On the Cauchy problem for a nonlocal perturbation of the KdV equation*

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Abstract

Let $\mathcal{H}$ denote the Hilbert transform and $\eta \geq 0$. We show that the initial value problems $u_t + uu_x + u_{xxx} + \eta(\mathcal{H}u_x + \mathcal{H}u_{xxx}) = 0$, $u(\cdot,0) = \phi(\cdot)$ and $u_t + \frac{1}{2}(u_x)^2 + u_{xxx} + \eta(\mathcal{H}u_x + \mathcal{H}u_{xxx}) = 0$, $u(\cdot,0) = \phi(\cdot)$ are globally well-posed in $H^s(\mathbb{R})$, $s \geq 1$, $\eta > 0$. We study the limiting behavior of the solutions of the first equation as η tends to zero in $H^s(\mathbb{R})$ and $s \geq 2$. Moreover, we prove a unique continuation theorem for the first equation in $\mathcal{F}_{3,3}(\mathbb{R}) = H^3(\mathbb{R}) \cap L^2_3(\mathbb{R})$, $\eta > 0$, which implies that the persistence property does not hold.

1 Introduction

In this paper we will consider real valued solutions of the Cauchy problems associated to the following equations

$$u_t + \frac{1}{2}(u_x)^2 + u_{xxx} + \eta(\mathcal{H}u_x + \mathcal{H}u_{xxx}) = 0,\ u(\cdot,0) = \phi(\cdot), \tag{1}$$

$$w_t + ww_x + w_{xxx} + \eta(\mathcal{H}w_x + \mathcal{H}w_{xxx}) = 0,\ w(\cdot,0) = \psi(\cdot), \tag{2}$$

where $\mathcal{H}$ denotes the Hilbert transform $\mathcal{H}f(x) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{f(y)}{y-x}dy$, $\ f \in \mathcal{S}(\mathbb{R})$, $\mathcal{P}$ represents the principal value of the integral and the parameter η is an arbitrary positive number. Some of the properties of the Hilbert transform $\mathcal{H}$ (see [6]) are:

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$\mathcal{H} \in B(L^p(\mathbb{R}))$, $p \in (1, \infty)$ and $\widehat{(\mathcal{H}f)}(\xi) = ih(\xi)\hat{f}(\xi)$, for all $f \in H^s(\mathbb{R})$, where

$$h(\xi) = \begin{cases} -1, & \xi < 0, \\ \ \ 1, & \xi > 0. \end{cases} \tag{3}$$

Equation (2) was derived by Ostrovsky et al. to describe the radiational instability of long waves in a stratified shear flow [16]. It deserves remark that the fourth and fifth terms represent, respectively, amplification and damping. B.F. Feng and T. Kawahara ([5]) studied periodic solutions and solitary waves associated with equation (2), from a numerical standpoint. For every $\eta > 0$, a family of solitary wave solutions whose members are distinguished by the number of "humps" is identified numerically. The tails of these waves decay as $O(1/|x|^2)$ when $|x| \to \infty$ independent of the value of η and the number of humps, consequently these waves decay to zero algebraically, which is consistent with Theorem 5.3 which shows that there is an upper limit for the rate of decay of the solution of equation (2). Theorem 5.3 implies that solutions do not fall off arbitrarily fast even if the initial condition has this property.

Equation (2) can be regarded also as a non local perturbation of a modified version of the generalized Ott-Sudan equation ($w_t + ww_x + w_{xxx} + \mu Bu = 0$, where $\mu > 0$ and $B = -\partial_x^2$, $B = 1$ or $B = -\mathcal{H}\partial_x$) which belongs to a class of nonlinear dissipative equations and it models the unidirectional propagation of small-amplitude nonlinear waves (see [1], [2]). Moreover, it is worth noting that if the Fourier transform of the solution of equation (2) has support contained in $\mathbb{R} \setminus [-1, 1]$ for all times, then (2) is a nonlinear dissipative equation, as we can see from equation (31) below.

The plan of this paper is as follows: In Section 2 we study the Cauchy problem associated to the linear part of equation (1), and we obtain some useful properties of the semigroup associated to the solution. The local theory for equation (1) in H^s, $s \geq 1$ and for equation (2) in H^s, $s > 1/2$ is developed in Section 3. Section 4 is devoted to prove global well-posedness of equations (1) and (2) in H^s, $s \geq 1$ and to study the behavior of the limit of solutions of (2) in H^s, $s \geq 2$, when η goes to 0. Moreover, it is worth to say that if u is a solution of problem (1) then u_x is a solution of problem (2), with $\psi = \phi'$. Therefore global existence of the solution of equation (1) in H^s, $s \geq 1$ implies global existence of the solution of equation (2) for initial data belonging to the set $\{\psi \in L^2(\mathbb{R}) / \exists \phi \in H^1(\mathbb{R}); \psi = \phi'\}$. Finally, in Section 5 we obtain a unique continuation theorem for the solution of equation (2). We prove that if the solution $w(t)$ is sufficiently smooth ($w(t) \in H^3(\mathbb{R})$) and falls off sufficiently fast as $|x| \to \infty$ ($w(t) \in L^2_3(\mathbb{R})$) for all $t \in [0, T]$, then $w(t) = 0$, for all $t \in [0, T]$.

Notation:

- $\|\cdot\|_s$: the norm in $H^s(\mathbb{R})$, $s \in \mathbb{R}$.

- $\mathcal{H}$: the Hilbert transform.
- $B(X,Y)$: the set of bounded linear operators from X into Y. If $X = Y$ we write $B(X)$.
- $L^2_r(\mathbb{R}) = (H^r(\mathbb{R}))^\wedge$ and $\mathcal{F}_{s,r}(\mathbb{R}) = H^s(\mathbb{R}) \cap L^2_r(\mathbb{R})$.

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2 The Linear Equation.

In this section we consider the Cauchy problem associated to the linear part of equation (1), namely

$$v_t + v_{xxx} + \eta(\mathcal{H}v_x + \mathcal{H}v_{xxx}) = 0, \;\; v(\cdot,0) = \phi(\cdot), \tag{4}$$

where $\phi \in H^s$, for $s \in \mathbf{R}$ and $\eta \geq 0$. Let $t \geq 0$ and $\xi \in \mathbb{R}$. Taking the Fourier transform in (4) we get $\hat{v}(\xi,t) = \exp[(i\xi^3 + \eta(|\xi| - |\xi|^3))\, t]\; \hat{\phi}(\xi)$.

For $t \geq 0$ and $\xi \in \mathbb{R}$, let

$$F_\eta(t,\xi) = \exp[t\left(i\xi^3 - \eta(|\xi|^3 - |\xi|)\right)]. \tag{5}$$

$$E_\eta(t)f = \mathcal{F}^{-1}(F_\eta(t,\cdot)\hat{f}), \;\; f \in L^2(\mathbb{R}). \tag{6}$$

Lemma 2.1. *Let $\eta > 0$. Then $(E_\eta(t))_{t\geq 0}$ is a C^0 semigroup in $H^s(\mathbb{R})$, $s \in \mathbb{R}$ and*

$$\|E_\eta(t)\|_{B(H^s;H^s)} \leq e^{\eta t}. \tag{7}$$

When $\eta = 0$, we obtain the unitary group associated with the KdV equation, in this case $\|E_\eta(t)\|_{B(H^s,H^s)} = 1$, for all $t \geq 0$. Let $\phi \in H^s(\mathbb{R})$, then $v(\cdot,t) = E_\eta(t)\phi$ is the unique solution of (4), in the class $C(0,\infty; H^s(\mathbb{R})) \cap C^1(0,\infty; H^{s-3}(\mathbb{R}))$.

Proof. The continuity of the semigroup is a consequence of the inequality $|F_\eta(\tau,\xi)| \leq e^{\eta(t+1/2)}$, $0 \leq \tau \leq t + 1/2$, $\xi \in \mathbb{R}$, and the Dominated Convergence Theorem. Relation (7) follows easily from the fact that $e^{-2\eta t(|\xi|^3 - |\xi|)} \leq e^{2\eta t}$, for all $\xi \in \mathbb{R}$. The last assertion in Lemma 2.1 is a consequence of the Dominated Convergence Theorem. ■

Lemma 2.2. *Let $t>0$, $s\in\mathbb{R}$, $\lambda\geq 0$ and $\eta>0$ be given. Then $E_\eta(t)\in B(H^s(\mathbb{R}),H^{s+\lambda}(\mathbb{R}))$. Moreover*

$$\|E_\eta(t)\phi\|_{s+\lambda}\leq c_\lambda[e^{\eta t}+(1+\frac{1}{(\eta t)^{\lambda/2}})e^{\frac{\eta t}{8}(\sqrt{3}+\sqrt{3+\frac{8\sqrt{3}\lambda}{\eta t}})}]\|\phi\|_s, \tag{8}$$

where $\phi\in H^s(\mathbb{R})$ and c_λ is a constant depending only on λ.

Proof. Let $\eta>0$, $t>0$, $s\in\mathbb{R}$ and $\lambda\geq 0$. We have that

$$\|E_\eta(t)\phi\|^2_{s+\lambda}\leq c_\lambda[e^{2\eta t}+\sup_{\xi\in\mathbb{R}}\xi^{2\lambda}e^{-2\eta t(|\xi|^3-|\xi|)}]\|\phi\|^2_s. \tag{9}$$

Since $\xi^3-\xi\geq 1/\sqrt{3}\xi^2-\xi$, for $\xi\geq 1/\sqrt{3}$, we get $\xi^{2\lambda}e^{-2\eta t(|\xi|^3-|\xi|)}\leq \xi^{2\lambda}e^{-2\eta t(1/\sqrt{3}\xi^2-\xi)}=:g(\xi)$, for $\xi\geq 1/\sqrt{3}$. Note that $g'(\xi)=0$ iff $\xi=\frac{1}{4}(\sqrt{3}+\sqrt{3+\frac{8\sqrt{3}\lambda}{\eta t}})$, for $\xi>0$. Let $\xi_0:=\frac{1}{4}(\sqrt{3}+\sqrt{3+\frac{8\sqrt{3}\lambda}{\eta t}})\geq\frac{\sqrt{3}}{2}\geq\frac{1}{\sqrt{3}}$, then $-2\eta t(\frac{\xi_0^2}{\sqrt{3}}-\xi_0)=\frac{\sqrt{3}}{4}\eta t-\lambda+\frac{\eta t}{4}\sqrt{3+\frac{8\sqrt{3}\lambda}{\eta t}}$. To finish the proof it is enough to see that

$$\sup_{\xi\geq 0}\xi^{2\lambda}e^{-2\eta t(\xi^3-\xi)}\leq c_\lambda\big[e^{2\eta t}+(1+\frac{1}{(\eta t)^{\lambda}})e^{\frac{\eta t}{4}(\sqrt{3}+\sqrt{3+\frac{8\sqrt{3}\lambda}{\eta t}})}\big].\blacksquare \tag{10}$$

Lemma 2.3. *Let $t>0$, $s>-1/2$ and $\eta>0$. Then $E_\eta(t)\in B(L^1(\mathbb{R}),H^s(\mathbb{R}))$ and*

$$\|E_\eta(t)\psi\|_s\leq c_s(e^{\eta t}+\frac{1}{(\eta t)^{\frac{2s+1}{6}}})\|\psi\|_{L^1}, \tag{11}$$

for all $\psi\in L^1(\mathbb{R})$.

Proof: Let $t>0$, $s>-1/2$, $\eta>0$ and $\psi\in L^1(\mathbb{R})$. Then

$$\|E_\eta(t)\psi\|^2_s\leq(2\pi)^{-1}\|\psi\|^2_{L^1}\int_{-\infty}^{+\infty}(1+\xi^2)^se^{-2\eta t(|\xi|^3-|\xi|)}d\xi. \tag{12}$$

Let I denote the integral on the right hand side of (12). We have that

$$\begin{aligned} I &= 2\int_0^{+\infty}(1+\xi^2)^se^{-2\eta t(\xi^3-\xi)}d\xi\leq 2^{3/2}3^se^{2\eta t}+2^{s+1}\int_{\sqrt{2}}^{+\infty}\xi^{2s}e^{-\eta t\xi^3}d\xi\\ &\leq 2^{3/2}3^se^{2\eta t}+\frac{2^{s+1}}{3(\eta t)^{\frac{2s+1}{3}}}\int_0^{+\infty}x^{\frac{2s-2}{3}}e^{-x}dx\\ &= 2^{3/2}3^se^{2\eta t}+\frac{2^{s+1}}{3(\eta t)^{\frac{2s+1}{3}}}\Gamma\big(\frac{2s+1}{3}\big)\leq c_s\big(e^{2\eta t}+\frac{1}{(\eta t)^{\frac{2s+1}{3}}}\big).\blacksquare \end{aligned}$$

3 Local Theory in $H^s(\mathbb{R}), s \geq 1, \eta > 0$.

In this section we use Banach's fixed point theorem in a suitable function space, to find a local solution to the integral equation (13) in $H^s(\mathbb{R}), s \geq 1$. Proposition 3.1 will give us a local solution for equation (1) in $H^s(\mathbb{R}), s \geq 1$. Similar considerations are made for equation (2) in $H^s(\mathbb{R}), s > 1/2$.

Theorem 3.1. *Let $\eta > 0$ be fixed and let $\phi \in H^s(\mathbb{R})$, where $s \geq 1$. Then, there exist $T_s(s, \|\phi\|_s, \eta) > 0$ and a unique function $u := u_\eta \in C(0, T_s; H^s(\mathbb{R}))$ satisfying the integral equation*

$$u(\cdot, t) = E_\eta(t)\phi(\cdot) - \frac{1}{2}\int_0^t E_\eta(t-t')(u_x)^2(\cdot, t')dt', \tag{13}$$

where $E_\eta(t)$ is defined by (6).

Proof: Let $M, T > 0$ be fixed, but arbitrary. T will be conveniently chosen later. Let us start analysing the case $s \in [1, 5/2)$. Consider

$$(Af)(t) = E_\eta(t)\phi - \frac{1}{2}\int_0^t E_\eta(t-t')(\partial_x f)^2(t')dt', \tag{14}$$

defined on the complete metric space

$$\Xi_s(T) = \big\{f \in C(0, T; H^s(\mathbb{R}));\ \sup_{t\in[0,T]} \|f(t) - E_\eta(t)\phi\|_s \leq M\big\}. \tag{15}$$

i.) Let $f \in \Xi_s(T)$. We will prove that $Af \in C(0, T : H^s(\mathbb{R}))$. Let $\tau > t > 0$.

$$\|(Af)(t) - (Af)(\tau)\|_s \leq \|(E_\eta(t) - E_\eta(\tau))\phi\|_s + F(t, \tau),$$

where

$$F(t,\tau) = \int_0^t \|(E_\eta(t-t') - E_\eta(\tau - t'))(f_x)^2(t')\|_s dt' + \int_t^\tau \|E_\eta(\tau - t')(f_x)^2(t')\|_s dt'.$$

Denote by $I_1(t,\tau)$ and $I_2(t,\tau)$ the first and second terms respectively on the right hand side of the last expression. By Lemma 2.3 and since $\|(f_x)^2(t')\|_{L^1} \leq \|f(t')\|_1^2$, we get

$$I_2(t,\tau) \leq c_s\eta^{-1}(M^2 + e^{2\eta T}\|\phi\|_s^2)\int_0^{\eta(\tau-t)} \big(e^x + x^{-\frac{2s+1}{6}}\big)dx.$$

It follows easily from the last inequality that $I_2(t,\tau)$ tends to zero as $\tau \to t$.
On the other hand, using triangular inequality and Lemma 2.3 again we get $\|(E_\eta(t-t') - E_\eta(\tau - t'))(f_x)^2(t')\|_s \leq 2c_s\big(e^{\eta(T-t')} + (\eta(t-t'))^{-\frac{2s+1}{6}}\big)\sup_{t'\in[0,T]}\|f(t')\|_1^2$, where the last term is an integrable function of $t' \in [0,t]$. Applying the Dominated Convergence Theorem we find that $\lim_{\tau\to t} I_1(t,\tau) = 0$.

ii.) Next, we prove that we can choose $T = \tilde{T} > 0$ sufficiently small such that $A(\Xi_s(\tilde{T})) \subset \Xi_s(\tilde{T})$. Let $u \in \Xi_s(T)$. By Lemma 2.3 we find that

$$\begin{aligned}\|(Au)(t) - E_\eta(t)\phi\|_s &\leq c_s \sup_{[0,T]} \|u(t)\|_1^2 \int_0^t \big(e^{\eta(t-t')} + \frac{1}{(\eta(t-t'))^{\frac{2s+1}{6}}}\big)dt' \\ &\leq c_s(M^2 + e^{2\eta T}\|\phi\|_s^2)\big(\frac{e^{\eta T}-1}{\eta} + \frac{6\eta^{-\frac{2s+1}{6}}}{-2s+5}T^{\frac{-2s+5}{6}}\big). \quad (16)\end{aligned}$$

Choose $T = \tilde{T} > 0$ small enough such that the right hand side of (16) is less than M.

iii.) Finally, let us prove that there exists $\hat{T} \in (0, \tilde{T}]$ such that A is a contraction in $\Xi_s(\hat{T})$. Let $t \in [0,T]$, $u, v \in \Xi_s(\tilde{T})$. Defining

$$g(T) := \frac{e^{\eta T}-1}{\eta} + \frac{6}{-2s+5}\eta^{-\frac{2s+1}{6}}T^{\frac{-2s+5}{6}}$$

we get

$$\begin{aligned}\|Au(t) - Av(t)\|_s &\leq \frac{c_s}{2}\sup_{0\leq t\leq\tilde{T}}\big(\|u(t)-v(t)\|_1(\|u(t)\|_1 + \|v(t)\|_1)\big)g(\tilde{T}) \\ &\leq c_s(M + e^{\eta\tilde{T}}\|\phi\|_s)g(\tilde{T})\sup_{0\leq t\leq\tilde{T}}\|u(t)-v(t)\|_s. \quad (17)\end{aligned}$$

Taking $\hat{T} \in (0,\tilde{T}]$ such that $c_s(M + e^{\eta\hat{T}}\|\phi\|_s)g(\hat{T}) < 1$, implies **iii.)**.

From **i.)**, **ii.)** and **iii.)**, it follows that A has a unique fixed point u in $\Xi_s(\hat{T})$ which satisfies equation (13) with $\hat{T} = T_s(s,\eta,\|\phi\|_s) > 0$. That u is the unique solution of equation (13) in the class $C(0,T_s : H^s(\mathbb{R}))$ is a consequence of Proposition 3.2 below.

If $s \geq 5/2$, $\phi \in H^s \subset H^2$. We consider equation (13) satisfied by $u \in C(0,T_2; H^2(\mathbb{R}))$, where $T_2 = T_2(s=2,\eta,\|\phi\|_2) > 0$ is the corresponding existence time to ϕ regarded as an element of $H^2(\mathbb{R})$. $E_\eta(t)\phi \in H^s(\mathbb{R})$ and by Lemma 2.2 $E_\eta(t-t')(u_x)^2(\cdot,t') \in H^s(\mathbb{R})$, for all $t' \in [0,t]$. Then $u(t) \in H^s(\mathbb{R})$, for all $t \in [0,T_2]$. We need to prove that $u \in C(0,T_2; H^s(\mathbb{R}))$, for that we analyse the cases $s \in [5/2, 3)$, $s \in [3, 7/2)$, and so on.

Let $s \in [5/2, 3)$ and $t, \tau \in [0, T_2]$. Assume $\tau > t > 0$.

$$\begin{aligned}
\|u(t) - u(\tau)\|_s \;\leq\; & \frac{1}{2}\int_0^t \|(E_\eta(t-t') - E_\eta(\tau - t'))(u_x)^2(\cdot, t')\|_s dt' \\
& + \frac{1}{2}\int_t^\tau \|E_\eta(\tau - t')(u_x)^2(\cdot, t')\|_s dt' + \|(E_\eta(t) - E_\eta(\tau))\phi\|_s. \quad (18)
\end{aligned}$$

The first term in the right hand side of (18) is treated as $I_1(t, \tau)$, using Lemma 2.2 with $\lambda = s - 1 \in [3/2, 2)$. Let $G(t, \tau)$ be the second term, then

$$\begin{aligned}
G(t,\tau) &\leq c_\lambda \int_t^\tau \big[e^{\eta(\tau - t')} + \big(1 + \frac{1}{(\eta(\tau - t'))^{\lambda/2}}\big) e^{\frac{\eta(\tau - t')}{8}\left(\sqrt{3} + \sqrt{3 + \frac{8\sqrt{3}\lambda}{\eta(\tau - t')}}\right)}\big] \|(u_x)^2(\cdot, t')\|_1 dt' \\
&\leq \frac{c_\lambda}{\eta} \sup_{t' \in [0, T_2]} \|u(\cdot, t')\|_2^2 \int_0^{\eta(\tau - t)} \big[e^x + \big(1 + \frac{1}{x^{\frac{\lambda}{2}}}\big) e^{\frac{x}{8}\left(\sqrt{3} + \sqrt{3 + \frac{8\sqrt{3}\lambda}{x}}\right)}\big] dx.
\end{aligned}$$

The last inequality implies that $G(t, \tau) \to 0$, as $\tau \to t$. For $s \in [3, 7/2)$, $s \in [7/2, 4)$, ... we proceed similarly. So we get $u \in C(0, T_2; H^s)$, for $s \geq 5/2$. When $s \in (2, 5/2)$ proceeding in a similar way to the case $s \geq 5/2$ we find that $u \in C(0, T_2; H^s)$. ■

Theorem 3.2. *Let $\eta > 0$ be fixed and let $\phi \in H^s(\mathbb{R})$, where $s > 1/2$. Then, there exist $T_s(s, \|\phi\|_s, \eta) > 0$ and a unique function $u := u_\eta \in C(0, T_s; H^s(\mathbb{R}))$ satisfying the integral equation*

$$u(\cdot, t) = E_\eta(t)\phi(\cdot) - \frac{1}{2}\int_0^t E_\eta(t - t')(u^2)_x(\cdot, t')dt', \tag{19}$$

where $E_\eta(t)$ is defined by (6).

Proof: Similar to the proof of Theorem 3.1. Use Lemma 2.2 (with $\lambda = 1$) and the fact that $\|(u^2)_x(t)\|_{s-1} \leq \|u^2(t)\|_s \leq \|u(t)\|_s^2$, for $s > 1/2$. ■

Proposition 3.1. *Problem (1) (resp. (2)) is equivalent to the integral equation (13) (resp. (19)). More precisely, if $u \in C([0, T]; H^s(\mathbb{R}))$ is a solution of (1) then u satisfies (13). Conversely, if $u \in C([0, T]; H^s(\mathbb{R}))$ is a solution of (13) then $u \in C^1([0, T]; H^{s-3}(\mathbb{R}))$ and satisfies (1).*

Proof: The first part was proved in Theorem 3.1 (resp. 3.2). The second part is similar to the proof of Theorem 4.19 (with $\lambda = 1$) in [11]. ■

Proposition 3.2 will prove the continuous dependance on the initial data for problems (1) and (2). Let us mention a known result we will use.

Lemma 3.1. *Suppose $\beta > 0$, $\gamma > 0$, $\beta + \gamma > 1$, $a \geq 0$, $b \geq 0$, u is nonnegative and $t^{\gamma-1}u(t)$ is locally integrable on $0 \leq t < T$. If*

$$u(t) \leq a + b\int_0^t (t-s)^{\beta-1}s^{\gamma-1}u(s)ds$$

a.e. in $(0,T)$, then

$$u(t) \leq aE_{\beta,\gamma}((b\Gamma(\beta))^{1/\nu}t),$$

where $\nu = \beta + \gamma - 1 > 0$, $E_{\beta,\gamma}(s) = \sum_{m=0}^{\infty} c_m s^{m\nu}$ with $c_0 = 1$ and $c_{m+1}/c_m = \Gamma(m\nu+\gamma)/\Gamma(m\nu+\gamma+\beta)$ for $m \geq 0$. As $s \to \infty$, $E_{\beta,\gamma}(s) = O(s^{1/2(\nu/\beta-\gamma)} \cdot \exp(\frac{\beta}{\nu}s^{\nu/\beta}))$.

Proof: See Lemma 7.1.2 in [7]. ■

Proposition 3.2. *Let $\phi, \psi \in H^s(\mathbb{R})$ and let $u, v \in C(0,T;H^s(\mathbb{R}))$ be the corresponding solutions of equation (13).*
1.) *If $s \in [1,2)$ then*

$$\|u(t) - v(t)\|_s \leq e^{\eta T}\|\phi - \psi\|_s E_{\frac{5-2s}{6},1}(\gamma t), \tag{20}$$

where $\gamma = \left(\frac{c_s M e^{\eta T}}{2}\frac{(\eta T)^{\frac{2s+1}{6}}+1}{\eta^{\frac{2s+1}{6}}}\Gamma(\frac{5-2s}{6})\right)^{\frac{6}{5-2s}}$ and $M = \sup_{t'\in[0,T]}(\|u(t')\|_s + \|v(t')\|_s)$.
2.) *If $s \geq 2$ then*

$$\|u(t) - v(t)\|_s \leq e^{\eta T}\|\phi - \psi\|_s E_{1/2,1}(\gamma t), \tag{21}$$

where $\gamma = (c_1 M F(T,\eta)\Gamma(1/2))^2$, $M = \sup_{t'\in[0,T]}(\|u(t')\|_s + \|v(t')\|_s)$ and $F(T,\eta) = (\frac{2\sqrt{\eta T}+1}{2\sqrt{\eta}})e^{\eta T + \frac{1}{8}\sqrt{3(\eta T)^2 + 8\sqrt{3}\eta T}}$.
Inequality (21) remains true if $\phi, \psi \in H^s(\mathbb{R}), s > 1/2$ and if $u, v \in C(0,T;H^s(\mathbb{R}))$, $s > 1/2$ are the corresponding solutions of equation (19).

Proof: Let ϕ, ψ, u, v be as above. Let us prove 1.). Let $s \in [1, 5/2)$, $w(t) = u(t) - v(t)$ and $M = \sup_{t'\in[0,T]}(\|u(t')\|_s + \|v(t')\|_s)$. By (11), (7) and the Cauchy-Schwarz inequality we obtain

$$\|w(t)\|_s \leq e^{\eta T}\|\phi - \psi\|_s + M\frac{c_s}{2}e^{\eta T}\int_0^t \frac{(\eta T)^{\frac{2s+1}{6}}+1}{\eta^{\frac{2s+1}{6}}}\frac{\|w(\cdot,t')\|_s}{(t-t')^{\frac{2s+1}{6}}}dt'. \tag{22}$$

Applying Lemma 3.1 to (22), (20) follows.

To prove 2.), let $s \geq 2$. Let $w(t)$ and M be as before. Since $s-1 \geq 1 > 1/2$, we obtain from (8) (with $\lambda = 1$)

$$\begin{aligned} \|w(t)\|_s &\leq e^{\eta T}\|\phi - \psi\|_s + \frac{c_1 M}{2}\int_0^t [e^{\eta(t-t')} + \big(1 + \frac{1}{(\eta(t-t'))^{1/2}}\big) e^{\frac{\eta(t-t')}{8}(\sqrt{3}+\sqrt{3+\frac{8\sqrt{3}}{\eta(t-t')}})}] \\ &\quad \cdot \|w(\cdot, t')\|_s dt'. \end{aligned} \tag{23}$$

Let $F(\eta, t, t')$ be the quantity under the integral and before $\|w(\cdot, t')\|_s$ in (23), then

$$\frac{1}{2}F(\eta, t, t') \leq \big(1 + \frac{1}{2\sqrt{\eta(t-t')}}\big) e^{\eta T + \frac{1}{8}\sqrt{3(\eta T)^2 + 8\sqrt{3}\eta T}}. \tag{24}$$

Combining (23) and (24) we get

$$\|w(t)\|_s \leq e^{\eta T}\|\phi - \psi\|_s + c_1 M F(T, \eta)\int_0^t \frac{\|w(\cdot, t')\|_s}{(t-t')^{1/2}} dt'. \tag{25}$$

Applying Lemma 3.1 to the last expression, (21) follows.

Consider now equation (19). Inequality (21) is obtained similarly to item 2.) for equation (13). Use the inequalities

$$\|(v^2(\cdot, t') - u^2(\cdot, t'))_x\|_{s-1} \leq \|v^2(\cdot, t') - u^2(\cdot, t')\|_s \leq M\|(v-u)(\cdot, t')\|_s$$

to obtain the same estimate (23) for $\|w(t)\|_s$. ■

4 Global Theory in $H^s(\mathbb{R}), s \geq 1, \eta > 0$.

In this section we prove that problems (1) and (2) are globally well posed in $H^s(\mathbb{R})$, $s \geq 1, \eta > 0$. Lemma 4.1 gives us some a priori estimates to prove the global Theorems 4.1 and 4.2. Lemma 4.2 will be used in Theorem 4.3 to study the limit of solutions of (2) when η goes to 0 in H^s, $s \geq 2$.

Lemma 4.1. *Let $\phi \in H^k(\mathbb{R}), k \geq 3$ and let $u \in C([0,T]; H^k)$ be a solution of (1), for some $T > 0$. Then, there exists a constant $c > 0$ independent of $\eta > 0$ such that:*

$$\begin{aligned} \|u\| &\leq (\|\phi\| + \sqrt{2T}\|\phi'\|^{5/3} e^{5/3\eta T}) e^{(1+\eta)T}, & (26) \\ \|u_x\| &\leq \|\phi'\| e^{\eta T}, & (27) \\ \|u_{xx}\| &\leq \|\phi''\| \cdot \exp[cT(\eta^{-3}\|\phi'\|^4 e^{4\eta T} + \eta)], & (28) \\ \|u_{xxx}\| &\leq \|\phi'''\| \cdot \exp[cT(\eta^{-3}\|\phi'\|^4 e^{4\eta T} + \eta)], & (29) \\ \|u_x\|_j^2 &\leq \|\phi'\|_j^2 \cdot \exp[cT(\sup_{[0,T]}\|u_x(t)\|_2 + \eta)], & (30) \end{aligned}$$

where $0 \leq t \leq T$, and $j \geq \max\{3, k-1\}$.

Proof. Let us prove (27). Defining $w := u_x$, it follows that w satisfies (2) with $\psi = \phi'$. Multiplying equation (2) by w and integrating over $\mathbb{R}$ we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w\|^2 &= -\eta(w, \mathcal{H}w_x) - \eta(w, \mathcal{H}w_{xxx}) = \eta(\int (|\xi| - |\xi|^3)|\hat{w}(\xi)|^2 d\xi) \\
&= \eta(\int_{|\xi|\leq 1} (|\xi| - |\xi|^3)|\hat{w}(\xi)|^2 d\xi + \int_{|\xi|>1} (|\xi| - |\xi|^3)|\hat{w}(\xi)|^2 d\xi) \qquad (31) \\
&\leq \eta(\int_{|\xi|\leq 1} (|\xi| - |\xi|^3)|\hat{w}(\xi)|^2 d\xi) \leq \eta(\int_{|\xi|\leq 1} |\hat{w}(\xi)|^2 d\xi) \leq \eta\|w\|^2. \qquad (32)
\end{aligned}$$

Integrating (32) between 0 and t and then applying Gronwall's inequality gives (27).

We will now prove (26). From equation (1) we have that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u\|^2 &= -\frac{1}{2}((u_x)^2, u) - \eta((\mathcal{H}u_x, u) + (\mathcal{H}u_{xxx}, u)) \\
&\leq \frac{1}{2}\|u\|_{L^\infty}\|u_x\|^2 + \eta\|u\|^2 \leq \|u\|^{1/2}\|u_x\|^{5/2} + \eta\|u\|^2 \\
&\leq (\|u\|^2 + \|u_x\|^{10/3}) + \eta\|u\|^2 \\
&\leq (1+\eta)\|u\|^2 + \|\phi'\|^{10/3}e^{10/3\eta T}, \qquad (33)
\end{aligned}$$

where we used (107), (109) and (27). Gronwall's inequality yields (26).

To prove (28), let $v := w_x = u_{xx}$. Then

$$v_t + (v^2 + wv_x) + v_{xxx} + \eta(\mathcal{H}v_x + \mathcal{H}v_{xxx}) = 0, \quad v(\cdot, 0) = \phi''(\cdot). \qquad (34)$$

Since $x - x^3 \leq -x^2$, for $x \geq 2$, we find that

$$\begin{aligned}
-\eta(v, \mathcal{H}v_x) - \eta(v, \mathcal{H}v_{xxx}) &\leq \eta\big(\int_{|\xi|<1} (|\xi| - |\xi|^3)|\hat{v}(\xi)|^2 d\xi + \int_{|\xi|>2} (|\xi| - |\xi|^3)|\hat{v}(\xi)|^2 d\xi\big) \\
&\leq \eta\big(\|v\|^2 - \int_{|\xi|>2} \xi^2|\hat{v}(\xi)|^2 d\xi\big). \qquad (35)
\end{aligned}$$

Using (35) and the fact that $(v, v^2) = -2(v, wv_x)$, we have that

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \leq (v, wv_x) + \eta(\|v\|^2 - \int_{|\xi|>2} \xi^2|\hat{v}(\xi)|^2 d\xi).$$

On the other hand,

$$(v, wv_x) \leq \|v\|_{L^\infty}\|w\|\|v_x\| \leq \|v\|^{1/2}\|v_x\|^{3/2}\|w\| \leq \|w\|(\epsilon\|v\|^2 + \epsilon^{-1/3}\|v_x\|^2),$$

for any $\epsilon > 0$. Then

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \leq (\eta + \epsilon\|\phi'\|e^{\eta T})\|v\|^2 + \|\phi'\|e^{\eta T}\epsilon^{-1/3}\|v_x\|^2 - \eta\int_{|\xi|>2}\xi^2|\hat{v}(\xi)|^2 d\xi, \tag{36}$$

where in (36) we employed (27). Choosing $\epsilon := (\frac{\|\phi'\|e^{\eta T}}{\eta})^3 > 0$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v\|^2 &\leq (\eta + \eta^{-3}\|\phi'\|^4 e^{4\eta T})\|v\|^2 + \eta\|v_x\|^2 - \eta\int_{|\xi|>2}\xi^2|\hat{v}(\xi)|^2 d\xi \\ &\leq (5\eta + \eta^{-3}\|\phi'\|^4 e^{4\eta T})\|v\|^2. \end{aligned} \tag{37}$$

We will now prove (29). Let $r := v_x = w_{xx} = u_{xxx}$. From (34) we obtain

$$r_t + 3vr + wr_x + r_{xxx} + \eta(\mathcal{H}r_x + \mathcal{H}r_{xxx}) = 0, \quad r(\cdot, 0) = \phi'''(\cdot). \tag{38}$$

Since $(w, rr_x) = 1/2(w, (r^2)_x) = -1/2(v, r^2) = -1/2(vr, r)$, we find that

$$\frac{1}{2}\frac{d}{dt}\|r\|^2 = 5(w, rr_x) - \eta((\mathcal{H}r_x, r) + (\mathcal{H}r_{xxx}, r)).$$

Following the same ideas used to prove (28) we get

$$\frac{1}{2}\frac{d}{dt}\|r\|^2 \leq 5\|w\|(\epsilon\|r\|^2 + \epsilon^{-1/3}\|r_x\|^2) + \eta(\|r\|^2 - \int_{|\xi|>2}\xi^2|\hat{v}(\xi)|^2 d\xi).$$

By (27) and choosing $\epsilon := (\frac{5\|\phi'\|e^{\eta T}}{\eta})^3 > 0$, we get $\frac{1}{2}\frac{d}{dt}\|r\|^2 \leq (\eta^{-3}(5\|\phi'\|e^{\eta T})^4 + 5\eta)\|r\|^2$.

Now we will prove (30). From equation (2) we have

$$\frac{1}{2}\frac{d}{dt}\|w\|_j^2 = -(w, ww_x)_j - \eta((w, \mathcal{H}w_x)_j + (w, \mathcal{H}w_{xxx})_j).$$

On the other hand,

$$-(w, \mathcal{H}w_x)_j - (w, \mathcal{H}w_{xxx})_j \leq \|w\|_j^2 - \int_{|\xi|>2}\xi^2((1+\xi^2)^j|\hat{w}(\xi)|^2 d\xi \tag{39}$$

$$\leq \|w\|_j^2. \tag{40}$$

Using (40) and (106), $j \geq 3$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|_j^2 \leq |(w, ww_x)_j| + \eta\|w\|_j^2 \leq (c\|w\|_2 + \eta)\|w\|_j^2. \blacksquare \tag{41}$$

Remark: Let $\psi \in H^k(\mathbb{R}), k \geq 2$. Let $w \in C([0,T]; H^k)$ be the solution of (2), for some $T > 0$. By Lemma 4.1 there exists a constant $c > 0$ independent of $\eta > 0$ such that (27) - (30) are satisfied with u_x replaced by w, ϕ' replaced by ψ and $j \geq 3$ in (30).

Theorem 4.1. *Let $\phi \in H^s(\mathbb{R})$, $s \geq 1$. Then, for each $\eta > 0$, there exists a unique $u_\eta \in C([0,\infty); H^s(\mathbb{R}))$ solution to the problem (1), such that $\partial_t u_\eta \in C([0,\infty); H^{s-3}(\mathbb{R}))$.*

Proof. If $s \in \mathbb{Z}^+$, then the result follows from the local theory and Lemma 4.1. If $s \geq 1$ is not an integer we will use nonlinear interpolation theory. More precisely we will use Theorems 1 and 2 in [4]. A similar proof was given in [3] for the KdV-Kuramoto-Sivashinsky equation. Let $k \geq 2$ be an integer, $k-1 < s < k$, $B_0^1 = L^2$, $B_0^2 = C(0,T;L^2)$, $B_1^1 = H^k$ and $B_1^2 = C(0,T;H^k)$. Let $\lambda = \frac{k-1}{k} \in (0,1)$ and $\theta = \frac{s}{k} \in (0,1)$. Then

$$\begin{aligned} B^1_{\lambda,2} &= [B_0^1, B_1^1]_{\lambda,2} = [L^2, H^k]_{\lambda,2} \approx H^{\lambda k} = H^{k-1}, \\ B^1_{\theta,2} &= [B_0^1, B_1^1]_{\theta,2} = [L^2, H^k]_{\theta,2} \approx H^{\theta k} = H^s. \end{aligned} \tag{42}$$

Since $\theta \geq \lambda$, we get $(\theta,2) \geq (\lambda,2)$, as in [4]. Let A be the mapping sending the initial data $\phi \in H^k$ into the unique solution $u \in C(0,T;H^k)$ to problem (1) obtained in Theorem 3.1. This mapping satisfies:

(1.) $A : H^k \to C(0,T;H^k)$ is continuous.

(2.) $A : H^{k-1} \to C(0,T;L^2)$ is Lipschitz in the following sense

$$\|A\phi - A\psi\|_{C(0,T;L^2)} \leq c_0(\|\phi\|_{H^{k-1}} + \|\psi\|_{H^{k-1}})\|\phi - \psi\|_{L^2}, \;\; \phi,\psi \in H^{k-1}. \tag{43}$$

(3.) $A : H^k \to C(0,T;H^k)$ is such that

$$\|A\phi\|_{C(0,T;H^k)} \leq c_1(\|\phi\|_{H^{k-1}})\|\phi\|_{H^k}, \;\; \phi \in H^k, \tag{44}$$

where $c_i : \mathbb{R}^+ \to \mathbb{R}^+$, $i = 0,1$ are continuous nondecreasing functions.

We will prove conditions **(1.)**, **(2.)** and **(3.)** below. Thus, all the hypotheses of Theorems 1 and 2 in [4] are valid in this context. It follows from Theorem 2 in [4] that

$$A : [L^2, H^k]_{\theta,2} \to [C(0,T;L^2), C(0,T;H^k]_{\theta,2}$$

continuously. But $[L^2,H^k]_{\theta,2} \approx H^s$ and by Proposition 3 in [4],

$$[C(0,T;L^2), C(0,T;H^k]_{\theta,2} \subset C(0,T;[L^2,H^k]_{\theta,2}) \approx C(0,T;H^s),$$

where the last inclusion is continuous. Hence $A : H^s \to C(0,T;H^s)$ continuously.

Since $k \geq 2$, condition **(1.)** follows directly from Proposition 3.2.

Next, we turn to condition **(2.)**. Let ϕ, $\psi \in H^{k-1}(\mathbb{R})$, $u = A\phi$, $v = A\psi$ and $w = u - v$.

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w\|^2 &= -\frac{1}{2}((u_x + v_x)w_x, w) - \eta(w, \mathcal{H}w_x + \mathcal{H}w_{xxx}) \\ &\leq \frac{1}{2}\|w_x\|(\|u_x\| + \|v_x\|)\|w\|_{L^\infty} + \eta(\|w\|^2 - \int_{|\xi|>2} \xi^2 |\hat{w}(\xi)|^2 d\xi). \end{aligned}$$

Since $\quad \|w_x\|\|w\|_{L^\infty} \leq \|w\|^{1/2}\|w_x\|^{3/2} \leq (\epsilon\|w\|^2 + \epsilon^{-1/3}\|w_x\|^2), \quad$ we find that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|^2 \;&\leq\; \big(\frac{1}{2}(\|\phi'\| + \|\psi'\|)e^{\eta T}\epsilon + \eta\big)\|w\|^2 + \frac{1}{2}\big(\|\phi'\| + \|\psi'\|\big)e^{\eta T}\epsilon^{-1/3}\|w_x\|^2 \\ &\quad -\eta\int_{|\xi|>2}\xi^2|\hat{w}(\xi)|^2 d\xi.\end{aligned}$$

Taking $\epsilon := \big(\frac{1}{2}\frac{(\|\phi'\|+\|\psi'\|)e^{\eta T}}{\eta}\big)^3 > 0$ into the last expression gives

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 \leq \big[\eta^{-3}\big(\frac{1}{2}(\|\phi'\| + \|\psi'\|)e^{\eta T}\big)^4 + 5\eta\big]\|w\|^2.$$

Then

$$\|w(\cdot,t)\| \leq e^{cT\left(\eta^{-3}(\|\phi'\|^4+\|\psi'\|^4)e^{4\eta T}+\eta\right)}\|\phi - \psi\|.$$

For $k = 1,2,3$, condition **(3.)** follows from Lemma 4.1 , formulas (26)-(29). For $k \geq 4$, we use (30) and we apply induction to $\|u\|_k^2 = \|u\|_{k-1}^2 + \|u_x\|_{k-1}^2$. ■

Theorem 4.2. *Let $\psi \in H^s(\mathbb{R})$, $s \geq 1$. Then, for each $\eta > 0$, there exists a unique $w_\eta \in C([0,\infty); H^s(\mathbb{R}))$ solution to the problem (2), such that $\partial_t w_\eta \in C([0,\infty); H^{s-3}(\mathbb{R}))$.*

Proof. The proof is the same as that of Theorem 4.1, the only difference being that

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\|w\|^2 &= -\tfrac{1}{2}((u^2 - v^2)_x, w) - \eta(w, \mathcal{H}w_x + \mathcal{H}w_{xxx}) \\ &\leq \tfrac{1}{2}\|w_x\|(\|u\| + \|v\|)\|w\|_{L^\infty} + \eta(\|w\|^2 - \textstyle\int_{|\xi|>2}\xi^2|\hat{w}(\xi)|^2 d\xi)\end{aligned}$$

is used now to prove condition **(2.)**. ■

Lemmas 4.2 and 4.4 will be used to prove Theorem 4.3 concerning the convergence of the solutions of equation (2) to the solution of the KdV equation as η tends to 0.

Lemma 4.2. *Consider the initial value problem (1), with $\phi \in H^s$, $s = 2,3$. Let $u \in C([0,T]; H^s)$ be the solution of (1) for some $T > 0$. Then, there exists a constant $c > 0$ independent of $\eta > 0$ such that*

$$\|u_{xx}\|^2 \leq \big[(1 + \eta T)\cdot e^{c\eta T}P_1(\|\phi\|_1) + P_2(\|\phi\|_2)\big]\cdot e^{c\eta T(1+\|\phi'\|^4 e^{4\eta T})}, \tag{45}$$

$$\begin{aligned}\|u_{xxx}\|^2 \;&\leq\; \big[Q_3(\|\phi\|_3) + (1+\eta T)e^{c\eta T}Q_1(\|\phi\|_1)\big]\cdot\exp\big\{\eta T\big[e^{c\eta T}Q_2(\|\phi\|_1) \\ &\quad +(1 + (1+\eta T)^{2/3})R_2(\|\phi\|_2)e^{c\eta T e^{c\eta T}R_1(\|\phi\|_1)}\big]\big\},\end{aligned} \tag{46}$$

for $t \in [0,T]$, and P_i, $i = 1,2$, Q_j, $j = 1,2,3$ and R_k, $k = 1,2$ are nondecreasing functions of their arguments.

Proof. Let us prove (45). Defining $w = u_x$, it follows that w is a solution of the problem (2), with $w(\cdot, 0) = \phi'(\cdot)$. As is well known, the functional

$$\Phi_2(w) = \frac{1}{2}\int_{-\infty}^{+\infty} [\frac{1}{3}w^3 - (w_x)^2]dx$$

is conserved by the KdV flow and

$$w_t = -\partial_x(\Phi_2'(w)) - \eta(\mathcal{H}w_x + \mathcal{H}w_{xxx}),$$

where $\Phi_2'(w) = 1/2w^2 + w_{xx}$. Then

$$\partial_t\Phi_2(w) = (\Phi_2'(w), w_t) = -\eta(\Phi_2'(w), \mathcal{H}w_x + \mathcal{H}w_{xxx}),$$

therefore

$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{+\infty}(\frac{1}{3}w^3 - (w_x)^2)dx = -\frac{\eta}{2}(w^2, \mathcal{H}w_x) - \frac{\eta}{2}(w^2, \mathcal{H}w_{xxx}) - \eta(w_{xx}, \mathcal{H}w_x) - \eta(w_{xx}, \mathcal{H}w_{xxx}).$

Integrating the last equation between 0 and t, we get

$$\begin{aligned}&\int_{-\infty}^{+\infty}(\frac{1}{3}w^3 - (w_x)^2)dx - \int_{-\infty}^{+\infty}(\frac{1}{3}(\phi')^3 - (\phi'')^2)dx \\ &= -\eta\int_0^t [(w^2, \mathcal{H}w_x) + (w^2, \mathcal{H}w_{xxx})]d\tau - 2\eta\int_0^t [(w_{xx}, \mathcal{H}w_x) + (w_{xx}, \mathcal{H}w_{xxx})]d\tau.\end{aligned}$$

From the last equality we obtain

$$\begin{aligned}\|w_x\|^2 &= \int_{-\infty}^{+\infty}[\frac{1}{3}w^3 - \frac{1}{3}(\phi')^3 + (\phi'')^2]dx + \eta\int_0^t (w^2, \mathcal{H}w_x)d\tau \\ &\quad +2\eta\int_0^t [-(ww_x, \mathcal{H}w_{xx}) - (w_x, \mathcal{H}w_{xx}) - (w_x, \mathcal{H}w_{xxxx})]d\tau. \qquad (47)\end{aligned}$$

On the other hand, we have

$$|\int_{-\infty}^{+\infty} w^3dx| \leq \|w\|_{L^\infty}\|w\|^2 \leq \|w\|^{5/2}\|w_x\|^{1/2} \leq \epsilon_1\|w_x\|^2 + \epsilon_1^{-1/3}\|w\|^{10/3}. \qquad (48)$$

$$\int_{-\infty}^{+\infty}(\phi')^3dx \leq \|\phi'\|_{L^\infty}\|\phi\|_2^2 \leq c\|\phi\|_2^3, \qquad \|\phi''\|^2 \leq \|\phi\|_2^2. \qquad (49)$$

$$(w^2, \mathcal{H}w_x) \leq \|w\|_{L^\infty}\|w\|\|w_x\| \leq \|w\|^{3/2}\|w_x\|^{3/2} \leq \|w\|^6 + \|w_x\|^2. \qquad (50)$$

Using estimates (48) - (50), it follows that the left hand side of (47) is bounded by

$$\begin{aligned}
\|w_x\|^2 &\leq \frac{\epsilon_1}{3}\|w_x\|^2 + \frac{1}{3\epsilon_1^{1/3}}\|w\|^{10/3} + c\|\phi\|_2^3 + \|\phi\|_2^2 + \eta\int_0^t (\|w\|^6 + \|w_x\|^2)d\tau \\
&\quad +2\eta\int_0^t [-(ww_x, \mathcal{H}w_{xx}) - (w_x, \mathcal{H}w_{xx}) - (w_x, \mathcal{H}w_{xxxx})]d\tau. \qquad (51)
\end{aligned}$$

Denote by $B(\tau)$ the integrand in the second integral in (51) and let $P_2(\|\phi\|_2) := c\|\phi\|_2^3 + \|\phi\|_2^2$. Taking $\epsilon_1 = 1$ in (51) and using inequality (27) we have that

$$\begin{aligned}
\frac{2}{3}\|w_x\|^2 &\leq ce^{c\eta T}\|\phi'\|^{10/3} + P_2(\|\phi\|_2) + \eta T(\|\phi'\|e^{\eta T})^6 + \eta\int_0^t \big(\|w_x\|^2 + 2B(\tau)\big)d\tau, \\
&\leq (1+\eta T)e^{c\eta T}P_1(\|\phi\|_1) + P_2(\|\phi\|_2) + \eta\int_0^t \big(\|w_x\|^2 + 2B(\tau)\big)d\tau, \qquad (52)
\end{aligned}$$

where $P_1(\|\phi\|_1) := c(1 + \|\phi\|_1^4 + \|\phi\|_1^6)$. Definig $v := w_x$, it follows that

$$|(wv, \mathcal{H}v_x)| \leq \|v\|_{L^\infty}\|w\|\|v_x\| \leq \|v\|^{1/2}\|w\|\|v_x\|^{3/2} \leq \|w\|(\epsilon\|v\|^2 + \epsilon^{-1/3}\|v_x\|^2).$$

Then

$$\begin{aligned}
B(\tau) &\leq |(wv, \mathcal{H}v_x)| - (v, \mathcal{H}v_x) - (v, \mathcal{H}v_{xxx}) \\
&\leq (5 + \|\phi'\|^4 e^{4\eta T})\|v\|^2. \qquad (53)
\end{aligned}$$

Inequality (53) was obtained in the same way as (37). Using (53) in (52), we get

$$\|w_x\|^2 \leq (1+\eta T)e^{c\eta T}P_1(\|\phi\|_1) + P_2(\|\phi\|_2) + c\eta(1 + \|\phi'\|^4 e^{4\eta T})\int_0^t \|w_x\|^2 d\tau.$$

Finally, applying Gronwall's inequality we obtain (45).

The next step will be to prove (46). To this end note that

$$\Phi_4(w) = \int_{-\infty}^{+\infty} (\frac{5}{12}w^4 - 5ww_x^2 + 3w_{xx}^2)dx,$$

is also conserved by the KdV flow. It follows that

$$\Phi_4'(w) = \frac{5}{3}w^3 + 5(w_x)^2 + 10ww_{xx} + 6w_{xxxx}.$$

Calculating $\partial_t(\Phi_4(w))$, we obtain

$$\partial_t(\Phi_4(w)) = (\Phi_4'(w), w_t) = (\Phi_4'(w), -(\Phi_2'(w))_x) - \eta(\Phi_4'(w), \mathcal{H}w_x + \mathcal{H}w_{xxx})$$
$$= -\eta\big(\frac{5}{3}w^3 + 5(w_x)^2 + 10ww_{xx} + 6w_{xxxx}, \mathcal{H}w_x + \mathcal{H}w_{xxx}\big)$$
$$= -\eta\big[\frac{5}{3}(w^3, \mathcal{H}w_x) + \frac{5}{3}(w^3, \mathcal{H}w_{xxx}) + 5((w_x)^2, \mathcal{H}w_x) + 5((w_x)^2, \mathcal{H}w_{xxx})$$
$$+10(ww_{xx}, \mathcal{H}w_x) + 10(ww_{xx}, \mathcal{H}w_{xxx}) + 6(w_{xxxx}, \mathcal{H}w_x) + 6(w_{xxxx}, \mathcal{H}w_{xxx})\big]. \quad (54)$$

On the other hand, we find that

$$|(w^3, \mathcal{H}w_x)| \le \|w\|_{L^\infty}^2\|w\|\|w_x\| \le \|w\|^2\|w_x\|^2 \le \|w\|^3\|w_{xx}\| \le \|w\|^6 + \|w_{xx}\|^2, \quad (55)$$

$$\begin{aligned} |(w^3, \mathcal{H}w_{xxx})| &= 3|(w^2w_x, \mathcal{H}w_{xx})| \le 3\|w\|_{L^\infty}^2\|w_x\|\|w_{xx}\| \\ &\le 3\|w\|\|w_x\|^2\|w_{xx}\| \le 3\|w\|^2\|w_{xx}\|^2, \quad (56) \\ |((w_x)^2, \mathcal{H}w_x)| &\le \|w_x\|_{L^\infty}\|w_x\|^2 \le \|w_x\|^{1/2}\|w_{xx}\|^{1/2}\|w_x\|^2 = \|w_x\|^{5/2}\|w_{xx}\|^{1/2} \\ &\le \|w\|^{5/4}\|w_{xx}\|^{7/4} \le \|w\|^{10} + \|w_{xx}\|^2, \quad (57) \end{aligned}$$

$$\begin{aligned} |((w_x)^2, \mathcal{H}w_{xxx})| &= 2|(w_xw_{xx}, \mathcal{H}w_{xx})| \le 2\|w_{xx}\|_{L^\infty}\|w_{xx}\|\|w_x\| \le 2\|w_x\|\|w_{xx}\|^{3/2}\|w_{xxx}\|^{1/2} \\ &\le 2\|w_x\|(\epsilon_1\|w_{xx}\|^2 + \epsilon_1^{-3}\|w_{xxx}\|^2), \quad (58) \\ |(ww_{xx}, \mathcal{H}w_x)| &\le \|w\|_{L^\infty}\|w_{xx}\|\|w_x\| \le \|w\|^{1/2}\|w_x\|^{3/2}\|w_{xx}\| \\ &\le \|w\|^{1/2}(\|w\|^{1/2}\|w_{xx}\|^{1/2})^{3/2}\|w_{xx}\| \le \|w\|^{10} + \|w_{xx}\|^2. \quad (59) \end{aligned}$$

Using (55)-(59), the left hand side of (54) is bounded by

$$\partial_t(\Phi_4(w)) \le c\eta\big[\|w\|^6 + \|w_{xx}\|^2 + \|w\|^2\|w_{xx}\|^2 + \|w\|^{10} + \|w_{xx}\|^2 + \|w_x\|(\epsilon_1\|w_{xx}\|^2$$
$$+\epsilon_1^{-3}\|w_{xxx}\|^2)\big] - 10\eta(ww_{xx}, \mathcal{H}w_{xxx}) - 6\eta\big[(w_{xx}, \mathcal{H}w_{xxx}) + (w_{xx}, \mathcal{H}w_{xxxxx})\big]. \quad (60)$$

As a consequence of (27), we obtain

$$\partial_t(\Phi_4(w)) \le \eta e^{c\eta T}Q_1(\|\phi\|_1) + \eta e^{c\eta T}Q_2'(\|\phi\|_1)\|w_{xx}\|^2 + c\eta\|w_x\|(\epsilon_1\|w_{xx}\|^2 + \epsilon_1^{-3}\|w_{xxx}\|^2)$$
$$-10\eta(ww_{xx}, \mathcal{H}w_{xxx}) - 6\eta\big[(w_{xx}, \mathcal{H}w_{xxx}) + (w_{xx}, \mathcal{H}w_{xxxxx})\big], \quad (61)$$

where $Q_1(\|\phi\|_1) = c(\|\phi\|_1^2 + \|\phi\|_1^{10})$ and $Q_2'(\|\phi\|_1) = c(1 + \|\phi\|_1^2)$. On the other hand, defining $r := w_{xx} = u_{xxx}$, we have that

$$\begin{aligned} (ww_{xx}, \mathcal{H}w_{xxx}) &\le |(ww_{xx}, \mathcal{H}w_{xxx})| = |(wr, \mathcal{H}r_x)| \le \|r\|_{L^\infty}\|w\|\|r_x\| \\ &\le \|w\|\|r\|^{1/2}\|r_x\|^{3/2} \le \|w\|(\epsilon_2\|r\|^2 + \epsilon_2^{-1/3}\|r_x\|^2), \quad (62) \end{aligned}$$

$$-(w_{xx},\mathcal{H}w_{xxx})-(w_{xx},\mathcal{H}w_{xxxxx})=-(r,\mathcal{H}r_x)-(r,\mathcal{H}r_{xxx})\le\|r\|^2-\int_{|\xi|>2}\xi^2|\hat{r}(\xi)|^2d\xi.$$

Hence, using (62) and the last inequality in (61), it follows that

$$\begin{aligned}\partial_t(\Phi_4(w)) &\le \eta e^{c\eta T}Q_1(\|\phi\|_1)+\eta e^{c\eta T}Q_2''(\|\phi\|_1)\|r\|^2+6\eta\big[\frac{c}{6}(\|u_{xx}\|\epsilon_1+\|u_x\|\epsilon_2)\|r\|^2\\ &\quad+\frac{c}{6}(\|u_{xx}\|\epsilon_1^{-3}+\|u_x\|\epsilon_2^{-1/3})\|r_x\|^2-\int_{|\xi|>2}\xi^2|\hat{r}(\xi)|^2d\xi\big],\end{aligned}\tag{63}$$

where $Q_2''(\|\phi\|_1):=Q_2'(\|\phi\|_1)+6$. Taking the square root in (45) we have

$$\|u_{xx}\|\le(1+\sqrt{1+\eta T})R_2(\|\phi\|_2)e^{c\eta Te^{c\eta T}R_1(\|\phi\|_1)},\tag{64}$$

where $R_1(\|\phi\|_1):=2+\|\phi\|_1^4$ and $R_2(\|\phi\|_2):=\sqrt{P_1(\|\phi\|_2)}+\sqrt{P_2(\|\phi\|_2)}$ are nondecreasing functions of their arguments. Combining (64), (27) and (63) we obtain

$$\begin{aligned}\partial_t(\Phi_4(w)) &\le \eta e^{c\eta T}Q_1(\|\phi\|_1)+\eta e^{c\eta T}Q_2''(\|\phi\|_1)\|r\|^2\\ &\quad+c\eta\big[\big((1+\sqrt{1+\eta T})R_2(\|\phi\|_2)e^{c\eta Te^{c\eta T}R_1(\|\phi\|_1)}\big)\epsilon_1+\|\phi'\|e^{\eta T}\epsilon_2\big]\|r\|^2\\ &\quad+6\eta\{\frac{c}{6}\big[\big((1+\sqrt{1+\eta T})R_2(\|\phi\|_2)e^{c\eta Te^{c\eta T}R_1(\|\phi\|_1)}\big)\epsilon_1^{-3}+\|\phi'\|e^{\eta T}\epsilon_2^{-1/3}\big]\|r_x\|^2\\ &\quad-\int_{|\xi|>2}\xi^2|\hat{r}(\xi)|^2d\xi\}\end{aligned}\tag{65}$$

Choosing $\epsilon_1=\big(\frac{c}{3}((1+\sqrt{1+\eta T})R_2(\|\phi\|_2)e^{c\eta Te^{c\eta T}R_1(\|\phi\|_1)})\big)^{1/3}>0$, and
$\epsilon_2=\big(\frac{c}{3}\|\phi'\|e^{\eta T}\big)^3>0$, it follows from (65) that

$$\begin{aligned}\partial_t(\Phi_4(w)) &\le \eta e^{c\eta T}Q_1(\|\phi\|_1)+\eta e^{c\eta T}Q_2''(\|\phi\|_1)\|r\|^2\\ &\quad+c\eta\big[\big((1+\sqrt{1+\eta T})R_2(\|\phi\|_2)e^{c\eta Te^{c\eta T}R_1(\|\phi\|_1)}\big)^{4/3}+\|\phi'\|^4e^{4\eta T}\big]\|r\|^2\\ &\quad+6\eta\big(\|r_x\|^2-\int_{|\xi|>2}\xi^2|\hat{r}(\xi)|^2d\xi\big).\end{aligned}\tag{66}$$

Since

$$\|r_x\|^2-\int_{|\xi|>2}\xi^2|\hat{r}(\xi)|^2d\xi=\int_{|\xi|\le2}\xi^2|\hat{r}(\xi)|^2d\xi\le4\|r\|^2,$$

we find that

$$\begin{aligned}\partial_t(\Phi_4(w)) &\le \eta e^{c\eta T}Q_1(\|\phi\|_1)+\eta e^{c\eta T}Q_2(\|\phi\|_1)\|r\|^2\\ &\quad+c\eta\big((1+\sqrt{1+\eta T})R_2(\|\phi\|_2)e^{c\eta Te^{c\eta T}R_1(\|\phi\|_1)}\big)^{4/3}\|r\|^2,\end{aligned}\tag{67}$$

where $Q_2(\|\phi\|_1) := Q_2''(\|\phi\|_1) + 24 + c\|\phi\|_1^4$. Integrating (67) in $[0,t]$ we obtain

$$\begin{aligned}
& \int_{-\infty}^{+\infty} \big(\frac{5}{12}w^4 - 5ww_x^2 + 3w_{xx}^2\big)dx \\
\leq \quad & \int_{-\infty}^{+\infty} \big(\frac{5}{12}(\phi')^4 - 5\phi'(\phi'')^2 + 3(\phi''')^2\big)dx + \eta T e^{c\eta T} Q_1(\|\phi\|_1) + \big\{\eta e^{c\eta T} Q_2(\|\phi\|_1) \\
& + c\eta\big((1+\sqrt{1+\eta T})R_2(\|\phi\|_2)e^{c\eta T e^{c\eta T} R_1(\|\phi\|_1)}\big)^{4/3}\big\} \cdot \int_0^t \|r(\tau)\|^2 d\tau. \qquad (68)
\end{aligned}$$

On the other hand:

$$\begin{aligned}
\int (\phi')^4 dx &\leq \|\phi'\|_{L^\infty}^2 \|\phi\|_1^2 \leq c\|\phi'\|_1^2\|\phi\|_2^2 \leq c\|\phi\|_2^4. && (69)\\
\Big|\int \phi'(\phi'')^2 dx\Big| &\leq \|\phi'\|_{L^\infty}\|\phi\|_2^2 \leq c\|\phi\|_2^3. && (70)\\
\int (\phi''')^2 dx &\leq \|\phi\|_3^2. && (71)\\
\int w^4 dx &\leq \|w\|_{L^\infty}^2\|w\|^2 \leq \|w\|^3\|w_x\| \leq c\|w\|^{7/2}\|w_{xx}\|^{1/2} \\
&\leq c(\epsilon_1\|w_{xx}\|^2 + \epsilon_1^{-1/3}\|w\|^{14/3}), && (72)\\
\int w(w_x)^2 dx &\leq \|w_x\|_{L^\infty}\|w\|\|w_x\| \leq \|w\|\|w_x\|^{3/2}\|w_{xx}\|^{1/2} \\
&\leq c\|w\|^{7/4}\|w_{xx}\|^{5/4} \leq c(\epsilon_2\|w_{xx}\|^2 + \epsilon_2^{-5/3}\|w\|^{14/3}). && (73)
\end{aligned}$$

Using (69)-(73) in (68) we obtain

$$\begin{aligned}
3\|r\|^2 \leq \; & c\big[\|\phi\|_2^4 + \|\phi\|_2^3 + \|\phi\|_3^2 + \epsilon_1\|r\|^2 + \epsilon_1^{-1/3}(\|\phi'\|e^{\eta T})^{14/3} + \epsilon_2\|r\|^2 \\
& + \epsilon_2^{-5/3}(\|\phi'\|e^{\eta T})^{14/3}\big] + \eta T e^{c\eta T}Q_1(\|\phi\|_1) + c\eta\big\{e^{c\eta T}Q_2(\|\phi\|_1) \\
& + \big((1+\sqrt{1+\eta T})R_2(\|\phi\|_2)e^{c\eta T e^{c\eta T} R_1(\|\phi\|_1)}\big)^{4/3}\big\}\int_0^t \|r(\tau)\|^2 d\tau. \qquad (74)
\end{aligned}$$

Taking $\epsilon_1 = 1/c = \epsilon_2 > 0$, defining $Q_3(\|\phi\|_3) := c(\|\phi\|_3^2 + \|\phi\|_3^3 + \|\phi\|_3^4)$ and $Q_4(\|\phi\|_1) := c\|\phi\|_1^{14/3}$, we get

$$\begin{aligned}
\|r\|^2 \leq \; & Q_3(\|\phi\|_3) + e^{c\eta T}Q_4(\|\phi\|_1) + \eta T e^{c\eta T}Q_1(\|\phi\|_1) + \eta\big\{e^{c\eta T}Q_2(\|\phi\|_1) \\
& + \big((1+(1+\eta T)^{2/3})R_2(\|\phi\|_2)e^{c\eta T e^{c\eta T}R_1(\|\phi\|_1)}\big)\big\}\int_0^t \|r(\tau)\|^2 d\tau. \qquad (75)
\end{aligned}$$

Gronwall's inequality applied to (75) implies (46), where $Q_1(\|\phi\|_1)$ now stands for $Q_1(\|\phi\|_1) + Q_4(\|\phi\|_1)$. This concludes the proof. ■

Lemma 4.3. *Let $s > 3/2$ be fixed, $\phi \in H^s$, and let $w_\eta \in C(0, T_s : H^s)$ be the solution of (2) with $\eta > 0$. Then, there exists a $T'_s > 0$ depending on s and $\|\phi\|_s$, but not on $0 < \eta < 1$, such that w_η can be extended to the interval $[0, T'_s]$, and there is a function $\rho \in C([0, T'_s]; \mathbb{R})$ such that*

$$\|w_\eta(t)\|_s^2 \leq \rho(t), \quad \rho(0) = \|\phi\|_s^2, \quad t \in [0, T'_s]. \tag{76}$$

Proof. Using $(w, w_{xxx})_s = 0$, (105), (40) and since $\eta \in (0, 1)$, it follows from equation (2) that

$$\partial_t \|w_\eta(t)\|_s^2 \leq C_s(\|w_\eta(t)\|_s^2 + \|w_\eta(t)\|_s^3).$$

Then $\|w_\eta(t)\|_s^2 \leq \rho(t)$ in $[0, T'_s]$, where $\rho(t)$ satisfies the differential equation

$$\partial_t \rho(t) = C_s(\rho(t) + \rho(t)^{3/2}), \quad \rho(0) = \|\phi\|_s^2.$$

Solving the last differential equation we have $\rho(t) = e^{C_s t}\|\phi\|_s^2 \cdot (1 + \|\phi\|_s - e^{C_s/2 \cdot t}\|\phi\|_s)^{-2}$, for $t \in [0, T'_s]$, where $T'_s < 2/C_s \cdot ln((1 + \|\phi\|_s)/(\|\phi\|_s))$. ■

Lemma 4.4. *Let $s \geq 2$ be fixed, $\phi \in H^s$, and let $w_\eta \in C(0, T : H^s)$ be the solution of (2) with $0 < \eta < 1$. Then, there exists a constant $C = C(s, T, \|\phi\|_2)$, such that*

$$\|w_\eta(t)\|_s^2 \leq \|\phi\|_s^2 \cdot \exp(C(s, T, \|\phi\|_2)T), \tag{77}$$

Proof. Using $(w, w_{xxx})_s = 0$, (106) and (40) we obtain easily from equation (2)

$$\partial_t \|w_\eta(t)\|_s^2 \leq 2\eta \|w_\eta(t)\|_s^2 + C_s \|w_\eta(t)\|_2 \cdot \|w_\eta(t)\|_s^2.$$

By Lemma 4.2, the H^2 norm of $w_\eta(t)$ is bounded by a function of T and $\|\phi\|_2$ but independently of $\eta < 1$. Since $\eta \in (0, 1)$, we obtain from the last inequality $\partial_t \|w_\eta(t)\|_s^2 \leq C(s, T, \|\phi\|_2) \cdot \|w_\eta(t)\|_s^2$. Gronwall's inequality then leads to (77). ■

Theorem 4.3. *Let $\eta > 0$ and $\phi \in H^s, s \geq 2$, be given, and let w_η be the solution of the equation (2) such that $w_\eta(\cdot, 0) = \phi(\cdot)$. Then, the limit $w_0 = \lim_{\eta \to 0} w_\eta$ exists in $C(0, T; H^s)$ and is the unique solution of (2) with $\eta = 0$. Moreover, the map $\phi \in H^s \mapsto w_0 \in C(0, T; H^s)$ is continuous with respect to the corresponding topologies.*

Proof: To prove this theorem we proceed as in [8] or [12].

Let $w^{(j)} = w_{\eta j}, j = 1, 2$, be the solutions of (2) with the same initial condition $\phi \in H^s$. Let $w = w^{(1)} - w^{(2)}$, then we obtain
$w_t + w^{(1)}w_x^{(1)} - w^{(2)}w_x^{(2)} + w_{xxx} + \eta_1(\mathcal{H}w_x + \mathcal{H}w_{xxx}) + (\eta_1 - \eta_2)(\mathcal{H}w_x^{(2)} + \mathcal{H}w_{xxx}^{(2)}) = 0.$
Multiplying the last equation by w and integrating over $\mathbb{R}$ we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w(t)\|^2 &= -(w, ww_x^{(1)}) - (w, w_x w^{(2)}) - \eta_1(w, \mathcal{H}w_x + \mathcal{H}w_{xxx}) \\
&\quad -(\eta_1 - \eta_2)(w, \mathcal{H}w_x^{(2)} + \mathcal{H}w_{xxx}^{(2)}) \\
&\le \|w_x^{(1)}\|_{L^\infty}\|w\|^2 + \frac{1}{2}\|w_x^{(2)}\|_{L^\infty}\|w\|^2 + \eta_1\|w\|^2 \\
&\quad +c|\eta_1 - \eta_2|(\|w\|\|w_x^{(2)}\| + \|w_{xx}\|\|w_x^{(2)}\|) \\
&\le c(\|w^{(1)}\|_2 + \|w^{(2)}\|_2 + \eta_1)\|w\|^2 + c|\eta_1 - \eta_2|(\|w^{(1)}\|_2 + \|w^{(2)}\|_2)\|w^{(2)}\|_1.
\end{aligned}$$

Applying Gronwall's inequality to the last relation we find

$$\begin{aligned}
\|w\|^2 &\le cT|\eta_1 - \eta_2| \sup_{t\in[0,T]}[(\|w^{(1)}\|_2 + \|w^{(2)}\|_2)\|w^{(2)}\|_1] \\
&\quad \cdot \exp[cT \sup_{t\in[0,T]}(\|w^{(1)}\|_2 + \|w^{(2)}\|_2 + \eta_1)].
\end{aligned}$$

By Lemma 4.2 the H^2 norms of $w^{(1)}$ and $w^{(2)}$ are bounded by a function of T and $\|\phi\|_2$ but independent of $\eta < 1$. From the last inequality and because L^2 is complete, there exists the limit $w_0(t) = \lim_{\eta\to 0} w_\eta(t)$ in L^2 uniformly with respect to $t \in [0, T]$, i.e. $\lim_{\eta\to 0}\sup_{t\in[0,T]}\|w_\eta(t) - w_0(t)\| = 0$, and therefore $w_0 \in C(0, T; L^2)$. Let us prove that $w_0(t) \in H^s$ for all $t \in [0, T]$. Let $t \in [0, T]$ and take a sequence $(\eta_n)_{n\in\mathbb{N}}$ with $\eta_n > 0$ and $\eta_n \to 0$ when $n \to \infty$. Since $L^2 - lim_{n\to\infty} w_{\eta_n}(t) = w_0(t)$, there exists a subsequence which we again call $(w_{\eta_n}(t))$ such that $\lim_{n\to\infty}\widehat{w_{\eta_n}}(t,\xi) = \widehat{w_0}(t,\xi), \quad \xi - a.e.$ Applying Fatou's Lemma, we find that

$$\int (1+\xi^2)^s|\widehat{w_0}(t,\xi)|^2 d\xi \le \liminf_{n\to\infty}\int(1+\xi^2)^s|\widehat{w_{\eta_n}}(t,\xi)|^2 d\xi \le \|\phi\|_s^2 e^{(C(s,T,\|\phi\|_s)T)},$$

where the last inequality is a consequence of inequality (77). We claim that

$$w_\eta(t) \rightharpoonup w_0(t), \ \ in \ \ H^s \tag{78}$$

uniformly over $[0, T]$ as $\eta \to 0$. In fact, defining $C^2 := \|\phi\|_s^2 \cdot \exp(C(s, T, \|\phi\|_s)T)$, as on the right hand side of (77), let $\varphi \in H^{-s}$ and let $\epsilon > 0$ be fixed but arbitrary then by density, there exists an element $\varphi_\epsilon \in L^2$ such that $\|\varphi - \varphi_\epsilon\|_{-s} < \epsilon/(3 \cdot C)$, then

$$|\varphi(w_\eta(t) - w_0(t))| \le \|\varphi - \varphi_\epsilon\|_{-s}\|w_\eta(t) - w_0(t)\|_s + |\varphi_\epsilon(w_\eta(t) - \ w_0(t))|. \tag{79}$$

Since $\varphi_\epsilon \in L^2$, there exists $\delta_{(\epsilon)} > 0$ such that $0 < \eta < \delta_{(\epsilon)}$ implies $|\varphi_\epsilon(w_\eta(t) - w_0(t))| < \epsilon/3$, for all $t \in [0,T]$. Using (77) and $\|w_0(t)\|_s \leq \liminf_{\eta\to 0} \|w_\eta(t)\|_s \leq C$ in inequality (79) we have that $\varphi(w_\eta(t) - w_0(t)) \to 0$ as $\eta \to 0$, uniformly in $t \in [0,T]$.
Since $w_\eta \in C(0,T;H^s)$, it follows that $w_\eta \in C^\omega(0,T;H^s)$. Then $w_0 \in C^\omega(0,T;H^s)$.

Let $\psi \in H^{s-3}$, then

$$\begin{aligned}(w_\eta(t),\psi)_{s-3} &= (\phi,\psi)_{s-3}\\ &- \int_0^t (w_\eta(t')\partial_x w_\eta(t') + \partial_x^3 w_\eta(t') + \eta(\mathcal{H}\partial_x w_\eta(t') + \mathcal{H}\partial_x^3 w_\eta(t')),\psi)_{s-3}dt', \end{aligned} \tag{80}$$

for all $t \in [0,T]$. From (78) we have that

$$\begin{aligned} \partial_x^3 w_\eta &\rightharpoonup \partial_x^3 w_0,\ \ in\ \ H^{s-3},\\ \mathcal{H}\partial_x w_\eta &\rightharpoonup \mathcal{H}\partial_x w_0,\ \ in\ \ H^{s-1},\\ \mathcal{H}\partial_x^3 w_\eta &\rightharpoonup \mathcal{H}\partial_x^3 w_0,\ \ in\ \ H^{s-3} \end{aligned} \tag{81}$$

uniformly over $[0,T]$ as $\eta \to 0$. On the other hand, $f_n \rightharpoonup f$ in H^r, $g_n \rightharpoonup g$ in H^r implies $f_n g_n \rightharpoonup fg$ in H^r, for $r > 1/2$. Then $w_\eta(t')\partial_x w_\eta(t') \rightharpoonup w_0(t')\partial_x w_0(t')$ in H^{s-1}, for all $t' \in [0,T]$. By (77) we see that

$$\|w_\eta(t')\partial_x w_\eta(t')\|_{s-3} \leq \|w_\eta(t')\partial_x w_\eta(t')\|_{s-1} \leq \|w_\eta(t')\|_s^2 \leq \|\phi\|_s^2 \cdot e^{C(s,T,\|\phi\|_2)T},$$

for all $t' \in [0,T]$. Then, using the Dominated Convergence Theorem, we get

$$\lim_{\eta\to 0}\int_0^t (w_\eta(t')\partial_x w_\eta(t'),\psi)_{s-3}dt' = \int_0^t (w_0(t')\partial_x w_0(t'),\psi)_{s-3}dt'. \tag{82}$$

Using (81),(82) and letting $\eta \to 0$ in (80), we obtain

$$(w_0(t),\psi)_{s-3} = (\phi,\psi)_{s-3} - \int_0^t (w_0(t')\partial_x w_0(t') + \partial_x^3 w_0(t'),\psi)_{s-3}dt', \tag{83}$$

for all $\psi \in H^{s-3}$ and $t \in [0,T]$. Since $t \in [0,T] \longmapsto w_0(t)\partial_x w_0(t) + \partial_x^3 w_0(t) \in H^{s-3}$ is weakly continuous, it is strongly measurable, thus

$$w_0(t) = \phi - \int_0^t w_0(t')\partial_x w_0(t') + \partial_x^3 w_0(t')dt' \tag{84}$$

exists as a Bochner integral. Then $w_0 \in AC([0,T];H^{s-3})$. As the equation (83) is valid for all $\psi \in H^{s-3}$, using the Fundamental Theorem of Calculus we have that

$$\partial_t w_0(t) = -w_0(t)\partial_x w_0(t) - \partial_x^3 w_0(t),\ \ t \in [0,T], \tag{85}$$

with the derivative respect to t calculated in the H^{s-3} norm. By Theorem 6.11 in [11], there is a unique solution for the KdV equation in the class $w_0 \in C([0,T] : L^2) \cap C^{\omega}([0,T]; H^s) \cap AC([0,T]; H^{s-3})$ and this unique solution belongs to $w_0 \in C([0,T]; H^s)$, as we asserted. To establish that the solution depends continuously on the initial data, we proceed as in [3]. This completes the proof of the theorem. ■

5 Global Theory in $\mathcal{F}_{r,r}(\mathbb{R})$ and Decay Properties of the Solution of (2).

Some decay properties of the solution of equation (2) for $\eta > 0$ are obtained, similar to those obtained for the Benjamin Ono equation (see [8]). Theorem 5.3 is a unique continuation theorem for equation (2). It implies loss of persistence for equation (2) in $\mathcal{F}_{3,3}$, while for BO this occurs in $\mathcal{F}_{4,4}$.

Lemma 5.1. *Let $F_\eta(t,\xi)$ be defined by (5), where $t \geq 0$, $\xi \in \mathbb{R}$ and $\eta \geq 0$. Then,*

$$\partial_\xi F_\eta(t,\xi) = F_\eta(t,\xi) t \big[3i\xi^2 - \eta h(\xi)(3\xi^2 - 1)\big], \tag{86}$$

$$\partial_\xi^2 F_\eta(t,\xi) = F_\eta(t,\xi)\big(t^2\big[3i\xi^2 - \eta h(\xi)(3\xi^2-1)\big]^2 + 6t\xi(i - \eta h(\xi))\big) + 2\eta t\delta \tag{87}$$

and

$$\begin{aligned}\partial_\xi^3 F_\eta(t,\xi) = {} & F_\eta(t,\xi)t^3\big[3i\xi^2 - \eta h(\xi)(3\xi^2-1)\big]^3 + 6t(i - \eta h(\xi))F_\eta(t,\xi) \\ & + F_\eta(t,\xi)t^2\big[36\xi^3(\eta^2 - 1) - 72i\eta h(\xi)\xi^3 + 12i\eta h(\xi)\xi - 12\eta^2\xi\big] \\ & + 6t^2\xi(i - \eta h(\xi))\big[3i\xi^2 - \eta h(\xi)(3\xi^2-1)\big]F_\eta(t,\xi) + 2\eta t\delta',\end{aligned} \tag{88}$$

where δ is the Dirac delta distribution and $h(\xi)$ is the sign function.

Proof: This result follows easily from the chain rule. ■

Lemma 5.2. *Let $\eta \geq 0$ be fixed. Then,*

1. *$E_\eta(t) : \mathcal{F}_{r,r} \to \mathcal{F}_{r,r}$, for $r = 0, 1$ is a C^0 semigroup and we have that*

$$\|E_\eta(t)\phi\|_{\mathcal{F}_{r,r}} \leq e^{c\eta t}\Theta_{\eta,r}(t)\|\phi\|_{\mathcal{F}_{r,r}}, \tag{89}$$

for all $\phi \in \mathcal{F}_{r,r}$, where $\Theta_{\eta,r}(t)$ is a polynomial of degree r with positive coefficients depending only on η and r.

2. *If $r \geq 2$ and $\phi \in \mathcal{F}_{r,r}$, the function $E_\eta(t)\phi \in C([0,\infty); \mathcal{F}_{r,r})$ if and only if*

$$(\partial_\xi^j \hat{\phi})(0) = 0, \quad j = 0, 1, ..., r-2. \tag{90}$$

In this case we obtain also an expression like (89).

Proof: Similar to the proof of Theorem 2.4 in [8]. ■

Next theorem is similar to Theorem 5.2.10 in [1], where it is proved that the regularized Ott-Sudan equation $w_t + ww_x + w_{xxx} - \mu\mathcal{H}w_x + \epsilon\mathcal{H}w_{xxx} = 0$, $\epsilon, \mu > 0$ is globally well possed in $\mathcal{F}_{2,1}$. In that case the amplification and damping terms give, because of their signs, $(xw, \mathcal{H}\partial_x(xw)) \leq 0$ and $-(xw, \mathcal{H}\partial_x^3(xw)) \leq 0$ so (see Theorem 5.1 below)

$$\frac{1}{2}\frac{d}{dt}\|xw(t)\|^2 \leq -(xw, xww_x) - (xw, xw_{xxx}) - \eta(xw, \mathcal{H}w) + 3\eta(xw, \mathcal{H}w_{xx}),$$

but in our case that can not be done.

Theorem 5.1. *Let $\phi \in \mathcal{F}_{2,1}$. Then, for each $\eta > 0$ there exists a unique solution $u = u_\eta \in C([0,\infty); \mathcal{F}_{2,1})$ of equation (2) such that $\partial_t u \in C([0,\infty); \mathcal{F}_{-1,1})$.*

Proof: Local theory is similar to the proof of Theorem 3.1 by using inequality (10) and the Fourier transform to prove that $\|xE_\eta(t-t')u(t')u_x(t')\| \leq F(t,t',\eta)\|u(t')\|^2_{\mathcal{F}_{2,1}}$, where $F(t,t',\eta)$ is an integrable function of t' in $[0,t]$. We remark that $u(t')u_x(t') \in H^1 \cap L^2_1$, where $\|\partial_x(u(t')u_x(t')\| \leq c\|u(t')\|^2_{\mathcal{F}_{2,1}}$ and $\|xu(t')u_x(t')\| \leq c\|u(t')\|_2\|u(t')\|_{L^2_1}$, for all $t' \in [0,t]$.
Since $\xi^J e^{-\eta t(\xi^3-\xi)} \leq \xi^J e^{-\eta t/2\xi^3}$, for $J \in \mathbb{N}$ and $\xi \geq \sqrt{2}$, we have the inequality

$$\sup_{\xi\geq 0} \xi^J e^{-\eta t(\xi^3-\xi)} \leq c(J)\big((\frac{1}{\eta t})^{J/3} + e^{\eta t}\big), \quad J \in \mathbb{N}. \tag{91}$$

Then $t\xi^3 e^{-\eta t(\xi^3-\xi)} \leq c\big(\frac{1}{\eta} + te^{\eta t}\big)$ and $t\xi^4 e^{-\eta t(\xi^3-\xi)} \leq c\big((\frac{1}{\eta^4 t})^{1/3} + te^{\eta t}\big)$, for all $\xi \geq 0$. By using the integral equation (19), the Fourier transform and the last three inequalities, it is not difficult to prove that if $\eta > 0$, and $u \in C([0,T]; \mathcal{F}_{2,1})$ is the solution of (2), then $\partial_x^J u(t) \in L^2_1(\mathbb{R})$, for $J = 0,1,2,3$, and $t \in [0,T]$.

Global existence in $\mathcal{F}_{2,1}$ follows combining Theorem 4.2 with the next computations.

$$\frac{1}{2}\frac{d}{dt}\|xu(t)\|^2 = -(xu, xuu_x) - (xu, xu_{xxx}) - \eta(xu, x(\mathcal{H}u_x + \mathcal{H}u_{xxx})),$$

$$\begin{aligned}
|(xu, xuu_x)| &\leq \|u_x\|_{L^\infty}\|xu\|^2 \leq c\|u\|_2\|u\|^2_{L^2_1}, \\
-(xu, xu_{xxx}) &= -(xu, [x, \partial_x^3]u) = 3(xu, u_{xx}) \leq 3\|u\|_{L^2_1}\|u\|_2 \leq c\|u\|^2_{\mathcal{F}_{2,1}},
\end{aligned}$$

where by Lemma 4.1, $\|u(t)\|_2 \leq F(T, \eta, \|\phi\|_2)$, for all $t \in [0,T]$. Moreover,

$$\begin{aligned}
-(xu, x\mathcal{H}u_x) &= -(xu, [x, \mathcal{H}\partial_x]u) - (xu, \mathcal{H}\partial_x(xu)) = (xu, Hu) - (xu, \mathcal{H}\partial_x(xu)), \\
-(xu, x\mathcal{H}u_{xxx}) &= -(xu, [x, \mathcal{H}\partial_x^3]u) - (xu, \mathcal{H}\partial_x^3(xu)) = 3(xu, \mathcal{H}u_{xx}) - (xu, \mathcal{H}\partial_x^3(xu)),
\end{aligned}$$

where we used $[x, \mathcal{H}\partial_x]u = [x, \mathcal{H}]\partial_x u - \mathcal{H}[\partial_x, x]u$, $[x, \mathcal{H}]\partial_x u = 0$, $[\partial_x, x]u = u$ and $[x, \mathcal{H}\partial_x^3]u = [x, \mathcal{H}]\partial_x^3 u - \mathcal{H}[\partial_x^3, x]u$, $[x, \mathcal{H}]\partial_x^3 u = 0$, $[\partial_x^3, x]u = 3\partial_x^2 u$. Hence

$$-(xu, x(\mathcal{H}u_x + \mathcal{H}u_{xxx})) \leq \|xu\|\|u\| + \|xu\|\|u\|_2 + \|xu\|^2 \leq 3\|u\|^2_{\mathcal{F}_{2,1}}. \blacksquare$$

Theorem 5.2. *Let $\eta > 0$ be fixed and let $T > 0$. Assume that $u \in C([0,T]; \mathcal{F}_{2,2})$ is the solution of (2). Then, $\hat{u}(t,0) = 0$, for all $t \in [0,T]$.*

Proof: Multiplying (2) by x^2 we obtain

$$\partial_t(x^2 u) = -x^2 u\partial_x u - x^2\partial_x^3 u - \eta x^2(\mathcal{H}\partial_x u + \mathcal{H}\partial_x^3 u). \tag{92}$$

By assumption $x^2 u(t) \in L^2(\mathbb{R})$, for all $t \in [0,T]$. Then,

$$\|x^2 u\partial_x u\| \leq \|\partial_x u\|_{L^\infty}\|x^2 u\| \leq \|u\|_2\|x^2 u\|,$$

hence $\gamma(t) := x^2(u\partial_x u)(t) \in L^2(\mathbb{R})$, for all $t \in [0,T]$. It follows easily that $\gamma \in C([0,T]; L^2(\mathbb{R}))$, in fact

$$\begin{aligned} \|\gamma(t) - \gamma(t_0)\| &\leq \|x^2 u(t)\|\|\partial_x(u(t) - u(t_0))\|_{L^\infty} + \|\partial_x u(t_0)\|_{L^\infty}\|x^2(u(t) - u(t_0))\| \\ &\leq \|u(t)\|_{\mathcal{F}_{2,2}}\|u(t) - u(t_0)\|_2 + \|u(t_0)\|_2\|u(t) - u(t_0)\|_{\mathcal{F}_{2,2}}. \end{aligned} \tag{93}$$

Applying the Fourier transform to (92) we get

$$\partial_t\partial_\xi^2\hat{u}(t,\xi) = \widehat{\gamma(t)}(\xi) + i\partial_\xi^2(\xi^3\hat{u}(t,\xi)) - \eta\partial_\xi^2[h(\xi)(-\xi + \xi^3)\hat{u}(t,\xi)]. \tag{94}$$

Let $\widehat{\beta(t)}(\xi) := \partial_\xi^2(\xi^3\hat{u}(t,\xi))$. Since $u(t) \in \mathcal{F}_{2,2}$ for all $t \in [0,T]$, it is easy to see that

$$\widehat{\beta(t)}(\xi) = 6\xi\hat{u}(t,\xi) + 6\xi^2\partial_\xi\hat{u}(t,\xi) + \xi^3\partial_\xi^2\hat{u}(t,\xi) \quad \in \quad C([0,T]; L^2_{-3}(\mathbb{R})), \tag{95}$$

and

$$\partial_\xi^2[h(\xi)(-\xi + \xi^3)\hat{u}(t,\xi)] = -2\delta(\xi)\hat{u}(t,0) + \widehat{\Gamma(t)}(\xi), \tag{96}$$

where $\widehat{\Gamma(t)}(\xi) = -2h(\xi)\partial_\xi\hat{u}(t,\xi) - \xi h(\xi)\partial_\xi^2\hat{u}(t,\xi) + 6\xi h(\xi)\hat{u}(t,\xi) + 6\xi^2 h(\xi)\partial_\xi\hat{u}(t,\xi) + \xi^3 h(\xi)\partial_\xi^2\hat{u}(t,\xi) \in C([0,T]; L^2_{-3}(\mathbb{R}))$.
From (94) - (96) we obtain

$$\partial_t\partial_\xi^2\hat{u}(t,\xi) = \widehat{\gamma(t)}(\xi) + i\widehat{\beta(t)}(\xi) + 2\eta\delta(\xi)\hat{u}(t,0) - \eta\widehat{\Gamma(t)}(\xi). \tag{97}$$

Integrating (97) between 0 and t we find that

$$2\eta\delta(\xi)\int_0^t \hat{u}(t',0)dt' \in C([0,T]; L^2_{-3}(\mathbb{R})).$$

Then $\int_0^t \hat{u}(t',0)dt' = 0$, for all $t \in [0,T]$, and therefore $\hat{u}(t,0) = 0$, for all $t \in [0,T]$. $\blacksquare$

Theorem 5.3. *Let $\eta > 0$ be fixed and let $T > 0$. Assume that $u \in C([0,T];\mathcal{F}_{3,3})$ is the solution of (2). Then, $u(t) = 0$, for all $t \in [0,T]$.*

Proof: Multiplying (2) by x^3 we obtain

$$\partial_t(x^3 u) = -x^3 u\partial_x u - x^3\partial_x^3 u - \eta x^3(\mathcal{H}\partial_x u + \mathcal{H}\partial_x^3 u). \tag{98}$$

By assumption $x^3 u \in L^2(\mathbb{R})$, then $\|x^3 u\partial_x u\| \leq \|\partial_x u\|_{L^\infty}\|x^3 u\| \leq \|u\|_2\|x^3 u\|$. So $\gamma(t) = x^3(u\partial_x u)(t) \in L^2(\mathbb{R})$, for all $t \in [0,T]$. Similar to Theorem 5.2 we have that $\gamma \in C([0,T];L^2(\mathbb{R}))$. Taking the Fourier transform in (98) we find that

$$\partial_t\partial_\xi^3\hat{u}(t,\xi) = -i\widehat{\gamma(t)}(\xi) + i\partial_\xi^3(\xi^3\hat{u}(t,\xi)) - \eta\partial_\xi^3(h(\xi)(-\xi+\xi^3)\hat{u}(t,\xi)). \tag{99}$$

Let $\widehat{\alpha(t)}(\xi) := \partial_\xi^3(\xi^3\hat{u}(t,\xi))$. We easily see that

$$\widehat{\alpha(t)}(\xi) = 6\hat{u}(t,\xi) + 18\xi\partial_\xi\hat{u}(t,\xi) + 9\xi^2\partial_\xi^2\hat{u}(t,\xi) + \xi^3\partial_\xi^3\hat{u}(t,\xi) \in C([0,T];L^2_{-3}(\mathbb{R})) \tag{100}$$

and
$$\begin{aligned}\partial_\xi^3(h(\xi)(-\xi+\xi^3)\hat{u}(t,\xi)) &= \partial_\xi\big(-2\delta(\xi)\hat{u}(t,0) - 2h(\xi)\partial_\xi\hat{u}(t,\xi) - \xi h(\xi)\partial_\xi^2\hat{u}(t,\xi)\\ &\quad +6\xi h(\xi)\hat{u}(t,\xi) + 6\xi^2 h(\xi)\partial_\xi\hat{u}(t,\xi) + \xi^3 h(\xi)\partial_\xi^2\hat{u}(t,\xi)\big)\\ &= -2\delta'(\xi)\hat{u}(t,0) - 4\delta(\xi)\partial_\xi\hat{u}(t,0) + \widehat{\Gamma(t)}(\xi),\end{aligned} \tag{101}$$

where

$$\begin{aligned}&\widehat{\Gamma(t)}(\xi) = -h(\xi)\partial_\xi^2\hat{u}(t,\xi) - \xi h(\xi)\partial_\xi^3\hat{u}(t,\xi) + 6h(\xi)\hat{u}(t,\xi) + 18\xi h(\xi)\partial_\xi\hat{u}(t,\xi)\\ &+9\xi^2 h(\xi)\partial_\xi^2\hat{u}(t,\xi) + \xi^3 h(\xi)\partial_\xi^3\hat{u}(t,\xi) \in C([0,T];L^2_{-3}(\mathbb{R})).\end{aligned}$$

From (99) - (101) we get

$$-i\partial_t\partial_\xi^3\hat{u}(t,\xi) = -\widehat{\gamma(t)}(\xi) + \widehat{\alpha(t)}(\xi) + i\eta\widehat{\Gamma(t)}(\xi) - 2i\eta\delta'(\xi)\hat{u}(t,0) - 4i\eta\delta(\xi)\partial_\xi\hat{u}(t,0).$$

Integrating the last equation in $[0,t]$ we get

$$-2i\eta\delta'(\xi)\int_0^t \hat{u}(t',0)dt' - 4i\eta\delta(\xi)\int_0^t \partial_\xi\hat{u}(t',0)dt' \in C([0,T];L^2_{-3}(\mathbb{R})).$$

Then $\int_0^t \partial_\xi\hat{u}(t',0)dt' = \int_0^t \hat{u}(t',0)dt' = 0$, for all $t \in [0,T]$. Hence $\partial_\xi\hat{u}(t,0) = \hat{u}(t,0) = 0$, for all $t \in [0,T]$.

On the other hand, denoting $v := u^2$, $w := \partial_x v$ and taking the Fourier transform in (19) we get

$$\hat{u}(t,\xi) = F_\eta(t,\xi)\hat{\phi}(\xi) - \frac{1}{2}\int_0^t F_\eta(t-t',\xi)\hat{w}(t',\xi)dt'.$$

Derivating three times the last expression, we obtain

$$\begin{aligned}\partial_\xi^3\hat{u}(t,\xi) &= \partial_\xi^3 F_\eta(t,\xi)\hat{\phi}(\xi) + 3\partial_\xi^2 F_\eta(t,\xi)\partial_\xi\hat{\phi}(\xi) + 3\partial_\xi F_\eta(t,\xi)\partial_\xi^2\hat{\phi}(\xi) + F_\eta(t,\xi)\partial_\xi^3\hat{\phi}(\xi)\\ &\quad -\frac{1}{2}\int_0^t \partial_\xi^3 F_\eta(t-t',\xi)\hat{w}(t',\xi)dt' - \frac{3}{2}\int_0^t \partial_\xi^2 F_\eta(t-t',\xi)\partial_\xi\hat{w}(t',\xi)dt'\\ &\quad -\frac{3}{2}\int_0^t \partial_\xi F_\eta(t-t',\xi)\partial_\xi^2\hat{w}(t',\xi)dt' - \frac{1}{2}\int_0^t F_\eta(t-t',\xi)\partial_\xi^3\hat{w}(t',\xi)dt'. \quad (102)\end{aligned}$$

Since $\eta > 0$, $\phi, u, v \in \mathcal{F}_{3,3}$ and using Lemma 5.1, it follows easily that

$$\partial_\xi F_\eta(t,\xi)\partial_\xi^2\hat{\phi}(\xi) = F_\eta(t,\xi)t\big(3i\xi^2 - \eta h(\xi)(3\xi^2-1)\big)\partial_\xi^2\hat{\phi}(\xi) \in C([0,T];L^2(\mathbb{R}))$$

and

$$F_\eta(t,\xi)\partial_\xi^3\hat{\phi}(\xi) \in C([0,T];L^2(\mathbb{R})).$$

On the other hand, we can easily see that
$\partial_\xi^2\hat{w}(t',\xi) = i\big(2\partial_\xi\hat{v}(t'\xi) + \xi\partial_\xi^2\hat{v}(t',\xi)\big)$ and $\partial_\xi^3\hat{w}(t',\xi) = i\big(3\partial_\xi^2\hat{v}(t',\xi) + \xi\partial_\xi^3\hat{v}(t',\xi)\big)$.
Then,

$$-\frac{3}{2}\int_0^t \partial_\xi F_\eta(t-t',\xi)\partial_\xi^2\hat{w}(t',\xi)dt' \in C([0,T];L^2(\mathbb{R})),$$

and

$$-\frac{1}{2}\int_0^t F_\eta(t-t',\xi)\partial_\xi^3\hat{w}(t',\xi)dt' \in C([0,T];L^2(\mathbb{R})).$$

Similarly, we find that

$$\partial_\xi^3 F_\eta(t,\xi)\hat{\phi}(\xi) = f_1(t,\xi) + 2\eta t\delta'(\xi)\hat{\phi}(\xi),$$

where $f_1(t,\xi) \in C([0,T];L^2(\mathbb{R}))$.
Making similar considerations to the other terms in (102), we obtain from (102)

$$\begin{aligned}\partial_\xi^3\hat{u}(t,\xi) &= f(t,\xi) + 2\eta t\delta'(\xi)\hat{\phi}(\xi) + 6\eta t\delta(\xi)\partial_\xi\hat{\phi}(\xi) - \frac{1}{2}\int_0^t 2\eta(t-t')\delta'(\xi)\hat{w}(t',\xi)dt'\\ &\quad -\frac{3}{2}\int_0^t 2\eta(t-t')\delta(\xi)\partial_\xi\hat{w}(t',\xi)dt',\end{aligned}$$

where $f(\cdot,\xi) \in C([0,T];L^2(\mathbb{R}))$. Since

$$\delta(\xi)\partial_\xi\hat{\phi}(\xi) = \delta(\xi)\partial_\xi\hat{\phi}(0) = 0,$$

we have that
$\partial_\xi^3\hat{u}(t,\xi) = f(t,\xi) + \big(2\eta t\hat{\phi}(\xi) - \eta\int_0^t (t-t')\hat{w}(t',\xi)dt'\big)\delta'(\xi) - 3\eta\int_0^t (t-t')\partial_\xi\hat{w}(t',0)dt'\delta(\xi)$.
Since $\partial_\xi^3\hat{u}(t,\xi)$ and $f(t,\xi)$ are measurable functions for all $t\in[0,T]$, it follows from the last equation that

$$\int_0^t (t-t')\partial_\xi\hat{w}(t',0)dt' = 0, \text{ for all } t\in[0,T]. \tag{103}$$

Let $t\in[0,T]$. $u(t)\in\mathcal{F}_{3,3}$ implies $xu(t)\in L_2^2(\mathbb{R})$. Then $\widehat{xu(t)}\in H^2(\mathbb{R})$ and therefore $xu(t,\cdot)\in L^1(\mathbb{R})$. On the other hand, we have that

$$\int |x\partial_x u(t)^2|dx \le 2\|\partial_x u(t)\|_{L^\infty}\int |xu(t,x)|dx \le 2\|u(t)\|_2\|xu(t,\cdot)\|_{L^1}.$$

Then,

$$\partial_\xi\hat{w}(t,0) = -\frac{i}{\sqrt{2\pi}}\int_{\mathbb{R}} x\partial_x u^2(t,x)dx = \frac{i}{\sqrt{2\pi}}\|u(t)\|^2. \tag{104}$$

Combining (103) and (104) we get $\int_0^t (t-t')\|u(t')\|^2dt' = 0$, for all $t\in[0,T]$. Then $\|u(t)\| = 0$, for all $t\in[0,T]$ and this concludes the proof. ■

Appendix

Kato's Inequalities:

1. Let $s > 3/2$. If u is real valued, then

$$|(u,uDu)_s| \le C_s\|Du\|_{s-1}\|u\|_s^2. \tag{105}$$

2. If $k\ge 2$ and u is real valued, then

$$|(u,uDu)_k| \le C_k\|u\|_2\|u\|_k^2. \tag{106}$$

Gagliardo-Nirenberg's inequalities:

$$\|u\|_{L^\infty} \le \|u\|^{1/2}\|Du\|^{1/2}, \quad u\in H^s(\mathbb{R}),\ \ s\ge 1, \tag{107}$$

$$\|D^j u\| \le C\|u\|^{1-j/m}\|D^m u\|^{j/m}, \quad 0\le j\le m,\ \ u\in H^m(\mathbb{R}). \tag{108}$$

Young's inequality: Let $a,b\ge 0$, $\epsilon>0$, $1/p+1/p'=1, 1<p<\infty$. Then,

$$ab \le \epsilon\cdot a^p + \epsilon^{-\frac{1}{p-1}}\cdot b^{p'}. \tag{109}$$

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