GLOBAL EXISTENCE FOR THE CRITICAL GENERALIZED KDV EQUATION

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ABSTRACT. We discuss results regarding global existence of solutions for the critical generalized Korteweg-de Vries equation,

$$u_t + u_{xxx} + u^4\, u_x = 0, \quad x,\, t \in \mathbb{R}.$$

The theory established shows the existence of global solutions in Sobolev spaces with order below the one given by the energy space $H^1(\mathbb{R})$, i.e. solutions corresponding to data $u_0 \in H^s(\mathbb{R})$, $s > 3/4$, with $\|u_0\|_{L^2} < \|Q\|_{L^2}$, where Q is the solitary wave solution of the equation.

1. Introduction

Consider the initial value problem

$$\begin{cases} u_t + u^4\, u_x + u_{xxx} = 0, & x \in \mathbb{R}, \quad t \in \mathbb{R} \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

We study the global existence of solutions for data in $H^s(R)$, $s \in (0,1)$.

In [8] it was showed the local (subcritical) well-posedness of IVP (1.1) for data in $H^s(\mathbb{R})$, $s > 0$, i.e. for any $u_0 \in H^s(\mathbb{R})$, there exist $T(\|u_0\|_s, s) > 0$ and a unique solution u of (1.1) in $C([0,T] : H^s(\mathbb{R}))$. It was also proved that for any data $u_0 \in L^2(\mathbb{R})$ there exist $T(u_0) > 0$ and a unique strong solution u in $C([0,T] : L^2(\mathbb{R}))$. Moreover, the solution is global for small data. The later follows from the local theory in [8] and the fact that L^2 is a critical space, (see argument below).

Notice that scaling also works for complex valued functions where the conservation of the $\|u(t)\|_{L^2}$ does not hold. More precisely, if u solves (1.1), then, for $\lambda > 0$, so does

$$u_\lambda(x,t) = \lambda^{1/2}\, u(\lambda x, \lambda^3 t),$$

with data $u_\lambda(x,0) = \lambda^{1/2}\, u(\lambda x, 0)$. It follows that

$$\|u_\lambda(\cdot,0)\|_{\dot H^s} = \lambda^s\, \|u(\cdot,0)\|_{\dot H^s}.$$

This suggests that the optimal Sobolev index s is $s = 0$. Indeed, this result was established in [1]. Hence, the results in [8] imply that complex solutions corresponding to data

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$u_0 \in L^2(\mathbb{R})$ with

$$\|u_0\|_{L^2} \leq \epsilon_0 \tag{1.2}$$

are global in time.

On the other hand, Weinstein [14] showed the following Gagliardo-Nirenberg estimate for $v \in H^1(\mathbb{R})$ and $Q(x) = \{3c\,\mathrm{sech}^2(2\sqrt{c}\,x)\}^{1/4}$ the solitary wave solution of (1.1)

$$\frac{1}{6}\int v^6 \leq \frac{1}{2}\Big(\frac{\int v^2}{\int Q^2}\Big)^2 \int v_x^2. \tag{1.3}$$

This estimate combined with the conservation laws for real solutions of (1.1)

$$\int u^2(t) = \int u_0^2, \tag{1.4}$$

and

$$\frac{1}{2}\int u_x^2(t) - \frac{1}{6}\int u^6(t) = \frac{1}{2}\int {u_0'}^2 - \frac{1}{6}\int u_0^6, \tag{1.5}$$

give an *a priori* estimate in $H^1(\mathbb{R})$ provided $\|u_0\|_{L^2} < \|Q\|_{L^2}$ and therefore the global existence of solutions in $H^1(\mathbb{R})$ for initial data satisfying that condition. From this it is clear that $\epsilon_0 < \|Q\|_{L^2}$. Thus if we want to show global existence of solutions in $H^s(\mathbb{R})$, $s \in (0,1)$ we must assume that the initial data satisfy $\epsilon_0 < \|u_0\|_{L^2} < \|Q\|_{L^2}$.

Several interesting results have been lately obtained for solutions of IVP (1.1). Merle [12] proved the existence of real valued solutions of (1.1) in $H^1(\mathbb{R})$ corresponding to data $u_0 \in H^1$ satisfying $\|u_0\|_{L^2} > \|Q\|_{L^2}$ that blow-up. There are also various results concerning instability of solitary wave solutions as well as the structure of the blow-up formation obtained by Martel and Merle [10], [11].

In the last few years there have been a great deal regarding results of global existence of solutions for several nonlinear dispersive equations below the energy space. The first result in this direction was obtained by Bourgain [2] for the two-dimensional Schrödinger equation. His method was used to obtain such kind of results, for instance, for the mKdV equation [5], the KdV equation [3], see also [7], [13]. Recently, a new variant of Bourgain's method was introduced to obtain sharp global results for the KdV and mKdV equations, see [4]. In [6] for the gKdV-3 equation it was obtained a sharp local result which implies L^2 global well-posedness.

Our main result in this paper reads as follows:

Theorem 1.1. *Let $s > 3/4$. Then for any initial data $u_0 \in H^s(\mathbb{R})$ satisfying $\epsilon_0 < \|u_0\|_{L^2} < \|Q\|_{L^2}$, see* (1.2)*, the unique solution of the IVP* (1.1) *given by the local theory (see Theorem 2.1) extends to any time interval* $[0,T]$.

The idea of the proof is similar the one we developed in [5]. That is a combination of sharp smoothing effects present in solutions of the linear problem associated to (1.1) and the iteration process introduced by Bourgain. There are two new elements that can bring some extra difficulty. First, we need to be careful controlling the L^2-norm of the initial data. We must be in the conditions of the global well-posedness in $H^1(\mathbb{R})$. This situation is similar to the one for the derivative Schrödinger equation treated by Takaoka in [9]. Second, in the local well-posedness result, the time of existence depends on the Sobolev space H^s (see below).

We organized this paper as follows: In section 2, we give the statement of the local result obtained in [8] and the main estimates used in the subsequent sections. To prove Theorem 1.1 we need to study two auxiliary IVP's and the properties of their solutions, this will be done in section 3. Finally, the proof of Theorem 1.1 is given in section 4.

2. Preliminary results

We begin this section with the local sharp result obtained in [8].

Theorem 2.1. *([8]) Let $s>0$. Then for any $u_0 \in H^s(\mathbb{R})$ there exist $T = T(\|u_0\|_{s,2})$ (with $T(\rho, s) \to \infty$ as $\rho \to 0$) and a unique strong solution $u(\cdot)$ of the IVP (1.1) satisfying*

$$u \in C([-T,T] : H^s(\mathbb{R})), \tag{2.1}$$

$$\|u\|_{L^5_x L^{10}_T} + \|D^s_x\, u\|_{L^5_x L^{10}_T} + \|D^{s/3}_t\, u\|_{L^5_x L^{10}_T} < \infty, \tag{2.2}$$

and

$$\|\partial_x u\|_{L^\infty_x L^2_T} + \|D^s_x\, \partial_x u\|_{L^\infty_x L^2_T} + \|D^{s/3}_t\, \partial_x u\|_{L^\infty_x L^2_T} < \infty. \tag{2.3}$$

Given $T' \in (0,T)$ there exists a neighborhood V of u_0 in $H^s(\mathbb{R})$ such that the map $\tilde{u}_0 \to \tilde{u}(t)$ from V into the class defined by (2.1), (2.2) and (2.3) with T' instead of T is Lipschitz.

Remark 2.2. *From the proof of Theorem 2.1 it follows that the time of existence T satisfies*

$$T \simeq c\,\|u_0\|_s^{-3/s}. \tag{2.4}$$

This can also be deduced from the scaling argument.

The proof of Theorem 2.1 uses as main tools

1. The smoothing effect of Kato's type associated to the linear problem

$$\begin{cases} \partial_t u + \partial_x^3 u = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{2.5}$$

More precisely, let $U(t)$ be the linear group describing the solution of (2.5), then

$$\Big(\int_{-\infty}^{\infty} |\partial_x U(t) u_0(x)|^2\, dt\Big)^{1/2} = \|u_0\|, \quad \text{for any } x \in \mathbb{R}. \tag{2.6}$$

2. Its dual version, that is,

$$\|\partial_x \int_{-\infty}^{\infty} U(-\tau) g(x,\tau)\, d\tau\| \le c\,\|g\|_{L^1_x L^2_t}. \tag{2.7}$$

3. The double smoothing effect, solutions of the non-homogeneous linear equation

$$\begin{cases} \partial_t u + \partial_x^3 u = f, \\ u(x,0) = 0, \end{cases} \tag{2.8}$$

satisfy the estimate

$$\|\partial_x^2 u\|_{L^\infty_x L^2_T} \le c\, \|f\|_{L^1_x L^2_T}. \tag{2.9}$$

4. The fractional derivative commutators.

Theorem 2.3. *Let $\alpha \in (0,1)$, $\alpha_1, \alpha_2 \in [0,\alpha]$, $\alpha_1 + \alpha_2 = \alpha$. Let $p, p_1, p_2, q, q_1, q_2 \in (1,\infty)$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then*

$$\|D_x^\alpha (fg) - f D_x^\alpha g - g D_x^\alpha f\|_{L^p_x L^q_T} \le c\, \|D_x^{\alpha_1} f\|_{L^{p_1}_x L^{q_1}_T}\, \|D_x^{\alpha_2} g\|_{L^{p_2}_x L^{q_2}_T}. \tag{2.10}$$

Moreover, for $\alpha_1 = 0$ the value $q_1 = \infty$ is allowed.

5. The chain rule for fractional derivatives

Theorem 2.4. *Let $\alpha \in (0,1)$, and $p, p_1, p_2, q, q_2 \in (1,\infty)$, $q_1 \in (1,\infty]$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, Then*

$$\|D_x^\alpha F(f)\|_{L^p_x L^q_T} \le c\, \|F'(f)\|_{L^{p_1}_x L^{q_1}_T}\, \|D_x^\alpha f\|_{L^{p_2}_x L^{q_2}_T}. \tag{2.11}$$

6. Several interpolation theorems.

These tools plus the contraction mapping principle give the result.

3. Local well-posedness for the auxiliary problems

In this section we describe the properties of two auxiliary IVP's we will use in the proof of Theorem 1.1. We first set some notation.

Let N be a sufficiently large positive number to be chosen later.

Define the norm $|\!|\!| \cdot |\!|\!|_s$ by

$$\begin{aligned} |\!|\!| u |\!|\!|_s = {} & \|u\|_{L^\infty_T H^s} + \|u\|_{L^5_x L^{10}_T} + \|D_x^s u\|_{L^5_x L^{10}_T} + \|D_t^{s/3} u\|_{L^5_x L^{10}_T} \\ & + \|\partial_x u\|_{L^\infty_x L^2_T} + \|D_x^s \partial_x u\|_{L^\infty_x L^2_T} + \|D_t^{s/3} \partial_x u\|_{L^\infty_x L^2_T}. \end{aligned} \tag{3.1}$$

Proposition 3.1. *Consider the IVP*

$$\begin{cases} v_t + v^4 v_x + v_{xxx} = 0, \quad x \in \mathbb{R}, \ \ t > 0 \\ v(x,0) = v_0(x) \in H^1(\mathbb{R}), \end{cases} \tag{3.2}$$

If v_0 satisfies

$$\|v_0\|_{L^2} < \|Q\|_{L^2}, \quad \text{and} \quad \|v_0\|_{H^1} \simeq N^{1-s}. \tag{3.3}$$

Then for $\Delta T \simeq \|v_0\|_{H^1}^{-3} \simeq c\, N^{-3(1-s)}$, the existence time given by Theorem 2.1.

1. *The solution v of (3.2) satisfies*

$$\sup_{[0,\Delta T]} \|v(t)\|_{H^1} \le cN^{1-s}. \tag{3.4}$$

2. *For any $\rho \in (0,1)$, the solution v of (3.2) satisfies*

$$|\!|\!| v |\!|\!|_\rho \simeq N^{\rho(1-s)}. \tag{3.5}$$

Proof. To prove (3.4) we use (1.5) and (1.3). The estimate (3.5) is established by using Theorem 2.1 and $\|v_0\|_{H^\rho} \le c\, N^{\rho(1-s)}$. □

Proposition 3.2. *Let $w_0 \in H^s(\mathbb{R})$, $s > 0$, and v the solution of the IVP (3.2). Then there exists a unique solution w of the IVP*

$$\begin{cases} \partial_t w + \partial_x^3 w + \partial_x(v^4 w + 2v^3 w^2 + 2v^2 w^3 + w^4 v + \frac{1}{5}\, w^5) = 0, \\ w(x,0) = w_0(x), \end{cases} \tag{3.6}$$

defined in the same interval of existence of v, $[0,\Delta T]$, such that,

$$w \in C([0,\Delta T] : H^s(\mathbb{R})),$$
$$\|w\|_{L^5_x L^{10}_{\Delta T}} + \|D_x^s\, w\|_{L^5_x L^{10}_{\Delta T}} + \|D_t^{s/3}\, w\|_{L^5_x L^{10}_{\Delta T}} < \infty,$$

and

$$\|\partial_x w\|_{L^\infty_x L^2_{\Delta T}} + \|D_x^s\, \partial_x w\|_{L^\infty_x L^2_{\Delta T}} + \|D_t^{s/3}\, \partial_x w\|_{L^\infty_x L^2_{\Delta T}} < \infty.$$

Proof. The argument of proof is the same used in [8] to show Theorem 2.1. For the sake of clearness we will sketch it.

We use the integral equation form of (3.6), that is,

$$w(t) = U(t)w_0(x) - \int_0^t U(t-t')\, \partial_x(v^4 w + 2v^3 w^2 + 2v^2 w^3 + w^4 v + \frac{1}{5}\, w^5)(t')\, dt'. \tag{3.7}$$

Defining

$$X_{a,T_o} = \{w \in C([0,T_o] : H^\rho(\mathbb{R})) : |\!|\!| w |\!|\!|_\rho \le a\}, \tag{3.8}$$

where $|\!|\!| \cdot |\!|\!|$ is as in (3.1), and

$$\begin{aligned} \Phi(w) &= \Phi_{v,w_0}(w) \\ &\equiv U(t)w_0(x) - \int_0^t U(t-t')\, \partial_x(v^4 w + 2v^3 w^2 + 2v^2 w^3 + w^4 v + \frac{1}{5}\, w^5)(t')\, dt'. \end{aligned} \tag{3.9}$$

Next we shall show that Φ is a contraction. We will work in detail the case $\|D_x^\rho \Phi\|_{L^2}$. The others estimates follow similar arguments.

Using the dual of the smoothing effect (2.7), the chain and Leibniz rules for fractional derivatives (2.10) and (2.11) it follows that

$$
\begin{aligned}
\|D_x^\rho \int_0^t U(t-t')\,\partial_x(v^4w)(t')\,dt'\| &= \|\partial_x \int_0^t U(t-t')\,D_x^\rho(v^4w)(t')\,dt'\| \\
&\le c\|D_x^\rho\,(v^4w)\|_{L^1_xL^2_{T_0}} \\
&\le c\|D_x^\rho(v^4)w\|_{L^1_xL^2_{T_0}} + c\|v^4\|_{L_x^{5/4}L_{T_0}^{5/2}}\|D_x^\rho w\|_{L^5_xL^{10}_{T_0}} \\
&\le c\|D^\rho(v^4)\|_{L_x^{5/4}L_{T_0}^{5/2}}\|w\|_{L^5_xL^{10}_{T_0}} + c\|v\|^4_{L^5_xL^{10}_{T_0}}\|D_x^\rho w\|_{L^5_xL^{10}_{T_0}} \\
&\le c\|v\|^3_{L^5_xL^{10}_{T_0}}\|D_x^\rho v\|_{L^5_xL^{10}_{T_0}}\|w\|_{L^5_xL^{10}_{T_0}} + c\|v\|^4_{L^5_xL^{10}_{T_0}}\|D_x^\rho w\|_{L^5_xL^{10}_{T_0}} \\
&\le cT_0^{4\rho/3}\|D_t^{\rho/3}v\|^3_{L^5_xL^{10}_{T_0}}\|D_x^\rho v\|_{L^5_xL^{10}_{T_0}}\|D_t^{\rho/3}w\|_{L^5_xL^{10}_{T_0}} \\
&\quad + cT_0^{4\rho/3}\,\|D_t^{\rho/3}v\|^4_{L^5_xL^{10}_{T_0}}\|D_x^\rho w\|_{L^5_xL^{10}_{T_0}} \\
&\le cT_0^{4\rho/3}|\!|\!|v|\!|\!|^4_\rho |\!|\!|w|\!|\!|_\rho,
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
&\|D_x^\rho \int_0^t U(t-t')\,\partial_x(v^3w^2)(t')\,dt'\| \le c\|D_x^\rho\,(v^3w^2)\|_{L^1_xL^2_{T_0}} \\
&\le c\|D_x^\rho\,v^3\|_{L_x^{5/3}L_{T_0}^{10/3}}\,\|w^2\|_{L_x^{5/2}L^5_{T_0}} + c\|v^3\|_{L_x^{5/3}L_{T_0}^{10/3}}\|D_x^\rho\,w^2\|_{L_x^{5/2}L^5_{T_0}} \\
&\le c\|v\|^2_{L^5_xL^{10}_{T_0}}\|D_x^\rho\,v\|_{L^5_xL^{10}_{T_0}}\,\|w\|^2_{L^5_xL^{10}_{T_0}} + c\|v\|^3_{L^5_xL^{10}_{T_0}}\|D_x^\rho\,w\|_{L^5_xL^{10}_{T_0}}\|\,w\|_{L^5_xL^{10}_{T_0}} \\
&\le cT_0^{4\rho/3}|\!|\!|v|\!|\!|^3_\rho |\!|\!|w|\!|\!|^2_\rho,
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
&\|D_x^\rho \int_0^t U(t-t')\,\partial_x(v^2w^3)(t')\,dt'\| \le c\|D_x^\rho\,(v^2w^3)\|_{L^1_xL^2_{T_0}} \\
&\le c\|D_x^\rho\,v\|_{L^5_xL^{10}_{T_0}}\|v\|_{L^5_xL^{10}_{T_0}}\|w\|^3_{L^5_xL^{10}_{T_0}} + c\|v\|^2_{L^5_xL^{10}_{T_0}}\|D_x^\rho\,w\|_{L^5_xL^{10}_{T_0}}\|w\|^2_{L^5_xL^{10}_{T_0}} \\
&\le cT_0^{4\rho/3}|\!|\!|v|\!|\!|^2_\rho |\!|\!|w|\!|\!|^3_\rho,
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
\|D_x^\rho \int_0^t U(t-t')\,\partial_x(vw^4)(t')\,dt'\| &\le c\|D_x^\rho\,(w^4v)\|_{L^1_xL^2_{T_0}} \\
&\le c\|D_x^\rho\,v\|_{L^5_xL^{10}_{T_0}}\|w\|^4_{L^5_xL^{10}_{T_0}} + c\|v\|_{L^5_xL^{10}_{T_0}}\|D_x^\rho\,w\|_{L^5_xL^{10}_{T_0}}\|w\|^3_{L^5_xL^{10}_{T_0}} \\
&\le cT_0^{4\rho/3}|\!|\!|v|\!|\!|_\rho |\!|\!|w|\!|\!|^4_\rho,
\end{aligned}
\tag{3.13}
$$

and finally

$$
\begin{aligned}
\|D_x^\rho \int_0^t U(t-t')\,\partial_x(w^5)(t')\,dt'\| &\le c\|D_x^\rho\,(w^4 v)\|_{L^1_x L^2_{T_\mathrm{o}}} \\
&\le c\|D_x^\rho\,(w^4)\|_{L^{5/4}_x L^{5/2}_{T_\mathrm{o}}}\|w\|_{L^5_x L^{10}_{T_\mathrm{o}}} + c\|w\|^4_{L^5_x L^{10}_{T_\mathrm{o}}}\|D_x^\rho\, w\|_{L^5_x L^{10}_{T_\mathrm{o}}} \\
&\le c\|w\|^4_{L^5_x L^{10}_{T_\mathrm{o}}}\|D_x^\rho\, w\|_{L^5_x L^{10}_{T_\mathrm{o}}} \\
&\le cT_\mathrm{o}^{4\rho/3}|\!|\!|w|\!|\!|^5_\rho.
\end{aligned}
\tag{3.14}
$$

Combining (3.7) and (3.10)–(3.14) we have

$$
\begin{aligned}
\|D_x^\rho \Phi(w)\| &\le c\|D_x^\rho w_0\| + T_\mathrm{o}^{4\rho/3}\Big\{|\!|\!|v|\!|\!|^4_\rho + |\!|\!|v|\!|\!|^3_\rho|\!|\!|w|\!|\!|_\rho \\
&\qquad + |\!|\!|v|\!|\!|^2_\rho|\!|\!|w|\!|\!|^2_\rho + |\!|\!|v|\!|\!|_\rho|\!|\!|w|\!|\!|^3_\rho + |\!|\!|w|\!|\!|^4_\rho\Big\}|\!|\!|w|\!|\!|_\rho \\
&\le c\|D_x^\rho w_0\| + T_\mathrm{o}^{4\rho/3}\Big\{|\!|\!|v|\!|\!|^{4\rho}_1 + |\!|\!|v|\!|\!|^{3\rho}_1|\!|\!|w|\!|\!|_\rho \\
&\qquad + |\!|\!|v|\!|\!|^{2\rho}_1|\!|\!|w|\!|\!|^2_\rho + |\!|\!|v|\!|\!|^{\rho}_1|\!|\!|w|\!|\!|^3_\rho + |\!|\!|w|\!|\!|^4_\rho\Big\}|\!|\!|w|\!|\!|_\rho.
\end{aligned}
\tag{3.15}
$$

Similarly we can obtain estimates for the other norms involving (3.1) to get

$$
|\!|\!|\Phi(w)|\!|\!|_\rho \le \|w_0\|_{H^\rho} + T_\mathrm{o}^{4\rho/3}F(|\!|\!|v|\!|\!|_1, |\!|\!|w|\!|\!|_\rho)|\!|\!|w|\!|\!|_\rho. \tag{3.16}
$$

Choosing $a = \max\{\|v_0\|_1, \|w_0\|_\rho\}(= \|v_0\|_1)$ and taking $T_\mathrm{o} = \Delta T$ we have

$$
c\,(\Delta T)^{4\rho/3}\,a^4 < \frac{1}{2}.
$$

Thus Φ is well defined since $F(|\!|\!|v|\!|\!|_1, |\!|\!|w|\!|\!|_\rho) \le c\,a^4$.

The same argument applies to show that Φ is a contraction. Hence the contraction mapping principle gives the result. □

Next we record an L^2-norm growth estimate for solutions of the IVP (3.6).

Lemma 3.3. *Let v and w be solutions of the IVP* (3.2) *and* (3.6) *with data v_0 in $H^1(\mathbb{R})$ and w_0 in $H^s(\mathbb{R})$ satisfying $\|v_0\|_{H^1} \simeq N^{1-s}$ and $\|w_0\|_2 \simeq N^{-s}$, $0 < s < 1$, respectively. Define*

$$
|\!|\!|w|\!|\!|_0 = \sup_{[0,\Delta T]} \|w(t)\|_{L^2} + \|w\|_{L^5_x L^{10}_{\Delta T}} + \|\partial_x w\|_{L^\infty_x L^2_{\Delta T}}.
$$

Then

$$
|\!|\!|w|\!|\!|_0 \simeq N^{-s}. \tag{3.17}
$$

4. Proof of Theorem 1.1

Consider the solution of the IVP

$$\begin{cases} u_t + u^4\, u_x + u_{xxx} = 0, & x \in \mathbb{R}, \quad t \in \mathbb{R} \\ u(x,0) = u_0(x). \end{cases} \tag{4.1}$$

For $u_0 \in H^s(\mathbb{R})$, $0 < s < 1$, with $\|u_0\|_{L^2} < \|Q\|_{L^2}$. We split u_0 as

$$u_0(x) = v_0(x) + w_0(x) = (\hat{u}_0\, \chi_{|\xi|\leq N})^{\vee} + (\hat{u}_0\, \chi_{|\xi|>N})^{\vee}.$$

$N \gg 1$ to be determined later. Observe that

$$\|v_0\|_{L^2} < \|Q\|_{L^2}, \qquad \|v_0\|_{H^1} \simeq N^{1-s}, \tag{4.2}$$

and

$$\|w_0\|_{\rho} \simeq N^{\rho - s}, \quad \text{for} \quad 0 < \rho \leq s < 1. \tag{4.3}$$

Using Propositions 3.1 and 3.2 we have solutions $v(t) \in H^1(\mathbb{R})$ and $w(t) \in H^{\rho}(\mathbb{R})$ such that $u(t) = v(t) + w(t)$, for $t \in [0, \Delta T]$ where $\Delta T \simeq \|v_0\|_{H^1}^{-3} \simeq N^{-3(1-s)}$.

Our goal is to extend the time of existence of (4.1) up to any time $T > 0$. To do so we use an iteration scheme that we will describe below.

First notice that from Proposition 3.2 we have

$$w(t) = U(t)w_0(x) + z(t)$$

where

$$z(t) = -\int_0^t U(t-t')\,\partial_x(v^4 w + 2v^3 w^2 + 2v^2 w^3 + w^4 v + \frac{1}{5}\, w^5)(t')\, dt', \quad t \in [0, \Delta T].$$

Hence the solution u of (4.1) can be written as

$$u(t) = v(t) + U(t)w_0 + z(t)$$

for any $t \in [0, \Delta T]$. In the process of iteration we need to check how is the growth of the solution at each interval of size ΔT.

We observe that at the point $t = \Delta T$

$$u(\Delta T) = v(\Delta T) + U(\Delta T)w_0 + z(\Delta T),$$

and that $U(\Delta T)w_0$ stays in $H^s(\mathbb{R})$ by group properties. Hence we shall show that for $v(\Delta T) + z(\Delta T)$ the conditions (4.2) hold.

From Proposition 3.1

$$\|v\|_{L_t^{\infty} H^1} \leq c\, N^{(1-s)}.$$

On the other hand,

$$\|z\|_{L_{\Delta T}^{\infty} H^1} \leq c\, N^{(1-2s)}. \tag{4.4}$$

In fact, the worst term in $z(t)$ is $\partial_x(v^4\, w)$. Thus we estimate

$$\|\partial_x \int_0^{\Delta T} U(t-t')\partial_x(v^4\, w)(t')\, dt'\|_{L^2},$$

and

$$\| \int_0^{\Delta T} U(t-t')\partial_x (v^4\, w)(t')\, dt' \|_{L^2}.$$

The dual version of the smoothing effect (2.7), Hólder's inequality and Sobolev's embedding give

$$\begin{aligned}
\|\partial_x \int_0^{\Delta T} U(t-t')\partial_x (v^4\, w)(t')\, dt' \|_{L^2} &\le \|v^4\, \partial_x w\|_{L^1_x L^2_{\Delta T}} + \|v^3\, \partial_x v\, w\|_{L^1_x L^2_{\Delta T}}\\
&\le \|v\|^4_{L^4_x L^\infty_{\Delta T}} \|\partial_x w\|_{L^\infty_x L^2_{\Delta T}} + \|v\|^3_{L^5_x L^{10}_{\Delta T}} \|\partial_x v\|_{L^5_x L^{10}_{\Delta T}} \|w\|_{L^5_x L^{10}_{\Delta T}}\\
&\le \|v\|^4_{L^4_x L^\infty_{\Delta T}} \|\partial_x w\|_{L^\infty_x L^2_{\Delta T}} + \Delta T \|D_t^{1/3} v\|^3_{L^5_x L^{10}_{\Delta T}} \|\partial_x v\|_{L^5_x L^{10}_{\Delta T}} \|w\|_{L^5_x L^{10}_{\Delta T}}\\
&\le c\, \|v\|^4_{1/4} \|w\|_0 + \Delta T\, \|v\|^4_1\, \|w\|_0\\
&\le c\, \|v\|_1 \|w\|_0 + \Delta T \|v\|^4_1\, \|w\|_0\\
&\le c N^{(1-s)}\, N^{-s} + c\, N^{-3(1-s)}\, N^{4(1-s)}\, N^{-s}\\
&\le c N^{(1-2s)}.
\end{aligned} \tag{4.5}$$

A similar argument as in (4.5) yields

$$\begin{aligned}
\| \int_0^{\Delta T} U(t-t')\partial_x (v^4 w)(t')\, dt' \|_{L^2} &= \|\partial_x \int_0^{\Delta T} U(t-t')(v^4 w)(t')\, dt' \|_{L^2}\\
&\le \|v\|^4_{L^5_x L^{10}_T}\, \|w\|_{L^5_x L^{10}_T} \le (\Delta T)^{4/3} \|D_t^{1/3}\, v\|^4_{L^5_x L^{10}_T}\, \|w\|_{L^5_x L^{10}_T}\\
&\le (\Delta T)^{4/3} \|v\|^4_1\, \|w\|_0 \le c\, N^{-4(1-s)}\, N^{4(1-s)}\, N^{-s}\\
&\le c\, N^{-s}.
\end{aligned} \tag{4.6}$$

From (4.5) and (4.6) we obtain (4.4).

We also have that

$$\begin{aligned}
\|v(\Delta T) + z(\Delta T)\|_{L^2} &= \|u(\Delta T) - U(\Delta T) w_0\|_{L^2}\\
&\le \|u_0\|_{L^2} + \|w_0\|_{L^2}\\
&\le \|u_0\|_{L^2} + N^{-s},
\end{aligned} \tag{4.7}$$

where we have used the conservation law (1.4) and group properties. So if N is sufficiently large we will have

$$\|v(\Delta T) + z(\Delta T)\|_{L^2} < \|Q\|_{L^2}.$$

Thus at each step there is a contribution of $N^{(1-2s)}$ from $\|z\|_{L^\infty_{\Delta T} H^1}$.

Hence, to reach the time T, the needed number of iterations is $\frac{T}{\Delta T}$, so if we want to guarantee that the H^1-norm grows on the interval $[0,T]$ as $N^{(1-s)}$, that is,

$$\frac{T}{\Delta T}\, N^{(1-2s)} \sim T\, N^{3(1-s)} N^{(1-2s)} \le c\, N^{(1-s)}, \tag{4.8}$$

we must choose $N(T) \sim T^{\frac{1}{3-4s}}$ for $3/4 < s < 1$.

From (4.7) we also have a contribution at each step from the L^2-norm of N^{-s}. Therefore to reach T we have

$$T\, N^{3(1-s)}\, N^{-s} \leq c,$$

which is satisfied by the $N(T)$ previously chosen. This completes the proof of the theorem. $\square$

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