Geometrical *versus* Topological Properties of Manifolds

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Abstract

Given a compact n-dimensional immersed Riemannian manifold M^n in some Euclidean space we prove that if the Hausdorff dimension of the singular set of the Gauss map is small, then M^n is homeomorphic to the sphere S^n.

Also, we define a concept of finite geometrical type and prove that finite geometrical type hypersurfaces with small set of points of zero Gauss-Kronecker curvature are topologically the sphere minus a finite number of points. A characterization of the $2n$-catenoid is obtained.

1 Introduction

Let $f : M^n \to N^m$ be a C^1 map. We denote by

$$rank(f) := \min_{p \in M} rank(D_p f).$$

If $n = \dim M = \dim N = m$, let $C := \{p \in M : \det D_p f = 0\}$ the set of *critical points* of f and $S := f(C)$ the set of *critical values* of f.

Now, let M^n a compact, connected, boundary less, n-dimensional manifold. Denote by H_s the s-dimensional Hausdorff measure and $\dim_H(A)$ the Hausdorff dimension of $A \subset M^n$. For definitions see section 2 below. Let x be an immersion $x : M^n \to \mathbb{R}^{n+1}$. In this case, let $G : M^n \to S^n$ the Gauss map associated to x, C the critical points of G and S the critical values of G. We denote by $dim_H(x) := dim_H(S)$. By Moreira's improvement of Morse-Sard theorem (see [Mo]), since G is a smooth map, we have that $\dim_H(S) \leq n-1$.

In other words, if $\mathcal{I}mm = \{x : M \to \mathbb{R}^{n+1} : x \text{ is an immersion}\}$, then $\sup\limits_{x \in \mathcal{I}mm} dim_H(x) \leq n-1$. Clearly, this supremum could be equal to $n-1$, as some immersions of S^n in $\mathbb{R}^{n+1}$ show (e.g., immersions with "cylindrical pieces"). Our interest here is the number $\inf dim_H(x)$. Before discuss this, we introduce some definitions.

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Definition 1.1. Given an immersion $x : M^n \to \mathbb{R}^{n+1}$ we define $rank(x) := rank(G)$, where G is the Gauss map for x.

Definition 1.2. We denote by $\mathcal{R}(k)$ the set $\mathcal{R}(k) = \{x \in \mathcal{I}mm : rank(x) \geq k\}$. Define by $\alpha_k(M)$ the numbers:

$$\alpha_k(M) = \inf_{x \in \mathcal{R}(k)} dim_H(x), k = 0, \ldots, n$$

If $\mathcal{R}(k) = \emptyset$ we define $\alpha_k(M) = n - 1$.

Now, we are in position to state our first result:

Theorem A. *If M^n is a compact manifold with $n \geq 3$ such that $\alpha_k(M^n) < k - [\frac{n}{2}]$, for some integer k, then M^n is homeomorphic to S^n ($[r]$ is the integer part of r).*

The proof of this theorem in the cases $n = 3$ and $n \geq 4$ are quite different. For higher dimensions, we can use the generalized Poincaré Conjecture (Smale and Freedman) to obtain that the given manifold is a sphere. Since the Poincaré Conjecture is not available in three dimensions, the proof, in this case, is a little bit different. We use a characterization theorem due to Bing to compensate the loss of Poincaré Conjecture, as commented before.

To prove this theorem in the case $n = 3$, we proceed as follows:

- By a theorem of Bing (see [B]), we just need to prove that every piecewise smooth simple curve γ in M^3 lies in a topological cube $\mathcal{R}$ of M^3;
- In order to prove it, we shall show that it is enough to prove for $\gamma \subset M - G^{-1}(S)$ and that $G : M - G^{-1}(S) \to S^3 - S$ is a diffeomorphism;
- Finally, we produce a cube $\tilde{\mathcal{R}} \supset G(\gamma)$ in $S^3 - S$ and we obtain $\mathcal{R}$ pulling back this cube by G

Observe that by [C], in three dimensions always there are Euclidean codimension 1 immersions. In particular, it is reasonable to consider the following consequence of the Theorem A:

Corollary 1.3. *The following statement is equivalent to Poincaré Conjecture : "Simply connected 3-manifolds admits Euclidean codimension one immersions with rank at least 2 and Hausdorff dimension of the singular set for this Gauss map less than 1".*

For a motivation of this conjecture and some comments about three dimensional manifolds see the section 7.

Our motivation behind proving this theorem are results by do Carmo, Elbert [dCE] and Barbosa, Fukuoka, Mercuri [BFM]. Roughly speaking, they obtain topological results about certain manifolds provided that there are special codimension 1 immersions of them. These results motivate the question : how the space of immersions (extrinsic information) influences the topology of

M (intrinsic information)? The Theorems A and B below are a partial answer to this question. The proofs of the theorems depends on the concept of Hausdorff dimension. Essentially, Hausdorff dimension is a fractal dimension that measures how "small" is a given set with respect to usual "regular" sets (e.g., smooth submanifolds, that always have integer Hausdorff dimension).

In section 6 of this paper we obtain the following generalizations of Theorem A and B:

Definition 1.4. Let $\overline{M}^n$ a compact (oriented) manifold and $p_1, \dots, p_k \in \overline{M}^n$. Let $M = \overline{M}^n - \{p_1, \dots, p_k\}$. An immersion $x : M^n \to \mathbb{R}^{n+1}$ is of *finite geometrical type* (in a weaker sense than that of [BFM]) if M^n is complete in the induced metric, the Gauss map $G : M^n \to S^n$ extends continuously to a function $\overline{G} : \overline{M}^n \to S^n$ and the set $G^{-1}(S)$ has $H_{n-1}(G^{-1}(S)) = 0$ (this last condition occurs if $rank(x) \geq k$ and $H_{k-1}(S) = 0$).

The conditions in the previous definition are satisfied by complete hypersurfaces with finite total curvature whose Gauss-Kronecker curvature $H_n = k_1 \dots k_n$ does not change of sign and vanish in a small set, as showed by [dCE]. Recall that a hypersurface $x : M^n \to \mathbb{R}^{n+1}$ has total finite curvature if $\int_M |A|^n dM < \infty$, $|A| = (\sum_i k_i^2)^{1/2}$, k_i are the principal curvatures. With these observations, one has :

Theorem B. *If $x : M^n \to \mathbb{R}^{n+1}$ is a hypersurface with finite geometrical type and $H_{k-[\frac{n}{2}]}(S) = 0$, $rank(x) \geq k$. Then M^n is topologically a sphere minus a finite number of points, i.e., $\overline{M}^n \simeq S^n$. In particular, this result holds for complete hypersurfaces with finite total curvature and $H_{k-[\frac{n}{2}]}(S) = 0$, $rank(x) \geq k$.*

For even dimensions, we follow [BFM] and improve Theorem B. In particular, we obtain the following characterization of $2n$-catenoids, as the unique minimal hypersurfaces of finite geometrical type.

Theorem C. *Let $x : M^{2n} \to \mathbb{R}^{2n+1}$, $n \geq 2$ an immersion of finite geometrical type with $H_{k-n}(S) = 0$, $rank(x) \geq k$. Then M^{2n} is topologically a sphere minus two points. If M^{2n} is minimal, M^{2n} is a $2n$-catenoid.*

2 Notations and Statements

Let M^n be a smooth manifold. Before starting the proofs of the statements we fix some notations and collect some (useful) standard propositions about Hausdorff dimension (and limit capacity, another fractal dimension). For the proofs of these propositions we refer [Fa].

Let X a compact metric space and $A \subset X$. We define the s-dimensional Hausdorff measure of A by

$$H_s(A) := \lim_{\varepsilon \to 0} \inf\{\sum_i (diam\, U_i)^s : A \subset \bigcup U_i, U_i \text{ is open and } diam(U_i) \leq \varepsilon, \forall i \in \mathbb{N}\}.$$

The *Hausdorff dimension* of A is $\dim_H(A) := \sup\{d \geq 0 : H_d(A) = \infty\} = \inf\{d \geq 0 : H_d(A) = 0\}$. A remarkable fact is that H_n coincides with Lebesgue measure for a smooth manifold M^n.

A related notion are the lower and upper *limit capacity* (sometimes called box counting dimension) defined by

$$\underline{\dim_B}(A) := \liminf_{\varepsilon \to 0} \log n(A, \varepsilon)/(-\log \varepsilon), \ \overline{\dim_B}(A) := \limsup_{\varepsilon \to 0} \log n(A, \varepsilon)/(-\log \varepsilon),$$

where $n(A, \varepsilon)$ is the minimum number of ε-balls that cover A. If $d(A) = \underline{\dim_B}(A) = \overline{\dim_B}(A)$, we say that the limit capacity of A is $\dim_B(A) = d(A)$.

These fractal dimensions satisfy the properties expected for "natural" notions of dimensions. For instance, $\dim_H(A) = m$ if A is a smooth m-submanifold.

Proposition 2.1. *The properties listed below hold :*

1. $\dim_H(E) \leq \dim_H(F)$ *if* $E \subset F$*;*
2. $\dim_H(E \cup F) = \max\{\dim_H(E), \dim_H(F)\}$*;*
3. *If f is a Lipschitz map with Lipschitz constant C, then $H_s(f(E)) \leq C \cdot H_s(E)$. As a consequence,* $\dim_H(f(E)) \leq \dim_H E$*;*
4. *If f is a bi-Lipschitz map (e.g., diffeomorphisms),* $\dim_H(f(E)) = \dim_H(E)$*;*
5. $dim_H(A) \leq \underline{dim_B}(A)$.

Analogous properties holds for lower and upper limit capacity. If E is countable, $\dim_H(E) = 0$ *(although we may have* $\dim_B(E) > 0$*).*

When we are dealing with product spaces, the relationship between Hausdorff dimension and limit capacity are the product formulae :

Proposition 2.2. $\dim_H(E) + \dim_H(F) \leq \dim_H(E \times F) \leq \dim_H(E) + \overline{\dim_B}(F)$. *Moreover, $c \cdot H_s(E) \cdot H_t(F) \leq H_{s+t}(E \times F) \leq C \cdot H_s(E)$, where c depends only on s and t, C depends only on s and* $\overline{\dim_B}(F)$*.*

Before starting the necessary lemmas to prove the central results, we observe that it follows from lemma above that if M and N are diffeomorphic n-manifolds then $\alpha_k(M) = \alpha_k(N)$. This proves :

Lemma 2.3. *The numbers*

$$\alpha_k(M) = \inf_{x \in \mathcal{R}(k)} dim_H(x), \ for \ k = 0, \ldots, n$$

are smooth invariants of M.

In particular, if $n = 3$ we also have that α_k are topological invariants. It is a consequence of a theorem due to Moise [M], which state that if M and N are homeomorphic 3-manifolds then they are diffeomorphic. Then, the following conjecture arises from the Theorem A

Conjecture 1. *If M^3 is simply connected, then*

$$\alpha_2(M^3) = \inf_{x \in \mathcal{R}(2)} dim_H(x) < 1$$

R. Cohen's theorem ([C]) says that there are immersions of compact n-manifolds M^n in $\mathbb{R}^{2n-\alpha(n)}$ where $\alpha(n)$ is the number of 1's in the binary expansion of n. This implies, for the case $n = 3$, that we always have that $\mathcal{I}mm \neq \emptyset$. In particular, the implicit hypothesis of existence of codimension 1 immersions in Theorem A is not too restrictive and our conjecture is reasonable. We point out that conjecture 1 is true if Poincaré conjecture holds and, in this case, $\sup_{x \in \mathcal{I}mm} rank(x) = 3$ and $\inf_{x \in \mathcal{R}(k)} dim_H(x) = 0$, for all $0 \leq k \leq 3$. A corollary of the theorem A and this observation is:

Corollary 2.4. *The Poincaré Conjecture is equivalent to the conjecture 1.*

From this, a natural approach to conjecture 1 is a deformation and desingularization argument for metrics given by pull-back of immersions in $\mathcal{I}mm$. We observe that Moreira's theorem give us $\alpha_2(M^3) \leq 2$. This motivates the following question, which is a kind of step toward Poincaré Conjecture. However, this question is of independent interest, since it can be true even if Poincaré Conjecture is false :

Question 1. *For simply connected 3-manifolds, is true that $\alpha_2(M^3) < 2$?*

3 Some lemmas

In this section, we prove some useful facts on the way to establish the Theorems A, B. The first one relates the Hausdorff dimension of subsets of smooth manifolds and rank of smooth maps :

Proposition 3.1. *Let $f : M^m \to N^n$ a C^1-map and $A \subset N$. Then $\dim_H f^{-1}(A) \leq \dim_H(A) + n - rank(f)$.*

Proof. The computation of Hausdorff dimension is a local problem. So, we can consider $p \in f^{-1}(S)$, coordinate neighborhoods $p \in U$, $f(p) \in V$ fixed and $f = (f_1, \dots, f_n) : U \to V$. Making a change of coordinates (which does not change Hausdorff dimensions), we can suppose that $\widetilde{f} = (f_1, \dots, f_r)$ is a submersion, where $r = rank(f)$. By the local form of submersions, there is φ a diffeomorphism such that $\widetilde{f} \circ \varphi(y_1, \dots, y_m) = (y_1, \dots, y_r)$. This implies that $f \circ \varphi(y_1, \dots, y_m) = (y_1, \dots, y_r, g(\varphi(y_1, \dots, y_m))$. Then, if π denotes the projection in the r first variables, $x \in f^{-1}(S) \Rightarrow \pi\varphi^{-1}(x) \in \pi(S)$, i.e., $f^{-1}(S) \subset \varphi(\pi(S) \times \mathbb{R}^{n-r})$. By properties of Hausdorff dimension (see section 2), we have $dim_H f^{-1}(S) \leq dim_H(\pi(S) \times \mathbb{R}^{n-r}) \leq dim_H \pi(S) + \overline{dim_B}(\mathbb{R}^{n-r}) \leq dim_H(S) + n - r$. This concludes the proof. $\square$

The second proposition relates Hausdorff dimension with topological results.

Proposition 3.2. *Let $n \geq 3$ and F is a closed subset of a n-dimensional connected (not necessarily compact) manifold M^n. If the Hausdorff dimension of F is strictly less than $n-1$ then $M^n - F$ is connected. If $M^n = \mathbb{R}^n$ or $M^n = S^n$, F is compact and the Hausdorff dimension of F is strictly less than $n-k-1$ then $M^n - F$ is k-connected (i.e., its homotopy groups π_i vanishes for $i \leq k$).*

Proof. First, if F is a closed subset of M^n with Hausdorff dimension strictly less than $n-1$, $x, y \in M^n - F$, take γ a path from x to y in M^n. Since $n \geq 3$, we can suppose γ a smooth simple curve (by transversality). In this case, γ admits some compact tubular neighborhood $\mathcal{L}$. For each $p \in \gamma$, denote $\mathcal{L}_p$ the $\mathcal{L}$-fiber passing through p. By hypothesis, $dim_H(F \cap \mathcal{L}_p) < n-1 \ \forall p$. In this case, the tubular neighborhood $\mathcal{L}$ is diffeomorphic to $\gamma \times D^{n-1}$, the fibers $\mathcal{L}_p$ are $p \times D^{n-1}$ (D^{n-1} is the $(n-1)$-dimensional unit disk centered at 0) and γ is $\gamma \times 0$. Then, since F is closed, it is easy that every $x \in \gamma$ admits a neighborhood $V(x)$ such that for some sequence $v_n = v_n(x) \to 0$ holds $(V(x) \times v_n) \cap F = \emptyset$. Moreover, again by the fact that F is closed, any vector v sufficiently close to some v_n satisfies $(V(x) \times v) \cap F = \emptyset$. With this in mind, by compactness of γ, we get some finite cover of γ by neighborhoods as described before. This guarantees the existence of v_0 arbitrarily small such that $(\gamma \times v_0) \cap F = \emptyset$. This implies that $M - F$ is connected.

Second, if F is a compact subset of $M^n = \mathbb{R}^n$, $dim_H F < n-k-1$, let $[\Gamma] \in \pi_i(\mathbb{R}^n - F)$ a homotopy class for $i \leq k$. Choose a smooth representative $\Gamma \in [\Gamma]$. Define $f : \Gamma \times F \to S^{n-1}$, $f(x,y) := (y-x)/||y-x||$. We will consider in $\Gamma \times F$ the sum norm, i.e., if $p, q \in \Gamma \times F$, $p = (x,y), q = (z,w)$ then $||p-q|| := ||x-z|| + ||y-w||$. For this choice of norm we have

$$||f(p) - f(q)|| = \frac{1}{||y-x|| \cdot ||z-w||} \cdot \Big\| \big\{ (y-x) \cdot ||z-w|| + ||y-x|| \cdot (z-w) \big\} \Big\| \Rightarrow$$

$$||f(p) - f(q)|| \leq \frac{\Big\| (y-x) \cdot ||z-w|| - ||z-w|| \cdot (w-z) \Big\|}{||y-x|| \cdot ||z-w||} + \frac{\Big\| ||z-w|| \cdot (w-z) - ||y-x|| \cdot (w-z) \Big\|}{||y-x|| \cdot ||z-w||} \cdot \Rightarrow$$

$$||f(p) - f(q)|| \leq \frac{1}{||y-x||} \cdot \Big\{ ||(z-x) + (y-w)|| \Big\} + \frac{1}{||y-x||} \cdot \big| \{ ||(z-w)|| - ||(y-x)|| \} \big| \Rightarrow$$

$$||f(p) - f(q)|| \leq 2 \cdot C \cdot ||p-q||$$

where $C = 1/d(\Gamma, F)$. We have $d(\Gamma, F) > 0$ since these are compact disjoint sets. This computation shows that f is Lipschitz.

Then, we have (Proposition 2.1, 2.2) $dim_H f(\Gamma \times F) \leq dim_H(\Gamma \times F) \leq \overline{dim_B}(\Gamma) + dim_H(F) < i + n - k - 1 \leq n - 1 \Rightarrow \exists \ v \notin f(\Gamma \times F)$. Now, F is compact implies that there is a real N such that $F \subset B_N(0)$. Then, making a translation of Γ at v direction, we can put, using this translation as homotopy, Γ outside B_N. Since $\mathbb{R}^n - B_N$ is n-connected (for $n \geq 3$), $\pi_i(\mathbb{R}^n - F) = 0$. This concludes the proof. □

Remark 3.3. We remark that the hypothesis F is closed in the previous proposition is necessary. For example, take $F = \mathbb{Q}^n$, $M^n = \mathbb{R}^n$. We have $dim_H(F) = 0$ (F is a countable set) but $M^n - F$ is not connected.

We can think of Proposition 3.2 as a weak type of transversality. In fact, if F is a compact $(n-2)$-submanifold of M^n then $M - F$ is connected and if F is a compact $(n-3)$-submanifold of $\mathbb{R}^n$ (or S^n) then $\mathbb{R}^n - F$ is simply connected. This follows from basic transversality. However, our previous proposition does not assume regularity of F, but allows us to conclude the same results. It is natural these results are true because Hausdorff dimension translates the fact that F is, in some sense, "smaller" than a $(n-1)$-submanifold N which has optimal dimension in order to disconnect M^n.

For later use, we generalize the first part of Proposition 3.2 as follows :

Lemma 3.4. *Suppose that $\Gamma \in \pi_i(M^n)$ is Lipschitz (e.g., if $i = 1$ and Γ is a piecewise smooth curve) and let $K \subset M^n$ compact, $dim_H K < n - i$. Then there are diffeomorphisms h of M, arbitrarily close to identity map, such that $h(\Gamma) \cap K = \emptyset$. In particular, if $[\Gamma] \in \pi_i(M^n)$ a homotopy class, $K \subset M^n$ a compact set, $dim_H(K) < n - i$, there is a smooth representative $\Gamma \in [\Gamma]$ such that $\Gamma \cap K = \emptyset$, i.e., $\Gamma \in \pi_i(M^n - K)$.*

Proof. First, consider a parametrized neighborhood $\phi : U \to B_3(0) \subset \mathbb{R}^n$ and suppose that Γ lies in $\overline{V_1}$, where $V_1 = \phi^{-1}(B_1(0))$. Let $K_1 = \phi(K) \subset \mathbb{R}^n$ and $\Gamma_1 = \phi(\Gamma) \subset \mathbb{R}^n$. Consider the map:

$$F : \Gamma_1 \times K_1 \to \mathbb{R}^n \text{ defined by } F(x, y) = x - y$$

Observe that, since Γ is Lipschitz and ϕ is a diffeomorphism, $\overline{dim_B}\Gamma = \overline{dim_B}\Gamma_1 \leq i$. This implies that $dim_H(F(\Gamma_1 \times K_1)) < n$, since $dim_H(K) < n - i$. This implies, in particular, that $\mathbb{R}^n - F(\Gamma_1 \times K_1)$ is an open and dense subset, since K is compact. Then, we may choose a vector $v \in \mathbb{R}^n - F(\Gamma_1 \times K_1)$ arbitrarily close to 0 such $(\Gamma_1 + v) \subset B_2(0)$. Since, $v \in \mathbb{R}^n - F(\Gamma_1 \times K_1)$ we have that $(\Gamma_1 + v) \cap K_1 = \emptyset$.

To construct h we consider a bump function $\beta : \mathbb{R}^n \to [0, 1]$, such that $\beta(x) = 1$ if $x \in B_1(0)$ and $\beta(x) = 0$ for every $x \in \mathbb{R}^n - B_2(0)$. It is easy to see that h defined by:

$$h(y) = y \text{ if } x \in M - U \text{ and } h(y) = \phi^{-1}(\beta(\phi(y))v + \phi(y)),$$

is a diffeomorphism that satisfies $h(\Gamma) \cap K = \emptyset$, since $(\Gamma_1 + v) \cap K_1 = \emptyset$.

In the general case, we proceed as follows : first, considering a finite number of parametrized neighborhoods $\phi_i : U_i \to B_3(0), i \in \{1, \ldots, n\}$ and $V_i = \phi_i^{-1}(B_1(0))$ covering Γ, by the previous case, there exists h_1 arbitrarily close to the identity such $h_1(\Gamma) \subset \bigcup_{i=1}^{n} V_i$ and such that $h_1(\Gamma \cap \overline{V_1}) \cap K = \emptyset$. Observe that, $d(h_1(\Gamma \cap \overline{V_1}), K) > \epsilon_1 > 0$, since $h_1(\Gamma \cap \overline{V_1})$ is a compact set.

The next step is to repeat the previous argument considering h_2 arbitrarily close to the identity, in such way that $h_2(h_1(\Gamma) \cap V_2) \cap K = \emptyset$ and $h_2(h_1(\Gamma)) \subset \bigcup_{i=1}^{n} V_i$. If $d(h_2, id) < \frac{\epsilon_1}{2}$ then $h_2(h_1(\Gamma) \cap V_1) \cap K = \emptyset$. Repeating this argument by induction, we obtain that $h = h_n \circ \cdots \circ h_1$ is a diffeomorphism such that $h(\Gamma) \cap K = \emptyset$. This concludes the proof. $\square$

4 Proof of Theorem A in the case $n = 3$

Before giving a proof for theorem A, we mention a lemma due to Bing [B] :

Lemma 4.1 (Bing). *A compact, connected, 3-manifold M is topologically S^3 if and only if each piecewise smooth simple closed curve in M lies in a topological cube in M.*

A modern proof of this lemma can be found in [R]. In modern language, Bing's proof shows that the hypothesis above imply that *Heegaard* splitting of M is in two balls. This is sufficient to conclude the result.

In fact, Bing's theorem is not stated in [B], [R] as above. But the lemma holds. Actually, to prove that M is homeomorphic to S^3, Bing uses only that, if a triangulation of M is fixed, every simple *polyhedral* closed curve lies in a topological cube. Observe that *polyhedral* curves are piecewise smooth curves, if we choose a smooth triangulation (smooth manifolds always can be smooth triangulated, see [T], page 194; see also [W], page 124).

Proof of Theorem A in the case $n = 3$. If $\alpha_2(M) < 1$, there is an immersion $x : M^3 \to \mathbb{R}^4$ such that $rank(x) \geq 2, dim_H(x) < 1$. Let G the Gauss map associated to x. By Propositions 3.2, 3.1, since $dim_H(S) < 1$, $M - G^{-1}(S), S^3 - S$ are connected manifolds. Consider $G : M - G^{-1}(S) \to S^3 - S$. This is a proper map between connected manifolds whose Jacobian never vanishes. So it is a surjective and covering map (see [WG]). Since, moreover, $S^3 - S$ is simply connected (by Proposition 3.2), $G : M - G^{-1}(S) \to S^3 - S$ is a diffeomorphism. To prove that M^3 is homeomorphic to S^3, it is necessary and sufficient that every piecewise smooth simple closed curve $\gamma \subset M^3$ is contained in a topological cube $Q \subset M^3$ (by Lemma 4.1).

In order to prove that every piecewise smooth curve γ lies in a topological cube, observe that we may suppose that $\gamma \cap K = \emptyset$ (here $K = G^{-1}(S)$). Indeed, by lemma 3.4 there exists a diffeomorphism h of M such $h(\gamma) \cap K = \emptyset$. Then, if $h(\gamma)$ lies in a topological cube R, the γ itself lies in the topological cube $h^{-1}(R)$ too, thus we can, in fact, make this assumption.

Now, since $\gamma \subset M - K$ and $M - K$ is diffeomorphic to $S^3 - S$, we may consider $\gamma \subset \mathbb{R}^3 - S$, S a compact subset of $\mathbb{R}^3$ with Hausdorff dimension less than 1 via identification by the diffeomorphism G and stereographic projection. In this case, we can follow the proof of Proposition 3.2 to conclude that $f : \gamma \times S \to S^2$, $f(x, y) = (x - y)/||x - y||$ is Lipschitz. Because $\overline{\dim}_B \gamma \leq 1, \dim_H S < 1$ (here we are using that γ is piecewise smooth), we obtain a direction $v \in S^2$ such that $F := \bigcup_{t \in \mathbb{R}} (L_t(\gamma))$ is disjoint from S, where $L_t(p) := p + t \cdot v$. By compactness of γ it is easy that F is a closed subset of $\mathbb{R}^n$. This implies that $3\ \epsilon = d(F, S) > 0$. Consider $F_\epsilon = \{x : d(x, F) \leq \epsilon\}$ and $S_\epsilon = \{x : d(x, S) \leq \epsilon\}$. By definition of $\epsilon > 0$, $F_\epsilon \cap S_\epsilon = \emptyset$, then we can choose $\varphi : \mathbb{R}^3 \to \mathbb{R}$ a smooth function such that $\varphi|_{F_\epsilon} = 1$, $\varphi|_{S_\epsilon} = 0$. Consider the vector field $X(p) = \varphi(p) \cdot v$ and let X_t, $t \in \mathbb{R}$ the X-flow. We have $X_t(p) = p + tv\ \forall p \in \gamma$ and $X_t(p) = p\ \forall p \in S$, for any $t \in \mathbb{R}$. Choosing N real such that $S \subset B_N(0)$ and T such that $t \geq$

$T \Rightarrow L_t(\gamma) \cap B_N(0) = \emptyset$, we obtain a global homeomorphism X_t which sends γ outside $B_N(0)$ and keep fixed S, $\forall\, t \geq T$.

Observe that $X_t(\gamma)$ is contained in the interior of a topological cube $Q \subset \mathbb{R}^3 - B_N(0)$. Then, observing that X_t is a diffeomorphism and that $X_t(x) = x$ for every $x \in S$ and $t \in \mathbb{R}$, we have that $\gamma \subset X_{-t}(Q) \subset \mathbb{R}^3 - S$, $\forall\, t \geq T$. This concludes the proof. $\square$

5 Proof of Theorem A in the case $n \geq 4$

We start this section with the statement of generalized Poincaré Conjecture :

Theorem 5.1. *A compact simply connected homological sphere M^n is homeomorphic to S^n, if $n \geq 4$ (diffeomorphic for $n = 5, 6$).*

The proof of generalized Poincaré Conjecture is due to Smale [S] for $n \geq 5$ and to Freedman [F] for $n = 4$. This theorem makes the proof of the Theorem B a little bit easier than the proof of Theorem A.

Proof of Theorem A in the case $n \geq 4$. If $k = n$, there is nothing to prove. Indeed, in this case, $G : M^n \to S^n$ is a diffeomorphism, by definition. I.e., without loss of generality we can suppose $k \leq n-1$; $\alpha_k(M) < k - [\frac{n}{2}] \Rightarrow \exists\, x : M^n \to \mathbb{R}^{n+1}$ immersion, $rank(x) \geq k$, $dim_H(x) < k - [\frac{n}{2}]$. The hypothesis implies that $M - G^{-1}(S)$ is connected, $S^n - S$ is simply connected and G is a proper map whose jacobian never vanishes. By [WG], G is a surjective, covering map. So, we conclude that $G : M - G^{-1}(S) \to S^n - S$ is diffeomorphism. But $S^n - S$ is $(n - 1 - k + [\frac{n}{2}])$-connected, by Proposition 3.2. In particular, because $k \leq n-1$, $S^n - S$ is $[\frac{n}{2}]$-connected and so, using the diffeomorphism G, $M - K$ is $[\frac{n}{2}]$-connected, where $K = G^{-1}(S)$. It is sufficient to prove that M^n is a simply connected homological sphere, by Theorem 5.1. By Lemma 3.4, $M - K$ is $[\frac{n}{2}]$-connected and $dim_H(K) < n - [\frac{n}{2}]$ (by Proposition3.2) implies M itself is $[\frac{n}{2}]$-connected. It is know that $H^i(M) = L(H_i(M)) \oplus T(H_{i-1}(M))$, L and T denotes the free part and the torsion part of the group. By Poincaré duality, $H_{n-i}(M) \simeq H^i(M)$. The fact that M is $[\frac{n}{2}]$-connected and the other facts give us $H_i(M) = 0$, for $0 < i < n$. This concludes the proof. $\square$

6 Proof of Theorems B and C

In this section we make some comments on extensions of Theorem A. Although these extensions are quite easy, they were omitted so far to make the presentation of the paper more clear. Now, we are going to improve our previous results. First, all preceding arguments works with assumption that $H_{k-[\frac{n}{2}]}(S) = 0$ and $rank(x) \geq k$ in Theorems A, B (where H_s is the s-dimensional Hausdorff measure). We prefer to consider the hypothesis as its stands in these theorems because it is more interesting to define the invariants $\alpha_k(M)$. The reason to this "new" hypothesis works is that our proofs, essentially, depend on the existence of special directions $v \in S^{n-1}$. But these directions exist if the singular sets have Hausdorff measure 0. Secondly, M need not to be compact. It is sufficient that M is of *finite geometric type* (here our definition of finite geometrical type is a little bit different from [BFM]). We will make more precise these comments in proof of Theorem 6.2 below, after recalling the definition :

Definition 6.1. Let $\overline{M}^n$ a compact (oriented) manifold and $q_1, \dots, q_k \in \overline{M}^n$. Let $M^n = \overline{M}^n - \{q_1, \dots, q_k\}$. An immersion $x : M^n \to \mathbb{R}^{n+1}$ is of finite geometrical type if M^n is complete in the induced metric, the Gauss map $G : M^n \to S^n$ extends continuously to a function $\overline{G} : \overline{M}^n \to S^n$ and the set $G^{-1}(S)$ has $H_{n-1}(G^{-1}(S)) = 0$ (this last condition occurs if $rank(x) \geq k$ and $\dim_H(x) < k-1$, or more generally, if $rank(x) \geq k$ and $H_{k-1}(S) = 0$).

As pointed out in the introduction, the conditions in the previous definition are satisfied, for example, by complete hypersurfaces with finite total curvature whose Gauss-Kronecker curvature $H_n = k_1 \dots k_n$ does not change of sign and vanish in a small set, as showed by [dCE]. Recall that a hypersurface $x : M^n \to \mathbb{R}^{n+1}$ has total finite curvature if $\int_M |A|^n dM < \infty$, $|A| = (\sum_i k_i^2)^{1/2}$, k_i are the principal curvatures. Then, there are examples satisfying the definition. With these observations, we now prove our Theorem B

Proof of Theorem B. To avoid unnecessary repetitions, we will only indicate the principal modifications needed in proof of Theorems A and B by stating "new" propositions, which are analogous to the previous ones, and making a few comments in their proofs. The details are left to reader.

Proposition 6.2 (proposition 3.1'). *Let $f : M^m \to N^n$ a C^1-map and $A \subset N$. If $H_s(A) = 0$, then $H_{s+n-rank(f)}(f^{-1}(A)) = 0$.*

Proof. It suffices to show that for any $p \in f^{-1}(A)$, there is an open set $U = U(p) \ni p$ such that $H_{s+n-r}(f^{-1}(A) \cap U) = 0$. However, if U is chosen as in proof of Proposition 3.1, we have $f^{-1}(A) \cap U \subset \varphi(\pi(A) \times \mathbb{R}^{n-r})$, where φ is a diffeomorphism, $r = rank(f)$ and π is the projection in first r variables. By Propositions 2.1, 2.2, $H_{s+n-r}(f^{-1}(A) \cap U) \leq C_1 \cdot H_{s+n-r}(\pi(A) \times \mathbb{R}^{n-r}) \leq C_1 \cdot C_2 \cdot H_s(A) = 0$, where C_1 depends only on φ and C_2 depends only on $(n-r)$. This finishes the proof. □

Proposition 6.3 (Proposition 3.2'). *Let $n \geq 3$ and F a closed subset of M^n such that $H_{n-1}(F) = 0$ then $M - F$ is connected. If $M^n = \mathbb{R}^n$ or $M^n = S^n$, F is compact and $H_{n-k}(F) = 0$ then $M^n - F$ is k-connected.*

Proof. First, if γ is a path in M^n from x to y, $x, y \notin F$, we can suppose γ a smooth simple curve. So, there is a compact tubular neighborhood $\mathcal{L} = \gamma \times D^{n-1}$ of γ. Since $dim(\mathcal{L}_p) = n-1$, $F \cap \mathcal{L}_p$ has Lebesgue measure 0 for any p. Thus, using that F is closed and γ is compact, we obtain some arbitrarily small vector v such that $(\gamma \times v) \cap F = \emptyset$. Then, $M^n - F$ is connected.

Second, if $[\Gamma] \in \pi_i(\mathbb{R}^n - F)$, $i \leq k$ is a homotopy class and Γ is a smooth representative, define $f : \Gamma \times F \to S^{n-1}$, $f(x,y) = (x-y)/||x-y||$. Following the proof of Proposition 3.2, f is Lipschitz. Now, since $H_{n-k-1}(F) = 0$, we have, by Proposition 2.2, $H_{n-1}(\Gamma \times F) = 0$. Thus, Proposition 2.1 implies $H_{n-1}(f(\Gamma \times F)) = 0$. This concludes the proof. $\square$

Lemma 6.4 (Lemma 3.4'). *Suppose that $\Gamma \in \pi_i(M^n)$ is Lipschitz and $K \subset M^n$ is compact, $H_{n-i}(K) = 0$. Then there are diffeomorphisms h of M, arbitrarily close to identity map, such that $h(\Gamma) \cap K = \emptyset$.*

Proof. If Γ is Lipschitz and Γ lies in a parametrized neighborhood, we can take $F : \Gamma \times K \to \mathbb{R}^n$, $F(x, y) = x - y$ a Lipschitz function. Because $H_n(\Gamma \times K) = 0$, then $H_n(F(\Gamma \times K)) = 0$. In the general case we proceed as in proof of Lemma 3.4. Take, by compactness, a finite number of parametrized neighborhoods and apply the previous case. By finiteness of number of parametrized neighborhoods and using that K is compact, an induction argument achieves the desired diffeomorphisms h. This concludes the proof. $\square$

Returning to proof of Theorem B, observe that in Theorem A, we need $\overline{G} : \overline{M}^n - \overline{G}^{-1}(\widetilde{S}) \to S^n - \widetilde{S}$ to be a diffeomorphism, where $\widetilde{S} = S \cup \{\overline{G}(q_i) : i = 1, \ldots, k\}$. This remains true because $(*)$ $H_{k-[\frac{n}{2}]}(S) = 0$ implies $S^n - \widetilde{S}$ is $(n - 1 - k + [\frac{n}{2}])$-connected. In fact, this is a consequence of $(*)$, Proposition 6.3 and $\{p_i : i = 1, \ldots, k\}$ is finite ($p_i := \overline{G}(q_i)$). Moreover, $rank(x) \geq k$ imply, by Proposition 6.2, 6.3, $\overline{M} - G^{-1}(\widetilde{S})$ is connected. Indeed, these propositions says that $rank(x) \geq k \Rightarrow H_{n-[\frac{n}{2}]}(G^{-1}(S)) = 0$ and $H_{n-1}(G^{-1}(S)) = 0 \Rightarrow M - G^{-1}(S)$ is connected. However, if $\overline{G}^{-1}(\widetilde{S}) - (G^{-1}(S) \cup \{q_i : i = 1, \ldots, k\}) := A$, then, for all $x \in A$, $(**)$ $\det D_x G \neq 0$. In particular, since $G(A) \subset \{p_i : i = 1, \ldots, k\}$, $(**)$ implies $\dim_H(A) = 0$. Then, $H_{n-[\frac{n}{2}]}(\overline{G}^{-1}(\widetilde{S})) = H_{n-[\frac{n}{2}]}(G^{-1}(S)) = 0$. Thus, by [WG], G is surjective and covering map (because it is proper and its jacobian never vanishes). In particular, by simple connectivity, G is a diffeomorphism. At this point, using the previous lemma and propositions, it is sufficient to follow proof of Theorem A, if $n = 3$, and proof of Theorem B, if $n \geq 4$, to obtain $\overline{M}^n \simeq S^n$. This concludes the proof. $\square$

For even dimensions, we can follow [BFM] and improve Theorem B :

Theorem 6.5 (Theorem C). *Let $x : M^{2n} \to \mathbb{R}^{2n+1}$, $n \geq 2$ an immersion of finite geometrical type with $H_{k-n}(S) = 0$, $rank(x) \geq k$. Then M^{2n} is topologically a sphere minus two points. If M^{2n} is minimal, M^{2n} is a $2n$-catenoid.*

For sake of completeness we present an outline of proof of Theorem C.

Outline of proof of Theorem C. Barbosa, Fukuoka, Mercuri define to each *end* p of M a geometric index $I(p)$ that is related with the topology of M by the formula (see theorem 2.3 of [BFM]):

$$\chi(\overline{M}^{2n}) = \sum_{i=1}^{k} (1 + I(p_i)) + 2\sigma m \tag{1}$$

where σ is the sign of Gauss-Kronecker curvature and m is the degree of $G : M^n \to S^n$. Now, the hypothesis $2n > 2$ implies (see [BFM]) $I(p_i) = 1, \forall\ i$. Since we know, by theorem 6.2, $\overline{M}^{2n}$ is a sphere, we have $2 = 2k + 2\sigma m$. But, it is easy to see that $m = deg(G) = 1$ because G is a diffeomorphism outside the singular set. Then, $2 = 2k + 2\sigma \Rightarrow k = 2,\ \sigma = -1$. In particular, M is a sphere minus two points.

If M is minimal, we will use the following theorem of Schoen : *The only minimal immersions, which are regular at infinity and have two ends, are the catenoid and a pair of planes.* The regularity at infinity in our case holds if the ends are embedded. However, $I(p) = 1$ means exactly this. So, we can use this theorem in the case of minimal hypersurfaces of finite geometric type. This concludes the outline of proof. $\square$

Remark 6.6. We can extend theorem A in a different direction (without mention of $rank(x)$). In fact, using only that G is Lipschitz, it suffices assume that $H_{n-[\frac{n}{2}]}(C) = 0$ (C is the set of points where the Gauss-Kronecker curvature vanishes). This is essentially the hypothesis of Barbosa, Fukuoka and Mercuri [BFM]. We prefer to state Theorems B and C as before since the classical theorems concerning estimates for Hausdorff dimension (Morse-Sard, Moreira) deal only with the critical values S and, in particular, our Corollary 2.4 will be more difficult if the hypothesis is changed to $H_1(C) = 0$ for some immersion $x : M^3 \to \mathbb{R}^4$ (although, in this assumption, we have no problems with $rank(x)$, i.e., this assumption has some advantages).

Remark 6.7. It would be interesting to know if there are examples of codimension 1 immersions with singular set which is not in the situation of Barbosa-Fukuoka-Mercuri and do Carmo-Elbert but instead satisfies our hypothesis. This question was posed to the second author by Walcy Santos during the Differential Geometry seminar at IMPA. In fact, these immersions can be constructed with some extra work. Some examples will be presented in another work to appear elsewhere.

7 Final Remarks

The Corollary 1.3 is motivated by Anderson's program for Poincaré Conjecture. In order to coherently describe this program, we briefly recall some facts about topology of 3-manifolds.

An attempt to better understand the topology of 3-manifolds (in particular, give an answer to Poincaré Conjecture) is the so called "Thurston Geometrization Conjecture". Thurston's Conjecture goes beyond Poincaré Conjecture (which is a very simple corollary of this conjecture). In fact, its goal is the understanding of 3-manifolds by decomposing them into pieces which could be "geometrizated", i.e., one could put complete locally homogeneous metric in each of this pieces. Thurston showed that, in three dimensions, there are exactly *eight* geometries, all of which are realizable. Namely, they are : the constant curvature spaces $\mathbb{H}^3$, $\mathbb{R}^3$, S^3, the products $\mathbb{H}^2 \times \mathbb{R}$, $S^2 \times \mathbb{R}$ and the twisted products $\widetilde{SL}(2, \mathbb{R})$, Nil, Sol (for details see [T]). Thurston proved his conjecture in some particular cases (e.g., for *Haken* manifolds). These particular cases are not easy. To prove the result Thurston developed a wealth of new geometrical ideas and machinery to carry this out. In a few words, Thurston's proof is made by induction. He decomposes the manifold M in an appropriate hierarchy of submanifolds $M_k = M \supset \cdots \supset$ union of balls $= M_0$ (this is possible if M is *Haken*). Then, if M_{i-1} has a metric with some properties, it is possible glue certain ends of M_{i-1} to obtain M_i. Moreover, by a deformation and isometric gluing of ends argument, M_i has a metric with the same properties of that from M_{i-1}. This is the most difficult part of the proof. So, the induction holds and M itself satisfies the Geometrization Conjecture.

Recently, M. Anderson [A] formulated three conjectures that imply Thurston's Conjecture. Morally, these three conjectures says that information about the *sigma* constant give us information about geometry and topology of 3-manifolds. We recall the definition of sigma constant. If $S(g) := \int_M s_g \, dV_g$ is the total scalar curvature functional (g is a metric with unit volume, i.e., $g \in \mathcal{M}_1$, dV_g is volume form determined by g and s_g is the scalar curvature) and $[g] := \{\widetilde{g} \in \mathcal{M}_1 : \widetilde{g} = \psi^2 g$, for some smooth positive function $\psi\}$ is the conformal class of g, then S is a bounded from below functional in $[g]$. Thus, we can define $\mu[g] = \inf_{g \in [g]} S(g)$ called *Yamabe* constant of $[g]$. An elementary comparison argument shows $\mu[g] \leq \mu(S^n, g_{can})$, where g_{can} is the canonical metric of S^n with unit 1 and positive constant curvature. Then makes sense to define the sigma constant :

$$\sigma(M) = \sup_{[g] \in \mathcal{C}} \mu[g] \tag{2}$$

where $\mathcal{C}$ is the space of all conformal classes. The sigma constant is a smooth invariant defined by a minimax principle (see equation 2). The first part of this minimax procedure was solved by Yamabe [Y]. More precisely, for any conformal class $[g] \in \mathcal{C}$, $\mu[g]$ is realized by a (smooth) metric $g_\mu \in [g]$ such that $s_{g_\mu} \equiv \mu[g]$ (a such g_μ is called *Yamabe* metric). The second part of this procedure is more difficult since it depends on the underlying topology. The sigma constant is important since is know that critical points of the scalar curvature functional S are Einstein metrics. But is not know if $\sigma(M)$ is a critical value of S (partially by non-uniqueness of Yamabe metrics). Then, if one show that is possible to realize the second part of minimax procedure and that $\sigma(M)$ is a critical value of S, we obtain the Geometrization Conjecture.

The approach above is very difficult. To see this, we remark that all of three Anderson's Conjectures are necessary to obtain the "Elliptization Conjecture" (the particular case of Thurston's Conjecture which implies Poincaré Conjecture). In others words, we have to deal with all cases of Thurston Conjecture to obtain Poincaré Conjecture. This inspires our definition of other minimax smooth invariants. The advantage in these invariants is that they do not requires construction of metrics with positive constant curvature. But the disadvantage is we always work extrinsically.

To finish the paper, we comment that there are many others attacks and approachs to Poincaré Conjecture. For example, see [G] for an accessible exposition of V. Poénaru's program and [P] for recent proof of one step of this program. In the other hand, some authors (e.g., Bing [B]) believes that only simple connectivity is not sufficient for a manifold be S^3. Recentely, Perelman has proposed a proof of the geometrization conjecture.

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References

[A] M. T. Anderson, Scalar Curvature and Geometrization Conjectures for 3-manifolds, *MSRI Publications*, Vol.30, 49–82, 1997

[B] R. H. Bing, Necessary and Sufficient Conditions that a 3-manifold be S^3, *Annals of Math.*, Vol.68, 17–37, 1958

[BFM] J. L. Barbosa, R. Fukuoka, F. Mercuri, Immersions of Finite Geometric Type in Euclidean Spaces. Annals of Global Analysis and Geometry, vol. 23 (2002), pp. 301-315.

[C] R. Cohen, The immersion conjecture for differentiable manifolds, *Annals of Math.*, Vol.122, 237–328, 1985

[dCE] M. do Carmo, M. Elbert, Complete hypersurfaces in Euclidean spaces with finite total curvature. Preprint (www.impa.br)

[F] M. H. Freedman, The topology of four-manifolds, *J. Diff. Geom.*, Vol.17, 357–453, 1982

[Fa] K. Falconer, Fractal Geometry : Mathematical foundations and applications, John Wiley & Sons Ltd., 1990

[G] D. Gabai, Valentin Poenaru's program for the Poincaré conjecture, *Geometry, topology & physics for Raoul Bott*, 139–166, Conf. Proc. Lecture Notes Geom. Topology, IV, Internat. Press, Cambridge, 1995

[M] E. Moise, Geometric Topology in Dimensions 2 and 3, Springer-Verlag, GTM, 47

[Mo] C. G. Moreira, Hausdorff measures and the Morse-Sard theorem, *Publ. Matematiques Barcelona*, Vol.45, 149–162, 2001

[P] V. Poénaru, Geometric simple connectivity in four-dimensional differential topology Part A. Preprint - Bures-sur-Yvette: IHES, 2001.-320 p

[R] D. Rolfsen, Knots and Links, Mathematics Lecture Series 7, Publish or Perish, Berkeley, Calif., 1976

[S] S. Smale, Generalized Poincaré's conjecture for dimension greater than four, *Annals of Math.*, Vol.74, 391–406, 1961

[T] W. P. Thurston, Three-Dimensional Geometry and Topology, Vol.1, Princeton Univ. Press, 1997

[W] H. Whitney, Geometric Integration Theory Princeton Univ. Press, 1957

[WG] J. Wolf, P. Griffiths, Complete maps and differentiable coverings, *Michigan Math. J.*, Vol.10, 253–255, 1963

[Y] H. Yamabe, On a deformation of Riemannian structures on compact manifolds, *Osaka Math. J.*, Vol.12, 21–37, 1960

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