BOUNDARY LAYERS IN PARABOLIC PERTURBATIONS OF SCALAR CONSERVATION LAWS

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Abstract. We consider the problem of estimating the boundary layer thickness for vanishing viscosity solutions of boundary value problems for parabolic perturbations of a scalar conservation law in a space strip in R^d. For the boundary layer thickness $\delta(\varepsilon)$ we obtain that one can take $\delta(\varepsilon)=\varepsilon^r$, for any $r<1/2$, arbitrarily close to $1/2$.

1. Introduction

The boundary-layer effect is a well known phenomenon in hydrodynamics. It occurs when solutions of the Navier-Stokes equations do not tend asymptotically to solutions of the Euler equations near boundaries as the viscosity ε vanishes. The same effect is also observed when the transition from a parabolic equation to an associated hyperbolic one is performed. The rigorous mathematical description of the asymptotic behaviour of viscous solutions near boundaries for the small viscosity ε is a challenging problem. R.E. Caflisch and M. Sanmartino [2] justified locally in time the Prandtl-type asymptotic near the boundary for the Navier-Stokes equations in a class of analytic solutions. On the other hand their result is no longer true in $L^\infty(0,T;W^{1,2}_{loc})$ for any T arbitrary small, as established by E. Grenier [9]. The asymptotic expansion method applied in [2] and [9] is based on the use of the inner scaled variable $Y=y/\varepsilon^q$, where $y>0$ is the flow domain, $(x,y)\in\mathbb{R}^d\times\mathbb{R}_+$. (The name of Y is borrowed from the A.M. Il'in paper [12].) The comparison between [2] and [9] shows that the choice $q=1/2$ is good for the analytic solutions but does not correspond with Sobolev-class solutions. Thus, to apply the asymptotic expansion method, one should first guess how to choose the inner variable $Y=y/\varepsilon^q$ or the boundary layer thickness $\delta=\varepsilon^q$.

The similar problem arises in the context of parabolic-hyperbolic equations. M. Gisclon and D. Serre [8] developed an asymptotic technique and constructed a boundary-layer profile with the inner variable $Y=y/\varepsilon$ for a class of one-dimensional perturbed hyperbolic systems when the boundary $y=0$ is not characteristic. Their results were generalized by E. Grenier and O. Guès [11] for the multi-dimensional case. Z. Xin [20] studied the boundary-layer profile for a one-dimensional perturbed scalar conservation law under the assumption that this law has a global smooth solution and the boundary $y=0$ is not characteristic, with the inner variable being $Y=y/\varepsilon$. When a boundary is totally characteristic, E. Grenier [10] constructed an asymptotic expansion using the inner variable $Y=y/\sqrt{\varepsilon}$ for a special class of multi-dimensional scalar conservation laws.

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Here, we are concerned with the boundary layer thickness for parabolic perturbations of a scalar conservation law with no specific concern about characteristics at boundaries. We address the case where the vanishing viscosity limit solution is identically zero.

Given the space strip

$$\Omega = \{ x \in \mathbb{R}^d : |y| < 1, x = (x_1, \cdots, x_{d-1}, y) \equiv (x', y) \},$$

we consider the initial boundary-value problem

$$U_t + \operatorname{div} f(U) = 0, \quad (t,x) \in Q_T = (0,T) \times \Omega, \tag{1.1}$$

$$U|_{y=\pm 1} = U_{\pm}(t, x'), \quad U|_{t=0} = u_0(x), \tag{1.2}$$

where $f = (f_1, \cdots, f_d)$. The parabolic perturbation of this problem is given by

$$u_t + \operatorname{div} f^{\varepsilon}(u) = \varepsilon \Delta u, \quad (t,x) \in Q_T, \tag{1.3}$$

$$u|_{y=\pm 1} = u^{\varepsilon}_{\pm}(t, x'), \quad u|_{t=0} = u^{\varepsilon}_0(x), \tag{1.4}$$

where $u^{\varepsilon}_{\pm}$, u^{ε}_0 are smooth approximations of $u_{\pm}$, u_0 in the respective L^1-norms, and f^{ε} is a smooth approximation of f locally in the C^2-norm. Let u^{ε} denote the solution of (1.3) and (1.4). One expects that an entropy solution U of (1.1) and (1.2) can be obtained as a limit of u^{ε} as $\varepsilon \downarrow 0$. But the convergence $u^{\varepsilon} \to U$ can not be very strong due to possible shocks of U and boundary layers near the surfaces $y = \pm 1$. Assuming that U has no shocks, we focus on the convergence $u^{\varepsilon} \to U$ in the presence of boundaries. Our analysis is based on the following definition of boundary layer.

Definition 1.1. A positive function $\delta(\varepsilon) \in C((0,1])$ is called a boundary layer thickness (BL-thickness) if $\delta(\varepsilon) \downarrow 0$, as $\varepsilon \downarrow 0$, and

$$\lim_{\varepsilon \to 0} \|u^{\varepsilon} - U\|_{L^{\infty}(0,T; W^{1,1}_g(\Omega_{\delta(\varepsilon)}))} = 0, \tag{1.5}$$

$$\liminf_{\varepsilon \to 0} \|u^{\varepsilon} - U\|_{L^{\infty}(0,T; W^{1,1}_g(\Omega))} > 0, \tag{1.6}$$

where Ω_{δ} is the subdomain $\{x \in \Omega : |y| < 1 - \delta\}$ and $W^{1,1}_g(\Omega)$ is a weighted Sobolev space with the norm

$$\|v\|_{W^{1,1}_g(\Omega)} = \|v\|_{1,g} + \|\nabla v\|_{1,g}, \quad \|v\|_{1,g} = \int_{\Omega} |v| e^{-|x'|} dx.$$

Remark 1.1. Given a BL-thickness $\delta(\varepsilon)$, any function such that $\delta_1(\varepsilon) \geq \delta(\varepsilon)$ and $\delta_1(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ is also a BL-thickness. Thus, there should exist a minimal BL-thickness $\delta_{\star}(\varepsilon)$, which may be considered true boundary-layer thickness.

Remark 1.2. The similar definition of $\delta(\varepsilon)$ was proposed by the authors in [7], with $W^{1,1}_g(\Omega)$ substituted by $L^{\infty}(\Omega)$, to justify the Stokes-Blasius law $\delta(\mu) \sim \sqrt{\mu}$ of the laminar boundary-layer thickness for axially symmetric flows of a compressible fluid between two cylinders when the shear viscosity μ is small and the dilatational viscosity is kept constant. In the present paper, the choice of $W^{1,1}_g(\Omega)$ is due to the fact that Ω is unbounded.

Remark 1.3. The use of the expanding subdomains $\Omega_{\delta_{\varepsilon}}$ in the condition (1.5) of Definition 1.1 yields the asymptotic nature of the boundary-layer theory. The violation of the restriction (1.6) whould mean the non-existence of the boundary-layer effect.

Our goal is to prove that $\delta_n(\varepsilon) = \varepsilon^{1/2-1/n}$ is a BL-thickness for any $n > 2$, in the sense of Definition 1.1. Clearly, $\inf_n \delta_n(\varepsilon) = \sqrt{\varepsilon}$ in agreement with the above mentioned results if no restrictions are imposed on characteristics of equation (1.1) at the boundary.

In order to give exact formulations of the results, we start with the notion of entropy solution to problem (1.1) ,(1.2). For bounded domains, this notion was first introduced in [1], in the context of BV solutions, then extended in [17] to allow L^∞ solutions. (See also [3, 4, 18, 5, 19] for some generalizations to hyperbolic systems and applications.) We recall this definition now.

Definition 1.2. Assume that

$$f \in C^1(\mathbb{R};\mathbb{R}^d), \quad u_0 \in L^\infty(\mathbb{R}), \quad u_\pm \in L^\infty((0,T)\times\mathbb{R}^{d-1}). \tag{1.7}$$

We say that $u \in L^\infty(Q_T)$ is an entropy solution of (1.1), (1.2) if it satisfies the following:

(i)

$$\int\limits_{Q_T} \eta(u)\varphi_t + q(u)\nabla\varphi \, dxdt \geq 0 \tag{1.8}$$

for all $\varphi \in C_0^\infty(Q_T)$, $\varphi \geq 0$, and for any convex entropy pair $(\eta, q) \in C^1(\mathbb{R})$, that is, $\eta(u)$ is convex and $q_j'(u) = \eta'(u) f_j'(u)$;

(ii)

$$\operatorname{ess}\lim_{y\to\pm1} \int\limits_0^T \int\limits_{\mathbb{R}^{d-1}} Q_d(u(t,x',y), u_\pm(t,x'))\beta(t,x')\, dx'\, dt \geq 0 \ (\leq 0) \tag{1.9}$$

for all $0 \leq \beta \in L^1((0,T)\times\mathbb{R}^{d-1})$ and for any boundary entropy pair $(H(u,v), Q(u,v))$. The latter means that, for each fixed v, $(H(u,v), Q(u,v))$ is a convex entropy pair for the variable u and $H(v,v) = Q(v,v) = \frac{\partial}{\partial u}H(v,v) = 0$;

(iii)

$$\operatorname{ess}\lim_{t\to0+} \int\limits_\Omega |u(t,x) - u_0(x)|\beta(x)\, dx = 0 \tag{1.10}$$

for all $0 \leq \beta \in L^1(\Omega)$.

In Section 2, we prove that there exists a unique entropy solution u of (1.1), (1.2) and that it can be obtained as the limit $u^\varepsilon \to u$, $\varepsilon \downarrow 0$, in $C([0,T]; L_g^1(\Omega))$, where u^ε is the solution of (1.3), (1.4), $\varepsilon > 0$, for smooth data $u_\pm^\varepsilon$, u_0^ε which satisfy suitable compatibility conditions at $y = \pm1$ and approximate the data $u_\pm$, u_0 in the respective L_g^1-norms, and $f^\varepsilon \to f$ locally in the C^2-norm, as $\varepsilon \downarrow 0$.

In Section 3, we then study the question of estmating the boundary layer thickness when $u = 0$, *i.e.*, $u^\varepsilon \to 0$ in $C([0,T]; L_g^1(\Omega))$ as $\varepsilon \downarrow 0$. We consider the following three cases:

$$f_d'(u) \leq 0, \quad u_+ = 0, \tag{1.11}$$

$$f_d'(u) \geq 0, \quad u_- = 0, \tag{1.12}$$

$$u f_d'(u) \geq 0, \quad f_d''(u) \geq 0, \quad u_- \leq 0, \quad u_+ \geq 0. \tag{1.13}$$

We remark that, since $Q_d(u,v)=\int_v^u H_u'(s,v)f_d'(s)ds$, the function $u\equiv 0$ and the data u_-,u_+ satisfy the condition (1.9) in each of the cases (1.11), (1.12), and (1.13) due to the properties of the boundary entropy pair (H,Q).

In what follows, the space $W_2^{2s,s}(Q_T)$ consists of functions $u(t,x)\in L^2(Q_T)$ such that

$$\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}\frac{\partial^{\beta}}{\partial t^{\beta}}u\in L^2(Q_T),\quad \alpha_1+\cdots+\alpha_d+2\beta\le 2s,\quad s\in\mathbb{N}.$$

Let us denote $\|f,g,\cdots\|=\|f\|+\|g\|\cdots$, where $\|\cdot\|$ is the norm in a functional space.

We also impose the following regularity assumption on the data $u_\pm$:

$$\begin{aligned}&u_\pm,\nabla u_\pm,\frac{\partial}{\partial t}u_\pm,\Delta u_\pm\in L^\infty((0,T)\times\mathbb{R}^{d-1}),\quad u_\pm\in W_2^{1,1}(Q_T),\\ &\nabla^2u_\pm,\nabla^3u_\pm,\frac{\partial}{\partial t}\nabla u_\pm,\frac{\partial}{\partial t}\nabla^2u_\pm,\frac{\partial^2}{\partial t^2}u_\pm\in L_g^1((0,T)\times\mathbb{R}^{d-1}).\end{aligned}\tag{1.14}$$

We now state the main result of this paper, whose proof will be given along Sections 2 and 3.

Theorem 1.1. *Assume $f^\varepsilon\equiv f\in C^{2s-1}(\mathbb{R};\mathbb{R}^d)$, $s\ge 2$, and let $u^\varepsilon\in W_2^{2s,s}(Q_T)$, $\varepsilon>0$, be the solutions of* (1.3) *and* (1.4) *corresponding to the zero initial data ($u_0^\varepsilon\equiv 0$) and the smooth boundary data independent of ε, $u_\pm^\varepsilon\equiv u_\pm$, satisfying* (1.14)*, and suitable compatibility conditions at $y=\pm1$ so that $u^\varepsilon\in W_2^{2s,s}(Q_T)$. Then $u^\varepsilon\to 0$ in $C([0,T];L_g^1(\Omega))$, as $\varepsilon\downarrow 0$, provided that one of the conditions* (1.11), (1.12) *or* (1.13) *is valid. Moreover, given a positive function $\delta(\varepsilon)\in C((0,1])$, we have*

$$\lim_{\varepsilon\to0}\|u^\varepsilon\|_{L^\infty(0,T;W_g^{1,1}(\Omega_{\delta(\varepsilon)})}=0,$$

as long as $\sqrt{\varepsilon}/\delta(\varepsilon)\to 0$, when $\varepsilon\to 0$. The inequality

$$\liminf_{\varepsilon\to0}\|u^\varepsilon\|_{L^\infty(0,T;W_g^{1,1}(\Omega))}>0$$

holds whenever the function $|u_-|^2+|u_+|^2$ is not identically zero.

Remark 1.4. The last inequality in Theorem 1.1 is a simple consequence of the continuity of the trace operator $W^{1,1}(G)\to L^1(\partial G,\mathcal{H}^{d-1})$ for any bounded domain $G\subseteq\mathbb{R}^d$ with the Lipschitz boundary [6].

2. Entropy solution

The existence and uniqueness of entropy solutions for a scalar equation is well studied both for the Cauchy problem and initial boundary-value problems in bounded domains [13, 1, 17]. To make the presentation self-contained and having in mind that some results of this section will be useful for the next one, we discuss existence and uniqueness schematically, paying more attention to the unboundedness of Ω.

First, we discuss uniqueness for problem (1.1), (1.2) under the assumptions (1.7). Let $\rho:\mathbb{R}^+\to\mathbb{R}$ be a non-increasing C^2 function such that $\rho(s)=1$ for $s\in[0,1/2]$, $\rho(s)$ is a non-negative polynomial for $s\in[1/2,1]$ and $\rho(s)=\exp(-s)$ for $s\ge 1$. Define $g(x')=\rho(|x'|)$. One can verify that $|\nabla g|\le cg$ and $|\Delta g|\le cg$ for some positive constant c.

Two solutions $u_i(t,x)$ with the data $u_i^0(x)$, $u_i^{\pm}(t,x')$ satisfy the inequality

$$
\begin{aligned}
(2.1)\qquad & -\int\limits_{Q_T} |u_1-u_2|\frac{\partial\beta}{\partial t} + F_j(u_1,u_2)\frac{\partial\beta}{\partial x_j}\,dx\,dt \\
& \le \int\limits_{\Omega} |u_1^0-u_2^0|\beta(0)\,dx + \int\limits_0^t \int\limits_{\mathbb{R}^{d-1}} \beta(t,x^{'},1)K_+ + \beta(t,x^{'},-1)K_-\,dx\,dt
\end{aligned}
$$

for all $0 \le \beta \in \mathcal{D}((\infty,T)\times\mathbb{R}^d)$, where

$$
\begin{gathered}
F_j(u_1,u_2) = \operatorname{sgn}(u_1-u_2)(f_j(u_1)-f_j(u_2)),\\
K_{\pm}(t,x') = \sup\{\,|f_d(a)-f_d(b)| : a,b\in[\min\{u_1^{\pm},u_2^{\pm}\},\max\{u_1^{\pm},u_2^{\pm}\}]\,\}.
\end{gathered}
$$

The derivation of (2.1) is performed in [16], following [17], for a bounded domain and a half-space. The proof of (2.1) for the space strip Ω is the same.

In (2.1), we choose $\beta = g\alpha_n(t)$, where $\alpha_n(t)=1$ for $1/n\le t\le\tau$, $\alpha_n(t)=nt$ for $0\le t\le 1/n$, $\alpha_n(t) = -n(t-\tau-1/n)$ for $\tau\le t\le\tau+1/n$, and $\alpha_n(t)=0$ for $\tau+1/n\le t\le T$. By the standard continuity argument, inequality (2.1) is also valid with $\beta = g\alpha_n$. Letting $n\to\infty$ and observing that $K_{\pm}\le M|u_1^{\pm}-u_2^{\pm}|$, $M=\sup|f'(u)|$, we obtain

$$
\|u_1(\tau)-u_2(\tau)\|_{1,g} \le c(\tau)\big\{\|u_1^0-u_2^0\|_{1,g} + M\int\limits_0^{\tau} \|u_1^+-u_2^+\|_{1,g} + \|u_1^- - u_2^-\|_{1,g}\,dt\big\},
$$

provided that τ is a Lebesgue point of $\|u_1(t)-u_2(t)\|_{1,g}$, and so for a.e. $\tau>0$. Therefore, the uniqueness is proved.

Next, we discuss the solution of (1.3), (1.4) and the vanishing viscosity limit as $\varepsilon\to 0$, yielding an entropy solution for (1.1), (1.2). We start with the linear parabolic problem

$$
u_t = \varepsilon\Delta u - \lambda u + F(t,x) \quad u|_{y=\pm1} = u_{\pm}(t,x^{'}),\quad u|_{t=0}=u_0(x),\quad \lambda\ge 0.
$$

The unique solvability in the class $W_2^{2s,s}(Q_T)$ is demonstrated in [14, 15] when $u_0\in W^{2s-1,s}(\Omega)$, $u_{\pm}\in W_2^{2s,s}((0,T)\times\mathbb{R}^{d-1})$, $F\in W_2^{2s,s}(Q_T)$ and corresponding compatibility conditions are satisfied. Then the fixed-point argument as in [16] can be applied to prove existence in the class $W_2^{2s,s}(Q_T)$ for the non-linear problem (1.3), (1.4), under the same regularity of the data $u_0^{\varepsilon}, u_{\pm}^{\varepsilon}$ if $f^{\varepsilon}\in C^{2s-1}(\mathbb{R})$ and appropriate compatibility conditions are satisfied. Here, $f^{\varepsilon}\to f$ in $C^2_{loc}(\mathbb{R})$, as $\varepsilon\to 0$.

Let us derive some estimates for the solutions u^{ε} to problem (1.3), (1.4), uniformly in ε. In what follows, c stands for different positive constants independent of ε, and we drop the superscripts ε since ε will be kept fixed for the moment. If both $\|u_0^{\varepsilon}|L^{\infty}(\Omega)\|\le c$ and $\|u_{\pm}^{\varepsilon}|L^{\infty}((0,T)\times\mathbb{R}^{d-1})\|\le c$ then, by the generalized maximum principle [16], $u\in L^{\infty}(Q_T)$ and

$$
\|u\|_{L^{\infty}(Q_T)} \le \max\{\|u_0^{\varepsilon}\|_{L^{\infty}(\Omega)}, \|u_+^{\varepsilon}\|_{L^{\infty}((0,T)\times\mathbb{R}^{d-1})}, \|u_-^{\varepsilon}\|_{L^{\infty}((0,T)\times\mathbb{R}^{d-1})}\} \le c.
$$

Let $u^D = u_+(1+y)/2 - u_-(y-1)/2$ be an extension of u_+ and u_- into Ω. Denote

$$\|u_0, u_\pm\|_D = \|\Delta u, \frac{\partial u^D}{\partial t}, \nabla u^D, u^D\|_{L^\infty(Q_T)} + \|u_0\|_{L^\infty(\Omega)} + \|\Delta u_0, \nabla u_0, u_0\|_{1,g}$$
$$+ \int_0^T \|\nabla^2 \frac{\partial u^D}{\partial t}, \nabla^3 u^D, \nabla^2 u^D, \frac{\partial^2 u^D}{\partial t^2}, \nabla \frac{\partial u^D}{\partial t}\|_{1,g}\, dt.$$

Lemma 2.1. *The solution u of* (1.3), (1.4) *satisfies the following estimate*

$$\sup_{t\in(0,T)} \|\nabla u(t), \frac{\partial u(t)}{\partial t}\|_{1,g} \le c,$$

where c depends only on $\|u_0^\varepsilon, u_\pm^\varepsilon\|_D$ and $\sup\{|f^{\varepsilon\prime\prime}(s)| : |s| \le \|u\|_\infty\}$.

Proof. Define $\omega = u - u^D$, $w_i = \frac{\partial\omega}{\partial x_i}$, $x_d = y$. Clearly, $\omega = 0$, $w_i = 0$, $1 \le i \le d-1$, at $y = \pm 1$, and

$$\frac{\partial w}{\partial t} + \operatorname{div}(f^\varepsilon(u) - f^\varepsilon(u^D)) - \varepsilon\Delta w = g, \tag{2.2}$$

$$\frac{\partial w_i}{\partial t} + \operatorname{div}(f^{\varepsilon\prime}(u) w_i) - \varepsilon\Delta w_i = -h_i, \quad 1 \le i \le d, \tag{2.3}$$

where

$$g \equiv \frac{\partial u^D}{\partial t} + \operatorname{div} f^\varepsilon(u^D) - \varepsilon\Delta u^D,$$
$$h_i \equiv \frac{\partial^2 u^D}{\partial t\partial x_i} + \operatorname{div}(f^{\varepsilon\prime}(u)\frac{\partial u^D}{\partial x_i}) - \varepsilon\Delta\frac{\partial u^D}{\partial x_i}.$$

In what follows we will frequently use the functions $\varphi_\eta(r) = (r^2 + \eta^2)^{1/2}$ which have the properties

$$r\varphi_\eta''(r) \le \eta, \quad \varphi_\eta(r) \to |r|, \quad r\varphi_\eta'(r) \to |r|, \quad \eta \downarrow 0.$$

We multiply (2.3) by $g\frac{\partial}{\partial v_i}\psi_\eta(\mathbf{w})$, $i = 1, \dots, d$, where $\psi_\eta(\mathbf{v}) = \varphi_\eta(|\mathbf{v}|)$, sum up and integrate over $(0,t)\times\Omega$. Using the standard sumation convention for repeated indices (as it will be done many times in what follows without explicit mention) and the notation $f(x)|_a^b = f(b) - f(a)$, we write the result as

$$\int_\Omega g\psi_\eta(\mathbf{w}(\mathbf{s}))|_0^t \quad dx + \int_0^t\int_\Omega g\frac{\partial^2\psi_\eta}{\partial v_i\partial v_k}(\mathbf{w})\{\varepsilon\frac{\partial w_i}{\partial x_j}\frac{\partial w_k}{\partial x_j} - w_i\frac{\partial w_k}{\partial x_j}f_j^{\varepsilon\prime}(u)\} \quad dxdt$$

$$+ \int_0^t\int_{\mathbb{R}^{d-1}} g\{f_d^{\varepsilon\prime}(u)w_d - \varepsilon\frac{\partial^2\omega}{\partial y^2}\}\frac{\partial\psi_\eta(\mathbf{w})}{\partial v_d}|_{y=-1}^{y=1} \quad dx'\, dt - \varepsilon\int_0^t\int_\Omega \Delta g\psi_\eta(\mathbf{w})dxdt$$

$$- \int_0^t\int_\Omega f_j^{\varepsilon\prime}(u)\frac{\partial g}{\partial x_j}w_i\frac{\partial\psi_\eta(\mathbf{w})}{\partial v_i}dxdt - \int_0^t\int_\Omega h_i g\frac{\partial\psi_\eta(\mathbf{w})}{\partial v_i}dxdt \equiv \sum_1^6 J_k = 0.$$

Observe that

$$\sum_{i,j,k}(w_k^2\frac{\partial w_i}{\partial x_j}\frac{\partial w_i}{\partial x_j} - w_i\frac{\partial w_i}{\partial x_j}w_k\frac{\partial w_k}{\partial x_j}) \ge 0.$$

Thus,

$$J_2 \geq \int_0^t \int_\Omega \frac{\eta^2 g}{(|\mathbf{w}|^2+\eta^2)^{3/2}}\{\varepsilon \frac{\partial w_i}{\partial x_j}\frac{\partial w_i}{\partial x_j} - w_i \frac{\partial w_i}{\partial x_j} {f_j^\varepsilon}'(u)\} dxdt \geq -\frac{M^2\eta}{4\varepsilon}\int_0^t\int_\Omega g dxdt.$$

It follows from (2.2) that

$${f_d^\varepsilon}'(u)w_d - \varepsilon\frac{\partial^2\omega}{\partial y^2} = -a \equiv -(u_t^D + \operatorname{div} f^\varepsilon(u^D) - \varepsilon\Delta u^D)$$

over $y=\pm 1$. Therefore,

$$J_3 \geq -\int_0^t \int_{\mathbb{R}^{d-1}} (|a(t,x',1)| + |a(t,x',-1)|) g\, dx'\, dt \equiv -J_3'.$$

Letting $\eta \to 0$, we get

$$\|\mathbf{w}(t)\|_{1,g} \leq \|\mathbf{w}(0)\|_{1,g} + J_3' + c\int_0^t \|\mathbf{w}(s)\|_{1,g} + \|\mathbf{h}(s)\|_{1,g}\, ds.$$

This implies

$$\|\nabla u(t)\|_{1,g} \leq \lambda(1 + \int_0^t \|\nabla u(s)\|_{1,g}\, ds),$$

where λ depends on T and $\|u_0^\varepsilon, u_\pm^\varepsilon\|_D$. The first part of the lemma now follows from Gronwall's lemma.

The function $v = u_t - u_t^D$ satisfies the equation

$$\frac{\partial v}{\partial t} + \operatorname{div}({f^\varepsilon}'(u)v) - \varepsilon\Delta v = b,$$

$$b \equiv \frac{\partial^2 u^D}{\partial t^2} + \operatorname{div}({f^\varepsilon}'(u)\frac{\partial u^D}{\partial t}) - \varepsilon\Delta\frac{\partial u^D}{\partial t}.$$

We multiply it by $g\varphi_\eta'(v)$ and integrate over $(0,t)\times\Omega$, obtaining

$$\|\frac{\partial u(t)}{\partial t}\|_{1,g} \leq \lambda(1 + \int_0^t \|\nabla u(s)\|_{1,g} ds).$$

This implies the last assertion of the lemma. □

Lemma 2.2. *Let u_1 and u_2 be solutions of* (1.3), (1.4) *corresponding to the smooth initial data u_1^0, u_2^0 and boundary data $u_1^\pm$, $u_2^\pm$ respectively. Then*

$$\|\xi_\varepsilon(u_1(t)-u_2(t))\|_{1,g} \leq (M+M^2)c\int_0^t \|u_1^+ - u_2^+\|_{1,g} + \|u_1^- - u_2^-\|_{1,g}\, ds$$
$$+ e^{cMt}\|\xi_\varepsilon(u_1^0 - u_2^0)\|_{1,g}.$$

The constant c depends on g only. Here, $\xi_\varepsilon(y) = 1 - \exp(-Ms_\delta(y)/\varepsilon)$, where $s_\delta(y)$ is piecewise linear, $s_\delta(-1) = s_\delta(1) = 0$, and $s_\delta(y) = \delta$ if $|y| \leq 1-\delta$.

Proof. Setting

$$\omega = u_1 - u_2, \quad u^{\pm} = u_1^{\pm} - u_2^{\pm}, \quad u^0 = u_1^0 - u_2^0,$$

we have

$$\omega_t + \operatorname{div}(f^\varepsilon(u_1) - f^\varepsilon(u_2)) = \varepsilon \Delta\omega.$$

We multiply this equation by $g\xi_\varepsilon \varphi_\eta'(\omega)$ and integrate over $(0,t)\times\Omega$. Let us discuss the resulting equality

$$\int\limits_\Omega g\xi_\varepsilon \varphi_\eta(\omega)|_0^t \, dx - \int\limits_0^t \int\limits_\Omega g\xi_\varepsilon \varphi_\eta''(\omega)\nabla\omega(f^\varepsilon(u_1) - f^\varepsilon(u_2) - \varepsilon\nabla\omega)\, dx\, ds$$

$$-\int\limits_0^t \int\limits_\Omega (f^\varepsilon(u_1) - f^\varepsilon(u_2))\nabla g\xi_\varepsilon \varphi_\eta'(\omega)\, dx\, ds - \int\limits_0^t \int\limits_\Omega (f_d^\varepsilon(u_1) - f_d^\varepsilon(u_2)) g\xi_\varepsilon' \varphi_\eta'(\omega)\, dx\, dt$$

$$+\varepsilon \int\limits_0^t \int\limits_\Omega \nabla\omega\nabla g\xi_\varepsilon \varphi_\eta'(\omega)\, dx\, ds - \varepsilon \int\limits_0^t \int\limits_\Omega \frac{\partial\omega}{\partial y} g\xi_\varepsilon' \varphi_\eta'(\omega)\, dx\, ds \equiv \sum_1^6 J_j = 0.$$

As in Lemma 2.1,

$$J_2 \geq -\frac{M^2\eta}{4\varepsilon} \int\limits_0^t \int\limits_\Omega g\xi_\varepsilon \quad dxds.$$

We also have

$$\lim_{\eta\to 0} J_5 = -\lim_{\eta\to 0} \varepsilon \int\limits_0^t \int\limits_\Omega \Delta g\xi_\varepsilon \varphi_\eta(\omega)\, dx\, ds \geq -c\varepsilon \int\limits_0^t \|\xi_\varepsilon \omega(s)\|_{1,g}\, ds.$$

Similarly,

$$\lim_{\eta\to 0} J_3 \geq -Mc \int\limits_0^t \|\xi_\varepsilon \omega(s)\|_{1,g}\, ds.$$

Due to the choice of ξ_ε,

$$|J_4| \leq M \int\limits_0^t \int\limits_\Omega g|\xi_\varepsilon'|\varphi_\eta(\omega)\, dx\, ds \tag{2.4}$$

$$\leq -J_6 + M \int\limits_0^t \int\limits_{\mathbb{R}^{d-1}} (\, \varphi_\eta(\omega)|_{y=1} + \varphi_\eta(\omega)|_{y=-1}\,)\, g\, dx'\, ds.$$

Sending η to zero, we obtain

$$z(t) \leq z(0) + c(M+\varepsilon) \int\limits_0^t z(s)\, ds + M \int\limits_0^t \int\limits_{\mathbb{R}^{d-1}} (|u_1^+ - u_2^+| + |u_1^- - u_2^-|)\, g\, dx'\, ds,$$

where

$$z(t) = \|\xi_\varepsilon g(u_1(t) - u_2(t))\|_{1,g}.$$

Therefore, applying Gronwall's estimate one obtains the desired conclusion. ∎

Theorem 2.1. *Let $f, f^\varepsilon \in C^2(\mathbb{R})$, $f^\varepsilon \to f$ in $C_{loc}(\mathbb{R})$, with $f^{\varepsilon\prime\prime}$ bounded, uniformly in ε. Let $u^\varepsilon \in W_2^{4,2}(Q_T)$ be the solutions of problem (1.3), (1.4) with the data*

$$u_0^\varepsilon \in W^{3,2}(\Omega) \cap L^\infty(\Omega), \quad u_\pm^\varepsilon \in W^{4,2}((0,T)\times\mathbb{R}^{d-1}) \cap L^\infty((0,T)\times\mathbb{R}^{d-1}),$$
$$\|\Delta u_\pm^\varepsilon, \frac{\partial u_\pm^\varepsilon}{\partial t}\|_{L^\infty(\Omega)} < \infty,$$

satisfying suitable compatibility conditions at $y = \pm 1$. Let $u_0^\varepsilon, u_\pm^\varepsilon$ be uniformly bounded in the respective L^∞-norms, as $\varepsilon \to 0$, and $u_0^\varepsilon \to u_0$ in $L_g^1(\Omega)$, $u_\pm^\varepsilon \to u_\pm$ in $L_g^1((0,T)\times\mathbb{R}^{d-1})$, where $u_0 \in L^\infty(\Omega)$, $u_\pm \in L^\infty((0,T)\times\mathbb{R}^{d-1})$. Then the sequence u^ε is also uniformly bounded in ε and converges in $C([0,T];L_g^1(\Omega))$ to the entropy solution $u \in L^\infty(\Omega)$ of problem (1.1), (1.2).

Remark 2.1. It can easily be verified that any function $v \in L^\infty(\Omega)$ can be approximated in $L_g^1(\Omega)$ by functions from $L^\infty(\Omega)\cap W^{s,2}(\Omega)$ for any $s > 1$.

Proof. Define an extention u_Γ^ε of the data u_0^ε, $u_\pm^\varepsilon$ by

$$u_\Gamma^\varepsilon(t,x',\pm1+s) = u_\pm^\varepsilon(t,x'), \quad 0<t<T, \quad |s|<\min\{t,\delta\},$$
$$u_\Gamma^\varepsilon(t,x) = u_0^\varepsilon(x), \quad -\delta < t < \min\{\delta, 1-|y|\},$$

and $u_\Gamma^\varepsilon(t,x) = 0$ elsewhere. Given a standard mollifier $\rho_h(t,x)$, we set

$$u_\Gamma^{\varepsilon,h}(t,x) = u_\Gamma^\varepsilon * \rho_h, \quad u_{\Gamma,0}^{\varepsilon,h}(x) = u_\Gamma^{\varepsilon,h}(0,x), \quad u_{\Gamma,\pm}^{\varepsilon,h} = u_\Gamma^{\varepsilon,h}(t,x',\pm1).$$

Clearly, $u_{\Gamma,0}^{\varepsilon,h} \to u_0^\varepsilon$ in $L_g^1(\Omega)$, $u_{\Gamma,\pm}^{\varepsilon,h} \to u_\pm^\varepsilon$ in $L_g^1((0,T)\times\mathbb{R}^{d-1})$ uniformly in ε as $h \to 0$. Moreover, the functions $u_{\Gamma,0}^{\varepsilon,h}$ and $u_{\Gamma,\pm}^{\varepsilon,h}$ are bounded in the respective L^∞-norms uniformly in ε and h. Denote by $u_{\varepsilon,h}$ the solutions of (1.3), (1.4) with the data $u_{\Gamma,0}^{\varepsilon,h}$ and $u_{\Gamma,\pm}^{\varepsilon,h}$. By Lemma2.2, using the fact that $u_{\varepsilon,h}$ is uniformly bounded, we get $u_{\varepsilon,h} \to u^\varepsilon$ in $C([0,T];L_g^1(\Omega))$, uniformly in ε as $h \to 0$. Because of the boundedness of u_0^ε, and $u_\pm^\varepsilon$ in the respective L_g^1-norms, we have

$$\|u_{\Gamma,0}^{\varepsilon,h}, u_{\Gamma,\pm}^{\varepsilon,h}\|_D \le ch^{-3}.$$

By Lemma 2.1 and the theorem of Arzelà-Ascoli, given $\varepsilon_k \downarrow 0$, the sequence $u_{\varepsilon_k,h}$ is compact in $C([0,T];L_g^1(\Omega))$. Therefore, any sequence u^{ε_k}, of solutions of (1.3), (1.4), with $\varepsilon = \varepsilon_k$, is compact in $C([0,T];L_g^1(\Omega_\delta))$, for any $0<\delta<1$, and there exists $u \in L^\infty(Q_T)$ such that $u^{\varepsilon_k} \to u$ in $C([0,T];L_g^1(\Omega))$.

Let us verify that u is an entropy solution of (1.1), (1.2). Given a boundary entropy pair for (1.3),

$$(H(u,k), Q^\varepsilon(u,k)), \quad Q^\varepsilon(u,k) = \int_k^u H'(v,k) f^{\varepsilon\prime}(v)dv,$$

we have

$$\frac{\partial}{\partial t} H(u_\varepsilon, l) + \operatorname{div} Q^\varepsilon(u_\varepsilon, l) \le \varepsilon \Delta H(u_\varepsilon, l).$$

Given $0 \le \beta \in \mathcal{D}((-\infty,T)\times\Omega)$, we multiply this inequality by $\beta\xi_\varepsilon(y)$ integrate over Q_T , and obtain

$$-\int\limits_{Q_T}\xi_\varepsilon(H\frac{\partial\beta}{\partial t}+Q^\varepsilon\nabla\beta+\varepsilon H\Delta\beta)\,dx\,dt-\int\limits_{\Omega}H\beta\xi_\varepsilon dx|_{t=0}-\int\limits_{Q_T}Q^\varepsilon_d\beta\xi'_\varepsilon\,dx\,dt$$

$$+\varepsilon\int\limits_{Q_T}\xi'_\varepsilon\frac{\partial(H\beta)}{\partial y}dxdt-2\varepsilon\int\limits_{Q_T}H\xi'_\varepsilon\frac{\partial\beta}{\partial y}dxdt\equiv\sum_1^5 J_i\le 0.$$

Observe that $|Q^\varepsilon_d(u,k)|\le MH(u,k)$ and

$$J_3+J_4\ge -M\int\limits_{Q_T}H\beta|\xi'_\varepsilon|\,dx\,dt+J_4\ge -M\int\limits_0^t\int\limits_{\mathbb{R}^{d-1}}(H\beta)|_{y=1}+(H\beta)|_{y=-1}\,dx'\,ds,$$

where the last inequality follows from (2.4) if one substitutes g by 1 and φ_η by $H\beta$. Since $\|\xi_\varepsilon,\varepsilon\xi'_\varepsilon|\,L^1(-1,1)\|\to 0$, as $\varepsilon\to 0$, we get in the limit, as $\varepsilon_k\to 0$,

$$-\int\limits_{Q_T}H(U,l)\frac{\partial\beta}{\partial t}+Q(U,l)\nabla\beta\,dx\,dt\le\int\limits_{\Omega}H(u_0,l)\beta(0)\,dx$$

$$+M\int\limits_0^t\int\limits_{\mathbb{R}^{d-1}}(H(u_+,l)\beta)|_{y=1}+(H(u_-,l)\beta)|_{y=-1}\,dx'\,ds,$$

where $Q(u,k)=\int_k^u H'(v,k)f'(v)\,dv$. Hence, u is an entropy solution of (1.1), (1.2) since this inequality is equivalent to Definition 1.2 (see [16] for details). By uniqueness, $u^\varepsilon\to u$ in $C([0,T];L^1_g(\Omega))$, as $\varepsilon\to 0$. The theorem is proved. □

3. Boundary layer

Consider problem (1.3), (1.4) and assume that its solution u converges to zero in $C([0,T];L^1_g(\Omega))$, as $\varepsilon\to 0$. It will be the case due to the uniqueness of entropy solutions for problem (1.1), (1.2) provided the data $f(u)$, u_0, u_+, and u_- satisfy the conditions of Theorem 1.1.

Given $\delta\in(0,1/2)$, let $\zeta_\delta(y)$ be an even function such that $\zeta_\delta(y)=1$ if $0\le y\le 1-\delta$ and

$$\zeta_\delta(y)=1-2\frac{(y-1+\delta)^2}{\delta^2},\quad 1-\delta\le y\le 1-\frac{\delta}{2},$$

$$\zeta_\delta(y)=\frac{2(y-1)^2}{\delta^2},\quad 1-\frac{\delta}{2}\le y\le 1.$$

Clearly,

$$0\le\zeta_\delta\le 1,\quad |\zeta_\delta^{'}|\le\frac{2}{\delta},\quad |\zeta_\delta^{''}|\le\frac{4}{\delta^2},\quad \zeta_\delta(\pm1)=\zeta_\delta^{'}(\pm1)=0.$$

The function ζ_δ will be used in the case (1.9). As for the case (1.7), we shall use the function ζ_δ^- such that $\zeta_\delta^-(y)=\zeta_\delta(y)$ for $-1\le y\le 0$ and $\zeta_\delta^-(y)=1$ for $0\le y\le 1$. In the case (1.8), we shall use the function ζ_δ^+ such that $\zeta_\delta^+(y)=1$ for $-1\le y\le 0$ and $\zeta_\delta^+(y)=\zeta_\delta(y)$ for $0\le y\le 1$.

Henceforth, to simplify notation, we drop the superscript ε of u^ε, the solutions of (1.3), (1.4), $\varepsilon>0$, referring to them simply as u.

Lemma 3.1. *Let the function $f(u)$ and the data u_+ and u_- meet the conditions* (1.13). *Then, there exists a positive constant c_0 independent of ε such that the solution u of problem* (1.3), (1.4) *satisfy the estimates*

$$c_0(y-1) \le f_d'(u) \le c_0(y+1).$$

Proof. Clearly, there is a constant $c > 0$ such that the function $w = u - c(y+1)$ is non-positive at $y = \pm 1$ and at $t = 0$. Since

$$w_t + f'(u) \cdot \nabla w + c f_d'(u) - \varepsilon \Delta w = 0,$$

we get, by the maximum principle, $w \le 0$, *i.e.*, $u \le c(y+1)$. Note that $f_d'' \ge 0$, therefore,

$$f_d'(u) \le f_d'(c(y+1)) \le \sup_v f_d''(v) c(y+1).$$

The boundedness from below for $f_d'(u)$ can be obtained similarly. The lemma is proved. ☐

Lemma 3.2. *Under the conditions of Theorem 1.1, all the solutions of problem* (1.3), (1.4) *satisfy the inequality*

$$\int_\Omega |\nabla u| g\zeta_\delta \, dx \le \int_0^t \int_\Omega |\nabla u| f'(u) \cdot \nabla(g\zeta_\delta)\, dx\, ds + \varepsilon \int_0^t \int_\Omega |\nabla u| \Delta(g\zeta_\delta)\, dx\, ds, \tag{3.1}$$

with ζ_δ replaced by ζ_δ^- and ζ_δ^+ in the cases (1.11) *and* (1.12), *respectively.*

Proof. Denoting $w_i = \partial u/\partial x_i$, we differentiate equation (1.3) with respect to x_i, multiply by $g\zeta_\delta \partial\psi_\eta(\mathbf{w})/\partial v_i$, sum up, and integrate over $(0,t)\times\Omega$. Let us discuss the resulting terms:

$$\int_0^t \int_\Omega \frac{\partial w_i}{\partial t} g\zeta_\delta \frac{\partial \psi_\eta(\mathbf{w})}{\partial v_i} dx dt = \int_\Omega \Phi \varphi_\eta(|\nabla u(t)|) dx, \quad \Phi(x) = g(x')\zeta_\delta(y),$$

$$\int_0^t \int_\Omega \Phi \frac{\partial \psi_\eta(\mathbf{w})}{\partial v_i} \operatorname{div}(f' w_i - \varepsilon \nabla w_i) dx dt =$$

$$\int_0^t \int_{\mathbb{R}^{d-1}} \{(f'w_i - \varepsilon\nabla w_i)\Phi \frac{\partial\psi_\eta(\mathbf{w})}{\partial v_i}\}|_{y=-1}^{y=1} \, dx\, dt$$

$$- \int_0^t \int_\Omega \Phi (f' w_i - \varepsilon \nabla w_i) \cdot \nabla \frac{\partial \psi_\eta(\mathbf{w})}{\partial v_i}\, dx\, dt$$

$$- \int_0^t \int_\Omega w_i \frac{\partial \psi_\eta(\mathbf{w})}{\partial v_i} f' \cdot \nabla \Phi dx dt - \varepsilon \int_0^t \int_\Omega \varphi_\eta(|\nabla u|) \Delta\Phi dx dt \equiv \sum_1^4 J_j.$$

In the case (1.9), we have $J_1 = 0$ since ζ_δ vanishes at $y = \pm 1$. In the case (1.11), we have $J_1 = 0$ since $\zeta_\delta^-(-1) = 0$ and $f'w_i - \varepsilon\nabla w_i = 0$ at $y = 1$ due to equation (1.3) and the fact that $u_+ = 0$. Similarly, we have $J_1 = 0$ in the case (1.12). As in

Lemma 2.1,

$$J_2 \geq -\frac{M^2\eta}{4\varepsilon}\int_0^t\int_\Omega \Phi dxdt.$$

Sending η to 0, we arrive at inequality (3.1).

$\square$

Proof of Theorem 1.1. First, consider the cases (1.11) and (1.12). Taking into account the properties of the functions $g(x')$ and $\zeta_\delta(y)$, the inequality $f_d'(u)\partial\zeta_\delta^\pm/\partial y \leq 0$, and Lemma 2.1, we get from (3.1) that

$$Y' \leq cY + \frac{c\varepsilon}{\delta^2}, \quad Y(t) = \int_0^t \|\nabla u(s)\|_{1,g}\, ds.$$

By Gronwall's lemma,

$$\|\nabla u(t)\|_{1,g} \leq \frac{c\varepsilon}{\delta^2}.$$

Notice that $\|u(t)\|_{1,g} \to 0$ uniformly in t as $\varepsilon \to 0$. Hence. Theorem 1.1 is proved in the cases (1.11) and (1.12).

Assume now that the data f, u_+, and u_- meet the conditions (1.13). In that case

$$f_d^{'}(u)\zeta_\delta^{'} \leq 2c_0\zeta_\delta$$

by Lemma 3.1. Again, the assertion of Theorem 1.1 follows from inequality (3.1).

$\square$

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