

Real Option Pricing with Mean-Reverting Investment and Project Value

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Abstract

In this work we are concerned with valuing the option to invest in a project when the project value and the investment value are both mean-reverting. Previous works which dealt with stochastic project and investment value concentrate on geometric Brownian motions for driving the values. However, when the project involved is linked to commodities, mean-reverting assumptions are more meaningful. Here, we introduce a model and prove that the optimal exercise strategy is not a function of ratio of project value to investment V/I – as it is in the Brownian case. We further apply the Fourier space time-stepping algorithm of Jaimungal and Surkov (2009) to numerically investigate the option to invest. The optimal exercise policies are found to be approximately linear in V/I ; however, the intercept is not zero.

Key-words: Real Options; Mean-Reverting; Stochastic Investment; Investment under Uncertainty

JEL Classification: C6 (Mathematical Methods and Programming); C61; C67;

1 Introduction

Quantitative methods to analyze the option to invest in a project enjoy a long and distinguished history. The classical work of McDonald and Siegel (1986) (see also Dixit and Pindyck (1994)) investigates the problem from the point of view of derivative pricing and assigns the value of the option to invest as

$$\text{value} = e^{-rT} \mathbb{E} [(V_T - I_T)_+] . \quad (1.1)$$

Here, the expected value is taken under an appropriate risk-adjusted measure. Furthermore, V_T and I_T represent the project's value and the amount to be invested, respectively, at time T .

If the project can be started at anytime, then (1.1) is modified to its American counterpart. In this case, the maturity date T is replaced by a stopping time τ ($0 \leq \tau \leq T$) and the investor chooses the stopping time to maximize the option's value, i.e.,

$$\text{value} = \sup_{\tau \in \mathcal{T}} e^{-r\tau} \mathbb{E} [(V_\tau - I_\tau)_+] , \quad (1.2)$$

where \mathcal{T} denotes the family of all stopping times in $[0, T]$. As such, the problem becomes a free boundary problem in which the optimal strategy is computed simultaneously with the option's value.

Traditionally, the project value is assumed to be a geometric Brownian motion (GBM) and the investment amount is constant or deterministic, as in the pioneering work of Tourinho (1979). Stochastic investment amounts have also been investigated previously: the case of the perpetual option and GBM investment is treated in McDonald and Siegel (1986) (see also Berk, Green, and Naik (1999)). More recently, Elliott, Miao, and Yu (2007) have investigated the case of regime switching investment costs for the option in

perpetuity. Perpetuities have also been investigated with a mean-reverting CIR project value (but constant investment) by Ewald and Wang (2007). It is interesting to note that the uncertain investment problems are similar to those found in exchange options, as in Margrabe (1978), and in uncertain payoffs, as in Fischer (1978).

Much of the traditional works which account for stochastic investment amounts – e.g. McDonald and Siegel (1986) and Blenman and Clark (2005) – assume that the investment amount is a GBM. This may be a good model for the project value in certain circumstances; however, as already noticed in McDonald and Siegel (1986), investment costs are typically linked to certain commodity prices, and thus are expected to revert to an equilibrium level. Furthermore, in situations where the cash-flows of the project are directly linked to commodities, the project value is also expected to fluctuate about an equilibrium level. One case where this connection is abundantly clear is the option to the invest in an oil field. Like most commodities, oil prices tend to mean-revert, and as a direct result the value of investment in an oil field is also mean-reverting. Consequently, it would not be appropriate to use GBM models for such projects. Of course, several authors have noticed this and mean-reverting processes have been considered, such as Metcalf and Hasset (1995) and Sarkar (2003). However, combining mean-reverting project value with mean-reverting investment amount has not been considered up to now. In the next section we provide a modeling framework which naturally extends the mean-reverting project value to account for mean-reverting investment.

2 A Mean-Reverting Value and Investment Model

The difficulty with allowing both project value V_t and investment amount I_t to be stochastic lies in the fact that the problem becomes two-dimensional and the optimal policy will, in general, depend on both V_t and I_t . However, since the payoff $(V_T - I_T)_+$ of the option to

invest is homogeneous in (V_T, I_T) , when both the project value and investment amount are GBMs, it turns out the optimal policy depends only on the ratio V_t/I_t and the option's value inherits the payoff's homogeneity. This was observed quite early in McDonald and Siegel (1986) and it seems that this *trigger ratio* policy has become a paradigm in Real Options pricing. See Dixit and Pindyck (1994) for a review of these triggers for perpetual options with both GBM and mean-reverting project values but constant investment amount. We will see that this very appealing property is not inherited by mean-reverting processes.

To this end, our model first assumes that the project value satisfies the SDE:

$$V_t = \exp\{\theta + X_t - \delta t\}, \tag{2.1a}$$

$$dX_t = -\alpha X_t dt + \sigma_X dW_t^X, \tag{2.1b}$$

Here, W_t^X is a standard Brownian motion and δ is a dividend yield representing the effect decay in value of the project due to waiting. In this manner, the project value is a mean-reverting diffusion process which reverts to the equilibrium value $\exp\{\theta - \delta t\}$. Similar models have been proposed in the literature for commodity prices as in Cartea and Figueroa (2005) and more generally in Jaimungal and Surkov (2009).

Now, we model the investment amount as another correlated mean-reverting processes. Specifically,

$$I_t = \exp\{\phi + Y_t\}, \tag{2.2a}$$

$$dY_t = -\beta Y_t dt + \sigma_Y dW_t^Y. \tag{2.2b}$$

Here, W_t^Y is a standard Brownian motion correlated to W_t^X with correlation ρ . Notice, that the investment I_t available for the project has an equilibrium level of $\exp\{\phi\}$.

To illustrate the flexibility of the model, in Figure 2.1 two sample paths for the value

and investment are presented. The sample paths were both generated from the the same uncorrelated Brownian sample paths to highlight the effect of the correlation. The volatility of the project was assumed to be 80% while the investment processes was assumed to have a volatility of 50%. Panel (a) contains no correlation in the investment amount whereas panel (b) illustrates the behavior when the investment and project value are perfectly correlated. Notice that, as expected, the positive correlation case has a much lower variability in the ratio of value to investment.

3 The European Option to Invest

We now investigate the option to invest under the modeling framework (2.1)-(2.2). The European option to invest has value equal to the discounted expectation in (1.1). In the Appendix we prove the following result

Theorem 3.1 European Option Price. *The value of the European option to invest in the project at a fixed date T under the modeling assumptions (2.1)-(2.2) is*

$$value = \mathbb{E}[V_T]\Phi(d_+) - \mathbb{E}[I_T]\Phi(d_-) . \quad (3.1)$$

Here, $\Phi(\cdot)$ denotes the normal cumulative cdf, the effective total variance

$$\tilde{\sigma}^2 = \sigma_X^2 \frac{1 - e^{-2\alpha T}}{2\alpha} - 2\rho\sigma_X\sigma_Y \frac{1 - e^{-(\alpha+\beta)T}}{\alpha + \beta} + \sigma_Y^2 \frac{1 - e^{-2\beta T}}{2\beta} , \quad (3.2)$$

and

$$d_{\pm} = \frac{\ln(\mathbb{E}[V_T]/\mathbb{E}[I_T]) \pm \frac{1}{2}\tilde{\sigma}^2}{\tilde{\sigma}} , \quad (3.3)$$

while the expectations are given by

$$\mathbb{E}[V_T] = \exp \left\{ \theta - \delta T + e^{-\alpha T} X_0 + \frac{\sigma_X^2}{4\alpha} (1 - e^{-2\alpha T}) \right\}, \quad (3.4)$$

$$\mathbb{E}[I_T] = \exp \left\{ \phi + e^{-\beta T} Y_0 + \frac{\sigma_Y^2}{4\beta} (1 - e^{-2\beta T}) \right\}. \quad (3.5)$$

In Figure 3.2, the value of the European option to invest in the project is shown together with the payoff function for two different mean-reversion rates of investment. Notice that, unlike in the purely GBM case, it is optimal to invest immediately if the project value is large enough, or the investment cost is small enough. Another interesting observation is the fact that the trigger curve itself is clearly not linear and does not go through the point $V = 0, I = 0$. This indicates that, even in the European case, the optimal strategy for investing is not provided by monitoring when the ratio V/I rises above a critical trigger. Instead, both processes must be monitored simultaneously. It is precisely the mean-reverting nature of both processes which causes such interesting results. We will see that similar features flow through to the Bermudan option case.

To further assist in understanding the modeling implications, Figure 3.3 explores how the trigger curve is affected by the various model parameters.

4 The Early Investment Option

Companies generally have the ability to invest early in a project, consequently, the value of the option to invest in a project is truly of American type as in (1.2). Analytical solutions for GBM and mean-reverting project value (with constant investment), with and without regime switching, are restricted to the perpetual option. For finite time horizons, no analytical solution are known, not even for the one dimensional case with GBM project value, therefore we do not attempt to find analytic solutions. Instead we will develop an

efficient numerical scheme and investigate the consequences of our model on the trigger curves.

Rather than focusing on tree approximations, finite difference schemes or invoking least squares Monte Carlo (Longstaff and Schwartz (2001)), we make use of the mean-reverting Fourier space time-stepping algorithm of Jaimungal and Surkov (2009). More details are to follow.

The Bermudan option to invest, where the project can only be invested in at the discrete times $\{t_0, t_1, \dots, t_n\}$ (e.g. quarterly, monthly, weekly or daily), can be priced recursively on the exercise dates as follows:

$$\begin{cases} p_{t_n}(V_{t_n}, I_{t_n}) = (V_{t_n} - I_{t_n})_+ \\ p_{t_m}(V_{t_m}, I_{t_m}) = \max \left\{ e^{-r\Delta t_m} \bar{p}_{t_m}^{(m)}(V_{t_m}, I_{t_m}) ; (V_{t_m} - I_{t_m})_+ \right\}, \\ \bar{p}_t^{(m)} = \mathbb{E} [p_{t_{m+1}}(V_{t_{m+1}}, I_{t_{m+1}}) | \mathcal{F}_t], \end{cases} \quad (4.1)$$

for $m = \{1, 2, \dots, n-1\}$. Notice that on the exercise date t_m , the discounted process $\bar{p}_t^{(m)}$, i.e. $e^{-r\Delta t_m} \bar{p}_{t_m}^{(m)}$, is the holding (or continuation) value of the option to invest on that date. Without loss of generality, the option value can be written in terms of the log processes X_t and Y_t , in which case we will write $\hat{p}_t^{(m)}(X_t, Y_t) = \bar{p}_t^{(m)}(e^{\theta+X_t}, e^{\phi+Y_t-\delta t})$. Jaimungal and Surkov (2009) show that this continuation price can be computed via Fourier transforms, resulting in

$$\hat{p}^{(m)}(t, X, Y) = \mathcal{F}^{-1} \left[\mathcal{F} [\tilde{p}_{t_{m+1}}(X, Y)](\omega_1, \omega_2) e^{\Psi((t_{m+1}-t), \omega_1, \omega_2)} \right]. \quad (4.2)$$

Here, $\mathcal{F}[\cdot]$ and $\mathcal{F}^{-1}[\cdot]$ represent Fourier and inverse Fourier transforms respectively,

$$\tilde{p}_{t_{m+1}}(X, Y) = \hat{p}_{t_{m+1}} \left(X e^{-\alpha(t_m - t_{m+1})}, Y e^{-\beta(t_m - t_{m+1})} \right), \quad (4.3)$$

Ψ is related to the characteristic function of the generating process

$$\Psi(s, \omega_1, \omega_2) = -\frac{1}{2}\sigma_X^2 \frac{e^{2\alpha s} - 1}{2\alpha} \omega_1^2 - \rho\sigma_X\sigma_Y \frac{e^{(\alpha+\beta)s} - 1}{\alpha + \beta} \omega_1\omega_2 - \frac{1}{2}\sigma_Y^2 \frac{e^{2\beta s} - 1}{2\beta} \omega_2^2. \quad (4.4)$$

By comparison with the intrinsic value, the optimal strategy can be computed numerically through two fast Fourier transforms which approximately evaluate the Fourier and inverse transforms. This procedure is far more efficient than a tree or finite-difference scheme as it requires $O(N \log N)$ computations per exercise date, while finite difference schemes will require $O(MN)$ where M is the number of steps required between exercise dates. For more details see Jaimungal and Surkov (2009).

In Figure 4.4, we plot the sequence of trigger curves for a ten year option to invest assuming investment can be made only once a year. As maturity approaches, the trigger curves move toward the exercise trigger of $V = I$, however, due to the mean-reversion point lying well within the exercise trigger region, the early trigger curves lie significantly above the line $V^* = I$. Furthermore, the trigger curves are not described by a line of the form $V^* = b I$, instead, close to the equilibrium point they appear as if they can be approximated by $V^* = a + b I$. As one moves away from the equilibrium, in the direction of smaller values, the non-linearities become apparent. For large values however, the non-linearities are harder to spot visually, but they do in fact persist. The linear approximation appears to only be valid near the equilibrium. This is in contrast to the case of GBM drives where the trigger curves will be described by $V^* = b I$.

In Figure 4.5, we plot the trigger surface for a ten year option to invest assuming investment can occur daily. The solid blue line indicates the mean-reversion level, while the black random path is a sample of the process. Interestingly, when viewed as in panel (a), the sample paths appear to move mostly in a plane almost perpendicular to the trigger curve. When the sample path crosses the surface, investment in the project should occur.

From the panel (a) viewpoint it is clear that investment should have occurred near year 2, while from the panel (b) viewpoint, this threshold crossing is not so evident. This suggests that using the V/I versus V perspective is advantageous.

5 A Co-integrated model

Since the seminal works of C. Granger and R. Engle, the concept of co-integration became crucial for the econometric study of financial time series (Engle and Granger 1987). In the analysis of investment decisions, it is natural to expect that project values and investment costs have co-integrated factors. Thus, it is natural to seek pairs of processes that co-integrate and also lead to a trigger curve.

The results presented above show that, even when the processes for the project value and the investment have the same mean-reversion, and then the ratio is also mean-reverting, we cannot characterize the investment exercise curve using a trigger curve.

In what follows we construct a class of mean-reverting process pairs (V, I) for which the ratio V/I is also a mean-reverting by considering co-integrated models.

We replace (2.2) by

$$I_t = \exp\{\phi + Y_t\}, \tag{5.1a}$$

$$dY_t = -((\alpha - \beta)X_{t-} + \beta Y_t) dt + \sigma_Y dW_t^Y. \tag{5.1b}$$

In this case, it is a simple matter to check that the modeling assumption (2.1)-(5.1) implies that the ratio V_t/I_t of the project's value and the amount invested is also a mean-reverting processes *and* the dynamics of this ratio depends only on the ratio itself. Specifically, notice that $(V_t/I_t) = \exp\{(\theta - \phi) + (X_t - Y_t)\}$, and that the difference process $Z_t = X_t - Y_t$

satisfies the SDE

$$dZ_t = -\beta Z_t dt + \sigma_X dW_t^X - \sigma_Y dW_t^Y . \quad (5.2)$$

Consequently, the ratio can be modeled directly as a mean-reverting process with rate β , effective instantaneous variance of $\sigma^2 := \sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y$.

The value for the European option to invest is still given by (3.1) together with (3.2) and (3.3). However, the expectations are now given by

$$\mathbb{E}[V_T] = \exp \left\{ \theta + e^{-\alpha T} X_0 + \frac{\sigma_X^2}{4\alpha} (1 - e^{-2\alpha T}) \right\} , \quad (5.3)$$

$$\begin{aligned} \mathbb{E}[I_T] = \exp \left\{ \phi + e^{-\beta T} Y_0 + (e^{-\alpha T} - e^{-\beta T}) X_0 \right. \\ \left. + \frac{1}{2}\sigma_X^2 \left[\frac{1 - e^{-2\alpha T}}{2\alpha} + \frac{1 - e^{-2\beta T}}{2\beta} - 2\frac{1 - e^{-(\alpha+\beta)T}}{\alpha + \beta} \right] \right. \\ \left. + \frac{1}{2}\sigma_Y^2 \left[\frac{1 - e^{-2\beta T}}{2\beta} \right] \right. \\ \left. + \rho\sigma_X\sigma_Y \left[\frac{1 - e^{-(\alpha+\beta)T}}{\alpha + \beta} - \frac{1 - e^{-2\beta T}}{2\beta} \right] \right\} . \end{aligned} \quad (5.4)$$

The value of the early investment option can be computed as described in section 4, but taking into account that the characteristic function for pair (X_t, Y_t) is now given by

$$\begin{aligned} \Psi(s, \omega_1, \omega_2) = -\frac{1}{2}\sigma_X^2 \left(\frac{e^{2\alpha s} - 1}{2\alpha} (\omega_1 + \omega_2)^2 + \frac{e^{2\beta s} - 1}{2\beta} \omega_2^2 - 2\frac{e^{(\alpha+\beta)s} - 1}{\alpha + \beta} \omega_2 (\omega_1 + \omega_2) \right) \\ - \frac{1}{2}\sigma_Y^2 \frac{e^{2\beta s} - 1}{2\beta} \omega_2^2 - \rho\sigma_X\sigma_Y \left(\frac{e^{(\alpha+\beta)s} - 1}{\alpha + \beta} \omega_1 - \frac{e^{2\beta s} - 1}{2\beta} \omega_2 \right) \omega_2 . \end{aligned} \quad (5.5)$$

6 Conclusions

In this work, we have addressed the problem of the decision of investing, when both the value of the project and the investment follows a mean-reverting dynamics. In this case, the optimal policy depends on both the value of the project and the investment level,

rather than just on their ratio. The former is known to be the case when the value and the investment follow GBM dynamics. This phenomenon precludes the use of a trigger curve for determining the investment frontier, which has been recognized, since the work by McDonald and Siegel (1986), as a specially convenient representation. For a particular class of mean-reverting dynamics, we are able to show that such an investment frontier can be represented just by the ratio between the project value and the investment level. In particular, the dynamics of the ratio is also mean-reverting. Nonetheless, the Fourier Space Time-Stepping method, developed by Jackson, Jaimungal, and Surkov (2008) and Jaimungal and Surkov (2009), can be used to numerically explore the trigger levels in such models.

A European Option Pricing Formulae

In this appendix we derive the value of the European option to invest in a project with stochastic investment and project value. The value is

$$\begin{aligned}
Opt_0 &= e^{-rT} \mathbb{E}[(V_T - I_T)_+ | \mathcal{F}_0] \\
&= e^{-rT} \mathbb{E}^T \left[\left(\frac{V_T}{I_T} - 1 \right)_+ \middle| \mathcal{F}_0 \right] \mathbb{E}[I_T | \mathcal{F}_0] \\
&= e^{-rT} \mathbb{E}^T [(\xi_T - 1)_+ | \mathcal{F}_0] \mathbb{E}[I_T | \mathcal{F}_0]
\end{aligned} \tag{A.1}$$

where, $\mathbb{E}^T[\cdot]$ represents expectations with respect to a new measure \mathbb{P}^T defined via the Radon-Nikodym derivative process

$$\eta_t^T \triangleq \left(\frac{d\mathbb{P}^T}{d\mathbb{P}} \right)_t = \frac{\mathbb{E}[I_T | \mathcal{F}_t]}{\mathbb{E}[I_T | \mathcal{F}_0]}, \quad \text{and} \quad \xi_t \triangleq \frac{\mathbb{E}[V_T | \mathcal{F}_t]}{\mathbb{E}[I_T | \mathcal{F}_t]}.$$

Note that (i) $\xi_T = V_T/I_T$ and (ii) ξ_t is a \mathbb{P}^T -martingale under any modeling assumptions for V_t and I_t (as long as I_t is strictly positive). Property (ii) can be seen from the following simple computation ($0 \leq s \leq t$):

$$\begin{aligned}\mathbb{E}^T[\xi_t|\mathcal{F}_s] &= \mathbb{E}\left[\frac{\mathbb{E}[V_T|\mathcal{F}_t]}{\mathbb{E}[I_T|\mathcal{F}_t]} \cdot \frac{\mathbb{E}_t[I_T]}{\mathbb{E}_0[I_T]} \middle| \mathcal{F}_s\right] \bigg/ \frac{\mathbb{E}[I_T|\mathcal{F}_s]}{\mathbb{E}[I_T|\mathcal{F}_0]} = \frac{\mathbb{E}[\mathbb{E}[V_T|\mathcal{F}_t]|\mathcal{F}_s]}{\mathbb{E}[I_T|\mathcal{F}_s]} \\ &= \frac{\mathbb{E}[V_T|\mathcal{F}_s]}{\mathbb{E}[I_T|\mathcal{F}_s]} = \xi_s.\end{aligned}$$

For our model (2.1)-(2.2), we have

$$\begin{aligned}X_T &= e^{-\alpha(T-t)} X_t + \sigma_X \int_t^T e^{-\alpha(T-u)} dW_u^X, \\ Y_T &= e^{-\beta(T-t)} Y_t + \sigma_Y \int_t^T e^{-\beta(T-u)} dW_u^Y,\end{aligned}$$

so that,

$$\begin{aligned}\mathbb{E}[V_T|\mathcal{F}_t] &= \exp\left\{\theta + \delta(T-t) + e^{-\alpha(T-t)} X_t + \frac{\sigma_X^2}{4\alpha} (1 - e^{-2\alpha(T-t)})\right\}, \\ \mathbb{E}[I_T|\mathcal{F}_t] &= \exp\left\{\phi + e^{-\beta(T-t)} Y_t + \frac{\sigma_Y^2}{4\beta} (1 - e^{-2\beta(T-t)})\right\}.\end{aligned}$$

These expressions provide an explicit formula for the ξ_t process. Further, using Ito's lemma and the fact that ξ_t is a \mathbb{P}^T -martingale, implies

$$\frac{d\xi_t}{\xi_{t-}} = \sigma_X e^{-\alpha(T-t)} dW_t^{T,X} - \sigma_Y e^{-\beta(T-t)} dW_t^{T,Y},$$

where $W_t^{T,X}$ and $W_t^{T,Y}$ are correlated standard \mathbb{P}^T -Brownian motions. Consequently,

$$\xi_T \stackrel{d}{=} \xi_0 \exp\left\{-\frac{1}{2}\tilde{\sigma}^2 + \tilde{\sigma}Z\right\},$$

where Z is a standard normal random variable and $\tilde{\sigma}$ is provided in (3.2). Since ξ_T is log-normally distributed, the remaining unknown expectation in (A.1) is

$$\mathbb{E}^T [(\xi_T - 1)_+ | \mathcal{F}_t] = \xi_t \Phi(d_+) - \Phi(d_-),$$

with d_{\pm} defined in (3.3). The final pricing result (3.1) is now an easy consequence.

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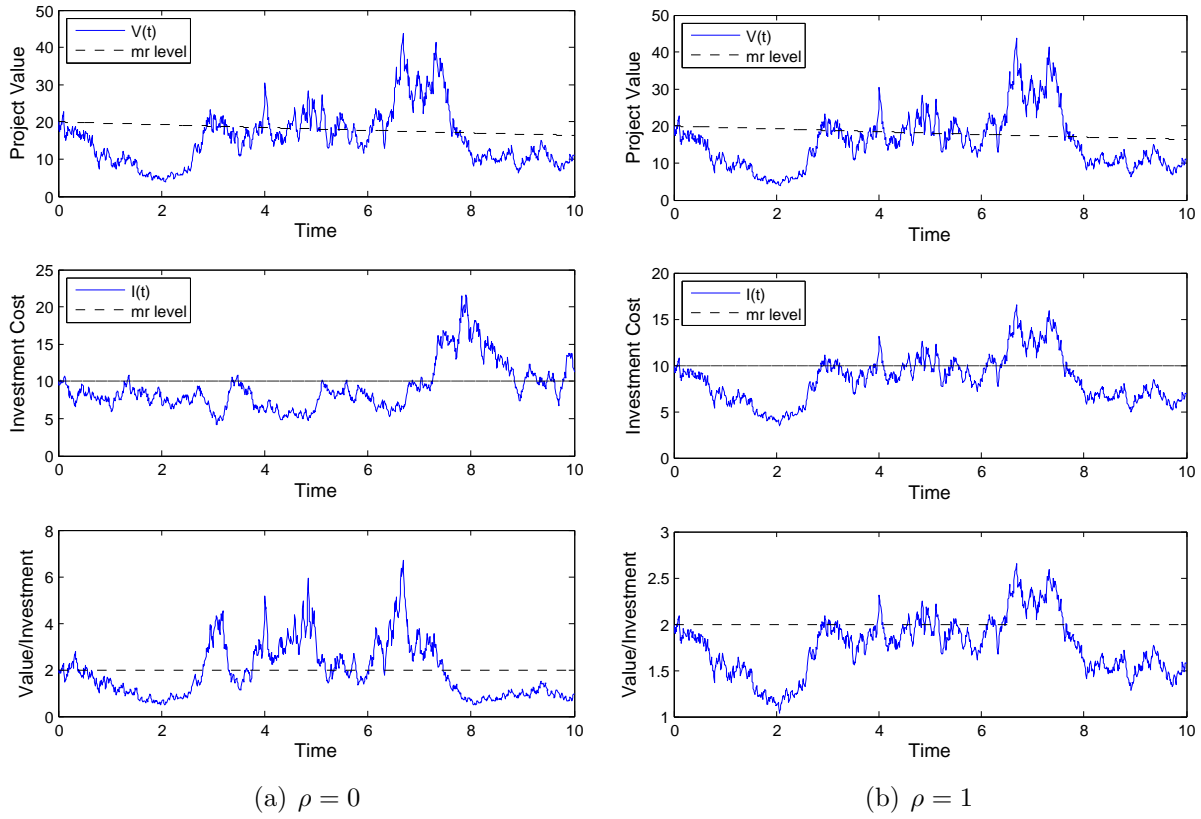


Figure 2.1: A sample path of project value and investment amount. The lines label *mr level* are the equilibrium mean-reverting levels for the value and investment, while *stock level* represents the stochastic level η_t . The model parameters are: $\alpha = 1$; $\theta = \ln(20)$; $\sigma_X = 0.8$; $\beta = 1$; $\phi = \ln(10)$; $\sigma_Y = 50\%$, $\delta = 2\%$ and $\lambda = 0$.

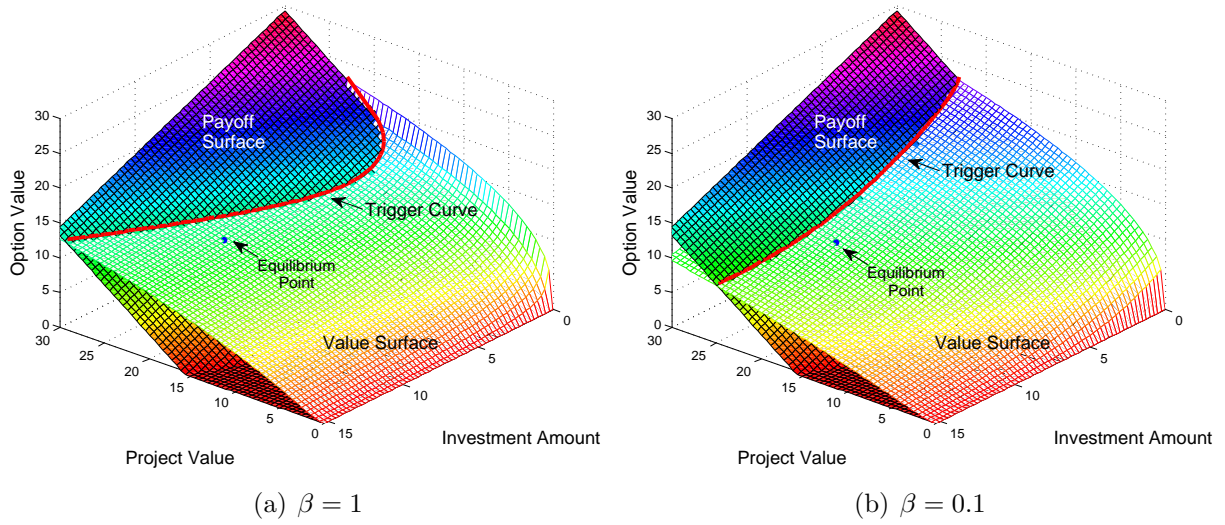


Figure 3.2: The price surface and optimal exercise trigger for a 1 year European option to invest in a project with two differing rates of investment mean-reversion. The model remaining parameters are as follows: $\alpha = 1$, $\theta = \ln(20)$, $\sigma_X = 80\%$, $\phi = \ln(10)$, $\sigma_Y = 50\%$, $\rho = 0.5$, $\delta = 2\%$, $r = 5\%$.

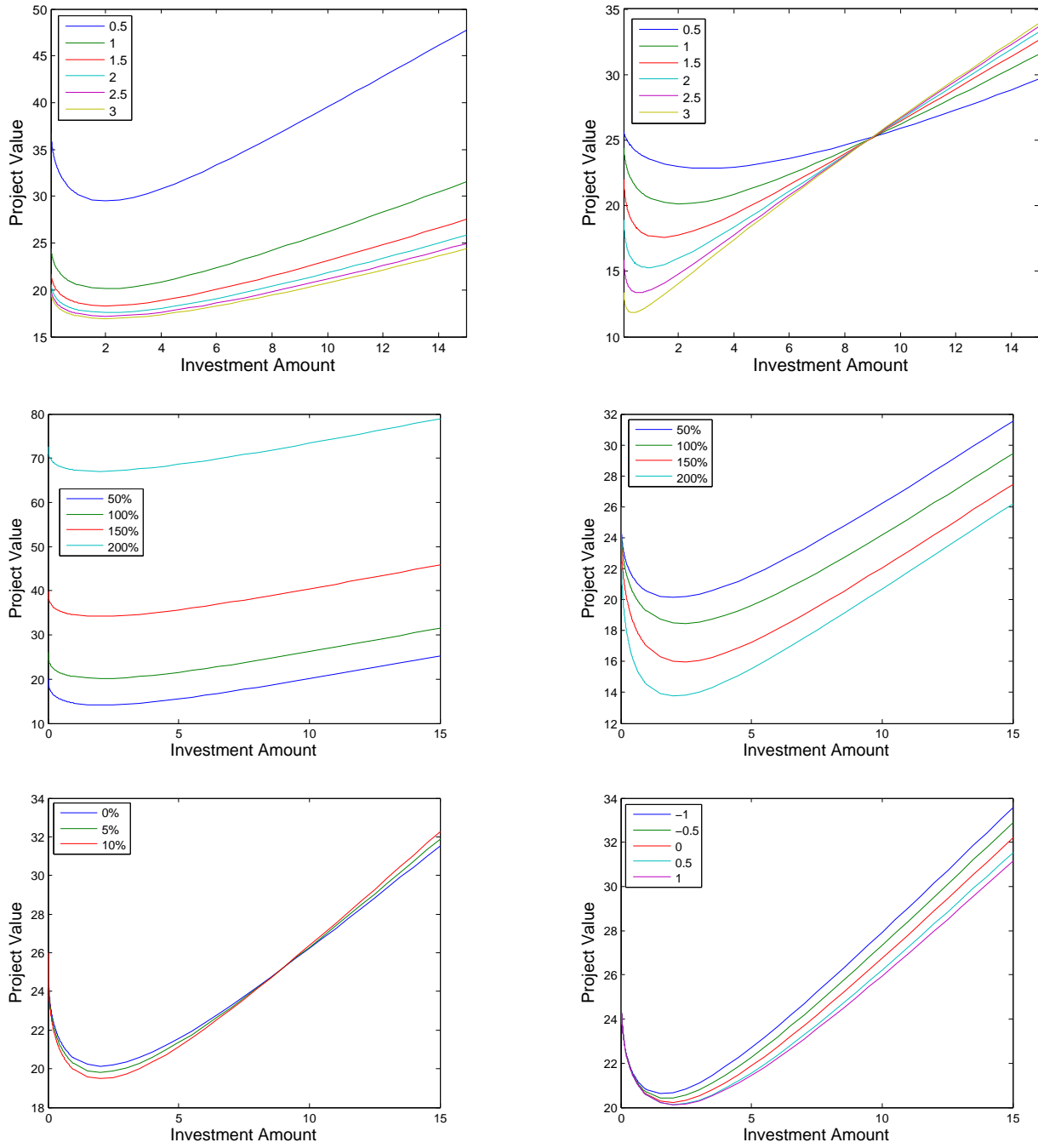


Figure 3.3: The α and β sensitivity of the optimal exercise trigger for a 1 year European option to invest in a project. The remaining model parameters are as in Figure 3.2 .

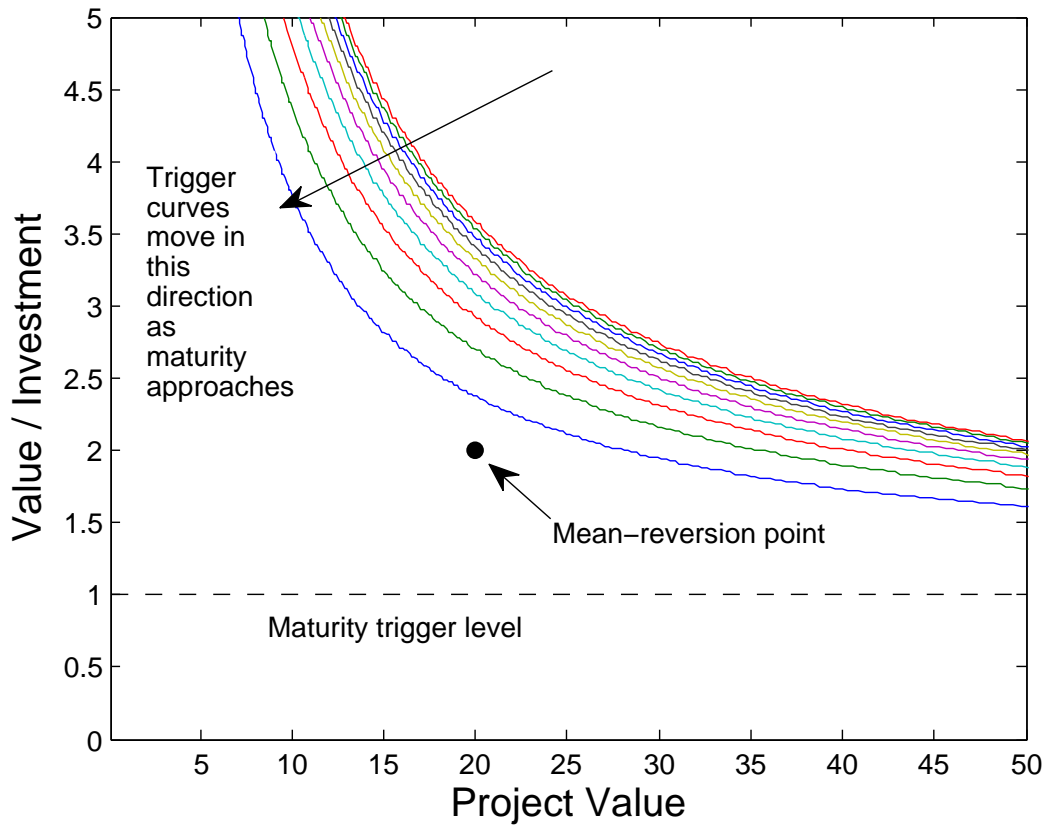
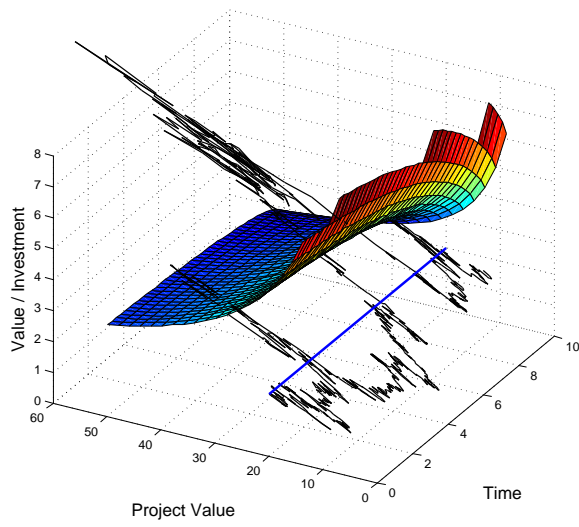
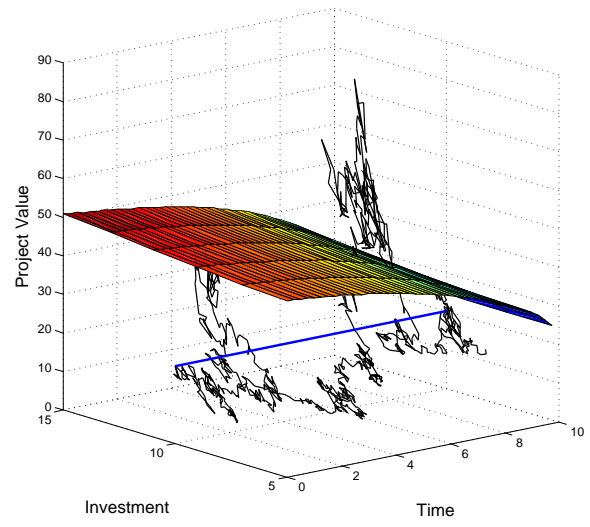


Figure 4.4: Trigger curves assuming yearly exercise dates. The dashed line indicates the maturity trigger curve of $V = I$. Each curve above the dash represents the trigger with one more year remaining to maturity. The model parameters are as in Figure 3.2.



(a) V/I versus V



(b) V versus I

Figure 4.5: The trigger surface together with a sample path for a ten year option to invest with daily exercise decisions.