_\pi \circ \pi^\sharp$ holds on elements $f$ and $df$, for $f \in \mathcal{O}(M)$. The fact that $\pi^\sharp(df) = -\partial_\pi f$ directly follows from the first equation in (3.15). On the other hand, since $[\pi,\pi]_{SN} = 0$, we have

$$-\partial_\pi(\pi^\sharp(df)) = \partial_\pi([\pi, f]_{SN}) = \partial_\pi^2(f) = 0,$$

which agrees with $\pi^\sharp(d^2 f) = 0$. □

To describe the action by closed 2-forms, it is convenient to have an alternative viewpoint to formal Poisson structures, in the spirit of Dirac geometry [14]. We consider the bundle

$$E := TM \oplus T^*M,$$

equipped with the symmetric $\mathcal{O}(M)$-bilinear pairing $\langle \cdot, \cdot \rangle : \Gamma(M, E) \times \Gamma(M, E) \to \mathcal{O}(M)$,

$$\langle (X, \xi), (Y, \eta) \rangle = \eta(X) + \xi(Y), \tag{3.16}$$

and the $\mathbb{C}$-bilinear bracket $[\![\cdot,\cdot]\!] : \Gamma(M,E) \times \Gamma(M,E) \to \Gamma(M,E)$,

$$[\![(X,\xi),(Y,\eta)]\!] = ([X,Y], \mathcal{L}_X\eta - i_Y d\xi), \tag{3.17}$$

known as the *Courant bracket*. Here $X, Y \in \mathcal{X}^1(M)$ and $\xi, \eta \in \Omega^1(M)$. Using $\hbar$-linearity, we extend these operations to

$$\begin{aligned} \langle\cdot,\cdot\rangle &: \Gamma(M,E)[[\hbar]] \times \Gamma(M,E)[[\hbar]] \to \mathcal{O}(M)[[\hbar]], \\ [\![\cdot,\cdot]\!] &: \Gamma(M,E)[[\hbar]] \times \Gamma(M,E)[[\hbar]] \to \Gamma(M,E)[[\hbar]]. \end{aligned} \tag{3.18}$$

The following is a straightforward observation.

**Lemma 3.4** *A $\mathcal{O}(M)[[\hbar]]$-linear map $T : \Omega^1(M)[[\hbar]] \to \mathcal{X}^1(M)[[\hbar]]$ is of the form $\pi^\sharp$ for a formal bivector field $\pi \in \mathcal{X}^2(M)[[\hbar]]$ if and only if*

$$\langle (T(\xi),\xi), (T(\eta),\eta)\rangle = 0, \quad \forall\ \xi,\eta \in \Omega^1(M).$$

We can characterize formal Poisson structures using (3.18) as follows.

**Lemma 3.5** *Given a formal bivector field $\pi \in \hbar\mathcal{X}^2(M)[[\hbar]]$, we have*

$$\mathrm{Jac}_\pi(f,g,h) = \Big\langle [\![(\pi^\sharp(df), df), (\pi^\sharp(dg), dg)]\!], (\pi^\sharp(dh), dh)\Big\rangle, \tag{3.19}$$

*for all $f, g, h \in \mathcal{O}(M)[[\hbar]]$.*

Proof. It suffices to verify the lemma for $f, g, h \in \mathcal{O}(M)$. It is clear from (3.9) that $\mathcal{L}_{\pi^\sharp(df)}g = \{f,g\}_\pi$, and it immediately follows from the definitions (3.16) and (3.17) (extended to formal power series) that the right-hand side of (3.19) is

$$dh([\pi^\sharp(df), \pi^\sharp(dg)]) + d(\mathcal{L}_{\pi^\sharp(df)}g)(\pi^\sharp(dh)) = \Big(\mathcal{L}_{\pi^\sharp(df)}\mathcal{L}_{\pi^\sharp(dg)} - \mathcal{L}_{\pi^\sharp(dg)}\mathcal{L}_{\pi^\sharp(df)}\Big)\, h + \mathcal{L}_{\pi^\sharp(dh)}\mathcal{L}_{\pi^\sharp(df)}g,$$

which is $\mathrm{Jac}_\pi(f,g,h)$. □

Given any $B = B_0 + \hbar B_1 + \cdots \in \Omega^2(M)[[\hbar]]$, there is an associated automorphism of $\mathcal{O}(M)[[\hbar]]$-modules given by

$$\lambda_B : \Gamma(M,E)[[\hbar]] \to \Gamma(M,E)[[\hbar]], \quad \lambda_B(X,\xi) = (X, \xi + i_X B). \tag{3.20}$$

The following properties of $\lambda_B$ are proven analogously as e.g. in [39].

**Lemma 3.6** *The following holds:*

1. *$\langle \lambda_B(X,\xi), \lambda_B(Y,\eta)\rangle = \langle (X,\xi),(Y,\eta)\rangle$ for all $(X,\xi),(Y,\eta) \in \Gamma(M,E)[[\hbar]]$.*
2. *$[\![\lambda_B(X,\xi), \lambda_B(Y,\eta)]\!] = \lambda_B([\![(X,\xi),(Y,\eta)]\!])$ for all $(X,\xi),(Y,\eta) \in \Gamma(M,E)[[\hbar]]$ if and only if $dB = 0$.*

Let us consider the $\mathcal{O}(M)[[\hbar]]$-linear map

$$B^\sharp : \mathcal{X}^1(M)[[\hbar]] \to \Omega^1(M)[[\hbar]], \quad B^\sharp(X) = i_X B,$$

associated with $B \in \Omega^2(M)[[\hbar]]$. For any formal bivector field $\pi \in \hbar\mathcal{X}^1(M)[[\hbar]]$, the operator

$$\mathrm{id} + B^\sharp\pi^\sharp : \Omega^1(M)[[\hbar]] \to \Omega^1(M)[[\hbar]]$$

is necessarily invertible; its inverse is given by

$$\Big(\mathrm{id} + B^\sharp\pi^\sharp\Big)^{-1} = \sum_{n=0}^{\infty}(-1)^n \Big(B^\sharp\pi^\sharp\Big)^n,$$

which gives a well-defined formal series in $\hbar$ since $\pi = 0 \mod \hbar$.

**Proposition 3.7** *Let $\pi \in \hbar\mathcal{X}^2(M)[[\hbar]]$ and $B \in \Omega^2(M)[[\hbar]]$. Then:*

1. *There exists a unique $\mathfrak{a}(B,\pi) \in \hbar\mathcal{X}^2(M)[[\hbar]]$ such that $\mathfrak{a}(B,\pi)^\sharp = \pi^\sharp \circ (\mathrm{id} + B^\sharp\pi^\sharp)^{-1}$.*

2. *If $dB = 0$ and $[\pi,\pi]_{SN} = 0$, then $[\mathfrak{a}(B,\pi), \mathfrak{a}(B,\pi)]_{SN} = 0$.*

Proof. Let $T = \pi^\sharp \circ (\mathrm{id} + B^\sharp\pi^\sharp)^{-1}$ and set $\xi' = (\mathrm{id} + B^\sharp\pi^\sharp)^{-1}(\xi)$, for $\xi \in \Omega^1(M)$. Then

$$(T(\xi), \xi) = (\pi^\sharp(\xi'), \xi' + i_{\pi^\sharp(\xi')}B) = \lambda_B(\pi^\sharp(\xi'), \xi').$$

Lemma 3.6 implies that

$$\langle (T(\xi), \xi), (T(\eta), \eta) \rangle = \Big\langle (\pi^\sharp(\xi'), \xi'), (\pi^\sharp(\eta'), \eta') \Big\rangle = 0,$$

so the first statement follows from Lemma 3.4. If $dB = 0$, we can use Lemmas 3.5 and 3.6 to conclude that $\mathrm{Jac}_\pi(f,g,h) = \mathrm{Jac}_{\mathfrak{a}(B,\pi)}(f,g,h)$ for all $f,g,h \in \mathcal{O}(M)$. The second statement easily follows from Prop. 3.2. □

Since $\lambda_{B+B'} = \lambda_B(\lambda_{B'})$, an immediate consequence of Prop. 3.7 is that the operation

$$\pi \mapsto \mathfrak{a}(B,\pi),$$

for $\pi \in \hbar\mathcal{X}^2(M)[[\hbar]]$ and $B \in \Omega^2(M)[[\hbar]]$, defines an action of the abelian group of closed formal 2-forms $\Omega^2_{cl}(M)[[\hbar]]$ on formal Poisson structures. We will now see that this action descends to an action of $H^2(M,\mathbb{C})[[\hbar]]$ on the set $\mathrm{FPoiss}(M)$. For that, it will be convenient to view $\mathfrak{a}(B,\pi)$ as a solution of a formal differential equation.

Let us consider the space $\hbar(\mathcal{X}^\bullet(M)[t])[[\hbar]]$ of formal power series in $\hbar$ with coefficients being polynomials in $t$. An element $\pi_t \in \hbar(\mathcal{X}^2(M)[t])[[\hbar]]$ defines, as in (3.12), a map $\pi_t^\sharp : \Omega^\bullet(M)[[\hbar]] \to \hbar(\mathcal{X}^\bullet(M)[t])[[\hbar]]$.

**Lemma 3.8** *Given a formal bivector field $\pi \in \hbar\mathcal{X}^2(M)[[\hbar]]$ and a 2-form $B \in \Omega^2(M)[[\hbar]]$, then $\pi_t = \mathfrak{a}(tB,\pi) \in \hbar(\mathcal{X}^2(M)[t])[[\hbar]]$ is the unique solution to the formal differential equation*

$$\frac{d}{dt}\pi_t = \pi_t^\sharp(B), \quad \pi_t\Big|_{t=0} = \pi, \tag{3.21}$$

*In particular, $\mathfrak{a}(B,\pi) = \pi_t|_{t=1}$, where $\pi_t$ is the unique solution to (3.21).*

Proof. The fact that (3.21) admits a unique solution follows from Prop. A.1 in Appendix A. Note that $\pi_t$ is a solution to (3.21) if and only if $\pi_t^\sharp$ satisfies

$$\frac{d}{dt}\pi_t^\sharp = -\pi_t^\sharp \circ B^\sharp \circ \pi_t^\sharp,$$

with initial condition $\pi_t^\sharp\Big|_{t=0} = \pi^\sharp$. A direct computation shows that

$$\frac{d}{dt}\pi^\sharp(\mathrm{id} + tB^\sharp\pi^\sharp)^{-1} = -\pi^\sharp(\mathrm{id} + tB^\sharp\pi^\sharp)^{-1}B^\sharp\pi^\sharp(\mathrm{id} + tB^\sharp\pi^\sharp)^{-1},$$

so the result follows. □

Let $\pi \in \hbar\mathcal{X}^2(M)[[\hbar]]$ be a formal Poisson structure, let $X(t) \in \hbar(\mathcal{X}^1(M)[t])[[\hbar]]$, and consider the equation

$$\frac{d}{dt}\pi(t) = [\pi(t), X(t)]_{SN}, \quad \pi(0) = \pi \tag{3.22}$$

in $\hbar(\mathcal{X}^2(M)[t])[[\hbar]]$. The following result is proven (in more generality) in Section B.3 of Appendix B.

**Lemma 3.9** *If $\pi(t) \in \hbar(\mathcal{X}^2(M)[t])[[\hbar]]$ is the solution to* (3.22)*, then $\pi(1)$ satisfies $[\pi(1), \pi(1)]_{SN} = 0$, and $\pi$ and $\pi(1)$ are equivalent formal Poisson structures (i.e., they lie in the same orbit of* (2.44)*).*

We can now prove the main result of this section.

**Proposition 3.10** *The action of closed 2-forms $\Omega^2_{cl}(M)[[\hbar]]$ on the space of formal Poisson structures, $(B, \pi) \mapsto \mathfrak{a}(B, \pi)$, descends to an action*

$$H^2(M, \mathbb{C})[[\hbar]] \times \mathrm{FPoiss}(M) \to \mathrm{FPoiss}(M), \quad ([B], [\pi]) \mapsto [B] \cdot [\pi] := [\mathfrak{a}(B, \pi)], \tag{3.23}$$

*on the set of equivalence classes of formal Poisson structures.*

Proof. Let us first show that for cohomologous 2-forms $B, B' \in \Omega_{cl}(M)[[\hbar]]$, the formal Poisson structures $\mathfrak{a}(B, \pi)$ and $\mathfrak{a}(B', \pi)$ are equivalent. It suffices to show that if $B = d\xi$ is exact, then $\mathfrak{a}(B, \pi)$ is equivalent to $\pi$.

Suppose that $\xi \in \Omega^1(M)[[\hbar]]$, and let $B = d\xi$. We know that $\mathfrak{a}(tB, \pi)$ is a path of formal Poisson structures connecting $\mathfrak{a}(B, \pi)$ and $\pi$, and that it satisfies (3.21). By Corollary 3.3,

$$\mathfrak{a}(tB, \pi)^\sharp(d\xi) = -[\mathfrak{a}(tB, \pi), \mathfrak{a}(tB, \pi)^\sharp(\xi)]_{SN}.$$

So, in this case, equation (3.21) can be rewritten as

$$\frac{d}{dt}\mathfrak{a}(tB, \pi) = -[\mathfrak{a}(tB, \pi), \mathfrak{a}(tB, \pi)^\sharp(\xi)]_{SN}.$$

Now Lemma 3.9 implies that $\pi$ and $\mathfrak{a}(B, \pi)$ are equivalent.

Next, we should prove that if $B \in \Omega^2_{cl}(M)[[\hbar]]$ and the Poisson structures $\pi$ and $\tilde{\pi}$ are equivalent, then so are the formal Poisson structures $\mathfrak{a}(B, \pi)$ and $\mathfrak{a}(B, \tilde{\pi})$. Let us assume that

$$\tilde{\pi} = \exp([\cdot, X]_{SN})\pi, \tag{3.24}$$

for $X \in \hbar\mathcal{X}^1(M)[[\hbar]]$. Since $\pi_t = \mathfrak{a}(tB, \pi)$ is the solution to (3.21), $\exp([\cdot, X]_{SN})\pi_t$ satisfies the differential equation

$$\frac{d}{dt}\exp([\cdot, X]_{SN})\pi_t = \exp([\cdot, X]_{SN})\frac{d}{dt}\pi_t = (\exp([\cdot, X]_{SN})\pi_t)^\sharp(\exp(-\mathcal{L}_X)\, B). \tag{3.25}$$

This equation implies that

$$\mathfrak{a}(\exp(-\mathcal{L}_X)\, B, \tilde{\pi}) = \exp([\cdot, X]_{SN})\mathfrak{a}(B, \pi).$$

On the other hand, the 2-form $\exp(-\mathcal{L}_X)B$ is always cohomologous to $B$, as a consequence of the Cartan-Weil formula for the Lie derivative,

$$\exp(-\mathcal{L}_X)B = \sum_{k=0}^{\infty}\frac{1}{k!}(-di_X)^k B = B - d\left(\sum_{k=1}^{\infty}\frac{1}{k!}i_X(-di_X)^{k-1}B\right).$$

Hence $\mathfrak{a}(\exp(\mathcal{L}_X)B, \tilde{\pi})$ is equivalent to both $\mathfrak{a}(B, \tilde{\pi})$ and $\mathfrak{a}(B, \pi)$. This concludes the proof of the proposition. □

### 3.3 Kontsevich's classes of Morita equivalent star products

As discussed in Section 3.1, the groups $\mathrm{Diff}(M)$ and $\mathrm{Pic}(M)$ naturally act on the space $\mathrm{Def}(M)$ of equivalence classes of star products on a manifold $M$, and their orbits characterize Morita equivalent star products. We now describe the corresponding actions on the moduli space of formal Poisson structure $\mathrm{FPoiss}(M)$, making the bijection

$$\mathcal{K}_* : \mathrm{FPoiss}(M) \to \mathrm{Def}(M)$$

equivariant. In other words, we will describe the equivalence relation in $\mathrm{FPoiss}(M)$ which is quantized to Morita equivalence under the Kontsevich map $\mathcal{K}_*$.

The group $\mathrm{Diff}(M)$ acts on formal Poisson structures in a natural way by push-forward,

$$(\varphi, \pi) \mapsto \varphi_* \pi = \hbar \varphi_* \pi_1 + \hbar^2 \varphi_* \pi_2 + \ldots,$$

and it descends to an action of $\mathrm{Diff}(M)$ on $\mathrm{FPoiss}(M)$. As a result of [18, Thm. 1], the map $\mathcal{K}_*$ respects this action, i.e.,

$$\mathcal{K}_*([\varphi_*]) = [*_\varphi].$$

So we focus on the description of the action $\Phi$ of $\mathrm{Pic}(M)$ on $\mathrm{Def}(M)$ (3.4) in terms of Kontsevich's classes, which is given by the next result.

**Theorem 3.11** *Let $L$ be a line bundle over $M$ representing an element in $\mathrm{Pic}(M)$, and suppose that $[*] = \mathcal{K}_*([\pi])$. The action $\Phi : \mathrm{Pic}(M) \times \mathrm{Def}(M) \to \mathrm{Def}(M)$ satisfies*

$$\Phi_L([*]) = \mathcal{K}_*([\mathfrak{a}(B, \pi)]) \tag{3.26}$$

*where $B \in \Omega^2(M)$ is a curvature 2-form of $L$ (i.e., $B$ represents $2\pi i c_1(L)$, where $c_1(L)$ is the Chern class of $L$).*

This result extends the semi-classical description of $\Phi$ in [4] and, as we will see in Section 4, agrees with [7] in the case of symplectic star products.

Before moving to the proof, we need an auxiliary technical statement. Let us consider an open subset $U \subset M$ for which

$$B\Big|_U = d\theta\,, \qquad \theta \in \Omega^1(U)\,.$$

Due to Corollary 3.3, the restriction of $\pi_t = \mathfrak{a}(tB, \pi)$ to $U$ is the unique solution to

$$\frac{d}{dt}\pi_t = [\pi_t, v^t]_{SN}, \quad \pi_t\Big|_{t=0} = \pi, \tag{3.27}$$

where $v^t = -\pi_t^\sharp(\theta) \in \hbar(\mathcal{X}^1(U)[t])[[\hbar]]$. We use the formality $\mathcal{K}$ to quantize $v^t$ to a series of differential operators $V^t \in \hbar(C^1(\mathcal{O}_U)[t])[[\hbar]]$ given by

$$V^t = \sum_{n=0}^{\infty} \frac{1}{n!}\mathcal{K}_{n+1}(\pi_t, \ldots, \pi_t, v^t). \tag{3.28}$$

Let us define the family of transformations $T^t : \mathcal{O}(U)[[\hbar]] \to \mathcal{O}(U)[[\hbar]]$ as the solution to the differential equation (see Proposition A.1 in Appendix A)

$$\frac{d}{dt}T^t(f) = T^t(V^t(f)), \quad T^t\Big|_{t=0} = \mathrm{id}. \tag{3.29}$$

It is not hard to see that

$$T^t \in \mathrm{id} + \hbar(C^1(\mathcal{O}_U)[t])[[\hbar]]$$

and, in particular,

$$T^t\Big|_{\hbar=0} = \mathrm{id}. \tag{3.30}$$

**Lemma 3.12** *Let $*$ and $*_t$ be the Kontsevich star products associated with $\pi$ and $\pi_t$, as in (2.46). If $T^t$ is the solution of the initial value problem (3.29) then $T^t$ is an equivalence between the star-product algebras $(\mathcal{O}(U)[[\hbar]], *_t)$ and $(\mathcal{O}(U)[[\hbar]], *)$, i.e., $T^t = \mathrm{id} \mod \hbar$ and*

$$T^t(f *_t g) = T^t(f) * T^t(g), \quad \textit{for all} \quad f, g \in \mathcal{O}(U)[[\hbar]].$$

Proof. By definition, $f *_t g = fg + \Pi_t(f, g)$, where

$$\Pi_t = \sum_{n=1}^{\infty} \frac{1}{n!} \mathcal{K}_n(\pi_t, \pi_t, \ldots, \pi_t). \tag{3.31}$$

Using (3.27), we have

$$\frac{d}{dt}\Pi_t = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{K}_{n+1}(\pi_t, \ldots, \pi_t, [\pi_t, v^t]_{SN}) = \partial_{*_t}^{\mathrm{Hoch}}(V^t), \tag{3.32}$$

where the last equality follows from the identity (B.21) in Appendix B; here $\partial_{*_t}^{\mathrm{Hoch}}$ is the Hochschild coboundary operator corresponding to $*_t$. It follows from (3.32) that, for all $f, g \in \mathcal{O}(U)[[\hbar]]$, we have

$$\frac{d}{dt}(f *_t g) = \partial_{*_t}^{\mathrm{Hoch}}(V^t)(f, g) = V^t(f) *_t g + f *_t V^t(g) - V^t(f *_t g). \tag{3.33}$$

Combining this equation with (3.29), we get

$$\frac{d}{dt} T^t(f *_t g) = T^t(V^t(f) *_t g) + T^t(f *_t V^t(g)). \tag{3.34}$$

Therefore the cochain $D^t \in (C^2(\mathcal{O}_U)[t])[[\hbar]]$,

$$D^t(f, g) = T^t(f *_t g) - T^t(f) * T^t(g), \tag{3.35}$$

satisfies the following differential equation:

$$\frac{d}{dt} D^t = D^t(V^t \otimes \mathrm{id} + \mathrm{id} \otimes V^t).$$

Taking into account the initial condition $D^t|_{t=0} = 0$ we deduce that $D^t$ is identically zero. This completes the proof of the lemma. $\square$

Proof of Theorem 3.11. Let $*$ be a Kontsevich star product on $M$ associated with the formal Poisson structure $\pi$. We consider a complex line bundle $L \to M$ equipped with a connection $\nabla^L$, and let $B \in \Omega^2(M)$ be the curvature of $\nabla^L$. We denote by $*_t$ the Kontsevich star product of $\pi_t = \mathfrak{a}(tB, \pi)$. We must show that

$$\Phi_L([*]) = [*_1],$$

and for that we will use the local criterium proved in Proposition 3.1.

Let us consider a cover $\{U_\alpha\}$ of $M$ by contractible open subsets with contractible intersections $U_\alpha \cap U_\beta$. We fix a set of local trivializations of $L$, defining transition functions $g_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$. Then $\nabla^L$ is described by a collection of connection 1-forms $\theta_\alpha \in \Omega^1(U_\alpha)$, satisfying

$$\theta_\beta - \theta_\alpha = g_{\alpha\beta}^{-1} dg_{\alpha\beta}, \qquad d\theta_\alpha = B|_{U_\alpha}. \tag{3.36}$$

By Lemma 3.12, we know that over each $U_\alpha$ there is an equivalence

$$T_\alpha^t(f *_t g) = T_\alpha^t(f) * T_\alpha^t(g)$$

for $f, g \in \mathcal{O}(U_\alpha)[[\hbar]]$. By (3.36), the difference between the local vector fields $v_\alpha^t = -\pi_t^\sharp(\theta_\alpha)$ and $v_\beta^t = -\pi_t^\sharp(\theta_\beta)$ is hamiltonian:

$$v_\alpha^t - v_\beta^t = \pi_t^\sharp(d\log(g_{\alpha\beta})) = -[\pi_t, \log(g_{\alpha\beta})]_{SN}.$$

Note that $\log(g_{\alpha\beta})$ is well-defined since $U_\alpha \cap U_\beta$ is contractible. It follows that the corresponding operators (3.28) satisfy

$$V_\alpha^t - V_\beta^t = -\sum_{n=0}^{\infty} \frac{1}{n!}\mathcal{K}_{n+1}(\pi_t, \ldots, \pi_t, [\pi_t, \log(g_{\alpha\beta})]_{SN}).$$

By (B.21), we can write

$$V_\alpha^t - V_\beta^t = -\partial_{*_t}^{\mathrm{Hoch}} h_{\alpha\beta}(t) = \mathrm{ad}_{*_t}(h_{\alpha\beta}(t)), \tag{3.37}$$

where $\partial_{*_t}^{\mathrm{Hoch}}$ denotes the Hochschild coboundary operator (1.4) of $*_t$, $\mathrm{ad}_{*_t}(f)(g) := f *_t g - g *_t f$, and

$$h_{\alpha\beta}(t) = \sum_{n=0}^{\infty} \frac{1}{n!}\mathcal{K}_{n+1}(\pi_t, \ldots, \pi_t, \log(g_{\alpha\beta})). \tag{3.38}$$

We will be interested in the operators

$$T_\alpha = T_\alpha^t|_{t=1},$$

which are local equivalences between $*_1$ and $*$. According to Proposition 3.1, to prove the theorem it suffices to define deformed transition functions $G_{\alpha\beta} = g_{\alpha\beta} \mod \hbar$, satisfying the cocycle conditions (3.5) as well as

$$T_\alpha(T_\beta)^{-1}(f) = G_{\alpha\beta} * f * G_{\alpha\beta}^{-1} \quad \text{for all} \quad f \in \mathcal{O}(U_\alpha \cap U_\beta)[[\hbar]]. \tag{3.39}$$

We will do that by constructing a family of functions $G_{\alpha\beta}(t)$ that will satisfy the desired properties for $t = 1$.

**Claim 3.13** *The operator $T_\alpha^t(T_\beta^t)^{-1}$ is a self-equivalence of $*$ satisfying*

$$\frac{d}{dt}T_\alpha^t(T_\beta^t)^{-1} = \mathrm{ad}_*(T_\alpha^t h_{\alpha\beta})T_\alpha^t(T_\beta^t)^{-1}, \quad T_\alpha^t(T_\beta^t)^{-1}|_{t=0} = \mathrm{id}.$$

Proof. It is clear that $T_\alpha^t(T_\beta^t)^{-1} = \mathrm{id} \mod \hbar$, that it is an automorphism of $(\mathcal{O}(U_\alpha \cap U_\beta)[[\hbar]], *)$, and that it equals id when $t = 0$. Since $T_\alpha^t$ satisfies (3.29), differentiating the identity $T_\alpha^t(T_\alpha^t)^{-1} = \mathrm{id}$ implies that

$$\frac{d}{dt}(T_\alpha^t)^{-1} = -V_\alpha^t(T_\alpha^t)^{-1}.$$

Using (3.37), we obtain

$$\frac{d}{dt}T_\alpha^t(T_\beta^t)^{-1} = T_\alpha^t V_\alpha^t(T_\beta^t)^{-1} - T_\alpha^t V_\beta^t(T_\beta^t)^{-1} = T_\alpha^t(\mathrm{ad}_{*_t}(h_{\alpha\beta}))(T_\beta^t)^{-1}.$$

Since $T_\alpha^t$ is an algebra homomorphism from $(\mathcal{O}(U_\alpha)[[\hbar]], *_t)$ to $(\mathcal{O}(U_\alpha)[[\hbar]], *)$, we have the identity $T_\alpha^t \mathrm{ad}_{*_t}(a) = \mathrm{ad}_*(T_\alpha^t a)T_\alpha^t$, from which the claim follows. $\triangledown$

Using Prop. A.4 in Appendix A, we can define the family of functions $G_{\alpha\beta}(t)$ as the unique solution to the differential equation

$$\frac{d}{dt}G_{\alpha\beta}(t) = (T_\alpha^t h_{\alpha\beta}(t)) * G_{\alpha\beta}(t), \quad G_{\alpha\beta}(0) = 1. \tag{3.40}$$

The solution of (3.40) is $*$-invertible, and its $*$-inverse satisfies

$$\frac{d}{dt}(G_{\alpha\beta}(t))^{-1} = -(G_{\alpha\beta}(t))^{-1} * (T^t_\alpha h_{\alpha\beta}(t)). \tag{3.41}$$

From (3.38), we see that $h_{\alpha\beta}(t) = c_{\alpha\beta} \mod \hbar$, where $c_{\alpha\beta} = \log(g_{\alpha\beta})$. When $\hbar = 0$, the differential equation (3.40) becomes

$$\frac{d}{dt}G^{(0)}_{\alpha\beta}(t) = c_{\alpha\beta}G^{(0)}_{\alpha\beta}(t),$$

where

$$G^{(0)}_{\alpha\beta}(t) = G_{\alpha\beta}(t)\Big|_{\hbar=0}.$$

Hence $G^0_{\alpha\beta}(t) = e^{tc_{\alpha\beta}}$ and, in particular

$$G_{\alpha\beta}(1) = g_{\alpha\beta} \mod \hbar. \tag{3.42}$$

**Claim 3.14** *If* $G_{\alpha\beta}(t)$ *is a solution to* (3.40) *then*

$$T^t_\alpha(T^t_\beta)^{-1}(f) = G_{\alpha\beta}(t) * f * G^{-1}_{\alpha\beta}(t) \quad \textit{for all} \quad f \in \mathcal{O}(U_\alpha \cap U_\beta)[[\hbar]]. \tag{3.43}$$

Proof. Let $\mathrm{Ad}_*(G_{\alpha\beta}(t))$ be the conjugation operator with respect to $*$,

$$\mathrm{Ad}_*(G_{\alpha\beta}(t))(f) = G_{\alpha\beta}(t) * f * G_{\alpha\beta}(t)^{-1},$$

for $f \in \mathcal{O}(U_\alpha \cap U_\beta)[[\hbar]]$. Then from (3.40) and (3.41) we see that

$$\begin{aligned}\frac{d}{dt}\mathrm{Ad}_*(G_{\alpha\beta}(t))(a) &= (T^t_\alpha h_{\alpha\beta}(t)) * G_{\alpha\beta}(t) * f * G_{\alpha\beta}(t)^{-1} - G_{\alpha\beta}(t) * f * (G_{\alpha\beta}(t))^{-1} * (T^t_\alpha h_{\alpha\beta}(t)) \\ &= \mathrm{ad}_*(T^t_\alpha h_{\alpha\beta}(t))\mathrm{Ad}_*(G_{\alpha\beta}(t))(f)\end{aligned}$$

Since $\mathrm{Ad}_*(G_{\alpha\beta}(t))|_{t=0} = \mathrm{id}$, we conclude from Claim 3.13 that $T^t_\alpha(T^t_\beta)^{-1} = \mathrm{Ad}_*(G_{\alpha\beta}(t))$. $\nabla$

The next claim implies that the functions

$$G_{\alpha\beta} := G_{\alpha\beta}(1)$$

satisfy the desired cocycle conditions.

**Claim 3.15** *The following identities hold:* $G_{\alpha\alpha} = 1$, $G_{\alpha\beta} * G_{\beta\alpha} = 1$, $G_{\alpha\beta} * G_{\beta\gamma} * G_{\gamma\alpha} = 1$.

Proof. Since $h_{\alpha\alpha}(t) = 0$, it is clear from (3.40) that $G_{\alpha\alpha} = 1$. For the second identity, we have

$$\begin{aligned}\frac{d}{dt}(G_{\alpha\beta}(t) * G_{\beta\alpha}(t)) &= (T^t_\alpha h_{\alpha\beta}(t)) * G_{\alpha\beta}(t) * G_{\beta\alpha}(t) + \\ &\quad G_{\alpha\beta}(t) * (T^t_\beta h_{\beta\alpha}(t)) * G_{\beta\alpha}(t) \\ &= (T^t_\alpha h_{\alpha\beta}(t)) * G_{\alpha\beta}(t) * G_{\beta\alpha}(t) + \\ &\quad G_{\alpha\beta}(t) * G_{\beta\alpha}(t) * (T^t_\alpha h_{\beta\alpha}(t)) * G^{-1}_{\beta\alpha}(t) * G_{\beta\alpha}(t) \\ &= \mathrm{ad}_*(T^t_\alpha h_{\alpha\beta}(t))(G_{\alpha\beta}(t) * G_{\beta\alpha}(t)),\end{aligned}$$

where we have used (3.40), (3.43) and $h_{\alpha\beta}(t) = -h_{\beta\alpha}(t)$. Note that $G_{\alpha\beta}(t) * G_{\beta\alpha}(t) = 1$ is the unique solution with initial condition $G_{\alpha\beta}(0) * G_{\beta\alpha}(0) = 1$.

We proceed similarly to prove that $G_{\alpha\beta} * G_{\beta\gamma} * G_{\gamma\alpha} = 1$. Using (3.40) and (3.43), we obtain:

$$\begin{aligned}\frac{d}{dt}G_{\alpha\beta}(t) * G_{\beta\gamma}(t) * G_{\gamma\alpha}(t) &= (T^t_\alpha h_{\alpha\beta}(t)) * G_{\alpha\beta}(t) * G_{\beta\gamma}(t) * G_{\gamma\alpha}(t) \\ &\quad + G_{\alpha\beta}(t) * (T^t_\beta h_{\beta\gamma}(t)) * G_{\beta\gamma}(t) * G_{\gamma\alpha}(t) \\ &\quad + G_{\alpha\beta}(t) * G_{\beta\gamma}(t) * (T^t_\gamma h_{\gamma\alpha}(t)) * G_{\gamma\alpha}(t) \\ &= (T^t_\alpha h_{\alpha\beta}(t)) * G_{\alpha\beta}(t) * G_{\beta\gamma}(t) * G_{\gamma\alpha}(t) \\ &\quad + G_{\alpha\beta}(t) * G^{-1}_{\alpha\beta}(t) * (T^t_\alpha h_{\beta\gamma}(t)) * G_{\alpha\beta}(t) * G_{\beta\gamma}(t) * G_{\gamma\alpha}(t) \\ &\quad + G_{\alpha\beta}(t) * G_{\beta\gamma}(t) * G^{-1}_{\beta\gamma}(t) * (T^t_\beta h_{\gamma\alpha}(t)) * G_{\beta\gamma}(t) * G_{\gamma\alpha}(t) \\ &= (T^t_\alpha h_{\alpha\beta}(t)) * G_{\alpha\beta}(t) * G_{\beta\gamma}(t) * G_{\gamma\alpha}(t) \\ &\quad + (T^t_\alpha h_{\beta\gamma}(t)) * G_{\alpha\beta}(t) * G_{\beta\gamma}(t) * G_{\gamma\alpha}(t) \\ &\quad + (T^t_\alpha h_{\gamma\alpha}(t)) * G_{\alpha\beta}(t) * G_{\beta\gamma}(t) * G_{\gamma\alpha}(t) \\ &= \left(T^t_\alpha \left(h_{\alpha\beta}(t) + h_{\beta\gamma}(t) + h_{\gamma\alpha}(t)\right)\right) * G_{\alpha\beta}(t) * G_{\beta\gamma}(t) * G_{\gamma\alpha}(t)\end{aligned} \tag{3.44}$$

Using Proposition 2.3, we see that

$$h_{\alpha\beta}(t) + h_{\beta\gamma}(t) + h_{\gamma\alpha}(t) = \sum_{n=0}^{\infty} \frac{1}{n!}\mathcal{K}_{n+1}\left(\pi_t, \ldots, \pi_t, c_{\alpha\beta} + c_{\beta\gamma} + c_{\gamma\alpha}\right) = c_{\alpha\beta} + c_{\beta\gamma} + c_{\gamma\alpha},$$

where $c_{\alpha\beta} = \log(g_{\alpha\beta})$. Since $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$, we have that $c_{\alpha\beta} + c_{\beta\gamma} + c_{\gamma\alpha} = 2\pi i n_{\alpha\beta\gamma}$, for $n_{\alpha\beta\gamma} \in \mathbb{Z}$ (note that $n_{\alpha\beta\gamma}$ is the Čech cocycle in $\check{H}^2(M,\mathbb{Z})$ representing the line bundle $L$). Hence the unique solution of (3.44) with initial condition 1 is

$$G_{\alpha\beta}(t) * G_{\beta\gamma}(t) * G_{\gamma\alpha}(t) = e^{2\pi i n_{\alpha\beta\gamma} t}. \tag{3.45}$$

In particular, for $t = 1$ we have $G_{\alpha\beta} * G_{\beta\gamma} * G_{\gamma\alpha} = 1$. ▽
This finishes the proof of Theorem 3.11. □

# 4 Fedosov's classes versus Kontsevich's classes

In this section we focus on formal Poisson structures $\pi = \hbar\pi_1 + \hbar^2\pi_2 + \ldots$ on a manifold $M$ for which the leading term $\pi_1 \in \mathcal{X}^2(M)$ is a nondegenerate bivector field, i.e., we assume that the associated vector-bundle map $\pi_1^\sharp : T^*M \to TM$, $\pi^\sharp(\xi) = i_\xi\pi$, is an isomorphism. In this case, $\pi_1$ corresponds to a symplectic form $\omega_{-1} \in \Omega^2(M)$, uniquely defined by

$$i_{\pi_1^\sharp(\xi)}\omega_{-1} = \xi, \quad \forall\, \xi \in \Omega^1(M),$$

and the Kontsevich star product (2.46) defines a deformation quantization of the symplectic manifold $(M, \omega_{-1})$.

The fact that $\pi_1$ is nondegenerate implies, more generally, that the $\mathcal{O}(M)[[\hbar]]$-linear map $\pi^\sharp : \Omega^1(M)[[\hbar]] \to \hbar\mathcal{X}^1(M)[[h]]$ (see (3.11)) is an isomorphism, and its inverse uniquely defines a formal series of 2-forms

$$\omega = \frac{1}{\hbar}\omega_{-1} + \omega_0 + \hbar\omega_1 + \hbar^2\omega_2 + \cdots. \tag{4.1}$$

The integrability condition $[\pi,\pi]_{SN} = 0$ is equivalent to $d\omega_j = 0$, $\forall\, j = -1, 0, 1, \ldots$. This gives us a 1-1 correspondence between formal Poisson structures $\pi = \hbar\pi_1 + \ldots$ for which $\pi_1$ is nondegenerate and series of closed 2-forms $\omega$ as in (4.1) for which $\omega_{-1}$ is symplectic (cf. [40, Sec. 3]). Furthermore, under this correspondence, the action of the group (2.43) boils down to the action

$$\omega \mapsto \omega + d\,\varepsilon,$$

where $\varepsilon = \varepsilon_0 + \hbar\varepsilon_1 + \ldots \in \Omega^1(M)[[\hbar]]$ is an arbitrary formal power series of 1-forms. In particular, for a fixed nondegenerate Poisson structure $\pi_1$, the set of equivalence classes $[\pi] \in \mathrm{FPoiss}(M)$ such that $\pi = \hbar\pi_1 + \ldots$ is in bijective correspondence with $H^2(M, \mathbb{C})[[\hbar]]$ (cf. [40, Prop. 13]).

On the other hand, Fedosov's construction [27] leads to a parametrization of the set of equivalence classes of star products on a given symplectic manifold $(M, \omega)$ by elements $\sum_{j=0}^{\infty} \hbar^j [\omega_j] \in H^2(M, \mathbb{C})[[\hbar]]$ (see e.g. [3, 15, 50]); the elements $\frac{1}{\hbar}[\omega] + \sum_{j=0}^{\infty} \hbar^j [\omega_j]$ are known as *Fedosov classes.*

**Theorem 4.1** *Let $\pi = \hbar\pi_1 + \hbar^2\pi_2 + \ldots$ be a formal Poisson structure such that $\pi_1$ is a nondegenerate bivector field, and let $\omega$ be the associated formal series of closed 2-forms as in* (4.1)*. Then the Fedosov class of the Kontsevich star product (2.46) of $\pi$ is represented by $\omega$.*

It directly follows from this result that the description of Morita equivalent star products in terms of Kontsevich's classes of Theorem 3.11 reduces, in the symplectic case, to the description of [7, Theorem 3.1] in terms of Fedosov's classes.

The plan of the proof is depicted on the following diagram:

$$\begin{array}{ccccc} \text{Kontsevich's} & & \text{modified Fedosov's} & & \text{original Fedosov's} \\ \text{star product } *_K & \sim & \text{star product } \tilde{*} & = & \text{star product } *_F \end{array} \tag{4.2}$$

This diagram will be turned into a proof of Theorem 4.1 in this section. In Subsection 4.1 we construct the modified Fedosov's star product $\tilde{*}$. The difference between the constructions of $\tilde{*}$ and the original Fedosov's star product $*_F$ is that for $\tilde{*}$ we use the fiberwise multiplication (4.3), which involves the whole series $\pi$, whereas to construct $*_F$ we use only the first term $\hbar\pi_1$. In Subsection 4.2 we introduce a version of the Emmrich-Weinstein differential. Using this differential in Subsection 4.3, we show that Kontsevich's star product $*_K$ (2.46) is equivalent to $\tilde{*}$. Finally, in Subsection 4.4, we prove that $\tilde{*}$ coincides with the original Fedosov's star product $*_F$ whose equivalence class is represented by $\omega$ (4.1). In following these steps, it will be important to recall (see Subsection 2.3) that the bijective correspondence between equivalence classes of Maurer-Cartan elements induced from the direct $L_\infty$ quasi-isomorphism $\mathcal{K}$ (2.32) and the sequence of quasi-isomorphisms in (2.31) coincide and are independent of the choice of the connection/Fedosov's differential.

## 4.1 The modified Fedosov construction

In this subsection we consider a modification of Fedosov's construction based on the following associative product on the sheaf $\mathcal{S}M[[\hbar]]$ of $\mathcal{O}_M[[\hbar]]$-modules:

$$a_1 \,\tilde{\diamond}\, a_2 = a_1 \exp\left( \pi^{ij}(x) \frac{\overleftarrow{\partial}}{\partial y^i} \frac{\overrightarrow{\partial}}{\partial y^i} \right) a_2, \tag{4.3}$$

where $\pi = \hbar\pi_1 + \ldots$ is a formal Poisson structure (in particular, the coefficients $\pi^{ij}$ are series in $\hbar$) and $\pi_1$ is nondegenerate. Recall that the sheaf $\mathcal{S}M[[\hbar]]$ is equipped with the descending filtration of Remark 2.5, and one can check that the product (4.3) is compatible with this filtration. We may view the product (4.3) as a quantization of the fiberwise Poisson structure

$$\pi_{\mathrm{fib}} = \pi^{ij}(x) \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j}. \tag{4.4}$$

Since $\pi_1$ is a non-degenerate Poisson bivector field, there exists a torsion-free connection form

$$dx^j (\Gamma_\hbar)^i_{jk}(x) = dx^j \Gamma^i_{jk}(x) + dx^j \hbar (\Gamma_1)^i_{jk}(x) + dx^j \hbar^2 (\Gamma_2)^i_{jk}(x) + \cdots,$$

satisfying the compatibility condition

$$\nabla\{a_1, a_2\}_{\pi_{\mathrm{fib}}} = \{\nabla a_1, a_2\}_{\pi_{\mathrm{fib}}} + \{a_1, \nabla a_2\}_{\pi_{\mathrm{fib}}}, \tag{4.5}$$

where

$$\nabla = dx^i \frac{\partial}{\partial x^i} - dx^i (\Gamma_\hbar)_{ij}^k(x) y^j \frac{\partial}{\partial y^k}, \tag{4.6}$$

and, for $a_1, a_2$ local sections of $\mathcal{S}M[[\hbar]]$,

$$\{a_1, a_2\}_{\pi_{\text{fib}}} = \pi^{ij}(x) \frac{\partial a_1}{\partial y^i} \frac{\partial a_2}{\partial y^j} \tag{4.7}$$

is the fiberwise Poisson bracket on $\mathcal{S}M[[\hbar]]$ coming from the fiberwise Poisson structure $\pi_{\text{fib}}$. Note that (4.5) implies that the connection $\nabla$ is also compatible with the product (4.3), i.e.,

$$\nabla(a_1 \mathbin{\tilde{\diamond}} a_2) = (\nabla a_1) \mathbin{\tilde{\diamond}} a_2 + a_1 \mathbin{\tilde{\diamond}} (\nabla a_2). \tag{4.8}$$

In general, the connection $\nabla$ is not flat. In fact,

$$\nabla^2 = \frac{1}{\hbar}[R, \, ]_{\tilde{\diamond}},$$

where

$$R = \frac{1}{2} dx^i \, dx^j R_{ij\,kl}(x) y^k y^l, \tag{4.9}$$

$R_{ij\,kl}(x) = \hbar \omega_{km}(x) R_{ij\,\,l}^{\,\,\,m}(x)$, and $R_{ij\,\,l}^{\,\,\,m}(x)$ are the components (possibly depending on $\hbar$) of the curvature tensor. Even though $R$ is not vanishing in general, we can modify $\nabla$ to the following flat connection:

$$D_\hbar^F = \nabla - \delta + \frac{1}{\hbar}[r, \cdot]_{\tilde{\diamond}}, \tag{4.10}$$

where $r$ is an element of $\Omega^1(M, \mathcal{F}^3\mathcal{S}M[[\hbar]])$ obtained by iterating the equation

$$r = \delta^{-1} R + \delta^{-1} \left( \nabla r + \frac{1}{2\hbar}[r, r]_{\tilde{\diamond}} \right). \tag{4.11}$$

It can be shown that by iterating (4.11) we get an element $r \in \Omega^1(M, \mathcal{F}^3\mathcal{S}M[[\hbar]])$ satisfying

$$R + \nabla r - \delta r + \frac{1}{2\hbar}[r, r]_{\tilde{\diamond}} = 0, \tag{4.12}$$

and this equation implies that $(D_\hbar^F)^2 = 0$. Notice that the derivation $\delta$ (2.5) of the algebra $\Omega^\bullet(M, \mathcal{S}M[[\hbar]])$ is inner. More precisely,

$$\delta = [dx^i \omega_{ij}(x, \hbar) y^j, \cdot]_{\tilde{\diamond}}. \tag{4.13}$$

As a consequence, the differential (4.10) can be rewritten as

$$D_\hbar^F = \nabla + \frac{1}{\hbar}[b, \cdot]_{\tilde{\diamond}}, \tag{4.14}$$

where

$$b = r - \hbar \, dx^i \omega_{ij}(x, \hbar) y^j. \tag{4.15}$$

It follows from (4.12) that the element $b$ satisfies

$$\frac{1}{\hbar} R + \frac{1}{\hbar} \nabla b + \frac{1}{2\hbar^2}[b, b]_{\tilde{\diamond}} = -\omega. \tag{4.16}$$

As already used in Section 2, we have the obvious map

$$\sigma : \Gamma(M, \mathcal{S}M)[[\hbar]] \cap \ker D_\hbar^F \longrightarrow \mathcal{O}(M)[[\hbar]], \quad \sigma(c) = c\Big|_{y^i=0}, \tag{4.17}$$

from the $\mathbb{C}[[\hbar]]$-module of $D^F_\hbar$-flat sections of $\mathcal{S}M[[\hbar]]$ to the $\mathbb{C}[[\hbar]]$-module $\mathcal{O}(M)[[\hbar]]$. The map (4.17) turns out to be an isomorphism, and the inverse map

$$\tilde{\tau} : \mathcal{O}(M)[[\hbar]] \longrightarrow \Gamma(M, \mathcal{S}M)[[\hbar]] \cap \ker D^F_\hbar . \tag{4.18}$$

is defined by the following iterative procedure:

$$\tilde{\tau}(f) = f + \delta^{-1}(\nabla\tilde{\tau}(f) + [r, \tilde{\tau}(f)]_{\tilde{\diamond}}), \tag{4.19}$$

where $f \in \mathcal{O}(M)[[\hbar]]$ and the iteration in (4.19) goes with respect to the filtration (2.37).

Using the isomorphism (4.18), we obtain the *modified Fedosov star product* $\tilde{*}$:

$$f_1 \,\tilde{*}\, f_2 = \sigma(\tilde{\tau}(f_1) \,\tilde{\diamond}\, \tilde{\tau}(f_2)), \tag{4.20}$$

where $f_1, f_2 \in \mathcal{O}(M)[[\hbar]]$. In Subsection 4.4 we will show that $\tilde{*}$ coincides with the original Fedosov's star product whose equivalence class is represented by $\omega$ (4.1).

## 4.2 The Emmrich-Weinstein differential

The compatibility between the "deformed" connection $\nabla$ (4.6) and the fiberwise Poisson bracket $\{\cdot, \cdot\}_{\pi_{\text{fib}}}$ (4.7) allows us to construct the Emmrich-Weinstein differential [25]

$$D^{\text{EW}} = \nabla - \delta + \frac{1}{\hbar}\{r^{cl}, \cdot\}_{\pi_{\text{fib}}}, \tag{4.21}$$

where $r^{cl}$ is the element of $\Omega^1(M, \mathcal{F}^3\mathcal{S}M[[\hbar]])$ obtained by iterating the equation

$$r^{cl} = \delta^{-1}R + \delta^{-1}\left(\nabla r^{cl} + \frac{1}{2\hbar}\{r^{cl}, r^{cl}\}_{\pi_{\text{fib}}}\right), \tag{4.22}$$

where $R$ is defined in (4.9). By iterating (4.22), we obtain an element $r^{cl} \in \Omega^1(M, \mathcal{F}^3\mathcal{S}M[[\hbar]])$ satisfying the equation

$$R + \nabla r^{cl} - \delta r^{cl} + \frac{1}{2\hbar}\{r^{cl}, r^{cl}\}_{\pi_{\text{fib}}} = 0, \tag{4.23}$$

and this equation implies that $(D^{\text{EW}})^2 = 0$. Similarly to equation (4.13), we have

$$\delta = \{dx^i\omega_{ij}(x, \hbar)y^j, \cdot\}_{\pi_{\text{fib}}}.$$

Therefore we can rewrite (4.21) as

$$D^{\text{EW}} = \nabla + \frac{1}{\hbar}\{b^{cl}, \cdot\}_{\pi_{\text{fib}}}, \tag{4.24}$$

where

$$b^{cl} = -\hbar dx^i\omega_{ij}(x, \hbar)y^j + r^{cl}. \tag{4.25}$$

Equation (4.23) implies that

$$\frac{1}{\hbar}R + \frac{1}{\hbar}\nabla b^{cl} + \frac{1}{2\hbar^2}\{b^{cl}, b^{cl}\}_{\pi_{\text{fib}}} = -\omega\,. \tag{4.26}$$

We remark that the differential $D^{\text{EW}}$ (4.21) differs from the original one introduced by Emmrich and Weinstein in [25, Sect. 8]. The fiberwise Poisson bracket considered in [25] does not involve $\hbar$, whereas $\pi_{\text{fib}}$ (4.4) is a series in $\hbar$. So, even though the recursion for $r^{cl}$ looks "classical", the element $r^{cl}$ does contain higher orders of $\hbar$. In particular, $r^{cl}$ is not just obtained by setting $\hbar = 0$ in the element $r$ (4.11). However we have the following:

**Proposition 4.2** *Let $r$ and $r^{cl}$ be the elements of $\Omega^1(M, \mathcal{F}^3\mathcal{S}M[[\hbar]])$ defined by iterating equations (4.11) and (4.22), respectively. Then*

$$r - r^{cl} = 0 \mod \hbar^2. \tag{4.27}$$

Proof. Let $r_k$ (resp. $r^{cl}_k$) be the approximation of $r$ (resp. $r^{cl}$) which we obtain on the $k$-th step of the iterative procedure (4.11) (resp. (4.22)). Namely, $r_0 = r^{cl}_0 = \delta^{-1}R$ and $r_k$ is related to $r_{k-1}$ via the equation

$$r_k = \delta^{-1}R + \delta^{-1}\left(\nabla r_{k-1} + \frac{1}{2\hbar}[r_{k-1}, r_{k-1}]_{\tilde{\circ}}\right), \tag{4.28}$$

while $r^{cl}_k$ is related to $r^{cl}_{k-1}$ via the equation

$$r^{cl}_k = \delta^{-1}R + \delta^{-1}\left(\nabla r^{cl}_{k-1} + \frac{1}{2\hbar}\{r^{cl}_{k-1}, r^{cl}_{k-1}\}_{\pi_{\text{fib}}}\right). \tag{4.29}$$

Let us show by induction that

$$r_k - r^{cl}_k = 0 \mod \hbar^2 \tag{4.30}$$

for all $k$. For $k = 0$, this is obvious. To perform the inductive step, we observe that

$$[a_1, a_2]_{\tilde{\circ}} - \{a_1, a_2\}_{\pi_{\text{fib}}} = 0 \mod \hbar^3 \tag{4.31}$$

for all $a_1, a_2 \in \Gamma(M, \mathcal{S}M)[[\hbar]]$. This observation and the inductive hypothesis imply that

$$\begin{aligned} r_k - r^{cl}_k &= \delta^{-1}\left(\nabla(r_{k-1} - r^{cl}_{k-1}) + \frac{1}{2\hbar}[r_{k-1}, r_{k-1}]_{\tilde{\circ}} - \frac{1}{2\hbar}\{r^{cl}_{k-1}, r^{cl}_{k-1}\}_{\pi_{\text{fib}}}\right) \\ &= \delta^{-1}\left(\frac{1}{2\hbar}[r_{k-1}, r_{k-1}]_{\tilde{\circ}} - \frac{1}{2\hbar}[r^{cl}_{k-1}, r^{cl}_{k-1}]_{\tilde{\circ}}\right) \mod \hbar^2. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\frac{1}{2\hbar}[r_{k-1}, r_{k-1}]_{\tilde{\circ}} - \frac{1}{2\hbar}[r^{cl}_{k-1}, r^{cl}_{k-1}]_{\tilde{\circ}} \\ &\quad = \frac{1}{2\hbar}[r_{k-1}, r_{k-1}]_{\tilde{\circ}} - \frac{1}{2\hbar}[r_{k-1}, r^{cl}_{k-1}]_{\tilde{\circ}} + \frac{1}{2\hbar}[r_{k-1}, r^{cl}_{k-1}]_{\tilde{\circ}} - \frac{1}{2\hbar}[r^{cl}_{k-1}, r^{cl}_{k-1}]_{\tilde{\circ}} \\ &\quad = \frac{1}{2\hbar}[r_{k-1}, (r_{k-1} - r^{cl}_{k-1})]_{\tilde{\circ}} + \frac{1}{2\hbar}[(r_{k-1} - r^{cl}_{k-1}), r^{cl}_{k-1}]_{\tilde{\circ}} = 0 \mod \hbar^2. \end{aligned}$$

Therefore Equation (4.30) holds for all $k$ and the proof is concluded.

□

### 4.3 The Kontsevich star product $*_K$ is equivalent to $\tilde{*}$

Since the differential $D^{\text{EW}}$ (4.21) has the form[4] (2.3), we may use it to construct the Kontsevich star product as in [20]. The class of the star product does not depend on this particular choice of differential due to Theorem 2.6. In particular, we denote by $\tau^{\text{EW}}$ the corresponding map (2.22).

We will use the sequence of $L_\infty$ quasi-isomorphisms (2.31) with $D = D^{\text{EW}}$ to the obtain the Kontsevich star product $*_K$ corresponding to $\pi$. Going through details of this construction, we will produce an equivalence transformation between $*_K$ and $\tilde{*}$ (4.20).

Let us consider the (fiberwise) Poisson-Lichnerowicz differential on $\Omega^\bullet(M, \mathcal{T}^\bullet_{poly})[[\hbar]]$,

$$\partial_{\pi_{\text{fib}}} = [\pi_{\text{fib}}, \cdot]_{SN}, \tag{4.32}$$

[4] See Remark 2.4.

corresponding to the fiberwise Poisson structure $\pi_{\rm fib}$ (4.4). We can then rewrite $D^{\rm EW}$ (4.24) as

$$D^{\rm EW} = \nabla - \frac{1}{\hbar}\partial_{\pi_{\rm fib}}(b^{cl}). \tag{4.33}$$

Combining this last equation with the compatibility between $\nabla$ and $\pi_{\rm fib}$, we conclude that

$$D^{\rm EW}\pi_{\rm fib} = 0. \tag{4.34}$$

In other words, the lift $\tau^{\rm EW}(\pi)$ of the formal Poisson structure (2.41) to a $D^{\rm EW}$-flat section of $\mathcal{T}^2_{poly}[[\hbar]]$ takes the following simple form:

$$\tau^{\rm EW}(\pi) = \pi^{ij}(x)\partial_{y^i}\wedge\partial_{y^j}.$$

An important consequence of this observation is that the components of $\tau^{\rm EW}(\pi)$ do not depend on the fiber coordinates $y$'s.

Following Subsection 2.2, we consider the "tail" of the differential $D^{\rm EW}$,

$$\mu^{EW}_U = -\Gamma_\hbar - \frac{1}{\hbar}\,\partial_{\pi_{\rm fib}} b^{cl}, \tag{4.35}$$

as a Maurer-Cartan element of the DGLA

$$(\Omega^\bullet(U,\mathcal{T}^{\bullet+1}_{poly})[[\hbar]], d, [\cdot,\cdot]_{SN}), \tag{4.36}$$

where $U$ is a coordinate open subset of $M$. Then twisting the $L_\infty$ quasi-isomorphism $K$ (2.28) by $\mu^{EW}_U$, we obtain the $L_\infty$ quasi-isomorphism

$$K^{\mu^{EW}_U} : (\Omega^\bullet(U,\mathcal{T}^{\bullet+1}_{poly})[[\hbar]], D^{\rm EW}, [\,,\,]_{SN}) \succ\!\!\rightarrow (\Omega^\bullet(U, C^{\bullet+1}(\mathcal{S}M))[[\hbar]], D^{\rm EW} + \partial^{\rm Hoch}, [\,,\,]_G). \tag{4.37}$$

As explained in Subsection 2.2, the $L_\infty$ quasi-isomorphism $K^{\mu^{EW}_U}$ does not depend on the choice of local coordinates on $U$. Hence we get a global $L_\infty$ quasi-isomorphism

$$K^{tw} : (\Omega^\bullet(M,\mathcal{T}^{\bullet+1}_{poly})[[\hbar]], D^{\rm EW}, [\,,\,]_{SN}) \succ\!\!\rightarrow (\Omega^\bullet(M, C^{\bullet+1}(\mathcal{S}M))[[\hbar]], D^{\rm EW} + \partial^{\rm Hoch}, [\,,\,]_G). \tag{4.38}$$

Let us denote by $\mu^K$ the Maurer-Cartan element of the DGLA

$$\left(\Omega^\bullet(M, C^{\bullet+1}(\mathcal{S}M))[[\hbar]], D^{\rm EW} + \partial^{\rm Hoch}, [\cdot,\cdot]_G\right) \tag{4.39}$$

obtained from $\tau^{\rm EW}(\pi) = \pi_{\rm fib}$ via the $L_\infty$ quasi-isomorphism $K^{tw}$:

$$\mu^K = \sum_{n=1}^{\infty}\frac{1}{n!}K^{tw}_n(\tau^{\rm EW}(\pi), \tau^{\rm EW}(\pi), \dots, \tau^{\rm EW}(\pi)). \tag{4.40}$$

A simple degree bookkeeping shows that $\mu^K = \mu^K_0 + \mu^K_1 + \mu^K_2$, where $\mu^K_0$ is a 0-form with values in $C^2(\mathcal{S}M)[[\hbar]]$, $\mu^K_1$ is a 1-form with values in $C^1(\mathcal{S}M)[[\hbar]]$, and $\mu^K_2$ is a 2-form with values in $C^0(\mathcal{S}M)[[\hbar]] = \mathcal{S}M[[\hbar]]$. More precisely,

$$\mu^K_0 = \sum_{n=1}^{\infty}\frac{1}{n!}K_n(\pi_{\rm fib}, \pi_{\rm fib}, \dots, \pi_{\rm fib}), \tag{4.41}$$

$$\mu^K_1 = \sum_{n=1}^{\infty}\frac{1}{n!}K_{n+1}\left(\pi_{\rm fib}, \pi_{\rm fib}, \dots, \pi_{\rm fib}, -\Gamma_\hbar - \frac{1}{\hbar}\partial_{\pi_{\rm fib}}(b^{cl})\right), \tag{4.42}$$

$$\mu^K_2 = \sum_{n=1}^{\infty}\frac{1}{n!\,2}K_{n+2}\left(\pi_{\rm fib}, \pi_{\rm fib}, \dots, \pi_{\rm fib}, -\Gamma_\hbar - \frac{1}{\hbar}\partial_{\pi_{\rm fib}}(b^{cl}), -\Gamma_\hbar - \frac{1}{\hbar}\partial_{\pi_{\rm fib}}(b^{cl})\right). \tag{4.43}$$

It is known that Kontsevich's star product corresponding to the constant Poisson structure coincides with the Moyal star product, see e.g. [62]. Therefore, since the components of $\pi_{\rm fib}$ do not depend on the fiber coordinates $y$'s, we conclude that[5]

$$\mu_0^K(a_1, a_2) = a_1 \sum_{n=1}^{\infty} \frac{1}{n!} \left( \pi^{ij}(x) \frac{\overleftarrow{\partial}}{\partial y^i} \frac{\overrightarrow{\partial}}{\partial y^j} \right)^n a_2 \tag{4.44}$$

where $a_1, a_2$ are local sections of $\mathcal{S}M[[\hbar]]$.

Using property P 4 from Subsection 2.2, we simplify $\mu_1^K$ and $\mu_2^K$ as follows:

$$\mu_1^K = -\sum_{n=1}^{\infty} \frac{1}{n!} K_{n+1}\left(\pi_{\rm fib}, \pi_{\rm fib}, \dots, \pi_{\rm fib}, \frac{1}{\hbar}\partial_{\pi_{\rm fib}}(b^{cl})\right), \tag{4.45}$$

$$\mu_2^K = \sum_{n=1}^{\infty} \frac{1}{n!\,2} K_{n+2}\left(\pi_{\rm fib}, \pi_{\rm fib}, \dots, \pi_{\rm fib}, \frac{1}{\hbar}\partial_{\pi_{\rm fib}}(b^{cl}), \frac{1}{\hbar}\partial_{\pi_{\rm fib}}(b^{cl})\right). \tag{4.46}$$

Due to property P 2 from Subsection 2.2, we can write

$$-\frac{1}{\hbar}\partial_{\pi_{\rm fib}}(b^{cl}) + \mu_1^K = -\frac{1}{\hbar}K_1(\partial_{\pi_{\rm fib}}(b^{cl})) - \sum_{n=1}^{\infty} \frac{1}{n!} K_{n+1}\left(\pi_{\rm fib}, \pi_{\rm fib}, \dots, \pi_{\rm fib}, \frac{1}{\hbar}\partial_{\pi_{\rm fib}}(b^{cl})\right)$$
$$= -\sum_{n=0}^{\infty} \frac{1}{\hbar n!} K_{n+1}\left(\pi_{\rm fib}, \pi_{\rm fib}, \dots, \pi_{\rm fib}, \partial_{\pi_{\rm fib}}(b^{cl})\right).$$

In other words,

$$-\frac{1}{\hbar}\partial_{\pi_{\rm fib}}(b^{cl}) + \mu_1^K = -\frac{1}{\hbar}K_1^{\pi_{\rm fib}}(\,\partial_{\pi_{\rm fib}}(b^{cl})\,), \tag{4.47}$$

where $K^{\pi_{\rm fib}}$ is the $L_\infty$ quasi-isomorphism obtained from

$$K : (\Omega^\bullet(M, \mathcal{T}_{poly}^{\bullet+1})[[\hbar]], 0) \succ\!\!\!\longrightarrow (\Omega^\bullet(M, C^{\bullet+1}(\mathcal{S}M))[[\hbar]], \partial^{\rm Hoch})$$

via twisting by $\pi_{\rm fib}$ (4.4). Using (B.21) from Appendix B, the map $K_1^{\pi_{\rm fib}}$ intertwines the Poisson-Lichnerowicz differential $\partial^{\pi_{\rm fib}}$ (4.32) with the Hochschild differential $\partial^{\rm Hoch}_{\tilde\diamond}$ corresponding to the product (4.3). Hence (4.47) can be rewritten as

$$-\frac{1}{\hbar}\partial_{\pi_{\rm fib}}(b^{cl}) + \mu_1^K = -\frac{1}{\hbar}\partial^{\rm Hoch}_{\tilde\diamond} K_1^{\pi_{\rm fib}}(b^{cl}) = \frac{1}{\hbar}[K_1^{\pi_{\rm fib}}(b^{cl}), \cdot]_{\tilde\diamond}. \tag{4.48}$$

Since the components of $\pi_{\rm fib}$ do not depend on the fiberwise coordinates $y$'s, it follows that, for every $n \geq 1$,

$$K_{n+1}(\pi_{\rm fib}, \pi_{\rm fib}, \dots, \pi_{\rm fib}, b^{cl}) = 0$$

Hence

$$K_1^{\pi_{\rm fib}}(b^{cl}) = b^{cl} \tag{4.49}$$

and

$$-\frac{1}{\hbar}\partial_{\pi_{\rm fib}}(b^{cl}) + \mu_1^K = \frac{1}{\hbar}[b^{cl},\ ]_{\tilde\diamond}. \tag{4.50}$$

Let us now find a simpler expression for $\mu_2^K$ (4.46). Property P 5 from Subsection 2.2 implies that

$$K_2\left(\frac{1}{\hbar}\partial_{\pi_{\rm fib}}(b^{cl}), \frac{1}{\hbar}\partial_{\pi_{\rm fib}}(b^{cl})\right) = 0.$$

[5]Due to (4.44), the product $\tilde\diamond$ can be written as $a_1 \,\tilde\diamond\, a_2 = a_1 a_2 + \mu_0^K(a_1, a_2)$.

Hence

$$\mu_2^K = K_2\left(\frac{1}{\hbar}\partial_{\pi_{\rm fib}}(b^{cl}), \frac{1}{\hbar}\partial_{\pi_{\rm fib}}(b^{cl})\right) + \sum_{n=1}^{\infty}\frac{1}{n!\,2}K_{n+2}\left(\pi_{\rm fib}, \pi_{\rm fib}, \ldots, \pi_{\rm fib}, \frac{1}{\hbar}\partial_{\pi_{\rm fib}}(b^{cl}), \frac{1}{\hbar}\partial_{\pi_{\rm fib}}(b^{cl})\right)$$

$$= \sum_{n=0}^{\infty}\frac{1}{2\hbar^2 n!}K_{n+2}\left(\pi_{\rm fib}, \pi_{\rm fib}, \ldots, \pi_{\rm fib}, \partial_{\pi_{\rm fib}}(b^{cl}), \partial_{\pi_{\rm fib}}(b^{cl})\right).$$

In other words,

$$\mu_2^K = \frac{1}{2\hbar^2}K_2^{\pi_{\rm fib}}\left(\partial_{\pi_{\rm fib}}(b^{cl}), \partial_{\pi_{\rm fib}}(b^{cl})\right). \tag{4.51}$$

Since $K_2^{\pi_{\rm fib}}(b^{cl}, \partial_{\pi_{\rm fib}}(b^{cl}))$ vanishes for degree reasons, the component $\mu_2^K$ can be written as

$$\mu_2^K = \frac{1}{2\hbar^2}\partial_{\tilde{\diamond}}^{\rm Hoch}K_2^{\pi_{\rm fib}}(b^{cl}, \partial_{\pi_{\rm fib}}(b^{cl})) + \frac{1}{2\hbar^2}K_2^{\pi_{\rm fib}}\left(\partial_{\pi_{\rm fib}}(b^{cl}), \partial_{\pi_{\rm fib}}(b^{cl})\right) + \frac{1}{2\hbar^2}K_2^{\pi_{\rm fib}}\left(b^{cl}, (\partial_{\pi_{\rm fib}})^2(b^{cl})\right).$$

Using Equation (B.16) from Appendix B, we obtain

$$\mu_2^K = \frac{1}{2\hbar^2}\left(K_1^{\pi_{\rm fib}}\left(\left[b^{cl}, \partial_{\pi_{\rm fib}}(b^{cl})\right]_{SN}\right) - \left[K_1^{\pi_{\rm fib}}(b^{cl}), K_1^{\pi_{\rm fib}}(\partial_{\pi_{\rm fib}}(b^{cl}))\right]_G\right). \tag{4.52}$$

Thus, due to Equation (4.49),

$$\mu_2^K = \frac{1}{2\hbar^2}\{b^{cl}, b^{cl}\}_{\pi_{\rm fib}} - \frac{1}{2\hbar^2}[b^{cl}, b^{cl}]_{\tilde{\diamond}}. \tag{4.53}$$

Combining equations (4.16) and (4.26), we deduce that

$$\frac{1}{2\hbar^2}[b, b]_{\tilde{\diamond}} - \frac{1}{2\hbar^2}\{b^{cl}, b^{cl}\}_{\pi_{\rm fib}} + \frac{1}{\hbar}\nabla(b - b^{cl}) = 0.$$

Therefore, Equation (4.53) can be rewritten as

$$\mu_2^K = \frac{1}{\hbar}\nabla(b - b^{cl}) + \frac{1}{2\hbar^2}[b, b]_{\tilde{\diamond}} - \frac{1}{2\hbar^2}[b^{cl}, b^{cl}]_{\tilde{\diamond}}. \tag{4.54}$$

Combining this equation with (4.50), we obtain

$$-\Gamma_\hbar - \frac{1}{\hbar}\partial_{\pi_{\rm fib}}b^{cl} + \mu^K = -\Gamma_\hbar + \frac{1}{\hbar}[b^{cl}, \cdot]_{\tilde{\diamond}} + \frac{1}{\hbar}\nabla(b - b^{cl}) + \frac{1}{2\hbar^2}[b, b]_{\tilde{\diamond}} - \frac{1}{2\hbar^2}[b^{cl}, b^{cl}]_{\tilde{\diamond}} + \mu_0^K. \tag{4.55}$$

The left-hand side of (4.55) is a Maurer-Cartan element of the DGLA

$$(\Omega^\bullet(U, C^{\bullet+1}(\mathcal{S}M))[[\hbar]], d + \partial^{\rm Hoch}, [\cdot, \cdot]_G), \tag{4.56}$$

where $U$ is a coordinate open subset of $M$.

The next step in the construction of the star product on $M$ is to eliminate the components $\mu_2^K$ and $\mu_1^K$ via an equivalence transformation. Our goal is to show that the component $\mu_2^K$ can be eliminated in a way that gives us a Maurer-Cartan element which combines both the defining part

$$-\Gamma_\hbar + \frac{1}{\hbar}[b, \cdot]_{\tilde{\diamond}}$$

of the quantum Fedosov differential $D_\hbar^F$ (4.14) and the defining part $\mu_0^K$ (4.44) of the fiberwise product $\tilde{\diamond}$ (4.3). We have the following result:

**Proposition 4.3** *There exists an element* $\xi \in \hbar\Omega^1(M, \mathcal{S}M)[[\hbar]]$ *such that*

$$\left(-\Gamma_\hbar - \frac{1}{\hbar}\partial_{\pi_{\rm fib}} b^{cl} + \mu^K\right)^{\exp(\xi)} = -\Gamma_\hbar + \frac{1}{\hbar}[b, \cdot]_{\tilde{\diamond}} + \mu_0^K \tag{4.57}$$

*in the DGLA (4.56) for every coordinate open subset* $U$ *of* $M$.

Proof. Let us denote by $\widetilde{\mu}^K$ the left-hand side of (4.55):

$$\widetilde{\mu}^K = -\Gamma_\hbar - \frac{1}{\hbar}\partial_{\pi_{\rm fib}} b^{cl} + \mu^K. \tag{4.58}$$

As we have remarked, $\widetilde{\mu}^K$ is a Maurer-Cartan element of the DGLA (4.56) for every open coordinate subset $U$. According to the formula in (B.4) in Appendix B, we have

$$\left(\widetilde{\mu}^K\right)^{\exp(\xi)} = \widetilde{\mu}^K + \frac{\exp([\cdot,\xi]_G) - 1}{[\cdot,\xi]_G}\left(d\xi + \partial^{\rm Hoch}\xi + [\widetilde{\mu}^K, \xi]_G\right). \tag{4.59}$$

Since the Gerstenhaber bracket is zero if both arguments take values in $C^0(\mathcal{S}M)$, we conclude that only the first two terms $1 + \frac{1}{2}[\cdot,\xi]_G$ of the series $\frac{\exp([\cdot,\xi]_G)-1}{[\cdot,\xi]_G}$ contribute to the right-hand side of (4.59). Hence,

$$\left(\widetilde{\mu}^K\right)^{\exp(\xi)} = \widetilde{\mu}^K + \left(1 + \frac{1}{2}[\cdot,\xi]_G\right)\left(d\xi + \partial^{\rm Hoch}\xi + [\widetilde{\mu}^K, \xi]_G\right). \tag{4.60}$$

Using (4.55), we write (4.60) as follows:

$$\begin{aligned}\left(\widetilde{\mu}^K\right)^{\exp(\xi)} = &-\Gamma_h + \frac{1}{\hbar}[b^{cl}, \cdot]_{\tilde{\diamond}} + \frac{1}{\hbar}\nabla(b - b^{cl}) + \frac{1}{2\hbar^2}[b,b]_{\tilde{\diamond}} - \frac{1}{2\hbar^2}[b^{cl}, b^{cl}]_{\tilde{\diamond}} + \mu_0^K \\ &+ \nabla\xi + \partial^{\rm Hoch}\xi + \frac{1}{\hbar}[b^{cl}, \xi]_{\tilde{\diamond}} + [\mu_0^K, \xi]_G + \frac{1}{2}\left[\partial^{\rm Hoch}\xi + [\mu_0^K, \xi]_G, \xi\right]_G.\end{aligned}$$

Using (1.5) and (4.44), we combine $\partial^{\rm Hoch}\xi + [\mu_0^K, \xi]_G$ into $-[\xi, \cdot]_{\tilde{\diamond}}$. Thus $\left(\widetilde{\mu}^K\right)^{\exp(\xi)}$ can be further simplified as

$$\begin{aligned}\left(\widetilde{\mu}^K\right)^{\exp(\xi)} = &-\Gamma_h + \frac{1}{\hbar}[b^{cl}, \cdot]_{\tilde{\diamond}} + \frac{1}{\hbar}\nabla(b - b^{cl}) + \frac{1}{2\hbar^2}[b,b]_{\tilde{\diamond}} - \frac{1}{2\hbar^2}[b^{cl}, b^{cl}]_{\tilde{\diamond}} + \mu_0^K \\ &+ \nabla\xi - [\xi, \cdot]_{\tilde{\diamond}} + \frac{1}{\hbar}[b^{cl}, \xi]_{\tilde{\diamond}} - \frac{1}{2}[\xi, \xi]_{\tilde{\diamond}}.\end{aligned} \tag{4.61}$$

One can now show that, by plugging

$$\xi = \frac{1}{\hbar}(b^{cl} - b) \tag{4.62}$$

into (4.61), we obtain the desired identity

$$\left(\widetilde{\mu}^K\right)^{\exp(\xi)} = -\Gamma_\hbar + \frac{1}{\hbar}[b, \cdot]_{\tilde{\diamond}} + \mu_0^K.$$

Equations (4.15) and (4.25) imply that

$$\xi = \frac{1}{\hbar}(r^{cl} - r).$$

By Proposition 4.2, it follows that $\xi = 0 \mod \hbar$, and this concludes the proof. □

Let us denote the right-hand side of (4.57) by $\tilde{\mu}_U$:

$$\tilde{\mu}_U = -\Gamma_\hbar + \frac{1}{\hbar}[b, \,]_{\tilde{\diamond}} + \mu_0^K, \tag{4.63}$$

where $b$ is defined in (4.15), and an expression for $\mu_0^K$ is given in (4.44). As we remarked above, $\tilde{\mu}_U$ is a Maurer-Cartan element of the DGLA (4.56), and Proposition 4.3 says that $\tilde{\mu}_U$ is equivalent to the Maurer-Cartan element $\widetilde{\mu}^K$.

Twisting the DGLA (4.56) by $\tilde{\mu}_U$ (4.63), we get the DGLA $\Omega^\bullet(U, C^{\bullet+1}(\mathcal{S}M))[[\hbar]]$ with differential $D_\hbar^F + \partial_{\tilde{\diamond}}^{\mathrm{Hoch}}$, where $D_\hbar^F$ is given in (4.14). Since the differential $D_\hbar^F + \partial_{\tilde{\diamond}}^{\mathrm{Hoch}}$ does not depend on the choice of coordinates on $U$, twisting by $\tilde{\mu}_U$ (4.63) gives us the DGLA

$$\left(\Omega^\bullet(M, C^{\bullet+1}(\mathcal{S}M))[[\hbar]], D_\hbar^F + \partial_{\tilde{\diamond}}^{\mathrm{Hoch}}, [\cdot,\cdot]_G\right). \tag{4.64}$$

To obtain Kontsevich's star product we need to further modify the Maurer-Cartan element (4.63) by an equivalence transformation of the form

$$T = \exp(\xi_1), \quad \text{with} \quad \xi_1 \in \hbar\,\Omega^0(M, C^1(\mathcal{S}M))[[\hbar]], \tag{4.65}$$

to get the Maurer-Cartan element[6]

$$\mu_U = -\Gamma_\hbar - \delta + \frac{1}{\hbar}\{r^{cl}, \}_{\pi_{\mathrm{fib}}} + \Pi^K, \tag{4.66}$$

where $\Pi^K \in \Omega^0(M, C^2(\mathcal{S}M))[[\hbar]]$ . Then the element $\Pi^K \in \Omega^0(M, C^2(\mathcal{S}M))[[\hbar]]$ is a Maurer-Cartan element of the DGLA

$$\left(\Omega^\bullet(M, C^{\bullet+1}(\mathcal{S}M))[[\hbar]], D^{\mathrm{EW}} + \partial^{\mathrm{Hoch}}, [\cdot,\cdot]_G\right), \tag{4.67}$$

which is, in turn, quasi-isomorphic to $C^{\bullet+1}(\mathcal{O}_M)[[\hbar]]$ via the map $\tau^{\mathrm{EW}}$ in (2.22). Since $\Pi^K$ has zero exterior degree, the Maurer-Cartan equation for $\Pi^K$ is equivalent to two equations:

$$\partial^{\mathrm{Hoch}}\Pi^K + \frac{1}{2}[\Pi^K, \Pi^K]_G = 0, \tag{4.68}$$

$$D^{\mathrm{EW}}\Pi^K = 0. \tag{4.69}$$

The last equation implies that $\Pi^K$ lies in the image of $\tau^{\mathrm{EW}}$ (2.21), i.e., $\Pi^K = \tau^{\mathrm{EW}}(\Pi)$ for a unique $\Pi \in \hbar C^2(\mathcal{O}_M)[[\hbar]]$. Then (4.68) implies that the product

$$f_1 * f_2 = f_1 f_2 + \Pi(f_1, f_2) \tag{4.70}$$

on $\mathcal{O}(M)[[\hbar]]$ is associative. This is exactly Kontsevich's star product corresponding to the formal Poisson structure $\pi$ (2.41).

A more explicit way to get the star product (4.70) from the element $\Pi^K$ is to use the isomorphism of $\mathbb{C}[[\hbar]]$-modules $\tau^{\mathrm{EW}} : \mathcal{O}(M)[[\hbar]] \to \Gamma(M, \mathcal{S}M)[[\hbar]] \cap \ker D^{\mathrm{EW}}$. This isomorphism is obtained by iterating the following equation in degrees in the fiber coordinates $y$'s:

$$\tau^{\mathrm{EW}}(f) = f + \delta^{-1}\left(\nabla\tau^{\mathrm{EW}}(f) + \{r^{cl}, \tau^{\mathrm{EW}}(f)\}_{\pi_{\mathrm{fib}}}\right), \quad \text{for} \quad f \in \mathcal{O}(M)[[\hbar]]. \tag{4.71}$$

Equation (4.68) implies that the formula

$$a_1 \diamond a_2 = a_1 a_2 + \Pi^K(a_1, a_2), \quad \text{for} \quad a_1, a_2 \in \Gamma(M, \mathcal{S}M)[[\hbar]], \tag{4.72}$$

defines an associative product on $\mathcal{S}M[[\hbar]]$. Equation (4.69), in turn, implies that the differential $D^{\mathrm{EW}}$ is a derivation of the product (4.72). Thus the formula

$$f_1 * f_2 = \sigma\left(\tau^{\mathrm{EW}}(f_1) \diamond \tau^{\mathrm{EW}}(f_2)\right) \quad \text{for} \quad f_1, f_2 \in \mathcal{O}(M)[[\hbar]] \tag{4.73}$$

[6]Recall that the differential $D$ (2.3) we use for the construction of Kontsevich's star product is $D^{\mathrm{EW}}$ (4.21).

defines an associative product on $\mathcal{O}(M)[[\hbar]]$. According to the construction of the map $\tau^{\mathrm{EW}}$ (see Subsection 2.1) the star product (4.73) coincides with (4.70).

To construct an equivalence transformation between the star products (4.20) and (4.73), we recall that the Maurer-Cartan elements $\tilde{\mu}_U$ (4.63) and $\mu_U$ (4.66) are connected by the equivalence transformation (4.65):

$$\mu_U = \tilde{\mu}_U + \frac{e^{[\cdot,\xi_1]_G} - 1}{[\cdot,\xi_1]_G}\Big(d\xi_1 + \partial^{\mathrm{Hoch}}\xi_1 + [\tilde{\mu}_U, \xi_1]_G\Big)\,. \tag{4.74}$$

This equation can be rewritten as

$$-\Gamma_\hbar + \frac{1}{\hbar}\{b^{cl},\cdot\}_{\pi_{\mathrm{fib}}} + \Pi^K = -\Gamma_\hbar + \frac{1}{\hbar}[b,\cdot]_{\tilde{\diamond}} + \mu_0^K + \frac{e^{[\cdot,\xi_1]_G} - 1}{[\cdot,\xi_1]_G}\Big(D_\hbar^F\xi_1 + \partial_{\tilde{\diamond}}^{\mathrm{Hoch}}\xi_1\Big)\,. \tag{4.75}$$

Since $\xi_1$ has zero exterior degree, (4.75) is equivalent to the pair of equations

$$\frac{1}{\hbar}\{b^{cl},\cdot\}_{\pi_{\mathrm{fib}}} - \Gamma_\hbar = -\Gamma_\hbar + \frac{1}{\hbar}[b,\cdot]_{\tilde{\diamond}} + \frac{e^{[\ ,\xi_1]_G} - 1}{[\cdot,\xi_1]_G}\, D_\hbar^F\xi_1, \tag{4.76}$$

$$\Pi^K = \mu_0^K + \frac{e^{[\cdot,\xi_1]_G} - 1}{[\cdot,\xi_1]_G}\,\partial_{\tilde{\diamond}}^{\mathrm{Hoch}}\xi_1. \tag{4.77}$$

Equation (4.76) says that the transformation (4.65) intertwines the differentials $D^{\mathrm{EW}}$ and $D_\hbar^F$:

$$D_\hbar^F e^{\xi_1}(a) = e^{\xi_1}\, D^{\mathrm{EW}}(a), \quad \text{for} \quad a \in \Gamma(M, \mathcal{S}M)[[\hbar]], \tag{4.78}$$

and Equation (4.77) implies that the transformation (4.65) intertwines the fiberwise products $\tilde{\diamond}$ from (4.3) and $\diamond$ as in (4.72):

$$e^{\xi_1}(a_1)\,\tilde{\diamond}\, e^{\xi_1}(a_2) = e^{\xi_1}(a_1 \diamond a_2), \quad \text{with} \quad a_1, a_2 \in \Gamma(M, \mathcal{S}M)[[\hbar]]. \tag{4.79}$$

Let us consider the map $E : \mathcal{O}(M)[[\hbar]] \to \mathcal{O}(M)[[\hbar]]$,

$$E(f) = \sigma\Big(e^{\xi_1}\tau^{\mathrm{EW}}(f)\Big) \quad \text{for} \quad f \in \mathcal{O}(M)[[\hbar]]\,. \tag{4.80}$$

Since $e^{\xi_1}$ intertwines the differentials $D_\hbar^F$ and $D^{\mathrm{EW}}$ we conclude that

$$e^{\xi_1} \circ \tau^{\mathrm{EW}}(f) = \tilde{\tau} \circ E(f), \quad \text{for} \quad f \in \mathcal{O}(M)[[\hbar]], \tag{4.81}$$

where the map $\tilde{\tau}$ is defined in (4.19). Using the definition of $\tilde{*}$ (4.20) and equations (4.79) and (4.81), we get the following identities:

$$\begin{aligned}
E(f_1 * f_2) &= \sigma\Big(e^{\xi_1}\tau^{\mathrm{EW}}(f_1 * f_2)\Big) = \sigma\Big(e^{\xi_1}\left(\tau^{\mathrm{EW}}(f_1) \diamond \tau^{\mathrm{EW}}(f_2)\right)\Big) \\
&= \sigma\Big(e^{\xi_1}(\tau^{\mathrm{EW}}(f_1))\,\tilde{\diamond}\, e^{\xi_1}(\tau^{\mathrm{EW}}(f_2))\Big) = \sigma\left(\tilde{\tau}(E(f_1))\,\tilde{\diamond}\,\tilde{\tau}(E(f_2))\right) \\
&= E(f_1)\,\tilde{*}\,E(f_2)
\end{aligned}$$

for all $f_1, f_2 \in \mathcal{O}(M)[[\hbar]]$. Since $E$ starts with the identity in the zeroth order in $\hbar$, Kontsevich's star product (4.73) is indeed equivalent to the modified Fedosov star product (4.20) via $E$.

## 4.4 The star product $\tilde{*}$ coincides with the original Fedosov star product

The construction of the original Fedosov star product $*_F$ is based on a fiberwise product different from $\tilde{\diamond}$ (4.3), namely, one uses the following product on $\mathcal{S}M[[\hbar]]$:

$$a_1 \diamond_{\mathrm{F}} a_2 = a_1 \exp\left(\hbar\pi_1^{ij}(x)\frac{\overleftarrow{\partial}}{\partial y^i}\frac{\overrightarrow{\partial}}{\partial y^i}\right) a_2. \tag{4.82}$$

We will show that the product $\diamond_{\mathrm{F}}$ can be connected to the product $\tilde{\diamond}$ (4.3) by an equivalence transformation of a specific form:

**Lemma 4.4** *The products $\tilde{\diamond}$ and $\diamond_{\mathrm{F}}$ are fiberwise equivalent,*

$$P(a_1) \diamond_{\mathrm{F}} P(a_2) = P(a_1 \,\tilde{\diamond}\, a_2), \tag{4.83}$$

*via an equivalence transformation of the form*

$$P = \exp(\chi) \quad \textit{with} \quad \chi = \left(\hbar\chi_1 + \hbar^2\chi_2 + \cdots\right)^i_j\, y^j \partial_{y^i}, \tag{4.84}$$

*where $\chi_r \in \Gamma(M, TM \otimes T^*M)$ for each $r$.*

Proof. The statement is fiberwise so it suffices to consider the following situation on $\mathbb{R}^{2n}$. Let $\star$ be the ordinary Weyl-Moyal star product, i.e.,

$$f_1 \star f_2 = \mu_0 \circ \exp\left(\hbar\pi_1^{ij}\partial_{y^i} \otimes \partial_{y^j}\right)(f_1 \otimes f_2),$$

where $\mu_0(f_1 \otimes f_2) = f_1 f_2$ denotes the ordinary commutative product on $\mathcal{O}(\mathbb{R}^{2n})[[\hbar]]$, and $\pi_1^{ij}$ is a constant antisymmetric nondegenerate $2n \times 2n$-matrix (with complex entries). We must construct a specific equivalence transformation intertwining the product $\star$ with the star product

$$f_1 \,\tilde{\star}\, f_2 = \mu_0 \circ \exp\left(\pi^{ij}\partial_{y^i} \otimes \partial_{y^j}\right)(f_1 \otimes f_2), \tag{4.85}$$

where $\pi$ is a formal power series of constant antisymmetric $2n \times 2n$-matrices (with complex entries) starting with $\hbar\pi_1$

$$\pi^{ij} = \hbar\pi_1^{ij} + \hbar^2\pi_2^{ij} + \hbar^3\pi_3^{ij} + \cdots .$$

Let us consider the following sets of operators,

$$\mathcal{B} = \left\{B^{ij}\partial_{y^i} \otimes \partial_{y^j} \;\middle|\; B^{ij} \in \hbar\mathbb{C}[[\hbar]]\right\}, \quad \mathcal{A} = \left\{\chi \otimes \mathrm{id} + \mathrm{id} \otimes \chi \;\middle|\; \chi = \chi^i_j y^j \partial_{y^i} \text{ with } \chi^i_j \in \hbar\mathbb{C}[[\hbar]]\right\},$$

acting on the tensor product of two copies of $\mathcal{O}(\mathbb{R}^{2n})[[\hbar]]$. Note that $\mathcal{A}$ is a subalgebra, while $\mathcal{B}$ is an abelian subalgebra of the Lie algebra of all endomorphisms of $\mathcal{O}(\mathbb{R}^{2n})[[\hbar]] \otimes_{\mathbb{C}[[\hbar]]} \mathcal{O}(\mathbb{R}^{2n})[[\hbar]]$. The property

$$[\mathcal{A}, \mathcal{B}] \subseteq \mathcal{B}$$

implies that

$$\exp(A) \circ \exp(B) \circ \exp(-A) = \exp(e^{[A,\cdot]}(B)) = \exp(\tilde{B}) \tag{4.86}$$

with $\tilde{B} = B + [A, B] + \cdots \in \mathcal{B}$. Let us suppose that the matrix $\pi^{ij}$ in (4.85) has the form ($m \geq 2$)

$$\pi^{ij} = \hbar\pi_1^{ij} + \hbar^m\pi_m^{ij} + \hbar^{m+1}\pi_{m+1}^{ij} + \hbar^{m+2}\pi_{m+2}^{ij} + \cdots .$$

In other words, $\pi^{ij} - \hbar\pi_1^{ij} = 0 \mod \hbar^m$. Consider an equivalence transformation $P_m$ of the form

$$P_m = \exp\left(\hbar^m(\chi_m)^i_j y^j \partial_{y^i}\right), \tag{4.87}$$

where $(\chi_m)^i_j$ is a constant $2n \times 2n$ matrix. Due to (4.86), we have

$$P_m(P_m^{-1} f_1 \,\tilde{\star}\, P_m^{-1} f_2) = \mu_0 \circ \exp(\chi \otimes \mathrm{id} + \mathrm{id} \otimes \chi) \circ \exp(\pi) \circ \exp(-\chi \otimes \mathrm{id} - \mathrm{id} \otimes \chi)(f_1 \otimes f_2)$$
$$= \mu_0 \circ \exp(\pi') \circ (f_1 \otimes f_2),$$

where

$$(\pi')^{ij} = \hbar \pi_1^{ij} + \hbar^m \left( \pi_m^{ij} - \pi_1^{kj} (\chi_m)^i_k + \pi_1^{ki} (\chi_m)^j_k \right) + \hbar^{m+1} \pi_{m+1}^{ij} + \hbar^{m+2} \pi_{m+2}^{ij} + \cdots .$$

Since the matrix $\pi_1^{ij}$ is nondegenerate, we can choose the matrix $(\chi_m)^i_j$ in such way that

$$\pi_m^{ij} - \pi_1^{kj} (\chi_m)^i_k + \pi_1^{ki} (\chi_m)^j_k = 0.$$

The new product $f_1, f_2 \mapsto P_m(P_m^{-1} f_1 \,\tilde{\star}\, P_m^{-1} f_2)$ has the form

$$P_m(P_m^{-1} f_1 \,\tilde{\star}\, P_m^{-1} f_2) = \mu_0 \circ \exp(\pi') \circ (f_1 \otimes f_2)\,,$$

where $\pi' = \hbar \pi_1 \mod \hbar^{m+1}$. Thus the desired equivalence transformation $P$ is obtained as an infinite product

$$P = \ldots P_4 \, P_3 \, P_2,$$

where the $m$-th transformation $P_m$ has the form (4.87). This infinite product converges in the $\hbar$-adic topology, and it is clear that $P$ has the form

$$P = \exp(\chi)$$

where $\chi$ is a linear vector field. This concludes the proof. □

Since the exponent of $P$ is a vector field, it is also an automorphism of the undeformed product:

$$P(a_1) P(a_2) = P(a_1 a_2), \tag{4.88}$$

for all $a_1, a_2 \in \Gamma(M, \mathcal{S}M)[[\hbar]]$. Furthermore, $P$ preserves the degree in $y$'s and transforms the differential $D^F_\hbar$ (4.14) to

$$D^{\mathrm{F}} = P \, D^F_\hbar \, P^{-1} = P \, \nabla \, P^{-1} + \frac{1}{\hbar} [P(b), \cdot]_{\diamond_{\mathrm{F}}} \tag{4.89}$$

with the element $b$ defined in (4.15). Since $P$ has the form (4.84), the operator

$$\nabla_0 = P \, \nabla \, P^{-1}$$

is again a connection (possibly with Christoffel symbols depending on $\hbar$). Furthermore, $\nabla_0$ is a derivation of $\diamond_{\mathrm{F}}$ because $\nabla$ is a derivation of $\tilde{\diamond}$. As for the curvature form (4.9), we have

$$(\nabla_0)^2 = P \, \nabla^2 \, P^{-1} = \frac{1}{\hbar} P \, [R, \cdot]_{\tilde{\diamond}} \, P^{-1} = \frac{1}{\hbar} [P(R), \,]_{\diamond_{\mathrm{F}}}.$$

Since the operator $P$ preserves the degree in $y$'s, we have

$$P(R) = \frac{1}{2} dx^i \, dx^j R^0_{ij\;kl}(x) y^k y^l, \tag{4.90}$$

where $R^0_{ij\;kl}(x)$ are components (possibly depending on $\hbar$) of the curvature tensor for $\nabla_0$.

Applying the operator $P$ to Equation (4.16), and using the fact that the components of $\omega$ (4.1) do not depend on the fiber coordinates $y$'s, we have that

$$\frac{1}{\hbar}P(R) + \frac{1}{\hbar}\nabla_0 P(b) + \frac{1}{2\hbar^2}[P(b), P(b)]_{\diamond_{\mathrm{F}}} = -\omega, \tag{4.91}$$

Thus $D^{\mathrm{F}}$ in (4.89) is the quantum Fedosov differential from the original construction in [27], and the Fedosov class of the resulting star product is represented by $\omega$ (4.1).

Let $\tau^{\mathrm{F}}$ be the isomorphism

$$\tau^{\mathrm{F}} : \mathcal{O}(M)[[\hbar]] \to \Gamma(M, \mathcal{S}M)[[\hbar]] \cap \ker D^{\mathrm{F}} \tag{4.92}$$

which lifts functions on $M$ to $D^{\mathrm{F}}$-flat sections of the sheaf $\mathcal{S}M[[\hbar]]$. Similarly to $\tilde{\tau}$, the isomorphism $\tau^{\mathrm{F}}$ is defined by iterating the equation

$$\tau^{\mathrm{F}}(a) = a + \delta^{-1}\left(\nabla_0 \tau^{\mathrm{F}}(a) + [r^{\mathrm{F}}, \tau^{\mathrm{F}}(a)]_{\diamond_{\mathrm{F}}}\right), \tag{4.93}$$

where $a \in \Gamma(M, \mathcal{S}M)[[\hbar]]$ and $r^{\mathrm{F}} = P(b) + \hbar\, dx^i \omega_{ij}(x, \hbar) y^j$. The (original) Fedosov star product is defined in terms of $\tau^{\mathrm{F}}$ and the fiberwise product $\diamond_{\mathrm{F}}$ for $f_1, f_2 \in \mathcal{O}(M)[[\hbar]]$ as

$$f_1 *_F f_2 = \sigma\left(\tau^F(f_1) \diamond_{\mathrm{F}} \tau^F(f_2)\right). \tag{4.94}$$

Since the transformation $P$ (4.84) intertwines the differentials $D^{\mathrm{F}}$ and $D^F_\hbar$, we conclude that $P \circ \tau^m(f)$ is $D^{\mathrm{F}}$-flat for every $f \in \mathcal{O}(M)[[\hbar]]$. On the other hand, since the transformation $P$ preserves the degree in the fiber coordinates $y$'s, we have that $P \circ \tilde{\tau}(f)\Big|_{y=0} = f$. Hence

$$P \circ \tilde{\tau}(f) = \tau^{\mathrm{F}}(f). \tag{4.95}$$

Combining the last equation with (4.83), and using the fact that $P$ preserves the degree in fiber coordinates $y$'s, we get the following series of identities: for $f_1, f_2 \in \mathcal{O}(M)[[\hbar]]$,

$$\begin{aligned} f_1 *_F f_2 &= \sigma\left(\tau^{\mathrm{F}}(f_1) \diamond_{\mathrm{F}} \tau^{\mathrm{F}}(f_2)\right) = \sigma\left(P \circ \tilde{\tau}(f_1) \diamond_{\mathrm{F}} P \circ \tilde{\tau}(f_2)\right) \\ &= \sigma\left(P(\tilde{\tau}(f_1) \,\tilde{\diamond}\, \tilde{\tau}(f_2))\right) = \sigma\left((\tilde{\tau}(f_1) \,\tilde{\diamond}\, \tilde{\tau}(f_2))\right) \\ &= f_1 \,\tilde{*}\, f_2. \end{aligned}$$

Thus the star product $\tilde{*}$ (4.20) coincides with the original Fedosov product $*_F$ (4.94), and Theorem 4.1 is proved.

# A Formal differential equations

In this appendix we collect some results on differential equations in $\mathbb{C}[[\hbar]]$-modules which are needed throughout Section 3. Most of the material is well-known or can be easily reconstructed from well-known results, see e.g. the textbook [59, Sect. 3].

Let us consider the following purely algebraic situation. We fix a commutative ring $\mathsf{C}$ containing $\mathbb{Q}$, let $V$ be a $\mathsf{C}$-module, and let $\mathcal{D} \subseteq \mathrm{End}_{\mathsf{C}}(V)$ be a unital sub-algebra. In our case we usually have $\mathsf{C} = \mathbb{C}$, $V = \mathcal{O}(M)$ or $\Gamma(M, E)$ for some vector bundle $E \longrightarrow M$, and $\mathcal{D}$ being the differential operators on $V$. Let us also consider the $\hbar$-adically complete $\mathsf{C}$-module $(V[t])[[\hbar]]$, i.e., in each order of $\hbar$ we have a polynomial in $t$ with coefficients in $V$. Note that this is different from $(V[[\hbar]])[t]$, which is a proper sub-module of $(V[t])[[\hbar]]$. Let $D(t) \in \hbar(\mathcal{D}[t])[[\hbar]]$ and $w(t) \in \hbar(V[t])[[\hbar]]$ be given, and consider the differential equation

$$\frac{d}{dt}v(t) = w(t) + D(t)v(t) \tag{A.1}$$

with initial condition $v(0) = v_0 \in V[[\hbar]]$.

**Proposition A.1** *For each initial condition $v(0) = v_0$ equation* (A.1) *has a unique solution $v(t) \in (V[t])[[\hbar]]$. Moreover, if $w = 0$ then the flow map $v_0 \mapsto v(t)$ is a formal series* $\mathrm{id} + \sum_{r=0}^{\infty} \hbar^r D_r(t)$ *with $D_r(t) \in \mathcal{D}[t]$.*

Proof. First we rewrite (A.1) as the integral equation

$$v(t) = v(0) + \int_0^t (w(\tau) + D(\tau)v(\tau))d\tau, \tag{A.2}$$

incorporating the initial condition. Since in each order of $\hbar$, $w(t)$ and $D(\tau)v(\tau)$ are polynomials in $\tau$, the integral operator is a purely algebraic gadget defined by linear extension of $\int_0^t \tau^n d\tau = \frac{1}{n+1}\tau^{n+1}$ (this is also the reason why we require $\mathbb{Q} \subseteq \mathsf{C}$). Since by assumption $D(t)$ and $w(t)$ are at least of order $\hbar$, the right-hand side of (A.2) is directly shown to be a *contracting* endomorphism in the $\hbar$-adic topology of the complete module $(V[t])[[\hbar]]$. It follows from the usual fixed point argument that there is a unique solution of (A.2) which is the unique solution of (A.1) with correct initial condition, see e.g. [59, Sect. 6.2.1]. When $w = 0$, the iteration clearly produces a flow map of the specified type. □

**Example A.2** Let $\mathcal{A}$ be a $\mathsf{C}$-algebra and let $\star$ be an associative deformation of $\mathcal{A}$, so that $\mathcal{A}[[\hbar]]$ is a $\mathsf{C}[[\hbar]]$-algebra with respect to $\star$. The product $\star$ extends to $(\mathcal{A}[t])[[\hbar]]$ in the obvious way, making it a $\mathsf{C}[[\hbar]]$-algebra. Let $d(t) \in \hbar(\mathcal{A}[t])[[\hbar]]$ be given. Then for every $a_0 \in \mathcal{A}[[\hbar]]$ the differential equation

$$\frac{d}{dt}a(t) = d(t) \star a(t) \quad \text{with} \quad a(0) = a_0 \tag{A.3}$$

has a unique solution by Proposition A.1.

**Example A.3** Let $\mathcal{A} = \mathrm{DiffOp}(\Gamma(M, E))$ be the differential operators on some vector bundle $E \longrightarrow M$ and let $\star$ be the undeformed multiplication of differential operators. Then for any $D(t) \in \hbar(\mathrm{DiffOp}(\Gamma(M, E))[t])[[\hbar]]$ the equation

$$\frac{d}{dt}A(t) = D(t) \star A(t) \tag{A.4}$$

has a unique solution for every initial condition $A(0) = A_0 \in \mathrm{DiffOp}(\Gamma(M, E))[[\hbar]]$. The important point here is that the solution is again in $(\mathrm{DiffOp}(\Gamma(M, E))[t])[[\hbar]]$. This is the situation which we encountered in Section 3 frequently.

Another situation refers to $\mathsf{C} = \mathbb{C}$ and smooth functions on a manifold only. Let $D(t) \in \hbar(\mathrm{DiffOp}(M)[t])[[\hbar]]$ be a formal series of differential operators depending polynomially on $t$ at each order of $\hbar$. Let $d_0 \in \mathcal{O}(M)$ be a function and consider the differential equation

$$\frac{d}{dt}f(t) = (d_0 + D(t))f(t) \quad \text{with} \quad f(0) = h \tag{A.5}$$

with some *invertible* $h \in \mathcal{O}(M)[[\hbar]]$. Note that $h$ is invertible iff the zeroth order $h_0$ is invertible. Proposition A.1 does not directly apply in this case due to the non-trivial zeroth order contribution coming from $d_0$. But the following holds.

**Proposition A.4** *For any invertible $h \in \mathcal{O}(M)[[\hbar]]$,* (A.5) *has a unique solution $f(t)$, for all $t \in \mathbb{R}$, of the form*

$$f(t) = \mathrm{e}^{td_0} h_0 g(t), \tag{A.6}$$

*where $g(t) = 1 + \sum_{n=1}^{\infty} \hbar^n g_n(t)$ with $g_n(t) \in \mathcal{O}(M)[t]$. In particular, $f(t)$ is invertible for all $t \in \mathbb{R}$.*

Proof. We write $f(t) = \sum_{n=0}^{\infty} \hbar^n f_n(t)$. Then in order $\hbar^0$ (A.5) reads

$$\frac{d}{dt} f_0(t) = d_0 f_0(t) \quad \text{with} \quad f_0(0) = h_0,$$

so $f_0 = \mathrm{e}^{td_0} h_0$ is the unique solution. Making the Ansatz $f(t) = \mathrm{e}^{td_0} h_0 g(t)$, we see that $f(t)$ is a solution of (A.5) if and only if $g(t)$ satisfies

$$\frac{d}{dt} g(t) = \mathrm{e}^{-td_0} \frac{1}{h_0} D(t) \left( \mathrm{e}^{td_0} h_0 g(t) \right) = \tilde{D}(t) g(t),$$

with initial condition $g(0) = \frac{h}{h_0}$. Note that $\tilde{D} \in \hbar(\mathrm{DiffOp}(M)[t])[[\hbar]]$ since every differentiation in $D(t)$ reproduces the exponential function, which in the end cancels. Thus only polynomials in $t$ remain. We can now apply Proposition A.1 and obtain a unique solution $g(t) \in (\mathcal{O}(M)[t])[[\hbar]]$. Since the solution is obtained by iteration, we have $g_0(t) = 1$, for all $t$, in zeroth order. The invertibility of $f(t)$ follows since its zeroth order is invertible. □

**Example A.5** Let $d_0 \in \mathcal{O}(M)$ and $d_+(t) \in \hbar(\mathcal{O}(M)[t])[[\hbar]]$ be given. Then we have a unique invertible solution $f(t)$ to the equation

$$\frac{d}{dt} f(t) = d(t) \star f(t) \quad \text{with} \quad f(0) = 1, \tag{A.7}$$

where $d(t) = d_0 + d_+(t)$ and $\star$ is a star product on $M$. If $d(t) \equiv d$ is time independent then $f(t)$ is the $\star$-exponential $\mathrm{Exp}_\star(td)$ as in [1], [7, App. A], and [59, Thm. 6.3.4]. Moreover, $f(t)$ is $\star$-invertible for all $t$ and the $\star$-inverse $f(t)^{-1}$ is determined by the equation

$$\frac{d}{dt} f(t)^{-1} = -f(t)^{-1} \star d(t) \quad \text{with} \quad f(0) = 1, \tag{A.8}$$

so we also can apply Proposition A.4 to this situation.

# B Maurer-Cartan elements and the twisting procedure

In this section we recall some general facts about Maurer-Cartan elements and the twisting procedure. Further details can be found in Sections 2.3 and 2.4 in [18] and in [19].

## B.1 Maurer-Cartan elements in DGLAs

Recall that every DGLA $(\mathcal{L}, d_\mathcal{L}, [\cdot,\cdot]_\mathcal{L})$ in this paper is equipped with a complete descending filtration

$$\cdots \supset \mathcal{F}^{-2}\mathcal{L} \supset \mathcal{F}^{-1}\mathcal{L} \supset \mathcal{F}^{0}\mathcal{L} \supset \mathcal{F}^{1}\mathcal{L} \supset \dots , \qquad \mathcal{L} = \lim_n \mathcal{L}/\mathcal{F}^n\mathcal{L}\,, \tag{B.1}$$

which means, in particular, that $\mathcal{F}^1\mathcal{L}$ is a projective limit of nilpotent DGLAs.

By definition, $\alpha$ is a *Maurer-Cartan element* of $\mathcal{L}$ if $\alpha \in \mathcal{F}^1\mathcal{L}^1$ (i.e., $\alpha \in \mathcal{F}^1\mathcal{L}$ and has degree 1 in $\mathcal{L}$) and satisfies the equation

$$d_\mathcal{L}\alpha + \frac{1}{2}[\alpha,\alpha]_\mathcal{L} = 0\,. \tag{B.2}$$

Notice that $\mathfrak{g}(\mathcal{L}) = \mathcal{F}^1\mathcal{L}^0$ forms an ordinary (not graded) Lie algebra which is the projective limit of nilpotent Lie algebras. Hence $\mathfrak{g}(\mathcal{L})$ can be exponentiated to the group

$$\mathfrak{G}(\mathcal{L}) = \exp(\, \mathcal{F}^1\mathcal{L}^0\,), \tag{B.3}$$

and this group acts on Maurer-Cartan elements of $\mathcal{L}$ via

$$\alpha \mapsto \alpha^{\exp(\xi)} = \exp([\cdot, \xi]_{\mathcal{L}})\alpha + \frac{\exp([\cdot, \xi]_{\mathcal{L}}) - 1}{[\cdot, \xi]_{\mathcal{L}}} d_{\mathcal{L}}\xi \,, \tag{B.4}$$

where $\xi \in \mathcal{F}^1\mathcal{L}^0$, and the expression

$$\frac{\exp([\cdot, \xi]_{\mathcal{L}}) - 1}{[\cdot, \xi]_{\mathcal{L}}}$$

is defined via the Taylor expansion of the function $\frac{e^x - 1}{x}$ around the point $x = 0$. Both terms on the right-hand side of (B.4) are well defined because the filtration on $\mathcal{L}$ is complete. We remark that (B.4) defines a right action, i.e., for all $\xi, \eta \in \mathcal{F}^1\mathcal{L}^0$ we have

$$\left(\alpha^{\exp(\xi)}\right)^{\exp(\eta)} = \alpha^{\exp(\mathrm{CH}(\xi,\eta))}\,, \tag{B.5}$$

where $\mathrm{CH}(\xi, \eta)$ is the Campbell-Hausdorff series:

$$\mathrm{CH}(\xi, \eta) = \log(e^{\xi}e^{\eta}) = \xi + \eta + \frac{1}{2}[\xi, \eta] + \dots \,. \tag{B.6}$$

We let $\mathrm{MC}(\mathcal{L})$ denote the transformation groupoid of the action (B.4), called the *Goldman-Millson groupoid* [38]: its objects are the Maurer-Cartan elements of $\mathcal{L}$ and morphisms between two Maurer-Cartan elements $\alpha_1$ and $\alpha_2$ are elements of the group $\mathfrak{G}$ (B.3) which transform $\alpha_1$ to $\alpha_2$. We call Maurer-Cartan elements *equivalent* if they are isomorphic in $\mathrm{MC}(\mathcal{L})$ and denote by $\pi_0(\mathrm{MC}(\mathcal{L}))$ the set of equivalence classes of Maurer-Cartan elements.

Every morphism $f : \mathcal{L} \to \widetilde{\mathcal{L}}$ of DGLAs defines a functor

$$f_* : \mathrm{MC}(\mathcal{L}) \to \mathrm{MC}(\widetilde{\mathcal{L}}). \tag{B.7}$$

According to [35, 38, 53], we have the following result.

**Theorem B.1** *If $f : \mathcal{L} \to \widetilde{\mathcal{L}}$ is a quasi-isomorphism of DGLAs, then the functor (B.7) induces a bijection from $\pi_0(\mathrm{MC}(\mathcal{L}))$ to $\pi_0(\mathrm{MC}(\widetilde{\mathcal{L}}))$.*

Every Maurer-Cartan element $\alpha$ of $\mathcal{L}$ can be used to modify the DGLA structure on $\mathcal{L}$. This modified structure is called the DGLA structure *twisted by the Maurer-Cartan* $\alpha$ [52]. The Lie bracket of the twisted DGLA structure is unchanged, and the differential is given by

$$d^{\alpha}_{\mathcal{L}} = d_{\mathcal{L}} + [\alpha, \cdot]_{\mathcal{L}}\,. \tag{B.8}$$

The DGLA resulting from twisting $\mathcal{L}$ by $\alpha$ will denote by $\mathcal{L}^{\alpha}$.

## B.2 $L_\infty$-morphisms of DGLAs

Two DGLAs $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are called *quasi-isomorphic* if there is a sequence of quasi-isomorphisms $f$, $f_1$, $f_2$, ..., $f_n$ connecting $\mathcal{L}$ with $\widetilde{\mathcal{L}}$:

$$\mathcal{L} \stackrel{f}{\to} \mathcal{L}_1 \stackrel{f_1}{\leftarrow} \mathcal{L}_2 \stackrel{f_2}{\to} \dots \stackrel{f_{n-1}}{\to} \mathcal{L}_n \stackrel{f_n}{\leftarrow} \widetilde{\mathcal{L}}\,. \tag{B.9}$$

It follows from Theorem B.1 that a sequence of quasi-isomorphisms (B.9) between the DGLAs $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ defines a bijection between the sets of equivalence classes of Maurer-Cartan elements.

We will need to extend the class of morphisms between DGLAs to $L_\infty$ morphisms. To this end, we need the Chevalley-Eilenberg complex $C(\mathcal{L})$ of a DGLA $\mathcal{L}$. As a graded vector space, $C(\mathcal{L})$ is the direct sum of all symmetric powers of the desuspension (see Subsection 1.1) $\mathbf{s}^{-1}\mathcal{L}$ of $\mathcal{L}$:

$$C(\mathcal{L}) = \bigoplus_{k=1}^{\infty} S^k(\mathbf{s}^{-1}\mathcal{L})\,. \tag{B.10}$$

The space $C(\mathcal{L})$ is equipped with the following cocommutative comultiplication:

$$\Delta \,:\, C(\mathcal{L}) \longrightarrow C(\mathcal{L}) \otimes C(\mathcal{L}), \tag{B.11}$$

defined by

$$\begin{aligned} &\Delta(v_1) = 0\,,\\ &\Delta(v_1, v_2, \dots, v_n) = \sum_{k=1}^{n-1} \sum_{\sigma \in \mathrm{Sh}(k,n-k)} \pm (v_{\sigma(1)}, \dots, v_{\sigma(k)}) \otimes (v_{\sigma(k+1)}, \dots, v_{\sigma(n)})\,, \end{aligned} \tag{B.12}$$

where $v_1, \dots, v_n$ are homogeneous elements of $\mathbf{s}^{-1}\mathcal{L}$, $\mathrm{Sh}(k, n-k)$ is the set of $(k, n-k)$-shuffles in $S_n$, and the signs are determined using the Koszul rule.

It can be shown that every coderivation $Q$ of the coalgebra $C(\mathcal{L})$ is uniquely determined by its composition $p \circ Q$ with the natural projection

$$p : C(\mathcal{L}) \to \mathbf{s}^{-1}\mathcal{L}\,. \tag{B.13}$$

This statement follows from the fact that $C(\mathcal{L})$ is a cofree cocommutative coalgebra[7]. A similar statement holds for cofree coalgebras of other types, see [36, Prop. 2.14].

We define the coboundary operator $Q$ of the complex $C(\mathcal{L})$ by requiring that $Q$ is a coderivation of the coalgebra structure and by setting

$$p \circ Q(v) = -d_{\mathcal{L}} v\,, \quad p \circ Q(v_1, v_2) = (-1)^{|v_1|+1}[v_1, v_2]_{\mathcal{L}}\,, \quad p \circ Q(v_1, v_2, \dots, v_k) = 0 \quad (k > 2),$$

where $v, v_1, \dots, v_k$ are homogeneous elements of $\mathcal{L}$. The equation $Q^2 = 0$ readily follows from the Leibniz rule and the Jacobi identity. Thus to every DGLA $(\mathcal{L}, d_{\mathcal{L}}, [\cdot,\cdot]_{\mathcal{L}})$ we assign a DG cocommutative coalgebra

$$(C(\mathcal{L}), Q)$$

without counit.

**Definition B.2** *An $L_\infty$ morphism*

$$F : \mathcal{L} \succ\!\!\!\rightarrow \widetilde{\mathcal{L}}$$

*from a DGLA $\mathcal{L}$ to a DGLA $\widetilde{\mathcal{L}}$ is a (degree zero) morphism of the corresponding DG cocommutative coalgebras:*

$$F : (C(\mathcal{L}), Q) \to (C(\widetilde{\mathcal{L}}), \widetilde{Q})\,.$$

The compatibility of $F$ with the comultiplication $\Delta$ (B.11) implies that $F$ is uniquely determined by its composition $p \circ F$ with the projection

$$p : C(\widetilde{\mathcal{L}}) \to \mathbf{s}^{-1}\widetilde{\mathcal{L}}\,.$$

[7] In fact, $C(\mathcal{L})$ is a cofree cocommutative coalgebra without counit.

We denote by $F_n$ the following restriction of $p \circ F$:

$$F_n = p \circ F\Big|_{S^n(\mathbf{s}^{-1}\mathcal{L})} S^n(\mathbf{s}^{-1}\mathcal{L}) \to \mathbf{s}^{-1}\widetilde{\mathcal{L}}\,. \tag{B.14}$$

The maps $F_n$'s are the *structure maps* of the $L_\infty$ morphism $F$. The presence of the desuspensions in (B.14) simply means that the map $F_n$ can be thought of as a map from $\mathcal{L}^{\otimes n}$ to $\widetilde{\mathcal{L}}$ of degree $1-n$ with the following symmetry in the arguments:

$$F_n(\dots,\gamma_1,\gamma_2,\dots) = -(-1)^{|\gamma_1||\gamma_2|} F_n(\dots,\gamma_2,\gamma_1,\dots)\,,$$

where $|\gamma_i|$ is the degree of $\gamma_i$ in $\mathcal{L}$. We tacitly use this identification in our paper.

The compatibility of $F$ with the codifferentials $Q$ and $\widetilde{Q}$ is equivalent to a sequence of quadratic relations on $F_n$. The first of these relations says that the map

$$F_1 : \mathcal{L} \to \widetilde{\mathcal{L}}$$

intertwines the differentials $d_{\mathcal{L}}$ and $d_{\widetilde{\mathcal{L}}}$:

$$F_1(d_{\mathcal{L}}\gamma) = d_{\widetilde{\mathcal{L}}} F_1(\gamma)\,, \qquad \gamma \in \mathcal{L}\,. \tag{B.15}$$

The second relation says that $F_1$ is compatible with the brackets up to homotopy:

$$d_{\widetilde{\mathcal{L}}} F_2(\gamma_1,\gamma_2) + F_2(d_{\mathcal{L}}\gamma_1,\gamma_2) + (-1)^{|\gamma_1|} F_2(\gamma_1, d_{\mathcal{L}}\gamma_2) = F_1([\gamma_1,\gamma_2]_{\mathcal{L}}) - [F_1(\gamma_1), F_1(\gamma_2)]_{\widetilde{\mathcal{L}}}\,, \tag{B.16}$$

where $\gamma_1, \gamma_2 \in \mathcal{L}$.

Condition (B.15) motivates the following definition:

**Definition B.3** *An $L_\infty$ morphism $F : \mathcal{L} \succ\!\!\to \widetilde{\mathcal{L}}$ is an $L_\infty$ quasi-isomorphism if $F_1$ induces an isomorphism from $H^\bullet(\mathcal{L}, d_{\mathcal{L}})$ to $H^\bullet(\widetilde{\mathcal{L}}, d_{\widetilde{\mathcal{L}}})$.*

Just as ordinary quasi-isomorphisms, an $L_\infty$ quasi-isomorphism between DGLAs induces a bijection between the sets of equivalence classes of Maurer-Cartan elements. More precisely, if $F$ is an $L_\infty$ morphism from $\mathcal{L}$ to $\widetilde{\mathcal{L}}$ then, for every Maurer-Cartan element $\alpha$ of $\mathcal{L}$,

$$\beta = \sum_{n=1}^{\infty} \frac{1}{n!} F_n(\alpha,\alpha,\dots,\alpha) \tag{B.17}$$

is a Maurer-Cartan element[8] of the DGLA $\widetilde{\mathcal{L}}$. Furthermore, if $\alpha$ is equivalent to $\alpha'$ in $\mathcal{L}$, then $\beta$ is equivalent to the Maurer-Cartan element

$$\beta' = \sum_{n=1}^{\infty} \frac{1}{n!} F_n(\alpha',\alpha',\dots,\alpha')$$

of $\widetilde{\mathcal{L}}$. As a result, the correspondence

$$\alpha \mapsto \beta = \sum_{n=1}^{\infty} \frac{1}{n!} F_n(\alpha,\alpha,\dots,\alpha) \tag{B.18}$$

induces a map

$$F_* : \pi_0(\mathrm{MC}(\mathcal{L})) \to \pi_0(\mathrm{MC}(\widetilde{\mathcal{L}}))\,. \tag{B.19}$$

Due to [18, Prop. 4] we have:

[8] The infinite series in (B.17) is well defined since $\alpha \in \mathcal{F}^1\mathcal{L}$ and $\widetilde{\mathcal{L}}$ is complete with respect to its filtration.

**Proposition B.4** *If $F : \mathcal{L} \succ\!\!\rightarrow \widetilde{\mathcal{L}}$ is an $L_\infty$ quasi-isomorphism, then the map (B.19) is a bijection.*

Given a Maurer-Cartan element $\alpha \in \mathcal{L}$, any $L_\infty$ morphism $F : \mathcal{L} \succ\!\!\rightarrow \widetilde{\mathcal{L}}$ can be modified to an $L_\infty$ morphism $F^\alpha$ between the twisted DGLAs $\mathcal{L}^\alpha$ and $\widetilde{\mathcal{L}}^\beta$, where $\beta$ is as in (B.17), see [18, Prop. 1]. We say that the $L_\infty$ morphism $F^\alpha : \mathcal{L}^\alpha \succ\!\!\rightarrow \widetilde{\mathcal{L}}^\beta$ is *twisted* by the Maurer-Cartan element $\alpha$; its structure maps $F^\alpha_n$ are given by

$$F^\alpha_n(\gamma_1, \gamma_2, \dots, \gamma_n) = \sum_{k=0}^{\infty} \frac{1}{k!} F_{k+n}(\alpha, \alpha, \dots, \alpha, \gamma_1, \gamma_2, \dots, \gamma_n)\,, \tag{B.20}$$

where $\gamma_i \in \mathcal{L}$. In particular, $F^\alpha_1$ intertwines the differentials in $\mathcal{L}^\alpha$ and $\widetilde{\mathcal{L}}^\beta$:

$$\sum_{k=0}^{\infty} \frac{1}{k!} F_{k+1}(\alpha, \alpha, \dots, \alpha, d_{\mathcal{L}}\gamma + [\alpha, \gamma]_{\mathcal{L}}) = \sum_{k=0}^{\infty} \frac{1}{k!} \,(d_{\widetilde{\mathcal{L}}} + [\beta, \;]_{\widetilde{\mathcal{L}}})\, F_{k+1}(\alpha, \alpha, \dots, \alpha, \gamma)\,. \tag{B.21}$$

According to [18, Prop. 1], twisting an $L_\infty$ quasi-isomorphism by a Maurer-Cartan element gives an $L_\infty$ quasi-isomorphism.

One can identify $L_\infty$ morphisms from a DGLA $\mathcal{L}$ to a DGLA $\widetilde{\mathcal{L}}$ with Maurer-Cartan elements of another DGLA, denoted by $\mathcal{H}$. As a graded vector space,

$$\mathcal{H} = \mathrm{Hom}(C(\mathcal{L}), \widetilde{\mathcal{L}})\,. \tag{B.22}$$

The differential $d_{\mathcal{H}}$ and the bracket $[\cdot,\cdot]_{\mathcal{H}}$ are given by the formulas:

$$d_{\mathcal{H}}\Psi = d_{\widetilde{\mathcal{L}}}\Psi - (-1)^{|\Psi|}\Psi Q\,, \tag{B.23}$$

$$[\Psi, \Theta]_{\mathcal{H}}(X) = \sum_i (-1)^{|\Theta|\,|X_i|}[\Psi(X_i), \Theta(X'_i)]_{\widetilde{\mathcal{L}}}\,, \tag{B.24}$$

where $\Delta X = \sum_i X_i \otimes X'_i$, and $Q$ is the codifferential on $C(\mathcal{L})$. The DGLA $\mathcal{H}$ is equipped with the following descending filtration:

$$\begin{aligned} &\mathcal{H} = \mathcal{F}^1\mathcal{H} \supset \mathcal{F}^2\mathcal{H} \supset \dots \supset \mathcal{F}^k\mathcal{H} \supset \dots \\ &\mathcal{F}^k\mathcal{H} = \Big\{ f \in \mathrm{Hom}(C(\mathcal{L}), \widetilde{\mathcal{L}}) \;\big|\; f\big|_{S^{<k}(\mathbf{s}^{-1}\mathcal{L})} = 0 \Big\}\,. \end{aligned} \tag{B.25}$$

The DGLA structure defined by (B.23) and (B.24) is compatible with this filtration, and the DGLA $\mathcal{H}$ is complete with respect to this filtration. Thus the group $\mathfrak{G}(\mathcal{H})$ is defined for $\mathcal{H}$ and acts on Maurer-Cartan elements of $\mathcal{H}$ according to (B.4).

Following [19, 56], the correspondence

$$F \mapsto p \circ F\,. \tag{B.26}$$

identifies an $L_\infty$ morphism $F : \mathcal{L} \succ\!\!\rightarrow \widetilde{\mathcal{L}}$ with a Maurer-Cartan element of the DGLA $\mathcal{H}$. Moreover, for two $L_\infty$ morphisms $F$ and $\widetilde{F}$, the Maurer-Cartan elements $p \circ F$ and $p \circ \widetilde{F}$ are connected by the action (B.4) of the group $\mathfrak{G}(\mathcal{H})$, and the structure maps $F_1$ and $\widetilde{F}_1$ are chain homotopic. As a result, if the Maurer-Cartan elements $p \circ F$ and $p \circ \widetilde{F}$ are equivalent and $F$ is an $L_\infty$ quasi-isomorphism, then so is $\widetilde{F}$. We say that two $L_\infty$ morphisms $F$ and $\widetilde{F}$ are *homotopy equivalent* if the corresponding Maurer-Cartan elements $p \circ F$ and $p \circ \widetilde{F}$ are connected by the action (B.4) of the group $\mathfrak{G}(\mathcal{H})$.

It is natural to ask whether two homotopy equivalent $L_\infty$ morphisms induce the same map from $\pi_0(\mathrm{MC}(\mathcal{L}))$ to $\pi_0(\mathrm{MC}(\widetilde{\mathcal{L}}))$. The following lemma gives a positive answer to this question.

**Lemma B.5** *Let $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ be DGLAs, and let $F$ and $\widetilde{F}$ be two $L_\infty$ morphisms from $\mathcal{L}$ to $\widetilde{\mathcal{L}}$. If the corresponding Maurer-Cartan elements $p \circ F$ and $p \circ \widetilde{F}$ of the DGLA $\mathcal{H}$ (B.22) are equivalent, then $F$ and $\widetilde{F}$ induce the same map from $\pi_0(\mathrm{MC}(\mathcal{L}))$ to $\pi_0(\mathrm{MC}(\widetilde{\mathcal{L}}))$.*

Proof. We need to show that for every Maurer-Cartan element $\alpha$ of $\mathcal{L}$, the Maurer-Cartan elements

$$\beta = \sum_{n=1}^{\infty} \frac{1}{n!} F_n(\alpha, \alpha, \dots, \alpha), \quad \text{and} \quad \widetilde{\beta} = \sum_{n=1}^{\infty} \frac{1}{n!} \widetilde{F}_n(\alpha, \alpha, \dots, \alpha)$$

are connected by the action (B.4) of the group $\mathfrak{G}(\widetilde{\mathcal{L}})$. Let us denote by $f$ (resp. $\widetilde{f}$) the composition $p \circ F$ (resp. $p \circ \widetilde{F}$):

$$f = p \circ F\,, \qquad \widetilde{f} = p \circ \widetilde{F}\,.$$

We know that $f$ and $\widetilde{f}$ are equivalent Maurer-Cartan elements of the DGLA $\mathcal{H}$. Hence there exists an element $\psi \in \mathcal{F}^1\mathcal{H}^0$ such that

$$\widetilde{f} = \exp([\cdot, \psi]_{\mathcal{H}}) f + \frac{\exp([\cdot, \psi]_{\mathcal{H}}) - 1}{[\cdot, \psi]_{\mathcal{H}}} \, d_{\mathcal{H}} \psi. \tag{B.27}$$

Let us consider the element

$$\sum_{k=1}^{\infty} \frac{1}{k!} (\underbrace{\alpha, \alpha, \dots, \alpha}_{k}) \tag{B.28}$$

in the completion of the coalgebra $C(\mathcal{L})$ with respect to the natural filtration coming from $\mathcal{L}$. A direct computation shows that applying both sides of equation (B.27) to the element (B.28) and using the Maurer-Cartan equation (B.2), we obtain

$$\widetilde{\beta} = \exp([\cdot, \xi]_{\widetilde{\mathcal{L}}}) \beta + \frac{\exp([\cdot, \xi]_{\widetilde{\mathcal{L}}}) - 1}{[\cdot, \xi]_{\widetilde{\mathcal{L}}}} \, d_{\widetilde{\mathcal{L}}} \xi\,,$$

where the element $\xi \in \mathcal{F}^1\widetilde{\mathcal{L}}^0$ is defined by

$$\xi = \sum_{k=1} \frac{1}{k!} \psi(\underbrace{\alpha, \alpha, \dots, \alpha}_{k})\,.$$

It follows that $\widetilde{\beta}$ is connected to the Maurer-Cartan element $\beta$ by the action (B.4) of the group $\mathfrak{G}(\widetilde{\mathcal{L}})$, concluding the proof. □

### B.3 The case of $\hbar$-adic filtration

If $(\mathcal{L}, d, [\cdot, \cdot])$ is a DGLA[9] which is not equipped with a descending filtration then, extending the differential $d$ and the Lie bracket $[\cdot, \cdot]$ by $\mathbb{C}[[\hbar]]$-linearity, we get the DGLA $\mathcal{L}[[\hbar]]$ over the ring $\mathbb{C}[[\hbar]]$ with the obvious descending filtration

$$\mathcal{F}^k \mathcal{L} = \hbar^k \mathcal{L}[[\hbar]]. \tag{B.29}$$

The new DGLA $\mathcal{L}[[\hbar]]$ is clearly complete with respect to this filtration. This case is of central importance in our paper and, here, we will give an alternative description of (iso)morphisms in the Goldman-Millson groupoid $\mathrm{MC}(\mathcal{L}[[\hbar]])$.

Let $\alpha \in \hbar\mathcal{L}^1[[\hbar]]$ and $\xi \in \hbar(\mathcal{L}^0[t])[[\hbar]]$. Due to Proposition A.1, the differential equation

$$\frac{d}{dt}\alpha(t) = d\xi + [\alpha(t), \xi] \tag{B.30}$$

[9] In this subsection we omit the subscript $\mathcal{L}$ for the differential $d_{\mathcal{L}}$ and for the bracket $[\cdot, \cdot]_{\mathcal{L}}$.

with initial condition

$$\alpha(0) = \alpha \tag{B.31}$$

has a unique solution in $\hbar(\mathcal{L}^1[t])[[\hbar]]$. We claim that if $\alpha$ satisfies the Maurer-Cartan equation

$$d\alpha + \frac{1}{2}[\alpha, \alpha] = 0,$$

then so does $\alpha(t)$. Indeed, let

$$\Psi(t) = d\alpha(t) + \frac{1}{2}[\alpha(t), \alpha(t)].$$

Taking a derivative in $t$ and using (B.30), we have

$$\frac{d}{dt}(d\alpha(t) + \frac{1}{2}[\alpha(t), \alpha(t)]) = [d\alpha(t) + \frac{1}{2}[\alpha(t), \alpha(t)], \xi],$$

that is,

$$\frac{d}{dt}\Psi(t) = [\Psi(t), \xi]\,.$$

Note that $\Psi(0) = 0$, since $\alpha$ satisfies the Maurer-Cartan equation. Then Proposition A.1 implies that $\Psi(t) \equiv 0$, i.e., $\alpha(t)$ satisfies the Maurer-Cartan equation for all $t$.

If $\xi$ does not depend on $t$ (that is $\xi \in \hbar\mathcal{L}^0[[\hbar]]$ ) then the initial value problem (B.30), (B.31) can be solved explicitly. Indeed, in this case we have

$$\alpha(t) = \exp(t[\cdot, \xi])\alpha + \frac{\exp(t[\cdot, \xi]) - 1}{[\cdot, \xi]_{\mathcal{L}}}\, d\xi\,.$$

In other words, if $\xi$ does not depend on $t$, then the evaluation of $\alpha(t)$ at $t = 1$ is connected with $\alpha$ by the action (B.4) of the group $\mathfrak{G}(\mathcal{L}[[\hbar]])$.

We will now show that, for an arbitrary element $\xi \in \hbar(\mathcal{L}^0[t])[[\hbar]]$, the evaluation $\alpha(1)$ is also connected with $\alpha$ by the action of the group $\mathfrak{G}(\mathcal{L}[[\hbar]])$. We need the following technical statement:

**Lemma B.6** *Consider a Maurer-Cartan element $\alpha$ of $\mathcal{L}[[\hbar]]$, let $\xi$ be an element of $\hbar(\mathcal{L}^0[t])[[\hbar]]$, and $\alpha(t)$ be the unique solution of (B.30) with initial condition (B.31). Then for every $\eta \in \hbar\mathcal{L}^0[[\hbar]]$ and every nonnegative integer $k$, the element*

$$\lambda(t) = \exp\left(\frac{t^{k+1}}{k+1}[\cdot, \eta]\right)\alpha(t) + \frac{\exp\left(\frac{t^{k+1}}{k+1}[\cdot, \eta]\right) - 1}{[\cdot, \eta]}\, d\eta \tag{B.32}$$

*satisfies the differential equation*

$$\frac{d}{dt}\lambda(t) = d\widetilde{\xi} + [\lambda(t), \widetilde{\xi}], \tag{B.33}$$

*where*

$$\widetilde{\xi} = t^k\eta + \exp\left(\frac{t^{k+1}}{k+1}[\cdot, \eta]\right)\xi\,.$$

Proof. We compute the derivative explicitly and obtain

$$\frac{d}{dt}\lambda(t) = t^k[\cdot, \eta]\exp\left(\frac{t^{k+1}}{k+1}[\cdot, \eta]\right)\alpha(t) + \exp\left(\frac{t^{k+1}}{k+1}[\cdot, \eta]\right)d\xi$$
$$+ \exp\left(\frac{t^{k+1}}{k+1}[\cdot, \eta]\right)[\alpha(t), \xi] + t^k\exp\left(\frac{t^{k+1}}{k+1}[\cdot, \eta]\right)d\eta.$$

The latter can be rewritten as

$$\frac{d}{dt}\lambda(t) = \left[\exp\left(\frac{t^{k+1}}{k+1}[\cdot,\eta]\right)\alpha(t), t^k\eta + \exp\left(\frac{t^{k+1}}{k+1}[\cdot,\eta]\right)\xi\right]$$
$$+ d(t^k\eta) + [\cdot,\eta]\frac{\exp\left(\frac{t^{k+1}}{k+1}[\cdot,\eta]\right)-1}{[\cdot,\eta]}d(t^k\eta)$$
$$+ \exp\left(\frac{t^{k+1}}{k+1}[\cdot,\eta]\right) d\exp\left(-\frac{t^{k+1}}{k+1}[\cdot,\eta]\right)\exp\left(\frac{t^{k+1}}{k+1}[\cdot,\eta]\right)\xi$$
$$- d\exp\left(\frac{t^{k+1}}{k+1}[\cdot,\eta]\right)\xi + d\exp\left(\frac{t^{k+1}}{k+1}[\cdot,\eta]\right)\xi.$$

Hence

$$\frac{d}{dt}\lambda(t) = d\left(t^k\eta + \exp\left(\frac{t^{k+1}}{k+1}[\cdot,\eta]\right)\xi\right)$$
$$+ \left[\lambda(t), t^k\eta + \exp\left(\frac{t^{k+1}}{k+1}[\cdot,\eta]\right)\xi\right]$$
$$+ \left(\exp\left(\frac{t^{k+1}}{k+1}[\cdot,\eta]\right) d\exp\left(-\frac{t^{k+1}}{k+1}[\cdot,\eta]\right) - d\right)\exp\left(\frac{t^{k+1}}{k+1}[\cdot,\eta]\right)\xi$$
$$- \left[\frac{\exp\left(\frac{t^{k+1}}{k+1}[\cdot,\eta]\right)-1}{[\cdot,\eta]}d\eta, \exp\left(\frac{t^{k+1}}{k+1}[\cdot,\eta]\right)\xi\right].$$

Thus, in order to prove the proposition, we need to show that

$$\exp\left(\frac{t^{k+1}}{k+1}[\cdot,\eta]\right)\left[d, \exp\left(-\frac{t^{k+1}}{k+1}[\cdot,\eta]\right)\right] = \left[\frac{\exp\left(\frac{t^{k+1}}{k+1}[\cdot,\eta]\right)-1}{[\cdot,\eta]}d\eta, \cdot\right]. \tag{B.34}$$

One now verifies that both sides of (B.34) satisfy the same differential equation:

$$\frac{d}{dt}\Theta(t) = \left[t^k[\cdot,\eta], \Theta(t)\right] + t^k[d\eta,\cdot],$$

with the same initial condition

$$\Theta(0) = 0.$$

Therefore, by Proposition A.1, (B.34) holds and the result follows. □

We can now prove the main result of this subsection.

**Proposition B.7** *Let $\alpha$ be a Maurer-Cartan element of $\mathcal{L}[[\hbar]]$, $\xi$ be an element of $\hbar(\mathcal{L}^0[t])[[\hbar]]$, and $\alpha(t)$ be the unique solution of (B.30) with the initial condition (B.31). Then the Maurer-Cartan element $\alpha(1)$ is connected with $\alpha$ by the action (B.4) of the group $\mathfrak{G}(\mathcal{L}[[\hbar]])$.*

Proof. Let us denote by $\mathrm{E}(\xi,\alpha)$ the evaluation of $\alpha(t)$ at $t=1$:

$$\mathrm{E}(\xi,\alpha) = \alpha(t)\Big|_{t=1}, \tag{B.35}$$

where $\alpha(t)$ is the solution of the differential equation (B.30) satisfying $\alpha(0)=\alpha$. In general, we have $\xi \in \hbar^m(\mathcal{L}^0[t])[[\hbar]]$, where $m$ is a positive integer, and

$$\xi = t^k\hbar^m\xi_{m,k} + t^{k+1}\hbar^m\xi_{m,k+1} + t^{k+2}\hbar^m\xi_{m,k+2} + \cdots + t^N\hbar^m\xi_{m,N} \mod \hbar^{m+1}, \tag{B.36}$$

$\xi_{m,j} \in \mathcal{L}^0$, for some nonnegative integers $k$ and $N$ with $k \le N$. Lemma B.6 implies that

$$\big(\mathrm{E}(\xi,\alpha)\big)^{\exp\left[-\frac{\hbar^m \xi_{m,k}}{k+1}\right]} = \mathrm{E}(\xi',\alpha)\,, \qquad (B.37)$$

where $\xi' \in \hbar^m(\mathcal{L}^0[t])[[\hbar]]$ and

$$\xi' = t^{k+1}\hbar^m \xi'_{m,k+1} + t^{k+2}\hbar^m \xi'_{m,k+2} + \dots + t^N \hbar^m \xi'_{m,N} \qquad \mathrm{mod} \quad \hbar^{m+1}, \qquad (B.38)$$

with $\xi'_{m,j} \in \mathcal{L}^0$.

Repeating this argument $N-k-1$ times, we see that there exists elements $\widetilde{\xi}_1 \in \hbar^{m+1}(\mathcal{L}^0[t])[[\hbar]]$ and $\eta_m \in \hbar^m \mathcal{L}^0[[\hbar]]$ such that

$$\mathrm{E}(\widetilde{\xi}_1,\alpha) = \big(\mathrm{E}(\xi,\alpha)\big)^{\exp[\eta_m]}\,. \qquad (B.39)$$

Therefore we have an infinite series of elements $\eta_{m+n-1} \in \hbar^{m+n-1}\mathcal{L}^0[[\hbar]]$ and elements

$$\widetilde{\xi}_n \in \hbar^{m+n}(\mathcal{L}^0[t])[[\hbar]]\,, \qquad n \ge 1,$$

such that

$$\mathrm{E}(\widetilde{\xi}_n,\alpha) = \big(\mathrm{E}(\xi,\alpha)\big)^{\Lambda_{n,m}}\,, \qquad (B.40)$$

where

$$\Lambda_{n,m} = \exp[\eta_m]\exp[\eta_{m+1}]\dots\exp[\eta_{m+n-1}]\,.$$

Since for large $n$ the element $\eta_{m+n-1}$ lies in the deeper filtration subalgebra $\hbar^{m+n-1}\mathcal{L}^0[[\hbar]]$, the infinite product

$$\Lambda = \exp[\eta_m]\exp[\eta_{m+1}]\dots\exp[\eta_{m+n}]\dots$$

is a well defined element of $\mathfrak{G}(\mathcal{L}[[\hbar]])$. Furthermore, due to (B.40), we have

$$\alpha = \big(\alpha(1)\big)^{\Lambda}$$

and the proposition follows. □

# C Independence of Fedosov's differential: proof of Theorem 2.6

This section presents the proof of Theorem 2.6, asserting that the correspondence between equivalence classes of star products and equivalence classes of formal Poisson structures induced by the sequence of $L_\infty$ quasi-isomorphisms (2.31) does not depend on the choice of the connection/Fedosov differential.

We would like to emphasize that we prove (and use) Theorem 2.6 in the setting where the ground field $\mathbb{C}$ is replaced by the ground ring[10] $\mathbb{C}[[\hbar]]$. In particular, the connection form $\Gamma$ in (2.4) is replaced by a general formal Taylor power series in $\hbar$:

$$\Gamma_\hbar = \Gamma_0 + \hbar\Gamma_1 + \hbar^2\Gamma_2 + \dots,$$

and the element $A$ (2.6) is allowed to have the more general form:

$$A = \sum_{p=2,r=0}^{\infty} dx^k \hbar^r A^j_{r;ki_1\dots i_p}(x) y^{i_1}\dots y^{i_p}\frac{\partial}{\partial y^j} \in \Omega^1(M,\mathcal{T}^1_{poly})[[\hbar]]\,.$$

[10]See Remark 2.4.

Although the proof is long and technical, the general idea is simple. The key point is observing that changing the geometric Fedosov differential corresponds to twisting the DGLAs $\Omega^\bullet(M, \mathcal{T}^{\bullet+1}_{poly})[[\hbar]]$ and $\Omega^\bullet(M, C^{\bullet+1}(\mathcal{S}M))[[\hbar]]$ by a Maurer-Cartan element which is equivalent to zero. What makes the proof intricate is that one needs filtrations on these DGLAs which are more subtle than the $\hbar$-adic ones.

Proof of Theorem 2.6. Let us introduce the following descending filtrations on the DGLAs $\Omega^\bullet(M, \mathcal{T}^{\bullet+1}_{poly})[[\hbar]]$ and $\Omega^\bullet(M, C^{\bullet+1}(\mathcal{S}M))[[\hbar]]$: The $m$-th subspace $\mathfrak{F}^m\Omega^\bullet(M, C^k(\mathcal{S}M))[[\hbar]]$ of the filtration on $\Omega^\bullet(M, C^k(\mathcal{S}M))[[\hbar]]$ consists of the elements $P$ of $\Omega^\bullet(M, C^k(\mathcal{S}M))[[\hbar]]$ satisfying

$$P\left(\mathcal{F}^{p_1}\mathcal{S}M[[\hbar]] \otimes \mathcal{F}^{p_2}\mathcal{S}M[[\hbar]] \otimes \cdots \otimes \mathcal{F}^{p_k}\mathcal{S}M[[\hbar]]\right) \subset \bigoplus_{s+t=m+p_1+p_2+\cdots+p_k} \Omega^s(\mathcal{F}^t\mathcal{S}M[[\hbar]]), \quad \text{(C.1)}$$

where the filtration $\mathcal{F}^\bullet\mathcal{S}M[[\hbar]]$ is defined in Remark 2.5; the $m$-th subspace $\mathfrak{F}^m\Omega^\bullet(M, \mathcal{T}^k_{poly})[[\hbar]]$ of the filtration on $\Omega^\bullet(M, \mathcal{T}^k_{poly})[[\hbar]]$ is specified by the same condition: for $\gamma \in \Omega^\bullet(M, \mathcal{T}^k_{poly})[[\hbar]]$, viewed as a element of $\Omega^\bullet(M, C^k(\mathcal{S}M))[[\hbar]]$,

$$\gamma\left(\mathcal{F}^{p_1}\mathcal{S}M[[\hbar]] \otimes \mathcal{F}^{p_2}\mathcal{S}M[[\hbar]] \otimes \cdots \otimes \mathcal{F}^{p_k}\mathcal{S}M[[\hbar]]\right) \subset \bigoplus_{s+t=m+p_1+p_2+\cdots+p_k} \Omega^s(\mathcal{F}^t\mathcal{S}M[[\hbar]]). \quad \text{(C.2)}$$

The filtrations $\mathfrak{F}^\bullet\Omega^\bullet(M, \mathcal{T}^{\bullet+1}_{poly})[[\hbar]]$ and $\mathfrak{F}^\bullet\Omega^\bullet(M, C^\bullet(\mathcal{S}M))[[\hbar]]$ assign to $y^i$, $dx^i$, $\partial_{y^i}$ and $\hbar$ the degrees 1, 1, $-1$, and 2, respectively. For the filtration $\mathfrak{F}^\bullet$ on $\Omega^\bullet(M, \mathcal{T}^{\bullet+1}_{poly})[[\hbar]]$ we have

$$\Omega^\bullet(M, \mathcal{T}^{\bullet+1}_{poly})[[\hbar]] = \lim_m \Omega^\bullet(M, \mathcal{T}^{\bullet+1}_{poly})[[\hbar]] \,/\, \mathfrak{F}^m\Omega^\bullet(M, \mathcal{T}^{\bullet+1}_{poly})[[\hbar]] \quad \text{(C.3)}$$

and

$$\Omega^\bullet(M, \mathcal{T}^{\bullet+1}_{poly})[[\hbar]] = \mathfrak{F}^{-d}\Omega^\bullet(M, \mathcal{T}^{\bullet+1}_{poly})[[\hbar]]\,, \quad \text{(C.4)}$$

where $d$ is the dimension of $M$. Although the filtration $\mathfrak{F}^\bullet$ on $\Omega^\bullet(M, C^{\bullet+1}(\mathcal{S}M))[[\hbar]]$ is unbounded in both directions, we still have the following important properties:

$$\Omega^\bullet(M, C^{\bullet+1}(\mathcal{S}M))[[\hbar]] = \lim_m \Omega^\bullet(M, C^{\bullet+1}(\mathcal{S}M))[[\hbar]] \,/\, \mathfrak{F}^m\Omega^\bullet(M, C^{\bullet+1}(\mathcal{S}M))[[\hbar]]\,, \quad \text{(C.5)}$$

$$\Omega^\bullet(M, C^{\bullet+1}(\mathcal{S}M))[[\hbar]] = \bigcup_m \mathfrak{F}^m\Omega^\bullet(M, C^{\bullet+1}(\mathcal{S}M))[[\hbar]]\,. \quad \text{(C.6)}$$

Property (C.5) follows from the fact that the local sections of the sheaf $C^k(\mathcal{S}M)$ are continuous ($\mathcal{O}(U)$-polylinear) maps from $\Gamma(U, \mathcal{S}M)^{\otimes k}$ to $\Gamma(U, \mathcal{S}M)$ in the $y$-adic topology on $\Gamma(U, \mathcal{S}M)$.

Let us consider two different geometric Fedosov differentials

$$D = \nabla - \delta + A\,, \quad \widetilde{D} = \widetilde{\nabla} - \delta + \widetilde{A}, \quad \text{(C.7)}$$

and let $\tau$, $\tilde{\tau}$ be the corresponding isomorphisms (see (2.23)),

$$\begin{aligned} \tau &: \mathcal{X}^\bullet(M)[[\hbar]] \xrightarrow{\cong} \Gamma(M, \mathcal{T}^\bullet_{poly})[[\hbar]] \cap \ker D\,, \\ \tilde{\tau} &: \mathcal{X}^\bullet(M)[[\hbar]] \xrightarrow{\cong} \Gamma(M, \mathcal{T}^\bullet_{poly})[[\hbar]] \cap \ker \widetilde{D}\,. \end{aligned} \quad \text{(C.8)}$$

The geometric Fedosov differential $\widetilde{D}$ can be rewritten as

$$\widetilde{D} = D + H, \quad \text{where} \quad H \in \mathfrak{F}^1\Omega^1(M, \mathcal{T}^1_{poly})[[\hbar]]. \quad \text{(C.9)}$$

Since $\widetilde{D}^2 = 0$, the element $H$ satisfies the Maurer-Cartan equation

$$DH + \frac{1}{2}[H, H]_{SN} = 0.$$

Let us consider the natural extension of the map $\sigma$ (2.10) to $\Omega^\bullet(M, \mathcal{T}^\bullet_{poly})[[\hbar]]$,

$$\sigma(\gamma) = \gamma\Big|_{y^i = dx^i = 0}.$$

For example, the subspace $\Omega^0(M, \mathcal{T}^p_{poly})[[\hbar]] \cap \ker \sigma$ consists of fiberwise polyvectors of the form

$$\gamma = \sum_{k \geq 0} \sum_{q \geq 1} \hbar^k \, \gamma^{i_1 i_2 \dots i_p}_{k; j_1 j_2 \dots j_q}(x) \, y^{j_1} y^{j_2} \dots y^{j_q} \, \partial_{y^{i_1}} \wedge \partial_{y^{i_2}} \wedge \dots \wedge \partial_{y^{i_p}},$$

(with summation in $q$ starting with 1). The element $H$ (C.9) is a Maurer-Cartan element of the the following truncation of the DGLA $(\Omega^\bullet(M, \mathcal{T}^{\bullet+1}_{poly}), D, [\,,\,]_{SN})$:

$$\mathcal{L}_{\mathcal{T}} = \bigoplus_{k \geq 0} \Omega^0(M, \mathcal{T}^{k+1}_{poly})[[\hbar]] \cap \ker \sigma \quad \oplus \quad \bigoplus_{k \geq 0} \Omega^{\geq 1}(M, \mathcal{T}^{k+1}_{poly})[[\hbar]]. \tag{C.10}$$

At the level of the associated graded complex

$$\bigoplus_m \mathfrak{F}^m \mathcal{L}_{\mathcal{T}} / \mathfrak{F}^{m+1} \mathcal{L}_{\mathcal{T}},$$

the differential $D$ (C.7) boils down to $-\delta$. Due to (2.9) and Remark 2.2, the associated graded complex of $\mathcal{L}_{\mathcal{T}}$ is acyclic. Using properties (C.3) and (C.4), we conclude that, for all $m$, the sub DGLAs $\mathfrak{F}^m \mathcal{L}_{\mathcal{T}}$ and the sub DGLA $\mathcal{L}_{\mathcal{T}}$ are acyclic. Theorem B.1 then implies that every Maurer-Cartan element of the DGLA (C.10) can be brought to zero via the action of the group

$$\exp\Big(\mathfrak{F}^1 \, \Omega^0(M, \mathcal{T}^1_{poly})[[\hbar]] \cap \ker \sigma\Big). \tag{C.11}$$

Since $H$ is a Maurer-Cartan element, it follows that there exists an element

$$X \in \mathfrak{F}^1 \, \Omega^0(M, \mathcal{T}^1_{poly})[[\hbar]] \cap \ker \sigma \tag{C.12}$$

such that

$$H = \frac{\exp([\cdot, X]_{SN}) - 1}{[\cdot, X]_{SN}} DX. \tag{C.13}$$

Since components of $H$ have degrees in $y$ greater than or equal to 1 and the contracting homotopy $\delta^{-1}$ for $\delta$ raises the degree in $y$ by 1, we conclude that one can find the element $X$ (C.12) satisfying (C.13) as well as the additional property

$$\partial_{y^i} X\Big|_{y=0} = 0, \quad \text{for all} \quad i. \tag{C.14}$$

In other words, we can find $X$ whose components have degrees in fiber coordinates $y$ greater than or equal to 2.

It follows from (C.13) that the operator $e^X$ intertwines the differentials $D$ and $\widetilde{D}$:

$$\widetilde{D} = e^{-X} \circ D \circ e^X. \tag{C.15}$$

Furthermore, combining equation (C.15) with property (C.14) we deduce that, for every formal Poisson structure $\pi$,

$$\exp(-[\cdot, X]_{SN}) \tilde{\tau}(\pi) = \tau(\pi). \tag{C.16}$$

Indeed, (C.15) implies that both sides of (C.16) are $D$-flat. Then, (C.14) implies that

$$\sigma(\tau(\pi)) = \sigma(\exp(-[\cdot, X]_{SN})\tilde{\tau}(\pi)).$$

Therefore, since every $D$-flat section $\gamma$ is uniquely determined by its image $\sigma(\gamma)$, we conclude that (C.16) holds. Combining equations (C.13) and (C.16), we deduce that

$$H + \tilde{\tau}(\pi) = \exp([\cdot, X]_{SN})\tau(\pi) + \frac{\exp([\cdot, X]_{SN}) - 1}{[\cdot, X]_{SN}} DX. \tag{C.17}$$

Hence the Maurer-Cartan elements $\tau(\pi)$ and $H + \tilde{\tau}(\pi)$ of the DGLA $(\Omega^\bullet(M, \mathcal{T}^{\bullet+1}_{poly})[[\hbar]], D, [\,,\,]_{SN})$ are equivalent.

Let $K^{tw}$ be the $L_\infty$ quasi-isomorphism

$$K^{tw} : (\Omega^\bullet(M, \mathcal{T}^{\bullet+1}_{poly})[[\hbar]], D, [\,,\,]_{SN}) \succ\!\!\rightarrow (\Omega^\bullet(M, C^{\bullet+1}(\mathcal{S}M))[[\hbar]], D + \partial^{\mathrm{Hoch}}, [\,,\,]_G)$$

of Subsection 2.2, and let $\widetilde{K}^{tw}$ be the analogous $L_\infty$ quasi-isomorphism obtained by replacing $D$ with the other geometric Fedosov differential $\widetilde{D}$ (see (C.7)). Let $\mu$ be the Maurer-Cartan element of the DGLA

$$(\Omega^\bullet(M, C^{\bullet+1}(\mathcal{S}M))[[\hbar]], D + \partial^{\mathrm{Hoch}}, [\,,\,]_G) \tag{C.18}$$

corresponding to the Maurer-Cartan element $\tau(\pi)$ via the $L_\infty$ quasi-isomorphism $K^{tw}$, i.e.,

$$\mu = \sum_{n=1}^{\infty} \frac{1}{n!} K^{tw}_n(\tau(\pi), \tau(\pi), \dots, \tau(\pi)). \tag{C.19}$$

Similarly, we let $\widetilde{\mu}$ be the Maurer-Cartan element in $(\Omega^\bullet(M, C^{\bullet+1}(\mathcal{S}M))[[\hbar]], \widetilde{D} + \partial^{\mathrm{Hoch}}, [\,,\,]_G)$ corresponding to the Maurer-Cartan element $\tilde{\tau}(\pi)$ via $\widetilde{K}^{tw}$:

$$\widetilde{\mu} = \sum_{n=1}^{\infty} \frac{1}{n!} \widetilde{K}^{tw}_n(\tilde{\tau}(\pi), \tilde{\tau}(\pi), \dots, \tilde{\tau}(\pi)). \tag{C.20}$$

Combining $\widetilde{\mu}$ with $H = \widetilde{D} - D$, we get a Maurer-Cartan element $H + \widetilde{\mu}$ of the DGLA (C.18).

**Claim C.1** *The Maurer-Cartan element $H+\widetilde{\mu}$ corresponds to the Maurer-Cartan element $H+\tilde{\tau}(\pi)$ via the $L_\infty$ quasi-isomorphism $K^{tw}$, that is,*

$$H + \widetilde{\mu} = \sum_{n=1}^{\infty} \frac{1}{n!} K^{tw}_n(H + \tilde{\tau}(\pi), H + \tilde{\tau}(\pi), \dots, H + \tilde{\tau}(\pi)). \tag{C.21}$$

Proof. We notice that $\widetilde{K}^{tw}$ is obtained from $K^{tw}$ via twisting by the Maurer-Cartan element $H$. Therefore the right hand side of (C.21) can be rewritten as

$$\sum_{n=1}^{\infty} \frac{1}{n!} K^{tw}_n(H + \tilde{\tau}(\pi), H + \tilde{\tau}(\pi), \dots, H + \tilde{\tau}(\pi))$$
$$= \sum_{n=1}^{\infty} \frac{1}{n!} K^{tw}_n(H, H, \dots, H) + \sum_{n=1}^{\infty} \frac{1}{n!} \widetilde{K}^{tw}_n(\tilde{\tau}(\pi), \tilde{\tau}(\pi), \dots, \tilde{\tau}(\pi)).$$

Using the properties P 2 and P 5, we rewrite the first sum in the previous equation as

$$\sum_{n=1}^{\infty} \frac{1}{n!} K^{tw}_n(H, H, \dots, H) = H.$$

This proves Claim C.1. $\triangledown$

Since the Maurer-Cartan elements $\tau(\pi)$ and $H + \tilde{\tau}(\pi)$ are equivalent in the DGLA

$$(\Omega^{\bullet}(M, \mathcal{T}^{\bullet+1}_{poly})[[\hbar]], D, [\,,\,]_{SN}), \tag{C.22}$$

Claim C.1 and Proposition B.4 from Appendix B imply that the Maurer-Cartan elements $\mu$ and $H + \widetilde{\mu}$ are equivalent in the DGLA (C.18). Furthermore, since the $L_\infty$ quasi-isomorphism $K^{tw}$ is compatible with the filtrations (C.1) and (C.2), we conclude that the transformation connecting $\mu$ and $H + \widetilde{\mu}$ has the form

$$\exp(\eta), \tag{C.23}$$

where $\eta$ is an element of $\mathfrak{F}^1\Omega^{\bullet}(M, C^{\bullet+1}(\mathcal{S}M))[[\hbar]]$ of total degree 0.

In general the element $\mu$ (C.19) have components in exterior degrees 0, 1, and 2. Let us show that the components of exterior degrees 1 and 2 can be eliminated by a transformation of the form (C.23).

**Claim C.2** *There exists an element $\eta \in \mathfrak{F}^1\Omega^{\bullet}(M, C^{\bullet+1}(\mathcal{S}M))[[\hbar]]$ of total degree 0 such that the Maurer-Cartan element*

$$\Pi_{\text{fib}} = \exp([\cdot, \eta]_G)\mu + \frac{\exp([\cdot, \eta]_G) - 1}{[\cdot, \eta]_G}\,(D\eta + \partial^{\text{Hoch}}\eta) \tag{C.24}$$

*belongs to $\Omega^0(M, C^2(\mathcal{S}M))[[\hbar]]$ .*

Proof. Let us denote by $\mu_1$ (resp. by $\mu_2$) the component of $\mu$ of exterior degree 1 (resp. 2). According to the definition of $K^{tw}$ (2.30), we have

$$\mu_1 = \sum_{n=1} \frac{1}{n!} K_{n+1}\left(\mu^D_U, \tau(\pi), \tau(\pi), \dots, \tau(\pi)\right) \tag{C.25}$$

$$\mu_2 = \sum_{n=1} \frac{1}{n!} K_{n+2}\left(\mu^D_U, \mu^D_U, \tau(\pi), \tau(\pi), \dots, \tau(\pi)\right), \tag{C.26}$$

where $\mu^D_U$ is defined in (2.27). Since the series $\pi$ (2.41) starts with $\hbar$, we have

$$\mu_1 = \sum_{n=1} \frac{1}{n!} K_{n+1}\big(-dx^i\partial_{y^i}, \pi^y, \pi^y, \dots, \pi^y\big) \mod \mathfrak{F}^1\Omega^1(M, C^1(\mathcal{S}M))[[\hbar]] \tag{C.27}$$

$$\mu_2 = \sum_{n=1} \frac{1}{n!} K_{n+2}\big(-dx^i\partial_{y^i}, -dx^j\partial_{y^j}, \pi^y, \pi^y, \dots, \pi^y\big) \mod \mathfrak{F}^1\Omega^2(M, C^0(\mathcal{S}M))[[\hbar]], \tag{C.28}$$

where

$$\pi^y = \pi^{ij}(x)\frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j}.$$

(Note that since $\pi$ is a series in $\hbar$, the coefficients $\pi^{ij}$ are $\hbar$-dependent.) We claim that

$$K_{n+1}\big(-dx^i\partial_{y^i}, \pi^y, \pi^y, \dots, \pi^y\big) = 0\,, \quad K_{n+2}\big(-dx^i\partial_{y^i}, -dx^j\partial_{y^j}, \pi^y, \pi^y, \dots, \pi^y\big) = 0$$

for all $n \geq 1$. The latter equality follows from the fact that the components of the vector $dx^i\partial_{y^i}$ and the bivector $\pi^y$ do not depend on $y$. As for the former equality, we note that every term in

$$K_{n+1}\big(-dx^i\partial_{y^i}, \pi^y, \pi^y, \dots, \pi^y\big)(a)\,, \qquad a \in \Gamma(M, \mathcal{S}M)$$

contains a $y$-derivative of the expression $\pi^{ij}(x)\partial_{y^i}\partial_{y^j}a(x,y)$ as a factor, and $\pi^{ij}(x)\partial_{y^i}\partial_{y^j}a(x,y) = 0$ due to the antisymmetry of $\pi$. Thus both components $\mu_1$ and $\mu_2$ belong to $\mathfrak{F}^1\Omega^{\bullet}(M, C^{\bullet+1}(\mathcal{S}M))[[\hbar]]$.

On the other hand, the differential $D+\partial^{\rm Hoch}$ boils down to $-\delta+\partial^{\rm Hoch}$ at the level of the associated graded complex

$$\bigoplus_m \mathfrak{F}^m\Omega^\bullet(M, C^{\bullet+1}(\mathcal{S}M))[[\hbar]]/\mathfrak{F}^{m+1}\Omega^\bullet(M, C^{\bullet+1}(\mathcal{S}M))[[\hbar]]\,.$$

Thus Claim C.2 follows from the fact that the differential $\delta$ is acyclic in positive exterior degree. $\triangledown$

Since $\Pi_{\rm fib}$ (C.24) has exterior degree 0, the Maurer-Cartan equation for $\Pi_{\rm fib}$,

$$D\Pi_{\rm fib} + \partial^{\rm Hoch}\Pi_{\rm fib} + \frac{1}{2}[\Pi_{\rm fib}, \Pi_{\rm fib}]_G = 0,$$

is equivalent to the pair of equations

$$D\Pi_{\rm fib} = 0 \tag{C.29}$$

$$\partial^{\rm Hoch}\Pi_{\rm fib} + \frac{1}{2}[\Pi_{\rm fib}, \Pi_{\rm fib}]_G = 0. \tag{C.30}$$

Equation (C.30) implies that $\Pi_{\rm fib}$ gives us a new associative product on $\mathcal{S}M[[\hbar]]$:

$$a_1 \diamond a_2 = a_1 a_2 + \Pi_{\rm fib}(a_1, a_2), \tag{C.31}$$

where $a_1, a_2 \in \Gamma(M, \mathcal{S}M)[[\hbar]]$. Equation (C.29) implies that $D$ is a derivation of the product $\diamond$.

Similarly, the Maurer-Cartan element $\widetilde{\mu}$ is equivalent to a Maurer-Cartan element $\tilde{\Pi}_{\rm fib} \in \Omega^0(M, C^2(\mathcal{S}M))[[\hbar]]$ of the DGLA

$$(\Omega^\bullet(M, C^{\bullet+1}(\mathcal{S}M))[[\hbar]], \widetilde{D} + \partial^{\rm Hoch}, [\,,\,]_G). \tag{C.32}$$

Just as $\Pi_{\rm fib}$, the element $\tilde{\Pi}_{\rm fib}$ gives us an associative product on $\mathcal{S}M[[\hbar]]$ by

$$a_1 \,\tilde{\diamond}\, a_2 = a_1 a_2 + \tilde{\Pi}_{\rm fib}(a_1, a_2), \tag{C.33}$$

and the differential $\widetilde{D}$ is a derivation of $\tilde{\diamond}$.

To explain how $\Pi_{\rm fib}$ and $\tilde{\Pi}_{\rm fib}$ are related to the corresponding star products $*$ and $\widetilde{*}$ on $M$, recall that we have the isomorphisms

$$\tau : \mathcal{O}(M)[[\hbar]] \to \Gamma(M, \mathcal{S}M)[[\hbar]] \cap \ker D \quad \text{and} \quad \widetilde{\tau} : \mathcal{O}(M)[[\hbar]] \to \Gamma(M, \mathcal{S}M)[[\hbar]] \cap \ker \widetilde{D}. \tag{C.34}$$

These isomorphisms are constructed by iterating the following equations:

$$\tau(f) = f + \delta^{-1}(\nabla\tau(f) + A\cdot\tau(f))\,, \qquad f \in \mathcal{O}(M)[[\hbar]], \tag{C.35}$$

$$\widetilde{\tau}(f) = f + \delta^{-1}(\widetilde{\nabla}\widetilde{\tau}(f) + \widetilde{A}\cdot\widetilde{\tau}(f))\,, \qquad f \in \mathcal{O}(M)[[\hbar]], \tag{C.36}$$

in degrees in the fiber coordinates $y$'s, respectively. The star products $*$, corresponding to $\Pi_{\rm fib}$, and $*'$, corresponding to $\tilde{\Pi}_{\rm fib}$, are defined by

$$f_1 * f_2 = f_1 f_2 + \Pi_{\rm fib}(\tau(f_1), \tau(f_2))\Big|_{y=0} \quad \text{and} \quad f_1 \widetilde{*} f_2 = f_1 f_2 + \tilde{\Pi}_{\rm fib}(\widetilde{\tau}(f_1), \widetilde{\tau}(f_2))\Big|_{y=0}, \tag{C.37}$$

respectively, where $f_1, f_2 \in \mathcal{O}(M)[[\hbar]]$. Our final goal is to show that $*$ is equivalent to $\widetilde{*}$.

Let us combine $\tilde{\Pi}_{\rm fib}$ with the difference $H = \widetilde{D} - D$ to get a Maurer-Cartan element $H + \tilde{\Pi}_{\rm fib}$ of the DGLA (C.18). Next, we will show that the Maurer-Cartan elements $\Pi_{\rm fib}$ and $H + \tilde{\Pi}_{\rm fib}$ of the DGLA (C.18) are connected by an equivalence transformation of a special form.

**Claim C.3** *There exists an element* $\psi \in \mathfrak{F}^1\,\Omega^0(M, C^1(\mathcal{S}M))[[\hbar]]$ *such that*

$$H + \tilde{\Pi}_{\rm fib} = \exp([\cdot, \psi]_G)\Pi_{\rm fib} + \frac{\exp([\cdot,\psi]_G) - 1}{[\cdot, \psi]_G}\,(D\psi + \partial^{\rm Hoch}\psi). \qquad \text{(C.38)}$$

Proof. Since $\widetilde{\mu}$ is equivalent to $\tilde{\Pi}_{\rm fib}$ in the DGLA (C.32), one can check that the Maurer-Cartan element $H + \widetilde{\mu}$ is equivalent to $H + \tilde{\Pi}_{\rm fib}$ in the DGLA (C.18). Using that the Maurer-Cartan element $\mu$ is equivalent to $\Pi_{\rm fib}$ and $H + \widetilde{\mu}$ in (C.18), we conclude that the Maurer-Cartan elements $\Pi_{\rm fib}$ and $H + \tilde{\Pi}_{\rm fib}$ are also equivalent. In addition, using Claim C.2, we see that the equivalence transformation which connects the Maurer-Cartan elements $\Pi_{\rm fib}$ and $H + \tilde{\Pi}_{\rm fib}$ has the form

$$\exp(\psi),$$

where $\psi$ is an element of $\mathfrak{F}^1\Omega^\bullet(M, C^{\bullet+1}(\mathcal{S}M))[[\hbar]]$ of total degree 0.

In general $\psi$ may have two non-zero components,

$$\psi = \psi_0 + \psi_1,$$

where $\psi_0 \in \mathfrak{F}^1\Omega^0(M, C^1(\mathcal{S}M))[[\hbar]]$ and $\psi_1 \in \mathfrak{F}^1\Omega^1(M, C^0(\mathcal{S}M))[[\hbar]]$. Our purpose is to show that $\psi_1$ can be eliminated by adjusting $\psi$ via the following transformation[11]:

$$\psi \mapsto {\rm CH}\big(D\theta + \partial^{Hoch}\theta + [\Pi_{\rm fib}, \theta]_G\ ,\ \psi\big), \qquad \text{(C.39)}$$

where $\theta \in \mathfrak{F}^1\,\Omega^0(M, C^0(\mathcal{S}M))[[\hbar]]$ and CH is the Campbell-Hausdorff series (B.6). The key point is that the element $\exp(D\theta + \partial^{\rm Hoch}\theta + [\Pi_{\rm fib}, \theta]_G)$ leaves the Maurer-Cartan element $\Pi_{\rm fib}$ unchanged. Hence the element $\psi$ in (C.38) can always be replaced by the right-hand side of (C.39).

Let us suppose that

$$\psi_1 \in \mathfrak{F}^m\,\Omega^1(M, C^0(\mathcal{S}M))[[\hbar]], \qquad \text{(C.40)}$$

for $m \ge 1$. Combining the contributions to $\Omega^2(M, C^0(\mathcal{S}M))[[\hbar]]$ in (C.38) we see that

$$\delta\psi_1 = 0 \qquad \text{mod} \qquad \mathfrak{F}^{m+1}\,\Omega^2(M, C^0(\mathcal{S}M))[[\hbar]]. \qquad \text{(C.41)}$$

Using the acyclicity of $\delta$ in positive exterior degrees, we conclude that there exists a

$$\theta_m \in \mathfrak{F}^m\Omega^0(M, C^0(\mathcal{S}M))[[\hbar]]$$

such that

$$\psi_1 - \delta\theta_m \in \mathfrak{F}^{m+1}\,\Omega^1(M, C^0(\mathcal{S}M))[[\hbar]].$$

The latter means that the $\Omega^1$-component of ${\rm CH}\big(D\theta + \partial^{\rm Hoch}\theta + [\Pi_{\rm fib}, \theta]_G\,,\ \psi\big)$ lies in the "smaller" filtration subspace $\mathfrak{F}^{m+1}\,\Omega^1(M, C^0(\mathcal{S}M))[[\hbar]]$. Iterating this argument infinitely many times and using the completeness of the filtration $\mathfrak{F}^\bullet$, we conclude that there exists an element

$$\theta \in \mathfrak{F}^1\,\Omega^0(M, C^0(\mathcal{S}M))[[\hbar]]$$

such that

$${\rm CH}\big(D\theta + \partial^{\rm Hoch}\theta + [\Pi_{\rm fib}, \theta]_G\,,\ \psi\big) \in \Omega^0(M, C^1(\mathcal{S}M))[[\hbar]].$$

This completes the proof of Claim C.3. $\triangledown$

[11] Following E. Getzler [34, 35] the Goldman-Millson groupoid of the DGLA (C.18) can be upgraded to a 2-groupoid. The transformation (C.39) is an example of a 2-morphism in this 2-groupoid.

Since the element $\psi$ has zero exterior degree, equation (C.38) splits into its homogeneous exterior degree components:

$$H = \frac{\exp([\cdot,\psi]_G) - 1}{[\cdot,\psi]_G} D\psi, \tag{C.42}$$

$$\tilde{\Pi}_{\text{fib}} = \exp([\cdot,\psi]_G)\Pi_{\text{fib}} + \frac{\exp([\cdot,\psi]_G) - 1}{[\cdot,\psi]_G} \partial^{\text{Hoch}}\psi. \tag{C.43}$$

Equation (C.42) implies that the operator

$$T_\psi = \exp(\psi) : \Gamma(M, \mathcal{S}M)[[\hbar]] \to \Gamma(M, \mathcal{S}M)[[\hbar]]$$

intertwines the geometric Fedosov differentials $D$ and $\widetilde{D}$:

$$T_\psi \circ \widetilde{D} = D \circ T_\psi. \tag{C.44}$$

Similarly, (C.43) implies that $T_\psi$ intertwines the fiberwise products (C.31), (C.33):

$$a_1 \,\tilde{\diamond}\, a_2 = T_{-\psi}(T_\psi(a_1) \diamond T_\psi(a_2)) \quad \text{for} \quad a_1, a_2 \in \Gamma(M, \mathcal{S}M)[[\hbar]]. \tag{C.45}$$

Using $T_\psi$, we define a $\mathbb{C}[[\hbar]]$-linear map

$$T : \mathcal{O}(M)[[\hbar]] \to \mathcal{O}(M)[[\hbar]], \quad T(f) = \sigma(T_\psi \circ \widetilde{\tau}(f)), \tag{C.46}$$

for $f \in \mathcal{O}(M)[[\hbar]]$, where $\sigma$ is defined in (2.10) and $\widetilde{\tau}$ is defined in (C.36). Just as $\tau$, the map $\widetilde{\tau}$ satisfies property (2.16). Combining this observation with the fact that $\psi$ belongs to the first filtration subspace, it follows that

$$T = \text{id} + \hbar T_1 + \hbar^2 T_2 + \cdots,$$

where $T_1, T_2, \ldots$ are differential operators on $M$. Since $T_\psi$ intertwines the geometric Fedosov differentials $D$ and $\widetilde{D}$, we get that

$$D T_\psi \circ \widetilde{\tau}(f) = 0, \tag{C.47}$$

for all $f \in \mathcal{O}(M)[[\hbar]]$. On the other hand, every $D$-flat section $\gamma$ of $\mathcal{S}M[[\hbar]]$ is uniquely determined by its image $\sigma(\gamma)$. Hence (C.46) and (C.47) imply that

$$T_\psi \widetilde{\tau}(f) = \tau(T(f)) \tag{C.48}$$

for all $f \in \mathcal{O}(M)[[\hbar]]$. Combining this observation with (C.45), we conclude that $T$ intertwines the star products (C.37):

$$T(f_1) * T(f_2) = T(f_1 \,\widetilde{*}\, f_2), \qquad f_1, f_2 \in \mathcal{O}(M)[[\hbar]].$$

Thus we proved that the correspondence between equivalence classes of star products and equivalence classes of formal Poisson structures produced by the sequence of $L_\infty$ quasi-isomorphisms (2.31) does not depend on the choice of the connection/Fedosov differential.

□

## References


[1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, Deformation theory and quantization. I. Deformations of symplectic structures, Ann. Phys. (N.Y.), **111** (1978) 61; Deformation theory and quantization, II. Physical applications, Ann. Phys. (N.Y.), **110** (1978) 111.

[2] F.A. Berezin, Quantization, Izv. Akad. Nauk., **38** (1974) 1116-1175;
General concept of quantization, Commun. Math. Phys., **40** (1975) 153-174.

[3] M. Bertelson, M. Cahen, and S. Gutt, Equivalence of star products. Class. Quant. Grav. **14**, 1A (1997) A93–A107.

[4] H. Bursztyn, Semiclassical Geometry of Quantum Line Bundles and Morita Equivalence of Star Products, Internat. Math. Res. Notices, **16** (2002) 821-846; arXiv:math/0105001.

[5] H. Bursztyn and S. Waldmann, Deformation quantization of Hermitian vector bundles, Lett. Math. Phys., **54** (2001) 349-365.

[6] H. Bursztyn and S. Waldmann, Bimodule deformations, Picard groups and contravariant connections, K-theory **31** (2004) 1-37; arXiv:math/0207255.

[7] H. Bursztyn and S. Waldmann, The characteristic classes of Morita equivalent star products on symplectic manifolds, Commun. Math. Phys., **228** (2002) 103-121; arXiv:math/0106178.

[8] H. Bursztyn and A. Weinstein, Poisson geometry and Morita equivalence, in *Poisson geometry, deformation quantisation and group representations*, London Mathematical Society Lecture Notes series, 323, Cambridge Univ. Press (2005).

[9] A.S. Cattaneo, G. Felder, and L. Tomassini, From local to global deformation quantization of Poisson manifolds, Duke Math. J. **115**, 2 (2002) 329–352.

[10] A. S. Cattaneo, G. Felder, and T. Willwacher, On L-infinity morphisms of cyclic chains, arXiv:0812.5056.

[11] A. S. Cattaneo, G. Felder, and T. Willwacher, The character map in deformation quantization, arXiv:0906.3122.

[12] P. Chen and V.A. Dolgushev, A simple algebraic proof of the algebraic index theorem, Math. Res. Lett., **12**, 5-6 (2005) 655–671.

[13] A. Chervov and L. Rybnikov, Deformation quantization of submanifolds and reductions via Duflo-Kirillov-Kontsevich map, arXiv:hep-th/0409005.

[14] T. Courant, Dirac manifolds, Trans. Amer. Math. Soc. **319** (1990) 631–661.

[15] P. Deligne, Déformations de l'algèbre des fonctions d'une variété symplectique: comparaison entre Fedosov et De Wilde, Lecomte, Selecta Math. (N.S.) **1**, 4 (1995) 667–697.

[16] M. De Wilde and P.B.A. Lecomte, Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds, Lett. Math. Phys. **7**, 6 (1983) 487–496.

[17] V. A. Dolgushev, The Van den Bergh duality and the modular symmetry of a Poisson variety, Selecta Math. (N.S.) **14**, 2 (2009) 199–228. arXiv:math/0612288.

[18] V.A. Dolgushev, A Proof of Tsygan's formality conjecture for an arbitrary smooth manifold, PhD thesis, MIT; math.QA/0504420.

[19] V.A. Dolgushev, Erratum to: "A Proof of Tsygan's Formality Conjecture for an Arbitrary Smooth Manifold", arXiv:math/0703113.

[20] V.A. Dolgushev, Covariant and Equivariant Formality Theorems, Adv. Math., **191**, 1 (2005) 147–177; arXiv:math/0307212.

[21] V.A. Dolgushev and V.N. Rubtsov, An algebraic index theorem for Poisson manifolds, accepted to J. Reine Angew. Math.; arXiv:0711.0184.

[22] V. Dolgushev, D. Tamarkin and B. Tsygan, The homotopy Gerstenhaber algebra of Hochschild cochains of a regular algebra is formal, J. Noncommut. Geom. **1**, 1 (2007) 1–25; arXiv:math/0605141.

[23] J.-D. Dufour and N.T. Zung, Poisson structures and their normal forms, Progress Math., Vol. 242, Birkhäuser Verlag, Basel, 2005.

[24] V.G. Drinfeld, On quasitriangular quasi-Hopf algebras and on a group that is closely connected with Gal($\overline{\mathbb{Q}}/\mathbb{Q}$). (Russian) Algebra i Analiz **2**, 4 (1990) 149–181; translation in Leningrad Math. J. **2**, 4 (1991) 829–860.

[25] C. Emmrich and A. Weinstein, The differential geometry of Fedosov's quantization, *Lie theory and geometry,* 217–239, Progr. Math., 123, Birkhuser Boston, Boston, MA, 1994; arXiv:hep-th/9311094

[26] P. Etingof and V. Ginzburg, Noncommutative del Pezzo surfaces and Calabi-Yau algebras, arXiv:0709.3593.

[27] B.V. Fedosov, A simple geometrical construction of deformation quantization, J. Diff. Geom. **40** (1994) 213–238.

[28] B.V. Fedosov, Deformation quantization and index theory, Akademie Verlag, Berlin, 1996.

[29] B. Feigin, G. Felder, and B. Shoikhet, Hochschild cohomology of the Weyl algebra and trace in deformation quantization, Duke Math. J. **127**, 3 (2005) 487–517; math.QA/0311303.

[30] G. Felder and B. Shoikhet, Deformation quantization with traces, Lett. Math. Phys. **53**, 1 (2000) 75–86; math.QA/0002057.

[31] I.M. Gelfand and D.V. Fuchs, Cohomology of the algebra of formal vector fields, Izv. Akad. Nauk., Math. Ser. **34** (1970) 322–337 (In Russian).

[32] I.M. Gelfand and D.A. Kazhdan, Some problems of differential geometry and the calculation of cohomologies of Lie algebras of vector fields, Soviet Math. Dokl., **12**, 5 (1971) 1367-1370.

[33] M. Gerstenhaber, The cohomology structure of an associative ring, Annals of Math., **78** (1963) 267–288.

[34] E. Getzler, A Darboux theorem for Hamiltonian operators in the formal calculus of variations, Duke Math. J. **111**, 3 (2002) 535–560.

[35] E. Getzler, Lie theory for nilpotent L-infinity algebras, to appear in Ann. Math.; arXiv:math/0404003

[36] E. Getzler and J.D.S. Jones, Operads, homotopy algebra and iterated integrals for double loop spaces, hep-th/9403055.

[37] V. Ginzburg, Calabi-Yau algebras, math.AG/0612139.

[38] W. Goldman and J. Millson, The deformation theory of representation of fundamental groups in compact Kähler manifolds, Publ. Math. I.H.E.S., **67** (1988) 43–96.

[39] M. Gualtieri, Generalized Complex Geometry, D. Phil. thesis, Oxford University, 2003; math.DG/0401221.

[40] S. Gutt, Variations on deformation quantization, Conférence Moshé Flato 1999, Vol. I (Dijon), 217–254, Math. Phys. Stud., 21, Kluwer Acad. Publ., Dordrecht, 2000.

[41] V. Hinich, Tamarkin's proof of Kontsevich formality theorem, Forum Math. **15**, 4 (2003) 591–614; math.QA/0003052.

[42] N. Hitchin, Generalized Calabi-Yau manifolds, Q. J. Math. **54**, 3 (2003) 281–308.

[43] B. Jurco, P. Schupp, and J. Wess, Noncommutative gauge theory for Poisson manifolds, Nucl. Phys. **B584** (2000) 784–794; arXiv:hep-th/0005005.

[44] B. Jurco, P. Schupp, and J. Wess, Noncommutative line bundle and Morita equivalence, Lett. Math. Phys. **61** (2002) 171–186; arXiv:hep-th/0106110.

[45] M. Kontsevich, Formality conjecture, *Deformation theory and symplectic geometry* (Ascona, 1996), 139–156, Math. Phys. Stud., 20, Kluwer Acad. Publ., Dordrecht, 1997.

[46] M. Kontsevich, Deformation quantization of Poisson manifolds, Lett. Math. Phys., **66** (2003) 157–216; q-alg/9709040.

[47] M. Kontsevich, Deformation quantization of algebraic varieties. Moshé Flato memorial conference 2000, Part III (Dijon). Lett. Math. Phys. **56**, 3 (2001) 271–294.

[48] A. Lichnerowicz, Les variétés de Poisson et leurs algèbres de Lie associées, J. Diff. Geom., **12**, 2 (1977) 253–300.

[49] K. Morita, Duality for modules and its applications to the theory of rings with minimum condition, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **6** (1958) 83–142.

[50] R. Nest and B Tsygan, Algebraic index theorem, Commun. Math. Phys. **172**, 2 (1995), 223–262.

[51] H. Omori, Y. Maeda, A. Yoshioka, Weyl manifolds and deformation quantization, Adv. Math. **85**, 2 (1991) 224–255.

[52] D. Quillen, Rational homotopy theory, Annals of Math., **90**, 2 (1969) 205–295.

[53] M. Schlessinger and J. Stasheff, Deformation theory and rational homotopy type, University of North Carolina preprint, 1979.

[54] N. Seiberg and E. Witten, String Theory and Noncommutative Geometry, JHEP **9909**, 032 (1999); hep-th/9908142.

[55] P. Ševera and A. Weinstein, Poisson geometry with a 3-form background, *Noncommutative geometry and string theory* (Yokohama, 2001). Progr. Theoret. Phys. Suppl. No. 144 (2001) 145–154.

[56] B. Shoikhet, An explicit construction of the Quillen homotopical category of dg Lie algebras, arXiv:0706.1333.

[57] D. Tamarkin, Another proof of M. Kontsevich formality theorem, math.QA/9803025.

[58] M. Van Den Bergh, A Relation between Hochschild Homology and Cohomology for Gorenstein Rings, Proc. Amer. Math. Soc. **126**, 5 (1998) 1345-1348;
Erratum to "A Relation between Hochschild Homology and Cohomology for Gorenstein Rings", Proc. Amer. Math. Soc. **130**, 9 (2002) 2809–2810.

[59] S. Waldmann, Poisson-Geometrie und Deformationsquantisierung. Eine Einführung, Springer-Verlag, Heidelberg, Berlin, New York, 2007.

[60] A. Weinstein, The local structure of Poisson manifolds, J. Differential Geom. **18**, 3 (1983) 523–557.

[61] A. Yekutieli, Mixed resolutions, simplicial sections and unipotent group actions, Israel J. Math., **162** (2007) 1–27; math.AG/0502206.

[62] A. Zotov, On Relation Between Weyl and Kontsevich Quantum Products. Direct Evaluation up to the $\hbar^3$-Order, Modern Phys. Lett. **A16**, 10 (2001) 615–625; arXiv:hep-th/0007072.

Instituto Nacional de Matematica Pura e Aplicada
Estrada Dona Castorina 110
Rio de Janeiro, 22460-320, Brasil
*E-mail address:* **henrique@impa.br**

Department of Mathematics, University of California at Riverside,
900 Big Springs Drive,
Riverside, CA 92521, USA
*E-mail address:* **vald@math.ucr.edu**

Fakultät für Mathematik und Physik
Physikalisches Institut
Hermann Herder Strasse 3
D 79104 Freiburg, Germany
*E-mail address:* **Stefan.Waldmann@physik.uni-freiburg.de**