C^1-generic Pesin's entropy formula

Ali Tahzibi [*]
Departamento de Matemática
ICMC-USP São Carlos
Caixa Postal 668
13560-970, São Carlos, SP
(e-mail: tahzibi@icmsc.sc.usp.br)

October 3, 2002

Abstract

The metric entropy of a C^2-diffeomorphism with respect to an invariant smooth measure μ is equal to the average of sum of the positive Lyapunov exponents of μ. This is the celebrated Pesin's entropy formula, $h_\mu(f) = \int_M \sum_{\lambda_i > 0} \lambda_i$. The C^2 regularity (or $C^{1+\alpha}$) of diffeomorphism is essential to the proof of this equality. We show that at least in two dimensional case this equality is satisfied for a C^1-generic diffeomorphisms and in particular we gain a larger than $C^{1+\alpha}$ volume preserving diffeomorphisms such that satisfy Pesin's formula.

Version français abrégée

Les exposants de caractéristique de Lyapunov d'une application différentiable de M (une variété compacte) dans M sont défini par théorème d'Osseledets. Soit μ une mesure de probabilité invariante pour f; pour presque tous points x il existe de nombres $\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_{k(x)}(x)$ (exposants) et il y a une unique décomposition $T_xM = E_1(x) \oplus \cdots E_{k(x)}(x)$ tel que

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df_x^n(v)\| = \lambda_i(x)$$

[*]This Work was supported by FAPESP, Brazil. I also thank hospitality of Unesp de São Jose do Rio preto

pour tout $0 \neq v \in E_i(x), 1 \leq i \leq l$. ($\dim(M) = l$) Les exposants caractéristiques défini comme au-dessus ont relation avec l'entropie de f. Par exemple pour mesure invariante ν et $f \in C^1$, laisse $\chi(x) := \sum_{\lambda_i > 0} \lambda_i$, alors par un résultat de Ruelle:

$$h_\nu(f) \leq \int_M \chi dm$$

Le résultat est en général une inégalité stricte. Mais si m est absolument continue por rapport à la mesure de Riemann sur M, e $f \in \mathrm{Diff}_m^{1+\alpha}(M), \alpha > 0$, alors

$$h_m(f) = \int_M \chi dm$$

En fait, on peut obtenir cette formule pour une plus grande classe de mesures.(voir [3]) Mais la régularité de f est une condition nécessaire d'obtenir telle égalité.

Nous allons montrer que si $\dim(M) = 2$, existe alors un sous-ensemble générique dans $\mathrm{Diff}_m^1(M)$ tel que satisfait la formule d'entropie de Pesin.

Theorem 1. *Existe un sous-ensemble générique $\mathcal{G} \in \mathrm{Diff}_m^1(M)$, tel que $f \in \mathcal{G}$ satisfait la formule d'entropie de Pesin et $\mathcal{G}$ strictement contient $\cup_{\alpha>0} \mathrm{Diff}_m^{1+\alpha}$.*

L'étape clef de la démonstration sera prouver que les points de continuit du deux fonctions suivantes $h_m(.), L(.)$ sont génériques dans C^1 topology.

Comme nous considérons difféomorhismes conservatif dans la dimension deux, il existe tout au plus un exposant de Lyapunov positif. Définir

- $L(f) = \int_M \lambda_1 dm$ pour $f \in \mathrm{Diff}_m^1(M)$ et
- $h_m(f) =$ l'entropie métrique de f pour $f \in \mathrm{Diff}_m^1(M)$.

Maintenant nous procédons en utilisant la formule d'entropie pour difféomorphismes dans $\cup \mathrm{Diff}_m^{1+\alpha}(M)$. Soit f un point de continuité pour $L(.)$ et $h_m(.)$. Par la densité de $\mathrm{Diff}_m^{1+\alpha}(M)$ dans $\mathrm{Diff}_m^1(M)$ prouvé dans [4], il y a une séquence $f_n \in \mathrm{Diff}_m^{1+\alpha}(M)$ tel cela f_n converge f dans C^1 topology. Par la formule de Pesin, $h_m(f_n) = L(f_n)$ et par la continuité f, $h_m(f) = L(f)$.

The Lyapunov exponents of a diffeomorphism f of a compact manifold M are defined by Oseledecs theorem which states that, for any invariant probability measure μ, for almost all points $x \in M$ there exist numbers $\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_{k(x)}(x)$ (Lyapunov exponents) and a unique splitting $T_xM = E_1(x) \oplus \cdots \oplus E_{k(x)}(x)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df^n(x)v\| = \lambda_i(x)$$

for all $0 \neq v \in E_i(x), 1 \leq i \leq m$. The characteristic exponents defined as above are related to the entropy of f. For example for any invariant measure ν and $f \in C^1$, let $\chi(x) := \sum_{\lambda_i>0} \lambda_i$, then by a result of Ruelle:

$$h_\nu(f) \leq \int_M \chi d\nu.$$

An estimation from below in terms of positive Lyapunov exponents is not true for general invariant measures, but if the measure $m$is absolutely continuous with respect to the Lebesgue measure of M, Pesin's formula states that for m-preserving diffeomorphisms with Hölder continuous derivative

$$h_m(f) = \int_M \chi dm.$$

In fact this entropy formula holds for a larger class of measures [3]. But the regularity of f is a necessary condition to get results of lower bounds for entropy.

We are going to show that if $\dim(M) = 2$ then there exists a generic subset in $\mathrm{Diff}^1_m(M)$ such that satisfy the Pesin's entropy formula.

Theorem 2. *There exists a C^1-residual subset $\mathcal{G} \subset \mathrm{Diff}^1_m(M)$ such that any $f \in \mathcal{G}$ satisfy the Pesin's entropy formula and $\mathcal{G}$ strictly contains $\cup_{\alpha>0} \mathrm{Diff}^{1+\alpha}_m$.*

The key idea is to prove that the continuity points of the following two functions are generic in C^1 topology. As we are considering volume preserving diffeomorhisms in dimension two, there exists at most one positive Lyapunov exponent. Define

- $L(f) = \int_M \sum_{\lambda_i>0} \lambda_i(x)dm \quad \text{for} \quad f \in \mathrm{Diff}^1_m(M)$ and
- $h_m(f) =$ the metric entropy of f for any $f \in \mathrm{Diff}^1_m(M)$.

Now we proceed by using the entropy formula for diffeomorphisms in $\cup \mathrm{Diff}^{1+\alpha}_m(M)$. Let f be a continuity point for $L(.)$ and $h_m(.)$. By density of $\mathrm{Diff}^{1+\alpha}_m(M)$ in $\mathrm{Diff}^1_m(M)$ proved in [4], there is a sequence $f_n \in \mathrm{Diff}^{1+\alpha}_m(M)$ such that f_n converges to f in C^1 topology. By Pesin's formula, $h_m(f_n) = L(f_n)$ and by continuity at f, $h_m(f) = L(f)$.

1 Continuity points of $L(.)$ and $h_m(.)$

The continuous dependence of Lyapunov exponents on diffeomorphism is an important problem. In fact let $\lambda_1(x,f) \geq \lambda_2(x,f) \geq \cdots \geq \lambda_d(x,f)$ denotes all Lyapunov exponents of of f and $\Lambda_i(f) = \int_M \sum_{j=1}^{i} \lambda_j$ (average of the i-greatest exponents) then it is well known that $f \to \Lambda_i(f)$ is an upper semi-continuous function.

Lemma 1. *The aplication $f \to \Lambda_i(f)$ is upper semi-continuous for $f \in \mathrm{Diff}_m^1(M)$*

Proof: By an standard argument we see that

$$\Lambda_i(f) = \inf_{n \geq 1} \frac{1}{n} \int_M \log \| \wedge^i (Df^n(x)) \| dm(x).$$

In fact to see why the limit is substituted by infimum, observe that the sequence

$$a_n = \int_M \log \| \wedge^i (Df^n(x)) \| dm(x)$$

is subadditive ($a_{n+m} \leq a_n + a_m$) and consequently $\lim \frac{a_n}{n} = \inf \frac{a_n}{n}$. Now, as $a_n(f)$ varies contonuously with f in C^1 topology and the infimum of continuous functions is upper semi-continuous, the proof of the lemma is complete. $\square$

Let us show that $L(f)$ is an upper semi-continuous function in $\mathrm{Diff}_m^1(M)$ independent of the dimension of M.

Lemma 2. *Let $L(x,f) = \sum_{\lambda_i \geq 0} \lambda_i(x,f)$ then $f \to L(f) = \int_M L(x,f) dm(x)$ is upper semi-continuous.*

Proof: Observe that the proof of this lemma for two dimensional case is the direct consequence of the Lemma 1. In fact for any $x \in M$, $\lim_{n \to \infty} \frac{1}{n} \log \| \wedge^p Df^n(x) \|$ exists and is equal to $\lambda_1 + \lambda_2(x) + \cdots + \lambda_p(x)$. This functions varies upper semi-continuously with respect to f. From this we claim that for any x the function $f \to \sum_{\lambda_i \geq 0} \lambda_i(f,x)$ is upper semi-continuous. Because let $f \in \mathrm{Diff}_m^1(M)$ and for $x \in M$, $\lambda_1(x) \geq \cdots \geq \lambda_p(x) \geq 0 \geq \lambda_{p+1} \geq \cdots \geq \lambda_d(x)$. Take U_ϵ a neighborhood of f such that for all $1 \leq k \leq d$ and any $g \in u_\epsilon$

$$\sum_{i=1}^{k} \lambda_i(x,g) \leq \sum_{i=1}^{k} \lambda_i(x,g) + \epsilon \tag{1}$$

Now take any such g and let $\lambda_1(x) \geq \cdots \geq \lambda'_p(x) \geq 0 \geq \lambda_{p'+1} \geq \cdots \geq \lambda_d(x)$ for some $1 < p' < d$. Using (1) we see that $\sum_{i=1}^{p'} \lambda_i(x,g) \leq \sum_{i=1}^{p} \lambda_i(x,f) + \epsilon$ (consider the three cases $p' < p, p = p', p < p'$) and the claim is proved.

Now we prove the lemma. By definition of $L(x,g)$, $L(x,g) \leq C$ for some uniform C in a neighborhood of f. Define

$$A_n = \{x \in M; d(f,g) \leq \frac{1}{n} \Rightarrow L(x,g) - L(x,f) \leq \frac{\epsilon}{2}\}$$

As $m(\cup A_n) = 1$ then for some large n we have $m(A_n) \geq 1 - \frac{\epsilon}{4C}$. So,

$$\int_M L(x,g) - L(x,f)dm = \int_{A_n} L(x,g) - L(x,f)dm + \int_{A_n^c} L(x,g) - L(x,f)dm$$

$$\leq \frac{\epsilon}{2} + 2C\frac{\epsilon}{4C} = \epsilon$$

and the proof of the lemma is complete. □

The upper semi-continuity is the key for the proof of our main theorem, because by a classical theorem in Analysis we know that the continuity points of a semi-continuous function on a Baire epace is always a residual subset of the space. (see e.g [2].)

The upper semi-continuity of $h_m(f)$ for f varying in $\mathrm{Diff}^1_m(M)$ is not known. In fact using Ruelles inequality and Pesin's equality we can show upper semi-continuity of $h_m(f)$ in the C^2 topology. (In this paper all C^2 statements can be replaced by $C^{1+\alpha}$). Let $g \in \mathrm{Diff}^2_m(M)$ be near enough to f, by semi continuity of $L(.)$ and Pesin's equality in C^2 topology:

$$h_m(g) \leq L(g) \leq L(f) + \epsilon = h_m(f) + \epsilon$$

So, we pose the following question:

Question 1. *Is it true that $h_m(f)$ is an upper semi-continuous function with C^1 volume preserving diffeomorphisms as its domain.*

However we are able to show that at least in two dimensional case the continuity points of $h_m(f)$ is generic in C^1 topology.

Proposition 1. *The continuity points of the map $h_m\colon \mathrm{Diff}^1_m(M) \to \mathbb{R}$ is a residual set.*

Proof: We use the result of Bochi [1] which gives a C^1 generic subset $\mathcal{G}' = A \cup Z$ such that any $g \in A$ is Anosov and for $g \in Z$ both Lyapunov exponents vanish almost everywhere. We show that $\mathcal{G}'$ contains a generic subset $\mathcal{G}$ and diffeomorphisms in $\mathcal{G}$ are continuity points of h_m. Firstly we state the following lemma:

Lemma 3. *Any $f \in Z$ is a continuity point of h_m.*

Proof: Let g be near enough to f by Ruelle's inequality and upper semi-continuity of $L(.)$ we get

$$h_m(g) \leq L(g) \leq L(f) + \epsilon = \epsilon$$

□

Now we prove that $h_m : A \rightarrow \mathbb{R}$ is upper semi-continuous. As $A \subset \mathrm{Diff}^1_m(M)$ is open we conclude that the continuity points of $h_m|A$ is generic inside A.

Proposition 2. *h_m restricted to C^1 Anosov diffeomorphisms is upper semi-continuous.*

Proof: To prove upper semi-continuity of $h_m|A$ we recall the definition of $h_m(f)$. By a theorem of Sinai we know that that if $\mathcal{P}$ is a generating partition then

$$h_m(f) = h_m(f, \mathcal{P}) = \lim_{n \to \infty} \frac{1}{n} H_m(\mathcal{P} \vee f^{-1}(\mathcal{P}) \cdots \vee f^{-n+1}(\mathcal{P})) \qquad (2)$$

If f is Anosov then there is $\epsilon > 0$ such that any g in a C^1 neighborhood of f is expansive with ϵ as expansivity constant. By the definition of generating partition any partition with diameter less than ϵ is generating and so we can choose a unique generating partition for a neighborhood of f. As m is an smooth measure we see that the function

$$f \to \frac{1}{n} H_m(\mathcal{P} \vee f^{-1}(\mathcal{P}) \cdots \vee f^{-n+1}(\mathcal{P})) = \frac{1}{n} \sum_P m(P) \log(m(P))$$

is continuous where the sum is over all elements of $\mathcal{P} \vee f^{-1}(\mathcal{P}) \cdots \vee f^{-n+1}(\mathcal{P})$. The limit in (2) can be replaced by infimum and we know that infimum of continuous function is upper semi-continuous. □

So, up to know we have proved that there exists a generic subset $A' \subset A$ such that diffeomorphisms in $\mathcal{G} = A' \cup Z$ are continuity point of h_m. Now we claim that $\mathcal{G}$ is C^1 generic in $\mathrm{Diff}^1_m(M)$. To proof the above claim we show a general fact about generic subsets.

Lemma 4. *Let $A \cup Z$ is a generic subset of a topological space T where A is an open subset. If $A' \subset A$ is generic inside A then $A' \cup Z$ is also generic in T.*

Proof: As a countable intersection of generic subsets is also generic we may suppose that A' is open and dense in A. By hypothesis $A \cup Z = \cap_n C_n$ where C_n are open and dense. So, we have

$$A' \cup Z = A' \cup (\cap C_n \cap A^c) = \cap_n (A' \cup C_n) \cap (A' \cup A^c) \tag{3}$$

firstly observe that each $A' \cup C_n$ is an open and dense subset and their intersection is generic. to complete the proof it is enough to show that $A' \cup (A^c)^\circ \subseteq A' \cup A^c$ is open and dense. Openness is obvious and denseness is left to reader as an easy excercise of general topology. □

So, by the fact that intersection of generic subsets is again a generic set we conclude that there is generic subset of $\mathrm{Diff}^1_m(M)$ where the diffeomorphisms in this generic subset are the continuity point of both $L(.)$ and $h_m(.)$ and so for this generic subset the Pesin's entropy formula is satisfied. To finish the proof of the Theorem 2 we have to show that $\cup_{\alpha>0} \mathrm{Diff}^{1+\alpha}_m(M)$ is not a generic subset and so the generic subset of Theorem 2 gives us some more diffeomorphisms satisfying Pesin's formula than Hölder derivative ones.

Lemma 5. $\mathrm{Diff}^{1+}_n(M) := \bigcup_{\alpha>0} \mathrm{Diff}^{1+\alpha}_m(M)$ *is not generic with C^1 topology.*

Proof: We show that the complement of $\mathrm{Diff}^{1+}_m(M)$ is a generic subset and this implies that $\mathrm{Diff}^{1+}_m(M)$ can not be generic.
As in what follows we are working locally, one may suppose that $M = \mathbb{R}^2$. Let's define

$$\|f\|_\alpha = Sup_{x \neq y \in M} \frac{d(Df(x), Df(y))}{d(x,y)^\alpha}$$

and denote

$$H_{n,k} = \{f \in \mathrm{Diff}^1_m(M), \|f\|_{\frac{1}{n}} > k\}.$$

By the above definition we get $\mathrm{Diff}^1_m(M) \setminus \mathrm{Diff}^{1+}_m(M) = \bigcap_{n,m\in\mathbb{N}} H_{n,m}$. To prove the lemma We claim that for any n, each $H_{n,k}$ is an open dense subset.

1. Openness

Let $f \in H_{n,k}$, by definition there exist x, y and $\eta > 0$ such that $\dfrac{d(Df(x), Df(y))}{d(x,y)^\alpha} > k + \eta$. Take any g, ϵ-near to f in C^1 topology by the triangular inequality we get:

$$\|g\|_{\frac{1}{n}} > \frac{d(Dg(x), Dg(y))}{d(x,y)^{\frac{1}{n}}} > \frac{d(Df(x), Df(y))}{d(x,y)^{\frac{1}{n}}} - \frac{2\epsilon}{(d(x,y))^{\frac{1}{n}}}$$

Taking ϵ small enough the above inequality shows that $\|g\|_{\frac{1}{n}} > k$ and the openness is proved.

2. Density

Let $f \in \mathrm{Diff}^1_m(M)$ we are going to find $g \in \mathrm{Diff}^1_m(M) \setminus \mathrm{Diff}^{1+}_m(M)$ such that g is near enough to f. For this purpose we construct $h \in \mathrm{Diff}^1_m(M) \setminus \mathrm{Diff}^{1+}_m(M)$ near enough to identity and then put $g = h \circ f$.
Considering local charts, it is enough to construct a C^1 volume preserving diffeomorphism $\tilde{I}$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that

1. $\tilde{I}$ is C^1 near to identity inside $B(0, \epsilon)$ for small $\epsilon > 0$
2. $\tilde{I}$ is identity outside the ball $B(0, 2\epsilon)$
3. $\tilde{I} \in \mathrm{Diff}^1_m(\mathbb{R}^2) \setminus \mathrm{Diff}^{1+}_m(\mathbb{R}^2)$

Let us parametrize $\mathbb{R}^2$ with polar (r, θ) coordinates and $\xi : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 bump function which is equal to one inside the ball $\{r < \epsilon\}$ and vanishes outside the ball of radius 2ϵ. Consider the following C^1 but not $C^{1+\alpha}$ (for any α) real diffeomorphism:

$$\eta(r) = \begin{cases} r + \frac{r}{\log \frac{1}{r}} & \text{if } r > 0 \\ r & \text{if } r \leq 0 \end{cases}$$

and define $\tilde{I}(r, \theta) = (r, \theta + \xi(r)\eta(r)\theta_0)$ for small θ_0. The jacobian matrix of $\tilde{I}$ is

$$D\tilde{I} = \begin{pmatrix} 1 & 0 \\ \theta_0(\xi(r)\eta(r))' & 1 \end{pmatrix}$$

and it is obvious that $\tilde{I}$ is volume preserving and taking θ_0 and ϵ, small enough it is near enough to identity in C^1 topology. $\square$

References

[1] J. Bochi. Genericity of zero lyapunov exponents. 2001. Preprint, IMPA.

[2] K Kuratowski. *Topology, Vol. 1.* Academic Press, 1966.

[3] F. Ledrappier and L.-S. Young. The metric entropy of diffeomorphisms i. characterization of measures satisfying pesin's entropy formula. *Ann. of Math*, 122:509–539, 1985.

[4] E. Zehnder. A note on smoothing symplectic and volume preserving diffeomorphims. volume 597 of *Lecture notes in Math*, pages 828–854. Springer Verlag, 1977.