On generalized Benjamin type equations

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Abstract. We establish local and global well-posedness for the initial value problem associated to the generalized Benjamin equation and generalizations of this in the energy space. We also studied the limit process of solutions when the surface tension is becoming small. To establish these results we make use of sharp theory developed to the study of the generalized Korteweg-de Vries equation.

1. Introduction

This work is concerned with the Cauchy problem associated to the generalized Benjamin-type equation, that is,

$$\begin{cases} u_t + u^p u_x - \beta\, D^{2m}\, u_x - \alpha\, \mathcal{H}^{2r}\, u_x = 0, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where $\widehat{D^{2m}\, u}(\xi) = |\xi|^{2m}\widehat{u}(\xi)$, $m \geq 1$, $\widehat{\mathcal{H}^{2r}\, u}(\xi) = -|\xi|^{2r}\widehat{u}(\xi)$, $0 < r < m$, $p \geq 1$ is an integer and α, β non negative constants.

Note that when we set $r = 1/2, \beta = 0$ and $m = 1, \alpha = 0$ we obtain two well known families of nonlinear dispersive equations called the generalized Benjamin-Ono equation (gBO) and the generalized Korteweg-de Vries equation (gKdV), respectively.

The case $m = 1$, $r = 1/2$ and $p = 1$ in (1.1) is known as the Benjamin equation. Benjamin in ([5],[6]) derived this model to describe the evolution of waves on the interface

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of a two-fluid system when surface-tension effects are taking in consideration. The main purpose of his work was to establish the existence of solitary wave solutions. Existence and stability of solitary wave solutions as well the Cauchy problem have been studied in [1], [2], [3], [20]. In [1] there is an interesting discussion about Benjamin's model explaining the meaning of the parameters in front of the dispersion operators and other features of the model.

The generalized Benjamin equation, that is, the equation in (1.1) with $m = 1$, $r = 1/2$ and $p > 1$, also have been object of study in the recent years. For instance, the results obtained by Chen and Bona in [10] are concerned with existence and asymptotic properties of solitary wave solutions and Angulo in [3] considered the instability of solitary waves.

Our purpose is to study the initial value problem associated to the generalized Benjamin-type equations (1.1) with data in the energy space. The main interest to consider this situation is due to the fact that this is the natural space to study properties of the solitary-wave solutions. It is also interesting to study this equation from the point of view of nonlinear dispersive equations theory mainly because it can be thought, in the particular case of generalized Benjamin equation, as an hybrid between the gKdV equation and gBO equation. The signs in front of the third and fourth terms of (1.1) are such that the KdV-type dispersion relation arising from the third term competes against the Benjamin-Ono-type dispersion relation arising from the fourth term. We will see that we can recover for the generalized Benjamin equation some of the well-posedness results obtained for the gKdV equation. This result is natural because the Benjamin model considered waves in the interface of a two incompressible fluid system which consists of a layer of dept h_1 of light fluid of density ρ_1 bounded above by a rigid plane and resting upon a layer of heavier fluid of density $\rho_2 > \rho_1$ of depth h_2, also resting on a rigid plane. The waves were assumed to propagate in only one direction, to have long wavelength μ, small amplitude a relatively to h_1 and the lower layer was assumed to be very deep relative to the upper layer. So, the waves that propagate along the interface may have the same properties of surface waves.

In our analysis we will establish the existence of global solutions in $H^s(\mathbb{R})$, $s \geq 1$, when $m = 1$ and $r = 1/2$. To achieve this result we will set the stage with local well-posedness result in Sobolev spaces $H^s(\mathbb{R})$, $s \geq 1$ and then combine it with the conserved quantity

$$\int_{\mathbb{R}} \Big\{\beta(u_x)^2 - \alpha(u\mathcal{H}\,u_x)^2 - \frac{2}{(p+1)(p+2)}\,u^p\Big\}(x,t)\,dx \tag{1.2}$$

where $\mathcal{H}$ denotes the Hilbert transform.

To obtain the local results we will make use of the of smoothing effects present in solutions associated to linear dispersive equations. These smoothing properties allow us to overcome the loss of derivatives in the nonlinear problems. The main tools are the Kato smoothing effect estimate, maximal function estimates and $L^p - L^q$ estimates associated to solutions of the linear problem,

$$\begin{cases} u_t - \beta\, D^{2m}\, u_x - \alpha\, \mathcal{H}^{2r}\, u_x = 0, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1.3}$$

To establish these estimates we follow closely the ideas developed by Kenig, Ponce and Vega in [16], [17], [18]. We shall notice that the best result up to date regarding the gKdV ($p > 1$) were obtained in [18]. Our purpose here is not to match these results but it is likely to obtain similar results in our case.

In the general case, that is, $m > 1$, $\beta \neq 0$ in (1.1), the most interesting situation is that for $1 < m \leq 3/2$ because in that case we must have to employ the smoothing effect techniques to obtain results in the energy space. For $m > 3/2$, we will be able to use Kato's abstract theory to establish local well-posedness in $H^s(\mathbb{R})$, $s > 3/2$, and therefore global existence in those spaces, see Remarks 3.1 and 3.2. On the other hand, if we want to obtain better local results we shall use Kenig, Ponce, Vega theory and Bourgain techniques.

We also deal with the comparison between solutions of the generalized Benjamin type equation (1.1) and solutions of the generalized Benjamin-Ono type equation,

$$\begin{cases} u_t + u^p u_x - \alpha\, \mathcal{H}^{2r}\, u_x = 0, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1.4}$$

The analysis is similar to the one developed in [13]. To achieve our goal we begin with initial data

$$u_0(x) = \varepsilon \Phi(\lambda^{-1} x) \tag{1.5}$$

conveniently scaled, i.e., $\varepsilon = \frac{a}{h_1}$ is a representative value of the amplitude of the motions in question and $\lambda = \frac{\mu}{h_1}$ a representative value of the wavelength, where it is presumed that both these quantities have been non-dimensionalized with respect to an underlying length scales present in the problem which is a standard assumption when we are interested in solution of the initial value problem. Then the conditions that the nonlinear and dispersive effects in (1.1) are small and balanced would be the requirements that $\varepsilon^p \lambda^{2r}$ is an order one quantity, while ε and λ^{-1} are both small as it was discussed in [1], [12] and [13]. So if

u is the solution of the generalized Benjamin type equation (1.1) and v is the solution of (1.4), it is expected that both u and v will be small, but, depending on the difference of the symbols $m_1 - m_2$ where $m_1(\xi) = \beta|\xi|^{2m} - \alpha|\xi|^{2r}$ and $m_2(\xi) = -\alpha|\xi|^{2r}$, we will have that $u - v$ is still smaller, at least over certain time intervals. It is worth to mention that the Benjamin's model is physically relevant if $\frac{\alpha^2}{4\beta}$ is comparable in size to λ^{-1}.

This comparison is specially interesting, for instance, in the case of generalized Benjamin equation with $p = 3$. A recent numerical study (see [11]) shows that there are initial data such that the corresponding solution of (1.4) with $r = 1/2$ blows-up in finite time meanwhile the solution of (1.1) is global. It would be interesting to explain the physical reasons of this phenomena.

Other interesting problem treated here is concerning the limit when the surface-tension effects tends to zero as well as α tends to zero, α is related with the difference of the densities of the two flows considered in the model see [1] for details. In the first case, we show that solutions of the Benjamin equation ($p = 1$) in the limit case are solutions of the Benjamin-Ono equation in Sobolev spaces $H^s(\mathbb{R})$, $s > 3/2$, recovering the local theory for this later equation. For the generalized Benjamin equation we can prove that solutions of this equations solve the gBO equation for power $p \geq 5$ and small data in Sobolev spaces $H^s(\mathbb{R})$, $s > 2$. For powers $p = 2, 3, 4$ we do not have estimates independent of β for solution of the gB equation. The same analysis can be applied when the parameter α approaches to zero. In this case we show that solutions of the gB equation in the limit situation are solutions of the KdV in $H^s(\mathbb{R})$, $s > 3/2$ for $p = 1$ and for $p \geq 5$ solutions of the gKdV in $H^s(\mathbb{R})$, $s > 2$.

This paper is organized as follows. In the first section we give the statements of our results. Section 2 will be devoted to establish the local theory. The global existence of solutions will be proved in section 3. The comparison of solutions will be presented in section 4. Finally, section 5 will contain the proofs of the results regarding the limit of the physical parameters.

Notation. We use the standard notation in partial differential equations. In addition, we will denote the L^2-norm by $\|\cdot\|$ and c will be an arbitrary constant that may change from line to line. The norm on the Sobolev spaces $H^s(\mathbb{R})$ will be denoted either by $\|\cdot\|_s$ or $\|\cdot\|_{H^s}$. The $L^p_x L^q_T$ norm is defined by

$$\|f\|_{L^p_x L^q_T} = \Big(\int_{\mathbb{R}} \Big(\int_0^T |f(x,t)|^q\,dt\Big)^{p/q} dx\Big)^{1/p}.$$

Main Results

Our first two results regard local and global well-posedness for the generalized Benjamin equation. More precisely,

Theorem 1.1. *Let $u_0 \in H^s(\mathbb{R})$, $s \geq 1$. Then there exist $T = T(\|u_0\|_s) > 0$ and a unique solution of the IVP* (1.1) *with $m = 1$, $r = 1/2$ and $p \geq 2$ such that*

$$u \in C([0,T]; H^s(\mathbb{R})) \tag{1.6}$$

and

$$\|u\|_{L^4_x L^\infty_T} < \infty. \tag{1.7}$$

Moreover, for $T' \in (0,T)$ there exists a neighborhood U of u_0 such that the map $\tilde{u}_0 \to \tilde{u}(t)$ from U into the class defined by (1.6)-(1.7) *with T' instead of T is Lipschitz.*

In addition the solution u satisfies Kato's smoothing effect, i.e.

$$\Big(\int_0^T |\partial_x u(t)|^2\, dt\Big)^{1/2} < \infty \quad \text{for any } x \in \mathbb{R}. \tag{1.8}$$

Remark 1.2. *It would be interesting to obtain local results as those obtained for the gKdV equation. However we do not known what would be the best possible results because the scaling does not apply in this case.*

Theorem 1.3. *The IVP (1.1) with $m = 1$, $r = 1/2$ is globally well-posed in $H^1(\mathbb{R})$ under the following conditions on the initial data.*

1. *For $p = 2, 3$, there is no restriction on the data.*
2. *For $p = 4$, the initial data u_0 must be small enough.*
3. *For $p > 4$, the data must be small.*

Remark 1.4. *It seems reasonable to expect that the restriction on the data u_0 in case $p = 4$ is given by*

$$\|u_0\| \leq \|S\|,$$

where S is the solitary wave solution of (1.1).

Proposition 1.5. *The IVP (1.1) for $m > 1$ and $m > r > 0$ is globally well posed in its energy space.*

The comparison theorem reads as follows:

Theorem 1.6. *Let u be the solution of the gB equation ($m = 1$, $r = 1/2$, in (1.1)) and v be the solution of the gBO ($r = 1/2$ in (1.4)) with initial data $u_0(x) = \varepsilon\Phi(\varepsilon^\gamma x)$. Then there exists a constant C such that u and v verify*

$$\|u(\cdot,t) - v(\cdot,t)\| \leq C\varepsilon t, \quad 0 \leq t \leq \min\{1, T\}, \tag{1.9}$$

where T depends on p, ε and γ.

A more complete statement appears below as Theorem 4.1.

Regarding the limit of solutions when the surface tension tends to zero we have the following results,

Theorem 1.7. *Solutions of the IVP (1.1) with $m = 1$ and $r = 1/2$ and $p = 1$ converge uniformly in H^s norm, $s > 3/2$, to solutions of the Benjamin-Ono equation when $\beta \to 0$. If $p \geq 5$ then solutions of the generalized Benjamin equation converge uniformly in H^s norm, $s > 2$, to solutions of the generalized Benjamin-Ono equation when $\beta \to 0$.*

Corollary 1.8. *If $p \geq 5$ then there exist solutions $u_\pm$ of the associated linear problem of the generalized Benjamin equation such that*

$$\|u(t) - u_\pm(t)\|_1 \to 0 \quad as \quad t \to \pm\infty. \tag{1.10}$$

Remark 1.9. *In the proof of Theorem 1.7 we study the asymptotic behavior of solutions of the IVP (1.1) with $m = 1$ and $r = 1/2$ to get a priori estimates independent of the parameters α and β. The result obtained allows us to show the nonlinear scattering result above. These kind of results have been extensively studied for nonlinear dispersive models, in particular the result in Corollary 1.8 is similar to the one given for the gKdV see* [22].

Corollary 1.10. *Solutions of the IVP (1.1) with $m > 1$ and $m > r > 0$ and $p > p_0$ converge uniformly in the H^s–norm, $s > s_0$, to solutions of*

$$u_t + u^p u_x - \alpha\mathcal{H}^{2r} u_x = 0 \tag{1.11}$$

when $\beta \to 0$.

Remark 1.11. *A similar analysis as in Theorem 1.7 can be done to treat the limit $\alpha \to 0$ for $\beta > 0$ fixed.*

2. Local well-posedness

We begin this section considering the linear equation associated to generalized Benjamin-type equation, that is,

$$\begin{cases} u_t - \beta D^{2m} u_x - \alpha \mathcal{H}^{2r} u_x = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), \end{cases} \tag{2.1}$$

whose solution is given in terms of the unitary group $\{V^{m,r}(t)\}$. More precisely,

$$u(x,t) = V^{m,r}(t)u_0(x) = \int_{\mathbb{R}} e^{i(t\phi^{\alpha,\beta}(\xi)+x\xi)} \widehat{u}_0(\xi)\, d\xi \tag{2.2}$$

where $\phi^{\alpha,\beta}$is the real symbol associated to the linear equation, i.e.,

$$\phi^{\alpha,\beta}(\xi) = \beta|\xi|^{2m}\xi - \alpha|\xi|^{2r}\xi. \tag{2.3}$$

To pursue the local theory for (1.1) we will use linear estimates associated to solutions of (2.1). These estimates can be deduced using the results established by Kenig, Ponce and Vega [17] for solutions of the linear dispersive equation

$$\begin{cases} \partial_t v - iP(D)v = 0, & x, t \in \mathbb{R} \\ v(x,0) = v_0(x). \end{cases} \tag{2.4}$$

The solution of (2.4) is given by the oscillatory integral

$$v(x,t) = \int_{\mathbb{R}} e^{i\phi(\xi)t + ix\xi}\, \widehat{v}_0(\xi)\, d\xi \tag{2.5}$$

where the phase function ϕ is the real symbol associated to P.

Let us consider $\mathcal{P}$ the class of phase functions ϕ, satisfying

1. $\phi : \Omega \to \mathbb{R}$ for some open $\Omega \subset \mathbb{R}$, $\phi \in C^3(\Omega)$, Ω is a finite union of intervals.
2. the set $S_\phi = \{\xi \in \bar{\Omega} \cup \{\pm\infty\} : \phi''(\xi) = 0 \quad \text{or} \quad \lim_{\bar{\xi}\to\xi} \phi''(\bar{\xi}) = \phi''(\xi) = \pm\infty\}$ is finite.
3. If $\xi_0 \in S_\phi$ with $\xi_0 \neq \pm\infty$ then there exist constants ϵ, c_1, c_2, and $\alpha \neq 0$ such that for $|\xi - \xi_0| < \epsilon$,
$$c_1|\xi - \xi_0|^{\alpha-2} \leq |\phi''(\xi)| \leq c_1|\xi - \xi_0|^{\alpha-2}.$$
4. If $\xi_0 = \pm\infty \in S_\phi$ then there exist constants ϵ, c_1, c_2, and $\alpha \neq 0$ such that for $|\xi| > \frac{1}{\epsilon}$,
$$c_1|\xi|^{\alpha-2} \leq |\phi''(\xi)| \leq c_1|\xi|^{\alpha-2}.$$
5. ϕ'' has a finite number of changes of monotonicity.

Then we have the following results proved in [17].

Theorem 2.1. *Let ϕ in the class $\mathcal{P}$ with $\phi' \neq 0 \in \Omega$. Then the solution of* (2.4) *satisfies*

$$\sup_x \int_{\mathbb{R}} |v(x,t)|^2\, dt \leq c \int_\Omega \frac{|\widehat{v}_0(\xi)|^2}{|\phi'(\xi)|}\, d\xi, \tag{2.6}$$

and

$$\Big(\int_{\mathbb{R}} \sup_{[0,T]} |u(x,t)|^4\, dx \Big)^{1/4} \leq c_\phi \Big(\int_{\mathbb{R}} |\widehat{v}_0(\xi)|^2 \Big| \frac{\phi'(\xi)}{\phi''(\xi)} \Big|^2 d\xi \Big)^{1/2}. \tag{2.7}$$

It is easy to check that $\phi^{\alpha,\beta}$ defined in (2.3) belongs to the class $\mathcal{P}$. Thus the results in Theorem 2.1 are satisfied by solutions of (2.1). Now, we will specialize in the case $m = 1$, $r = 1/2$.

In the next lemma we list the linear estimates associated to the solution of (2.1) that we will use along this work. These estimates are the main tools to obtain the local results.

Lemma 2.2. *Let $u_0 \in \mathcal{S}(\mathbb{R})$, $r = 1/2$ and $m = 1$. Then*

$$\|\partial_x V^{m,r}(t) u_0\|_{L^\infty_x L^2_t} \leq c \|u_0\|, \tag{2.8}$$

and

$$\|V^{m,r}(t) u_0\|_{L^4_x L^\infty_T} \leq c \|D_x^{1/4} u_0\|. \tag{2.9}$$

Remark 2.3. *The dual version of the smoothing effect* (2.8) *is*

$$\|\partial_x \int_0^T V^{m,r}(-\tau) g(\cdot,\tau)\, d\tau\| \leq c \|g\|_{L^1_x L^2_T}. \tag{2.10}$$

This is the main ingredient in the arguments below.

Proof of Lemma 2.2. The estimate (2.8) follows applying (2.6) in Theorem 2.1. The estimate (2.9) is obtained by using (2.7). □

Now we are ready to show that the IVP (1.1) is locally well-posed in $H^s(\mathbb{R})$, $s \geq 1$.

Proof of Theorem 1.1. We use the equivalent integral equation associated to the IVP (1.1)

$$u(t) = V^{m,r}(t) u_0 + \int_0^t V^{m,r}(t-\tau)\, u^p\, u_x(\tau)\, d\tau, \tag{2.11}$$

where $\{V^{m,r}(t)\}$ is the unitary group defined in (2.2). For clearness we will give the details of the proof for data in H^1. The general case follows using fractional derivatives estimates.

Define the ball

$$M_T^a = \{w \in L^\infty([0,T] : H^1(\mathbb{R})) \quad \text{such that} \;\; \mathcal{A}_j(w) \leq a, \;\; j = 1,2\}$$

where

$$\begin{aligned} \mathcal{A}_1(w) &\equiv \sup_{[0,T]} \|w(t)\|_{H^1} \\ \mathcal{A}_2(w) &\equiv \|w\|_{L^4_x L^\infty_T}. \end{aligned} \tag{2.12}$$

and define for $u_0 \in H^1(\mathbb{R})$ the operator ψ_{u_0} on M_T^a,

$$\psi(u)(t) = \psi_{u_0}(u)(t) = V^{m,r}(t)u_0 + \int_0^t V^{m,r}(t-\tau)\, u^p\, u_x(\tau)\, d\tau.$$

To establish the first part of the theorem we have to show that there exist a and T such that

1. $\psi(u) : M_T^a \to M_T^a$ is well defined.
2. $\psi(u)$ is a contraction.

We start with $\mathcal{A}_1(\psi(u))$.

$$\begin{aligned} \mathcal{A}_1(\psi(u)) &\leq \|V^{m,r}(t)u_0\|_{H^1} + \|\int_0^t V^{m,r}(t-\tau)\; u^p\, u_x(\tau)\; d\tau\|_{H^1} \\ &\leq \|u_0\|_{H^1} + \|\int_0^t V^{m,r}(t-\tau)\; u^p\, u_x(\tau)\; d\tau\| + \|\partial_x \int_0^t V^{m,r}(t-\tau)\; u^p\, u_x(\tau)\; d\tau\|. \end{aligned} \tag{2.13}$$

Next we estimate the second term on the right hand side of (2.13). We use the Minkowski inequality, group properties, the Cauchy-Schwarz inequality and Sobolev's embedding to yield

$$\begin{aligned} \|\int_0^t V^{m,r}(t-\tau)\; u^p u_x(\tau)\; d\tau\| &\leq \int_0^T \|u^p\, u_x(\tau)\|\, d\tau \\ &\leq \int_0^T \|u(\tau)\|_{L^\infty}^p \|u_x(\tau)\|\, d\tau \\ &\leq cT \sup_{[0,T]} \|u(t)\|_{H^1}^p\, \|u_x\| \\ &\leq cT\,(\mathcal{A}_1(u))^{p+1}. \end{aligned} \tag{2.14}$$

The last term on the right hand side of (2.13) is estimated as follows. We apply first the dual version of the smoothing effect (2.10), the Cauchy-Schwarz inequality and Sobolev's embedding.

$$
\begin{aligned}
\|\partial_x \int_0^t V^{m,r}(t-\tau)\; u^p u_x(\tau)\; d\tau\| &\le c\|u^p u_x\|_{L^1_x L^2_T} \\
&\le \int_{\mathbb{R}} \sup_{[0,T]} |u|^2 \Big(\int_0^T |u^{p-2}\, u_x(\tau)|^2\; d\tau\Big)^{1/2} dx \\
&\le c\, T^{1/2} \sup_{[0,T]} \|u(t)\|_{H^1}^{p-1} \|u\|^2_{L^4_x L^\infty_T} \\
&\le c T^{1/2} (\mathcal{A}_1(u))^{p-1} (\mathcal{A}_2(u))^2.
\end{aligned} \tag{2.15}
$$

Plugging the estimates (2.14) and (2.15) in (2.13) give

$$
\mathcal{A}_1(\psi(u)) \le c\|u_0\|_{H^1} + c\, T^{1/2}\{T^{1/2}(\mathcal{A}_1(u))^{p+1} + (\mathcal{A}_1(u))^{p-1}\, (\mathcal{A}_2(u))^2\}. \tag{2.16}
$$

Now we estimate $\mathcal{A}_2(\psi(u))$. We proceed as follows. We apply the Minkowski inequality, the maximal function estimate (2.9), interpolation and the argument used in (2.13)-(2.15).

$$
\begin{aligned}
\|\psi(u)(t)\|_{L^4_x L^\infty_T} &\le \|D^{1/4} u_0\| + \int_0^T \|V^{m,r}(t-\tau)\; u^p\, u_x(\tau)\|_{L^4_x L^\infty_T} d\tau \\
&\le c\|u_0\|_{H^1} + c\, T^{1/2}\{T^{1/2}(\mathcal{A}_1(u))^{p+1} + (\mathcal{A}_1(u))^{p-2}\, (\mathcal{A}_2(u))^2\}.
\end{aligned} \tag{2.17}
$$

Let $|\!|\!| w |\!|\!| = \max\{\mathcal{A}_1(w), \mathcal{A}_2(w)\}$. Then from (2.16) and (2.17) it follows that

$$
|\!|\!| \psi(u) |\!|\!| \le c\|u_0\|_{H^1} + cT^{1/2}\,\{T^{1/2} |\!|\!| u |\!|\!|^p + |\!|\!| u |\!|\!|^p\} |\!|\!| u |\!|\!|. \tag{2.18}
$$

Then we need to choose values a and T such that the function ψ_{u_0} will be well defined and in fact a contraction in M^a_T. Choosing $a = 2c\,\|u_0\|_{H^1}$ and T such that $cT^{1/2}(T^{1/2}a^p + a^p) < 1/2$, then it follows from (2.18) that $\psi(u)$ is well defined. To see that it is a contraction we use the same arguments as before to show that

$$
|\!|\!| \psi(u) - \psi(v) |\!|\!| \le cT^{1/2}\,\{T^{1/2} |\!|\!| u |\!|\!|^p + |\!|\!| u |\!|\!|^p\} |\!|\!| u - v |\!|\!|. \tag{2.19}
$$

The previous choice of a and T gives the desired result. Then the contraction mapping principle guarantees the existence of a unique solution of (2.11). The uniqueness gives us the result in the class defined in (1.6)-(1.7). Using a similar argument we can prove that the map $\tilde{u}_0 \to \tilde{u}(t)$ is Lipschitz.

The last claim of the theorem follows applying the estimate (2.8) to the expression in (2.11), the solution of our problem, and the argument in (2.14) to have

$$
\sup_x \Big(\int_0^T |\partial_x u(t)|^2\, dt\Big)^{1/2} \le \|u_0\|_{H^1} + cT \sup_{[0,T]} \|u(t)\|_{H^1}^{p+1}.
$$

Therefore (1.8) follows and the theorem is proved. □

3. Global solution

In this section we aim to prove global existence of solutions in the Sobolev space $H^1(\mathbb{R})$. A straightforward extension of the H^1 proof shows that our result is actually valid in $H^s, s > 1$. The argument used here to obtain the global solutions is standard, so we restrict ourselves to the case of $m = 1$, $r = 1/2$ which is known as the generalized Benjamin-equation and then we give the indications how to get the general result.

We are looking for an *a priori* estimate for the H^1-norm of the solution. For this purpose we need the conserved quantities.

$$I_2(u) = \int u^2(x,t)dx \tag{3.1}$$

$$I_3(u) = \int \beta\, u_x^2 dx - \int \alpha (D_x^{1/2}u)^2 dx - \frac{2}{(p+1)(p+2)} \int u^{p+2} dx. \tag{3.2}$$

Proof of Theorem 1.3 . Using the conserved quantity (3.2), interpolation, the Gagliardo-Nirenberg inequality and the conservation of mass (3.1) we have

$$\begin{aligned} \|u_x\|^2 &= c(\beta)\, I_3(u_0) + \beta^{-1}\alpha\, \|D_x^{1/2}u\|^2 + \frac{2}{\beta(p+1)(p+2)} \|u\|_{p+2}^{p+2} \\ &\le c(\beta)\, I_3(u_0) + c(\beta,\alpha)\|u\|\|u_x\| + c(\beta)\|u\|^{(p+4)/2}\|u_x\|^{p/2} \\ &\le c(\beta) I_3(u_0) + c(\alpha,\beta)\|u_0\|\|u_x\| + c(\beta)\|u_0\|^{(p+4)/2}\|u_x\|^{p/2} \\ &\le c\, I_3(u_0) + c\,\|u_0\|^2 + \frac{1}{2}\|u_x\|^2 + c\,\|u_0\|^{(p+4)/2}\|u_x\|^{p/2} \end{aligned} \tag{3.3}$$

where $c = c(\alpha, \beta)$.

Now we have to distinguish three cases. If $p < 4$. Then from (3.3) and an elementary interpolation we have

$$\|u_x\|^2 \le 2c\, I_3(u_0) + c\,\|u_0\|^2 + 2c\,\|u_0\|^{(p+4)/(4-p)} + \frac{1}{2}\|u_x\|^2. \tag{3.4}$$

From this inequality the first part of Theorem 1.3 follows.

Case $p = 4$. From (3.3) we have

$$(1 - 2c\|u_0\|^4)\|u_x\|^2 \le 2c\, I_3(u_0) + c\,\|u_0\|^2.$$

So in order to get *a priori* estimate we have to require data $u_0 \in H^1(\mathbb{R})$ such that

$$1 - 2c\|u_0\|^4 > 0$$

holds. This shows the second claim in our theorem.

Finally, we consider the case $p > 4$. Defining $X(t) = \|u_x(t)\|^2$ we have from (3.3) the following equation:

$$X(t) \leq c\,(\|u_0\|_1) + c\|u_0\|^{(p+4)/2}\, X(t)^{1+\epsilon}.$$

A standard argument shows the existence of a bounded interval where $X(t)$ lies whenever $\|u_0\|$ is sufficiently small. This finishes the proof. □

Remark 3.1. *Solutions of the IVP* (1.1) *besides preserving the* L^2 *norm they satisfy*

$$I(u(t)) = \int \beta\,|D_x^m u|^2 dx - \int \alpha |D_x^r u|^2 dx - \frac{2}{(p+1)(p+2)} \int u^{p+2}\, dx = I(u_0). \tag{3.5}$$

Remark 3.2. *It is clear that the same argument used here apply to the general case. That is, for* $u_0 \in H^m(\mathbb{R})$, $m > r$, $m > 1$ *we will have global solutions for the IVP* (1.1) *under the following conditions:*

1. *no restriction on the data if* $p < 2m$,
2. *data small enough for* $p = 2m$,
3. *sufficient small data if* $p > 2m$.

Remark 3.3. *We observe that the most interesting case in the previous remark is when* $m > r$, $m \in (0, 3/2]$. *This is due to the fact we need to use the techniques in Section 3 to establish local well-posedness in that range. For* $m > 3/2$ *Kato's theory can be applied instead (see* [14]*).*

The proof of Proposition 1.5 follows from the previous remarks.

4. Comparison

In this section we follow the ideas developed in [13] to compare solutions of the IVP

$$\begin{cases} u_t + u^p u_x - \beta D^{2m} u_x - \alpha \mathcal{H}^{2r} u_x = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \epsilon \Phi(\epsilon^\gamma x) \end{cases} \tag{4.1}$$

with solutions of the IVP

$$\begin{cases} v_t + v^p v_x - \alpha \mathcal{H}^{2r} v_x = 0, & x \in \mathbb{R},\ t > 0, \\ v(x,0) = \epsilon \Phi(\epsilon^\gamma x), \end{cases} \tag{4.2}$$

where γ is a fixed positive number and the symbols are as before, i.e., $\widehat{D^{2m} u}(\xi) = |\xi|^{2m} \widehat{u}(\xi)$ and $\widehat{\mathcal{H}^{2r} u}(\xi) = -|\xi|^{2r} \widehat{u}(\xi)$ with $m \geq 1$, p a positive integer and $\frac{m}{r} \geq 1 + \frac{1}{p}$.

Let $m_1(\xi) = \beta|\xi|^{2m} - \alpha|\xi|^{2r}$ and $m_2(\xi) = -\alpha|\xi|^{2r}$. We start with the change of variables

$$\begin{aligned} u(x,t) &= \epsilon U(\epsilon^{\gamma} x, \epsilon^{\delta} t) \\ v(x,t) &= \epsilon V(\epsilon^{\gamma} x, \epsilon^{\delta} t), \end{aligned} \tag{4.3}$$

with $\delta = p + \gamma$ and $\gamma = p/2r$. This change of variables agrees with the scale discussed in (1.5). Thus, we obtain the following problems for U and V

$$\begin{cases} U_t + U^p U_x - \varepsilon^{\frac{p}{2r}(2m-2r)} \beta D^{2m} U_x - \alpha \mathcal{H}^{2r} U_x = 0, & x \in \mathbb{R},\ t > 0, \\ U(x,0) = \Phi(x), \end{cases} \tag{4.4}$$

$$\begin{cases} V_t + V^p V_x - \alpha \mathcal{H}^{2r} V_x = 0, & x \in \mathbb{R},\ t > 0, \\ V(x,0) = \Phi(x). \end{cases} \tag{4.5}$$

Note that the equation (4.2) is invariant by this change of variables. Therefore for the equations (4.4) and (4.5), we have the symbols m_1^{ε} and m_2^{ε} defined by $m_1^{\varepsilon}(\xi) = \varepsilon^{\frac{p}{2r}(2m-2r)}\beta\,|\xi|^{2m} - \alpha|\xi|^{2r}$ and $m_2^{\varepsilon} = -\alpha|\xi|^{2r}$ corresponding to the dispersive operators respectively.

Now we are in position to prove the following.

Theorem 4.1. *Suppose γ and p are such that for all ξ and sufficiently small ϵ, the symbols m_1^{ϵ} and m_2^{ϵ} defined above satisfy*

$$|m_1^{\epsilon}(\xi) - m_2^{\epsilon}(\xi)| \leq \epsilon\, P_{s-1}(\xi)|,$$

for all $\xi \in \mathbb{R}$, where $s \geq 2$ is an integer and P_{s-1} is a polynomial of degree $s-1$. Let $\Phi \in H^{k+s}$ where $k \geq 0$. Suppose that the initial-value problems (4.1) and (4.2) are locally well posed in $C(0,T;H^{k+s})$ for some $T > 0$ and that the H^{k+s}-norms of U and V are bounded in $[0,T]$ with a bound that depends only on the norm of Φ is H^{k+s}, and not on ϵ, at least for ϵ small. Then there are constants $B_j, 0 \leq j \leq k$, which depend only on $\|\Phi\|_{H^{k+s}}$ such that, for $0 \leq t \leq \min\{1,T\}$ and $0 < \epsilon$,

$$\|\partial_x^j(u(\cdot,t) - v(\cdot,t))\| \leq \epsilon^{2+\gamma(j-\frac{1}{2})} B_j \epsilon^{\delta}\, t,$$

provided $0 \leq t \leq T\epsilon^{\delta}$. By interpolation it follows that

$$\|\partial_x^j(u(\cdot,t) - v(\cdot,t))\|_{L^\infty} \leq \epsilon^{2+\gamma j} C_j \epsilon^{\delta} t,$$

where $C_j = (B_j B_{j-1})^{1/2}$.

Proof. The proof of this theorem follows the same lines as Lemma 3.1 and Theorem 3.2 in [13], therefore we will omit it. $\square$

Remark 4.2. *Note that in this theorem we need local existence of solution in* $H^{k+s}, s \geq 2, k \geq 0$. *Our global result solve the problem for (4.1) and for the equation (4.2) we refer to* [19] *and references therein. It is also interesting to observe that in the case of* $r = 1/2$, *the Benjamin type equation (4.1) has global results of existence of solutions which are similar to ones for gKdV type equation but this solution behaves locally as the solution of the generalized Benjamin-Ono equation (4.2) with the same initial data.*

Remark 4.3. *Theorem 1.6 is a particular case of Theorem 4.1*

Remark 4.4. *Typically, the model for the Benjamin-Ono equation describes deep water waves with long wavelength and small amplitude. This justifies the scale in (1.5) for the initial data which leaves the equation (4.5) invariant. In our comparison we consider the equation* (4.1) *a dispersive perturbation of* (4.2). *We can interpret it saying that, in some sense, the addition of the dispersive term smoothed out the solution of* (4.2).

5. Limits

In this section we will study the behavior of the solutions of the IVP (1.1) when the parameters α and β approaches to zero. First we need to establish *a priori* estimates for solutions of (1.1) independent on α and β. As in the previous sections we will treat the case $m = 1$ and $r = 1/2$ and then we explain how to deal with the general case. Notice that the energy (1.2) is not useful for our purpose.

We first treat the case $p = 1$. Thus using Kato's commutators theorem (see[14]) we obtain the following *a priori* estimate for solutions of the (1.1) in $H^s(\mathbb{R})$, $s > 3/2$, independent of α and β

$$\begin{aligned} \frac{d}{dt}\|u\|_s^2 &= (u, \partial_t u)_s = \beta(u, \partial_x^3 u)_s + \alpha(u, \mathcal{H}\partial_x^2 u)_s - (u, u^p \partial_x u)_s \\ &\leq |(u, u\partial_x u)_s| \leq c\|u\|_s^3 \end{aligned} \tag{5.1}$$

and therefore we will have a solution as long as the solution of the differential equation

$$\begin{cases} \rho'(t) \leq \rho^{3/2}(t) \\ \rho(0) = \|u_0\|_s^2 \end{cases} \tag{5.2}$$

has one.

We will study in detail the limit when β approach to zero and $p=1$. Let us consider solutions u and v of the IVP

$$
\begin{cases} U_t + UU_x - \beta\partial_x^3 U - \alpha\mathcal{H}\partial_x^2 U_x = 0, \quad x \in \mathbb{R},\ t>0, \\ U(x,0) = \varphi(x) \end{cases} \tag{5.3}
$$

with $\beta=\beta_1$ and $\beta=\beta_2$ for the solutions u and v respectively. Then $w := u - v$ satisfies the equation

$$
\begin{cases} w_t + w_x u + w v_x - (\beta_1-\beta_2)\partial_x^3 u - \beta_2\partial_x^3 w = 0 \\ w(\cdot,0) = 0. \end{cases} \tag{5.4}
$$

Next we multiply (5.4) by w and integrate over $\mathbb{R}$ to obtain

$$
\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w\|^2 &= -(w, w_x u) - (w, w v_x) - (w, \beta_2\partial_x^3 w) - (w, (\beta_1-\beta_2)\partial_x^3 u) \\ &\le \|u\|_{L^\infty}\|w\|^2 + \|v_x\|_{L^\infty}^2\|w\|^2 + C|\beta_1-\beta_2|(\|u\|_s + \|v\|_s)\|u\|_s \\ &\le (\|u\|_s + \|v\|_s)\|w\|^2 + C|\beta_1-\beta_2|\,(\|u\|_s + \|v\|_s)\|u\|_s. \end{aligned} \tag{5.5}
$$

Integrating over $[0,t]$ and applying Gronwall's lemma, we get

$$
\begin{aligned} \|w\|^2 \le CT\,|\beta_1-\beta_2| \sup_{[0,T]} (\|u(t)\|_s + \|v(t)\|_s)\|u(t)\|_s \\ \times \exp\{(CT)\sup_{[0,T]}(\|u(t)\|_s + \|v(t)\|_s)\}, \end{aligned} \tag{5.6}
$$

for $0 \le t \le T$. Since the H^s norms, $s > 3/2$, of u and v are bounded independent of $\beta < 1$, say, then if $\beta^1, \beta^2 \to 0$, the last term on (5.6) converges to 0 so that the limit $u_0 = \lim_{\beta\to 0} u$ exists at least in L^2, uniformly with respect to $t \in [0,T]$.

Now it is shown that u_0 satisfies (5.3) with $\beta = 0$. In fact, if u_β is the solution of (5.3), then,

$$
u_\beta(\cdot,t) - u_\beta(\cdot,\tau) = -\int_\tau^t \left(u_\beta\partial_x u_\beta + \beta\partial_x^3 u_\beta + \alpha\mathcal{H}\partial_x^2 u_\beta\right) dt'. \tag{5.7}
$$

This implies that at least in H^3, u_0 satisfies

$$
u_0(\cdot,t) - u_0(\cdot,\tau) = -\int_\tau^t \left(u_0\partial_x u_0 - \alpha\mathcal{H}\partial_x^2 u_0\right) dt'
$$

and so $u_0 \in AC(0,T;H^{s-3}) \cap L^\infty(0,T;H^s)$ from which it follows that u_0 satisfies equation (5.3) with $\beta = 0$ for almost every t. But the local existence result for $\beta = 0$ implies that (5.3) has a unique solution in $C(0,T;H^s)$ corresponding to initial data in H^s. Therefore u_0 coincides with the strong solution of (5.3) with $\beta = 0$, i.e., the BO equation.

We have showed the first part of Theorem 1.7.

Next we establish an *a priori* estimate for solutions the generalized Benjamin equation in case $p > 1$ independent of α and β. We proceed as follows. First we make use of the Kato-Ponce commutators [15] to obtain

$$\begin{aligned} \frac{d}{dt}\|u(t)\|_s^2 &= (u, \partial_t u)_s = \beta(u, \partial_x^3 u)_s + \alpha(u, \mathcal{H}\partial_x^2 u)_s - (u, u^p \partial_x u)_s \\ &\leq |(u, u^p\partial_x u)_s| \leq c\|u_x\|_{L^\infty}\|u\|_{L^\infty}^{p-2}\|u(t)\|_s^2. \end{aligned} \tag{5.8}$$

Gronwall's inequality yields

$$\|u(t)\|_s^2 \leq c\,\|u_0\|_s\,\exp\{\int_0^t c\|u_x\|_{L^\infty}\|u\|_{L^\infty}^{p-1}\,dt'\}. \tag{5.9}$$

Then we show that the solution of the IVP associate to the generalized Benjamin equation satisfies for $T > 0$ that

$$\sup_{[0,T]} c(1+t)^{1/3}\|u\|_{1,\infty} < \infty, \tag{5.10}$$

where $\|u\|_{1,\infty} = \|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty}$. To do this we study the decay properties of the linear problem.

We will use the following result due to Ben-Artzi and Saut ([7]). Let

$$I_\gamma(x,t) = \int_0^\infty \xi^\gamma\, e^{it(q(\xi)-x\xi)}\,d\xi, \quad t > 0,\ x \in \mathbb{R},$$

with $q(\xi) = \xi^m + r(\xi)$, $r(\xi) = \sum_{j=0}^{m-1} a_j \xi^j$, is a real polynomial of degree $m \geq 3$.

Lemma 5.1. *Fix $T > 0$ and let $q(\xi)$ as above and assume $0 \leq \gamma \leq \frac{m-2}{2}$. Then there exists a constant $C > 0$ depending only on T, γ, q such that*

$$\sup_{x\in\mathbb{R}} |I_\gamma(x,t)| \leq Ct^{-\frac{\gamma+1}{m}}, \quad 0 < t \leq T. \tag{5.11}$$

From Lemma 5.1 it can be deduced the next result.

Lemma 5.2. *Let $u_0 \in L^1(\mathbb{R})$, $m = 1$, $r = 1/2$, $T > 0$ and*

$$I_t(x) = \int_0^\infty e^{it(\xi^3 - |\xi|\xi - x\xi)}\,d\xi.$$

Assume that $0 \leq \gamma \leq \frac{1}{2}$. Then $V^{m,r}(t)u_0 = I_t * u_0$ satisfies

$$\|D_x^\gamma V^{m,r}(t)u_0\| \leq c\,t^{-\frac{\gamma+1}{3}}\|u_0\|_1, \quad 0 < t < T. \tag{5.12}$$

Proof. The estimate (5.12) follows combining (5.11) and Young's inequality. $\square$

Lemma 5.3. *Let $u_0 \in H^s(\mathbb{R}) \cap W_1^1(\mathbb{R})$, $s \geq 2$, and $p \geq 5$. There exists $\delta > 0$ such that*

$$\|u_0\|_{2,2} + \|u_0\|_{1,1} < \delta.$$

Then the IVP associated to the generalized Benjamin equation has a unique global strong solution $u(t)$ in the class $C_b(\mathbb{R}; H^s(\mathbb{R}))$ satisfying there exists a constant c such that

$$\|u(\cdot,t)\|_{1,\infty} \leq c\,(1+t)^{-1/3} \text{ holds for } \quad t > 0. \tag{5.13}$$

Proof. To simplify the notation we write $V(t)$ instead of $V^{m,r}(t)$, the unitary group associated to the linear Benjamin equation.

The local solution of the generalized Benjamin equation satisfies

$$u(t) = V(t)u_0 + c\int_0^t V(t-\tau)\partial_x\left(u^{p+1}(\tau)\right)d\tau. \tag{5.14}$$

Using Lemma 5.2 and the Sobolev inequality we get

$$\begin{aligned} \|u(t)\|_{1,\infty} &\leq c(1+t)^{-1/3}(\|u_0\|_2 + \|u_0\|_{1,1}) + \int_0^t (t-\tau)^{-1/3}\|u(\tau)\|_{L^\infty}^{p-1}\|u(\tau)\|_2^2\,d\tau \\ &\leq c\delta(1+t)^{-1/3} + c\delta\int_0^t (t-\tau)^{-1/3}\|u\|_{L^\infty}^{p-1}\,d\tau. \end{aligned} \tag{5.15}$$

Let define $M(T) = \sup_{[0,T]}(1+t)^{1/3}\|u(t)\|_{1,\infty}$. Then using this definition and (5.9) in (5.15) it follows

$$\begin{aligned} M(T) \leq & c\delta + \delta\,(1+t)^{1/3}\sup_{[0,T]}\, M(T)^{p-1}\,\delta^2 \times \exp\left\{cM(T)^p\int_0^t (1+\tau)^{-\frac{1}{3}p}\,d\tau\right\} \\ & \times\int_0^t (t-\tau)^{-\frac{1}{3}}(1+\tau)^{-\frac{1}{3}(p-1)}. \end{aligned} \tag{5.16}$$

Now if $p \geq 5$

$$M(T) \leq c\delta + c\delta^2\,M(T)^{p-1}\exp\{cM(T)^p\}. \tag{5.17}$$

Therefore, for δ sufficiently small we will have

$$M(T) \leq C.$$

for any $T > 0$, where C is the smallest positive zero of the function $f(x) = c\delta + c\delta^2 x^{p-1}\exp\{cx^p\} - x$. To complete the proof we use a standard argument. This will give us the desire result. □

Now with this estimate we can prove the second part of Theorem 1.7. Since the analysis uses a similar argument as the one given for the first part we will omit its proof.

The proof of Corollary 1.8 follows a standard argument once the estimate (5.13) is established.

To prove Corollary 1.10 we use the same argument used in the second part of Theorem 1.7.

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