BOUNDS ON LEAVES OF ONE-DIMENSIONAL FOLIATIONS

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Abstract. Let X be a variety over an algebraically closed field, $\eta\colon \Omega^1_X \to \mathcal{L}$ a one-dimensional singular foliation, and $C \subseteq X$ a projective leaf of η. We prove that $2p_a(C)-2 = \deg(\mathcal{L}|C)+\lambda(C)-\deg(C\cap S)$ where $p_a(C)$ is the arithmetic genus, where $\lambda(C)$ is the colength in the dualizing sheaf of the subsheaf generated by the Kähler differentials, and where S is the singular locus of η. We bound $\lambda(C)$ and $\deg(C\cap S)$, and then improve and extend some recent results of Campillo, Carnicer, and de la Fuente, and of du Plessis and Wall.

1. Introduction

In 1891, Poincaré [32], p. 161, considered, in effect, a foliation of the plane given by a polynomial vector field, and he posed the problem of deciding whether it is algebraically integrable or not. Poincaré observed that it is enough to find a bound on the degree of the integral.

Over the years, this problem has attracted a lot of attention. Recently, it has been interpreted as the problem of bounding the degrees of the algebraic leaves of the foliation, be it algebraically integrable or not. As such, the problem was addressed in [9], by local methods, and in [5], [7], and [23], using resolution of singularities. A bound depending only on the degree of the foliation was proved in [8] for foliations without diacritical singularities.

In general, Lins Neto [24], Main Thm., p. 234, showed that there is no bound depending only on the degree of the foliation and on the analytic type of its singularities. Bounds depending on the degree of the foliation and the analytic type of the singularities of the leaves were proved in [10], [12] and [38]. In [30], bounds depending on the degree and plurigenera of the foliation and the geometric genera of the leaves were proved for foliations of general type.

The problem was extended to surfaces with trivial Picard group in [2] and, more generally, to any smooth ambient variety in [6]. Bounds on numerical invariants of subvarieties saturated by leaves were considered in [13], [35] and [36]. Finally, the analogous problem for Pfaff differential equations was considered in [3] and [14].

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Here we address the following version of the problem. Let X be a variety over an algebraically closed field of arbitrary characteristic. Let $C \subseteq X$ be a *curve*, that is, a **reduced** subscheme of pure dimension 1; assume C is projective. Let $\eta\colon \Omega^1_X \to \mathcal{L}$ be a *one-dimensional singular foliation* of X; that is, η is nonzero, and $\mathcal{L}$ is invertible. Assume C is a *leaf*; that is, C contains only finitely many singularities of η, and $\eta|C$ factors through the standard map $\sigma\colon \Omega^1_X|C \to \Omega^1_C$. Say $\mu\colon \Omega^1_C \to \mathcal{L}|C$ is the induced map. We strive to relate the numerical invariants of C and μ.

The major global invariant of C is its arithmetic genus, $p_a(C) := 1 - \chi(\mathcal{O}_C)$. Notice that $p_a(C) = h^1(\mathcal{O}_C)$ when C is connected, and that $p_a(C)$ remains constant when C varies in a family.

The singularities P of C are measured by several invariants. One in particular arises naturally in the present work. It is denoted $\lambda(C,P)$ by Buchweitz and Greuel in [4], Def. 6.1.1, p. 265, and it is defined as the colength, in the dualizing module ω_P, of the $\mathcal{O}_{C,P}$-submodule generated by $\Omega^1_{C,P}$. Notice that $\lambda(C,P) > 0$ if and only if P is singular. So we may set $\lambda(C) := \sum \lambda(C,P)$.

Our key relation is the following simple formula, given in Proposition 5.2:

$$2p_a(C) - 2 - \deg(\mathcal{L}|C) = \lambda(C) - \deg(C \cap S) \qquad (1.1).$$

Here S is the *singular locus* of η, that is, the subscheme of X where η fails to be surjective; so $C \cap S$ is the singular locus of μ. We prove our formula by comparing Euler characteristics of certain torsion-free sheaves on C.

Under more restrictive hypotheses, versions of Formula (1.1) were proved by Cerveau and Lins Neto [9], Prop., p. 885, and by Lins Neto and Soares [25], Prop. 2.7, p. 659. In [35], p. 495, Soares suggested using the formula when C is smooth, to solve the Poincaré problem by bounding $\deg(C \cap S)$ from below. In the same vein, our main results, Theorem 5.3 and Theorem 6.1, follow from the general case of Formula (1.1) and from bounds we obtain on $\lambda(C,P)$ and $\deg(C \cap S)$.

Note that $\deg(C \cap S) \geq \iota(C)$ where $\iota(C) := \sum \iota(C,P)$ and $\iota(C,P)$ is the least length of the cokernel of a map $\Omega^1_{C,P} \to \mathcal{O}_{C,P}$. Hence, as is also stated in our Proposition 5.2,

$$2p_a(C) - 2 - \deg(\mathcal{L}|C) \leq \lambda(C) - \iota(C). \qquad (1.2)$$

In characteristic 0, if P is a singularity, then $\iota(C,P) \geq 1$ because $\Omega^1_{C,P}/\text{torsion}$ cannot be free by [26], Thm. 1, p. 879. Hence then $\iota(C)$ is at least the number of singularities.

Assume X is smooth. In [6], Thm. 2.7, p. 62, Campillo, Carnicer and De la Fuente gave an upper bound on $2p_a(C) - 2 - \deg(\mathcal{L}|C)$ in terms of multiplicities associated to C and η along a sequence of blowups of X resolving the singularities of C. As a consequence, they obtained in [6], Thm. 3.1, p. 64, an upper bound on $2p_a(C) - 2 - \deg(\mathcal{L}|C)$ that holds universally for all η having C as leaf. Our Theorem 5.3 provides a somewhat better bound; this bound follows from (1.2), given the bound on $\lambda(C,P)$ asserted in our Proposition 4.4. Thus (1.2) is the sharpest available bound on $2p_a(C) - 2 - \deg(\mathcal{L}|C)$.

Our proof of Proposition 4.4 uses the Hironaka–Noether bound, Proposition 3.1. It bounds the colength ℓ of a reduced one-dimensional Noetherian local ring A in the blowup at its maximal ideal in terms of its multiplicity e; namely, $\ell \leq e(e-1)/2$, with equality if and only if A has embedding dimension at most 2. Noether [29] considered, in effect, only the case where A is the local ring of a complex plane curve. Hironaka [20], p. 186, asserted the bound without proof when A is the local ring of an arbitrary complex curve. In the same setup, Stevens [37] proved a formula for ℓ, and then asserted the bound without proof. Inspired by the Stevens's work, we give a somewhat different proof, and obtain the general case.

Take $X := \mathbf{P}^n$ now, and set $d := \deg C$. Suppose d is not a multiple of the characteristic. Over $\mathbb{C}$, Jouanolou [21], Prop. 4.2, p. 130, proved $C \cap S$ is nonempty, even when C is smooth. In [14], Cor. 4.5, Jouanolou's result is refined: the Castelnuovo–Mumford regularity $\operatorname{reg}(C \cap S)$ is shown to be at least $m+1$ where $m := 1 + \deg \mathcal{L}$. Now, the regularity of any finite subscheme is at least its degree. Hence, (1.1) yields

$$2p_a(C) - (d-1)(m-1) \leq \lambda(C), \tag{1.3}$$

which our Theorem 6.1 asserts. It continues by asserting that, if equality holds, then $\deg(C \cap S) = m+1$; also then $C \cap S$ lies on a line M, and either $M \subseteq S$ or M is a leaf.

If, in addition, the singular locus S is finite, then, as our Proposition 6.3 asserts,

$$\lambda(C) \leq 2p_a(C) - (d-1)(m-1) + m^2 + \cdots + m^n. \tag{1.4}$$

This bound too results from (1.1); indeed, a simple Chern class computation evaluates $\deg(S)$, but $\deg(S) \geq \deg(C \cap S)$.

Another important global invariant of C is its geometric genus, $p_g(C) := \mathrm{h}^1(\mathcal{O}_{\overline{C}})$ where $\overline{C}$ is the normalization of C. Our Corollary 6.2 asserts that, if C is connected and the characteristic is 0, then

$$p_g(C) \leq (m-1)(d-1)/2 + (r(C)-1)/2$$

where $r(C)$ is the number of irreducible components. Notice that this bound is nontrivial for $m < d-1$ and that it does not depend in any way on the singularities of C or of η. The problem of bounding $p_g(C)$ was posed by Painlevé and has been considered by Lins Neto among others; see [24].

There are two better known singularity invariants, the δ-invariant $\delta(C,P)$ and the Tjurina number $\tau(C,P)$. The former is the colength of $\mathcal{O}_{C,P}$ in its normalization; the latter, the dimension of the tangent space of the miniversal deformation space of the singularity. These invariants are related to $\lambda(C,P)$. First, $\delta(C,P) \leq \lambda(C,P) \leq 2\delta(C,P)$, but the second inequality is valid only in characteristic 0; see Subsection 2.1. Second, $\tau(C,P) = \lambda(C,P)$ if C is a complete intersection at P; see Proposition 2.2.

Finally, take $X := \mathbf{P}^2$. Then $p_a(C) = (d-1)(d-2)/2$, and $\lambda(C) = \tau(C)$ where $\tau(C) := \sum \tau(C,P)$. Again suppose d is not a multiple of the characteristic. Then (1.3) and (1.4) hold, and reduce to the following lower and upper bounds on $\tau(C)$:

$$(d-1)(d-m-1) \leq \tau(C) \leq (d-1)(d-m-1) + m^2. \tag{1.5}$$

These bounds are the ones masterfully proved over $\mathbb{C}$ by du Plessis and Wall [12], Thm. 3.2, p. 263, in a more elementary way. However, they define m as the least degree of a nontrivial polynomial vector field ϕ annihilating the equation of C. Considering the foliation η defined by ϕ, we derive their lower bound in our Corollary 6.4. Their upper bound is also obtained there, under the additional assumption that the singular locus of η intersects C in finitely many points.

In fact, Du Plessis and Wall prove more: if $2m+1>d$, then

$$\tau(C) \leq (d-1)(d-m-1) + m^2 - (2m+2-d)(2m+1-d)/2.$$

The present authors have found a more conceptual version of the proof, which also works for other ambient spaces; this material is treated in [15].

The lower bound in (1.5) was rediscovered over $\mathbb{C}$ by Chavarriga and Llibre [10], Thm. 3, p. 12, and they gave yet a third proof.

The lower bound in (1.5) is improved in characteristic 0 via yet a fourth argument in [15], as follows: $(d-1)(d-m-1)+u \leq \tau(C)$ where u is the number of singularities *not* quasi-homogeneous (that is, at which a local analytic equation is not weighted homogeneous); moreover, if equality holds, then either $m=d-1$ and C is smooth, or $m<d-1$ and $\operatorname{reg}(\operatorname{Sing} C) = 2d-3-m$.

The Poincaré problem is to bound d given the invariants of η. As is well known, the difficulty lies in the possibility that C may be highly singular. In this connection, the lower bound in (1.5) says this: the higher its degree, the more singular is C. As noted above, our proof of (1.5) uses the lower bound $\operatorname{reg}(C\cap S)$ given in [14], Cor. 4.5. A result in [15] asserts that $\operatorname{reg}(\operatorname{Sing} C) \geq 2d-3-m$ if $m \leq d-2$ and that $\operatorname{reg}(\operatorname{Sing} C) = 2d-3-m$ if $m \leq (d-2)/2$, provided d is not a multiple of the characteristic. In other words, for high d, not only must C have many singularities, but also they must lie in special position in the plane.

In short, Section 2 of the present paper introduces some local and some global invariants of a curve C, and relates them. Section 3 treats the Hironaka–Noether bound. Section 4 uses this bound to help establish an upper bound on $\lambda(C,P)$. Section 5 establishes our bound (1.2) on $2p_a(C)-2-\deg(\mathcal{L}|C)$, and compares it favorably to the bound of Campillo, Carnicer and De la Fuente with the aid of our bound on $\lambda(C,P)$. Finally, Section 6 establishes the bounds (1.3) and (1.4) on $\lambda(C)$, and shows that they recover the bounds in (1.5) on $\tau(C)$ in the form treated by du Plessis and Wall and by Chavarriga and Llibre.

2. Invariants of curves

2.1. *Local invariants.* Let C be a curve, $\mathbf{n}\colon \overline{C} \to C$ the normalization map, and

$$\mathbf{n}^{\#}\colon \mathcal{O}_C \to \mathbf{n}_*\mathcal{O}_{\overline{C}} \quad\text{and}\quad d\,\mathbf{n}\colon \Omega^1_C \to \mathbf{n}_*\Omega^1_{\overline{C}}$$

the associated maps on sheaves of functions and differentials. Let ω_C be the dualizing sheaf (or canonical sheaf, or Rosenlicht's sheaf of regular differentials); see [34], or [19], Sec. III-7, or [4], pp. 243–244, or [1], for example. There is a natural map

$$\operatorname{tr}\colon \mathbf{n}_*\Omega^1_{\overline{C}} \to \omega_C;$$

it is known as the *trace*, and the composition

$$\gamma\colon \Omega^1_C \xrightarrow{d\,\mathbf{n}} \mathbf{n}_*\Omega^1_{\overline{C}} \xrightarrow{\operatorname{tr}} \omega_C$$

is known as the *class map*.

Fix a closed point $P \in C$. Taking lengths $\ell(-)$, set

$$\begin{aligned}\delta(C,P) &:= \ell(\mathrm{Cok}(\mathbf{n}_P^{\#})),\\ \tau(C,P) &:= \ell(\mathrm{Ext}^1_{\mathcal{O}_{C,P}}(\Omega^1_{C,P},\mathcal{O}_{C,P})),\\ \lambda(C,P) &:= \ell(\mathrm{Cok}(\gamma_P)).\end{aligned}$$

The first two invariants are known respectively as the *δ-invariant* or the *genus diminution,* and the *Tjurina number;* see [37], p. 98, and [16], pp. 142–143. The third invariant was formally introduced and studied by Buchweitz and Greuel [4], pp. 265–269, although it appears implicitly earlier, notably in Rim's paper [33].

By Rosenlicht's theorem (see [34], Thm. 8 and Cor. 1, pp. 177–178, or [1], Prop. 1.16(ii), p. 168), the cokernels of $\mathbf{n}^{\#}$ and tr are perfectly paired; so $\delta(C,P) = \ell(\mathrm{Cok}(\mathrm{tr}_P))$. Hence

$$\lambda(C,P) = \delta(C,P) + \ell(\mathrm{Cok}(d\,\mathbf{n}_P)). \tag{2.1.1}$$

Let $\alpha\colon \Omega^1_{C,P} \to \mathcal{O}_{C,P}$ range over all maps such that $\mathrm{Cok}\,\alpha$ has finite length, and set

$$\iota(C,P) := \min_{\alpha} \ell(\mathrm{Cok}\,\alpha).$$

This invariant is the local isomorphism defect of $\Omega^1_C/\text{torsion}$ in $\mathcal{O}_C$ at P, as defined by Greuel and Lossen in [18], p. 330, and as defined earlier, but with the opposite sign, by Greuel and Karras in [17], p. 103; however, the present invariant $\iota(C,P)$ itself is not explicitly considered in either of those papers.

Suppose that P is a singularity of C. In characteristic zero, $\iota(C,P) \geq 1$ because $\mathrm{Hom}(\Omega^1_{C,P},\mathcal{O}_{C,P})$ is not free by [26], Thm. 1, p. 879. In characteristic $p > 0$, sometimes $\iota(C,P) = 0$; for example (see [26], p. 892), in the plane, take $C : y^{p+1} - x^p = 0$ and take $P := (0,0)$.

Let $r(C,P)$ denote the number of branches, or analytic components, of C at P.

Let $d\colon \mathcal{O}_C \to \Omega^1_C$ be the universal derivation, and set

$$\mu(C,P) := \ell(\mathrm{Cok}(\gamma \circ d)_P).$$

Then $\lambda(C,P) \leq \mu(C,P)$. Also, it is not hard to see that $\mu(C,P) < \infty$ if and only if the characteristic is 0. (Over $\mathbb{C}$, Buchweitz and Greuel, generalizing work of Bassein, name $\mu(C,P)$ the Milnor number in Def. 1.1.1, p. 244, [4], and prove, in Thm. 4.2.2, p. 258, that, when C degenerates, $\mu(C,P)$ increases by the number of vanishing cycles.)

In characteristic 0, Buchweitz and Greuel [4], Prop. 1.2.1, p. 246, prove

$$\mu(C,P) = 2\delta(C,P) - r(C,P) + 1,$$

which extends the Milnor–Jung formula for plane curves. Now, $\lambda(C,P) \leq \mu(C,P)$. So

$$\lambda(C,P) \leq 2\delta(C,P) - r(C,P) + 1 \tag{2.1.2}$$

in characteristic 0. For an upper bound in positive characteristic, see Proposition 4.4.

Proposition 2.2 (Rim)**.** *Let C be a curve, and $P \in C$ a closed point. If C is a complete intersection at P, then $\tau(C,P) = \lambda(C,P)$.*

Proof. Over $\mathbb{C}$, the assertion follows directly from [4], Lem. 1.1.2, p. 245 and Cor. 6.1.6, p. 268. In arbitrary characteristic, the assertion follows directly from two formulas buried in the middle of p. 269 in Rim's paper [33]. The first formula says that τ is equal to the length of the torsion submodule of Ω^1_C. A cleaner version of the proof, which is based on local duality, was given by Pinkham [31], p. 76. The formula itself was originally proved when C is irreducible by Zariski, [39], Thm. 1, p. 781. The second formula says that this length is equal to $\lambda(C, P)$; here is another version of the proof of this formula.

Since the invariants in question are local, we may complete C and then normalize it off P. Thus we may assume that C is projective and that P is its only singularity.

The torsion submodule of Ω^1_C is equal to the kernel of the class map $\gamma\colon \Omega^1_C \to \omega_C$ since ω_C is torsion free. However, $\lambda(C, P) := \ell(\mathrm{Cok}(\gamma_P))$. Hence it suffices to prove $\chi(\Omega^1_C) = \chi(\omega_C)$.

Let $\mathcal{N}$ be the conormal sheaf of C in its ambient projective space, X say, and set $\mathcal{M} := \Omega^1_X|C$. Since C is a local complete intersection, $\mathcal{N}$ is locally free and we have an exact sequence of the form

$$0 \to \mathcal{N} \to \mathcal{M} \to \Omega^1_C \to 0.$$

So $\chi(\Omega^1_C) = \chi(\mathcal{M}) - \chi(\mathcal{N})$. Hence, by Riemann's theorem,

$$\chi(\Omega^1_C) = \deg \mathcal{M} - \deg \mathcal{N} + (\operatorname{rk} \mathcal{M} - \operatorname{rk} \mathcal{N})\chi(\mathcal{O}_C) = \deg \mathcal{M} - \deg \mathcal{N} + \chi(\mathcal{O}_C).$$

On the other hand, $\omega_C = \det(\mathcal{M}) \otimes (\det \mathcal{N})^*$ by [19], Thm. 7.11, p. 245. So, again by Riemann's theorem,

$$\chi(\omega_C) = \deg(\det \mathcal{M}) - \deg(\det \mathcal{N}) + \chi(\mathcal{O}_C).$$

Now, $\deg(\det \mathcal{M}) = \deg \mathcal{M}$ and $\deg(\det N) = \deg \mathcal{N}$. Hence $\chi(\Omega^1_C) = \chi(\omega_C)$. $\square$

2.3. *Global invariants.* Let C be a projective curve, $\mathbf{n}\colon \overline{C} \to C$ the normalization map, and $\mathbf{n}^{\#}\colon \mathcal{O}_C \to \mathbf{n}_*\mathcal{O}_{\overline{C}}$ the associated map.

If C is smooth at a closed point P, then the local invariants $\delta(C, P)$, $\tau(C, P)$, $\lambda(C, P)$, and $\iota(C, P)$ all vanish. So it makes sense to set

$$\delta(C) := \sum_{P \in C} \delta(C, P), \quad \lambda(C) := \sum_{P \in C} \lambda(C, P),$$
$$\tau(C) := \sum_{P \in C} \tau(C, P), \quad \iota(C) := \sum_{P \in C} \iota(C, P).$$

Let $r(C)$ denote the number of irreducible components of C.

Recall that the *arithmetic genus* and the *geometric genus* are defined by the formulas:

$$p_a(C) := 1 - \chi(\mathcal{O}_C) \quad \text{and} \quad p_g(C) := \mathrm{h}^1(\mathcal{O}_{\overline{C}}).$$

Extracting Euler characteristics from the short exact sequence

$$0 \to \mathcal{O}_C \to \mathbf{n}_*\mathcal{O}_{\overline{C}} \to \mathrm{Cok}(\mathbf{n}^{\#}) \to 0$$

yields Clebsch's formula

$$p_g(C) = p_a(C) - \delta(C) + r(C) - 1. \tag{2.3.1}$$

Suppose C is connected. Then $r(C) - 1 \leq \sum_P (r(C,P) - 1)$. In charactersitic 0, therefore, (2.1.2) yields

$$\lambda(C) \leq 2\delta(C) - r(C) + 1. \tag{2.3.2}$$

Proposition 2.4. *Let A and B be (reduced) plane curves of degrees a and b with no common components. Set $C := A \cup B$. Then*

$$\tau(A) + \tau(B) + ab \leq \tau(C),$$

with equality if A and B are transverse.

Proof. If A and B are transverse, then $\tau(C,P) = 1$ for $P \in A \cap B$. There are ab such P. Hence

$$\tau(A) + \tau(B) + ab = \tau(C).$$

By the theorem of transversality of the general translate for projective space [22], Cor. 11, p. 296, there is a dense open subset of automorphisms g of the plane such that the translate A^g is transversal to B. Set $C_g := A^g \cup B$. Then, by the preceding case,

$$\tau(A^g) + \tau(B) + ab = \tau(C_g).$$

The function $g \mapsto \tau(C_g)$ is upper semi-continuous. Indeed, $\tau(C_g) = \lambda(C_g)$ by Proposition 2.2. Furthemore, $g \mapsto \lambda(C_g)$ is upper semi-continuous, because $\lambda(C_g)$ is the length, on the fiber over g, of the restriction of the cokernel of a map between coherent sheaves on the total space of the C_g, namely, the relative class map.

Hence $\tau(C_g) \leq \tau(C)$. But $\tau(A^g) = \tau(A)$ since A^g and A are isomorphic. Therefore, the asserted bound holds. $\square$

3. The Hironaka–Noether bound

Proposition 3.1 (Hironaka–Noether bound)**.** *Let A be a reduced Noetherian local ring of dimension 1 and multiplicity $e \geq 2$. Let B be the blowup of A at its maximal ideal $\mathbf{m}$. Then the length of the A-module B/A satisfies the following inequality:*

$$\ell(B/A) \leq e(e-1)/2.$$

Furthermore, equality holds if and only if A has embedding dimension 2.

Proof. Set $k := A/\mathbf{m}$. Let's first reduce the question to the case where k is infinite; we'll use a well-known trick, found for instance in [27], p. 114. So, let x be an indeterminate, $A[x]$ the polynomial ring, and $\mathbf{p}$ the extension of $\mathbf{m}$. Set $A(x) := A[x]_{\mathbf{p}}$. Then $A(x)$ is a reduced Noetherian local ring of dimension 1. Its maximal ideal is the extension $\mathbf{m}A(x)$, and its residue field is the infinite field $k(x)$.

In addition, $A(x)$ is flat over A. Hence, the multiplicity of $A(x)$ is also e, and the blowup of $A(x)$ at its maximal ideal is $B \otimes_A A(x)$. Also,

$$\ell\big((B \otimes_A A(x))/A(x)\big) = \ell\big((B/A) \otimes_A A(x)\big) = \ell(B/A).$$

Therefore, replacing A by $A(x)$, we may assume k is infinite.

Since k is infinite and A is reduced and of dimension 1, there is an $f \in \mathbf{m}$ such that the equation $B = A[\mathbf{m}/f]$ holds in the total ring of fractions of A. Note that

$$\mathbf{m}B = f(1/f)\mathbf{m}B \subseteq fB;$$

whence $\mathbf{m}B = fB$. It follows that, for every $i \geq 0$, we have
$$\ell(\mathbf{m}^i B/\mathbf{m}^{i+1}B) = e. \tag{3.1.1}$$

For each $i \geq 0$, form the A-module
$$V_i := \mathbf{m}^i B/(\mathbf{m}^i + \mathbf{m}^{i+1}B).$$
Then V_i is the cokernel of the natural map
$$\mathbf{m}^i/\mathbf{m}^{i+1} \to \mathbf{m}^i B/\mathbf{m}^{i+1}B.$$
Hence, we get
$$\ell(V_i) \geq \ell(\mathbf{m}^i B/\mathbf{m}^{i+1}B) - \ell(\mathbf{m}^i/\mathbf{m}^{i+1}). \tag{3.1.2}$$

Let's now prove that, for some integer $q \geq 0$, we have
$$e - 1 = \ell(V_0) > \ell(V_1) > \cdots > \ell(V_{q-1}) > \ell(V_q) = \ell(V_{q+1}) = \cdots = 0. \tag{3.1.3}$$
Indeed, first observe that
$$\ell(V_0) = \ell(B/\mathbf{m}B) - \ell(A/(\mathbf{m}B \cap A)).$$
Now, $\ell(B/\mathbf{m}B) = e$ by (3.1.1). Also, $\mathbf{m}B \cap A = \mathbf{m}$. So $\ell(V_0) = e - 1$.

Next, notice that, for each $i \geq 0$, multiplication by f induces a map
$$h_i \colon V_i \xrightarrow{\times f} V_{i+1}.$$
This map h_i is surjective because $\mathbf{m}B = fB$. Moreover, $\mathrm{Ker}(h_i) = 0$ if and only if
$$\mathbf{m}^i B \cap (1/f)(\mathbf{m}^{i+1} + \mathbf{m}^{i+2}B) \subseteq \mathbf{m}^i + \mathbf{m}^{i+1}B.$$

However, $\mathbf{m}^{i+1} + \mathbf{m}^{i+2}B \subseteq \mathbf{m}^{i+1}B$. Also $(1/f)\mathbf{m}^{i+1}B = \mathbf{m}^i B$ since $\mathbf{m}B = fB$. Hence, $\mathrm{Ker}(h_i) = 0$ if and only if
$$(1/f)(\mathbf{m}^{i+1} + \mathbf{m}^{i+2}B) \subseteq \mathbf{m}^i + \mathbf{m}^{i+1}B.$$

Of course, we have
$$(1/f)(\mathbf{m}^{i+1} + \mathbf{m}^{i+2}B) = (1/f)\mathbf{m}(\mathbf{m}^i + \mathbf{m}^{i+1}B).$$
Since $B = A[\mathbf{m}/f]$, it follows that $\mathrm{Ker}(h_i) = 0$ if and only if $\mathbf{m}^i + \mathbf{m}^{i+1}B$ is a B-module; that is, if and only if
$$\mathbf{m}^i + \mathbf{m}^{i+1}B = \mathbf{m}^i B + \mathbf{m}^{i+1}B = \mathbf{m}^i B. \tag{3.1.4}$$
Therefore, $h_i \colon V_i \to V_{i+1}$ is injective if and only if $V_i = 0$. Since h_i is surjective, $\ell(V_i) \geq \ell(V_{i+1})$; moreover, if equality holds, then h_i is bijective, and therefore $V_i = 0$. Thus (3.1.3) holds for some q.

Next, let's prove that, for all $j \geq 0$, we have
$$\mathbf{m}^q + \mathbf{m}^{q+j}B = \mathbf{m}^q B. \tag{3.1.5}$$
This equation is trivial for $j = 0$. Now, given $j \geq 0$, suppose (3.1.5) holds. Since (3.1.3) holds, $V_{q+j} = 0$; so (3.1.4) holds for $i := q + j$. Hence, we have
$$\mathbf{m}^q + \mathbf{m}^{q+j+1}B = \mathbf{m}^q + \mathbf{m}^{q+j} + \mathbf{m}^{q+j+1}B = \mathbf{m}^q + \mathbf{m}^{q+j}B = \mathbf{m}^q B.$$
Thus, by induction, (3.1.5) holds for all $j \geq 0$.

Let's now improve (3.1.5) by showing it implies that

$$\mathbf{m}^q = \mathbf{m}^q B. \tag{3.1.6}$$

Indeed, the A-module $B/\mathbf{m}^q$ has finite length. Hence it is annihilated by $\mathbf{m}^{q+j}$ for some $j \geq 0$; in other words, $\mathbf{m}^{q+j}B \subseteq \mathbf{m}^q$. Thus (3.1.5) yields (3.1.6).

We can now prove the first assertion. Indeed, owing to (3.1.6), the sequence

$$0 \to A/\mathbf{m}^q \to B/\mathbf{m}^q B \to B/A \to 0$$

is exact. Filter the first term by $\mathbf{m}^i/\mathbf{m}^q$ for $i = 0, \dots, q$, and the second by $\mathbf{m}^i B/\mathbf{m}^q B$. Then we get

$$\ell(B/A) = \sum_{i=0}^{q-1} \bigl(\ell(\mathbf{m}^i B/\mathbf{m}^{i+1}B) - \ell(\mathbf{m}^i/\mathbf{m}^{i+1})\bigr). \tag{3.1.7}$$

Now, (3.1.3) yields $\ell(V_i) \leq (e-1-i)$ and $q \leq e-1$. Hence (3.1.2) yields

$$\ell(B/A) \leq \sum_{i=0}^{q-1} \ell(V_i) \leq \sum_{i=0}^{e-2} (e-1-i) = e(e-1)/2. \tag{3.1.8}$$

Thus the first assertion is proved.

To prove the second assertion, first assume $\ell(B/A) = e(e-1)/2$. Then the equalities hold in (3.1.8). So equality holds in (3.1.2), and $\ell(V_i) = e-1-i$ for $0 \leq i \leq e-1$. Hence (3.1.1) yields $\ell(\mathbf{m}^i/\mathbf{m}^{i+1}) = i+1$. In particular, $\ell(\mathbf{m}/\mathbf{m}^2) = 2$.

Conversely, assume $\ell(\mathbf{m}/\mathbf{m}^2) = 2$. Then $\mathbf{m}$ is generated by two elements. So $\mathbf{m}^i$ is generated by at most $i+1$ elements for all $i \geq 0$; whence,

$$\ell(\mathbf{m}^i/\mathbf{m}^{i+1}) \leq i+1. \tag{3.1.9}$$

Together, (3.1.1) and (3.1.6) and (3.1.9) yield

$$e = \ell(\mathbf{m}^q B/\mathbf{m}^{q+1}B) = \ell(\mathbf{m}^q/\mathbf{m}^{q+1}) \leq q+1.$$

Therefore, (3.1.7) and (3.1.1) and (3.1.9) yield

$$\ell(B/A) = \sum_{i=0}^{q-1} \bigl(e - \ell(\mathbf{m}^i/\mathbf{m}^{i+1})\bigr) \geq \sum_{i=0}^{e-2} (e-1-i) = e(e-1)/2.$$

Since $\ell(B/A) \leq e(e-1)/2$ by (3.1.8), equality holds. □

4. Infinitely near points

4.1. *Infinitely near points.* Let X be a smooth scheme of dimension 2 or more. An infinite sequence $P, P', P'', \dots$ is said to be a *succession of infinitely near points of* X if P is a closed point of X, if P' is a closed point of the exceptional divisor E' of the blowup X' of X at P, if P'' is a closed point of the exceptional divisor E'' of the blowup X'' of X' at P', and so forth.

In this case, whenever $m \leq n$, then $P^{(n)}$ is said to be *infinitely near to* $P^{(m)}$ *of order* $n-m$. In addition, $P^{(n)}$ is said to be *proximate* to $P^{(m)}$ if $m < n$ and if $P^{(n)}$ lies on the proper (or strict) transform of $E^{(m+1)}$ on $X^{(n)}$; given n, denote the number of these $P^{(m)}$ by $i(P, P^{(n)})$. Note that $i(P, P^{(n)}) = 0$ if and only if $n = 0$.

Let $C \subset X$ be a curve. Let $C^{(n)}$ be the proper transform of C on $X^{(n)}$. Denote by $e(C, P^{(n)})$, by $\delta(C, P^{(n)})$, and by $r(C, P^{(n)})$ the multiplicity, the δ-invariant, and the number of branches of $C^{(n)}$ at $P^{(n)}$; by convention, these numbers are 0 if $C^{(n)}$ does not contain $P^{(n)}$. Similarly, given a branch Γ of C at P, denote by $e(\Gamma, P^{(n)})$ and by $\delta(\Gamma, P^{(n)})$ the multiplicity and the δ-invariant at $P^{(n)}$ of the proper transform of Γ.

Note that $P^{(n)}$ determines its predecessors $P, P', \ldots, P^{(n-1)}$, but not its successors $P^{(n+1)}, P^{(n+2)}, \ldots$; the latter vary with the particular succession through $P^{(n)}$. Call $P^{(n-1)}$ the *immediate predecessor* of $P^{(n)}$. Denote the set of all predecessors of $P^{(n)}$, including $P^{(n)}$ and P, by $[P, P^{(n)}]$. Denote the set of all possible successors Q of $P^{(n)}$, including $P^{(n)}$, by $N(P^{(n)})$; denote the subset of those Q proximate to $P^{(n)}$ by $N^*(P^{(n)})$.

Lemma 4.2. *Let X be a smooth scheme of dimension 2 or more, $C \subset X$ a curve, and $P \in C$ a closed point. Then*

$$\sum_{Q \in N(P)} e(C, Q)\big(e(C, Q) - 2 + i(P, Q)\big) \geq 2\delta(C, P) - r(C, P),$$

with equality if and only if the embedding dimension of C at P is 1 or 2.

Proof. The sum in question is well defined. Indeed, if Q lies off the proper transform of C, then $e(C, Q) = 0$. Of the remaining Q, all but finitely many are such that $e(C, Q) = 1$ and $i(P, Q) = 1$ by the theorem of embedded resolution of singularities.

Let $t(C, P)$ be the greatest order of a $Q \in N(P)$ such that either $e(C, Q) > 1$ or $e(C, Q) = 1$ and $i(P, Q) > 1$. However, if no such Q exists, set $t(C, P) := -1$.

Suppose $t(C, P) = -1$. Then, for every $Q \in N(P) \setminus P$, either $e(C, Q) = 0$ or $e(C, Q) = 1$ and $i(P, Q) = 1$; moreover, $e(C, P) = 1$ and $i(P, P) = 0$. Hence the sum in question is equal to -1. Moreover, $\delta(C, P) = 0$ and $r(C, P) = 1$; also the embedding dimension of C at P is 1. Hence the assertion holds in this case.

Proceed by induction on $t(C, P)$. So suppose $t(C, P) \geq 0$. Let X' be the blowup of X at P, and C' the proper transform of C. Say $P'_1, \ldots, P'_n \in C'$ lie over P.

Fix j. If $t(C', P'_j) = -1$, then $t(C', P'_j) < t(C, P)$. Now, take $Q \in N(P'_j)$; say Q is of order m. Then $Q \in N(P)$ with order $m + 1$. Moreover, $e(C', Q) = e(C, Q)$. Also

$$i(P'_j, Q) = \begin{cases} i(P, Q), & \text{if } Q \text{ is not proximate to } P; \\ i(P, Q) - 1, & \text{if } Q \text{ is proximate to } P. \end{cases}$$

Therefore, if $t(C', P'_j) \geq 0$, then again $t(C', P'_j) < t(C, P)$.

So the induction hypothesis and the above formulas for $e(C', Q)$ and $i(P'_j, Q)$ yield

$$\sum_{Q \in N(P'_j)} e(C, Q)\big(e(C, Q) - 2 + i(P, Q)\big) - \sum_{Q \in N(P'_j) \cap N^*(P)} e(C, Q)$$
$$\geq 2\delta(C', P'_j) - r(C', P'_j), (4.2.1)]$$

with equality if the embedding dimension of C' at P'_j is at most 2. The latter holds, of course, if the embedding dimension of C at P is at most 2.

Let δ be the colength of $\mathcal{O}_{C,P}$ in its blowup. By Proposition 3.1,

$$e(C, P)(e(C, P) - 1) \geq 2\delta, \tag{4.2.2}$$

with equality if and only if the embedding dimension of C at P is at most 2. Moreover,

$$\delta(C,P)=\sum_{j=1}^{n}\delta(C',P'_j)+\delta. \tag{4.2.3}$$

Sum the inequalities in (4.2.1) over i, and use (4.2.2) and (4.2.3). We get

$$\begin{aligned}\sum_{Q\in N(P)} & e(C,Q)\big(e(C,Q)-2+i(P,Q)\big)\\ &= e(C,P)(e(C,P)-2)+\sum_{j=1}^{n}\sum_{Q\in N(P'_j)} e(C,Q)\big(e(C,Q)-2+i(P,Q)\big)\\ &\geq 2\delta-e(C,P)+\sum_{j=1}^{n}\Bigg(\sum_{Q\in N(P'_j)\cap N^*(P)} e(C,Q)+2\delta(C',P'_j)-r(C',P'_j)\Bigg)\\ &= 2\delta(C,P)-r(C,P)-e(C,P)+\sum_{Q\in N^*(P)} e(C,Q).\end{aligned}$$

with equality if and only if the embedding dimension of C at P is at most 2. However, the last two terms cancel by the proximity equality; see [11], Formula (2.18), p. 27, for example. Thus the assertion holds. □

Lemma 4.3. *Let X be a smooth scheme of dimension 2 or more in characteristic $p>0$. Let $C\subset X$ be a curve, and $P\in C$ a closed point. Given a branch Γ of C at P, let $Q(\Gamma)$ be the point infinitely near to P of least order such that $p\nmid e(\Gamma,Q(\Gamma))$. Then*

$$\lambda(C,P)\leq 2\delta(C,P)-r(C,P)+\sum_{\Gamma}v(\Gamma,P)\ \textit{where}\ v(\Gamma,P):=\sum_{R\in[P,Q(\Gamma)]}e(\Gamma,R).$$

Proof. Let $\mathbf{n}\colon\overline{C}\to C$ be the normalization map, $d\,\mathbf{n}\colon\Omega^1_C\to\mathbf{n}_*\Omega^1_{\overline{C}}$ its differential. Set

$$I:=\operatorname{Im}((d\,\mathbf{n})_P)\subseteq(\mathbf{n}_*\Omega^1_{\overline{C}})_P\ \text{and}\ \overline{I}:=(\mathbf{n}_*\mathcal{O}_{\overline{C}})_P I\subseteq(\mathbf{n}_*\Omega^1_{\overline{C}})_P;$$

so I is an $\mathcal{O}_{C,P}$-submodule, and $\overline{I}$ is the $(\mathbf{n}_*\mathcal{O}_{\overline{C}})_P$-submodule I generates. Take an $f\in I$ so that $\overline{I}=(\mathbf{n}_*\mathcal{O}_{\overline{C}})_P f$. Then $\overline{I}/(\mathcal{O}_{C,P}f)\cong(\mathbf{n}_*\mathcal{O}_{\overline{C}})_P/\mathcal{O}_{C,P}$. Hence

$$\ell(\overline{I}/I)\leq\delta(C,P). \tag{4.3.1}$$

Now, $\mathbf{n}^*\Omega^1_C\to\Omega^1_{\overline{C}}\to\Omega^1_{\overline{C}/C}\to 0$ is exact. So the Chinese remainder theorem yields

$$(\mathbf{n}_*\Omega^1_{\overline{C}})_P/\overline{I}=\bigoplus_{\overline{P}\in\mathbf{n}^{-1}P}(\Omega^1_{\overline{C}/C})_{\overline{P}}. \tag{4.3.2}$$

Fix a branch Γ of C at P, and set $v:=v(\Gamma,P)$. Say Γ corresponds to $\overline{P}\in\mathbf{n}^{-1}P$. Below, we'll find an $f\in\mathcal{O}_{C,P}$ of order v at $\overline{P}$. Now, $p\nmid v$. Hence the derivative of f with respect to any local parameter of $\overline{C}$ at $\overline{P}$ has order $v-1$. So $\ell((\Omega^1_{\overline{C}/C})_{\overline{P}})\leq v-1$.

Therefore, Equation (4.3.2) yields

$$\ell\big((\mathbf{n}_*\Omega^1_{\overline{C}})_P/\overline{I}\big)\leq\sum_{\Gamma}(v(\Gamma,P)-1)=-r(C,P)+\sum_{\Gamma}v(\Gamma,P). \tag{4.3.3}$$

On the other hand, Equation (2.1.1) yields

$$\lambda(C,P)=\delta(C,P)+\ell((\mathbf{n}_*\Omega^1_{\overline{C}})_P/I)=\delta(C,P)+\ell(\overline{I}/I)+\ell\big((\mathbf{n}_*\Omega^1_{\overline{C}})_P/\overline{I}\,\big).$$

Hence, Inequalities (4.3.1) and (4.3.3) yield the assertion, given the existence of an f.

To find an f, let X' be the blowup of X at P, and C' the proper transform of C. Say $P'\in C'$ is the image of $\overline{P}$. Let $y_1,\dots,y_m$ be generators of the maximal ideal $\mathbf{m}_{C,P}$. Rearranging the y_i, we may assume y_1 generates the extension $\mathbf{m}_{C,P}\mathcal{O}_{C',P'}$. Then the order of y_1 at $\overline{P}$ is $e(\Gamma,P)$. So, if $p\nmid e(\Gamma,P)$, that is, if $Q=P$, take $f:=y_1$.

Proceed by induction on the order n of Q/P. Suppose $n>0$. Then the order of Q/P' is $n-1$. Say $y_i=z_iy_1$ where $z_i\in\mathcal{O}_{C',P'}$. Let a_i be the value z_i takes at P'. Then $y_1,z_2-a_2,\dots,z_m-a_m$ are generators of the maximal ideal $\mathbf{m}_{C',P'}$.

By induction, we may assume that a certain scalar linear combination

$$f':=b_1y_1+b_2(z_2-a_2)+\cdots+b_m(z_m-a_m)$$

has order $v(\Gamma,P')$ at $\overline{P}$. Then $f'y_1$ has order $v(\Gamma,P)$ at $\overline{P}$. Furthermore, $f'y_1$ is a scalar linear combination of the y_i. So take $f:=f'y_1$. □

Proposition 4.4. *Let X be a smooth scheme of dimension 2 or more in characteristic $p\geq 0$. Let $C\subset X$ be a curve, and $P\in C$ a closed point. If $p=0$, then*

$$\lambda(C,P)\leq 1+\sum_{Q\in N(P)}e(C,Q)\big(e(C,Q)-2+i(P,Q)\big).$$

Suppose $p>0$. For each $Q\in N(P)$, set $\epsilon(C,Q):=0$ if $Q\neq P$ and if $e(C,R)\leq 1$ where R is the immediate predecessor of Q; otherwise, set $\epsilon(C,Q):=1$. Then

$$\lambda(C,P)\leq \sum_{Q\in N(P)}e(C,Q)\big(e(C,Q)-2+i(P,Q)+\epsilon(C,Q)\big),$$

Proof. If $p=0$, then the asserted bound follows directly from (2.1.2) and Lemma 4.2.

Suppose $p>0$. Fix $Q\in N(P)$. Notice, as Γ ranges over all the branches of C at P,

$$\sum_{\Gamma}e(\Gamma,Q)=e(C,Q). \tag{4.4.1}$$

Fix a Γ, and suppose Q is the point of least order such that $p\nmid e(\Gamma,Q)$. Let $R\in[P,Q]$. If $R\neq Q$, then $p\mid e(\Gamma,R)$, and so $e(\Gamma,R)>1$. Hence $\epsilon(C,R):=1$ for all $R\in[P,Q]$.

It now follows from Lemma 4.3 and Formula (4.4.1) that

$$\lambda(C,P)\leq 2\delta(C,P)-r(C,P)+\sum_{Q\in N(P)}e(C,Q)\epsilon(C,Q).$$

Hence Lemma 4.2 yields the asserted bound. □

5. Foliations

5.1. *Foliations.* Let X be a scheme, $\mathcal{L}$ an invertible sheaf, and $\eta\colon\Omega^1_X\to\mathcal{L}$ a nonzero map. Then η will be called a *(singular one-dimensional) foliation* of X.

Let $S \subseteq X$ be the zero scheme of η, that is, the closed subscheme whose ideal $\mathcal{I}_{S/X}$ is the image of the induced map $\Omega^1_X \otimes \mathcal{L}^{-1} \to \mathcal{O}_X$. Then S will be called the *singular locus* of η.

Let $C \subseteq X$ be a closed curve. Suppose for a moment (1) that $C \cap S$ is finite and (2) that the restricted map $\eta|C$ factors through the standard map $\sigma\colon \Omega^1_X|C \to \Omega^1_C$, in other words, that there is a commutative diagram

$$\begin{array}{ccc} \Omega^1_X & \xrightarrow{\eta} & \mathcal{L} \\ \big\downarrow & & \big\downarrow \\ \Omega^1_C & \xrightarrow{\mu} & \mathcal{L}|C \end{array} \tag{5.1.1}$$

Then C will be called a *leaf* of η.

Notice the following. Assume X is smooth. Let $P \in X - S$ be a closed point, and $\eta^*\colon \mathcal{L}^* \to \mathcal{T}_X$ the dual map. Then the image of $\eta^*(P)$ is a one-dimensional vector subspace, $F(P)$ say, of the fiber $\mathcal{T}_X(P)$. Moreover, if C is a leaf and if P is a simple point of C, then $F(P) \subseteq \mathcal{T}_C(P)$.

Conversely, assume $C \cap S$ is finite, and let $U \subseteq C - S$ be a dense open subset. Let's prove that, if $F(P) \subseteq \mathcal{T}_C(P)$ for every simple point $P \in U$, then C is a leaf.

Indeed, let $\mathcal{K}$ be the kernel of $\sigma\colon \Omega^1_X|C \to \Omega^1_C$, and $\kappa\colon \mathcal{K} \to \mathcal{L}|C$ the restriction of $\eta|C$ to $\mathcal{K}$. It follows from the hypothesis that $\kappa(P) = 0$ for every simple point $P \in U$. So, since U is dense in C, the image of κ has finite support. Now, C is reduced and $\mathcal{L}|C$ is invertible. Hence $\kappa = 0$. So there is a map $\mu\colon \Omega^1_C \to \mathcal{L}|C$ making the diagram (5.1.1) commute. Thus C is a leaf.

Proposition 5.2. *Let X be a scheme, $C \subseteq X$ a projective curve, $\eta\colon \Omega^1_X \to \mathcal{L}$ a foliation, and S its singular locus. If C is a leaf of η, then*

$$\begin{aligned} 2p_a(C) - 2 - \deg(\mathcal{L}|C) &= \lambda(C) - \deg(C \cap S) \\ &\leq \lambda(C) - \iota(C). \end{aligned}$$

Proof. Form the standard exact sequence

$$0 \to \mathcal{I}_{(C\cap S)/C} \to \mathcal{O}_C \to \mathcal{O}_{C\cap S} \to 0.$$

Twist it by $\mathcal{L}$, and take Euler characteristics; we get

$$\chi(\mathcal{I}_{(C\cap S)/C} \otimes \mathcal{L}) = \chi(\mathcal{L}|C) - \chi(\mathcal{L}|(C \cap S)).$$

Use Riemann's theorem to evaluate $\chi(\mathcal{L}|C)$. Then we get

$$\chi(\mathcal{I}_{(C\cap S)/C} \otimes \mathcal{L}) = \deg(\mathcal{L}|C) + 1 - p_a(C) - \chi(\mathcal{L}|(C \cap S)). \tag{5.2.1}$$

Since C is a leaf, there is a map $\mu\colon \Omega^1_C \to \mathcal{L}|C$ making the diagram (5.1.1) commute. Since S is the singular locus of η, the image $\mathrm{Im}(\eta)$ is equal to $\mathcal{I}_{S/X} \otimes \mathcal{L}$. Hence

$$\mathrm{Im}(\mu) = \mathcal{I}_{(C\cap S)/C} \otimes \mathcal{L}. \tag{5.2.2}$$

So $\mathrm{Cok}(\mu) = \mathcal{L}|(C \cap S)$. However, $\mathcal{L}$ is invertible. Hence

$$\iota(C) \leq \chi(\mathcal{L}|(C \cap S)) = \deg(C \cap S). \tag{5.2.3}$$

On the other hand, C is reduced. So $\mathcal{L}|C$ is torsion free. Hence $\operatorname{Im}(\mu)$ is equal to $\Omega^1_C/\text{torsion}$ because $C \cap S$ is finite. In addition, the canonical sheaf ω_C is torsion free. Hence the image of the class map $\gamma\colon \Omega^1_C \to \omega_C$ is also equal to $\Omega^1_C/\text{torsion}$. So

$$\operatorname{Im}(\gamma) = \operatorname{Im}(\mu). \tag{5.2.4}$$

Since $\lambda(C) = \chi(\operatorname{Cok}(\gamma))$, it follows that

$$\lambda(C) = \chi(\omega_C) - \chi(\operatorname{Im}(\gamma)).$$

Now, $\chi(\omega_C) = p_a(C) - 1$. Hence (5.2.1)–(5.2.4) yield the assertion. □

Theorem 5.3. *Let X be a smooth scheme of dimension 2 or more in characteristic $p \geq 0$, and $C \subset X$ a projective curve. Let P range over all the closed points of X. For each $Q \in N(P)$, set $\epsilon(C, Q) := 0$ either (i) if $e(C, Q) = 0$, or (ii) if $p = 0$, or (iii) if $p > 0$, if $Q \neq P$, and if $e(C, R) = 1$ where R is the immediate predecessor of Q; otherwise, set $\epsilon(C, Q) := 1$. Next, set*

$$\ell(C, Q) := e(C, Q) - 2 + i(P, Q) + \epsilon(C, Q).$$

(1) *Let $\eta\colon \Omega^1_X \to \mathcal{L}$ be a foliation, and assume C is a leaf. Then*

$$2p_a(C) - 2 - \deg(\mathcal{L}|C) \leq \sum_{P \in X} \sum_{Q \in N(P)} e(C, Q)\ell(C, Q).$$

(2) *Let $A \subset X$ be a divisor. For each P and $Q \in N(P)$, let $e(A, Q)$ be the multiplicity at Q of the proper transform of A on the successive blowup of X determined by Q. Assume that $e(A, Q) \geq \ell(C, Q)$ and that C is a leaf of $\eta\colon \Omega^1_X \to \mathcal{L}$. Then*

$$2p_a(C) - 2 - \deg(\mathcal{L}|C) \leq (A \cdot C).$$

Proof. To prove (1), recall that, if $p = 0$ and P is a singular point of C, then $\iota(C, P) \geq 1$. Hence Theorem 5.2 and Proposition 4.4 yield (1).

To prove (2), note that $(A \cdot C) = \sum_Q e(A, Q)e(C, Q)$ by Noether's formula; see [11], Formula (2.17), p. 27, for example. Hence (1) yields (2). □

6. Projective space

Theorem 6.1. *Let $X := \mathbf{P}^n$ with $n \geq 2$, and let $C \subset X$ be a closed curve of degree d. Assume d is not a multiple of the characteristic. Let $\eta\colon \Omega^1_X \to \mathcal{O}_X(m-1)$ be a foliation, S its singular locus. Assume C is a leaf. Then*

$$2p_a(C) - (d-1)(m-1) \leq \lambda(C),$$

with equality only if $C \cap S$ has degree $m+1$ and lies on a line M and either $M \subseteq S$ or M is a leaf.

Proof. It is well known, and reproved below, that $\deg(C \cap S)$ is at least the Castelnuovo–Mumford regularity $\operatorname{reg}(C \cap S)$. In turn, $\operatorname{reg}(C \cap S) \geq m+1$ owing to [14], Cor. 4.5. So Proposition 5.2 yields the asserted inequality.

Suppose equality holds in the assertion. Then the above reasoning yields

$$\deg(C \cap S) = \operatorname{reg}(C \cap S) = m + 1. \tag{6.1.1}$$

It follows, as is well known and reproved below, that the scheme $C\cap S$ lies on a line M.

Suppose that $M \not\subseteq S$ and that M is not a leaf. Then there is a point P in $M\setminus S$ at which the tangent "direction" $F(P)\subset T_{X,P}$ associated to η differs from that $T_{M,P}$ associated to M; see the end of Subsection 5.1. Take a hyperplane H containing M such that $T_{H,P}\not\supset F(P)$.

Let $\beta\colon \Omega^1_X|H\to\Omega^1_H$ be the natural map, and set $\xi := (\beta,\,\eta|H)$, so that

$$\xi\colon \Omega^1_X|H\to\Omega^1_H\oplus\mathcal{O}_H(m-1).$$

Set $\zeta := (\wedge^n\xi)(n+1)$. Now, $\eta|H$ factors through the twisted ideal $\mathcal{I}_{(H\cap S)/H}(m-1)$. So ζ factors through $\mathcal{I}_{(H\cap S)/H}(m)$. However, $\zeta(P)\neq 0$ because $T_{H,P}\not\supset F(P)$.

Form the zero scheme Z of ζ. It follows that $\mathcal{O}_H(Z)=\mathcal{O}_H(m)$; also, $Z\supset H\cap S$, but $Z\not\ni P$, whence $Z\not\supset M$. So $M\cap Z$ is finite, has degree m, and contains $M\cap S$. But $\deg(M\cap S)\geq m+1$ because $(M\cap S)\supseteq(C\cap S)$ and because of (6.1.1). A contradiction has been reached. So the proof is now complete, given the two well-known results.

Let's now derive these two results from Mumford's original work [28]. Let $W\subset X$ be a finite subscheme. Take a hyperplane H that misses W. Then the ideal $\mathcal{I}_{(H\cap W)/H}$ is trivial, so it is 0-regular. Hence, by the last display on p. 102 in [28], the ideal $\mathcal{I}_{W/X}$ is r-regular with $r := h^1(\mathcal{I}_{W/X}(-1))$. But $r=\deg W$ owing to the sequence

$$0\to\mathcal{I}_{W/X}(s)\to\mathcal{O}_X(s)\to\mathcal{O}_W(s)\to 0$$

with $s:=-1$. Thus $\operatorname{reg}W\leq\deg W$.

Suppose $\operatorname{reg}W=\deg W$. So $\mathrm{h}^1(\mathcal{I}_{W/X}(\deg W-2))\neq 0$. As $h^1(\mathcal{I}_{W/X}(-1))=\deg W$, it follows that $\mathrm{h}^1(\mathcal{I}_{W/X}(1))=\deg W-2$, by Display $(\#')$ on p. 102 in [28]. Hence $\mathrm{h}^0(\mathcal{I}_{W/X}(1))=n-1$ owing to the above sequence with $s:=1$. So W lies on $n-1$ linearly independent hyperplanes of X, whence on their line of intersection. □

Corollary 6.2. *Let $X:=\mathbf{P}^n$ with $n\geq 2$, and $C\subset X$ be a closed curve of degree d. Assume C is connected and the characteristic is 0. Let $\eta\colon\Omega^1_X\to\mathcal{O}_X(m-1)$ be a foliation. Assume C is a leaf. Then*

$$p_g(C)\leq(m-1)(d-1)/2+(r(C)-1)/2.$$

Proof. The assertion results from Theorem 6.1, Formula (2.3.1), and Bound (2.3.2). □

Proposition 6.3. *Let $X:=\mathbf{P}^n$ with $n\geq 2$, and $C\subset X$ a closed curve of degree d. Let $\eta\colon\Omega^1_X\to\mathcal{O}_X(m-1)$ be a foliation, S its singular locus. Assume S is finite and C is a leaf. Then*

$$\lambda(C)\leq 2p_a(C)-(d-1)(m-1)+m^2+\cdots+m^n.$$

Proof. Since S is finite, it represents the top Chern class of $(\Omega^1_X)^*(m-1)$. Hence

$$\deg(S)=1+m+m^2+\cdots+m^n.$$

Since $\deg(S)\geq\deg(C\cap S)$, Proposition 5.2 now yields the assertion. □

Corollary 6.4 (du Plessis and Wall)**.** *Let C be a (reduced) plane curve of degree d. Assume d is not a multiple of the characteristic. Let m be the least degree of a nonzero polynomial vector field ϕ annihilating the polynomial defining C. Then $m\leq d-1$ and*

$$(d-1)(d-m-1)\leq\tau(C).$$

If the foliation defined by ϕ has only finitely many singularities on C, then also

$$\tau(C) \leq (d-1)(d-m-1)+m^2.$$

Proof. Pick homogeneous coordinates x, y, z for the plane X. Say

$$\phi = f\,\frac{\partial}{\partial x} + g\,\frac{\partial}{\partial y} + h\,\frac{\partial}{\partial z} \quad \text{and} \quad C : u = 0$$

where f, g, h are polynomials in x, y, z of degree m and where u is one of degree d. By hypothesis, $\phi u = 0$. Also, $\phi \neq 0$; that is, $(f, g, h) \neq 0$.

In any case, u is annihilated by the three Hamilton fields

$$\frac{\partial u}{\partial y}\,\frac{\partial}{\partial z} - \frac{\partial u}{\partial z}\,\frac{\partial}{\partial y}, \quad \frac{\partial u}{\partial z}\,\frac{\partial}{\partial x} - \frac{\partial u}{\partial x}\,\frac{\partial}{\partial z}, \quad \frac{\partial u}{\partial x}\,\frac{\partial}{\partial y} - \frac{\partial u}{\partial y}\,\frac{\partial}{\partial x}.$$

Since d is not a multiple of the characteristic, at least two of the three are nonzero. Hence $m \leq d-1$.

Consider the Euler exact sequence,

$$0 \longrightarrow \Omega^1_X \longrightarrow \mathcal{O}_X(-1)^3 \xrightarrow{(x,y,z)} \mathcal{O}_X \longrightarrow 0.$$

The triple (f, g, h) defines a map $\mathcal{O}_X(-1)^3 \to \mathcal{O}_X(m-1)$. Let η be its restriction to Ω^1_X.

Owing to the exactness, $\eta = 0$ if and only if $(f, g, h) = p(x, y, z)$ for some polynomial p. But, if p exists, then $\phi u = 0$ yields $pdu = 0$; whence, $p = 0$ because d is not a multiple of the characteristic. Since $\phi \neq 0$, necessarily $\eta \neq 0$. Thus η is a foliation.

Diagram (5.1.1) exists as $\phi u = 0$. So, if $C \cap S$ is finite, then C is a leaf.

Since C is plane, $\tau(C) = \lambda(C)$ by Proposition 2.2; also, $2p_a(C) - 2 = d(d-3)$ by adjunction. Therefore, if S is finite, and so C is a leaf, then the asserted bounds follow from Theorem 6.1 and Proposition 6.3.

So assume S is infinite. Let $B \subseteq S$ be the effective divisor of largest degree, b say. Then $\mathcal{I}_{S/X} \subseteq \mathcal{O}_X(-B)$. So η factors through a foliation $\eta' \colon \Omega^1_X \to \mathcal{O}_X(m-1-b)$, whose singular locus has $\mathcal{I}_{S/X}(B)$ as its ideal. Hence the singular locus of η' is finite.

Set $\mathcal{L} := \mathcal{O}_X(m-1-b)$. The Euler sequence gives rise to the sequence

$$\operatorname{Hom}(\mathcal{O}_X(-1)^3, \mathcal{L}) \to \operatorname{Hom}(\Omega^1_X, \mathcal{L}) \to \operatorname{Ext}^1(\mathcal{O}_X, \mathcal{L}).$$

The third term is equal to $H^1(\mathcal{L})$, so vanishes. Hence η' lifts to a polynomial vector field ϕ' of degree $m-b$. Then $\phi' \neq 0$ simply because ϕ' is a lift.

Say $B : w = 0$ where w is a polynomial of degree b. Then $\phi - w\phi' = p\epsilon$ where p is a suitable polynomial and

$$\epsilon := x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y} + z\,\frac{\partial}{\partial z}$$

is the Euler, or radial, vector field. Now, $\phi u = 0$; hence,

$$-w\phi' u = pdu. \tag{6.4.1}$$

Let T be a component of B. Say $T : t = 0$ where t is a polynomial of degree e. Suppose T is not a component of C. Then $t \mid w$, but $t \nmid u$. So (6.4.1) implies $t \mid p$. Set $q := p/t$ and $r := w/t$. Then $r\phi' u = -qdu$. Set $\phi'' := r\phi' + q\epsilon$. Then $\phi'' u = 0$.

Moreover, $\phi'' \neq 0$ because $\eta \neq 0$. So ϕ'' is a nonzero polynomial vector field of degree $m-e$ annihilating u. But $m-e<m$, yet m is minimal—a contradiction! Thus T is a component of C.

Suppose T appears in B with multiplicity 2 or more. Set $r := w/t^2$. Since u is reduced, (6.4.1) implies $t \mid p$. Set $q := p/t$. Then $rt\phi' u = -q du$. Set $\phi'' := rt\phi' + q\epsilon$. Then ϕ'' is a nonzero polynomial vector field of degree $m-e$ annihilating u. But $m-e<m$, yet m is minimal—a contradiction! Thus B is reduced.

Set $A := C - B$ and $a := d-b$. Then A is a reduced effective divisor, so a curve of degree a. And $a>0$ as $b \le m \le d-1$. Moreover, A is a leaf of η', which has finite singular locus. As observed above, Proposition 2.2, adjunction, and Theorem 6.1 yield

$$(a-1)(a-(m-b)-1) \le \tau(A).$$

Now, $\tau(A)+\tau(B)+ab \le \tau(C)$ by Proposition 2.4. But $0 \le \tau(B)$. Hence

$$(a-1)(a-(m-b)-1)+ab \le \tau(C).$$

Now, $m \ge b$; so $b(a-(m-b)-1) < ab$. Hence

$$(a+b-1)(a+b-m-1) < (a-1)(a-(m-b)-1)+ab \le \tau(C).$$

Since $a+b=d$, the first assertion therefore holds.

As to the second assertion, suppose η has only finitely many singularities on C. But $B \subset C$. Hence $B=\emptyset$. So S is finite. Therefore, as was observed above, the upper bound holds. □

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