Well-posedness results for the modified Zakharov-Kuznetsov equation

H. A. Biagioni
IMECC-UNICAMP
Campinas, 13081-970, Brazil
and
F. Linares
IMPA
Estrada Dona Castorina 110,
Rio de Janeiro, 22460-320, Brazil

Abstract. We establish local and global well-posedness for the modified Zakharov-Kuznetsov equation for initial data in $H^1(\mathbb{R}^2)$. We use smoothing estimates for solutions of the linear problem plus a fixed point theorem to prove the local result.

1. Introduction

In this note we consider the initial value problem associated to the modified Zakharov-Kuznetsov equation,

mzk
$$\begin{cases} u_t + u_{xxx} + u_{xyy} + u^2 u_x = 0, & (x,y) \in \mathbb{R}^2, t \in \mathbb{R}, \\ u(x,y,0) = u_0(x,y), \end{cases} \tag{1.1}$$

where u is a real function.

This model is a two-dimensional generalization of the Korteweg-de Vries equation. More precisely, in the context of plasma physics, Zakharov and Kuznetsov [15] showed that the propagation of nonlinear ion–acoustic waves in magnetized plasma is governed by the equation

zk
$$u_t + u_{xxx} + u_{xyy} + u u_x = 0. \tag{1.2}$$

On the other hand, Kakutani and Ono [4] established that the equation describing the propagation of Alfvén waves at a critical angle to the undisturbed magnetic field was the modified Korteweg-de Vries equation. The two-dimensional equation in this physical situation is the modified Zakharov-Kuznetsov equation appearing in (1.1), see [11] for more details.

Our purpose here is to establish local and global well-posedness for the IVP (1.1).

In [3] Faminskii considered the IVP associated to the equation (1.2). He showed local and global well-posedness for initial data in $H^m(\mathbb{R}^2)$, $m \geq 1$, integer. His method of proof

This research was partially supported by CNPq–Brazil.

was inspired by the one given by Kenig, Ponce and Vega [8] to show local well-posedness for the IVP associated to the Korteweg-de Vries in $H^s(\mathbb{R})$, $s > 3/4$. More precisely, he established a series of estimates for solutions of the linear problem and then he used these estimates in a regularized problem to establish local well-posedness. To prove the global results he made use of the L^2 and H^1 quantities conserved by the Zakharov-Kuznetsov flow (1.2). Here we show that the IVP (1.1) is locally well-posed in $H^1(\mathbb{R}^2)$ and using the conserved quantities associated to solutions of the equation in (1.1), that is,

mass
$$I_1(u(t)) = \int\limits_{\mathbb{R}^2} u^2(t)\,dxdy \;=\; \int\limits_{\mathbb{R}^2} u_0^2\,dxdy, \tag{1.3}$$

energy
$$I_2(u(t)) = \int\limits_{\mathbb{R}^2} (u_x^2 + u_y^2 - \frac{1}{4}u^4)(t)\,dxdy \;=\; \int\limits_{\mathbb{R}^2} ({u_0}_x^2 + {u_0}_y^2 - \frac{1}{4}u_0^4)\,dxdy, \tag{1.4}$$

we extend the local result to any time interval $[0,T]$.

Our main results are:

t1
Theorem 1.1. *For any $u_0 \in H^1(\mathbb{R}^2)$, there exist $T = T(\|u_0\|_{H^1}) > 0$ and a unique solution of the IVP (1.1) $u(\cdot)$ defined in the interval $[0,T]$ satisfying*

1
$$u \in C([0,T] : H^1(\mathbb{R}^2)), \tag{1.5}$$
$$\|\partial_x^2 u\|_{L^\infty_x L^2_y L^2_T} + \|\partial_{xy}^2 u\|_{L^\infty_x L^2_y L^2_T} \;< \infty, \tag{1.6}$$
$$\|u\|_{L^3_T L^\infty_x L^\infty_y} + \|\partial_x u\|_{L^3_T L^\infty_x L^\infty_y} \;< \infty, \tag{1.7}$$

and

4
$$\|u\|_{L^2_x L^\infty_y L^\infty_T} < \infty. \tag{1.8}$$

Moreover, for $T' \in (0,T)$ there exists a neighborhood V of $u_0 \in H^1$ such that the map $\tilde{u}_0 \to \tilde{u}(t)$ from V into the class defined by (1.5)–(1.8) is Lipschitz.

To give the statement of the global result we will consider φ a positive radial solution of the equation

ground
$$\Delta\varphi - \varphi + \varphi^3 = 0. \tag{1.9}$$

(see [14] and references therein.)

t2
Theorem 1.2. *Let $u_0 \in H^1(\mathbb{R}^2)$. If $\|u_0\|_{L^2} < \|\varphi\|_{L^2}$, φ as in (1.9), then the solution u given by Theorem 1.1 can be extended to any interval of time $[0,T]$.*

To establish local well-posedness we use smoothing estimates for solutions of the linear problem and the contraction mapping principle, see [6]. The smoothing estimates employed in our argument of proof were shown in [3].

We notice that the method of proof given here can be used to obtain a similar result as Famisnskii's [3]. More precisely, it can be proved the following theorem.

t3 **Theorem 1.3.** *For any* $u_0 \in H^1(\mathbb{R}^2)$*, there exist* $T = T(\|u_0\|_{H^1}) > 0$ *and a unique solution* $u(\cdot)$ *of the IVP associated to* (1.2) *with* $u(x,y,0) = u_0(x,y)$ *defined in the interval* $[0,T]$ *satisfying*

10
$$u \in C([0,T] : H^1(\mathbb{R}^2)), \tag{1.10}$$

$$\|\partial_x^2 u\|_{L_x^\infty L_y^2 L_T^2} + \|\partial_{xy}^2 u\|_{L_x^\infty L_y^2 L_T^2} < \infty, \tag{1.11}$$

$$\|\partial_x u\|_{L_T^3 L_x^\infty L_y^\infty} < \infty, \tag{1.12}$$

and

11
$$\|u\|_{L_x^2 L_y^\infty L_T^\infty} < \infty. \tag{1.13}$$

In addition, for $T' \in (0,T)$ *there exists a neighborhood* W *of* $u_0 \in H^1$ *such that the map* $\tilde{u}_0 \to \tilde{u}(t)$ *from* W *into the class defined by* (1.10)-(1.13) *is Lipschitz.*

We will not give the proof of this theorem since it follows closely the argument explained in detail to establish Theorem 1.1.

Since solutions of the IVP (1.2) are conserved in L^2 space it is natural to ask whether or not it is possible to have local solutions in that space or more generally what would be the largest Sobolev space where local well-posedness can be hold. Using the scaling argument we can have an insight in this question.

Observe that if u solves the initial value problem (1.2) with data u_0 then $u_\lambda(x,y,t) = \lambda\, u(\lambda x, \lambda y, \lambda^3 t)$ also solves (1.2) with initial data $u_\lambda(x,y,0) = \lambda\, u_0(\lambda x, \lambda y)$. Thus the highest derivative that leaves invariant the $\dot{H}^s(\mathbb{R}^2)$-norm of u_λ is $s = 0$. In fact,

$$\|u(\cdot,0)\|_{\dot{H}^s} = \lambda^s \|u_0\|_{\dot{H}^s} \tag{1.14}$$

where $\dot{H}^s(\mathbb{R}^2)$ denotes the homogeneous Sobolev space of order s. ¿From this we can deduce that local well-posedness for (1.2) could be realized in Sobolev spaces of indices greater or equal than 0.

To establish better results than the one mentioned above for the KdV equation [6], Bourgain [1] introduced a new method to prove local and global well-posedness for data in L^2. That method combined with the bilinear estimate method of Kenig, Ponce, Vega [5] allowed them to show local well-posedness in H^s, $s > -3/4$. These two methods have been proved useful to study another well known generalization of the KdV equation, the Kadomtsev-Petviashvili (KP) equations. The difference between the treatment of KdV and KP equations is that one needs to use a sharp Strichartz estimate to handle the latter equations, see [2], [10], [12], [13] and references therein.

A final remark is regarding the global existence of solutions. The modified Zakharov-Kusnetsov equation has a similar critical character as the generalized Korteweg-de Vries equation,

cri
$$u_t + u_{xxx} + u^4 u_x = 0. \tag{1.15}$$

Weinstein in [14] showed that for initial data in $H^1(\mathbb{R})$ with $\|u_0\|_{L^2} < \|Q\|_{L^2}$ the IVP for (1.15) has global solutions in $H^1(\mathbb{R})$, where Q denotes the solitary wave solution of

(1.15). Recently, Merle [9] has shown that for data in $H^1(\mathbb{R})$ satisfying $\|u_0\|_{L^2} \geq \|Q\|_{L^2}$ solutions of the IVP for (1.15) blow-up in finite time. Thus it is natural to ask whether the same results are shared by solutions of the modified Zakharov–Kutnetsov equation. These questions and the ones mentioned above will be addressed somewhere else.

This note is organized as follows. In the next section we list the linear estimates needed in the proof of Theorem 1.1. The proof of Theorem 1.1 and 1.2 will be given in Section 3.

Before leaving this section we introduce some notation:

The mixed space-time norm is denoted by

$$\|f\|_{L^p_x L^q_y L^r_T} = \Big(\int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} \Big(\int_0^T |f(x,y,t)|^r \, dt \Big)^{q/r} dy \Big)^{p/q} dx \Big)^{1/p}.$$

When t appears instead of T, the notation above means integration on the whole real axis with respect to t.

The letter c denotes a constant that may change from line to line.

2. Preliminary results

Consider the initial value problem

lp (2.1)
$$\begin{cases} u_t + u_{xxx} + u_{xyy} = 0, & (x,y) \in \mathbb{R}^2, t \in \mathbb{R}, \\ u(x,y,0) = u_0(x,y). \end{cases}$$

Solutions of (2.1) are described by the unitary group $\{U(t)\}_{t=-\infty}^{\infty}$, that is,

(2.2)
$$u(t) = U(t)u_0 = \int_{\mathbb{R}^2} e^{i(t(\xi^3+\xi\eta^2)+x\xi+y\eta)} \, \hat{u}_0(\xi,\eta) \, d\xi d\eta.$$

Faminskii in [3] established the following estimates associated to solutions of (2.1).

l1 **Lemma 2.1.** *Let $\phi \in L^2(\mathbb{R}^2)$. Then*

se (2.3)
$$\|\nabla U(t)\,\phi\|_{L^\infty_x L^2_y L^2_t} \leq c\,\|\phi\|_{L^2},$$

ste (2.4)
$$\|U(t)\,\phi\|_{L^3_t L^\infty_x L^\infty_y} \leq c\,\|\phi\|_{L^2}.$$

In addition, if $\phi \in H^s(\mathbb{R}^2)$, $s > 3/4$, then

mfe (2.5)
$$\|U(t)\,\phi\|_{L^2_x L^\infty_y L^\infty_T} \leq c(T,s)\|\phi\|_{H^s}.$$

We also have the dual version of (2.3), i.e.

dse (2.6)
$$\|\nabla \int_0^t U(-\tau)\, f(\cdot,\tau)\, d\tau\|_{L^2_{xy}} \leq c\,\|f\|_{L^1_x L^2_y L^2_T}.$$

The estimate (2.3) is known as a smoothing effect of Kato type, see [7]. The inequality (2.4) establishes a $L^p - L^q$ or Strichartz estimate for solutions of (2.1). Finally, we have the maximal function estimate (2.5) for the unitary group $U(\cdot)$. The proof of these estimates are contained in [3]. These estimates are the main ingredients in the proof of Theorems 1.1 and 1.3.

3. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. As usual we consider the integral equation form of IVP (1.1), that is,

ie
$$u(t) = U(t)u_0 + \int_0^t U(t-t')\,(u^2\partial_x u)(\tau)\,d\tau. \tag{3.1}$$

We define the operator

iop
$$\Phi(u(t)) = \Phi_{u_0}(u(t)) = U(t)u_0 + \int_0^t U(t-t')\,(u^2\partial_x u)(\tau)\,d\tau, \tag{3.2}$$

and the metric space

$$Z_T^a = \{v \in C([0,T] : H^1(\mathbb{R}^2))\;:\;|||v||| \le a\} \tag{3.3}$$

where

$$\begin{aligned} |||v||| \;&=\; \|v\|_{L^\infty_T H^1} + \|v\|_{L^3_T L^\infty_x L^\infty_y} + \|\partial_x v\|_{L^3_T L^\infty_x L^\infty_y} \\ &\quad + \|\partial_x^2 v\|_{L^\infty_x L^2_y L^2_T} + \|\partial_{xy}^2 v\|_{L^\infty_x L^2_y L^2_T} + \|v\|_{L^2_x L^\infty_y L^\infty_T}. \end{aligned}$$ norma

We will show that there exist a and T such that

(1) $\Phi : Z_T^a \to Z_T^a$.
(2) Φ is a contraction.

We begin by estimating the H^1-norm of $\Phi(u)$. Thus using Minkowski's inequality, group properties and Hölder inequality we obtain

e1
$$\begin{aligned} \|\Phi(u)\|_{L^2_{xy}} &\le c\,\|u_0\|_{L^2} + \int_0^T \|(u^2\partial_x u)(\tau)\|_{L^2_{xy}}\,d\tau \\ &\le c\,\|u_0\|_{H^1} + c\int_0^T \|u\|^2_{L^\infty_{xy}}\|\partial_x u\|_{L^2_{xy}}\,d\tau \\ &\le c\,\|u_0\|_{H^1} + c\,T^{1/3}\sup_{[0,T]}\|\partial_x u(t)\|_{L^2_{xy}}\,\|u\|^2_{L^3_T L^\infty_{xy}}. \end{aligned} \tag{3.4}$$

By using group properties, Minkowski's inequality and Hölder's inequality we deduce

e2 (3.5)
$$
\begin{aligned}
&\|\partial_x\Phi(u)\|_{L^2_{xy}}\\
&\le c\,\|\partial_x u_0\|_{L^2}+\int_0^T \|\partial_x(u^2\partial_x u)(\tau)\|_{L^2_{xy}}\,d\tau\\
&\le c\,\|u_0\|_{H^1}+c\int_0^T \|u^2\partial_x^2 u\|_{L^2_{xy}}\,d\tau+\int_0^T \|u(\partial_x u)^2\|_{L^2_{xy}}\,d\tau\\
&\le c\,\|u_0\|_{H^1}+c\int_0^T \|u\|_{L^\infty_{xy}}\|u\partial_x^2 u\|_{L^2_{xy}}\,d\tau+\int_0^T \|u\|_{L^2_{xy}}\|(\partial_x u)^2\|_{L^\infty_{xy}}\,d\tau\\
&\le c\,\|u_0\|_{H^1}+c\,(\int_0^T \|u\|^2_{L^\infty_{xy}}\,d\tau)^{1/2}\|u\partial_x^2 u\|_{L^2_{xy}L^2_T}\\
&\qquad+\sup_{[0,T]}\|u(t)\|_{L^2_{xy}}\int_0^T \|\partial_x u\|^2_{L^\infty_{xy}}\,d\tau\\
&\le c\,\|u_0\|_{H^1}+c\,T^{1/6}\|u\|_{L^3_TL^\infty_{xy}}\,\|u\|_{L^2_xL^\infty_yL^\infty_T}\,\|\partial_x^2 u\|_{L^\infty_xL^2_yL^2_T}\\
&\qquad+c\,T^{1/3}\sup_{[0,T]}\|u(t)\|_{L^2_{xy}}\,\|\partial_x u\|^2_{L^3_TL^\infty_{xy}},
\end{aligned}
$$

and

e3 (3.6)
$$
\begin{aligned}
&\|\partial_y\Phi(u)\|_{L^2_{xy}}\\
&\le c\,\|\partial_y u_0\|_{L^2}+c\int_0^T \|u^2\partial_{xy}^2 u\|_{L^2_{xy}}\,d\tau+\int_0^T \|u\partial_y u\,\partial_x u\|_{L^2_{xy}}\,d\tau\\
&\le c\,\|u_0\|_{H^1}+c\int_0^T \|u\|_{L^\infty_{xy}}\|u\partial_{xy}^2 u\|_{L^2_{xy}}\,d\tau+\int_0^T \|\partial_y u\|_{L^2_{xy}}\|u\|_{L^\infty_{xy}}\|\partial_x u\|_{L^\infty_{xy}}\,d\tau\\
&\le c\,\|u_0\|_{H^1}+c\,(\int_0^T \|u\|^2_{L^\infty_{xy}}\,d\tau)^{1/2}\|u\partial_{xy}^2 u\|_{L^2_{xy}L^2_T}\\
&\qquad+\sup_{[0,T]}\|\partial_y u(t)\|_{L^2_{xy}}\int_0^T \|u\|_{L^\infty_{xy}}\|\partial_x u\|_{L^\infty_{xy}}\,d\tau\\
&\le c\,\|u_0\|_{H^1}+c\,T^{1/6}\|u\|_{L^3_TL^\infty_{xy}}\,\|u\|_{L^2_xL^\infty_yL^\infty_T}\,\|\partial_{xy}^2 u\|_{L^\infty_xL^2_yL^2_T}\\
&\qquad+c\,T^{1/3}\sup_{[0,T]}\|u(t)\|_{H^1_{xy}}\,\|\partial_x u\|_{L^3_TL^\infty_{xy}}\|u\|_{L^3_TL^\infty_{xy}}.
\end{aligned}
$$

The estimate (2.4), Minkowski's and Hölder's inequalities give

e4

$$\begin{aligned}
\|\Phi(u)\|_{L^3_T L^\infty_x L^\infty_y} &\le c\,\|U(t)u_0\|_{L^3_T L^\infty_x L^\infty_y} + \|U(t)\Big(\int_0^T U(-\tau)(u^2\partial_x u)(\tau)\,d\tau\Big)\|_{L^3_T L^\infty_x L^\infty_y}\\
&\le c\,\|u_0\|_{L^2} + \int_0^T \|(u^2\partial_x u)(\tau)\|_{L^2_{xy}}\,d\tau\\
&\le c\,\|u_0\|_{H^1} + c\int_0^T \|u\|^2_{L^\infty_{xy}}\|\partial_x u\|_{L^2_{xy}}\,d\tau\\
&\le c\,\|u_0\|_{H^1} + c\,T^{1/3}\sup_{[0,T]}\|\partial_x u(t)\|_{L^2_{xy}}\,\|u\|^2_{L^3_T L^\infty_{xy}}.
\end{aligned}\tag{3.7}$$

Applying estimate (2.4), group properties, Minkowski's and Hölder inequalities and the argument in (3.5) it follows that

e5

$$\begin{aligned}
&\|\partial_x\Phi(u)\|_{L^3_T L^\infty_x L^\infty_y}\\
&\quad\le c\,\|U(t)\partial_x u_0\|_{L^3_T L^\infty_x L^\infty_y} + \|U(t)\Big(\int_0^T U(-\tau)\partial_x(u^2\partial_x u)(\tau)\,d\tau\Big)\|_{L^3_T L^\infty_x L^\infty_y}\\
&\quad\le c\,\|u_0\|_{H^1} + c\int_0^T \|u\,(\partial_x u)^2\|_{L^2_x L^2_y} + \int_0^T \|u\|_{L^\infty_{xy}}\|u\,\partial_x^2 u\|_{L^2_x L^2_y}\\
&\quad\le c\,\|u_0\|_{H^1} + c\,T^{1/3}\,\|\partial_x u\|^2_{L^3_T L^\infty_{xy}}\sup_{[0,T]}\|u(t)\|_{H^1}\\
&\qquad\qquad + \,c\,T^{1/6}\|u\|_{L^3_T L^\infty_{xy}}\|u\|_{L^2_x L^\infty_y L^\infty_T}\|\partial_x^2 u\|_{L^\infty_x L^2_y L^2_T}.
\end{aligned}\tag{3.8}$$

Next we use the estimate (2.3), group properties and the argument in (3.6) to yield

e6

$$\begin{aligned}
\|\partial_x^2\Phi(u)\|_{L^\infty_x L^2_y L^2_T} &\le c\,\|\partial_x u_0\|_{L^2} + \|\partial_x U(t)\Big(\int_0^T U(-\tau)\partial_x(u^2\partial_x u)(\tau)\,d\tau\Big)\|_{L^\infty_x L^2_y L^2_T}\\
&\le c\,\|u_0\|_{H^1} + c\int_0^T \|\partial_x(u^2\partial_x u)\|_{L^2_{xy}}d\tau\\
&\le c\,\|u_0\|_{H^1} + c\,T^{1/3}\,\|\partial_x u\|^2_{L^3_T L^\infty_{xy}}\sup_{[0,T]}\|u(t)\|_{H^1}\\
&\qquad + \,c\,T^{1/6}\|u\|_{L^3_T L^\infty_{xy}}\|u\|_{L^2_x L^\infty_y L^\infty_T}\|\partial_x^2 u\|_{L^\infty_x L^2_y L^2_T},
\end{aligned}\tag{3.9}$$

and

e7

$$\begin{aligned}
\|\partial_{xy}^2\Phi(u)\|_{L^\infty_x L^2_y L^2_T} &\le c\,\|\partial_x u_0\|_{L^2} + \|\partial_y U(t)\Big(\int_0^T U(-\tau)\partial_x(u^2\partial_x u)(\tau)\,d\tau\Big)\|_{L^\infty_x L^2_y L^2_T}\\
&\le c\,\|u_0\|_{H^1} + c\int_0^T \|\partial_x(u^2\partial_x u)\|_{L^2_{xy}}d\tau\\
&\le c\,\|u_0\|_{H^1} + c\,T^{1/3}\,\|\partial_x u\|^2_{L^3_T L^\infty_{xy}}\sup_{[0,T]}\|u(t)\|_{H^1}\\
&\qquad + \,c\,T^{1/6}\|u\|_{L^3_T L^\infty_{xy}}\|u\|_{L^2_x L^\infty_y L^\infty_T}\|\partial_x^2 u\|_{L^\infty_x L^2_y L^2_T}.
\end{aligned}\tag{3.10}$$

Finally, the estimate (2.5), group properties and Sobolev spaces properties give

$$\begin{aligned}\|\Phi(u)\|_{L^2_xL^\infty_yL^\infty_T} &\le c(T,s)\,\|u_0\|_{H^s} + c(T,s)\|\Big(\int_0^T U(-\tau)(u^2\partial_x u)(\tau)\,d\tau\Big)\|_{H^s_{xy}}\\ &\le c(T,s)\,\|u_0\|_{H^1} + c(T,s)\,\|\int_0^T U(-\tau)(u^2\partial_x u)(\tau)\,d\tau\|_{H^1_{xy}}.\end{aligned}$$

Hence we can argue as in (3.6) to obtain the following estimate

e8

$$\begin{aligned}\|\Phi(u)\|_{L^2_xL^\infty_yL^\infty_T} &\le c(T)\,\|u_0\|_{H^1} + c\,T^{1/3}\,\|\partial_x u\|^2_{L^3_TL^\infty_{xy}}\sup_{[0,T]}\|u(t)\|_{H^1}\\ &\quad + c(T)\,T^{1/6}\|u\|_{L^3_TL^\infty_{xy}}\|u\|_{L^2_xL^\infty_yL^\infty_T}\|\partial_x^2 u\|_{L^\infty_xL^2_yL^2_T}.\end{aligned}\tag{3.11}$$

¿From the inequalities (3.4)–(3.11) and (3.4) it follows that

well

$$|||\Phi(u)||| \le c(T)\|u_0\|_{H^1} + c(T)(1+T^{1/6})T^{1/6}\,|||u|||^3\tag{3.12}$$

Choosing $a = 2c(T)\|u_0\|_{H^1}$ and T such that $a^2(1+T^{1/6})T^{1/6} < 1/2$ it follows that $\Phi : Z^a_T \to Z^a_T$. To show that Φ is a contraction we follow the same arguments as in (3.4)–(3.11) plus the choice of a and T. □

Proof of Theorem 1.2. The claim in our result follows using the conserved quantities (1.3), (1.4) and the following Gagliardo-Nirenberg interpolation inequality (see [14]),

$$\frac{1}{4}\|u(t)\|^4_{L^4} \le \frac{1}{2}\|\varphi\|^{-2}_{L^2}\,\|u(t)\|^2_{L^2}\,\|\nabla u(t)\|^2_{L^2}$$

where φ is a positive radial solution of (1.9).

Indeed,

glo

$$\begin{aligned}\|u(t)\|^2_{H^1} &= \|u(t)\|^2_{L^2} + I_2(u(t)) + \frac{1}{4}\|u(t)\|^4_{L^4}\\ &= \|u_0\|^2_{L^2} + I_2(u_0) + \frac{1}{4}\|u(t)\|^4_{L^4}\\ &\le \|u_0\|^2_{L^2} + c\,(\|u_0\|_{H^1}) + \frac{1}{2}\frac{\|u_0\|^2_{L^2}}{\|\varphi\|^2_{L^2}}\,\|\nabla u(t)\|^2_{L^2}\end{aligned}\tag{3.13}$$

Hence using the hypothesis we obtain

$$\|u(t)\|^2_{H^1} \le c(\|u_0\|_{H^1}).$$

This *a priori* estimate and a standard argument yield the desire result. □

References

b1 [1] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and application to nonlinear evolution equations II. The KdV equation*, GAFA **3** (1993), 209–262.

b2 [2] J. Bourgain, *On the Cauchy problem for the Kadomtsev-Petviashvili equation*, GAFA **3** (1993), 315–341.

F [3] A.V. Faminskii, *The Cauchy problem for the Zakharov-Kuznetsov equation*, Differential Equations **31** (6) (1995), 1002–1012.

ko [4] T. Kakutani and H. Ono, *Weak nonlinear hydromagnetic waves in a cold collision-free plasma*, J. Phys. Soc. Japan **26** (1965), 1305–1318

KPV1 [5] C.E. Kenig, G. Ponce and L. Vega, *A bilinear estimate with applications to the KdV equation*, J. Amer. Math. Soc. **9** (2) (1996), 573–603.

KPV2 [6] C.E. Kenig, G. Ponce and L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math. **46** (1993), 527–620.

KPV3 [7] C.E. Kenig, G. Ponce and L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J. **40** (1991), 33–69.

KPV4 [8] C.E. Kenig, G. Ponce and L. Vega, *Well-posedness of the initial value problem for the Korteweg-de Vries equation*, J. Amer. Math. Soc. **4** (1991), 323–347.

m [9] F. Merle, *Existence of blow-up solutions in the energy space for the critical generalized KdV equation*, J. Amer. Math. Soc. **14** (2001), 555–578.

mst [10] L. Molinet, J.-C. Saut and N. Tzetkov, *Well-posedness and ill-posedness results for the Kadomtsev-Petviashvili-I equation*, preprint 2001.

sb [11] R. Sipcic and D.J. Benney, *Lump interactions and collapse in the modified Zakharov-Kuznetsov equation*, Studies in Appl. Math. **105** (2000) 385–403.

st [12] J.-C. Saut and N. Tzvetkov, *The Cauchy problem for higher order KP equations*, J. Diff. Equation **153** (1999), 196–222.

t [13] N. Tzvetkov, *Global low regularity solutions for Kadomtsev-Petviashvili equation*, Differential Integral Equations **13** (2000), 1289–1320.

w [14] M. Weinstein, *Nonlinear Schródinger equations and sharp interpolation estimates*, Comm. Math. Phys. **87** (1983), 567–576.

zk [15] V.E. Zakharov and E.A. Kuznetsov, *On three-dimensional solitons*, Sov. Phys. JETP, **39** (1974), 285–286.

E-mail address: hebe@ime.unicamp.br, linares@impa.br