IRREDUCIBLE COMPONENTS OF THE SPACE OF FOLIATIONS ASSOCIATED TO THE AFFINE LIE ALGEBRA

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Abstract. In this paper, we give the explicit construction of certain components of the space of holomorphic foliations of codimension one, in complex projective spaces. These components are associated to some algebraic representations of the affine Lie algebra $\mathfrak{Aff}(\mathbb{C})$. Some of them, the so-called exceptional or Klein-Lie components, are rigid, in the sense that all generic foliations in the component are equivalent (example 1 of §2.2). In particular, we obtain rigid foliations of all degrees. Some generalizations and open problems are given the end of §1.

§1. Introduction

It is known that the space $\mathcal{F}(\nu;n)$ of singular holomorphic codimension one foliations of degree $\nu \geq 0$ on $\mathbb{CP}(n)$; $n \geq 3$; can be considered as an algebraic subset of the space of 1-forms on $\mathbb{C}^{n+1}$ whose coefficients are homogeneous polynomials of degree $\nu + 1$ (cf. [Ce-LN1], [Ce-LN3] and [CA]). Some of the irreducible components of this algebraic subset have been described; for example, the logarithmic components, which correspond to foliations defined by closed meromorphic 1-forms (cf. [CA]). Other components are the rational (cf. [Ce-LN1]) and the pull-back components (cf. [Ce-LN3]). For $\nu = 0;1;2$ the complete decomposition of $\mathcal{F}(\nu;n)$ in irreducible components was obtained in [Ce-LN1].

In this paper, we present new components of $\mathcal{F}(\nu;n)$, $n \geq 3$, related with some special representations of the affine Lie algebra $\mathrm{aff}(\mathbb{C}) := \{e_1;e_2;[e_1;e_2] = e_2\}$ in the algebra of polynomial vector fields of an affine chart $\mathbb{C}^3 \subset \mathbb{CP}(3)$. These new components include as a particular case the "exceptional component" of $\mathcal{F}(2;n)$, described in [Ce-LN1].

To obtain our result we follow three steps:

(1) We construct families of foliations $\mathcal{F}_P \subset \mathcal{F}(\nu;3)$, where P denotes a discrete invariant, arising from representations of the affine algebra.
(2) We find sufficient conditions in order to prove stability under deformations of some of these families, i.e. we prove that for certain values of P the deformation of a generic foliation $\mathcal{F} \in \mathcal{F}_P$ is still a foliation in $\mathcal{F}_P$.
(3) We get codimension one foliations in $\mathbb{CP}(n)$, $n \geq 4$, by pull-back of the foliations just constructed, and prove that we also have irreducible components in $\mathcal{F}(\nu;n)$.

The description of the families in the first step can be geometrically described. To do that, we consider the so called Klein–Lie curves. They are characterized by the fact of being the rational projective curves fixed by an infinite group of projective automorphism. In $\mathbb{CP}(3)$ such curves, up

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to an automorphism in $PGL(4,\mathbb{C})$, can be parameterized by $\Gamma(t:s) = (t^p : t^q s^{p-q} : t^r s^{p-r} : s^p)$, where $1 \le r < q < p$ are positive integers with $\gcd(p,q,r) = 1$.

For each $\lambda \neq 0$ such that $\lambda + r \in \{0\} \cup \mathbb{N}$, we have a representation of the affine Lie algebra $\rho_\lambda : \mathrm{aff}(\mathbb{C}) \to \mathfrak{X}(\mathbb{C})$, determined by the two vector fields $s_\lambda := \frac{1}{\lambda} t \frac{\partial}{\partial t}$, and $x_\lambda := t^{\lambda+1}\frac{\partial}{\partial t}$. Consider the linear semi-simple vector field on $\mathbb{C}^3$

$$S = px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y} + rz\frac{\partial}{\partial z}.$$

Suppose that there is another polynomial vector field X on $\mathbb{C}^3$ such that $[S,X] = \lambda X$, and so that

$$\gamma_*\big(s_\lambda\big) = \frac{1}{\lambda} S\big(\gamma(t)\big), \quad \gamma_*(x_\lambda) = X\big(\gamma(t)\big),$$

where $\gamma(t) = (t^p, t^q, t^r)$ is the affine curve $\Gamma \cap \mathbb{C}^3$. Then, the algebraic foliation $\mathcal{F} = \mathcal{F}(S,X)$ on $\mathbb{C}^3$, defined by the 1–form $\Omega = i_S i_X (dz_1 \wedge dz_2 \wedge dz_3)$ is associated to a representation of the affine algebra in the algebra of polynomial vector fields in $\mathbb{C}^3$, and it can be extended to a foliation on $\mathbb{CP}(3)$ of certain degree ν.

We give explicitly several examples in Section 2, all in the case $r = 1$. Note also that both s_λ and x_λ are complete vector fields on $\mathbb{C}$ just in case $\lambda = 1$. This is what happens in Example 1, where S and X are complete and the flow of S is periodic: both necessary conditions for the existence of an action of the affine group on $\mathbb{C}^3$ associated to the foliation.

We define

$$\mathcal{F}\big((p,q,r),\lambda,\nu\big) := \{\mathcal{F} \in \mathcal{F}(\nu,3) \mid \mathcal{F} = \mathcal{F}(S,X) \quad \text{in some affine chart}\}$$

and we will show that they are irreducible subvarieties of $\mathcal{F}(\nu,3)$. We also show that if $\mathcal{F} \in \mathcal{F}\big((p,q,r),\lambda,\nu\big)$ then the tangent sheaf $T_{\mathcal{F}}$ is isomorphic to $\mathcal{O} \oplus \mathcal{O}(2-\nu)$.

In order to carry on the second step, we will need some technical results. Let us first give some definitions.

Definition 1. Let ω be an integrable 1-form defined in a neighborhood of $p \in \mathbb{C}^3$. We say that p is a generalized Kupka (briefly g.K.) singularity of ω if $\omega_p = 0$ and, either $d\omega_p \neq 0$, or p is an isolated zero of $d\omega$.

The local structure of a foliation near a g.K. singularity is well known by now. When $d\omega_p \neq 0$ it is of Kupka type and it is locally the product of two foliations: a singular one in dimension two and a nonsingular one of dimension 1, as in fig. 1 (cf. [K, Me]). When p is an isolated singularity of $d\omega$, the singularity is quasi-homogeneous (cf. Theorem A and [LN1]) or logarithmic (cf. Remark 1 and [C–LN2]).

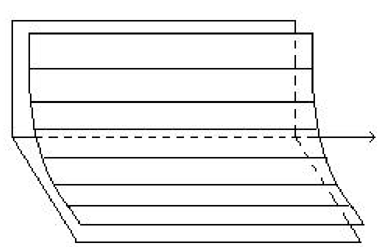

Fig. 1

We also prove that g.K. singularities are stable under deformations, (cf. [C-LN] and Proposition 1).

Definition 2. A codimension one holomorphic foliation $\mathcal{F}$ in a complex three manifold M is strongly generalized Kupka (briefly s.g.K.), if all the singularities of $\mathcal{F}$ are g.K.

We will show, as a consequence of the stability of g.K. singularities, that s.g.K foliations are stable under deformations. In fact, we first note that the local structure of g.K. singularities implies that the analytic tangent sheaf of a s.g.K foliation is locally free. Using well-known results on holomorphic vector bundle theory (Theorem B), we can prove the following

Theorem 1. Suppose that $\mathcal{F}\left((p,q,r),\ell,\nu\right)$ contains some s.g.K foliation. Then $\overline{\mathcal{F}\left((p,q,r),\ell,\nu\right)}$ is an irreducible component of $\mathcal{F}(\nu,3)$.

Theorem 1 and Example 1 in Section 2, give for any $\nu \geq 3$ a new irreducible component of the space of foliations of degree ν. This component is, in fact, the closure of a natural action of $PGL(4,\mathbb{C})$ on $\mathcal{F}(\nu,3)$. In particular, a foliation corresponding to a generic point in the component, is linearly stable. On the other hand, given (p,q,r) positive integers such that $p > q > r$, the set $\{(\ell,\nu)\}$ such that $\mathcal{F}\left((p,q,r),\ell,\nu\right)$ contains some s.g.K foliation is finite (Theorem 3). This motivates the following problem :

Problem 1 Given three positive integers $p > q > r \geq 1$, are there (ℓ,ν) such that $\mathcal{F}\left((p,q,r),\ell,\nu\right)$ contains a s.g.K foliation ?

The examples in §2.2 are s.g.K foliations in $CP(3)$, all of them belonging to some $\mathcal{F}\left((p,q,r),\ell,\nu\right)$. Consequently, the tangent sheaf for these examples splits. This motivates the following questions :

Problem 2 Is it true that $T_{\mathcal{F}}$ splits for any s.g.K foliation $\mathcal{F}$ on $CP(3)$? More generally, let $\mathcal{F}$ be a codimension one foliation on $CP(3)$ such that for any $p \in CP(3)$ the sheaf of germs of vector fields at p tangent to $\mathcal{F}$ is free with two generators. Does $T_{\mathcal{F}}$ split ?

We observe that all examples that we have of s.g.k. foliations on $CP(3)$ have at most two quasi-homogeneous singularities. A natural question is the following :

Problem 3. Are there s.g.K foliations on $CP(3)$ with more than two quasi-homogeneous singularities?

Finally, concerning the third step, in §3.2 we will consider foliations on $CP(n)$, $n \geq 4$, which are pull-back of s.g.K foliations on $CP(3)$ by a generic linear rational map $f: CP(n) \to CP(3)$. Denote by $\mathcal{F}\left((p,q,r),\ell,\nu,n\right) \subset \mathcal{F}(\nu,n)$ the set of foliations so obtained from $\mathcal{F}((p,q,r),\ell,\nu)$,

$$\mathcal{F}\left((p,q,r),\ell,\nu,n\right) := \{\mathcal{F} \mid \mathcal{F} = f^*\mathcal{G},\ \mathcal{G} \in \mathcal{F}((p,q,r),\ell,\nu)\}$$

We prove the following:

Theorem 2. Let $\mathcal{F}$ be a foliation on $CP(n)$, $n \geq 4$ and $i: CP(3) \to CP(n)$ be a linear embedding of a 3–plane in general position with respect to $\mathcal{F}$. Suppose that $\mathcal{G} = i^*(\mathcal{F})$ is a s.g.K foliation in $\mathcal{F}(\nu,3)$ and that it is generated by two one–dimensional foliations on $CP(3)$. Then there exists a linear rational map $f: CP(n) \to CP(3)$ such that $\mathcal{F} = f^*(\mathcal{G})$. In particular $\overline{\mathcal{F}\left((p,q,r),\ell,\nu,n\right)}$ is an irreducible component of $\mathcal{F}(\nu,n)$.

§2 Preliminary results and examples

Notation. Through out the paper, we will consider $(z_1 : z_2 : z_3 : z_4)$ as homogeneous coordinates in $CP(3)$. The basic affine open subsets, will be $E_1 = \{(1 : w : v : u) | (u,v,w) \in \mathbb{C}^3\}$; $E_2 = \{(r : 1 : s : t) | (r,s,t) \in \mathbb{C}^3\}$; $E_3 = \{(r : s : 1 : t) | (r,s,t) \in \mathbb{C}^3\}$ and $E_0 = \{(x : y : z : 1) | (x,y,z) \in \mathbb{C}^3\}$.

§2.1 **Generalized Kupka and quasi-homogeneous singularities.** Let $p \geq q \geq r > 0$ be relatively prime integers and S be the semi-simple vector field on $\mathbb{C}^3$ defined as in (1) by $S = px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y} + rz\frac{\partial}{\partial z}$. We say that a vector field X, holomorphic in a neighborhood of $0 \in \mathbb{C}^3$, is S-quasi-homogeneous of weight ℓ, if we have the following Lie bracket identity : $[S, X] = \ell X$. Remark that necessarily $\ell + r$ is a non-negative integer and X is a polynomial vector field. In fact, if $X = P_1\frac{\partial}{\partial x} + P_2\frac{\partial}{\partial y} + P_3\frac{\partial}{\partial z}$, the condition that X is S-quasi-homogeneous of weight ℓ is equivalent to the fact that, after giving weights p, q and r to the variables x, y and z, respectively, the polynomials P_1, P_2 and P_3 are weighted homogeneous of degrees $\ell + p$, $\ell + q$ and $\ell + r$, respectively.

Moreover, S and X give a representation of the affine Lie algebra in the algebra of polynomial vector fields. If we suppose that S and X are linearly independent at generic points, then these vector fields generate an algebraic foliation on $\mathbb{C}^3$, which is given by the integrable 1-form $\Omega = i_S\, i_X(dx \wedge dy \wedge dz)$. Since Ω is a polynomial 1-form, this foliation can be extended to a singular foliation of $\mathbb{CP}(3)$, which will be denoted by $\mathcal{F}(\Omega)$ or by $\mathcal{F}(S, X)$. Observe that S extends to a holomorphic vector field on $\mathbb{CP}(3)$ and that its trajectories are contained in the leaves of $\mathcal{F}(\Omega)$. On the other hand, in general, the vector field X is meromorphic in $\mathbb{CP}(3)$, but the foliation defined by it on $\mathbb{C}^3$ extends to a foliation on $\mathbb{CP}(3)$, which will be denoted by $\mathcal{G}(X)$, whose leaves are also contained in the leaves of $\mathcal{F}(\Omega)$. Remark that the singular set of $\mathcal{F}(\Omega)$, denoted by $\mathrm{sing}(\mathcal{F}(\Omega))$, is invariant by the flow of S, $\exp(tS) := S_t$. This follows from the relation

$$(2)\quad L_S(\Omega) = m.\Omega\ ;\ m = \ell + tr(S) = \ell + p + q + r\ ;$$

as the reader can check. Relation (2) implies also that, if $p_o \notin \mathrm{sing}(S)$, then $\mathcal{F}(\Omega)$ is, in a neighborhood of p_o, equivalent to the product of a foliation in dimension two by a one-dimensional disk, like in fig. 1. In fact, let $(U, (u, v, w))$ be a holomorphic coordinate system such that $S|_U = \frac{\partial}{\partial u}$. Then, it is not difficult to see that, the integrability condition and (2) imply that

$$\Omega(u, v, w) = e^{mu}.\Omega(0, v, w) = e^{mu}.(A(v, w)dv + B(v, w)dw)\ ;$$

which proves the assertion.

In the affine chart $\mathbb{C}^3 \subset \mathbb{CP}(3)$, where S is like in (1), the leaves of $\mathcal{F}(\Omega)$ are "S-cones" with vertex at $0 \in \mathbb{C}^3$, that is, immersed surfaces invariant by the flow of S. If $\mathrm{sing}(\mathcal{F}(\Omega))$ has codimension two, then each one of its components is the closure of an orbit of S. Now, we impose a condition which implies the local stability of this kind of singularity by small perturbations of the form defining the foliation.

Let ω be an integrable 1-form in a neighborhood of $p_o \in \mathbb{C}^3$ and μ be a holomorphic 3-form such that $\mu_{p_o} \neq 0$. Then $d\omega = i_Z(\mu)$, where Z is a holomorphic vector field. It is not difficult to see that p_o is a g.K. singularity of ω if, and only if, p_o is an isolated singularity of Z.

Definition 3. We say that p_o is a quasi-homogeneous (briefly q.h.) singularity of ω if p_o is an isolated singularity of Z and the germ of Z at p_o is nilpotent (as a derivation in the local ring of formal power series at p_o).

This definition is justified by the following result (cf. [LN]):

Theorem A. Let $p_o \in \mathbb{C}^3$ be a quasi-homogeneous singularity of an integrable 1-form ω. Then there exist two holomorphic vector fields S and X and a local chart $(U, (x, y, z))$ around p_o such that $x(p_o) = y(p_o) = z(p_o) = 0$ and :
(a). $\omega = i_S i_X(dx \wedge dy \wedge dz)$.
(b). $S = px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y} + rz\frac{\partial}{\partial z}$, where p, q and r are positive integers with $\gcd(p, q, r) = 1$.
(c). p_o is an isolated singularity for X, X is a polynomial in the chart $(U, (x, y, z))$ and $[S, X] = \ell.X$, where $\ell \geq 1$.

Definition 4. Let $p_o \in \mathbb{C}^3$ be a q.h. singularity of ω. We say that it is of type (p, q, r, ℓ), if for some local chart and vector fields S and X, then properties (a), (b) and (c) of Theorem A are satisfied.

Remark 1. If the singularity p_o is g.K. but the germ of Z at p_o is semi-simple, then the foliation $\mathcal{F}(\omega)$ can be defined locally by an action of $\mathbb{C}^2$. More precisely, there exists a germ of vector field X at p_o such that $[Z, X] = 0$ and

$$i_X i_Z(dx \wedge dy \wedge dz) = f.\omega ,$$

where $f(p_o) \neq 0$. This fact is a consequence of the results of [Ce-LN-2]. We call this type of singularity a logarithmic type singularity.

Remark 2. Let p_o be a q.h. singularity of type (p, q, r, ℓ) of an integrable 1-form ω. If S and X are as in Theorem A, then the multiplicity of X at the singularity p_o (the Milnor number) is given by

$$\mu(X, p_o) = \frac{(\ell + p)(\ell + q)(\ell + r)}{p.q.r}.$$

In particular, $p.q.r$ must divide $(\ell + p)(\ell + q)(\ell + r)$. The proof of this fact can be found in [LN].

We can now state the stability result :

Proposition 1. Let $(\Omega_s)_{s \in \Sigma}$ be a holomorphic family of integrable 1-forms defined in a neighborhood of a compact ball $B = \{z \in \mathbb{C}^3 ; |z| \leq \rho\}$, where Σ is a neighborhood of $0 \in \mathbb{C}^k$. Suppose that $0 \in B$ is a q.h. singularity of Ω_0 of type (p, q, r, ℓ). There exists $\epsilon > 0$ such that if $|s| < \epsilon$, then Ω_s has a q.h. singularity $z(s)$ in B, of type (p, q, r, ℓ). Moreover, the function $s \mapsto z(s)$ is holomorphic and $z(0) = 0$.

The arguments of the proof of Proposition 1 are contained in the proof of Lemma 6 of §4.3 of [Ce-LN-1]. We leave the details for the reader.

As a consequence of Proposition 1 and of Theorem 5 of [C-LN], we get the following :

Corollary. Let $\mathcal{F}_0$ be a codimension one s.g.K foliation on a compact complex threefold M. Then there exists a neighborhood $\mathcal{U}$ of $\mathcal{F}_0$ in the space of codimension one foliations, such that any $\mathcal{F} \in \mathcal{U}$ is s.g.K.

We use Theorem 5 of [C-LN] to guarantee the stability of the singularities of Kupka and logarithmic types.

Remark 3. If p_o is a g.K. singularity of a foliation $\mathcal{F}$, then the sheaf of germs of vector fields at p_o tangent to $\mathcal{F}$, is locally free and has two generators.

In fact, if $\mathcal{F}$ is defined by ω in a neighborhood of p_o and $d\omega = i_Z\,\mu$, where $\mu_{p_o} \neq 0$, then the germ of Z at p_o has an isolated singularity at p_o. The integrability of ω implies that $i_Z(\omega) = 0$, so that, by De Rham's division Theorem (cf.[DR] and [C-LN]), we can write $\omega = i_Z(\theta)$, where θ is a 2-form. Since we are in dimension three, we have $\theta = -i_Y(\mu)$, where Y is a vector field. This implies that $\omega = i_Y\, i_Z(\mu)$. Now, if X is a germ of vector field such that $i_X(\omega) = 0$, we have $X = a.Y + b.Z$ where a and b are holomorphic outside $\mathrm{sing}(\omega)$. Since $\mathrm{sing}(\omega)$ has codimension two, it follows from Hartog's Theorem that a and b can be extended to a neighborhood of p_o, which proves the assertion.

Remark 4. Let p_o be an isolated singularity of a codimension one foliation $\mathcal{F}$ on a threefold (for instance a Morse singularity). Then the sheaf of germs of vector fields at p_o tangent to $\mathcal{F}$ is not locally free. In fact, it follows from Malgrange's Theorem (cf. [M]), that $\mathcal{F}$ has a local holomorphic first integral. This implies the assertion, as the reader can check (see also [LN-1]).

Remark 5. If $\mathcal{F}$ is a s.g.K foliation on M, we can associate to $\mathcal{F}$ a rank two vector bundle over M, the tangent bundle of $\mathcal{F}$, which will be denoted by $T_{\mathcal{F}}$, as follows. Take a covering $(U_\alpha)_{\alpha\in A}$ of M by open sets such that for any $\alpha \in A$ there are two holomorphic vector fields on U, say X_α and Y_α, such that the sheaf of vector fields tangent to $\mathcal{F}|_{U_\alpha}$ is generated by these vector fields. If $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$, then in $U_{\alpha\beta}$ we can write

$$(3)\quad \begin{cases} X_\beta = a_{\alpha\beta}X_\alpha + b_{\alpha\beta}Y_\alpha \\ Y_\beta = c_{\alpha\beta}X_\alpha + d_{\alpha\beta}Y_\alpha \end{cases} ,\text{where the matrix } A_{\alpha\beta} = \begin{pmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ c_{\alpha\beta} & d_{\alpha\beta} \end{pmatrix} \text{ is in } SL(2,\mathcal{O}(U_{\alpha\beta})).$$

Clearly, $(A_{\alpha\beta})_{U_{\alpha\beta}\neq\emptyset}$ is a cocycle of matrices, that is, if $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, then $A_{\alpha\beta}.A_{\beta\gamma}.A_{\gamma\alpha} = Id$ on $U_{\alpha\beta\gamma}$.

Let W be the disjoint union $\uplus_\alpha (U_\alpha \times \mathbb{C}^2)$ and $\sim$ be the equivalence relation on W defined by

$$(4)\ U_\alpha \times \mathbb{C}^2 \ni (x_\alpha, v_\alpha) \sim (x_\beta, v_\beta) \in U_\beta \times \mathbb{C}^2 \quad \Leftrightarrow \quad x_\alpha = x_\beta = x \in U_{\alpha\beta} \text{ and } v_\alpha = v_\beta . A_{\alpha\beta}(x);$$

where in the above relation, we consider v_α and v_β as line vectors. We define $T_{\mathcal{F}} = W/\sim$ and $\pi : T_{\mathcal{F}} \to M$ by $\pi[x, v_\alpha] = x$, where $[x, v_\alpha]$ is the quotient class of $(x, v_\alpha) \in W$. It is not difficult to prove that $T_{\mathcal{F}}$ is a complex manifold and $T_{\mathcal{F}} \xrightarrow{\pi} M$ is a vector bundle.

We observe that to any holomorphic (resp. meromorphic) section of $T_{\mathcal{F}}$ on some open set $U \subset M$ corresponds an unique holomorphic (resp. meromorphic) vector field tangent to $\mathcal{F}$. In fact, given a section $\sigma : U \to T_{\mathcal{F}}$, we can write on $U \cap U_\alpha \neq \emptyset$, $\sigma|_{(U\cap U_\alpha)} = (\lambda_\alpha, \mu_\alpha) : U \cap U_\alpha \to \mathbb{C}^2$. Define $Z_\alpha = \lambda_a.X_\alpha + \mu_\alpha.Y_\alpha$. The reader can check, by using (3) and (4), that if $U \cap U_{\alpha\beta} \neq \emptyset$ then $Z_\alpha \equiv Z_\beta$ on $U \cap U_{\alpha\beta}$, which implies that there exists a vector field Z on U, tangent to $\mathcal{F}$, such that $Z|_{(U\cap U_\alpha)} = Z_\alpha$ for any $\alpha \in A$.

Conversely, to any vector field Z, holomorphic (resp. meromorphic) on U and tangent to $\mathcal{F}$, there exists a holomorphic (resp. meromorphic) section $\sigma : U \to T_{\mathcal{F}}$, such that the associated vector field is Z. We leave the details for the reader. Before stating the next result, we need a definition.

Definition 5. We say that a codimension one foliation $\mathcal{F}$ on a complex threefold M is generated by two foliations of dimension one, say $\mathcal{G}_1$ and $\mathcal{G}_2$, if for any $p \in M$ there exists a neighborhood U of p and holomorphic vector fields X_1 and X_2 on U such that :
(a). $\mathcal{G}_j$ is defined in U by X_j, $j = 1, 2$.
(b). $\mathcal{F}|_U$ is defined by the 1-form $\omega = i_{X_1} i_{X_2} \mu$, where μ is a nonvanishing 3-form on U. In particular, we have that $\mathcal{G}_1$ and $\mathcal{G}_2$ are tangent to $\mathcal{F}$ and that
(b.1). If $p \in M \setminus (sing(\mathcal{G}_1) \cup sing(\mathcal{G}_2))$ and $T_p\mathcal{G}_1 \neq T_p\mathcal{G}_2 \subset T_pM$, then $T_p\mathcal{F} = T_p\mathcal{G}_1 \oplus T_p\mathcal{G}_2$.
(b.2). $sing(\mathcal{F}) = sing(\mathcal{G}_1) \cup sing(\mathcal{G}_2) \cup D$, where

$$D = \{p \in M \setminus sing(\mathcal{G}_1) \cup sing(\mathcal{G}_2) \mid T_p\mathcal{G}_1 = T_p\mathcal{G}_2\}.$$

Proposition 2. Let $\mathcal{F}$ be a s.g.K foliation on M and $T_{\mathcal{F}}$ be its tangent bundle. Then :
(a). To any line sub-bundle L of $T_{\mathcal{F}}$, corresponds a foliation by curves $\mathcal{G}_L$ on M with the following properties :
(a.1). $\mathcal{G}_L$ is tangent to $\mathcal{F}$.
(a.2). $sing(\mathcal{G}_L) \subset sing(\mathcal{F})$.
(b). $T_{\mathcal{F}}$ splits as a sum of two line bundles if, and only if, $\mathcal{F}$ is generated by two foliations of dimension one.

The proof of the proposition is straightforward and is left for the reader.

In the next section we will see some examples of s.g.K foliations on $\mathbb{CP}(3)$. In all examples the bundle $T_{\mathcal{F}}$ splits. This has motivated problem 2 in §1.

§2.2 Examples. This section is devoted to describe some examples of strongly generalised Kupka foliations on $\mathbb{CP}(3)$. Each example will be generated by two foliations of dimension one, $\mathcal{G}_1$ and $\mathcal{G}_2$, in the sense of definition 5. One of these one-dimensional foliations, say $\mathcal{G}_1$, will be generated by a global vector field S on $\mathbb{CP}(3)$, which in some affine coordinate system $(x; y; z) \in \mathbb{C}^3 \subset \mathbb{CP}(3)$ is like in (1) : $S = px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y} + rz\frac{\partial}{\partial z}$; where $p; q; r \in \mathbb{N}$, g.c.d$(p; q; r) = 1$ and $p > q > r$. On the other hand, $\mathcal{G}_2$ will be of degree $d \geq 1$, so that the foliation will be of degree $\nu = d + 1$.

Being foliations in $\mathcal{F}(p; q; r; d + 1; l)$, all the examples that we give share a geometrical pattern that we now explain. As the singular locus of the foliation is invariant by a global vector field in $\mathbb{CP}(3)$, it is globally fixed by an infinite group of projective automorphisms: the one given by the flow of S. Each curve in the singular locus has to be of a very special type.

Klein and Lie showed (see, e.g. [E-C]) that a curve $\mathbb{CP}(n)$ fixed by the action of an infinite group of projective automorphisms is rational algebraic. If it is of degree p ($\geq n$), it is obtained as an adequate linear projection of the rational normal curve $\Gamma_p \subset \mathbb{CP}(p)$, i.e. $\mathbb{CP}(1)$ embedded as $\Gamma_p(s : t) := (t^p : t^{p-1}s : \dots : ts^{p-1} : s^p)$. For $n = 3$, they showed that the projected curve could be written, after a change of coordinates, as (in the affine open set E_4)

$$\gamma_{p;q;r}(t) := (t^p; t^q; t^r)$$

where $p > q > r \geq 1$ are positive integers. A curve so parametrized is fixed by the projective transformations $x' = \alpha^p x$, $y' = \alpha^q y$, $z' = \alpha^r z$ that correspond to changing t by αt, and fix the points $A = (1 : 0 : 0 : 0)$ and $B = (0 : 0 : 0 : 1)$. Finally, note that if the numbers $p; q; r$ admit a greatest common divisor $k > 1$, then the curve (KL) is a degree $\frac{p}{k}$ one, counted k times. One can in this case substitute the parameter t by a new parameter t'.

Let us write $\Gamma_{p;q;r} := \overline{\gamma_{p;q;r}} \subset \mathbb{CP}(3)$. Observe that, when $r = 1$, $\Gamma_{p;q;r}$ is smooth if and only if $p = 3$ (in this case it is the rational normal curve in $\mathbb{CP}(3)$), and it has the point B as its only (cuspidal) singularity if $p \geq 4$. On the other hand, if $r > 1$, A is also a singular point of $\gamma_{p;q;r}$.

Let us insist in the fact that not every cuspidal rational algebraic curve is a KL curve. In particular, not all the cuspidal rational curves with the same degree and number of cusps are projectively equivalent (see, e.g. [E-H]).

Let t be the coordinate on $\mathbb{C}$, and consider the vector field on $\mathbb{C}$, $t\frac{\partial}{\partial t}$. The vector field $(\gamma_{p;q;r})_*(t\frac{\partial}{\partial t})$ can be extended to $\mathbb{C}^3$ as : $S = px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y} + rz\frac{\partial}{\partial z}$: On the other hand, $(\gamma_{p;q;r})_*(t^{\ell+1}\frac{\partial}{\partial t})$, $\ell + r \geq 0$, can be extended as a polynomial vector field X which is S-quasi-homogeneous, if certain arithmetical relations hold among $p; q; r$ and ℓ. When $r = 1$, which is the case that we will consider in the examples, this extension can be done so that X is S-quasi-homogeneous of weight ℓ. Thus we can define a foliation generated by the subfoliations given by S and X, which will be of degree d if the foliation generated by X is of degree $\nu = d - 1$.

Example 1. Klein–Lie foliations with one quasi-homogeneous singularity. We give examples that extend one found in [Ce-LN-1], giving origin to the so-called exceptional components. They appear in a family that we will denote as Klein–Lie (KL, for short) foliations in $\mathbb{CP}(3)$. KL foliations are not always s.g.K, but for each degree there is exactly one which is s.g.K, and that has just one q.h. singularity.

KL foliations in $\mathbb{C}^3$ and actions of $Aff(\mathbb{C})$. Recall that if t is the coordinate on $\mathbb{C}$, the two basic complete vector fields on $\mathbb{C}$, that are the infinitesimal generators of the action of $Aff(\mathbb{C})$, are $t\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial t}$. As noted above, the vector fields $(\gamma_{p;q;1})_*(t\frac{\partial}{\partial t})$ and $(\gamma_{p;q;1})_*(\frac{\partial}{\partial t})$, can be extended as

$$S = px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$

and

$$X_\tau = p \sum_{i+aj=p-1} \tau_{ij} z^i y^j \frac{\partial}{\partial x} + q z^{q-1} \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \quad \text{where} \sum_{i+qj=p-1} \tau_{ij} = 1.$$

The vector fields S and X_τ are complete, linearly independent outside the curve $\gamma_{p,q,1}$, and they satisfy the relation $[S, X_\tau] = -X_\tau$, thus they generate a local action of $Aff(\mathbb{C})$. To define a foliation associated to it, we consider the polynomial 1–form $\omega^\tau_{p,q,1} = i_S i_{X_\tau} dz \wedge dy \wedge dx$, i.e. the 1–form

$$q(y - z^{q-1})dx + p\Big(\sum \tau_{ij} z^{i+1} y^j - x\Big)dy + pq\Big(z^{q-1}x - \sum \tau_{ij} z^i y^{j+1}\Big)dz.$$

The relation $d\omega^\tau_{p,q,1} = (p+q)\, i_{X_\tau} dx \wedge dy \wedge dz$ implies that $\gamma_{p,q,1}$ is the Kupka set of the foliation represented by $\omega^\tau_{p,q,1}$, and it has transversal type $\eta = -pv\,du + qu\,dv$. Moreover, the diffeomorphism

$$\phi_\tau(v,u,t) = \left(v + p \sum \tau_{ij} \int_0^t s^i (u+s^q)^j ds,\ u + t^q,\ t\right)$$

which is the time t of the flow of the vector field X_τ, with initial condition $(v,u,0)$, satisfies the relation $\phi_\tau^*(\omega^\tau_{p,q,1}) = -pv\,du + qu\,dv$. Therefore, the foliation has a rational first integral

$$H_\tau = \frac{(y - z^q)^p}{(x - \psi_\tau(z,y))^q}$$

where ψ_τ is a polynomial of degree p on the variable z and depending on the parameters τ_{ij}.

Now we study the extension to $\mathbb{CP}(3)$ of the foliations obtained above. It is given by the homogeneous 1–form $\Omega^\tau_{p,q,1} = \omega_1 dz_1 + \omega_2 dz_2 + \omega_3 dz_3 + \omega_4 dz_4$, obtained from $\omega^\tau_{p,q,1}$. Note that, by means of the action of $PGL(4,\mathbb{C})$ on $\Omega^\tau_{p,q,1}$, we get a family of foliations: we will refer to all of them as KL foliations in $\mathbb{CP}(3)$.

A natural question is, given an integer $d \geq 1$, are there Klein–foliations in $\mathbb{CP}(3)$ of degree $d+1$?

Note that the degree of the KL foliation defined by $\Omega^\tau_{p,q,1}$ is $d + 1 = \max\{q, i + j + 1 \mid \tau_{i,j} \neq 0\}$. Then we have

$$\begin{aligned}
\omega_1 &= q z_4 (z_4^d z_2 - z_4^{d-q+1} z_3^q)\\
\omega_2 &= p z_4 \Big(\sum \tau_{ij} z_4^{d-i-j} z_3^{i+1} z_2^j - z_4^d z_1\Big)\\
\omega_3 &= pq z_4 \Big(z_4^{d-q+1} z_3^{q-1} z_1 - \sum \tau_{ij} z_4^{d-i-j} z_3^i z_2^{j+1}\Big)\\
\omega_4 &= \Big(p(q-1) \sum \tau_{ij} z_4^{d-i-j} z_3^{i+1} z_2^{j+1} + (p-q) z_4^d z_2 z_1 - q(p-1) z_4^{d-q+1} z_3^q z_1\Big)
\end{aligned}$$

with $1 < q \leq d+1 < p \leq qd+1 \leq d(d+1)+1$, and one of the following possibilities holds:

(1) $q = d+1$, and $i + j < d$, if $\tau_{ij} \neq 0$;
(2) $q = d+1$, and there is a unique pair (i_0, j_0) with $\tau_{i_0 j_0} \neq 0$ and $j_0 = d - i_0$;
(3) $q < d$, and there is a unique pair (i_0, j_0) with $\tau_{i_0 j_0} \neq 0$ and $j_0 = d - i_0$.

Observe that the hyperplane $\{z_4 = 0\}$ is invariant by the foliation defined by $\Omega^\tau_{p,q,1}$. Concerning its singular locus, it is the union of $\Gamma_{p,q,1}$ and the set $\{z_4 = \omega_4(z_1, z_2, z_3, z_4) = 0\}$ which, according to the possibilities discussed above, is:

(1) $\{z_3^{d+1} = z_4 = 0\} \cup \{z_1 = z_4 = 0\}$;
(2) $\{z_3^{i_0+1} = z_4 = 0\} \cup \{z_4 = p(q-1)\tau_{i_0, d-i_0} z_2^{d-i_0+1} - q(p-1) z_1 z_3^{d-i_0} = 0\}$;
(3) $\{z_3^{i_0+1} = z_4 = 0\} \cup \{z_2^{j_0+1} = z_4 = 0\}$.

To study the foliation around the point $(1:0:0:0)$, we choose its affine open neighbourhood E_1 and calculate the rotational of the form which represents the foliation $\eta^{\tau}_{p,q,1} := \overline{\omega}^{\tau}_{p,q,1}|_{E_1}$

$$
\begin{aligned}
\eta^{\tau}_{p,q,1} = &\Big(-p(q-1)\sum \tau_{ij} u^{d-i-j} w^{i+1} v^{j+1} + (p-q)u^d v - q(p-1)u^{d-q+1}w^q \Big) du \\
&+ p\Big(\sum \tau_{ij} u^{d-i-j+1} w^{i+1} v^j - u^{d+1}\Big) dv + pq\Big(u^{d-q+2}w^{q-1} - \sum \tau_{ij} u^{d-i-j+1} w^i v^{j+1}\Big) dw.
\end{aligned}
$$

Its exterior derivative is $d\eta^{\tau}_{p,q,1} = Q^{(p,q,\tau)}_{uw} du \wedge dw + Q^{(p,q,\tau)}_{wv} dw \wedge dv + Q^{(p,q,\tau)}_{vu} dv \wedge du$, where

$$
\begin{aligned}
Q^{(p,q,\tau)}_{uw} &= q(p(d+2)-q)u^{d-q+1}w^{q-1} + p(p-q(d+1))\sum \tau_{ij} u^{d-i-j} w^i v^{j+1}; \\
Q^{(p,q,\tau)}_{wv} &= p(q+p-1)\sum \tau_{ij} u^{d-i-j+1} w^i v^j; \\
Q^{(p,q,\tau)}_{vu} &= (p-q+p(d+1))u^d - \sum p(d-p-q+3)\tau_{ij} u^{d-i-j} w^i v^j;
\end{aligned}
$$

and the rotational is given by

$$
R_{\eta^{\tau}_{p,q,1}} = Q^{(p,q,\tau)}_{wv} \frac{\partial}{\partial u} + Q^{(p,q,\tau)}_{vu} \frac{\partial}{\partial w} + Q^{(p,q,\tau)}_{uw} \frac{\partial}{\partial v}.
$$

The only case in which the rotational above has isolated singularities is when $q = d+1$ and there is just one τ_{ij} different from zero (case 2), the one corresponding to $i = 0$ and $j = d$, which is 1. In that case, the KL foliation is s.g.K. By changing to the affine coordinates $E_2 = \{(r:1:s:t) | (r,s,t) \in \mathbb{C}^3\}$ and $E_3 = \{(r:s:1:t) | (r,s,t) \in \mathbb{C}^3\}$, it can be shown that all points in $\mathbb{CP}(3) \setminus \{(1:0:0:0)\}$ are of Kupka type and that $\mathrm{sing}(\mathcal{F})$ is the union of $\overline{\Gamma_{p,q,1}}$ with the two curves $\{z_3^{i_0+1} = z_4 = 0\}$ and $\{z_4 = p(q-1)\tau_{i_0,d-i_0} z_2^{d-i_0+1} - q(p-1)z_1 z_3^{d-i_0} = 0\}$. We leave the details for the reader.

Recall that the foliation has a meromorphic first integral F, which in the affine chart E_0 can be written as

$$
F(x,y,z) = \frac{(y-z^q)^p}{(x+z^p h(y/z^q))^q}, \text{ where } h(t) = \sum_{j=0}^{d} h_j t^j
$$

is the solution of $q(t-1)h'(t) = p(t^d + h(t))$.

In all the other cases, one can check that there is a one dimensional set of singular points on which the rotational vanish, so the corresponding KL foliation is not s.g.K.

Finally, and motivated by the previous study, we now analyse when there is just one pair (i,j) with $\tau_{ij} \neq 0$: that is, there is a unique determination of vector field X_τ, and of the form $\overline{\omega}^{\tau}_{p,q,1}$. For this to be the case, certain relations must hold between p, q and the degree $d+1$:

(1) if $q = d+1$, and $d+1$ divides $p-1$, then $i = 0$ and $j = \frac{p-1}{d+1}$.
(2) if $q < d+1$, and $p-1 = qd$, then $i = 0$ and $j = d$.

Example 2. Let us consider the curve $\gamma_{3,2,1}$ and the extension of the vector field $(\gamma_{3,2,1})_*(t\frac{\partial}{\partial t})$ as $S = 3x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ and the polynomial vector field $X = P + z^3 R$, where $R = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ is the radial vector field on $\mathbb{C}^3$, and $P = P_1\frac{\partial}{\partial x} + P_2\frac{\partial}{\partial y} + P_3\frac{\partial}{\partial z}$, with

$$
(5) \quad \begin{cases} P_1(x,y,z) = ax^2 + bxyz + cy^3 \\ P_2(x,y,z) = dxy + exz^2 + fy^2 z \\ P_3(x,y,z) = gxz + hy^2 + iyz^2 \end{cases}
$$

We consider this set of polynomials parametrized by $(a;b;c;d;e;f;g;h;i) \in \mathbb{C}^9$. It is not difficult to see that $[S;X] = 3X$, so X is a weighted S-quasi homogeneous degree 3 polynomial vector field extending $(\gamma_{3;2;1})_*(t^4 \frac{\partial}{\partial t})$. The foliations defined by S and X on $\mathbb{CP}(3)$ generate a codimension one foliation of degree four on $\mathbb{CP}(3)$, which will be denoted by $\mathcal{F}(P)$.

We take P in such a way that $d(i_P(dx \wedge dy \wedge dz)) = 0$, which is equivalent to $\operatorname{div}(P) := P_{1x} + P_{2y} + P_{3z} = 0$, or to $2a + d + g = b + 2f + 2i = 0$. In this case, if $\Omega_P = i_S\, i_X\,(dx \wedge dy \wedge dz)$, then Ω_P defines $\mathcal{F}(P)$ in the affine chart E_0. A straightforward calculation (using $\operatorname{div}(P) = 0$), gives $d\Omega_P = i_{Z_P}(dx \wedge dy \wedge dz)$, where

$$Z_P = 9P + z^3R - 6S.$$

As the reader can check, the set

$$A_0 = \{P \mid 2a + d + g = b + 2f + 2i = 0 \text{ and } Z_P \text{ has a nonisolated singularity at } 0 \in E_0 \simeq \mathbb{C}^3\};$$

is an algebraic subset of codimension three of $\mathbb{C}^9$. Therefore, if $P \notin A_0$ then $\mathcal{F}(P)$ has a q.h. singularity at $0 \in E_0$. Moreover, $\operatorname{sing}(\mathcal{F}(P)) \cap E_0$ contains seven integral curves of S, say Γ_j, $j = 1,\dots,7$, where $\Gamma_6 = (y = z = 0)$, $\Gamma_7 = (x = y = 0)$ and the others are generic trajectories of S of the form $\Gamma_j = \{(\alpha_j t^3, \beta_j t^2, t) \mid t \in \mathbb{C}\}$, $\alpha_j, \beta_j \neq 0$.

Now, let us see how $\mathcal{F}_P$ looks like in the chart $E_1 = \{(1 : w : v : u) \mid (u,v,w) \in \mathbb{C}^3\}$. In this chart we have $S = -S_1$, where

$$(6)\quad S_1 = 3u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w}.$$

Since X has a pole of order two at $(u = 0)$, the foliation $\mathcal{F}(P)$ is generated in this chart by S_1 and $X_1 := u^2 . X$. Observe that

$$[S_1;X_1] = -[S; x^{-2}X] = -S(x^{-2})X - x^{-2}[S;X] = 3X_1.$$

This implies that X_1 is of the same type of X, that is $X_1 = Q + m . w^3 R$, where $Q = Q_1\frac{\partial}{\partial x} + Q_2\frac{\partial}{\partial y} + Q_3\frac{\partial}{\partial z}$ and $Q_1;Q_2;Q_3$ are as in (5) (by changing $x \to u$, $y \to v$, $z \to w$ and the parameters $(a;\dots;i) \to (a';\dots;i')$). In other words, the point $(1 : 0 : 0 : 0) \in E_1$ is a q.h. singularity of $\mathcal{F}(P)$ for a generic P. It is possible to verify, by taking other affine charts, that $\mathcal{F}(P)$ is a s.g.K foliation with two q.h. singularities, the points $p_0 := (0 : 0 : 0 : 1) \in E_0$ and $p_1 := (1 : 0 : 0 : 0) \in E_1$. Moreover, $\operatorname{sing}(\mathcal{F}(P)) = \cup_{j=0}^{7}\overline{\Gamma_j}$, where $\Gamma_0 = \{(1 : w : v : u) \in E_1 \mid u = v = 0\}$ and the points in $\operatorname{sing}(\mathcal{F}(P)) \setminus \{p_0;p_1\}$ are of Kupka type. We leave the details for the reader.

Example 3. In this example we take again the curve $\gamma_{3;2;1}$ and $S = 3x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$, as in the Example 2, and

$$(7)\quad X = (ay^2 + bxz)\frac{\partial}{\partial x} + (cx + dyz)\frac{\partial}{\partial y} + (ey + fz^2)\frac{\partial}{\partial z};$$

so that $[S;X] = X$.

The foliation generated by S and X on $\mathbb{CP}(3)$ has degree three in this case. It is defined in the chart E_0 by the form $\Omega = i_S\, i_X\,(dx \wedge dy \wedge dz)$. We will denote this foliation by $\mathcal{F}(S;X)$. If we take X in such a way that $\operatorname{div}(X) = 0$, that is $b + d + 2f = 0$, then $d\Omega = i_Z(dx \wedge dy \wedge dz)$, where $Z = 7X$. As the reader can verify, if we take $X \notin A$, where

$$A = \{X \mid X \text{ is as in (7) and } abcdef(acf + bde) = 0\};$$

then $0 \in E_0 \simeq \mathbb{C}^3$ is an isolated zero of $d\Omega$, that is a q.h. singularity of $\mathcal{F}(S;X)$. For generic $X \in \mathcal{A}$, $sing(\mathcal{F}(S;X)) \cap E_0$ has three components : $\Gamma_0 = (x = y = 0)$ and Γ_1, Γ_2, which are the closure of two trajectories of S, not contained in the coordinate planes.

If we change coordinates to the chart $E_1 = \{(1 : w : v : u) | (u;v;w) \in \mathbb{C}^3\}$, we find that $\mathcal{F}(S;X)$ is generated in E_1 by $S = -S_1$, where S_1 is like in (6), and

$$X_1 = u.X = (-buv - auw^2)\frac{\partial}{\partial u} + (euw + (f-b)v^2 - avw^2)\frac{\partial}{\partial v} + (cu + (d-b)vw - aw^3)\frac{\partial}{\partial w}.$$

Therefore, $\mathcal{F}(S;X)$ is represented in this chart by $\Omega_1 = i_{S_1} i_{X_1}(du \wedge dv \wedge dw)$. On the other hand, we have $d\Omega_1 = i_{Z_1}(du \wedge dv \wedge dw)$, where $Z_1 = 8X_1 - div(X_1).S_1$. As the reader can check, this implies that under generic assumptions on the coefficients $a;b;c;d;e;f$, the point $0 = p_1 \in E_1$ is an isolated singularity of Z_1, so that it is a q.h. singularity of $\mathcal{F}(S;X)$. In this chart, the plane $(u = 0)$ is invariant for $\mathcal{F}(S;X)$ and

$$sing(\mathcal{F}(S;X)) \cap E_1 = (\overline{\Gamma_1} \setminus \{x = 0\}) \cup (\overline{\Gamma_2} \setminus \{x = 0\}) \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5$$

where $\Gamma_3 = (u = v = 0)$, $\Gamma_4 = (u = w = 0)$ and Γ_5 is a parabola in the plane $(u = 0)$ of the form $\{(0;\alpha t^2;\beta t) | t \in \mathbb{C}\}$.

We observe that the curves $\overline{\Gamma_0}$, $\overline{\Gamma_4}$ and $\overline{\Gamma_5}$ meet at the point $(0 : 0 : 1 : 0)$, which is a singularity of logarithmic type for $\mathcal{F}(S;X)$. It can be proved, by changing variables to other affine charts, that $sing(\mathcal{F}(S;X)) = \cup_{j=0}^{5} \overline{\Gamma_j}$ and all points in $sing(\mathcal{F}(S;X)) \setminus \{(0 : 0 : 0 : 1); (1 : 0 : 0 : 0); (0 : 0 : 1 : 0)\}$ are of Kupka type.

§2.3 Some remarks about the construction of the examples. In this section we discuss the possibility of constructing families of foliations s.g.K in $CP(3)$, generated by two one-dimensional foliations, say $\mathcal{G}_1$ and $\mathcal{G}_2$, as in §2.2. We suppose that $\mathcal{G}_1$ is the foliation defined in the affine chart $E_0 = \{(x : y : z : 1) | (x;y;z) \in \mathbb{C}^3\}$ by the linear vector field $S = px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y} + rz\frac{\partial}{\partial z}$, where $p;q;r \in \mathbb{N}$, $p \geq q \geq r > 0$ and $gcd(p;q;r) = 1$. If $p = q = r = 1$, then it is possible to construct s.g.K foliations of any degree. Take a homogeneous vector field of degree d on E_0, say X, so that $[S;X] = (d-1)X$. The foliation generated by S and X in $CP(3)$ is defined on E_0 by the form $\Omega = i_S i_X (dx \wedge dy \wedge dz)$. This type of example is considered in [C-LN] and for generic X it is s.g.K. On the other hand, in the case where the integers p, q and r are not equal, the situation is not so clear and we don't have a complete picture of all possibilities, if we fix p,q,r. Nevertheless, in the case where $p > q > r$, the number of possible families of foliations is finite, as we will see.

Consider S as in (1) and $p > q > r > 0$. Let us suppose that there is a one-dimensional foliation $\mathcal{G}_2$ of degree d, which in the chart E_0 is defined by a polynomial vector field X such that $[S;X] = \ell.X$, where $\ell > 0$. We denote by $\mathcal{F}(S;X)$ the foliation on $CP(3)$, which in the chart E_0 is generated by S and X. Observe that $\mathcal{F}(S;X) \in \mathcal{F}(p;q;r;d+1;\ell)$.

Theorem 3. If $p > q > r > 0$ are fixed, then the set

$$\mathcal{P} = \{(d;\ell) | d \geq 0; \ell > 0 \text{ and } \mathcal{F}(p;q;r;d+1;\ell) \text{ contains a s.g.K foliation}\}$$

is finite.

Proof. Observe that S has four singularities in $CP(3)$, the points $p_0 = (0 : 0 : 0 : 1) \in E_0$, $p_1 = (1 : 0 : 0 : 0) \in E_1$, $p_2 = (0 : 1 : 0 : 0)$ and $p_3 = (0 : 0 : 1 : 0)$. The eigenvalues of S at these points are respectively $(p;q;r)$, $(-p;q-p;r-p)$, $(p-q;-q;r-q)$, $(p-r;q-r;-r)$. Note that only in the first two sets the eigenvalues have the same sign. As a consequence, the points p_2 and p_3 can not be quasi-homogeneous singularities for a foliation $\mathcal{F} \in \mathcal{F}(p;q;r;d+1;\ell)$.

The idea is to use the formula for the multiplicity of an isolated singularity of a q.h. vector field in Remark 2. We will prove that the existence of a s.g.K foliation $\mathcal{F} \in \mathcal{F}(p,q,r;d+1,\ell)$ implies the existence of a one-dimensional foliation $\mathcal{G}$ of degree d with the following properties :
(i). p_0 and p_1 are isolated singularities of $\mathcal{G}$.
(ii). $\mathcal{G}$ is defined in the chart E_0 by a vector field Y such that $[S,Y]=\ell.Y$.

Let us suppose the existence of $\mathcal{G}$ satisfying properties (i) and (ii) and prove the theorem. Since p_0 is an isolated singularity for Y, it follows from Remark 2 that

$$(8)\quad \mu_0=\mu_0(d,\ell):=\mu(Y,p_0)=\frac{(\ell+p)(\ell+q)(\ell+r)}{p.q.r}.$$

On the other hand, $\mathcal{G}$ is defined in the chart $E_1=\{(1:w:v:u)\,|\,(u,v,w)\in\mathbb{C}^3\}$, by the vector field Y_1, where $Y_1=u^{d-1}.Y=x^{-d+1}.Y$ in $E_0\cap E_1$. It follows that

$$[S,Y_1]=S(x^{-d+1}).Y+x^{-d+1}.[S,Y]=(\ell-p(d-1)).Y_1.$$

Note that, in the chart E_1, we have

$$S=-pu\frac{\partial}{\partial u}-(p-r)v\frac{\partial}{\partial v}-(p-q)w\frac{\partial}{\partial w},$$

so that, if we set $S_1=-S$ then $[S_1,Y_1]=(p(d-1)-\ell).Y_1$. Set $q_1=p-r$, $r_1=p-q$ and $\ell_1=p(d-1)-\ell$. We assert that $\ell_1\geq 0$.

In fact, suppose by contradiction that $\ell_1<0$. Let $Y_1=A\frac{\partial}{\partial u}+B\frac{\partial}{\partial v}+C\frac{\partial}{\partial w}$. Since $p_1=(0,0,0)$ is an isolated singularity of $\mathcal{G}$, we must have $C\not\equiv 0$, so that there is a non-zero monomial of the form $u^a v^b w^c$ in C. Now, the relation $[S_1,Y_1]=\ell_1.Y_1$ implies that $S_1(C)=(\ell_1+r_1).C$, and so

$$p.a+q_1.b+r_1.c=\ell_1+r_1<r_1.$$

But the above relation is not possible if $a,b,c\geq 0$ and $p>q_1>r_1\geq 1$. This contradiction implies that $\ell_1\geq 0$.

In this case, we get from Remark 2 that

$$(9)\quad \mu_1=\mu_1(d,\ell):=\mu(Y_1,p_1)=\frac{(\ell_1+p)(\ell_1+q_1)(\ell_1+r_1)}{p.q_1.r_1}.$$

Since $\mathcal{G}$ has degree d, we must have (cf. [LN-S]) :

$$(10)\quad \mu_0+\mu_1\leq d^3+d^2+d+1$$

Let us see how (10) implies the Theorem. First of all we write (10) as a function of ℓ and ℓ_1. Since $\ell+\ell_1=p(d-1)$ we have

$$d^3+d^2+d+1=(d-1)^3+4(d-1)^2+6(d-1)+4=$$

$$=\frac{1}{p^3}[(\ell+\ell_1)^3+4p(\ell+\ell_1)^2+6p^2(\ell+\ell_1)+4p^3]:=\frac{1}{p^3}G(\ell,\ell_1).$$

Therefore, (10) is equivalent to $F(\ell,\ell_1)\leq 0$, where

$$F(\ell,\ell_1)=p^2q_1r_1(\ell+p)(\ell+q)(\ell+r)+p^2qr(\ell_1+p)(\ell_1+q_1)(\ell_1+r_1)-qq_1rr_1.G(\ell,\ell_1)$$

Note that $F(\lambda,\lambda_1)$ is a degree three polynomial in (λ,λ_1) and its homogeneous term of degree three is

$$F_3(\lambda,\lambda_1) = p^2q_1r_1\lambda^3 + p^2qr\lambda_1^3 - qq_1rr_1(\lambda+\lambda_1)^3 .$$

Assertion. If $p > q > r > 0$, then there exists $C > 0$ (which depends only on p,q,r) such that $F_3(\lambda,\lambda_1) \geq C(\lambda+\lambda_1)^3$ if $\lambda,\lambda_1 \geq 0$.

Proof. Suppose that $\lambda_1 > 0$, $\lambda \geq 0$ and set $y = \lambda/\lambda_1$. Then $F_3(\lambda,\lambda_1) = \lambda_1^3.f(y)$, where $f(y) = p^2q_1r_1y^3 + p^2qr - qq_1rr_1(y+1)^3$. Observe that $f(0) = qr(p^2 - q_1r_1) > 0$ and

$$\frac{1}{3}f'(y) = p^2q_1r_1y^2 - qq_1rr_1(y+1)^2$$

so that $f'(0) < 0$ and $f'(y) = 0$ has an unique positive root : $y_0 = \frac{\sqrt{qr}}{p-\sqrt{qr}}$. As the reader can check, by calculating f'' and f''', the point y_0 is the positive minimum of $f(y)$. Since

$$f(y_0) = \frac{2p^3qr}{(p-\sqrt{qr})^2}\left(\frac{q+r}{2} - \sqrt{qr}\right) > 0 ;$$

we have $f(y) \geq f(y_0) = \alpha > 0$ for all $y \geq 0$, so that $F_3(\lambda,\lambda_1) \geq \alpha.\lambda_1^3$. Similarly, there exists $\beta > 0$ such that $F_3(\lambda,\lambda_1) \geq \beta.\lambda^3$, if and $\lambda,\lambda_1 \geq 0$. It follows that

$$F_3(\lambda,\lambda_1) \geq \frac{1}{2}\alpha.\lambda_1^3 + \frac{1}{2}\beta.\lambda^3 \geq C(\lambda+\lambda_1)^3$$

for some $C > 0$ and $\lambda,\lambda_1 \geq 0$. $\square$

Now, since $F(\lambda,\lambda_1) - F_3(\lambda,\lambda_1)$ is a degree two polynomial in (λ,λ_1), there exists $\rho > 0$ such that if $\lambda,\lambda_1 \geq 0$ and $\lambda+\lambda_1 \geq \rho$, then $|F(\lambda,\lambda_1) - F_3(\lambda,\lambda_1)| \leq \frac{C}{2}(\lambda+\lambda_1)^3$, which implies that $F(\lambda,\lambda_1) \geq \frac{C}{2}(\lambda+\lambda_1)^3$, if $\lambda,\lambda_1 \geq 0$ and $\lambda+\lambda_1 \geq \rho$. It follows that the number of pairs $(\lambda,\lambda_1) \in \mathbb{N}^2$ which are solutions of $F(\lambda,\lambda_1) \leq 0$ is finite. Since $\lambda+\lambda_1 = p(d-1)$, the number of pairs $(\lambda,d) \in \mathbb{N}^2$ which are solutions of (10) is also finite.

It remains to prove the existence of a foliation $\mathcal{G}$ satisfying (i) and (ii). We will prove that there are two foliations $\mathcal{G}_0$ and $\mathcal{G}_1$ of degree d such that :
(iii). p_j is an isolated singularity of $\mathcal{G}_j$, $j = 0,1$.
(iv). $\mathcal{G}_j$ is defined in the chart E_j by a vector field X_j such that $[S_j,X_j] = \lambda_j.X_j$, where $S_0 = S$ and $\lambda_0 = \lambda$.

If we have two foliations like above, then the generic foliation in the pencil $\mathcal{G}_\alpha = \mathcal{G}_0 + \alpha\mathcal{G}_1$ satisfies (i) and (ii), as the reader can check. Recall that $\mathcal{G}_\alpha$ is the foliation that in the chart E_0 is defined by $X_\alpha = X_0 + \alpha x^{d-1}.X_1$.

Let us construct $\mathcal{G}_0$. Consider a foliation $\mathcal{F} \in \mathcal{F}(p,q,r,d+1,\lambda)$. Then it has degree $d+1$ and is defined in the chart E_0 by an integrable 1-form Ω such that $d\Omega = i_Z(dx \wedge dy \wedge dz)$, $p_0 = 0$ is an isolated singularity of Z and $[S,Z] = \lambda.Z$. Since $\mathcal{F}$ has degree $d+1$, the form Ω has degree $d+2$, so that $d \leq dg(Z) \leq d+1$. If $dg(Z) = d$, then the foliation $\mathcal{G}(Z)$ on $CP(3)$ defined in the chart E_0 by Z has degree d and we take $\mathcal{G}_0 = \mathcal{G}(Z)$. Let us suppose that $dg(Z) = d+1$. In this case we must have $div(Z) = 0$, so that, if Z_{d+1} is the homogeneous part of Z of degree $d+1$, then $div(Z_{d+1}) = 0$ and $[S,Z_{d+1}] = \lambda.Z_{d+1}$. As the reader can check, these relations imply that $Z_{d+1} = g(mR - nS)$, where R is the radial vector field on $\mathbb{C}^3$, $m = \lambda+p+q+r$, $n = d+3$ and g is a homogeneous polynomial of degree d such that $S(g) = \lambda.g$. Let us write $Z = P + g(mR - nS)$, where $dg(P) \leq d$, $P = A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + C\frac{\partial}{\partial z}$ and

$$Z = (A + (m-np)xg)\frac{\partial}{\partial x} + (B + (m-nq)yg)\frac{\partial}{\partial y} + (C + (m-nr)zg)\frac{\partial}{\partial z} .$$

Observe that if λ is small then 0 is an isolated singularity of $Z + \lambda g R$. Take λ in such a way that $m - np + \lambda, m - nq + \lambda, m - nr + \lambda \neq 0$. In this case, the vector field

$$X_0 = \left(\frac{A}{m - np + \lambda} + g\,x\right)\frac{\partial}{\partial x} + \left(\frac{B}{m - nq + \lambda} + g\,y\right)\frac{\partial}{\partial y} + \left(\frac{A}{m - nr + \lambda} + g\,z\right)\frac{\partial}{\partial z}$$

has an isolated singularity at 0. Moreover, $[S, X_0] = \ell . X_0$ and the foliation defined by X_0 on $CP(3)$ has degree d. The construction of $\mathcal{G}_1$ is similar and this finishes the proof of Theorem 3. ∎

Remark 6. When $p = 3$, $q = 2$ and $r = 1$, then the unique possibilities are those of examples 1 (with $d = 1$), 2 and 3 of §2.2. In fact, in this case if we set $k = d - 1 \geq 0$, we have $\ell_1 = 3k - \ell$ and

$$(11)\quad F(\ell, 3k - \ell) = 3[A(k)\ell^2 - B(k)\ell + C(k)],$$

where $A(k) = 3k + 4$, $B(k) = 12k + 9k^2$ and $C(k) = 7k^3 + 10k^2 - k - 4$. On the other hand, the inequality $F(\ell, 3k - \ell) \leq 0$ implies that for a solution (k, ℓ) we must have $B^2 - 4AC \geq 0$. Since

$$B^2 - 4AC = -(k - 2)(k + 2)(k + 4)(3k + 4)$$

we get that the unique possible solutions are $k \in \{0, 1, 2\}$, that is $d \in \{1, 2, 3\}$. If we substitute these values of k in (11) we get the following possibilities for ℓ and ℓ_1

$$\begin{cases} k = 0 \implies \ell, \ell_1 \in \{0, 1\} \\ k = 1 \implies \ell, \ell_1 \in \{1, 2\} \\ k = 2 \implies \ell = \ell_1 = 3 \end{cases}$$

which give exactly the values of (d, ℓ, ℓ_1) of the examples.

The above result has motivated problem 1 in §1.

§3 Proofs of Theorems 1 and 2

§3.1 Proof of Theorem 1. Let $\mathcal{F} \in \mathcal{F}(p, q, r, \nu, \ell)$ be a s.g.K foliation on $CP(3)$. Observe that $\mathcal{F}$ is generated by two one-dimensional foliations of $CP(3)$, say $\mathcal{G}_1$ and $\mathcal{G}_2$, the foliations defined in the chart E_0 by the vector fields S and X, respectively. As we have seen in Proposition 2, this implies that its tangent bundle $T_{\mathcal{F}}$ splits as the sum of two line bundles : $T_{\mathcal{F}} = L_1 \oplus L_2$, where L_1 corresponds to the foliation $\mathcal{G}_1$ and L_2 to $\mathcal{G}_2$. Moreover, the Corollary of Proposition 1 implies that there exists a neighborhood $\mathcal{U}$ of $\mathcal{F}$ such that any foliation in $\mathcal{U}$ is s.g.K, so that its tangent bundle is well defined.

Remark 7. Since S is a global vector field in $CP(3)$, we have that L_1 is a trivial line bundle, that is $L_1 \simeq CP(3) \times \mathbb{C} = \mathcal{O}(0)$. On the other hand, if d is the degree of $\mathcal{G}_2$, we have $L_2 \simeq \mathcal{O}(1 - d)$ (cf [Br]) and that the degree of $\mathcal{F}$ is $\nu = d + 1$.

Since $\mathcal{F}(d + 1, 3)$ is finite dimensional, it is sufficient to prove that for any holomorphic curve $\Sigma \ni t \mapsto \mathcal{F}_t \in \mathcal{F}(d + 1, 3)$, such that $0 \in \Sigma \subset \mathbb{C}$ and $\mathcal{F}_0 = \mathcal{F}$, then $\mathcal{F}_t \in \mathcal{F}(p, q, r, d + 1, \ell)$ for small $|t|$.

Let $(\mathcal{F}_t)_{t \in \Sigma}$ be a holomorphic family of foliations on $\mathcal{F}(d + 1, 3)$, parametrized in an open set $0 \in \Sigma \subset \mathbb{C}$, where $\mathcal{F}_0 = \mathcal{F}$. We take Σ so small that for any $t \in \Sigma$, $\mathcal{F}_t$ is s.g.K and $T_{\mathcal{F}_t}$ is well defined. Moreover, $(T_{\mathcal{F}_t})_{t \in \Sigma}$ is a holomorphic family of rank two vector bundles over $CP(3)$. We will prove first that $T_{\mathcal{F}_t}$ is isomorphic to $T_{\mathcal{F}} = T_{\mathcal{F}_0}$, if $|t|$ is small. To do that, we essentially use

Theorem B. (Horrock's splitting criterion, see [O-S-S]) A holomorphic bundle E over $CP(n)$ splits precisely when $H^i(CP(n), E(k)) = 0 \quad$ for $i = 1, \ldots, n - 1$ and all $k \in \mathbb{Z}$.

Note that, as $T_{\mathcal{F}_0}$ splits, then $H^1(\mathbb{CP}(3); T_{\mathcal{F}_0}(k)) = H^2(\mathbb{CP}(3); T_{\mathcal{F}_0}(k)) = 0$ for every integer k. But, as $T_{\mathcal{F}_t}$ is a holomorphic family of vector bundles over $\mathbb{CP}(3)$, the dimension of the vector spaces $H^i(\mathbb{CP}(3); T_{\mathcal{F}_t}(k))$ is upper semicontinuous. We conclude, by using again the splitting criterion above, that $T_{\mathcal{F}_t}$ splits for small $|t|$.

In order to conclude that for small $|t|$, it is $T_{\mathcal{F}_t} \simeq T_{\mathcal{F}_0}$, we make use of the well known fact (see, [S]) that the infinitesimal deformations of $T_{\mathcal{F}_0} = \mathcal{O} \oplus \mathcal{O}(1 - d)$ are given by the vector space $H^1(\mathbb{CP}(3); \mathrm{End}\, T_{\mathcal{F}_0})$, where $\mathrm{End}\, T_{\mathcal{F}_0}$ is the sheaf of endomorphisms of $T_{\mathcal{F}_0}$. But, the dimension of that vector space is zero, as $\mathrm{End}\, T_{\mathcal{F}_0} = T^*_{\mathcal{F}_0} \otimes T_{\mathcal{F}_0}$, where $T^*_{\mathcal{F}_0} = \mathcal{O} \oplus \mathcal{O}(d - 1)$ is the dual bundle of $T_{\mathcal{F}_0}$.

Now, let $(\mathcal{F}_t)_{t\in\Sigma}$ be a holomorphic family of foliations such that $\mathcal{F} = \mathcal{F}_0 \in \mathcal{F}(p; q; r; d + 1; \ell)$ is s.g.K. It follows from Remark 7 and the results above that, if Σ is a small neighborhood of $0 \in \mathbb{C}$, then $T_{\mathcal{F}_t} \simeq \mathcal{O}(0) \oplus \mathcal{O}(1 - d)$ for all $t \in \Sigma$. On the other hand, (b) of Proposition 2, implies that $\mathcal{F}_t$ is generated by two foliations of dimension one, say $\mathcal{G}_1(t)$ and $\mathcal{G}_2(t)$, where $\mathcal{G}_1(t)$ corresponds to the factor $\mathcal{O}(0)$ and $\mathcal{G}_2(t)$ to the factor $\mathcal{O}(1 - d)$. As a consequence, $\mathcal{G}_1(t)$ is generated by a global vector field $S(t)$ on $\mathbb{CP}(3)$. Now, Proposition 1 of §2.1, implies that $S(t)$ has a singularity whose eigenvalues, say $\lambda_1; \lambda_2; \lambda_3$, are multiples of $p; q; r$, so that we can suppose without lost of generality that $\lambda_1 = p$, $\lambda_2 = q$ and $\lambda_3 = r$. Consider an affine coordinate system $(U(t) = \mathbb{C}^3; (x; y; z))$ where $S(t) = px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y} + rz\frac{\partial}{\partial z}$. Let $\Omega(t)$ be a polynomial integrable 1-form which defines $\mathcal{F}_t$ in this chart. We assert that

$$(12)\quad L_{S(t)}\,\Omega(t) = (\ell + p + q + r)\,\Omega(t).$$

In fact, since $\mathcal{G}_1(t)$ is tangent to $\mathcal{F}_t$, we have $i_{S(t)}\,\Omega(t) = 0$. This implies that $L_{S(t)}\,\Omega(t) = i_{S(t)}d\Omega(t)$. On the other hand, it follows from the integrability condition, $\Omega(t) \wedge d\Omega(t) = 0$, that $\Omega(t) \wedge i_{S(t)}\,d\Omega(t) = 0$, which implies that $L_{S(t)}\,\Omega(t) = \lambda(t).\Omega(t)$, where $\lambda: \mathbb{C}^3 \to \mathbb{C}^*$ is holomorphic. Now, the eigenvalues of the operator $\omega \mapsto L_{S(t)}\,\omega$ are integers, so that $\lambda(t)$ is a constant. Since $\Omega(0) = \Omega = i_S\, i_X(dx \wedge dy \wedge dz)$, where $[S; X] = \ell.X$, we have $L_S\,\Omega = (\ell + tr(S))\,\Omega = (\ell + p + q + r)\,\Omega$, which proves that $\lambda(0) = \ell + p + q + r \equiv \lambda$, and the assertion.

Now, let $Z(t)$ be the vector field in $\mathbb{C}^3 = U(t)$ defined by $i_{Z(t)}(dx \wedge dy \wedge dz) = d\Omega(t)$. It follows from (12) that

$$\lambda.i_{Z(t)}(dx \wedge dy \wedge dz) = \lambda.d\Omega(t) = L_{S(t)}\,d\Omega(t) = L_{S(t)}(i_{Z(t)}(dx \wedge dy \wedge dz)) =$$

$$= i_{[S(t);Z(t)]}(dx \wedge dy \wedge dz)) + i_{Z(t)}(L_{S(t)}(dx \wedge dy \wedge dz)) = i_{[S(t);Z(t)]}(dx \wedge dy \wedge dz) + tr(S(t))d\Omega(t)$$

$$\Longrightarrow \quad [S(t); Z(t)] = (\lambda - tr(S(t))).Z(t) = \ell.Z(t)$$

This implies that $\mathcal{F}_t \in \mathcal{F}(p; q; r; d + 1; \ell)$ for small $|t|$ and finishes the proof of Theorem 1 as $\overline{\mathcal{F}(p; q; r; d + 1; \ell)}$ is an irreducible algebraic subset of $\mathcal{F}(d+1; 3)$. Indeed, recall from the description of the foliations in $\mathcal{F}(p; q; r; d + 1; \ell)$ that in order to define such a foliation we need choosing an affine open $\mathbb{C}^3 \subset \mathbb{CP}(3)$ (or equivalently a point in the dual projective space $\mathbb{CP}^*(3)$), fixing linear coordinates on it and choosing (up to multiplication by the same constant) the coefficients of the vector field X. This shows that there is a surjective map from a dense open subset $U \subset \mathbb{CP}^*(3) \times GL(3; \mathbb{C}) \times \mathbb{C}^N$ onto $\mathcal{F}(p; q; r; d + 1; \ell)$, for a certain N. So the irreducibility of the last algebraic subset follows from that of U.

Furthermore, to parametrize $\mathcal{F}(p; q; r; d + 1; \ell)$, we should analyse the map above in order to detect which elements in U give rise to the same foliation. Note that for a fixed affine open, a linear change of coordinates of the form $x' = \alpha x$, $y' = \beta y$, $z' = \gamma z$ takes S to $S' = px'\frac{\partial}{\partial x'} + qy'\frac{\partial}{\partial y'} + rz'\frac{\partial}{\partial z'}$ and X to an S'-quasi-homogeneous vector field X' of weight $\ell + 1$. As the open affine $\mathbb{C}^3$, the coordinates (x', y', z') and the vector fields S', X' define the same foliation, we should factor the group $GL(3, \mathbb{C})$ by the subgroup of diagonal invertible matrices. ∎

For KL foliations we have the following result, extending the existence of the exceptional component in [Ce-LN1], that corresponds to the case $d = 1$:

Corollary. Let $d \geq 1$ be an integer. There is a 13-dimensional irreducible component

$$\overline{\mathcal{F}(d(d+1)+1, d+1, 1, d+1, -1)}$$

of the space $\mathcal{F}(d+1, 3)$ whose general point corresponds to a s.g.K Klein–Lie foliation with exactly one q.h. singularity. Moreover, this component is the closure of a $PGL(4, \mathbb{C})$ orbit on $\mathcal{F}(d+1, 3)$.

Proof. It is an immediate consequence of Theorem 1, the study of KL foliations in Example 1, and the analysis of the parametrizations of the sets $\mathcal{F}(p, q, r, d+1, \ell)$. Indeed, if $\mathcal{F}$ is a foliation in $\mathcal{F}(d(d+1)+1, d+1, 1, d+1, -1)$, then in an affine open subset we have that it is determined by the vector fields

$$S = (d(d+1)+1)x\frac{\partial}{\partial x} + (d+1)y\frac{\partial}{\partial y} + \frac{\partial}{\partial z} \quad \text{and} \quad X_{\alpha\beta} = \alpha(d(d+1)+1)y^d\frac{\partial}{\partial x} + \beta(d+1)z^d\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

Note that X, the S-quasi-homogeneous vector field of weight 0, is uniquely defined up to the choice of the nonzero constants α and β (we take the last coordinate, which is necessarily a constant, to be 1). The dependence locus of S and X, which is the singular set of the foliation $\mathcal{F}$ in $\mathbb{C}^3$, is the Klein–Lie curve $(\alpha t^{d(d+1)+1}, \beta t^{d+1}, t)$. After the linear change of coordinates given by $x = \alpha x'$, $y = \beta y'$, $z = z'$, the foliation in $\mathbb{C}^3$ is exactly the one described in Example 1, whose singular locus is the curve $\gamma_{d(d+1)+1, d+1, 1}(t) = (t^{d(d+1)+1}, t^{d+1}, t)$. The extended foliation in $\mathbb{CP}(3)$ is s.g.K was studied in Example 1, it has just one q.h. singularity, an invariant hyperplane (that at infinity, $\mathbb{CP}(3) \setminus \mathbb{C}^3$), and we also know its singular locus. ∎

§3.2 Proof of Theorem 2. We observe that the second statement of the Theorem is a direct consequence of the first and of Theorem 1, so that we will prove only the first.

We will do the arguments in homogeneous coodinates. Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}(n)$ be the natural projection. Given a codimension one holomorphic foliation $\mathcal{F}$ on $\mathbb{CP}(n)$ of degree d, then the foliation $\mathcal{F}^* = \pi^*(\mathcal{F})$, on $\mathbb{C}^{n+1} \setminus \{0\}$, extends to a foliation on $\mathbb{C}^{n+1}$, which can be defined by a polynomial 1-form $\Omega = \sum_{j=0}^{n} A_j(z)\, dz_j$ satifying the following properties (cf. [Ce-LN-1]) :

(i). A_j is a homogeneous polynomial of degree $\nu = d + 1$ for all $j = 0, \dots, n$.
(ii). $\sum_{j=0}^{n} z_j . A_j(z) \equiv 0$.
(iii). $\Omega \wedge d\Omega = 0$ (integrability condition).
(iv). $\pi(sing(\Omega)) = sing(\mathcal{F})$ and $cod_{\mathbb{C}}(sing(\Omega)) \geq 2$.
(v). If U_α is the affine chart $(z_\alpha = 1)$ then $\mathcal{F}|_{U_\alpha}$ is defined by $\Omega_\alpha = \Omega|_{U_\alpha}$.

Moreover, if $\mathbb{CP}(k) \simeq E \subset \mathbb{CP}(n)$ is a linearly embedded k-plane, $2 \leq k < n$, non-invariant for $\mathcal{F}$, where $\pi^{-1}(E) = E^*$, then

(vi). $\pi^*(\mathcal{F}|_E) = \mathcal{F}^*|_{E^*}$ is defined by $\Omega|_{E^*}$.

Now, suppose that $n = 3$ and that $\mathcal{F}$ is generated by two one dimensional foliations, say $\mathcal{G}_j$ of degree d_j, $j = 1, 2$. We have the following :

Lemma 1. In the above hypothesis, let Ω be as before. Then there exist polynomial vector fields X_j on $\mathbb{C}^4$, $j = 1, 2$, with the following properties :

(a). The components of X_j are homogeneous of degree d_j.
(b). The two-dimensional foliation on $\mathbb{C}^4 \setminus \{0\}$, $\pi^*(\mathcal{G}_j)$, extends to $\mathbb{C}^4$ and is generated by X_j and the radial vector field on $\mathbb{C}^4$: $R = \sum_{j=0}^{3} z_j \frac{\partial}{\partial z_j}$.

(c). $\Omega = i_R\, i_{X_1}\, i_{X_2}(dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3)$.

Proof. The existence of vector fields X_j, $j = 1,2$, satisfying (a) and (b), is well known (cf. [LN-S]). Since G_1 and G_2 generate F, we must have $i_{X_j}\Omega = 0$, $j = 1,2$. We have also $i_R(\Omega) = 0$ (from (ii)). Let $\Theta = i_R\, i_{X_1}\, i_{X_2}(dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3)$. It follows from Definition 5 and (b), that $cod_{\mathbb{C}}(sing(\Theta)) \geq 2$ and that for any $p \in \mathbb{C}^4 \setminus sing(\Theta)$ we have $T_p(F^*) = ker(\Theta(p)) = ker(\Omega(p))$, where $T_p(F^*)$ denotes the tangent space to the leaf of F^* through p. This implies that $\Theta = \lambda.\Omega$ outside $sing(\Theta)$, where $\lambda \neq 0$ is some holomorphic function on $\mathbb{C}^4 \setminus sing(\Theta)$. Since $cod(sing(\Theta)) \geq 2$, λ extends to a holomorphic function on $\mathbb{C}^4$, which of course is a homogeneous polynomial. Now, it follows from $dg(G_j) = d_j$, that $dg(F) = d_1 + d_2$, and so $dg(\Omega) = d_1 + d_2 + 1 = dg(\Theta)$. This implies that λ is a constant. Now, if $\tilde{X}_1 = \lambda^{-1}.X_1$, then $\Omega = i_R\, i_{\tilde{X}_1}\, i_{X_2}(dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3)$, which proves the Lemma. ∎

We have the following consequences :

Corollary 1. Let F, F^* and $\Omega = i_R\, i_{X_1}\, i_{X_2}(dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3)$ be as in Lemma 1. Then for any $p \in \mathbb{C}^4$ the sheaf of germs of holomorphic vector fields at p which are tangent to F^* is free and generated by the germs of R, X_1 and X_2 at p.

The proof is similar to the proof of Remark 3 of §2.1 and is left for the reader.

Corollary 2. Let F, F^* and Ω be as in Lemma 1. Let $(V_\alpha)_{\alpha\in A}$ be a covering of $\mathbb{C}^4 \setminus \{0\}$ by Stein open sets and $(X_{\alpha\beta})_{V_{\alpha\beta}\neq\emptyset}$ be an additive cocycle of holomorphic vector fields such that for any $V_{\alpha\beta} \neq \emptyset$, $X_{\alpha\beta}$ is tangent to F^*, that is $i_{X_{\alpha\beta}}\Omega = 0$. Then for any $\alpha \in A$ there exists a holomorphic vector field X_α on V_α such that X_α is tangent to F^* and $X_{\alpha\beta} = X_\beta - X_\alpha$ on $V_\alpha \cap V_\beta := V_{\alpha\beta} \neq \emptyset$.

Proof. Let X_1 and X_2 be as in Lemma 1, so that $\Omega = i_R\, i_{X_1}\, i_{X_2}(dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3)$. It follows from Corollary 1 that if $V_{\alpha\beta} \neq \emptyset$ then there exist $f^j_{\alpha\beta} \in O(V_{\alpha\beta})$, $j = 0,1,2$, such that

$$X_{\alpha\beta} = f^0_{\alpha\beta} R + f^1_{\alpha\beta} X_1 + f^2_{\alpha\beta} X_2 .$$

Clearly, $(f^j_{\alpha\beta})_{V_{\alpha\beta}\neq\emptyset}$ is an additive cocycle for $j = 0,1,2$. Since $H^1(\mathbb{C}^4 \setminus \{0\}, O) = 0$, there exist collections $(f^j_\alpha)_{\alpha\in A}$, where $f^j_\alpha \in O(V_\alpha)$, $j = 0,1,2$, such that $f^j_{\alpha\beta} = f^j_\beta - f^j_\alpha$ on $V_{\alpha\beta} \neq \emptyset$. If we set $X_\alpha = f^0_\alpha R + f^1_\alpha X_1 + f^2_\alpha X_2$, then X_α is tangent to F^* and $X_{\alpha\beta} = X_\beta - X_\alpha$. ∎

Now, we consider the case in which $F|_E$ is s.g.K.

Lemma 2. Let F be a codimension one foliation of degree d on $\mathbb{CP}(n)$. Suppose that there exists a 3-plane E like in (vi) before Lemma 1 and that $F|_E$ is s.g.K. Let F^*, E^* and Ω be as before. Then, for any $p \in E^* \setminus \{0\}$, there exists a local coordinate system around p, say $(U,(t,u,v))$, where $t\colon U \to \mathbb{C}$, $u = (u_1,u_2,u_3)\colon U \to \mathbb{C}^3$ and $v = (v_1,\ldots,v_{n-2})\colon U \to \mathbb{C}^{n-3}$, such that $t(p) = 0$, $u(p) = 0$, $v(p) = 0$ and

(a). $E^* = (v = 0)$.

(b). $\Omega|_U = e^{t(d+2)} \sum_{j=1}^{3} \alpha_j(u)\, du_j$.

In particular, $F^*|_U$ is locally equivalent to the product of a codimension one foliation on $\mathbb{C}^4$ by a non-singular foliation, say P, of dimension $n - 3$, which is given in this chart by $(t,u) = cte$.

Proof. The Lemma is a consequence of [K] and [C-LN]. First of all, observe that $L_R(\Omega) = (d+2)\Omega$, because Ω is homogeneous of degree $d + 1$. This implies that

$$(13)\ R^*_s(\Omega) = e^{s(d+2)}.\Omega ,$$

where $R_s(q) = e^s.q$ is the flow of R. Let $p = (p_0,\ldots,p_n) \in E^* \setminus \{0\}$. After a linear change of variables in $\mathbb{C}^{n+1}$, we can suppose that $E^* = (z_4 = \ldots = z_n = 0)$ and $p = (1,0,\ldots,0) \in E^*$. Let H

be the hyperplane $(z_0 = 1)$ of $\mathbb{C}^{n+1}$. Since R is transversal to H, there exists coordinate system $(t;x)\colon V \to D \times \mathbb{C}^n$, where $V = \{R_s(q) \mid s \in D;\ q \in H\}$, such that $R = \frac{\partial}{\partial t}$, $H = (t = 0)$ and $p = 0$, in this chart. It follows from (13) that

$$(14)\quad \Omega(t;x) = e^{t(d+2)}.\omega\ ,\ \text{where}\ \omega = \sum_{j=1} \omega_j(x)\,dx_j$$

depends only on $x = (x_1;\dots;x_n)$. We can suppose also that $E \cap H = E^* \cap H$ is the plane $E_0 = (x_4 = \dots = x_n = 0)$. Note that (v) and the hypothesis, imply that all singularities of $\omega|_{E_0}$ are generalized Kupka. We have three possibilities :

(I). $\Omega(p) = \omega(0) \neq 0$. In this case, we have $\omega|_{E_0}(0) \neq 0$, that is F^* is transversal to E_0 at 0. In fact, since $\omega(0) \neq 0$, F has a holomorphic first integral in a neighborhood of 0, say f, so that $\omega = g.df$, where $g(0) \neq 0$. Now, $\omega|_{E_0}(0) = 0$ implies that $df|_{E_0}(0) = 0$, and so $f|_{E_0}$ has an isolated singularity at 0, which is not possible (see Remark 4 of §2.1). As the reader can check, this implies the Lemma in this case.

(II). $\omega|_{E_0}(0) = 0$ and $d\omega|_{E_0}(0) \neq 0$. In this case, 0 is a Kupka singularity of $\omega|_{E_0}$ and of ω. The Lemma follows from the arguments in [K] or in [Me], in this case.

(III). $\omega|_{E_0}(0) = 0$, $d\omega|_{E_0}(0) = 0$ and 0 is an isolated zero of $d\omega|_{E_0}$. In this case, the Lemma follows from Theorem 4 of [C-LN]. ∎

Now, Lemma 2 implies that there exists an open covering $(U_\alpha)_{\alpha\in A}$ of $E^* \setminus \{0\}$ with the following properties :

(vii). $U_\alpha = V_\alpha \times W_\alpha$, where V_α is a Stein open subset of E^*, and W_α is a polydisk in $\mathbb{C}^{n-3}$.

(viii). $F^*|_{U_\alpha}$ is the product of a codimension one foliation on V_α by a non-singular foliation P_α of dimension $n-3$, transversal to E^*.

We will suppose that $E^* = (z_4 = \dots = z_n = 0)$ and use the notation $z = (x;y)$, where $x = (x_1;\dots;x_4) = (z_0;\dots;z_3)$ and $y = (y_1;\dots;y_{n-3}) = (z_4;\dots;z_n)$. Since P_α is non-singular of dimension $n-3$ and transversal to E^*, by taking a smaller U_α if necessary, we can suppose that it is generated by $n-3$ holomorphic vector fields, say $Y^1_\alpha;\dots;Y^{n-3}_\alpha$, of the form

$$(15)\quad Y^j_\alpha(x;y) = \frac{\partial}{\partial y_j} + X^j_\alpha(x;y)\ ,\ \text{where}\ X^j_\alpha(x;y) = \sum_{i=1} A^j_{\alpha;i}(x;y)\frac{\partial}{\partial x_i}\ \text{and}\ A^j_{\alpha;i} \in \mathcal{O}(U_\alpha)\ :$$

Lemma 3. For any $j = 1;\dots;n-3$, there exists a constant vector field Z_j on $\mathbb{C}^{n+1}$ of the form

$$(16)\quad Z_j = \frac{\partial}{\partial y_j} + \sum_{i=1} a^j_i \frac{\partial}{\partial x_i}$$

such that $i_{Z_j}\Omega(q) = 0$ for any $q \in E^*$ and any $j \in \{1;\dots;n-3\}$.

Proof. Fix $j \in \{1;\dots;n-3\}$ and consider the covering $(U_\alpha = V_\alpha \times W_\alpha)_{\alpha\in A}$ and the vector fields Y^j_α as in (15). Consider the additive cocycle of vector fields $(X_{\alpha;\beta})_{V_\alpha\cap V_\beta\neq\emptyset}$ on $E^* \setminus \{0\}$, where $X_{\alpha;\beta}(x) = Y^j_\beta(x;0) - Y^j_\alpha(x;0) = X^j_\beta(x;0) - X^j_\alpha(x;0)$. Clearly, $X_{\alpha\beta}$ is tangent to $F^*|_{E^*}$ if $V_\alpha\cap V_\beta \neq \emptyset$. It follows from Corollary 2 of Lemma 1 that we can write $X_{\alpha;\beta} = T_\beta - T_\alpha$, where T_α is holomorphic on V_α and tangent to $F^*|_{E^*}$. Since $Y^j_\alpha(x;0) + T_\alpha(x) = Y_\beta(x;0) + T_\beta(x)$ on $V_\alpha\cap V_\beta \neq \emptyset$, there exists a holomorphic vector field Z along $E^* \setminus \{0\}$, such that $Z(x) = Y^j_\alpha(x;0) + T_\alpha(x)$ if $x \in V_\alpha$. It follows from Hartog's Theorem that we can extend Z to a vector field on E^*, which we shall denote by Z again. Let $Z(x) = \sum_{k=0}^{\infty} Z^k(x)$ be Taylor series of Z at $0 \in E^*$, where $Z^k(x)$ is a vector field with polynomial coefficients homogeneous of degree k. Since Y^j_α is tangent to $\mathcal{F}^*$ and Z_α is tangent to $\mathcal{F}^*|_{V_\alpha}$, we have $i_{Z(q)}\,\Omega(q) = 0$ for any $q \in E^*$. Now, since the coefficients of Ω are homogeneous of the same degree, we get that $i_{Z^0}\,\Omega(q) = 0$ for any $q \in E^*$. Finally, observe that Z^0 is a constant vector field as in (16), which proves the lemma. □

Let us finish the proof of the first part of Theorem 2. We will prove that there exists a linear change of variables on $\mathbb{C}^{n+1}$ of the form $(x, y) = L(u, v) = (u + b(v), v)$ such that

$$L^*(\Omega) = \sum_{j=1}^{4} \omega_j(u)\, du_j\ .$$

This clearly implies the first part of Theorem 2.

Let Z_j, $j = 1, \dots, n-3$, be as in (16). Consider the linear change of variables $(x, y) = L(u, v)$ as above, given by $y = v$ and $x_j = u_j + \sum_{i=1}^{n-3} a^i_j v_i$, $j = 1, \dots, 4$. As the reader can check, we have $L^*(Z_j) = \frac{\partial}{\partial v_j}$ for all $j = 1, \dots, n-3$. Therefore, returning to the old notation, we can suppose that $Z_j = \frac{\partial}{\partial y_j}$.

Assertion. Let $(x, y) \in \mathbb{C}^4 \times \mathbb{C}^{n-3}$ be a linear coodinate system such that $E^* = (y = 0)$ and $Z_j = \frac{\partial}{\partial y_j}$, $j = 1, \dots, n-3$. Then $\Omega = \sum_{j=1}^{4} \omega_j(x)\, dx_j$ in this coordinate system.

Proof. Let us suppose first that $n = 4$, so that $y \in \mathbb{C}$ and $Z_1 = \frac{\partial}{\partial y}$. Write

$$\Omega(x, y) = \sum_{k=0}^{\nu} y^k\, \Omega_k(x)$$

where ν is the degree of Ω and the coefficients of Ω_k are homogeneous polynomials of degree $\nu - k$ in x. We can write

$$\Omega_k(x) = \Omega^0_k(x) + f_k(x)\, dy\ , \text{ where } \Omega^0_k(x) = \sum_{i=1}^{4} g^i_k(x)\, dx_i\ :$$

and f_k, g^i_k are homogeneous polynomials of degree $\nu - k$, $i = 1, \dots, 4$. We want to prove that $\Omega = \Omega^0_0$. First of all, observe that $f_0 = 0$, because $f_0(x) = i_{Z_1}\Omega(x, 0) = 0$. Let us suppose by induction that $\Omega_j = 0$ for $j = 1, \dots, k-1$, $k < \nu$, and prove that $\Omega_k = 0$. In this case, we have

$$\Omega = \Omega^0_0 + y^k(\Omega^0_k + f_k\, dy) \pmod{y^{k+1}} \text{ and } d\Omega = d\Omega^0_0 + k y^{k-1} dy \wedge \Omega^0_k \pmod{y^k}\ ;$$

so that, the integrability condition gives us

$$0 = \Omega \wedge d\Omega = \Omega^0_0 \wedge d\Omega^0_0 + k y^{k-1}\, \Omega^0_0 \wedge dy \wedge \Omega^0_k \pmod{y^k}\ :$$

Since $\Omega^0_0 = \Omega|_{E^*}$, it is integrable; $\Omega^0_0 \wedge d\Omega^0_0 = 0$, and we get $\Omega^0_0 \wedge dy \wedge \Omega^0_k = 0$. But, the forms Ω^0_j do not contain terms in dy, and so $\Omega^0_0 \wedge \Omega^0_k = 0$. This implies that $\Omega^0_k = \lambda.\Omega^0_0$, where λ is holomorphic, because $\operatorname{cod}(\operatorname{sing}(\Omega^0_0)) \geq 2$. On the other hand, the fact that the coefficients of Ω^0_k are homogeneous polynomials of degree $\nu - k$, while the coefficients of Ω^0_0 are of degree $\nu > \nu - k$, implies that $\lambda = 0$, and so $\Omega^0_k = 0$.

Let us prove that $f_k = 0$. We will use the vector fields $Y^1_\alpha = \frac{\partial}{\partial y} + X^1_\alpha$, $\alpha \in A$, as in (15). We can write for $(x, y) \in V_\alpha \times W_\alpha$ that

$$Y^1_\alpha(x, y) = Z_1 + \sum_{j=0} y^j\, X_{\alpha, j}(x)$$

where the vector fields $X_{\alpha,j}$ contain only terms in $\frac{\partial}{\partial x_i}$, $i = 1,\dots,4$. Since $i_{Y^1_\alpha}\Omega = 0$ and $i_{Z_1}\Omega^0_0 = 0$, we get

$$0 \equiv i_{Y^1_\alpha(x,y)}\Omega(x,y) = i_{Z_1}\Omega(x,y) + \sum_{j=0}^{k} y^j\, i_{X_{\alpha,j}(x)}\Omega(x,y) =$$

$$= y^k f_k(x) + \sum_{j=0}^{k} y^j\, i_{X_{\alpha,j}(x)}\Omega^0_0(x) \pmod{y^{k+1}}\ ;$$

as the reader can check. This implies that $i_{X_{\alpha,j}}\Omega^0_0 = 0$ for $j = 0,\dots,k-1$ and $f_k + i_{X_{\alpha,k}}\Omega^0_0 = 0$. For $V_{\alpha\beta} \neq \emptyset$, set $X_{\alpha\beta}(x) = X_{\beta,k}(x) - X_{\alpha,k}(x)$. Clearly, $(X_{\alpha\beta})_{V_{\alpha\beta}\neq\emptyset}$ is an additive cocycle of vector fields. Moreover, $i_{X_{\alpha\beta}}\Omega^0_0 = 0$, so that we can apply the Corollary 2 of Lemma 1 to obtain vector fields T_α on V_α such that $X_{\alpha\beta} = T_\beta - T_\alpha$ on $V_{\alpha\beta} \neq \emptyset$ and $i_{T_\alpha}\Omega^0_0 = 0$ for all $\alpha \in A$. This implies that there exists a vector field X on $E^* \setminus \{0\}$ such that $X|_{V_\alpha} = -(X_{\alpha,k} + T_\alpha)$ for all $\alpha \in A$. By Hartog's Theorem X can be extended to E^*. On the other hand, as the reader can check

$$(17)\quad i_X\Omega^0_0 = f_k$$

But, f_k is homogeneous of degree $\nu - k$ and Ω^0_0 homogeneous of degree $\nu > \nu - k$, so that (17) implies that $f_k = 0$. This finishes the case $n = 4$.

The general case can be reduced to the above one by taking sections. In fact, since $i_{Z_j}\Omega(x,0) = 0$, $j = 1,\dots,n-3$, we can write

$$\Omega(x,y) = \Omega^0_0(x) + \sum_{1\le|\sigma|\le\nu} y^\sigma\,\Omega^0_\sigma(x) + \sum_{i=1}^{n-3}\sum_{1\le|\sigma|\le\nu} y^\sigma f^i_\sigma(x)dy_i\ ;$$

where $\sigma = (\sigma_1,\dots,\sigma_{n-3})$, $y^\sigma = y_1^{\sigma_1}\dots y_{n-3}^{\sigma_{n-3}}$, $|\sigma| = \sigma_1 + \dots + \sigma_{n-3}$, f^i_σ and the coefficients of Ω^0_σ are homogeneous polynomials of degree $\nu - |\sigma|$ and Ω^0_σ contains only terms in $dx_1,\dots,dx_4$. Let $v = (v_1,\dots,v_{n-3})$ be a non-zero vector of $\mathbb{C}^{n-3}$ and consider the linear immersion $L : E^*\times\mathbb{C} \to E^*\times\mathbb{C}^{n-3} \simeq \mathbb{C}^{n+1}$ given by $L(x,w) = (x, w.v)$. We have

$$L^*(\Omega) = \Omega^0_0(x) + \sum_{k=1}^{\nu} w^k\left[\sum_{|\sigma|=k} v^\sigma\,\Omega^0_\sigma(x) + \left(\sum_{i=1}^{n-3}\sum_{|\sigma|=k} v^\sigma v_i f^i_\sigma(x)\right)dw\right]:$$

It follows from the case $n = 4$ that

$$\sum_{|\sigma|=k} v^\sigma\,\Omega^0_\sigma(x) = 0\ ,\forall v \in \mathbb{C}^{n-3}\ ;\ \forall 1 \le k \le \nu \implies \Omega^0_\sigma = 0\ ;\ \forall \sigma \neq 0\ :$$

This implies that

$$\Omega(x,y) = \Omega^0_0(x) + \sum_{i,\sigma} y^\sigma f^i_\sigma(x)\,dy_i \implies d\Omega(x,y) = d\Omega^0_0(x) + \sum_{i,\sigma} y^\sigma df^i_\sigma(x)\wedge dy_i + \sum_{i<j}\omega_{i,j}\,dy_i\wedge dy_j$$

Now, by using the integrability condition and collecting in $\Omega\wedge d\Omega = 0$ the coefficients of the terms containing only the factors $dx_i\wedge dx_j\wedge dy_\ell$, we get that

$$\sum_{i,\sigma} y^\sigma\left(\Omega^0_0\wedge df^i_\sigma + f^i_\sigma\, d\Omega^0_0\right)\wedge dy_i = 0 \implies df^i_\sigma\wedge\Omega^0_0 = f^i_\sigma\, d\Omega^0_0\ ;\forall i,\sigma\ ;1\le|\sigma|\le\nu\ ;1\le i\le n-3\ :$$

The last relation implies that, $f^i_\sigma = 0$, for all i, σ. In fact, we have seen in the proof of Lemma 2 that $L_R(\Omega'_0) = (\nu + 1)\Omega'_0$, so that $i_R(d\Omega'_0) = i_R(d\Omega'_0) + d(i_R\,\Omega'_0) = L_R(\Omega'_0) = (\nu + 1)\Omega'_0$. Hence

$$i_R(df^i_\sigma \wedge \Omega'_0) = i_R(f^i_\sigma\, d\Omega'_0) \implies (\nu - |\sigma|) f^i_\sigma = (\nu + 1) f^i_\sigma \implies f^i_\sigma = 0\,;$$

because f^i_σ is homogeneous of degree $\nu - |\sigma|$. This finishes the proof of the assertion and of the Theorem. □

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