HOFER-ZEHNDER CAPACITY AND HAMILTONIAN CIRCLE ACTIONS

LEONARDO MACARINI

Abstract. We introduce the Hofer-Zehnder G-semicapacity $c^G_{HZ}(M,\omega)$ of a symplectic manifold (M,ω) with respect to a subgroup $G \subset \pi_1(M)$ ($c_{HZ}(M,\omega) \leq c^G_{HZ}(M,\omega)$) and prove that if (M,ω) is tame and there exists an open subset $U \subset M$ admitting a Hamiltonian free circle action with order greater than two then U has bounded Hofer-Zehnder G-semicapacity, where $G \subset \pi_1(M)$ is the subgroup generated by the orbits of the action.

We give a lot of applications of this result. Using P. Biran's decomposition theorem, we prove the following: let (M^{2n},Ω) be a closed Kähler manifold ($n > 2$) with $[\Omega] \in H^2(M,\mathbb{Z})$ and Σ a complex hypersurface representing the Poincaré dual of $k[\Omega]$, for some $k \in \mathbb{N}$. Suppose either that Ω vanishes on $\pi_2(\Sigma)$ or that $k > 2$. Then there exists a decomposition of $M \setminus \Sigma$ into an open dense subset with finite Hofer-Zehnder G-semicapacity and an isotropic CW-complex, where $G \subset \pi_1(M\setminus\Sigma)$ is the subgroup generated by the obvious circle action on the normal bundle of Σ. Moreover, we prove that if (M,Σ) is subcritical then $M\setminus\Sigma$ has finite Hofer-Zehnder G-semicapacity.

We also show that given a hyperbolic surface M and TM endowed with the twisted symplectic form $\omega_0 + \pi^*\Omega$, where Ω is the area form on M, then the Hofer-Zehnder G-semicapacity of the domain bounded by the hypersurface of kinetic energy k minus the zero section M_0 is finite iff $k \leq 1/2$, where $G \subset \pi_1(TM \setminus M_0)$ is the subgroup generated by the fibers of SM.

Finally, we will consider the problem of the existence of periodic orbits on prescribed energy levels for magnetic flows. We prove that given any weakly exact magnetic field Ω on any compact Riemannian manifold M then there exists a sequence of contractible periodic orbits of energy arbitrarily small, extending a previous result of L. Polterovich.

1. Introduction

The problem of the existence of periodic orbits for Hamiltonian systems has a very far and rich history in the development of Mathematics and Physics. In fact, some of the major contributions on this question include the celebrated Poincaré-Birkhoff theorem on the existence of periodic orbits for area preserving twist maps on the annulus and the Morse-Lyusternik-Schnirelman theory on the existence of closed geodesics on compact Riemannian manifolds.

Recently, a special class of symplectic invariants was introduced by H. Hofer and E. Zehnder [22] as a link between rigidity phenomena of symplectic mappings and the

Date: current version: May 2002.

This work was partially supported by FAPERJ and Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Brazil.

existence of periodic orbits provided by classical variational methods. More precisely, they introduced an axiomatic definition of symplectic capacities in the following way:

Definition 1.1. Consider the class of all symplectic manifolds $\{(M,\omega)\}$ of fixed dimension equal to 2n. A symplectic capacity is a map $(M,\omega) \mapsto c(M,\omega)$ that associate to (M,ω) a non negative number or ∞ and that satisfies the following axioms:

- **Monotonicity:** if there exists a symplectic embedding $\varphi : (M,\omega) \to (N,\tau)$ then
$$c(M,\omega) \leq c(N,\tau);$$
- **Conformality:** $c(M,\alpha\omega) = |\alpha| c(M,\omega)\ \forall \alpha \in \mathbb{R} \setminus \{0\}$;
- **Non-triviality:** $c(B^{2n}(r),\omega_0) = c(Z^{2n}(r),\omega_0) = \pi r^2$,

where $B^{2n}(r) = \{(x,y) \in \mathbb{R}^{2n}; \sum_{i=1}^n x_i^2 + y_i^2 \leq r\}$ is the ball of radius r and $Z^{2n}(r) = \{(x,y) \in \mathbb{R}^{2n}; x_1^2 + y_1^2 \leq r\}$ is the cylinder over the (x_1,y_1)-plane.

A distinguished symplectic capacity was introduced by Hofer and Zehnder and is directly related to the question of the existence of periodic orbits on prescribed energy levels of Hamiltonian flows:

Definition 1.2. Given a symplectic manifold (M,ω) and an open subset $U \subset M$, define the Hofer-Zehnder capacity of U by
$$c_{HZ}(U,\omega) = \sup_{H \in \mathcal{H}_a(U,\omega)} \max H,$$
where $\mathcal{H}_a(U,\omega)$ is the set of *admissible Hamiltonians* H defined on U, that is,

- $H \in \mathcal{H}(U,\omega) \subset C^\infty(M,\mathbb{R})$, where $\mathcal{H}(U,\omega)$ is the set of *pre-admissible Hamiltonians* defined on U, that is, H satisfies the following properties: $0 \leq H \leq m(H) := \max H$, there exist an open set $V \subset U$ such that $H|_V \equiv 0$ and a compact set $K \subset U \setminus \partial U$ satisfying $H|_{U\setminus K} \equiv m(H)$;
- every non-constant periodic orbit of X_H has period greater than 1.

Intuitively, this capacity measure the sufficient oscillation for a pre-admissible Hamiltonian to have fast periodic orbits, that is, of period less than one. It is easy to see that it satisfies the monotonicity and conformality properties. Remarkably, by making use of classical variational methods, Hofer and Zehnder [22] showed that it also satisfies the non-triviality axiom, giving, in particular, a dynamical new proof of Gromov's nonsqueezing theorem [17].

Moreover, this symplectic invariant has important consequences on the question of the existence of periodic orbits on prescribed energy levels of Hamiltonian systems. In fact, it is easy to prove that if a symplectic manifold (M,ω) has *bounded Hofer-Zehnder capacity*, that is, if $c_{HZ}(U,\omega) < \infty$ for every open subset $U \subset M$ with compact closure, then given any Hamiltonian $H : M \to \mathbb{R}$ with compact energy levels, there exists a dense subset $\Sigma \subset H(M)$ such that for every $c \in \Sigma$ the energy hypersurface $H^{-1}(c)$ has a periodic solution.

This result was improved by M. Struwe in [38] showing that if the energy hypersurfaces bound a compact symplectic submanifold then Σ has total Lebesgue measure. Note that as an immediate consequence we have a proof of the Weinstein conjecture

[40] for manifolds with bounded Hofer-Zehnder capacity, since we can construct in a neighborhood of a contact hypersurface S a distinguished parametrized family of hypersurfaces with equivalent flows.

Unfortunately, it is, in general, a difficult task to determine when a symplectic manifold has bounded Hofer-Zehnder capacity. For $(\mathbb{R}^{2n}, \omega_0)$, where ω_0 denotes the canonical symplectic form, it follows by the Hofer-Zehnder theorem [22]. For surfaces, it was proved by K. Siburg [37], that the Hofer-Zehnder capacity of an open subset U coincides with the area of U. However, the situation is dramatically different for higher dimensional manifolds. In fact, Hofer and Viterbo showed in [20] that

$$c_{HZ}(M \times D^2(r), \omega_M \oplus \omega_0) = \pi r^2,$$

where (M, ω_M) is any closed symplectic manifold and $D^2(r)$ is the two dimensional disk of radius $r > 0$, with r satisfying the condition $r \leq m(M, \omega_M)$, where $m(M, \omega_M)$ is the *index of rationality* of (M, ω_M), that is,

$$m(M, \omega_M) = \inf\left\{ \int_{[u]} \omega_M; \quad [u] \in \pi_2(M) \text{ satisfies } \int_{[u]} \omega_M > 0 \right\}.$$

If the above set is empty, we define $m(M, \omega_M) = \infty$. Recall that a symplectic form ω_M is *weakly exact* iff $m(M, \omega_M) = \infty$.

This result was extended by G. Lu [27, 28] to noncompact symplectic manifolds satisfying certain conditions at the infinity. More precisely, he showed that

$$c_{HZ}(M \times D^2(r), \omega_M \oplus \omega_0) \leq \pi r^2,$$

where (M, ω_M) is a tame symplectic manifold. A symplectic manifold is called *tame* (or geometrically bounded) if there exists an almost complex structure J on P such that $\omega(\cdot, J\cdot)$ is a Riemannian metric whose sectional curvature is bounded and whose injectivity radius is bounded away from zero.

The closed symplectic manifolds are clearly tame; a product of two tame symplectic manifolds is also such a manifold. Moreover, it was proved in [27] that the cotangent bundle of a compact manifold endowed with any twisted symplectic form is tame. A *twisted symplectic structure* on a cotangent bundle T^*M is a symplectic form given by $\omega_0 + \pi^*\Omega$, where ω_0 is the canonical symplectic form, $\pi : T^*M \to M$ is the canonical projection and Ω is a closed 2-form on M.

Thus, using the monotonicity property, we can conclude that $(T^*(N \times S^1), \omega_0)$ has bounded Hofer-Zehnder capacity, where N is any compact manifold and ω_0 is the canonical symplectic form (because we can embed symplectically $T^*M \times T^*S^1$ into $T^*M \times \mathbb{R}^2$). In fact, by the same reason, we can prove that $(M \times T^*S^1, \omega_M \oplus \omega_0)$ has bounded Hofer-Zehnder capacity for any tame symplectic manifold M. It should be noted that it is necessary to consider product symplectic forms $\omega_M \oplus \omega_0$. In fact, there is an example of Herman-Zehnder [18, 41] where they construct a symplectic form on $T^3 \times \mathbb{R} \simeq T^2 \times T^*S^1$ with unbounded Hofer-Zehnder capacity. Note that any symplectic form on T^2 is weakly exact.

Others know examples of cotangent bundles with bounded Hofer-Zehnder capacity are (T^*M, ω_0), where M is a compact simply connected manifold [39], and T^*T^n

endowed with any twisted symplectic form $\omega_0 + \pi^*\Omega$, where Ω is a closed 2-form on T^n and $\pi : T^*T^n \to T^n$ is the canonical projection. This last result was proved by J. Jiang [23] for $\Omega \equiv 0$ and by V. Ginzburg and E. Kerman [16] for the general case.

We will consider here a reformulation of the Hofer-Zehnder capacity by considering periodic orbits whose homotopy class is contained in a given subgroup G of $\pi_1(M)$. More precisely, we define the Hofer-Zehnder semicapacity as follows; it may be considered as a generalization of the π_1-sensitive Hofer-Zehnder capacity defined by M. Schwarz [36] and Lu [27, 28] and the relative symplectic capacity introduced by P. Biran, L. Polterovich and D. Salamon [6]:

Definition 1.3. Given a symplectic manifold (M, ω) and a subgroup $G \subset \pi_1(M)$ define the *Hofer-Zehnder G-semicapacity* of a subset $U \subset M$ by

$$c^G_{HZ}(U, \omega) = \sup_{H \in \mathcal{H}^G_a(U,\omega)} \max H,$$

where $\mathcal{H}^G_a(U, \omega)$ is the set of *G-admissible Hamiltonians* defined on U, that is,

- $H \in \mathcal{H}(U, \omega)$, that is, H is pre-admissible (see the definition 1.2);
- every nonconstant periodic orbit of X_H whose homotopy class belongs to G has period greater than 1.

We call it a *semicapacity* because given a symplectic embedding $\psi : (N, \omega_N) \to (M, \omega_M)$ such that $\dim N = \dim M$ it cannot be expected, in general, that $c^G_{HZ}(N, \omega_N) \le c^{\psi_* G}_{HZ}(M, \omega_M)$, where $\psi_* : \pi_1(N) \to \pi_1(M)$ is the induced homomorphism on the fundamental group. However, we can state the following *weak monotonicity property* (for a proof see the Section (2)): given a symplectic embedding $\psi : (N, \omega_N) \to (M, \omega_M)$ such that $\dim N = \dim M$ then

$$c^{\psi_*^{-1}(\psi_* G)}_{HZ}(N, \omega_N) \le c^{\psi_* G}_{HZ}(M, \omega_M).$$

In particular, if ψ_* is injective we have that $c^G_{HZ}(N, \omega_N) \le c^{\psi_* G}_{HZ}(M, \omega_M)$. Note that obviously,

$$c_{HZ}(U, \omega_M) = c^{\pi_1(M)}_{HZ}(U, \omega_M) \le c^G_{HZ}(U, \omega_M),$$

for every open subset $U \subset M$ and any subgroup $G \subset \pi_1(M)$. Moreover, it is easy to see that if the Hofer-Zehnder G-semicapacity is bounded then there are periodic orbits with homotopy class in G on a dense subset of energies for every proper Hamiltonian.

We remark that the periodic orbits in $M \times D^2(r)$ given by the theorems of Hofer-Viterbo and Lu are *contractible*. Thus, in terms of symplectic semicapacities, their theorems state the stronger result that

$$c^0_{HZ}(M \times D^2(r), \omega_M \oplus \omega_0) \le \pi r^2$$

for any tame symplectic manifold (M, ω_M) and $0 < r \le m(M, \omega_M)$. By the weak monotonicity property, we can conclude then that $c^{\pi_1(S^1)}_{HZ}(M \times T^*S^1, \omega_M \oplus \omega_0)$ is bounded, where $\pi_1(S^1) \subset \pi_1(M \times T^*S^1)$ is the obvious subgroup generated by the circle in the cylinder T^*S^1. In fact, the kernel of the induced homomorphism on the fundamental group of the symplectic embedding of $M \times T^*S^1$ into $M \times \mathbb{R}^2$ is given by $\pi_1(S^1)$.

The aim of the first part of this paper is to generalize this result in the following way: note that $M \times T^*S^1$ has a natural Hamiltonian free circle action with Hamiltonian given by the angle form $d\theta$ viewed here as a function on $TS^1 \simeq T^*S^1$. We will consider the following:

Question: Is the existence of a Hamiltonian free circle action sufficient to ensure the boundedness of the Hofer-Zehnder semicapacity (with respect to the subgroup of $\pi_1(M)$ generated by the orbits of the action)?

Before we state our first result we need a previous topological definition related to free circle actions:

Definition 1.4. Let $\varphi : N \times S^1 \to N$ be a free circle action on an open subset N of a manifold M. We define the *order of the action* φ as the order of the cyclic subgroup of $\pi_1(M)$ generated by the homotopy class of the orbits of φ.

Now, we are able to state our first theorem. In what follows we will denote the oscillation of a Hamiltonian H by $\|H\| := \sup H - \inf H$.

Theorem 1.1. *Let (M, ω_M) be a tame symplectic manifold and $N \subset M$ an open subset that admits a free Hamiltonian circle action $\varphi : N \times S^1 \to N$ (with period equal to 2π) given by the Hamiltonian $H_1 : N \to \mathbb{R}$. Suppose that the order of this action n_φ (considered as an action on M) satisfies $n_\varphi > 2$. Given an open subset $U \subset N$ with compact closure, suppose that*

$$m(M, \omega_M) > \|H_1|_U\| \left(1 + \frac{2}{n_\varphi - 2}\right).$$

Then, we have that

$$c_{HZ}^{G_\varphi}(U, \omega_M) \leq 2\pi \|H_1|_U\| \left(1 + \frac{2}{n_\varphi - 2}\right) < \infty,$$

where $G_\varphi \subset \pi_1(M)$ is the subgroup generated by the orbits of φ.

We remark that the circle action above needs to be Hamiltonian. In fact, the example of Herman-Zehnder [18, 41] has unbounded Hofer-Zehnder capacity and it is easy to see that it admits a free symplectic circle action with infinite order. Before we explain the main ideas involved in the proof of the Theorem (1.1), let us state some corollaries and applications.

Corollary 1.1. *Let (M, ω_M) be a tame symplectic manifold. Suppose that M admits a free Hamiltonian circle action generated by $H_1 : M \to \mathbb{R}$ whose Euler class $[e] \in H^2(M/S^1, \mathbb{R})$ satisfies $[e]|_{\pi_2(M/S^1)} = 0$. Then, for any open subset $U \subset M$ with compact closure, we have that*

$$c_{HZ}^{G_\varphi}(U, \omega_M) \leq 2\pi \|H_1|_U\| < \infty,$$

provided that $m(M, \omega_M) > \|H_1|_U\|$.

In fact, it can be proved that given a free circle action on a manifold M such that the Euler class $[e] \in H^2(M/S^1, \mathbb{R})$ satisfies $[e]|_{\pi_2(M/S^1)} = 0$ then the order of the action is infinite (see the Section (3)).

Now, consider the finite subgroup $\mathbb{Z}_n \subset S^1$. Note that, since the circle action is Hamiltonian, the induced action of $\mathbb{Z}_n$ on M is symplectic. Since the action is free, the quotient manifold $M/\mathbb{Z}_n$ has a induced symplectic structure ω_M^n defined uniquely by the property that the pullback of ω_M^n by the quotient projection is ω_M. Moreover, $M/\mathbb{Z}_n$ has an induced free Hamiltonian circle action (with period $2\pi/n$) with order greater than or equal to n. Consequently, we have the following immediate corollary.

Corollary 1.2. *Let (M, ω_M) be a tame symplectic manifold. Suppose that M admits a free Hamiltonian circle action generated by $H_1 : M \to \mathbb{R}$ and let $U \subset M$ be an open subset with compact closure. If $n \geq 3$ and $m(M, \omega_M) \geq (1/n)\|H_1|_U\|$, then*

$$c_{HZ}^{G_\varphi}(\tau_n(U), \omega) \leq (2\pi/n)\|H_1|_U\|(1 + 2/(n-2)) < \infty,$$

where $\tau_n : M \to M/\mathbb{Z}_n$ is the quotient projection, ω is the induced symplectic form on $M/\mathbb{Z}_n$ and φ is the induced circle action.

There are a lot of non-trivial examples of symplectic manifolds admitting such free Hamiltonian circle actions. The first one that we will consider is the following: let M be a compact manifold admitting a free circle action. Then the lift of this action to the cotangent bundle of M is Hamiltonian with respect to the canonical symplectic form. In fact, it is Hamiltonian with respect to any twisted symplectic form $\omega_0 + \pi^*\Omega$, where Ω is a closed two-form on M given by the pullback of a closed two-form Ω_{M/S^1} on M/S^1. Thus, we have the following corollaries:

Corollary 1.3. *Let M be a compact manifold admitting a free circle action φ whose Euler class $[e] \in H^2(M/S^1, \mathbb{R})$ satisfies $[e]|_{\pi_2(M/S^1)} = 0$ and Ω a closed two-form on M given by the pullback of a closed two-form Ω_{M/S^1} on M/S^1 . Then the cotangent bundle $(T^*M, \omega_0 + \pi^*\Omega)$ has bounded Hofer-Zehnder G_φ-semicapacity.*

Corollary 1.4. *Let M be a compact manifold such that $\pi_2(M) = 0$ and P the total space of a circle bundle $S^1 \xrightarrow{\varphi} P \longrightarrow M$. Then (T^*P, ω_0) has bounded Hofer-Zehnder G_φ-semicapacity.*

Corollary 1.5. *Let M be a compact manifold admitting a free circle action φ. Then $(T^*(M/\mathbb{Z}_n), \omega_0 + \pi^*\Omega)$ has bounded Hofer-Zehnder G_φ-semicapacity for any $n \geq 3$ and any closed 2-form $\Omega = \tau_n^*\Omega_{M/S^1}$, where $\tau_n : M/\mathbb{Z}_n \to M/S^1$ is the quotient projection.*

It is easy to construct specific non-trivial examples of such circle bundles. For example, the lens spaces $S^3/\mathbb{Z}_n$ $(n > 2)$ over S^2, the Heisenberg manifold over the 2-torus and, more generally, any 2-step nilmanifold [33]. In fact, it can be proved that the group of equivalence classes of circle bundles over a manifold M is isomorphic to $H^2(M, \mathbb{Z})$ [25]. Thus, any non-trivial element of $H^2(M, \mathbb{Z})$ for a manifold M such that $\pi_2(M) = 0$ corresponds to a non-trivial circle bundle with infinite order. The Heisenberg manifold, for example, corresponds to the cohomology class of the area form on T^2.

In the Section (4), we show some applications of the Theorem (1.1) where the circle action is not a lifted action. We will consider circle actions given by the magnetic flow associated to monopoles on surfaces.

More precisely, let (M, g) be a closed Riemannian manifold, Ω a closed 2-form on M and TM endowed with the twisted symplectic form $\omega_0 + \pi^*\Omega$, where ω_0 is the pullback of the canonical symplectic form via the Riemannian metric. When M is a hyperbolic surface and Ω is the area form, this manifold has remarkable properties and is related with McDuff's example of a symplectic manifold with disconnected contact-type boundary [30] (see the Section (4), where we give a simple construction of such a manifold). The Hamiltonian flow ϕ given by the kinetic energy $H(x, v) = (1/2)g_x(v, v)$ with respect to $\omega_0 + \pi^*\Omega$ is called the magnetic flow generated by the Riemannian metric g and the magnetic field Ω.

It can be show that, when M is a surface of constant negative curvature and Ω is the area form, the magnetic flow defines a free circle action with infinite order on certain subsets of TM. Using this fact and the expanding completion of convex manifolds developed by Eliashberg and Gromov [11], we can prove the following theorem, proving, in particular, a conjecture of V. Ginzburg [15] in dimension two (taking out the zero section). This result also gives a characterization of the Mañé's critical value associated to the magnetic flow in terms of symplectic semicapacities (see [8] and [9]).

Theorem 1.2. *Let M be a surface of genus $g \geq 2$ endowed with the hyperbolic metric g_0 and the area form Ω_{g_0} (with normalized area equal to 1). Let $H_{g_0} : TM \to \mathbb{R}$ be the usual Hamiltonian given by the kinetic energy $H_{g_0}(x, v) = (1/2)g_0(v, v)$ and $k \in \mathbb{R}$ a positive real number. Consider the open subset $U_k = \cup_{0<\mu<k} H_{g_0}^{-1}(\mu)$. Then,*

$$c_{HZ}^G(U_k, \omega_0 + \pi^*\Omega_{g_0}) < \infty,$$

if and only if $k \leq 1/2$, where $G \subset \pi_1(TM \setminus M_0)$ is the subgroup generated by the fibers of the unitary bundle and M_0 is the zero section. In particular, the periodic orbits have contractible projection on M.

Remark. Actually, we prove that $c_{HZ}^G(U_k, \omega_0 + \pi^*\Omega_{g_0}) \leq k$ for $0 < k \leq 1/2$.

By Moser's theorem, we have the following corollary:

Corollary 1.6. *Let M be a surface of genus $g \geq 2$ and Ω a symplectic form on M. Then there exists a Riemannian metric g on M such that*

$$c_{HZ}^G(U_k, \omega_0 + \pi^*\Omega) < \infty,$$

if and only if $k \leq 1/2$, where $U_k = \cup_{0<\mu<k} H_g^{-1}(\mu)$ and $G \subset \pi_1(TM \setminus M_0)$ is the subgroup generated by the fibers of the unitary bundle as above.

Remark. When M is the 2-torus and Ω is given by a symplectic form, it is well know that

$$c_{HZ}^0(U, \omega_0 + \pi^*\Omega) < \infty,$$

for every open subset $U \subset TT^2$ with compact closure. In fact, it is easy to prove that $(TT^2, \omega_0 + \pi^*\Omega)$ is symplectomorphic to $(T^2 \times \mathbb{R}^2, \Omega \oplus \sigma)$, where σ is the canonical symplectic form on $\mathbb{R}^2$ [16]. This result can also be obtained by the Theorem (1.1).

Now, we will use the Theorem (1.1) to show that for closed Kähler manifolds (M, Ω) the possible obstruction to the boundedness of the Hofer-Zehnder capacity is a very thin subset of M which, in many situations, can be explicitly described.

To prove it, we will need the following nice decomposition result of P. Biran [3] which enable us to represent a Kähler manifold as a disjoint union of two basic components whose symplectic nature is very standard:

Theorem 1.3 (P. Biran [3])**.** *Let (M^{2n}, Ω) be a closed Kähler manifold with $[\Omega] \in H^2(M, \mathbb{Z})$ and $\Sigma \subset M$ a complex hypersurface whose homology class $[\Sigma] \in H_{2n-2}(M)$ is the Poincaré dual to $k[\Omega]$ for some $k \in \mathbb{N}$. Then, there exists an isotropic CW-complex $\Delta \subset (M, \Omega)$ whose complement - the open dense subset $(M \setminus \Delta, \Omega)$ - is symplectomorphic to a standard symplectic disc bundle $(E_0, \frac{1}{k}\omega_{can})$ modeled on the normal bundle N_Σ of Σ in M and whose fibers have area $1/k$.*

The symplectic form ω_{can} is given by

$$\omega_{\text{can}} = k\pi^*(\Omega|_\Sigma) + d(r^2\alpha),$$

where $\pi : E_0 \to \Sigma$ is the bundle projection, r is the radial coordinate using a Hermitian metric $\|\cdot\|$ and α is a connection form on E such that $d\alpha = -k\pi^*(\Omega|_\Sigma)$. The form $\frac{1}{k}\omega_{\text{can}}$ is uniquely characterized by the requirements that its restriction to the zero section Σ equals $\Omega|_\Sigma$, the fibers of $\pi : E_0 \to \Sigma$ are symplectic and have area $1/k$ and ω_{can} is invariant under the obvious circle action along the fibers. It is called standard because the symplectic type of $(E_0, \omega_{\text{can}})$ depends only on the symplectic type of $(\Sigma, \Omega|_\Sigma)$ and the topological type of the normal bundle N_Σ [3].

Let us recall that the pair (M, Σ) is called *subcritical* [4, 5] if $M \setminus \Sigma$ is a subcritical Stein manifold, that is, if there exists a plurisubharmonic Morse function φ on $M \setminus \Sigma$ such that $\text{index}_p(\varphi) < \dim_{\mathbb{C}} M$ for every critical point p of φ. It is equivalent to the condition that the dimension of Δ (that is, the maximal dimension of the cells of Δ) is strictly less than n.

Theorem 1.4. *Let (M^{2n}, Ω) be a closed Kähler manifold ($n > 2$) with $[\Omega] \in H^2(M, \mathbb{Z})$ and $\Sigma \subset M$ a complex hypersurface whose homology class $[\Sigma] \in H_{2n-2}(M)$ is the Poincaré dual to $k[\Omega]$ for some $k \in \mathbb{N}$. Then there exists an open dense subset E of M whose complement is an isotropic CW-complex $\Delta \subset M$ such that if $[\Omega]|_{\pi_2(\Sigma)} = 0$ then*

$$c^G_{HZ}(E \setminus \Sigma, \Omega) \leq 1/k,$$

where $G \subset \pi_1(M \setminus \Sigma)$ is the subgroup generated by the orbits of the obvious S^1-action on $N_\Sigma \setminus \Sigma$. If $[\Omega]|_{\pi_2(\Sigma)} \neq 0$ and $k > 2$, we have that

$$c^G_{HZ}(E \setminus \Sigma, \Omega) \leq \frac{1}{k} + \frac{2}{k^2 - 2k}.$$

Moreover, if (M,Σ) is subcritical and $[\Omega]|_{\pi_2(\Sigma)} = 0$, then

$$c^G_{HZ}(M \setminus \Sigma, \Omega) \leq 1/k.$$

Remark. Under the hypothesis that either $\dim_{\mathbb{R}} M \leq 6$ or $[\Omega]|_{\pi_2(M)} = 0$, it was proved by P. Biran [3] that the *Gromov capacity* of E satisfies $c_G(E,\Omega) \leq 1/k$. These assumptions are not necessary as was proved recently by G. Lu [29].

When (M,Σ) is subcritical we do not consider the case $k > 2$ because, as was proved in [12], every subcritical polarization has degree 1.

We remark that the CW-complex Δ above is given by the union of the stable manifolds of the gradient flow of a plurisubharmonic function φ defined on $M \setminus \Sigma$ such that it can be explicitly computed in many examples [3].

The next theorem shows that we can get a similar result of the Theorem (1.4) in the neighborhood of a symplectic hypersurface of Donaldson type [10]:

Theorem 1.5. *Let (M^{2n},Ω) be a closed symplectic manifold ($n > 2$) with $[\Omega] \in H^2(M,\mathbb{Z})$ and $\Sigma \subset M$ a symplectic hypersurface whose homology class $[\Sigma] \in H_{2n-2}(M)$ is the Poincaré dual to $k[\Omega]$. Then there exists $k_0 \geq 2$ such that if $k > k_0$ there exists an open neighborhood V of Σ such that if $[\Omega]|_{\pi_2(\Sigma)} = 0$ then*

$$c^G_{HZ}(V \setminus \Sigma, \Omega) \leq 1/k,$$

where $G \subset \pi_1(M \setminus \Sigma)$, as above, is the subgroup generated by the orbits of the obvious S^1-action on $N_\Sigma \setminus \Sigma$. If $[\Omega]|_{\pi_2(\Sigma)} \neq 0$, we have that

$$c^G_{HZ}(V \setminus \Sigma, \Omega) \leq \frac{1}{k} + \frac{2}{k^2 - 2k}.$$

It is important to remark that an extension of Biran's result for symplectic hypersurfaces considered above (which is generally expected to remains valid) will enable us to extend the Theorem (1.4) for general symplectic manifolds.

The essential idea in the proof of the Theorem (1.1) is to relate the Hofer-Zehnder capacity of a symplectic manifold endowed with a Hamiltonian circle action with the Hofer-Zehnder capacity of its reduced symplectic manifold (in the sense of Marsden-Weinstein) with respect to this action.

More precisely, we construct a Hamiltonian circle action on $P := M \times T^*S^1$ whose reduced symplectic manifold $(J^{-1}(\mu)/S^1, \sigma_\mu)$ is given by (M, ω_M). Then, given a pre-admissible Hamiltonian $H \in \mathcal{H}(U, \omega_M)$ we construct a S^1-invariant pre-admissible Hamiltonian $\widehat{H}$ on P whose reduced dynamics coincides with the Hamiltonian vector field of H on M. Thus, we can apply the results of Hofer-Viterbo and Lu to get a periodic orbit for $X_{\widehat{H}}$ and so, by reduction, a periodic orbit for X_H. The essential step is to show the non-triviality of the projected closed orbit and it is here that the hypothesis on the order of the action plays an essential role. The idea is to use the condition on the homotopy of the periodic orbit together with an upper bound on the period of the orbit.

The hypothesis on the order of the circle action seems to be artificial, but we do not know how to avoid this. An approach to this problem is the following: let M be a tame symplectic manifold endowed with a Hamiltonian free circle action. By the Corollary (1.2), we know that the finite quotient $M/\mathbb{Z}_n$ endowed with the induced symplectic form has bounded Hofer-Zehnder semicapacity. This naturally raises the following:

Question: Given a finite symplectic covering $(M, \omega_M) \to (N, \omega_N)$ is it true that the Hofer-Zehnder (semi)capacity of N is bounded if and only if the Hofer-Zehnder (semi)capacity of M is bounded?

It is true for surfaces and also for the cotangent bundle of the torus T^n because we can estimate such a capacity via two-dimensional methods. By now, we can just get the following result:

Theorem 1.6. *Let N be a compact manifold admitting a free circle action $S^1 \xrightarrow{\xi} N \xrightarrow{\tau} N/S^1$, β a connection form for this bundle and $H_1 : T^*N \to \mathbb{R}$ the Hamiltonian corresponding to the lifted action φ of ξ. Then, given $n \leq n_\varphi$ we have that*

$$c_{HZ}^{G_\varphi}(H_1^{-1}(-n/4, n/4), \omega_0 + \pi^* d\beta) < \infty.$$

Moreover, the same result holds if we take either $H^{-1}(0, n/2)$ or $H^{-1}(-n/2, 0)$.

Note that this result generalizes the Corollary (1.5). In fact, if $n_\varphi \geq 3$, we have that a neighborhood of $H_1^{-1}(1)$ has finite Hofer-Zehnder G_φ-semicapacity and consequently (since $\psi(q,p) = (q, p + \beta_q)$ defines a symplectomorphism between ω_0 and $\omega_0 + \pi^* d\beta$) a neighborhood V of the zero section satisfies $c_{HZ}^{G_\varphi}(V, \omega_0) < \infty$ and so (T^*M, ω_0) has bounded Hofer-Zehnder G_φ-semicapacity.

In the second part of this paper, we will consider the problem of the existence of periodic orbits on prescribed energy levels for the special class of Hamiltonian dynamical systems given by the magnetic flows. This problem was first considered by V. Arnold [1, 2] and S. Novikov [32] and for a discussion of the results in this area we refer to [9, 14].

It was proved by L. Polterovich [34] that for every nontrivial weakly exact magnetic field on a manifold whose Euler characteristic vanishes, there exist non-trivial contractible closed orbits of the magnetic flow in a sequence of arbitrarily small energy levels. The proof uses the geometry of the Hofer's metric in the group of Hamiltonian diffeomorphisms and the fundamental fact that the displacement energy of the zero section of TM, with respect to the twisted symplectic form given by a non-vanishing magnetic field, is equal to zero. Recently, E. Kerman [24] gives the same result for magnetic fields given by symplectic forms.

We will prove here an extension of Polterovich's theorem for any manifold, without the assumption on the Euler characteristic.

Theorem 1.7. *Let M be any closed Riemannian manifold and Ω a non-trivial weakly exact magnetic field. Then there exists a sequence of arbitrarily small energy levels containing contractible periodic orbits.*

The main idea in the proof is similar to that of the Theorem (1.1). In fact, we consider a lift of the magnetic flow to $T^*M \times T^*S^1$ and use the topological condition on the periodic orbit to ensure the non-triviality of the projected periodic orbit by symplectic reduction. The details are given in the Section (8).

Acknowledgements: I am very grateful to Gabriel Paternain for many helpful comments and suggestions and to the Centro de Investigación en Matemática, Guanajuato, Mexico for the hospitality while part of this work was in progress. I am also grateful to Dusa McDuff for useful comments.

2. Proof of the Theorem (1.1)

Before we begin the proof of the Theorem (1.1) we need some previous results.

Proposition 2.1. *Let (M,ω_M) be a tame symplectic manifold and $P := M \times T^*S^1$ endowed with the symplectic form $\omega_P := \omega_M \oplus \omega_0^{S^1}$, where $\omega_0^{S^1}$ is the canonical symplectic form on T^*S^1. Let $U \subset P$ be an open subset with compact closure and suppose that $m(M,\omega_M) \geq \left| \int_{\pi_2(U)} \omega_0^{S^1} \right|$, where $\pi_2 : P \to T^*S^1$ is the projection onto the second factor. Then given a Hamiltonian $H \in \mathcal{H}(U,\omega_P)$, there exists a nonconstant periodic orbit γ of X_H with period*

$$T < T_{max}(U,H) := \frac{\left| \int_{\pi_2(U)} \omega_0^{S^1} \right|}{m(H)}.$$

Moreover, the homotopy class $[\gamma]$ of γ belongs to the subgroup $\pi_1(S^1) \subset \pi_1(P)$.

Proof. Since $\overline{U}$ is compact, there exists a positive constant $a > 0$ such that $\pi_2(U) \subset S^1 \times [-a/2, a/2] \subset T^*S^1$. Let ω_0 be the canonical symplectic form on $\mathbb{R}^2$ and consider the symplectomorphism $\phi : S^1 \times [-a,a] \to A := \{(x,y) \in \mathbb{R}^2; a \leq x^2 + y^2 \leq 5a\}$ given by

$$\phi(\theta, r) = (\sqrt{3a+2r}\sin\theta, \sqrt{3a+2r}\cos\theta).$$

Note that $Ker(Id,\phi)_* = \pi_1(S^1)$, where $(Id,\phi)_* : \pi_1(T^*M \times S^1 \times [-a,a]) \simeq \pi_1(T^*P) \to \pi_1(T^*M \times \mathbb{R}^2) \simeq \pi_1(T^*M)$ is the homomorphism induced on the fundamental group by the transformation given by the identity and ϕ in the first and second factors respectively.

We can suppose, without loss of generality, that $\phi(\pi_2(\overline{U})) \subset \mathbb{R}^2$ is a connected two-dimensional compact submanifold with boundary, so that there exists an open disk of radius R with L distinct points $(0 \leq L < \infty)$ $y_j \in B^2(R)$ and an orientation preserving diffeomorphism

$$\psi : \phi(\pi_2(U)) \to B^2(R) \setminus \{y_1, ..., y_L\}$$

such that

$$\left| \int_{\phi(\pi_2(U))} \omega_0 \right| = \left| \int_{B^2(R)\setminus\{y_1,...,y_L\}} \omega_0 \right| = \pi R^2.$$

From a theorem of Bacorogna and Moser (see Lemma 2.2 in [37]) this ψ can be required to be symplectic. Thus, consider the Hamiltonian $\bar{H} : M \times B^2(R) \to \mathbb{R}$

given by

$$\bar{H}(x,y) = \begin{cases} m(H) \text{ if } y = y_j \text{ for some } j = 1, ..., L \\ H(x, \phi^{-1}\psi^{-1}y) \text{ otherwise} \end{cases}$$

which is obviously C^∞, since $H|_{U \setminus K} = m(H)$, where $K \subset U$ is a compact subset such that $K \subset U \setminus \partial U$.

By the Theorem C of [27], $X_{\bar{H}}$ has a contractible periodic orbit $\bar{\gamma}$ with period

$$T < \frac{\pi R^2}{m(\bar{H})} = \frac{|\int_{\phi(\pi_2(U))} \omega_0|}{m(\bar{H})} = \frac{|\int_{\pi_2(U)} \omega_0^{S^1}|}{m(H)}.$$

Finally, note that, by the remark above, the periodic orbit γ of X_H given by $\gamma = (Id, \psi \circ \phi)^{-1}\bar{\gamma}$ has homotopy class $[\gamma]$ contained in $\pi_1(S^1) \subset \pi_1(P)$, since $\bar{\gamma}$ is contractible. □

Lemma 2.1. *If $\psi : (U, \omega_1) \to (M_2, \omega_2)$ is a symplectic embedding with $U \subset M_1$ then*

$$c_{HZ}^{\psi_*^{-1}(\psi_* G)}(U, \omega_1) \leq c_{HZ}^{\psi_* G}(M_2, \omega_2),$$

where $G \subset \pi_1(M_1)$ is a subgroup of the fundamental group of M_1 generated by closed curves in U.

Proof. Define the map $\psi_* : \mathcal{H}(U, \omega_1) \to C^\infty(M_2)$ by

$$\psi_*(H) = \begin{cases} H \circ \psi^{-1}(x) \text{ if } x \in \psi(U); \\ m(H) \text{ if } x \notin \psi(U). \end{cases}$$

It is clear that $\psi_*(\mathcal{H}(U, \omega_1)) \subset \mathcal{H}(\psi(U), \omega_2)$ and that $m(\psi_* H) = m(H)$.

Then it is sufficient to prove that $\psi_*(\mathcal{H}_a^{\psi_*^{-1}(\psi_* G)}(U, \omega_1)) \subset \mathcal{H}_a^{\psi_* G}(\psi(U), \omega_2)$. To do it, suppose, by contradiction, that there exists $H \in \mathcal{H}_a^{\psi_*^{-1}(\psi_* G)}(U, \omega_1)$ such that $X_{\psi_* H}^{\omega_2}$ has a nonconstant periodic orbit γ such that $[\gamma] \in \psi_* G$ and the period of γ is less than 1. Then the image of γ by ψ^{-1} (note that the image of γ is contained in $\psi(U)$, since $\psi_* H \equiv m(H)$ outside $\psi(U)$) is a periodic orbit of $X_H^{\omega_1}$ of the same period and

$$[\psi^{-1} \circ \gamma] \in \psi_*^{-1}(\psi_* G),$$

a contradiction. □

Proof of the Theorem (1.1): Let $P = N \times T^*S^1 \subset M \times T^*S^1$, where $M \times T^*S^1$ is endowed with the symplectic form $\omega_P = \omega_M \oplus \omega_0^{S^1}$, where $\omega_0^{S^1}$ is the canonical symplectic form on T^*S^1. Note that T^*S^1 has a natural Hamiltonian circle action given by the Hamiltonian $d\theta$ (where $d\theta$ is the angle form viewed here as a function on $T^*S^1 \simeq TS^1$). Let $n \in \mathbb{R}$ be such that $n > 2\|H_1|_U\|/(n_\varphi - 2)$ and $m(M, \omega_M) \geq \|H_1|_U\| + n$.

Consider the diagonal action $\rho : P \times S^1 \to P$ on P given by the Hamiltonian

$$J(x, \theta, d\theta) = (1/n)(H_1(x) + d\theta),$$

and let $\psi : P \to P$ be the diffeomorphism given by

$$\psi(x, \theta, d\theta) = (\varphi_\theta(x), \theta, d\theta - H_1(x)),$$

where $\varphi : N \times S^1 \to N$ is the flow generated by H_1. The following lemma is a straightforward computation.

Lemma 2.2. *The diffeomorphism ψ is a symplectomorphism with respect to ω_P. Moreover, $J \circ \psi = n^{-1} d\theta$.*

Proof. It is clear that $\psi^* J = n^{-1} d\theta$. In effect,

$$\begin{aligned} J(\psi(x,\theta,\mu)) &= n^{-1}(H_1(\varphi_\theta(x)) + \mu - H_1(x)) \\ &= n^{-1}(H_1(x) - H_1(x) + \mu) \\ &= n^{-1}\mu. \end{aligned}$$

To show that $\psi^*\omega_P = \omega_P$, it is more convenient to write ψ as

$$\psi(z) = \varphi_{\theta(z)}(z) - H_1(z)\frac{\partial}{\partial\theta},$$

where $\theta : P \to S^1$ is the projection onto the circle and $\frac{\partial}{\partial\theta}$ is the unit vector field tangent to the fibers of T^*S^1 such that $d\theta(\frac{\partial}{\partial\theta}) = 1$. Consequently,

$$d\psi(z)\xi = (d\varphi)_{\theta(z)}(z)\xi + \pi_2^* d\theta(\xi) X_{H_1}(\varphi_{\theta(z)}(z)) - dH_1(z)\xi\frac{\partial}{\partial\theta},$$

where X_{H_1} is the Hamiltonian vector field generated by H_1 and $\pi_2^* d\theta$ is the pullback of the angle form on S^1 to P. Thus,

$$\begin{aligned} &(\psi^*\omega_P)_z(\xi,\eta) = (\omega_P)_{\psi(z)}(d\psi(z)\xi, d\psi(z)\eta) \\ &= \omega_P((d\varphi)_{\theta(z)}(z)\xi, (d\varphi)_{\theta(z)}(z)\eta) + \omega_P((d\varphi)_{\theta(z)}(z)\xi, \pi_2^* d\theta(\eta) X_{H_1}(\varphi_{\theta(z)}(z))) \\ &- \omega_P((d\varphi)_{\theta(z)}(z)\xi, dH_1(z)\eta\frac{\partial}{\partial\theta}(\varphi_{\theta(z)}(z))) + \omega_P(\pi_2^* d\theta(\xi) X_{H_1}(\varphi_{\theta(z)}(z)), (d\varphi)_{\theta(z)}(z)\eta) \\ &+ \omega_P(\pi_2^* d\theta(\xi) X_{H_1}(\varphi_{\theta(z)}(z)), \pi_2^* d\theta(\eta) X_{H_1}(\varphi_{\theta(z)}(z))) - \omega_P(\pi_2^* d\theta(\xi) X_{H_1}(\varphi_{\theta(z)}(z)), dH_1(z)\eta\frac{\partial}{\partial\theta}(\varphi_{\theta(z)}(z))) \\ &- \omega_P(dH_1(z)\xi\frac{\partial}{\partial\theta}(\varphi_{\theta(z)}(z)), (d\varphi)_{\theta(z)}(z)\eta) - \omega_P(dH_1(z)\xi\frac{\partial}{\partial\theta}(\varphi_{\theta(z)}(z)), \pi_2^* d\theta(\eta) X_{H_1}(\varphi_{\theta(z)}(z))) \\ &+ \omega_P(dH_1(z)\xi\frac{\partial}{\partial\theta}(\varphi_{\theta(z)}(z)), dH_1(z)\eta\frac{\partial}{\partial\theta}(\varphi_{\theta(z)}(z))). \end{aligned}$$

But note that

$$\omega_P(\pi_2^* d\theta(\xi) X_{H_1}(\varphi_{\theta(z)}(z)), \pi_2^* d\theta(\eta) X_{H_1}(\varphi_{\theta(z)}(z))) = 0$$

and

$$\omega_P(dH_1(z)\xi\frac{\partial}{\partial\theta}(\varphi_{\theta(z)}(z)), dH_1(z)\eta\frac{\partial}{\partial\theta}(\varphi_{\theta(z)}(z))) = 0,$$

since the vectors are colinear. On the other hand,

$$\omega_P(\pi_2^* d\theta(\xi) X_{H_1}(\varphi_{\theta(z)}(z)), dH_1(z)\eta\frac{\partial}{\partial\theta}(\varphi_{\theta(z)}(z))) = 0$$

and

$$\omega_P(dH_1(z)\xi\frac{\partial}{\partial\theta}(\varphi_{\theta(z)}(z)), \pi_2^* d\theta(\eta) X_{H_1}(\varphi_{\theta(z)}(z))) = 0,$$

because the vectors are orthogonal with respect to the product decomposition $P = M \times T^*S^1$ and by the definition of ω_P.

We have then that,

$$(\psi^*\omega_P)_z(\xi,\eta) = \omega_P((d\varphi)_{\theta(z)}(z)\xi, (d\varphi)_{\theta(z)}(z)\eta) + \omega_P((d\varphi)_{\theta(z)}(z)\xi, \pi_2^* d\theta(\eta) X_{H_1}(\varphi_{\theta(z)}(z)))$$
$$- \omega_P((d\varphi)_{\theta(z)}(z)\xi, dH_1(z)\eta\frac{\partial}{\partial\theta}(\varphi_{\theta(z)}(z))) + \omega_P(\pi_2^* d\theta(\xi) X_{H_1}(\varphi_{\theta(z)}(z)), (d\varphi)_{\theta(z)}(z)\eta)$$
$$- \omega_P(dH_1(z)\xi\frac{\partial}{\partial\theta}(\varphi_{\theta(z)}(z)), (d\varphi)_{\theta(z)}(z)\eta) =$$
$$= (\omega_P)z(\xi,\eta) + \pi_2^* d\theta(\eta)(i_{X_{H_1}(\varphi_{\theta(z)}(z))}\omega)(d\varphi_{\theta(z)}(z)\xi) - dH_1(z)\eta(i_{\frac{\partial}{\partial\theta}(\varphi_{\theta(z)}(z))}\omega)(d\varphi_{\theta(z)}(z)\xi)$$
$$+ \pi_2^* d\theta(\xi)(i_{X_{H_1}(\varphi_{\theta(z)}(z))}\omega)(d\varphi_{\theta(z)}(z)\eta) - dH_1(z)\xi(i_{\frac{\partial}{\partial\theta}(\varphi_{\theta(z)}(z))}\omega)(d\varphi_{\theta(z)}(z)\eta),$$

where the last equality follows from the fact that $\varphi_t^*\omega_P = \omega_P$. Now, note that

$$\pi_2^* d\theta(\eta)(i_{X_{H_1}(\varphi_{\theta(z)}(z))}\omega)(d\varphi_{\theta(z)}(z)\xi) = \pi_2^* d\theta(\eta) dH_1(z)\xi,$$

because $i_{X_{H_1}}\omega_P = dH_1$ and $\varphi^* dH_1 = dH_1$. On the other hand, we have that

$$dH_1(z)\eta(i_{\frac{\partial}{\partial\theta}(\varphi_{\theta(z)}(z))}\omega)(d\varphi_{\theta(z)}(z)\xi) = dH_1(z)\eta\pi_2^* d\theta(\xi),$$

because $i_{\frac{\partial}{\partial\theta}}\omega_P = \pi_2^* d\theta$ and $\varphi^*\pi_2^* d\theta = \pi_2^* d\theta$ (note that $\pi_2^* d\theta$ here is the pullback of the angle form $d\theta$ on S^1 to P). Consequently,

$$\begin{aligned}(\psi^*\omega_P)_z(\xi,\eta) = {} & (\omega_P)z(\xi,\eta) + \\ & + \pi_2^* d\theta(\eta) dH_1(z)\xi - dH_1(z)\eta\pi_2^* d\theta(\xi) + \pi_2^* d\theta(\xi) dH_1(z)\eta - dH_1(z)\xi\pi_2^* d\theta(\eta) \\ & = \omega_z(\xi,\eta),\end{aligned}$$

as desired. □

Lemma 2.3. *The Marsden-Weinstein reduced symplectic space at $J^{-1}(\mu)$ is given by M with the reduced symplectic form σ_μ equal to ω_M for every $\mu \in \mathbb{R}$.*

Proof. We claim that the quotient projection $\pi_\mu : J^{-1}(\mu) \to J^{-1}(\mu)/S^1$ is given by $\pi_1 \circ \psi^{-1}|_{J^{-1}(\mu)}$, where $\pi_1 : P \to M$ is the projection onto the first factor. In effect, note that, by the previous lemma, ψ sends the orbits of X_J to the orbits of $X_{n^{-1}d\theta}$. On the other hand, π_1 is the quotient projection at $(n^{-1}d\theta)^{-1}(\mu) = \psi^{-1}(J^{-1}(\mu))$ with respect to the trivial circle bundle given by the orbits of $X_{d\theta}$.

To show that σ_μ is equal to ω_M, note that

$$\begin{aligned}\pi_\mu^*\omega_M &= (\psi^{-1})^* i^*_{(n^{-1}d\theta)^{-1}(\mu)}\pi_1^*\omega_M \\ &= (\psi^{-1})^* i^*_{(n^{-1}d\theta)^{-1}(\mu)}\omega_P \\ &= i^*_{\psi((n^{-1}d\theta)^{-1}(\mu))}\omega_P \\ &= i_{J^{-1}(\mu)}\omega_P,\end{aligned}$$

where the third equality follows from the fact that $\psi^*\omega_P = \omega_P$. □

In view of the last lemma, consider the map $\widetilde{\pi} : P \to N$ defined by

$$\widetilde{\pi} = \pi_1 \circ \psi^{-1},$$

such that $\widetilde{\pi}|_{J^{-1}(\mu)} = \pi_\mu$. Now, let H be a pre-admissible Hamiltonian on U. We will construct from H a ρ-invariant Hamiltonian defined on P. Fix a sufficiently small constant $\delta > 0$ and define this new Hamiltonian by

$$\widehat{H}(z) = (H + \alpha(J(z))(m(H) - H))(\widetilde{\pi}(z)),$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is a C^∞ function such that $\alpha \geq 0$, $\alpha(\mu) = 1 \ \ \forall \mu \notin (\delta, 1-\delta)$, $\alpha(\mu) = 0 \ \ \forall \mu \in (\frac{1}{2} - \delta, \frac{1}{2} + \delta)$ and $|\alpha'(\mu)| \leq 2 + 2\delta$ for every $\mu \in [0,1]$.

The first obvious property of $\widehat{H}$ is that $m(\widehat{H}) = m(H)$. In effect, $m(\widehat{H}) = \sup_{\mu \in \mathbb{R}} m(H + \alpha(\mu)(m(H) - H)) = m(H)$, since $0 \leq \alpha \leq 1$ and $\alpha(\mu) = 1 \ \ \forall \mu \notin (\delta, 1-\delta)$.

It is easy to see that $\widehat{H} \in \mathcal{H}(\widehat{U}, \omega)$ where $\widehat{U}$ is given by

$$\widehat{U} = \bigcup_{0 \leq \mu \leq 1} \pi_\mu^{-1}(U).$$

In effect, it is clear that $0 \leq \widehat{H} \leq m(\widehat{H})$. Moreover, $\widehat{H}|_{\widehat{V}} \equiv 0$ where $\widehat{V}$ is the subset of P given by $\bigcup_{\frac{1}{2}-\delta \leq \mu \leq \frac{1}{2}+\delta} \pi_\mu^{-1}(V)$ and $V \subset M$ is an open set such that $H|_V \equiv 0$ (which exists because $H \in \mathcal{H}(U, \omega_M)$) since $\alpha(\mu) = 0 \ \ \forall \mu \in (\frac{1}{2} - \delta, \frac{1}{2} + \delta)$. Note that the interior of $\widehat{V}$ is not empty.

On the other hand, $\widehat{H} \equiv m(\widehat{H})$ outside the compact set $\widehat{K} = \bigcup_{\delta \leq \mu \leq 1-\delta} \pi_\mu^{-1}(K) \subset \widehat{U} \setminus \partial \widehat{U}$, where $K \subset U \setminus \partial U$ is the compact set such that $H|_{U \setminus K} \equiv m(H)$.

Let us now prove that there exists a periodic orbit γ of X_H of period less than $2\pi(\|H_1|_U\| + n)/m(H)$ and homotopy class $[\gamma] \in G_\varphi$. Repair firstly that

$$\left| \int_{\pi_2(\widehat{U})} \omega_0^{S^1} \right| = 2\pi(\|H_1|_U\| + n).$$

In effect, $\widehat{U} \cup J^{-1}(0)$ is just the graph of $-H_1$ restricted to U over $d\theta^{-1}(0)$ and $\widehat{U} \cup J^{-1}(1)$ is the graph of $-H_1|_U$ over $d\theta^{-1}(n)$.

Now, note that $H_\mu := H + \alpha(\mu)(m(H) - H) = (1 - \alpha(\mu))H + \alpha(\mu)m(H)$ has the Hamiltonian vector field with respect to ω_M given by $X_{H_\mu} = (1 - \alpha(\mu))X_H$. Since $0 \leq \alpha(\mu) \leq 1$, the nonconstant periodic orbits of X_{H_μ} have period greater than or equal to those of X_H.

Consequently, by the Lemma (2.3), it is sufficient to show that the periodic orbit of $\widehat{H}$ given by the Proposition (2.1) is not tangent to a fiber given by the action of φ (such that it is projected onto a singularity of X_H).

Thus, suppose that there exists a periodic orbit γ of X_H tangent to an orbit of ρ and whose homotopy class belongs to $\pi_1(S^1) \subset \pi_1(P)$. We will show that the period of γ is strictly greater than $T_{\max}(\widehat{U}, \widehat{H})$ (see the Proposition (2.1)).

Firstly, note that

$$T_{\max}(\widehat{U}, \widehat{H}) = \frac{\left| \int_{\pi_2(\widehat{U})} \omega_0^{S^1} \right|}{m(\widehat{H})} = \frac{2\pi(\|H_1|_U\| + n)}{m(H)}.$$

Now, let W be an almost complex structure on P compatible with ω_P and $\langle\cdot,\cdot\rangle = \omega_P(W\cdot,\cdot)$ be the corresponding Riemannian metric. We have that along γ,

$$X_{\widehat{H}} = \Big\langle X_{\widehat{H}}, \frac{X_J}{\|X_J\|^2}\Big\rangle X_J.$$

On the other hand, the period of the orbits of X_J is equal to $2\pi n$. Consequently, since $[\gamma] \in \pi_1(S^1)$, the period of γ is given by

$$T = \frac{2\pi n n_\varphi}{\big|\big\langle X_{\widehat{H}}, \frac{X_J}{\|X_J\|^2}\big\rangle\big|}.$$

But, by the definition of n,

$$T = \frac{2\pi n n_\varphi}{\big|\big\langle X_{\widehat{H}}, \frac{X_J}{\|X_J\|^2}\big\rangle\big|} > \frac{2\pi(2+2\delta)(\|H_1|_U\| + n)}{\big|\big\langle X_{\widehat{H}}, \frac{X_J}{\|X_J\|^2}\big\rangle\big|}.$$

In effect, note that the inequality

$$2\pi n n_\varphi > 2\pi(2+2\delta)(\|H_1|_U\| + n)$$

is equivalent to

$$2\pi n(n_\varphi - (2+2\delta)) > 2\pi(2+2\delta)\|H_1|_U\|,$$

that is,

$$n > (2+2\delta)\frac{\|H_1|_U\|}{n_\varphi - (2+2\delta)}.$$

Since $\delta > 0$ is arbitrarily small, this is exactly the condition on n. Thus, it is sufficient to prove that

$$\Big|\Big\langle X_{\widehat{H}}, \frac{X_J}{\|X_J\|^2}\Big\rangle\Big| \leq (2+2\delta)m(H),$$

such that

$$T > \frac{2\pi(\|H_1|_U\| + n)}{m(H)}.$$

To conclude this inequality, note that

$$\begin{aligned}\Big|\Big\langle X_{\widehat{H}}, \frac{X_J}{\|X_J\|^2}\Big\rangle\Big| &= \Big|\Big\langle W\nabla\widehat{H}, \frac{W\nabla J}{\|W\nabla J\|^2}\Big\rangle\Big| \\ &= \Big|\Big\langle \nabla\widehat{H}, \frac{\nabla J}{\|\nabla J\|^2}\Big\rangle\Big| \\ &= \Big|d\widehat{H}\Big(\frac{\nabla J}{\|\nabla J\|^2}\Big)\Big|\end{aligned}$$

where in the second equation we used the fact that W defines an isometry. On the other hand, we have that for every $\xi \in T_zP$,

$$\begin{aligned}d\widehat{H}(z)\xi &= dH(\widetilde{\pi}(z))d\widetilde{\pi}(z)\xi + m(H)\alpha'(J(z))dJ(z)\xi - H(\widetilde{\pi}(z))\alpha'(J(z))dJ(z)\xi \\ &\quad - \alpha(J(z))dH(\widetilde{\pi}(z))d\widetilde{\pi}(z)\xi \\ &= (m(H) - H\circ\widetilde{\pi}(z))\alpha'(J(z))dJ(z)\xi + (1-\alpha(J(z)))dH(\widetilde{\pi}(z))d\widetilde{\pi}(z)\xi.\end{aligned}$$

But note that, since γ is tangent to a fiber, it is projected onto a singularity of X_H, that is, $dH(\widetilde{\pi}(z)) = 0$ for every $z \in \gamma$. Consequently, we have that

$$\begin{aligned} \left|\left\langle X_{\widehat{H}}, \frac{X_J}{\|X_J\|^2} \right\rangle\right| &= \left| d\widehat{H}\left(\frac{\nabla J}{\|\nabla J\|^2} \right) \right| \\ &= (m(H) - H \circ \widetilde{\pi}(x))|\alpha'(J(z))| \\ &\leq (2 + 2\delta)m(H), \end{aligned}$$

as desired.

Finally, note that the periodic orbit $\bar{\gamma}$ of $X_{\widehat{H}}$ given by the Proposition (2.1) satisfies $[\bar{\gamma}] \in \pi_1(S^1)$. Consequently, the corresponding reduced periodic orbit γ satisfies $[\gamma] \in G_\varphi$ since $(\pi_\mu)_* \pi_1(S^1) = G_\varphi$.

Thus, we obtain the estimate

$$c_{HZ}^{G_\varphi}(U, \omega_M) \leq 2\pi(\|H_1|_U\| + n).$$

But, since n is arbitrarily close to $2\|H_1|_U\|/(n_\varphi - 2)$, we conclude that

$$c_{HZ}^{G_\varphi}(U, \omega_M) \leq 2\pi \|H_1|_U\| \left(1 + \frac{2}{n_\varphi - 2} \right).$$

$\square$

3. Proof of the Corollary (1.1)

The proof of the Corollary (1.1) follows immediately from the Theorem (1.1) and the following useful proposition:

Proposition 3.1. *Let $S^1 \xrightarrow{\varphi} P \xrightarrow{\pi_P} M$ be a principal circle bundle whose Euler class $[\Omega] \in H^2(M, \mathbb{R})$ satisfies $[\Omega]|_{\pi_2(M)} = 0$. Then φ has infinite order.*

Proof. Initially, note that $[\Omega]|_{\pi_2(M)} = 0$ iff $\tau_M^*[\Omega] = 0$, where $\tau_M : \widetilde{M} \to M$ is the universal cover. In effect, it is clear that if $\tau_M^*\Omega$ is exact, then $[\Omega]|_{\pi_2(M)} = 0$. To prove the converse, note that, by the Hurewicz's theorem, we have that $H_2(\widetilde{M}, \mathbb{Z}) \simeq \pi_2(\widetilde{M}) \simeq \pi_2(M)$. Thus, the condition that Ω is weakly exact is equivalent that the integral of $\tau_M^*[\Omega]$ over any 2-cycle is equal to zero. But, by de Rham's theorem, it is equivalent to $\tau_M^*\Omega$ be exact.

Now, suppose that the action φ has finite order. Consider the lift $\widetilde{\varphi}$ of φ to the universal cover $\widetilde{P}$ of P.

Claim 1. $\widetilde{P}/\widetilde{\varphi} = \widetilde{M}$, where $\widetilde{P}/\widetilde{\varphi}$ is the quotient of $\widetilde{P}$ by the action $\widetilde{\varphi}$ and $\widetilde{M}$ is the universal cover of M.

To prove the claim, note that, since the fiber is path connected, $\widetilde{P}/\widetilde{\varphi}$ is simply connected. Thus, it is sufficient to prove that $\widetilde{P}/\widetilde{\varphi}$ covers M. Now, repair that

$$M = \widetilde{P}/\Gamma_P/\varphi,$$

where Γ_P is the group of deck transformations on $\widetilde{P}$. On the other hand, $\widetilde{\varphi}$ commutes with the action of Γ_P, since it is a lifted action from P. Consequently, it induces an action of a discrete group Γ_M on $\widetilde{P}/\widetilde{\varphi}$ such that

$$M = \widetilde{P}/\Gamma_P/\varphi = \widetilde{P}/\widetilde{\varphi}/\Gamma_M,$$

proving the claim.

Claim 2. The following diagram is commutative:

$$\begin{array}{ccc} \widetilde{P} & \xrightarrow{\tau_P} & P \\ {\scriptstyle \pi_{\widetilde{P}}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_P} \\ \widetilde{M} & \xrightarrow{\tau_M} & M \end{array}$$

In fact, note that

$$\pi_P \circ \tau_P : \widetilde{P} \to \widetilde{P}/\Gamma_P/\varphi$$

and

$$\tau_M \circ \pi_{\widetilde{P}} : \widetilde{P} \to \widetilde{P}/\widetilde{\varphi}/\Gamma_M.$$

But, as remarked above, $\widetilde{\varphi} \circ \gamma_P = \gamma_P \circ \widetilde{\varphi}\ \forall \gamma_P \in \Gamma_P$ and Γ_M is the induced action on $\widetilde{P}/\widetilde{\varphi} = \widetilde{M}$ such that $\pi_P \circ \tau_P = \tau_M \circ \pi_{\widetilde{P}}$.

Claim 3. the Euler class of $S^1 \xrightarrow{\widetilde{\varphi}} \widetilde{P} \xrightarrow{\pi_{\widetilde{P}}} \widetilde{M}$ is given by $[\tau_M^*\Omega]$, where Ω is the curvature form of $S^1 \xrightarrow{\varphi} P \xrightarrow{\pi_P} M$.

In effect, let α be a connection form of φ. Then $\tau_P^*\alpha$ is a connection for $\widetilde{\varphi}$ and

$$\begin{aligned} d\tau_P^*\alpha &= \tau_P^* d\alpha \\ &= \tau_P^*\pi_P^*\Omega \\ &= \pi_{\widetilde{P}}^*\tau_M^*\Omega. \end{aligned}$$

Thus, if $\tau_M^*\Omega$ is exact, we have that $S^1 \xrightarrow{\widetilde{\varphi}} \widetilde{P} \xrightarrow{\pi_{\widetilde{P}}} \widetilde{M}$ defines a trivial circle bundle, contradicting the fact that $\widetilde{P}$ is simply connected. □

4. Proof of the Theorem (1.2)

Let M be a surface of genus $g \geq 2$ endowed with a Riemannian metric g of constant negative curvature. Let TM be the tangent bundle of M with the symplectic form ω_0 given by the pullback of the canonical symplectic form of T^*M by the bundle isomorphism induced by g.

Consider on M the almost complex structure J such that $\Omega = g(J\cdot, \cdot)$ is the area form (normalized such that $\int_M \Omega = 1$). As defined in the introduction, the magnetic flow associated to g and Ω is the Hamiltonian flow given by the kinetic energy $H(x,v) = (1/2)g_x(v,v)$ with respect to the twisted symplectic form $\omega_1 := \omega_0 + \pi^*\Omega$. It is easy to see that a curve $t \mapsto (\gamma(t), \dot\gamma(t))$ defines an orbit of this flow if and only if $\nabla_{\dot\gamma(t)}\dot\gamma(t) = J(\dot\gamma(t))$. In other words, the trajectories are the curves with constant geodesic curvature.

Consequently, the dynamics of this magnetic flow can be explicitly described: on the energy levels greater than one half, the magnetic flow is topologically equivalent to the geodesic flow (which has periodic orbits in every nontrivial homotopy class and does not have projected contractible periodic orbits, because there is no conjugate points); on the sphere bundle, it coincides with the horocycle flow and hence is minimal, that is, all the orbits are dense; and on the energy levels below than one half, all the orbits are closed with the same period and homotopic to the fibers of the sphere bundle.

In particular, it defines a Hamiltonian circle action with infinite order on $U_{1/2}$ (since $\pi_2(M) = 0$) such that we can apply the Theorem (1.1). The problem is the condition on $TM \setminus M_0$ to be tame. To circumvent this problem we need the following proposition:

Proposition 4.1. *Given $\epsilon > 0$ there exists an exact tame symplectic form ρ_ϵ on $TM \setminus M_0$ such that ρ_ϵ restricted to $(TM \setminus M_0) \setminus \overline{U}_\epsilon$ coincides with ω_1.*

Proof. The proof relies on the method of expanding completion of convex manifolds introduced by Eliashberg and Gromov [11]. Firstly, we need the following key lemma which has intrinsic interest:

Lemma 4.1. *The vector field X on $TM \setminus M_0$ given by (with respect to the horizontal and vertical subbundles)*

$$X = \left(-\frac{1}{\|v\|^2} Jv, \left(1 - \frac{1}{\|v\|^2}\right) v \right)$$

is conformally symplectic with respect to ω_1, that is, $L_X \omega_1 = \omega_1$.

Proof. To fix the notation, let $\omega_0(\xi, \eta) = \langle \pi_* \xi, K\eta \rangle - \langle K\xi, \pi_* \eta \rangle$, where K is the curvature map, be the pullback of the canonical symplectic form of T^*M by the Riemannian metric and $\pi^*\Omega(\xi, \eta) = \langle J\pi_*\xi, \pi_*\eta \rangle$. Define $X_1(x, v) = (-(1/\|v\|^2)Jv, 0)$ and $X_2(x, v) = (0, (1 - 1/\|v\|^2)v)$. Consider the Liouville form

$$\alpha_{(x,v)}(\xi) = \langle v, \pi_*\xi \rangle$$

and the 1-form

$$\beta_{(x,v)}(\xi) = \langle -Jv, K\xi \rangle.$$

Note that $d\alpha = \omega_0$ and $d((1/\|v\|^2)\beta) = \pi^*\Omega$, where the last equality follows from the fact that $(1/\|v\|^2)\beta$ defines a connection form on $H^{-1}((1/2)\|v\|^2)$ and that M is a hyperbolic surface.

Let us compute the Lie derivative of $\omega_1 = \omega_0 + \pi^*\Omega$ with respect to X_1. Initially, we have that,

$$\begin{aligned} i_{(-Jv,0)}\omega_1(\xi) &= \langle -Jv, K\xi \rangle + \langle v, \pi_*\xi \rangle \\ &= \beta + \alpha \end{aligned}$$

such that

$$\begin{aligned} L_{(-Jv,0)}\omega_1 &= d(i_{(-Jv,0)}\omega_1) \\ &= d(\beta + \alpha). \end{aligned}$$

But, we have that

$$\begin{aligned}\pi^*\Omega &= d((1/\|v\|^2)\beta)\\ &= \frac{1}{\|v\|^2}d\beta - \frac{2}{\|v\|^3}dv\wedge\beta.\end{aligned}$$

Consequently,

$$d\beta = \|v\|^2\pi^*\Omega + \frac{2}{\|v\|}dv\wedge\beta.$$

Thus,

$$d(i_{(-Jv,0)}\omega_1) = \omega_0 + \|v\|^2\pi^*\Omega + \frac{2}{\|v\|}dv\wedge\beta$$

and so

$$\begin{aligned}d(i_{(1/\|v\|^2)(-Jv,0)}\omega_1) &= \frac{1}{\|v\|^2}\omega_0 + \pi^*\Omega + \frac{2}{\|v\|^3}dv\wedge\beta - \frac{2}{\|v\|^3}dv\wedge i_{(-Jv,0)}\omega_1\\ &= \frac{1}{\|v\|^2}\omega_0 + \pi^*\Omega + \frac{2}{\|v\|^3}dv\wedge\beta - \frac{2}{\|v\|^3}dv\wedge(\beta+\alpha)\\ &= \frac{1}{\|v\|^2}\omega_0 + \pi^*\Omega - \frac{2}{\|v\|^3}dv\wedge\alpha.\end{aligned}$$

On the other hand, the Lie derivative of ω_1 with respect to X_2 is given by

$$\begin{aligned}d(i_{(0,(1-(1/\|v\|^2))v)}\omega_1) &= d(\alpha - \frac{1}{\|v\|^2}\alpha)\\ &= \omega_0 - \frac{1}{\|v\|^2}\omega_0 + \frac{2}{\|v\|^3}dv\wedge\alpha\end{aligned}$$

since

$$i_{(0,(1-(1/\|v\|^2))v)}\omega_1 = \alpha - \frac{1}{\|v\|^2}\alpha.$$

Consequently, we have that

$$\begin{aligned}L_X\omega_1 &= d(i_X\omega_1)\\ &= d(i_{X_1+X_2}\omega_1)\\ &= \frac{1}{\|v\|^2}\omega_0 + \pi^*\Omega - \frac{2}{\|v\|^3}dv\wedge\alpha + \omega_0 - \frac{1}{\|v\|^2}\omega_0 + \frac{2}{\|v\|^3}dv\wedge\alpha\\ &= \omega_0 + \pi^*\Omega\\ &= \omega_1.\end{aligned}$$

$\square$

Remark. The previous lemma shows that the symplectic manifold

$$\Big(\bigcup_{\epsilon\leq\mu\leq(1/2)+\delta} H^{-1}(\mu), \omega_1\Big)$$

has disconnected contact-type boundary for any $0 < \epsilon < 1/2$ and $\delta > 0$. It can be compared with McDuff's example [30].

It was proved by G. Lu [27] that the tangent bundle of a compact manifold endowed with any twisted symplectic form is tame. Thus, consider the almost complex structure J on TM such that $g(\cdot,\cdot) = \omega_1(J\cdot,\cdot)$ defines a Riemannian metric with bounded sectional curvature and positive injectivity radius.

By the previous lemma, given any $\epsilon > 0$ sufficiently small, the subset $M_\epsilon := (TM\setminus M_0)\setminus \overline{U}_\epsilon$ has contact type boundary, that is, its boundary is convex in the sense of Eliashberg-Gromov [11]. Thus, we can consider the set $N_\epsilon := M_\epsilon \sqcup \partial M_\epsilon \times [0,\infty)$ (that we can identify with $TM \setminus M_0$) and a symplectic form ρ_ϵ on N_ϵ such that ρ_ϵ coincides with ω_1 on M_ϵ and the vector field $\partial/\partial t$ is an extension of X to $\partial M_\epsilon \times [0,\infty)$ (that is, $-\partial/\partial t$ coincides with $-X$ on ∂M_ϵ) such that $L_{\partial/\partial t}\rho_\epsilon = \rho_\epsilon$. Note that ρ_ϵ is exact, since ω_1 on M_ϵ is exact and ρ_ϵ restricted to $\partial M_\epsilon \times [0,\infty)$ is obviously exact.

Consider now the almost complex structure J_ϵ on N_ϵ given by $J|_{M_\epsilon}$ on M_ϵ and by the pushforward $d\varphi_t \circ J \circ d\varphi_{-t}$ of J by the flow φ of $\partial/\partial t$ on $\partial M_\epsilon \times [0,\infty)$. By the construction, J_ϵ is invariant by φ on $\partial M_\epsilon \times [0,\infty)$.

Define $g_\epsilon(\cdot,\cdot) = \rho_\epsilon(J_\epsilon\cdot,\cdot)$. By our previous discussion, we have that g_ϵ defines a geometrically bounded Riemannian metric on M_ϵ. We need to prove that it also defines a geometrically bounded Riemannian metric on $\partial M_\epsilon \times [0,\infty)$.

But, we have that $\varphi_t^*\rho_\epsilon = e^t\rho_\epsilon$, that is,

$$(\rho_\epsilon)_{(x,t)} = e^t(\varphi_t)_*(\rho_\epsilon)_{(x,0)}.$$

Consequently, since J_ϵ is invariant by φ on $\partial M_\epsilon \times [0,\infty)$, we obtain that

$$(g_\epsilon)_{(x,t)} = e^t(\varphi_t)_*(g_\epsilon)_{(x,0)}.$$

Thus, the sectional curvature $K_{(x,t)}(v,w)$ of g_ϵ on (x,t) is given by

$$e^{-t}K_{(x,0)}(d\varphi_{-t}v, d\varphi_{-t}w)$$

and the injectivity radius $i(N_\epsilon,(x,t))$ is greater than or equal to

$$e^t i(N_\epsilon,(x,0)).$$

But, g_ϵ has bounded sectional curvature and positive injectivity radius on M_ϵ, proving the proposition. □

Proof of Theorem (1.2): By our previous discussion, the magnetic flow restricted to an energy level $k > 1/2$ is topologically conjugated to the geodesic flow such that it has no projected contractible periodic orbits. Consequently, we have that

$$c_{HZ}^G(U_k,\omega_1) = \infty$$

for every $k > 1/2$.

On the other hand, we have that for energies $k < 1/2$ the magnetic flow defines a free Hamiltonian circle action (of period 1) on U_k of infinite order on $TM\setminus M_0$. By the previous proposition, given $\epsilon > 0$ sufficiently small, there exists a tamed symplectic form ρ_ϵ on $(TM \setminus M_0)$ that coincides with ω_1 on $(TM \setminus M_0) \setminus \overline{U}_\epsilon$. Thus, by the Theorem (1.1) we have that

$$c^G_{HZ}(U_{1/2} \setminus \overline{U}_\epsilon, \omega_1) = c^G_{HZ}(U_{1/2} \setminus \overline{U}_\epsilon, \rho_\epsilon) \leq 1/2 - \epsilon.$$

But,

$$c^G_{HZ}(U_{1/2}, \omega_1) = \sup_{0<\epsilon<1/2} c^G_{HZ}(U_{1/2} \setminus \overline{U}_\epsilon, \omega_1) \leq 1/2.$$

□

5. Proof of the Theorem (1.4)

Before we give the proof of the Theorem (1.4), we need some previous lemmas. The first one can be derived by the arguments in the proof of the Proposition (4.1) and the fact $M \setminus \Sigma$ has a convex boundary. However, for the reader's convenience we present an independent proof using the structure of Stein manifolds:

Lemma 5.1. *Given a neighborhood U of Σ, there exists an exact tame symplectic form ρ on $M \setminus \Sigma$ such that ρ coincides with Ω on the complement of U.*

Proof. Note that Σ can be obtained as the zero set of a holomorphic section $s : M \to L$ of a complex line bundle defined by the divisor Σ. It can be show that the function $\varphi : M \setminus \Sigma \to \mathbb{R}$ given by

$$\varphi(x) = -\frac{1}{4\pi k} \log(\|s(x)\|^2)$$

is plurisubharmonic [4]. The next lemma proved in [5] shows that we can construct a new plurisubharmonic function on $M \setminus \Sigma$ that coincides with φ on $M \setminus U$ and whose gradient vector field is complete.

Lemma 5.2 (Lemma (3.1) of Biran-Cieliebak [5]). *Let (V, J, φ) be a Stein manifold. Then for every $R \in \mathbb{R}$ there exists an exhausting plurisubharmonic function $\varphi_R : V \to \mathbb{R}$ with the following properties:*

1. *$\varphi_R = \varphi$ on $V_{\varphi \leq R} := \{x \in V; \varphi(x) \leq R\}$;*
2. *the gradient vector field of φ_R is complete;*
3. *$Crit(\varphi_R) = Crit(\varphi)$ and for every $p \in Crit(\varphi_R)$, $index_p(\varphi_R) = index_p(\varphi)$.*

In particular, the inclusion $(V_{\varphi \leq R}, \omega_\varphi) \hookrightarrow (V, \omega_{\varphi_R})$ is a symplectic embedding.

Now, let $R > 0$ be sufficiently great and consider the Riemannian metric $g_{\varphi_R}(\cdot, \cdot) = \omega_{\varphi_R}(J\cdot, \cdot)$, where $\omega_{\varphi_R} = d(J^* d\varphi_R)$.

Since M is closed, we have that g_{φ_R} defines a geometrically bounded Riemannian metric on $V_{\varphi \leq R}$, that is, the sectional curvature of g_{φ_R} restricted to $V_{\varphi \leq R}$ is bounded from above and the injectivity radius is positive. We need to prove that it also defines a geometrically bounded Riemannian metric on all $M \setminus \Sigma$.

Given $x \in (M \setminus \Sigma) \setminus V_{\varphi \leq R}$, there exists $y \in \partial V_{\varphi \leq R}$ such that $X^t_{\varphi_R}(y) = x$ for some $t > 0$, where $X^t_{\varphi_R}$ is the gradient flow of φ_R with respect to g_{φ_R}. But, we have that $\varphi_t^* \Omega_{\varphi_R} = e^t \Omega_{\varphi_R}$, that is,

$$(\Omega_{\varphi_R})_{(x,t)} = e^t (\varphi_t)_* (\Omega_{\varphi_R})_{(x,0)}.$$

Consequently, since the complex structure J is invariant by $X^t_{\varphi_R}$, we obtain that

$$(g_{\varphi_R})_{(x,t)} = e^t(\varphi_t)_*(g_{\varphi_R})_{(x,0)}.$$

Thus, the sectional curvature $K_{(x,t)}(v,w)$ of g_{φ_R} on (x,t) is given by

$$e^{-t}K_{(x,0)}(d\varphi_{-t}v, d\varphi_{-t}w)$$

and the injectivity radius $i(N_\epsilon,(x,t))$ is greater than or equal to

$$e^t i(N_\epsilon,(x,0)).$$

But, g_{φ_R} has bounded sectional curvature and positive injectivity radius on $V_{\varphi\leq R}$, proving that Ω_{φ_R} is tame.

Finally, define $\rho = \Omega_{\varphi_R}$ for $R > 0$ sufficiently large. □

Lemma 5.3 (Lemma (3.2) of Biran-Cieliebak [5]). *Suppose that M is subcritical and let $U \subset M \setminus \Sigma$ be an open subset with compact closure. Let ρ be a tame symplectic form on $M \setminus \Sigma$ given by the Lemma (5.1). Then there exists a symplectomorphism isotopic to the identity $\psi : (M \setminus \Sigma, \rho) \to (M \setminus \Sigma, \rho)$ such that $\psi(U) \cap \Delta = \emptyset$.*

Proof. Remember that the isotropic CW-complex Δ is given by the union of the stable manifolds of the gradient vector field X_φ of a certain plurisubharmonic function φ with a complete gradient flow such that $d(J^*d\varphi) = \rho$ [3].

Since M is subcritical, we have that $\dim \Delta < \frac{1}{2}\dim_{\mathbb{R}} M$ and so there exists a Hamiltonian isotopy $k_t : (M\setminus\Sigma, \rho) \to (M\setminus\Sigma, \rho)$ compactly supported in an arbitrarily small neighborhood of Δ such that $k_1(\Delta) \cap \Delta = \emptyset$. Since Δ is compact, there exists a neighborhood V of Δ such that $k_1(V) \cap V = \emptyset$.

Fix T so large that $X_\varphi^{-T}(U) \subset V$. Since k_1 moves V away from itself, we have that

$$X^T_\varphi \circ k_1 \circ X^{-T}_\varphi(U) \cap \Delta = \emptyset,$$

because X^T_φ leaves $(M \setminus \Sigma) \setminus \Delta$ invariant. Since X_φ is conformally symplectic, we have that

$$\psi = X^T_\varphi \circ k_1 \circ X^{-T}_\varphi$$

is the desired symplectomorphism. □

Proof of the Theorem (1.4): By the Theorem (1.3), there exists an isotropic CW-complex $\Delta \subset M$ (given by the union of the stale manifolds of the gradient vector field associated to a plurisubharmonic function defined on $M \setminus \Sigma$) whose complement $E := M \setminus \Delta$ is symplectomorphic to a standard symplectic disc bundle $(E_0, \frac{1}{k}\omega_{\text{can}})$ over Σ modeled on the normal bundle N_Σ of Σ in M, where

$$\omega_{\text{can}} = k\pi^*(\Omega|_\Sigma) + d(r^2\alpha).$$

Moreover, this symplectomorphism $F : (E, \Omega) \to (E_0, \frac{1}{k}\omega_{\text{can}})$ sends Σ to the zero section of E_0.

Repair that $(E_0 \setminus \Sigma, \frac{1}{k}\omega_{\text{can}})$ has an obvious Hamiltonian free circle action (of period 2π) generated by the Hamiltonian $H_0(x) = (1/2\pi k)\|x\|^2$ such that $H_1 = H_0 \circ F$ also generates a Hamiltonian free circle action on $(E \setminus \Sigma, \Omega)$.

By the Lefschetz theorem [7], we have that the inclusion $i : H_1^{-1}(r) \to M \setminus \Sigma$ induces an injective map on the fundamental group such that the order of the circle action on $E \setminus \Sigma \subset M \setminus \Sigma$ is equal to the order of the circle action $X_{H_0}^{\omega_{\mathrm{can}}}$ generated by H_0 on $E_0 \setminus \Sigma$.

Since the Chern class of E_0 is given by $k[\Omega|_\Sigma]$, we have that if $[\Omega]_{\pi_2(\Sigma)} = 0$ then, by the Proposition (3.1), the order of the action generated by H_1 is infinite.

On the other hand, the order of the action is always grater than or equal to k. In effect, repair that the S^1-bundle $S^1 \longrightarrow H_1^{-1}(r) \longrightarrow \Sigma$ is equivalent to the quotient by $\mathbb{Z}_k$ of a S^1-bundle $S^1 \longrightarrow P \longrightarrow \Sigma$ with Euler class $[\Omega_\Sigma]$ such that the subgroup of $\pi_1(H_1^{-1}(r))$ generated by the orbits of the action has order greater than or equal to k.

Let ρ_ϵ be the exact tame symplectic form on $M \setminus \Sigma$ given by the Lemma (5.1) that coincides with Ω on the complement $M \setminus U_\epsilon$ of a tubular neighborhood of radius ϵ of Σ. Note that $(E \setminus \Sigma, \rho_\epsilon)$ also has a Hamiltonian free circle action that coincides with $X_{H_1}^{\Omega}$ on $M \setminus U_\epsilon$.

Now, let $U \subset E \setminus \Sigma$ be an open subset with compact closure and $\epsilon > 0$ sufficiently small such that $U \subset E \setminus U_\epsilon$. Then, by the Theorem (1.1),we have that if $[\Omega]_{\pi_2(\Sigma)} = 0$ then,

$$c_{HZ}^G(U, \Omega) = c_{HZ}^G(U, \rho_\epsilon) \leq 2\pi \|H_1|_U\| \leq 2\pi \|H_1|_E\| = 1/k.$$

where $G \subset \pi_1(M \setminus \Sigma)$ is the subgroup generated by the homotopy class of the orbits of $X_{H_1}^{\Omega}$. On the other hand, if $[\Omega]_{\pi_2(\Sigma)} \neq 0$ and $k > 2$ we have the inequality

$$c_{HZ}^G(U, \Omega) = c_{HZ}^G(U, \rho_\epsilon) \leq \frac{1}{k} + \frac{2}{k^2 - 2k}.$$

Finally, suppose that (M, Σ) is subcritical and let $U \subset M \setminus \Sigma$ be an open subset with compact closure and $\epsilon > 0$ sufficiently small such that $U \subset M \setminus U_\epsilon$. Then, by the Lemma (5.3), there exists a symplectomorphism isotopic to the identity $\psi : (M \setminus \Sigma, \rho_\epsilon) \to (M \setminus \Sigma, \rho_\epsilon)$ such that $\psi(U) \cap \Delta = \emptyset$. Take $\epsilon > 0$ sufficiently small such that $\psi(U) \subset M \setminus U_\epsilon$. Thus, by the weak monotonicity property, we conclude that

$$c_{HZ}^G(U, \Omega) = c_{HZ}^G(U, \rho_\epsilon) = c_{HZ}^G(\psi(U), \rho_\epsilon) \leq 1/k.$$

$\square$

6. Proof of the Theorem (1.5)

Proof of the Theorem (1.5): The proof of the Theorem (1.5) is very similar to the proof of the Theorem (1.4) but instead of complex hypersurfaces we consider symplectic hypersurfaces of Donaldson type [10].

In fact, by the symplectic neighborhood theorem, there exists a neighborhood V of Σ and a symplectomorphism $F : (V, \Omega) \to (W, \omega_{\mathrm{can}})$ between (V, Ω) and a neighborhood W of the zero section of the canonical disc bundle $(E_0, \frac{1}{k}\omega_{\mathrm{can}})$, where

$$\omega_{\mathrm{can}} = k\pi^*(\Omega|_\Sigma) + d(r^2\alpha).$$

Moreover, F sends Σ to the zero section of E_0.

We will need the following lemma which generalizes the Lemma (5.1) when the complement of Σ may not be a Stein manifold:

Lemma 6.1. *Given a neighborhood U of Σ, there exists a tame symplectic form ρ on $M \setminus \Sigma$ such that ρ coincides with Ω on the complement of U.*

Proof. Note that

$$(1/k)\omega_{\text{can}} = \pi^*(\Omega|_\Sigma) + (1/k)d(r^2\alpha) = d(((1/k)r^2 - 1)\alpha).$$

Consequently, we have that the vector field

$$X = \frac{(1/k)r^2 - 1}{(1/k)2r}\partial_r$$

is conformally symplectic with respect to ω_{can}.

Now, repair that X point towards the interior of the compact domains bounded by the hypersurfaces $r =$ constant, that is, the subset $E_\epsilon := E_0 \setminus \overline{U}_\epsilon$ has a convex boundary in the sense of Eliashberg-Gromov [11], where $U_\epsilon = \{x \in E_0;\ \ H_0(x) < \epsilon\}$ and $H_0(x)$ is the radial coordinate of x.

Thus, we can consider the set $N_\epsilon := E_\epsilon \sqcup \partial E_\epsilon \times [0, \infty)$ and a symplectic form ρ_ϵ on N_ϵ such that ρ_ϵ coincides with ω_{can} on E_ϵ and the vector field ∂_r is a complete extension of X to $\partial E_\epsilon \times [0, \infty)$ (that is, $-\partial_r$ coincides with $-X$ on ∂E_ϵ) such that $L_{\partial_r}\rho_\epsilon = \rho_\epsilon$.

Now, we can proceed as in the proof of the Proposition (4.1) to conclude that $g_\epsilon(\cdot, \cdot) := \rho_\epsilon(J_\epsilon \cdot, \cdot)$ defines a geometrically bounded Riemannian metric on N_ϵ. □

In what follows we will repeat many arguments of the proof of the Theorem (1.4):

Repair that $(W \setminus \Sigma, \frac{1}{k}\omega_{\text{can}})$ has an obvious Hamiltonian free circle action (of period 2π) generated by the Hamiltonian $H_0(x) = (1/2\pi k)\|x\|^2$ such that $H_1 = H_0 \circ F$ also generates a Hamiltonian free circle action on $(V \setminus \Sigma, \Omega)$.

By the Lefschetz theorem for symplectic hypersurfaces of Donaldson type [10], there exists $k_0 > 0$ such that if $k > k_0$ then the inclusion $i : H_1^{-1}(r) \to M \setminus \Sigma$ induces an injective map on the fundamental group and so the order of the circle action on $V \setminus \Sigma \subset M \setminus \Sigma$ is equal to the order of the circle action $X_{H_0}^{\omega_{\text{can}}}$ generated by H_0 on $W \setminus \Sigma$. Let $k_0 \geq 2$.

Since the Chern class of E_0 is given by $k[\Omega|_\Sigma]$, we have that, as in the proof of the Theorem (1.4), if $[\Omega]_{\pi_2(\Sigma)} = 0$ then the order of the action generated by H_1 is infinite. On the other hand, the order of the action is always grater than or equal to k.

Let ρ_ϵ be the tame symplectic form on $M \setminus \Sigma$ given by the Lemma (6.1) that coincides with Ω on the complement $M \setminus U_\epsilon$ of a tubular neighborhood of radius ϵ of Σ.

Note that $(M \setminus \Sigma, \rho_\epsilon)$ is weakly exact, because ρ_ϵ is exact on $V \setminus \Sigma$ and, by the Lefschetz theorem, the inclusion $V \setminus \Sigma \hookrightarrow M \setminus \Sigma$ induces a sobrejective application $\pi_2(V \setminus \Sigma) \to \pi_2(M \setminus \Sigma)$.

Now, let $U \subset V \setminus \Sigma$ be an open subset with compact closure and $\epsilon > 0$ sufficiently small such that $U \subset V \setminus U_\epsilon$. Then, by the Theorem (1.1), we have that if $[\Omega]_{\pi_2(\Sigma)} = 0$

then,

$$c^G_{HZ}(U,\Omega) = c^G_{HZ}(U,\rho_\epsilon) \leq 2\pi\|H_1|_U\| \leq 2\pi\|H_1|_E\| = 1/k.$$

where $G \subset \pi_1(M \setminus \Sigma)$ is the subgroup generated by the homotopy class of the orbits of $X^\Omega_{H_1}$. Analogously, if $[\Omega]_{\pi_2(\Sigma)} \neq 0$ and $k > k_0 \geq 2$ we have the inequality

$$c^G_{HZ}(U,\Omega) = c^G_{HZ}(U,\rho_\epsilon) \leq \frac{1}{k} + \frac{2}{k^2 - 2k}.$$

□

7. Proof of the Theorem (1.6)

The proof of the Theorem (1.6) is very similar to the proof of the Theorem (1.1) with an essential difference in the construction of the Hamiltonian $\widehat{H}$ on P from a pre-admissible Hamiltonian H on M.

In what follows we will use the same notation of the proof of the Theorem (1.1) for $M := T^*N$ endowed with the canonical symplectic form ω_0 and define $J(x,\theta,d\theta) = H_1(x) + d\theta$, where $H_1(q,p) = p(X)$ (X is the vector field on N that generates the circle action) is the Hamiltonian corresponding to the lifted circle action on T^*N .

Let $\epsilon, \lambda > 0$ sufficiently small, $U = H_1^{-1}(-n/4+\epsilon, n/4-\epsilon)$ or $U = H^{-1}(0, 1/2-2\epsilon)$ or $U = H^{-1}(1/2+2\epsilon, 0)$ and H be a pre-admissible Hamiltonian on U. As in the proof of the Theorem (1.1), we will construct from H a ρ-invariant Hamiltonian $\widehat{H}$ on P but now given by

$$\widehat{H}|_{J^{-1}(\mu)}(z) = (H + \alpha(\mu)(m(H) - H))(\phi^1_{\lambda+\mu}(\widetilde{\pi}(z)) - (\lambda+\mu)\beta_{\pi(\phi^1_{\lambda+\mu}(\widetilde{\pi}(z)))}),$$

where $\phi^\lambda_\mu : T^*N \to T^*N$ is given by $\phi^\lambda_\mu(q,p) = (q, \frac{\lambda}{\mu}p)$.

Lemma 7.1. *Let Ω be a closed 2-form on N and λ, μ real constants different from zero. Then, we have that $(\phi^\lambda_\mu)^*\omega_\lambda = \frac{\lambda}{\mu}\omega_\mu$, where $\omega_\lambda := \omega_0 + \lambda\pi^*\Omega$.*

Proof. Let $\Omega' = \mu\Omega$. Then, for all $\lambda' \in \mathbb{R}$ we have that

$$\begin{aligned}
\phi^*_{\lambda'}\omega'_{\lambda'} &= \phi^*_{\lambda'}(\omega_0 + \lambda'\mu\pi^*\Omega) \\
&= \lambda'\omega_0 + \lambda'\mu\pi^*\Omega \\
&= \lambda'\omega'_1 \\
&= \lambda'(\omega_0 + \mu\pi^*\Omega),
\end{aligned}$$

where $\omega'_\lambda = \omega_0 + \lambda\pi^*\Omega'$ and $\phi_\lambda : T^*M \to T^*M$ is given by $\phi_\lambda(q,p) = (q,\lambda p)$. The equalities follows from the fact that $\phi^*_{\lambda'}\omega_0 = \lambda'\omega_0$ and $\phi^*_{\lambda'}\pi^*\Omega = \pi^*\Omega'$. Taking then $\lambda' = \frac{\lambda}{\mu}$ we have

$$\begin{aligned}
\phi^*_{\lambda'}\omega'_{\lambda'} &= \phi^*_{\lambda'}(\omega_0 + \lambda\pi^*\Omega) \\
&= \frac{\lambda}{\mu}(\omega_0 + \mu\pi^*\Omega).
\end{aligned}$$

Note that $\phi^\lambda_\mu = \phi_{\frac{\lambda}{\mu}}$. □

Using the same arguments of the Theorem (1.1), it is easy to see that $m(\widehat{H}) = m(H)$ and $\widehat{H} \in \mathcal{H}(\widehat{U}, \omega_P)$, where

$$\widehat{U} = \bigcup_{0\leq\mu\leq 1} \bigcup_{x\in U} \pi^{-1}_{\lambda+\mu}((\phi^1_{\lambda+\mu})^{-1}(x) + (\lambda+\mu)\beta_{\pi(x)}).$$

Let us prove that there exists a periodic orbit γ of X_H with period less than $(1/\lambda)\pi n/m(H)$ and homotopy class $[\gamma] \in G_\varphi$. Consider initially the Hamiltonian $G_\mu : T^*N \to \mathbb{R}$ given by

$$G_\mu(x) = (H + \alpha(\mu)(m(H) - H))(x),$$

for $0 \leq \mu \leq 1$. By the Lemma (7.1), the Hamiltonian flow of G_μ with respect to $\omega_1 = \omega_0 + \pi^* d\beta$ is topologically equivalent to a reparametrization of the Hamiltonian flow of

$$F_\mu(x) = (H + \alpha(\mu)(m(H) - H))(\phi^1_{\lambda+\mu}(x))$$

with respect to $\omega_{\lambda+\mu}$. On the other hand, since the transformation $\zeta_\lambda(q,p) = (q, p - \lambda_q)$ defines a symplectomorphism between (T^*N, ω_λ) and (T^*N, ω_0), the Hamiltonian flow of F_μ with respect to $\omega_{\lambda+\mu}$ is equivalent to the Hamiltonian flow of

$$H_\mu(x) = (H + \alpha(\mu)(m(H) - H))((\phi^1_{\lambda+\mu}(x) - (\lambda+\mu)\beta_{\pi(\phi^1_{\lambda+\mu}(x))})$$

with respect to ω_0. Summarizing, we have the following equivalences

$$\frac{1}{\lambda+\mu} X^{\omega_1}_{G_\mu} \overset{\phi^1_{\lambda+\mu}}{\simeq} X^{\omega_{\lambda+\mu}}_{F_\mu} \overset{\zeta_{\lambda+\mu}}{\simeq} X^{\omega_0}_{H_\mu},$$

where X^ω_H is the Hamiltonian vector field of H with respect to the symplectic form ω.

Thus, we have that if $X^{\omega_0}_{H_\mu}$ has a periodic orbit of period less than T, then $X^{\omega_1}_{G_\mu}$ also has a periodic orbit of period less than $(1/\lambda)T$.

Thus, since $\widehat{H}$ is obviously ρ-invariant, it follows by the previous discussion, the arguments in the proof of the Theorem (1.1), and the Lemma (2.3), that, if $\widehat{H}$ has a non-trivial periodic orbit of period T not tangent to an orbit of ρ, then $X^{\omega_1}_H$ has a non-trivial periodic orbit of period greater than or equal to $(1/\lambda)T$.

To finish the proof, we claim that it is sufficient to prove that

$$\left| \int_{\pi_2(\widehat{U})} \omega_0^{S^1} \right| < \pi n.$$

In effect, proceeding as in the proof of the Theorem (1.1), suppose that the periodic orbit of $X_{\widehat{H}}$ given by the Proposition (2.1) is tangent to an orbit of the circle action generated by J. Then, we have that along γ, $X_{\widehat{H}} = \left\langle X_{\widehat{H}}, \frac{X_J}{\|X_J\|^2} \right\rangle X_J$, and, since $[\gamma] \in \pi_1(S^1)$, the period of γ is given by

$$T = \frac{2\pi n_\varphi}{\left|\left\langle X_{\widehat{H}}, \frac{X_J}{\|X_J\|^2} \right\rangle\right|} \geq \frac{2\pi n}{\left|\left\langle X_{\widehat{H}}, \frac{X_J}{\|X_J\|^2} \right\rangle\right|} \geq \frac{2\pi n}{(2+2\delta)m(H)}.$$

Since $\delta > 0$ is arbitrarily small, we have that

$$T \geq \frac{\pi n}{m(H)}.$$

But we know that the period of γ satisfies

$$T < \frac{|\int_{\pi_2(\widehat{U})} \omega_0^{S^1}|}{m(H)} < \frac{\pi n}{m(H)},$$

a contradiction. Note that by the same argument of the Theorem (1.1), $[\gamma] \in G_\varphi$ and

$$\begin{aligned} c_{HZ}^{G_\varphi}(H_1^{-1}(-n/2, n/2), \omega_1) &= \sup_{\epsilon > 0} c_{HZ}^{G_\varphi}(H_1^{-1}(-n/2 + \epsilon, n/2 - \epsilon), \omega_1) \\ &< (1/\lambda)\pi n \\ &< \infty. \end{aligned}$$

To prove that $\left| \int_{\pi_2(\widehat{U})} \omega_0^{S^1} \right| < \pi n$, note that, by the definition of a connection, we have that

$$H_1(x - (\lambda + \mu)\beta_{\pi(x)}) = H_1(x) - (\lambda + \mu).$$

Thus, since $J^{-1}(\mu)$ the graph of $-H_1$ over $d\theta^{-1}(\mu)$ and $H_1((\phi^1_{\lambda+\mu})^{-1}(x)) = (\lambda + \mu)H_1(x)$, we conclude that

$$\sup_{x \in J^{-1}(\mu) \cap \widehat{U}} d\theta(x) - \inf_{x \in J^{-1}(\mu) \cap \widehat{U}} d\theta(x) \leq (\lambda + \mu)\|H_1|U\|.$$

Moreover, by the translation factor $-(\lambda + \mu)\beta$, we have that

$$\sup_{x \in \widehat{U}} d\theta(x) = -(\lambda + 1) \inf_{x \in \widehat{U}} H_1(x)$$

and

$$\inf_{x \in \widehat{U}} d\theta(x) = -(\lambda + 1) \sup_{x \in \widehat{U}} H_1(x).$$

Consequently, we have that

$$\begin{aligned} \left| \int_{\pi_2(\widehat{U})} \omega_0^{S^1} \right| &\leq 2\pi(1 + \lambda)\|H_1|_U\| \\ &< 2\pi(1 + \lambda)((n/2) - 2\epsilon) \\ &< \pi n, \end{aligned}$$

for $\lambda > 0$ and $\epsilon > 0$ sufficiently small.

8. Proof of the Theorem (1.7)

A diffeomorphism of a symplectic manifold (M, ω) is called Hamiltonian if it is given by the time one map of a time-dependent Hamiltonian vector field. Let $Ham(M, \omega)$ be the group of Hamiltonian diffeomorphisms of (M, ω). It was discovered by Hofer [19] that this group carries a natural biinvariant Finsler metric with a non-degenerate distance function. This distance is defined as follows. The Lie algebra $\mathcal{A}$ of $Ham(M, \omega)$ consists of all smooth functions on M satisfying a normalization condition which ensures that, when M is open, $F \in \mathcal{A}$ iff F has compact support, and, when M is closed, $F \in \mathcal{A}$ iff F has zero mean with respect to the volume form induced by ω. The adjoint action of $Ham(M, \omega)$ on $\mathcal{A}$ is the standard action of diffeomorphisms on functions. Consider the L^∞-norm in $\mathcal{A}$,

$$\|F\| = \max_M F - \min_M F$$

This norm is invariant under the adjoint action, and thus defines a biinvariant Finsler metric on $Ham(M, \omega)$. Now, let $\{\gamma_t\}$, $t \in [0, 1]$, a path of Hamiltonian diffeomorphisms, that is, a curve $\gamma : [0, 1] \to Ham(M, \omega)$. Let $F(x, t)$ be its normalized Hamiltonian function, that is, $F(\cdot, t) \in \mathcal{A}$ for all t. We define then the length of this curve by

$$l(\gamma) = \int_0^1 \|F(\cdot, t)\| dt$$

The Hofer's metric is then defined by

$$\rho(\phi, \psi) = \inf_\gamma l(\gamma)$$

where the infimum is taken over all paths γ which join ϕ and ψ.

For a subset U of M define the displacement energy $e(U)$ as the measure of the distance between the identity map and the set of $\psi \in Ham(M, \omega)$ which displaces U from itself, in the sense that $\psi(U) \cap U \neq \emptyset$.

Using the geometry of the Hofer's metric and that, for a manifold whose Euler characteristic vanishes, the displacement energy of the zero section of T^*M, with the twisted symplectic form given by a non-vanishing magnetic field, is equal to zero, Polterovich [34] proved the following theorem:

Theorem 8.1 (Polterovich [34]). *Let M be a closed manifold whose Euler characteristic vanishes and Ω a non-vanishing weakly exact magnetic field. Then there exist contractible magnetic closed orbits on a sequence of arbitrarily small energy levels.*

The idea of the proof is the following [34]: take a smooth function $r(x)$, $x \in [0, \infty)$, which equals $x - 2\epsilon$ on $[0, \epsilon]$, vanishes on $[3\epsilon, \infty)$ and is strictly increasing on $[0, 3\epsilon)$. Thus, the closed orbits of the (normalized) Hamiltonian $F = r \circ H$ corresponds to reparametrized magnetic periodic orbits whose energy is less than 3ϵ. The key point is that the minimum set of F coincides with the zero section of T^*M and so its displacement energy vanishes [35]. It implies that the asymptotic non-minimality of F is strictly less than 1 and thus that F (as a vector in the Lie algebra $\mathcal{A}$) does not generate a minimal geodesic in $Ham(M, \omega)$ [34]. Then, using a result of Lalonde and McDuff [26], we conclude that the Hamiltonian F has a non-constant contractible closed orbit, since Ω is weakly exact.

Remark. It should be noted that the contractibility of the periodic orbits given by Lalonde and McDuff follows by the fact that the proof of the Theorem 5.4 of [26] relies on the Hofer-Zehnder capacity-area inequality (Theorem 1.17 of [26]) proved by Floer, Hofer and Viterbo [13], where the periodic orbits are contractible (see [31]).

Proof of the Theorem 1.5: Consider the trivial S^1-bundle $S^1 \longrightarrow P = M \times S^1 \stackrel{\pi}{\longrightarrow} M$, the flat connection $d\theta$ and the product metric $\bar{g} = \langle , \rangle$. Let Ω_P be the pullback of the magnetic field Ω to P. The lifted circle action to TP is Hamiltonian with respect to the twisted symplectic form $\omega_1 = \omega_o + \pi_P^* \Omega_P$, where $\pi_P : TP \to P$ is the canonical projection. In fact, let $\widehat{X}$ the vector field on TP that generates the circle action. Then,

$$\begin{aligned} i_{\widehat{X}}\omega_1(\xi) &= i_{\widehat{X}}\omega_0(\xi) + i_{\widehat{X}}\pi_P^*\pi^*\Omega(\xi) \\ &= i_{\widehat{X}}\omega_0(\xi) + \Omega(\pi_* X, \pi_*(\pi_P)_*\xi) \\ &= i_{\widehat{X}}\omega_0(\xi) = d\theta(\xi), \end{aligned}$$

where $d\theta$ above is viewed as a function in TP. Moreover, the form $d\theta$ is invariant by the magnetic flow of TP, because taking a magnetic orbit γ in P, we have that

$$\begin{aligned} \gamma' d\theta(\gamma') &= \gamma'\langle \gamma', X\rangle \\ &= \langle \gamma'', X\rangle + \langle \gamma', \nabla_{\gamma'} X\rangle \\ &= \langle Y_P(\gamma'), X\rangle \\ &= \Omega_P(\gamma', X) \\ &= \Omega(\pi_*\gamma', \pi_* X) = 0, \end{aligned}$$

where Y_P denotes the Lorentz force of the magnetic field Ω_P (defined as the bundle map $Y : TP \to TP$ uniquely characterized by the property that $(\Omega_P)_x(v,w) = \bar{g}_x(Y_x(v), w)$ $\forall v, w \in T_x P$) and in the third equation we used the fact that X is parallel.

Thus, the magnetic flow on TP is equivariant with respect to the Hamiltonian lifted action of S^1 to TP. On the other hand, it is easy to see that the Marsden-Weinstein reduced symplectic manifold with respect to the circle action on (TP, ω_1) is given by $(TM, \omega_0 + \pi_M^*\Omega)$.

Moreover, the pullback of the Hamiltonian metric H_g in TM by $d\pi|_{d\theta^{-1}(\xi)}$ is given by $H_{\bar{g}}|_{d\theta^{-1}(\xi)} - \frac{1}{2}|\xi|^2$. Consequently, the reduced dynamics of the magnetic flow ϕ_P of P on $d\theta^{-1}(\xi)$ is given by the magnetic flow ϕ_M on M.

Now, suppose that Ω is weakly exact. Since P is the product, the fibers of P are non-contractible and, in fact, the subgroup generated by the fibers in $\pi_1(P)$ is isomorphic to $\mathbb{Z}$. Since the Euler characteristic of P vanishes and $\pi^*\Omega$ is weakly exact (because $\pi_2(M \times S^1) \simeq \pi_2(M)$), we can apply Polterovich's theorem to conclude that ϕ_P has contractible closed orbits γ_n of arbitrarily small energy. Since the subgroup in $\pi_1(P)$ generated by the fibers of P is cyclic infinite, these orbits cannot be given by the fibers. This ensures that they are projected on non-constant contractible periodic orbits of ϕ_M. Finally, note that the energy of these projected orbits is equal to

$$H_{\bar{g}}(\gamma_n) - \frac{1}{2}|\xi|_n^2 \stackrel{n\to\infty}{\longrightarrow} 0$$

where $\xi_n = d\theta(\gamma_n') \stackrel{n\to\infty}{\longrightarrow} 0$, because $\|\gamma_n'\|_{\bar{g}} \to 0$ as n grows to infinite. $\square$

References

[1] V. Arnold, *Some remarks on flows of line elements and frames*, Soviet Math. Dokl. **2** (1961), 565–564.

[2] V. Arnold, *First steps of symplectic topology*, Russian Math. Surveys **41** (1986), 1–21.

[3] P. Biran, *Lagrangian barriers and symplectic embeddings*, GAFA **11** (2001), no. 3, 407–464.

[4] P. Biran, K. Cieliebak, *Symplectic topology on subcritical manifolds*, Comment. Math. Helv. **76** (2001), 712–753.

[5] P. Biran, K. Cieliebak, *Lagrangian embeddings into subcritical Stein manifolds*, to appear in Israel Journal of Mathematics.

[6] P. Biran, L. Polterovich, D. Salamon, *Propagation in Hamiltonian dynamics and relative symplectic homology*, preprint 2001, math.SG/0108134.

[7] R. Bott, *On a theorem of Lefschetz*, Michigan Math. J. **6** (1959), 211–216.

[8] K. Burns, G. Paternain, *Anosov magnetic flows, critical values and topological entropy*, to appear in Nonlinearity.

[9] G. Contreras, L. Macarini, G. Paternain, *Periodic orbits for exact magnetic flows on surfaces*, preprint 2002.

[10] S. Donaldson, *Symplectic submanifolds and almost-complex geometry*, J. Differential Geom. **44** (1996), no. 4, 666–705.

[11] Y. Eliashberg, M. Gromov, *Convex symplectic manifolds* Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989), 135-162, Proc. Sympos. Pure Math., 52, Part 2, Amer. Math. Soc., Providence, RI, 1991.

[12] Y. Eliashberg, A. Givental, H. Hofer, *Introduction to symplectic field theory*, GAFA - Special Volume, Part II (2000), 560–673.

[13] A. Floer, H. Hofer, C. Viterbo, *The Weinstein conjecture in $P \times \mathbb{C}^l$*, Math. Z. **203** (1990), 469–482.

[14] V. Ginzburg, *A charge in a magnetic field: Arnold's problems 1981-9, 1982-24, 1984-4, 1994-14, 1996-17, and 1996-18*, preprint

[15] V. Ginzburg, *The Hamiltonian Seifert conjecture: examples and open problems*, Preprint 2000 math.DG/0004020.

[16] V. Ginzburg, E. Kerman, *Periodic orbits in magnetic fields in dimensions greater than two*, in *Geometry and topology in dynamics*, Contemp. Math **246**, 119–121.

[17] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Inv. Math. **82** (1985), 307–347.

[18] M. Herman, *Exemples de flots Hamiltoniens dont aucune perturbation en topologie C^∞ n'a d'orbites periodiques sur un ouvert de surfaces d'energies*, Comptes Rendus de l'Académie des Sciences de Paris **412** (1991), 989–994.

[19] H. Hofer, *Estimates for the energy of a symplectic map*, Com. Math. Helvetici **68** (1993), 48–72.

[20] H. Hofer, C. Viterbo, *The Weinstein conjecture in the presence of holomorphic spheres*, Comm. Pure. Apl. Math. **45** (1992), 583–622.

[21] H. Hofer, C. Viterbo, *The Weinstein conjecture in cotangent bundles and related results*, Ann. Sc. Norm. Sup. Pisa **15** (1988), 411–445.

[22] H. Hofer, E. Zehnder, *Symplectic invariants and Hamiltonian dynamics*, Birkhäuser Advanced Texts; Basel-Boston-Berlin, 1994.

[23] M. Jiang, *Hofer-Zehnder symplectic capacity for 2-dimensional manifolds*, Proc. Royal Soc. Edinburgh **123A** (1993), 945–950.

[24] E. Kerman, *Symplectic geometry and the motion of a particle in a magnetic field*, PhD. thesis, University of California - Santa Cruz, 2000.

[25] S. Kobayashi, *Principal fibre bundles with the 1-dimensional toroidal group*, Tohoku J. Math. **8** (1956), 29-45.

[26] L. Lalonde, D. McDuff, *Hofer's L^∞ geometry: geodesics and stability I, II*, Inv. Math. **122** (1995), 1–33,35–69.

[27] G. Lu, *The Weinstein conjecture on some symplectic manifolds containing the holomorphic spheres*, Kyushu J. Math **52** (1998), 331–351.

[28] G. Lu, *Errata to the "The Weinstein conjecture on some symplectic manifolds containing the holomorphic spheres"*, Kyushu J. Math **54** (2000).

[29] G. Lu, *A note on Lagrangian barrier theorem by P. Biran*, Preprint 2001, math.SG/0111183.

[30] D. McDuff, *Symplectic manifolds with contact-type boundaries*, Inventiones Mathematicae **103** (1991), no. 3, 651–671.

[31] D. McDuff, J. Slimowitz *Hofer-Zehnder capacity and length minimizing Hamiltonian paths*, Geom. Topol. **5** (2001), 799–830

[32] S. Novikov, *The Hamiltonian formalism and many-valued analogue of Morse theory* Russian Math.surveys **37** (1982), 1–56.

[33] R. Palais, T. Stewart, *Torus bundles over a torus*, Proc. of AMS **42** (1961), 29-29.

[34] L. Polterovich, *Geometry on the group of Hamiltonian diffeomorphisms*, Proceedings of the ICM - 1998 (Vol. II), 401–410.

[35] L. Polterovich, *An obstacle to non-Lagrangian intersections*, pp. 575-586 in The Floer Memorial Volume edited by H. Hofer, C. Taubes, A. Weinstein and E. Zehnder, Birkhäuser, 1995.

[36] M. Schwarz, *On the action spectrum for closed symplectically aspherical manifolds*, Pacific J. Math. **193** (2000), no. 2, 419–461.

[37] K. Siburg, *Symplectic capacities in two dimensions*, Manuscripta Math. **78** (1993), 149–163.

[38] M. Struwe, *Existence of periodic solutions of Hamiltonian systems on almost every energy surface*, Bol. da Sociedade Brasileira de Matemática **20** (1990), 49–58.

[39] C. Viterbo, *Exact Lagrange submanifolds, periodic orbits and the cohomology of free loop spaces*, J. Differential Geom. **47** (1997), no. 3, 420–468.

[40] A. Weinstein, *On the hypothesis of Rabinowitz's periodic orbit theorems*, J. Diff. Equation **33** (1979), 353–358.

[41] E. Zehnder, *Remarks on periodic solutions on hypersurfaces*, NATO ASI Series, Series C, In: Periodic Solutions of Hamiltonian Systems and Related Topics, **209** (1987), 267–279.

Instituto de Matemática Pura e Aplicada - IMPA, Estrada Dona Castorina, 110 - Jardim Botânico, 22460-320 Rio de Janeiro RJ, Brasil

E-mail address: `leonardo@impa.br`