A SIMPLE PROOF OF A THEOREM OF HALPHEN AND GENERALIZATIONS

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Abstract. Let M be a germ at $0 \in \mathbb{C}^{n+1}$ of an irreducible hypersurface, where $n \geq 2$ and 0 is a singular point of M. We study the question : when there exists a germ of holomorphic map $\phi: (\mathbb{C}^n, 0) \to (M, 0)$ such that $\phi^{-1}(0) = \{0\}$? We prove essentialy three results. In Theorem 1 we consider the case where $M = \{F = 0\}$ is quasi-homogeneous, that is there exists a linear holomorphic vector field X on $\mathbb{C}^{n+1}$, with eigenvalues $\lambda_0, ..., \lambda_n \in \mathbb{Q}_+$ such that $X(F) = F$. We prove that if there exists a germ of holomorphic map ϕ as above and $dim_{\mathbb{C}}(sing(M)) \leq n-2$ then $\lambda_0 + ... + \lambda_n > 1$. In Theorem 2 we answer the question completely when $n = 2$ and 0 is an isolated singularity of M. In Theorem 3 we prove that, if there exists a map as above and $dim_{\mathbb{C}}(sing(M)) \leq n-2$, then $dim_{\mathbb{C}}(sing(M)) = n-2$. We observe that Theorems 1 and 2 are generalizations of some results due to Halphen [Ha].

§1. Introduction

Around 1884 Halphen proved the following result (cf. [Ha] or [Ha-1], chap. I, pg. 15):

Theorem. *Let f, g and h be three (non zero) homogeneous polynomials in $\mathbb{C}^3$, two by two without common factors. Suppose that $f^p + g^q + h^r \equiv 0$, where p, q, r are integers, $2 \leq p \leq q \leq r$ and $p.dg(f) = q.dg(g) = r.dg(h)$. Then*

$$(1) \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$$

Moreover, for each solution of the inequality (1), then

(a). *There exist homogeneous polynomials F, G, H in $\mathbb{C}^2$ such that $F^p + G^q + H^r \equiv 0$.*

(b). *If f, g, h are three homogeneous polynomials in $\mathbb{C}^n$ without common factors which satisfy $f^p + g^q + h^r \equiv 0$, then there exists a homogeneous map $\phi: \mathbb{C}^n \to \mathbb{C}^2$ such that $(f, g, h) = (F, G, H) \circ \phi$.*

In other words, we can say that for each solution (p, q, r) of the inequality (1), there exists a map $\psi = (F, G, H): \mathbb{C}^2 \to M$, where $M = \{(X, Y, Z) \in \mathbb{C}^3 \,|\, X^p + Y^q + Z^r = 0\}$, such that if $M^* = M \setminus \{0\}$ and $\psi_1 := \psi|_{\mathbb{C}^2 \setminus \{0\}}$, then $\psi_1: \mathbb{C}^2 \setminus \{0\} \to M^*$ is the holomorphic universal covering of M^*.

The purpose of this paper is to generalize this result in two ways. First of all, we will generalize inequality (1) for germs of holomorphic maps $\phi: (\mathbb{C}^n, 0) \to$

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$(M^n, 0)$, where $M^n \subset \mathbb{C}^{n+1}$ is a hypersurface defined by $F(z) = 0$ and F is a quasi-homogeneous polynomial of $n+1$ variables. In order to state our first result, we recall that F is a quasi-homogeneous polynomial in $\mathbb{C}^{n+1}$, if there exists a linear vector field, say $X(z) = \sum_{j=0}^{n} \lambda_j\, z_j\, \frac{\partial}{\partial z_j}$, such that $X(F) = F$ and the eigenvalues $\lambda_0, ..., \lambda_n \in \mathbb{Q}_+$. So, for instance, in the case of the hypersurface $M = \{x^p + y^q + z^r = 0\} \subset \mathbb{C}^3$, we can take $X(x,y,z) = \frac{1}{p}\, x\, \frac{\partial}{\partial x} + \frac{1}{q}\, y\, \frac{\partial}{\partial y} + \frac{1}{r}\, z\, \frac{\partial}{\partial z}$. We need a definition.

Definition 1. Let $M \subset B$ be an analytic irreducible hypersurface, where $B \subset \mathbb{C}^{n+1}$ is a ball. We will denote by $sing(M)$ the singular set of M. We will say that $dim_{\mathbb{C}}(sing(M)) \le k$ if, either $sing(M) = \emptyset$, or $sing(M) \ne \emptyset$ and all irreducible components of $sing(M)$ have complex dimension $\le k$. We will say that $dim_{\mathbb{C}}(sing(M)) = k$ if $sing(M) \ne \emptyset$ and all irreducible components of $sing(M)$ have complex dimension k. Let $p \in M$ and $\phi\colon (\mathbb{C}^n, q) \to (M, p)$ be a germ of holomorphic map. We will say that $\phi^{-1}(p) = \{q\}$ if there exists a representative of ϕ, denoted again by ϕ, say $\phi\colon V \to M$, such that $\phi^{-1}(p) \cap V = \{q\}$.

The first generalization is the following :

Theorem 1. *Let $n \ge 2$ and $M \subset \mathbb{C}^{n+1}$ be a hypersurface defined by a quasi-homogeneous reduced polynomial F which satisfies $X(F) = F$, where*

$$X(z) = \sum_{j=0}^{n} \lambda_j\, z_j\, \frac{\partial}{\partial z_j}$$

with $\lambda_0, ..., \lambda_n \in \mathbb{Q}_+$. Suppose that $dim_{\mathbb{C}}(sing(M)) \le n-2$ and that there exists a germ of map $\phi\colon (\mathbb{C}^n, 0) \to (M, 0)$ such that $\phi^{-1}(0) = \{0\}$. Then $\sum_{j=0}^{n} \lambda_j > 1$.

We observe that the above result is no longer true if $sing(M)$ has some component of dimension $n-1$ (see example 6).

As a consequence, we obtain a generalization of the first part of Halphen's Theorem :

Corollary . *Let $M \subset \mathbb{C}^3$ be the hypersurface*

$$M = \{(x,y,z) \in \mathbb{C}^3 \,|\, x^p + y^q + z^r = 0\}\ ,$$

where $p, q, r \ge 2$. Suppose that there exists a holomorphic map $\phi\colon U \to M$, where U is some neighborhood of $0 \in \mathbb{C}^n$, $n \ge 2$, such that $\phi(0) = 0 \in M$ and $\phi^{-1}(0) = \{0\}$. Then

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1\ .$$

In particular, we must have $(p,q,r) \in \{(2,2,r), (2,3,3), (2,3,4), (2,3,5)\}$.

Definition 2. Let M_1, M_2 be two germs at $0 \in \mathbb{C}^m$ of hypersurfaces. We will say that M_1 and M_2 are *equivalent* if there exists a germ of biholomorphism $\psi\colon (\mathbb{C}^m, 0) \to (\mathbb{C}^m, 0)$ such that $\psi(M_1) = M_2$.

The second generalization is the folowing :

Theorem 2. *Let M be a germ at $0 \in \mathbb{C}^3$ of hypersurface with an isolated singularity at 0. Suppose that there exists a germ of holomorphic map $\phi\colon (\mathbb{C}^2, 0) \to (M, 0)$, such that $\phi^{-1}(0) = \{0\}$. Then M is equivalent to one of the following surfaces :*

(a). $M_{(p,q,r)}$, *where*

$$M_{(p,q,r)} = \{(x,y,z) \in \mathbb{C}^3 \,|\, x^p + y^q + z^r = 0\}$$

and $(p,q,r) \in \{(2,2,r),(2,3,3),(2,3,4),(2,3,5)\}$.
(b). $M_b = \{(x,y,z) \in \mathbb{C}^3 \,|\, z^2 = xy(y - x^{m+1})\}$, *where* $m \geq 1$.
(c). $M_c = \{(x,y,z) \in \mathbb{C}^3 \,|\, z^2 = y(y^2 + x^3)\}$.
(d). $M_d = \{(x,y,z) \in \mathbb{C}^3 \,|\, z^2 = x(y^2 + x^{2m+1})\}$, *where* $m \geq 1$.

Moreover, the surfaces in the above statement are two by two non-equivalent.

Concerning the dimension of the singular set of M we have the following result :

Theorem 3. *Let M be a germ at $0 \in \mathbb{C}^{n+1}$ of hypersurface with a singularity at 0 and $dim_{\mathbb{C}}(sing(M)) \leq n-2$. Suppose that there exists a germ of holomorphic map $\phi\colon (\mathbb{C}^n,0) \to (M,0)$, such that $\phi^{-1}(0) = \{0\}$. Then $dim_{\mathbb{C}}(sing(M)) = n-2$.*

In the next four examples we show that for any one of the surfaces as in **(a)**, **(b)**, **(c)** or **(d)**, there exists a regular map ϕ like in Theorem 2. As the reader can check, in all the examples, the map $\phi_1\colon \mathbb{C}^2 \setminus \{0\} \to M^*$ is a universal covering of $M^* = M \setminus \{0\}$ (see also [Ha]).

Example 1. The possible solutions of inequality (1) are $(p,q,r) \in \{(2,2,r),(2,3,3),(2,3,4),$ $(2,3,5)\}$. In each case, the map $\phi = (F,G,H)\colon \mathbb{C}^2 \to M_{(p,q,r)}$ can be obtained by considering the finite subgroups of $PSL(2,\mathbb{C})$ (cf. [Ha] and [Mi] pg. 79). So, for instance, the case $(p,q,r) = (2,3,5)$, corresponds to the group of orientation preserving isometries of $\overline{\mathbb{C}} \simeq S^2 \subset \mathbb{R}^3$, say G, which leaves invariant the regular icosahedron (or dodecahedron) inscribed in S^2, say $\mathcal{I}$. Let us sketch the construction of ϕ in this case. Let us consider the following sets : V_0, the set of vertexes of $\mathcal{I}$, $E_0 = \{z \in \overline{\mathbb{C}} \,|\, z \text{ is the center of an edge of } \mathcal{I}\}$ and $F_0 = \{z \in \overline{\mathbb{C}} \,|\, z \text{ is the center of a face of } \mathcal{I}\}$. It is well known (cf. [F]), that $\#G = 60$, $\#V_0 = 12$, $\#E_0 = 30$ and $\#F_0 = 20$. We will take an affine coordinate system $z \in \mathbb{C} \subset \overline{\mathbb{C}}$, such that $\infty \notin V_0 \cup E_0 \cup F_0$. We will use the following known facts (cf. [F]):
(i). For each $a \in V_0$, its orbit $G(a) = \{\sigma(a) \,|\, \sigma \in G\} = V_0$. Moreover, the stabilizer of a, $G_a = \{\sigma \in G \,|\, \sigma(a) = a\}$ has five elements.
(ii). For each $a \in E_0$, we have $G(a) = E_0$ and $\#G_a = 2$.
(iii). For each $a \in F_0$, we have $G(a) = F_0$ and $\#G_a = 3$.

Given $a, b \in \mathbb{C} \subset \overline{\mathbb{C}}$, such that $G(a) \neq G(b)$ and $\infty \notin G(a) \cup G(b)$, consider rational map of degree 60

$$g_{a,b}(z) = \Pi_{\sigma \in G} \frac{\sigma(z) - a}{\sigma(z) - b} \;.$$

It is not difficult to prove that :
(iv). $g_{a,b}(z) = k.\Pi_{\sigma \in G} \frac{z - \sigma(a)}{z - \sigma(b)}$, for some constant $k \neq 0$.
(v). Given three distinct points $a, b, c \in \mathbb{C}$, then there are non zero constants $\alpha, \beta \in \mathbb{C}$, such that $g_{a,c}(z) = \alpha.g_{b,c}(z) + \beta$, for all $z \in \overline{\mathbb{C}}$.

Now, consider $g_{a,c}$, where $a \in V_0$ and $c \in E_0$. It follows from **(i)**, **(ii)** and **(iv)** that we can write $g_{a,c}(z) = \frac{(R(z))^5}{(P(z))^2}$, where $R(z)$ and $P(z)$ are polynomials with simple roots of degrees 12 and 30, respectively. Similarly, if $b \in F_0$ and c is as above, then $g_{b,c}(z) = \frac{(Q(z))^3}{(P(z))^2}$, where $Q(z)$ is a polynomial of degree 20. It follows from **(v)** that there are non zero constants α,β such that $g_{a,c}(z)=\alpha.g_{b,c}(z)+\beta$, so that $\beta.(P(z))^2+\alpha.(Q(z))^3-(R(z))^5\equiv 0$. If we take $H(u,v)=\beta_1.v^{12}.R(\frac{u}{v})$, $G(u,v)=\alpha_1.v^{20}.Q(\frac{u}{v})$ and $F(u,v)=i.v^{30}.P(\frac{u}{v})$, where $\beta_1^5=\beta$ and $\alpha_1^3=\alpha$, then we get $F^2+G^3+H^5\equiv 0$, which provides the desired map $\phi=(F,G,H)$.

The map ϕ can be obtained in a similar way in the other cases. We observe that, in all cases, the map ϕ, obtained as above, is such that $\phi|_{\mathbb{C}^2\setminus\{0\}}\colon\mathbb{C}^2\setminus\{0\}\to M_{(p,q,r)}\setminus\{0\}$ is a universal covering of $M_{(p,q,r)}\setminus\{0\}$.

In the cases $(p,q,r)\in\{(2,2,r),(2,3,3),(2,3,4)\}$, we have the following explicit maps :

(a). $p=q=2$, $r\geq 2$: $F(u,v)=u^r+v^r$, $G(u,v)=i(u^r-v^r)$, $H(u,v)=\alpha\,u\,v$, where $\alpha^r=-4$.

(b). $p=2$, $q=r=3$: $G(u,v)=g(u,v)=A^2(u,v)+j\,B^2(u,v)$, $H(u,v)=h(u,v)=\beta(A^2(u,v)+j^2\,B^2(u,v))$, $F(u,v)=f(u,v)=\gamma\,u\,v\,A(u,v)\,B(u,v)$, where $A(u,v)=u^2+v^2$, $B(u,v)=u^2-v^2$, $\beta^3=-1$, $\gamma^2=-12\sqrt{3}\,i$ and $j=e^{2\pi i/3}$.

(c). $p=2$, $q=3$, $r=4$: Let $f,g,h\colon\mathbb{C}^2\to\mathbb{C}$ be as in **(b)**, that is, such that $f^2+g^3+h^3=0$. Set $F(u,v)=i(g^3(u,v)-h^3(u,v))$, $H(u,v)=f(u,v)$, $G(u,v)=\lambda\,g(u,v)\,h(u,v)$, where $\lambda^3=-4$.

Example 2. We construct a holomorphic map $\phi\colon\mathbb{C}^2\to M_b$, where

$$M_b=\{(x,y,z)\in\mathbb{C}^3\mid z^2=xy^2-x^{m+1}y\}\ ,\ m\geq 1$$

such that $\phi^{-1}(0)=\{0\}$. Observe that M is quasi-homogeneous with respect to the vector field

$$X=\frac{1}{2m+1}\,x\,\frac{\partial}{\partial x}+\frac{m}{2m+1}\,y\,\frac{\partial}{\partial y}+\frac{1}{2}\,z\,\frac{\partial}{\partial z}$$

Let $f,g,h\colon\mathbb{C}^2\to\mathbb{C}$ be homogeneous polynomials in $\mathbb{C}^2$, without common factors, such that $g^2-f^{2m}=h^2$ (see **(a)** in Example 1). It is not difficult to see that, if $\phi=(X,Y,Z)$, where $X=f^2$, $Y=g^2$ and $Z=f.g.h$, then $Z^2=X.Y(Y-X^m)$. Therefore $\phi(\mathbb{C}^2)=M$. On the other hand $\phi|_{\mathbb{C}^2\setminus\{0\}}\colon\mathbb{C}^2\setminus\{0\}\to M^*$ is a holomorphic universal covering, as the reader can check.

Example 3. We construct a holomorphic map $\phi\colon\mathbb{C}^2\to M_c$, where

$$M_c=\{(x,y,z)\in\mathbb{C}^3\mid z^2=y(y^2+x^3)\}$$

such that $\phi^{-1}(0)=\{0\}$. Observe that M is quasi-homogeneous with respect to the vector field

$$X=\frac{2}{9}\,x\,\frac{\partial}{\partial x}+\frac{1}{3}\,y\,\frac{\partial}{\partial y}+\frac{1}{2}\,z\,\frac{\partial}{\partial z}\ .$$

Let $f,g,h\colon\mathbb{C}^2\to\mathbb{C}$ be homogeneous polynomials without common factors, such that $f^2=g^3+h^4$ (see **(c)** in Example 1). Set $\phi=(X,Y,Z)$, where $X=g$, $Y=h^2$ and $Z=f.h$. Then, it is not difficult to see that $Z^2=Y(Y^2+X^3)$ and $\phi^{-1}(0)=\{0\}$.

Example 4. We construct a holomorphic map $\phi\colon\mathbb{C}^2\to M_d$, where

$$M_d=\{(x,y,z)\in\mathbb{C}^3\mid z^2=x(y^2+x^{2m+1})\}\ ,m\geq 1\ ,$$

such that $\phi^{-1}(0)=\{0\}$. Observe that M is quasi-homogeneous with respect to the vector field

$$X=\frac{1}{2(m+1)}\,x\,\frac{\partial}{\partial x}+\frac{2m+1}{4(m+1)}\,y\,\frac{\partial}{\partial y}+\frac{1}{2}\,z\,\frac{\partial}{\partial z}$$

Let $f,g,h\colon\mathbb{C}^2\to\mathbb{C}$ be homogeneous polynomials without common factors, such that $h^2=g^2+f^{2(2m+1)}$ (see **(a)** in Example 1). Set $\phi=(X,Y,Z)$, where $X=f^2$, $Y=g$ and $Z=f.h$. Then, it is not difficult to see that $Z^2=X(Y^2+X^{2m+1})$ and $\phi^{-1}(0)=\{0\}$.

Let us see an example in bigger dimension.

Example 5. Let

$$M=\{(z_0,...,z_n)\in\mathbb{C}^{n+1}\,|\,z_0^p=z_1...z_n\}\,.$$

We have the following map $\phi\colon\mathbb{C}^n\to M$,

$$\phi=(\phi_0,...,\phi_n)\,,\ \text{where}\ \phi_0(u_1,...,u_n)=u_1...u_n\,,\ \text{and}\ \phi_j(u_1,...,u_n)=u_j^p\,,\ j=1,...,n\,. \blacksquare$$

Observe that $dim_{\mathbb{C}}(sing(M))=n-2$, $\phi^{-1}(0)=\{0\}$ and M is homogeneous, that is $X(z_0^n-z_1...z_n)=z_0^n-z_1...z_n$, where

$$X=\frac{1}{p}\,z_0\frac{\partial}{\partial z_0}+\frac{1}{n}\sum_{j=1}^{n}z_j\frac{\partial}{\partial z_j}$$

Example 6. In this example we show that the hypothesis $dim_{\mathbb{C}}(sing(M))\le n-2$ is the best possible in Theorem 1. Let $M=\{(x_0,x_1,x_2...,x_n)\in\mathbb{C}^{n+1}|\,F(x)=x_0^6-x_1^3.x_2^2...x_n^2=0\}$. Then M is irreducible and $sing(M)=\cup_{j=1}^n S_j$, where $S_j=\{x_0=x_j=0\}$, so that $dim_{\mathbb{C}}(S_j)=n-1$. The reader can easily verify that the map $\phi\colon\mathbb{C}^n\to M$ defined by

$$\phi(u_1,u_2,...,u_n)=(u_1...u_n,u_1^2,u_2^3,...,u_n^3)$$

satisfies $\phi^{-1}(0)=\{0\}$. On the other hand, let $X=\sum_{j=0}^n\lambda_j\,x_j\,\frac{\partial}{\partial x_j}$ be a vector field such that $X(F)=F$ and $\lambda_0,...,\lambda_n\in\mathbb{Q}_+$. Then we must have $\lambda_0=\frac{1}{6}$ and $3\lambda_1+2\lambda_2+...+2\lambda_n=1$. But this implies that $\lambda_1+...+\lambda_n<\frac{1}{2}$ and so $\sum_{j=0}^n\lambda_j<\frac{1}{6}+\frac{1}{2}<1$.

Remark 1. We would like to observe that the conclusion of Theorem 3 is not true if ϕ is not holomorphic. It is well known that there are examples of hipersurfaces of the form

$$M_p=\{(x_0,...,x_n)\in\mathbb{C}^{n+1}|\,x_0^{p_0}+...+x_n^{p_n}=0\}\,,$$

with $n\ge3$ and $p_0,...,p_n\ge2$ such that $K_r=M_p\cap S_r$ is homeomorphic to a sphere S^{2n-1} (cf. [Hi] and [Mi]), where $S_r=\{(x_0,...,x_n)|\,|x_0|^2+...+|x_n|^2=r^2\}$. Since M_p is homeomorphic to a cone over K_r (cf. [Mi]), then M_p is homeomorphic to $\mathbb{C}^n$ in these cases, so that there exists a continuous map $\phi\colon\mathbb{C}^n\to M_p$ satisfiyng the hypothesis of Theorem 3, but $dim_{\mathbb{C}}(sing(M_p))=0$. An example of such hypersurfaces is when $p_0=3$, $p_1=...=p_n=2$ and n is odd (cf. [Mi]).

We would like to state the following problems :

Problem 1. Let M be a germ at $0 \in \mathbb{C}^{n+1}$ of hypersurface, where $n \geq 3$ and $dim_{\mathbb{C}}(sing(M)) = n-2$. Suppose that there exists a germ of holomorphic map $\phi: (\mathbb{C}^n, 0) \to (M, 0)$. Is M (as germ) equivalent to a quasi-homogeneous hypersurface of $\mathbb{C}^{n+1}$? We would like to observe that when $n = 2$ the answer is yes. This fact will be proved in §3 and it is crucial in our proof of Theorem 2. However, our proof works only when the singularity of M is isolated and this is not the case if $n \geq 3$, by Theorem 3.

Problem 2. Is it possible to classify the germs at $0 \in \mathbb{C}^{n+1}$, of hypersurfaces M such that $dim_{\mathbb{C}}(sing(M)) = n-2$ and there exists a germ of holomorphic map $\phi: (\mathbb{C}^n, 0) \to (M, 0)$, when $n \geq 3$? This question seems easier when we restrict to the case where M is quasi-homogeneous.

The next section of this paper will be dedicated to the proof of Theorem 1. The proof of this Theorem will be based in the existence of a holomorphic n-form on $M^* = M \setminus sing(M)$, say η, such that $\eta_p \neq 0$ for any $p \in M^*$. This form will be used also in the proofs of Theorems 2 and 3, which will be done in §3 and in §4, respectively.

§2. Proof of Theorem 1.

Let M be a germ at $0 \in \mathbb{C}^{n+1}$ of an irreducible hypersurface. It is well known that in this case we have $M = F^{-1}(0)$, where $F: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is an irreducible germ of holomorphic function. We will suppose that 0 is effectively a singularity, so that $0 \in sing(M)$ and

$$sing(M) = \{z \in U \mid dF(z) = 0\} ,$$

where U is some small ball around 0 such that F is defined. We will use the notation $M^* = M \setminus sing(M)$. Let us consider a holomorphic coordinate system in U, say $(x_0, ..., x_n): U \to \mathbb{C}^n$. Set $U_j = \{z \in U \mid F_{x_j}(z) \neq 0\}$ and $M_j = U_j \cap M$. We observe that $(M_j)_{j=0}^n$ is a covering of M^* by Stein open sets. For $j = 0, ..., n$, let η_j be the n-form on U_j defined by

$$\eta_j = \frac{(-1)^j}{F_{x_j}} dx_0 \wedge ... \wedge \overset{\wedge}{dx_j} \wedge ... \wedge dx_n \ ,$$

where the symbol $\overset{\wedge}{dx_j}$ means the omission of dx_j in the product.

Assertion 1. *If $M_{ij} = M_i \cap M_j$ then $\eta_i|_{M_{ij}} = \eta_j|_{M_{ij}}$, so that there exists a holomorphic n-form η on M^* such that $\eta|_{M_j} = \eta_j|_{M_j}$ for all $j = 0, ..., n$.*

Proof. Let Z_j be the holomorphic vector field defined on U_j by

$$Z_j = \frac{1}{F_{x_j}} . \frac{\partial}{\partial x_j} .$$

Then $\eta_j = i_{Z_j}(\nu)$, where i denotes the interior product and $\nu = dx_0 \wedge ... \wedge dx_n$. Observe that $i_{Z_j}(dF) = 1$ for all $j = 0, ..., n$. Since $dF \wedge \nu = 0$, we get

$$0 = i_{Z_j}(dF \wedge \nu) = \nu - dF \wedge \eta_j \ , \forall j \implies DF \wedge (\eta_j - \eta_i) = 0 \ , \forall i, j \implies \eta_i|_{M_{ij}} = \eta_j|_{M_{ij}} . \blacksquare$$

□

Remark 2. We observe that $\eta_p \neq 0$ for all $p \in M^*$, as the reader can check. This implies that the (n,n)-form

$$\mu = c.\eta \wedge \overline{\eta}$$

where $c = i^n.(-1)^{\frac{n(n+1)}{2}}$, is a volume form on M^*. We will use this form in the proof of Theorem 1.

Now, suppose that F is a quasi-homogeneous polynomial with respect to the vector field $X(x) = \sum_{j=0}^{n} \lambda_j \, x_j \, \frac{\partial}{\partial x_j}$, where $\lambda_0, ..., \lambda_n \in \mathbb{Q}_+$, so that,

$$(2) \quad X(F) = F(x_0, ..., x_n) = \sum_{j=0}^{n} \lambda_j \, x_j \, F_{x_j}(x_0, ..., x_n) \ .$$

In particular, M is invariant by the flow X_t of X, that is, if $p \in M$ then $X_t(p) \in M$ for all $t \in \mathbb{C}$. Now, let η be the n-form on M^* considered before.

Assertion 2. *We have $L_X(\eta) = a.\eta$, where $a = -1 + \sum_{j=0}^{n} \lambda_j$ and L_X denotes the Lie derivative along X.*

Proof. Since $\eta|_{M_0} = \eta_0|_{M_0}$ and M_0 is dense in M^*, it is sufficient to prove that $L_X(\eta_0) = a.\eta_0$. We have

$$L_X(\eta_0) = -\frac{X(F_{x_0})}{(F_{x_0})^2} \, dx_1 \wedge ... \wedge dx_n \, + \frac{1}{F_{x_0}} \sum_{j=1}^{n} dx_1 \wedge ... \wedge d(X(x_j)) \wedge ... \wedge dx_n =$$

$$= -\frac{X(F_{x_0})}{F_{x_0}} \eta_0 + \frac{1}{F_{x_0}} \sum_{j=1}^{n} \lambda_j \, dx_1 \wedge ... \wedge dx_n = \big(-\frac{X(F_{x_0})}{F_{x_0}} + \sum_{j=1}^{n} \lambda_j \big) \eta_0 \ .$$

On the other hand, if $e_0 = \frac{\partial}{\partial x_0}$, we have

$$X(F_{x_0}) = X(e_0(F)) = [X, e_0](F) + e_0(X(F)) = -\lambda_0 \, e_0(F) + e_0(F) = (1 - \lambda_0) F_{x_0} \ .$$

This implies that $L_X(\eta_0) = a.\eta_0$. □

Let us prove Theorem 1. We observe first that the condition $dim_{\mathbb{C}}(sing(M)) \leq n-2$ implies that M is irreducible, so that F is also irreducible. From now on, we fix a representative of the germ $\phi \colon (\mathbb{C}^n, 0) \to (M, 0)$, denoted again by ϕ, and some open ball of $\mathbb{C}^n$, $W \ni 0$, such that $\phi^{-1}(0) \cap W = \{0\}$. We will use the following facts, that will be proved at the end.

Assertion 3. *If W is sufficiently small, then $V := \phi(W)$ is an open neighborhood of $0 \in M$. Moreover, $\phi \colon W \to V$ satisfies the following properties :*

(i). *There exists $d \in \mathbb{N}$ such that $\#(\phi^{-1}(p)) \leq d$ for any $p \in V$.*

(ii). *ϕ is a proper and open map.*

(iii). *If $A \subset M$ is an irreducible analytic subset of complex dimension k then any irreducible component of $\phi^{-1}(A)$ has complex dimension k. In particular $cod_{\mathbb{C}}(\phi^{-1}(sing(M))) \geq 2$.*

The main fact is the following :

Lemma 1. *Let $r > 0$ be such that*

$$M_r := \{z \in M \,|\, ||x|| := \big(\sum_{j=0}^{n} |x_j|^2 \big)^{1/2} \leq r\} = M \cap \overline{B}_r(0) \subset V \ .$$

Then $vol_\mu(M_r^*) < +\infty$, *where* $M_r^* = M_r \cap M^*$, $\mu = c.\eta \wedge \overline{\eta}$ *is like in Remark 2 and*

$$vol_\mu(M_r^*) = \int_{M_r^*} \mu \ .$$

Proof . Let $\nu = \phi^*(\eta)$, which is a holomorphic n-form on $W \setminus \phi^{-1}(sing(M)$. It follows from **(iii)** of Assertion 3 and Hartogs' Theorem, that ν can be extended to a holomorphic n-form on W. This implies that the (n,n)-form $\phi^*(\mu)$ can be extended to a real analytic (n,n)-form on W. Since ϕ is proper and M_r is a compact subset of V, it follows that $\phi^{-1}(M_r)$ is a compact subset of W. This implies that

$$\int_{\phi^{-1}(M_r)} \phi^*(\mu) \ < \ +\infty \ .$$

The Lemma will follow from the following fact :

$$(3) \ \ vol_\mu(M_r^*) \ \leq \ \int_{\phi^{-1}(M_r)} \phi^*(\mu) \ < \ +\infty \ .$$

Let us prove (3). Let $C(\phi) \subset W$ and $CV(\phi) = \phi(C(\phi))$ be the sets of critical points and critical values of ϕ, respectively. Since $CV(\phi)$ has measure zero (Sard's Theorem), we have

$$\int_{M_r^*} \mu = \int_{M_r^* \setminus CV(\phi)} \mu \ .$$

In order to prove (3), it is sufficient to prove that for any open subset $A \subset M_r^* \setminus CV(\phi)$, with closure $\overline{A} \subset M_r^* \setminus CV(\phi)$, then

$$(3') \ \ \int_A \mu \leq \int_{\phi^{-1}(M_r)} \phi^*(\mu) \, .$$

Let us fix A as above. Given $p \in A$, then $\phi^{-1}(p) = \{q_1, ..., q_r\}$, where $r \leq d$ and $q_1, ..., q_r$ are regular points of ϕ. Since ϕ is proper, there exist mutually disjoints neighborhoods $B_1, ..., B_r$ of $q_1, ..., q_r$, respectively, and a neighborhood $V_p \subset A$ such that $\phi^{-1}(V_p) = \cup_{j=1}^r B_j$ and $\phi|_{B_j} : B_j \to V_p$ is a biholomorphism. It follows that

$$\int_{V_p} \mu = \int_{B_j} \phi^*(\mu) \ , \forall j = 1, ..., r \ \ \Longrightarrow \ \ \int_{V_p} \mu \leq \int_{\phi^{-1}(V_p)} \phi^*(\mu) \ .$$

Now, consider a partition of the unity $(\varphi_p)_{p \in A}$ subordinated to the covering $(V_p)_{p \in A}$. We have,

$$\int_A \mu = \sum_p \int_{V_p} \varphi_p . \mu \ \leq \ \sum_p \int_{\phi^{-1}(V_p)} \varphi_p \circ \phi . \ \phi^*(\mu) = \int_{\phi^{-1}(A)} \phi^*(\mu) \ \leq \ \int_{\phi^{-1}(M_r)} \phi^*(\mu) \ .$$

This proves the Lemma. $\square$

The following Lemma implies the Theorem.

Lemma 2. *Let* M_r *and* a *be as before. If* $a \leq 0$, *then* $vol_\mu(M_r^*) = +\infty$.

Proof. Since $L_X(\eta) = a.\eta$ (Assertion 2), we have $X_t^*(\eta) = e^{at}.\eta$, where X_t denotes the flow of X, $X_t(x_0, ..., x_n) = (e^{\lambda_0 t}.x_0, ..., e^{\lambda_n t}.x_n)$. This implies that, $X_t^*(\mu) = e^{2at}.\mu$, for $t \in \mathbb{R}$, as the reader can check. Remark that the real flow X_t, $t \in \mathbb{R}$, is transversal and points outward the spheres of radius r, $S_r^{2n+1} \subset \mathbb{C}^{n+1}$, because $\lambda_0, ..., \lambda_n > 0$. Observe also that M^* is invariant for the flow X_t. This follows from the fact that $X(F_{x_j}) = (1 - \lambda_j).F_{x_j}$, for all $j = 0, ..., n$. This implies that, if $t > 0$, then $M_r^* \subset int(X_t(M_r^*)) \subset M^*$. Now, suppose by contradiction that $vol_\mu(M_r^*) < +\infty$. Since $M_r^* \subset int(X_t(M_r^*)) \subset M^*$, for $t > 0$, we should have

$$t > 0 \implies vol_\mu(M_r^*) < vol_\mu(X_t(M_r^*)) = \int_{X_t(M_r^*)} \mu = \int_{M_r^*} X_t^*(\mu) = e^{2at}.\int_{M_r^*} \mu \leq vol_\mu(M_r^*) ,$$

because $a \leq 0$. This contradiction implies the Lemma. $\square$

It remains to prove Assertion 3.

Proof of Assertion 3. It follows from Weierstrass preparation Theorem, that we can choose a local holomorphic coordinate system $(U, (z, x_1, ..., x_n) = (z, x))$, around $0 \in \mathbb{C}^{n+1}$ such that $F|_U = u(z, x).(z^k + a_1(x)\, z^{k-1} + ... + a_k(x))$, where $a_1(0) = ... = a_n(0) = 0$ and $u(z, x) \neq 0$ for all $(z, x) \in U$, so that we can suppose that $F(z, x) = z^k + a_1(x)\, z^{k-1} + ... + a_k(x)$. Let $\pi: \mathbb{C}^{n+1} \to \mathbb{C}^n$ be the second projection $\pi(x, z) = z$. We can chose U in such a way that $\pi|_{U_1}: U_1 \to \pi(U_1) \subset \mathbb{C}^n$ is a k-sheeted branched covering (cf. [G-H]), where $U_1 = U \cap M$. Let $W \subset \mathbb{C}^n$ be a ball around 0, such that $\phi(W) \subset U_1$ and $\phi^{-1}(0) \cap W = \{0\}$. Set $\phi = (\phi_0, \phi_1, ..., \phi_n) = (\phi_0, \psi)$. It follows from $a_1(0) = ... = a_n(0) = 0$ that $\psi^{-1}(0) \cap W = \{0\}$, so that the intersection number, say d_1, of $\phi_1, ..., \phi_n$, at $0 \in \mathbb{C}^n$, is finite and greater than zero. This implies that $\psi(W) := W_1$ is open and that for any $q \in W_1$, $\psi^{-1}(q)$ contains $d_1 \geq 1$ points in W, counted with multiplicity. It follows that for any $p \in \pi^{-1}(W_1) \cap M := V$, we must have $\#(\phi^{-1}(p)) \leq d_1$, which proves **(i)**. Observe that this implies also that $\psi: W \to W_1$ is proper. Since $\psi = \pi \circ \phi$ and ψ is proper and open, ϕ is also proper and open, which proves **(ii)**.

Now, let A be an irreducible analytic subset of V of dimension ℓ. Since $\pi|_{U_1}$ is a k-sheeted branched covering, $\pi(A)$ is an irreducible analytic subset of W_1 of dimension ℓ. It is well known that any irreducible component of $\psi^{-1}(\pi(A))$ has dimension ℓ. But an irreducible component of $\phi^{-1}(A)$ is also an irreducible component of $\psi^{-1}(\pi(A))$. This implies **(iii)**. Finally, since $dim_{\mathbb{C}}(sing(M)) \leq n - 2$, we must have $dim_{\mathbb{C}}(\phi^{-1}(sing(M))) \leq n - 2$ and so $codim_{\mathbb{C}}(\phi^{-1}(sing(M))) \geq 2$. $\square$

Remark 3. We observe that the above proof is also valid for germs of hypersurfaces at $0 \in \mathbb{C}^{n+1}$. In fact, it is possible to find arbitrarily small neighborhoods U of 0 in $\mathbb{C}^n$ and V of 0 in $\mathbb{C}^{n+1}$ such that $\phi: U \to V \cap M$ is defined, $\phi^{-1}(V \cap M) = U$ and $\phi|_U$ is a finite ramified covering, in the following sense : let $CP(\phi) \subset U$ be the set of critical points of ϕ. This set can be defined by $\{f = 0\}$, where f is given by $\phi^*(\eta) = f.du_1 \wedge ... \wedge du_n$, and $(u_1, ..., u_n)$ denotes an affine coordinate system of $\mathbb{C}^n$. Set $CV(\phi) = \phi(CP(\phi)) \cup sing(M)$ (the critical values of ϕ) and $W = \phi^{-1}(CV(\phi))$. Then $\phi|_{U \setminus W}: U \setminus W \to V \setminus CV(\phi)$ is a finite covering map. In particular, given $p \in V \setminus CV(\phi)$, the *local degree* of ϕ at $0 \in \mathbb{C}^n$ is

$$deg(\phi, 0) = \#\{q \in U \setminus W |\, \phi(q) = p\} .$$

We observe that $deg(\phi, 0)$ can be calculated as follows : Let $(U, (z, x_1, ..., x_n) = (z, x))$ be holomorphic coordinates around $0 \in \mathbb{C}^{n+1}$ such that $M \cap U = (F = 0)$, where $F(z,x) = z^k + a_1(x)\, z^{k-1} + ... + a_k(x)$, $a_1(0) = ... = a_k(0) = 0$ and $\pi|_{M\cap U}$ is a ramified covering of k-sheets, $\pi(z,x) = x$. Set $\pi \circ \phi = (\phi_1, ..., \phi_n)$ and let $d_1 = [\phi_1, ..., \phi_n]_0$ (the intersection number of $\phi_1, ..., \phi_n$ at $0 \in \mathbb{C}^n$). Then

$$(4)\ \ deg(\phi, 0) = \frac{d_1}{k}$$

In particular, if the components of ϕ are homogeneous polynomials we have

$$(4')\ \ deg(\phi, 0) = \frac{deg(\phi_1)...deg(\phi_n)}{k}$$

§3. Proof of Theorem 2.

The proof will be divided in three steps :

1^{st}-step. We will prove that there exists a germ of holomorphic vector field at $0 \in \mathbb{C}^3$, say X, such $X(F) = F$, where $F = 0$ is a reduced equation of M. In this case, it follows from a Theorem of Saito (cf. [S]), that there exists a germ of biholomorphism $\psi: (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ such that :

(I). $\psi_*(X) = \lambda_1\, x \frac{\partial}{\partial x} + \lambda_2\, y \frac{\partial}{\partial y} + \lambda_3\, z \frac{\partial}{\partial z}$, where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Q}_+$.

(II). $F \circ \psi^{-1}$ is a quasi-homogeneous polynomial with respect to the vector field $\psi_*(X)$.

2^{nd}-step. We will prove that if F is a quasi-homogeneous polynomial with respect to $X = \lambda_1\, x \frac{\partial}{\partial x} + \lambda_2\, y \frac{\partial}{\partial y} + \lambda_3\, z \frac{\partial}{\partial z}$, where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Q}_+$ and $\lambda_1 + \lambda_2 + \lambda_3 > 1$, then F is equivalent to one of the forms in **(a)**, **(b)**, **(c)** or **(d)** in the statement of Theorem 2.

3^{rd}-step. We will prove that the surfaces in **(a)**, **(b)**, **(c)** and **(d)** are two by two non-equivalent.

Proof of the first step. We will divide the proof in two Lemmas.

Let M be a germ of hypersurface at $0 \in \mathbb{C}^3$, with an isolated singularity at 0, given by a reduced equation $F = 0$, where $F \in \mathcal{O}_3$. Consider the 2-form η, as defined in §2. Suppose that there exists a germ of holomorphic map $\phi: (\mathbb{C}^2, 0) \to (M, 0)$ such that $\phi^{-1}(0) = \{0\}$. Given a neighborhood V of $0 \in \mathbb{C}^3$ such that F is defined (that is has a representative $F: V \to \mathbb{C}$) and $sing(F) \cap V = \{0\}$, we will use the notations $M_V = \{p \in V |\, F(p) = 0\}$ and $M_V^* = M_V \setminus \{0\}$. When there is no confusion, we will use the notations $M_V = M$ and $M_V^* = M^*$.

Lemma 3. *If V is sufficiently small, then there exists a holomorphic 1-form ω on M_V^*, such that $d\omega = \eta$.*

Proof. Fix neighborhoods U of $0 \in \mathbb{C}^2$ and V of $0 \in \mathbb{C}^3$ such that F has a representative $F: V \to \mathbb{C}$, ϕ has a representative $\phi: U \to \mathbb{C}^3$ and $\phi(U) \subset V$. As we have seen before the form $\phi^*(\eta)$ extends to a closed holomorphic 2-form on U, say η_1. Since η_1 is closed, it follows from Poincaré Lemma that $\eta_1 = d\alpha$ in a small neighborhood of $0 \in \mathbb{C}^2$, where α is holomorphic. Therefore, if we take U and V small enough, we can suppose that :

(i). α is defined in U.

(ii). ϕ has a representative $\phi: U \to M_V$.

We can choose U and V in such a way that :

(iii). $\phi^{-1}(M_V) = U$ and $\phi: U \setminus W \to M_V \setminus CV(\phi)$ is a covering map with d sheets, where W and $CV(\phi)$ are as in Remark 3 of §2. We will use the notation $\hat{M}$ for $M_V \setminus CV(\phi)$.

We will use α to construct the a form ω on M_V such that $d\omega = \eta$. Let us construct first ω on $\hat{M}$.

It follows from **(iii)** that, given a point $p \in \hat{M}$, there exists a neighborhood $V_p \subset \hat{M}$ of p and neighborhoods $U_p^1, ..., U_p^d$ of $q_1, ..., q_d$, where $\phi^{-1}(p) = \{q_1, ..., q_d\}$, such that

(iv). V_p is biholomorphic to a ball in $\mathbb{C}^2$.

(v). $U_p^i \cap U_p^j = \emptyset$, for $i \neq j$.

(vi). $\phi_p^j := \phi|_{U_p^j}: U_p^j \to V_p$ is a biholomorphism.

For each $j = 1, ..., d$, consider the 1-form β_p^j on V_p defined by $\beta_p^j = ((\phi_p^j)^{-1})^*(\alpha)$. Since $\phi^*(\eta) = d\alpha$, we have $d\beta_p^j = \eta|_{U_p}$. Define a 1-form ω_p on V_p by

$$(5) \quad \omega_p = \frac{1}{d} \sum_{j=1}^{d} \beta_p^j \ .$$

Observe that $d\omega_p = \eta$. By standards arguments, we can construct a covering $\mathcal{V} = \{V_p\}_{p \in \hat{M}}$ of $\hat{M}$ by connected open sets, and a collection of holomorphic 1-forms $\{\omega_p\}_{p \in \hat{M}}$, $\omega_p \in \Omega^1(V_p)$, such that

(vii). If $V_p \cap V_q \neq \emptyset$ then $V_p \cap V_q$ is contractible.

(viii). $d\omega_p = \eta|_{V_p}$ for all p.

By taking the $V_{p's}$ small, we can suppose also that

(ix). If $V_p \cap V_q \neq \emptyset$, $\phi^{-1}(V_p) = \cup_{j=1}^d U_p^j$ and $\phi^{-1}(V_q) = \cup_{j=1}^d U_q^j$, then for every $1 \leq j \leq d$, there exists an unique $k = k(j) \in \{1, ..., d\}$ such that $U_p^j \cap U_q^k \neq \emptyset$.

We assert that, if $V_p \cap V_q \neq \emptyset$ then $\omega_p = \omega_q$ on $V_p \cap V_q$. In fact, let $\phi^{-1}(p) = \{p_1, ..., p_d\}$, $\phi^{-1}(q) = \{q_1, ..., q_d\}$, $\phi^{-1}(V_p) = \cup_{j=1}^d U_p^j$ and $\phi^{-1}(V_q) = \cup_{j=1}^d U_q^j$, be as in **(ix)**. Given $1 \leq j \leq d$, let $k \in \{1, ..., d\}$ be such that $U_p^j \cap U_q^k \neq \emptyset$. Since $\phi_p^j = \phi_q^k = \phi$ on $U_p^j \cap U_q^k$, we get from the construction that $\beta_p^j = \beta_q^k$ on $V_p \cap V_q$. This implies that $\omega_p = \omega_q$ on $V_p \cap V_q$.

It follows that we can define a 1-form ω on $\hat{M}$ such that $d\omega = \eta$. It remains to prove that ω extends to M_V^*. We will use the local forms for ϕ near a singular point. Observe first, that if we take V sufficiently small, then $C := CV(\phi) \cap M_V$ and $W = \phi^{-1}(CV(\phi) \cap U)$ are curves such that $sing(C) = \{0\}$ and $sing(W) = \{0\}$. Remark also that, if $C = \cup_j C_j$ and $W = \cup_k W_k$ is the decomposition of C and W into irreducible components, then for each k there exists an unique j such that $\phi(W_k) = C_j$. Moreover, if $q \in W_k \setminus \{0\}$ and $p = \phi(q)$ then $D\phi|_{T_q W_k}: T_q W_k \to T_p C_j$ is an isomorphism. Therefore we can find holomorphic coordinate systems $(U_q, (u, v))$ and $(V_p, (x, y))$ around q and p respectively, such that

(x). $U_q = \{(u, v) \in \mathbb{C}^2 \,|\, |u| < 1, |v| < 1\}$, $V_p = \{(x, y) \in \mathbb{C}^2 \,|\, |x| < 1, |y| < 1\}$, $W \cap U_q = W_k \cap U_q = \{v = 0\}$ and $CV(\phi) \cap V_p = C_j \cap V_p = \{y = 0\}$.

(xi). $\phi(u, v) = (X(u, v), Y(u, v)) = (u, v^n)$, for some $n \geq 1$ (Whitney's local forms).

Observe that on V_p we have $\eta = h(x, y).dx \wedge dy = d(H(x, y)dx)$, where $H_y = -h$. Therefore, $\phi^*(\eta) = d(H(u, v^n)\, du) = d\alpha$ on U_q, so that $\alpha|_{U_q} = H(u, v^n)\, du + dg(u, v)$, where $g \in \mathcal{O}(U_q)$. Let $g(u, v) = \sum_{j=0}^{\infty} g_j(u) v^j$.

Now, fix $p_0 = (x_0, y_0) \in V_p \setminus \{y = 0\}$ and let $\phi^{-1}(p_0) = \{q_1, ..., q_d\}$. Since $\phi(u,v) = (u, v^n)$ for $(u,v) \in U_q$, then $U_q \cap \phi^{-1}(p_0)$ contains n points, say $q_1, ..., q_n$, where $q_j = (x_0, \delta^j.v_0)$, δ is a primitive n^{th}-root of the unity and $v_0^n = y_0$. Let $D \subset \{y|\, 0 < |y| < 1\}$ be a small disk centered at y_0 and $b(y) = y^{1/n}$ be the branch of the n^{th}-root of y, defined in D and such that $b(y_0) = v_0$. It follows from the definition of $\beta^k_{p_0}$ that, for $k = 1, ..., n$, we have $\beta^k_{p_0} = H(x,y)dx + dg_k(x,y)$, in a small neighborhood of p_0, where

$$g_k(x,y) = \sum_{j=0}^{\infty} g_j(x)\, \delta^{kj}\, (b(y))^j \ .$$

Hence,

$$\sum_{k=1}^{n} \beta^k_{p_0} = n\, H(x,y)\, dx + d\Big[\sum_{j=0}^{\infty} \Big(\sum_{k=1}^{n} \delta^{kj}\Big)\, g_j(x)\, (b(y))^j\Big] = n\, H(x,y)\, dx + n\, d\Big[\sum_{j=0}^{\infty} g_{jn}(x)\, y^j\Big] \ ,$$

because $\sum_{k=1}^{n} \delta^{kj} = 0$ if n does not divide j. This implies that the form $\sum_{k=1}^{n} \beta^k_{p_0}$ extends to a holomorphic 1-form on V_p, say β, such that $d\beta = n\,\eta$. Using the same argument in the other points of $\phi^{-1}(p) \subset W$, it is possible to prove that $\sum_{k=n+1}^{d} \beta^k_{p_0}$ (p_0 near p), extends to a holomorphic 1-form defined in a neighborhood of p, say β_1, such that $d\beta_1 = (d-n)\eta$. Since $\frac{1}{d}(\beta + \beta_1) = \omega$, we get that ω extends to a neighborhood of p. $\square$

Let us construct a germ at $0 \in \mathbb{C}^3$, of holomorphic vector field X, such that $X(F) = F$. Let α be as in Lemma 3. Since $\eta_p \neq 0$ for all $p \in M^*$, we get that $\alpha = i_Y(\eta)$, where Y is a holomorphic vector field on M^*. It is well known that Y can be extended to a holomorphic vector field defined in a neighborhood of $0 \in \mathbb{C}^3$ (cf. [LN]). We will denote this extension by the same letter. Since M is invariant for Y we have $Y(F) = h.F$, where $h \in \mathcal{O}_3$. If $h(0) \neq 0$, then we set $X = \frac{1}{h}.Y$, so that X is a germ of holomorphic vector field at $0 \in \mathbb{C}^3$ such that $X(F) = F$, and we are done. Therefore, we have only to prove that $h(0) = 0$ leads to a contradiction. Remark that $Y(0) = 0$, because 0 is a singular point of M. Let $L = DY(0)$ be the linear part of Y at 0.

Assertion 4. *Let $Y = Y_1 \frac{\partial}{\partial x} + Y_2 \frac{\partial}{\partial y} + Y_3 \frac{\partial}{\partial z}$, where $Y_1, Y_2, Y_3 \in \mathcal{O}_3$. Then :*

$$Y_{1x} + Y_{2y} + Y_{3z} = 1 + h$$

on M. In particular, if $h(0) = 0$, then $trace(L) = 1$ and L has at least one non zero eigenvalue.

Proof. Observe first that $L_Y(\eta) = i_Y(d\eta) + d(i_Y(\eta)) = d\alpha = \eta$. In $M_3 = \{p \in \mathbb{C}^3 |\, F_z(p) \neq 0\}$, we have $\eta = \frac{dx\wedge dy}{F_z}$, so that on M_3 we have

$$(6)\quad L_Y(\eta) = L_Y\Big(\frac{dx \wedge dy}{F_z}\Big) = \frac{1}{F_z}(dY_1 \wedge dy + dx \wedge dY_2) - \frac{Y(F_z)}{F_z}\,\eta =$$

$$= \Big(Y_{1x} + Y_{2y} - \frac{Y(F_z)}{F_z}\Big)\,\eta - \frac{1}{F_z}(Y_{1z}\, dy\wedge dz + Y_{2z} dz \wedge dx) = \Big(Y_{1x} + Y_{2y} - \frac{Y(F_z)}{F_z} - \frac{F_x\, Y_{1z}}{F_z} - \frac{F_y\, Y_{2z}}{F_z}\Big)\eta$$

because $\eta = \frac{dy \wedge dz}{F_x} = \frac{dz \wedge dx}{F_y}$ on $\{F_x \neq 0\} \cap \{F_y \neq 0\}$. On the other hand,

$$Y(F_z) = Y(\frac{\partial}{\partial z}(F)) = [Y, \frac{\partial}{\partial z}](F) + \frac{\partial}{\partial z}(Y(F)) = -Y_{1z}.F_x - Y_{2z}.F_y - Y_{3z}.F_z + h_z.F + h.F_z$$

By substituting the above expression in (6) and using that $M = \{F = 0\}$, we obtain

$$L_Y(\eta) = (Y_{1x} + Y_{2y} + Y_{3z} - h)\eta$$

which implies that $Y_{1x} + Y_{2y} + Y_{3z} - h \equiv 1$ on M. If $h(0) = 0$, we get

$$trace(L) = Y_{1x}(0) + Y_{2y}(0) + Y_{3z}(0) = 1 \ .$$

□

Let N be a a germ at $0 \in \mathbb{C}^3$ of a holomorphic submanifold of dimension k, $k \in \{1, 2, 3\}$. We will say that it is an *invariant manifold of Poincaré type* for the vector field Y (brifly i.m.P.t) if

(I). N is smooth at 0 and invariant for Y.

(II). $L|_{T_0 N}$ is in the Poincaré domain, in the sense that its eigenvalues are non zero (observe that $L(T_0 N) = T_0 N$) and there exists a line $\ell \subset \mathbb{C}$, $0 \in \ell$, such that all eigenvalues of $L|_{T_0 N}$ are contained in one of the components of $\mathbb{C} \setminus \ell$.

Lemma 4. *If $h(0) = 0$ and N is an i.m.P.t. for Y, then*

(a). *$N \subset M$. In particular, $dim(N) = 1$.*

(b). *If N has dimension one and the eigenvalue of $L|_{T_0 N}$ is $\lambda \neq 0$, then $-n\,\lambda$ is also eigenvalue of L, where $n \in \mathbb{N}$. In particular the eigenvalues of L are λ, $-n.\lambda$ and $1 + n.\lambda - \lambda$.*

Proof. Let us prove **(a)**. We denote the local flow of Y by $Y_T = (Y_T^1, Y_T^2, Y_T^3)$, $T \in \mathbb{C}$. Since N is smooth, we can suppose that $N \subset T_0\, N = \Sigma$, where Σ is a linear subspace of $\mathbb{C}^3$ and $L(\Sigma) = \Sigma$. Since the eigenvalues of $L|_\Sigma$ are in the Poincaré domain, there exists $\alpha \in \mathbb{C}^*$ such that the eigenvalues of $\alpha.L|_\Sigma$ have negative real part. Let $Z = \alpha.Y|_N$, $g = \alpha.h$ and $a > 0$ be such that $a < min\{-Re(\lambda) |\, \lambda$ is an eigenvalue of $Z\}$. It is well known (cf. [Ar]) that there exists a neighborhood U of $0 \in N$ such that

(i). For any $p \in U$, $Z_t(p)$ is defined for all $t > 0$.

(ii). If $t > 0$ and $p \in U$ then $||Z_t(p)|| \leq C.e^{-at}$, where $C > 0$.

Since $Z(F) = g.F$, it follows from **(i)** and **(ii)** that

(iii). If $p \in U$ and $t > 0$ then

$$F(Z_t(p)) = exp\Big(\int_0^t g(Z_s(p))ds\Big) \,.\, F(p) \ .$$

Observe that $g(0) = 0$ and **(ii)** imply that there exists $A > 0$ such that $|g(Z_s(p))| \leq A.e^{-at}$, so that $\int_0^\infty g(Z_s(p))ds$ is convergent, say $\int_0^\infty g(Z_s(p))ds = b \in \mathbb{C}$. It follows from **(iii)** that

$$e^b.F(p) = \lim_{t \to \infty} F(Z_t(p)) = 0 \implies F(p) = 0 \implies p \in M \implies N \subset M.$$

Since $dim_{\mathbb{C}}(M) = 2$ we must have $dim_{\mathbb{C}}(N) \leq 2$. On the other hand, since M is irreducible and is singular at $0 \in \mathbb{C}^3$, we must have $dim_{\mathbb{C}}(N) = 1$. This proves **(a)**.

Let us prove **(b)**. We can suppose that $N \subset \{(x,0,0) | x \in \mathbb{C}\}$. This implies that $F(x,y,z) = y.A(x,y,z) + z.B(x,y,z)$, where $A, B \in \mathcal{O}_3$. It follows from Poincaré's linearization Theorem (cf. [Ar]) that we can find a local coordinate system $x \in \mathbb{C}$ such that $Y(x,0,0) = \lambda\, x\, \frac{\partial}{\partial x}$, so that $Y_T(x,0,0) = (e^{\lambda\, T}.x, 0, 0)$. We can suppose also that $L = DY(0)$ is in Jordan's canonical form, so that the eigenvalues of L are $\frac{\partial Y_1}{\partial x}(0) = \lambda$, $\frac{\partial Y_2}{\partial y}(0)$ and $\frac{\partial Y_3}{\partial z}(0)$.

Observe that $Y(F) = h.F$ implies that $L_Y(dF) = d(h.F) = h.dF + F.dh$. Therefore, for $p = (x,0,0)$ we get

$$(7)\ \ L_Y(dF)(p) = h(p).dF(p) \implies [Y_T^*(dF)](p) = exp\Big(\int_0^T h(Y_s(p))ds\Big) \,.\, dF(p)\ .$$

On the other hand, $dF(x,0,0) = A(x,0,0)\, dy + B(x,0,0)\, dz = x^k.u(x)\, dy + x^\ell.v(x)\, dz$, where $k, \ell \geq 1$, $u(0).v(0) \neq 0$. We can suppose, without lost of generality, that $k \leq \ell$, and so

$$dF(x,0,0) = x^k.u(x)\,(dy + x^m\, v_1(x)\, dz)\ ,$$

where $m = \ell - k \geq 0$ and $v_1 = \frac{v}{u}$. Now, the change of variables $x_1 = x$, $y_1 = y + x^m\, v_1(x)\, z$, $z_1 = z$ is a biholomorphism near $0 \in \mathbb{C}^3$ and in these new coordinates we have $dF(x_1,0,0) = x_1^k\, u(x_1)\, dy_1$, so that, returning to the old notation, we can suppose that

$$(8)\ \ dF(x,0,0) = x^k\, u(x)\, dy\ .$$

Observe that after this change of variables $\frac{\partial Y_2}{\partial y}(0)$ is still an eigenvalue of L. We leave this computation for the reader. We are going to prove that $\frac{\partial Y_2}{\partial y}(0) = -k\,\lambda$.

If we set $Y_T = (Y_T^1, Y_T^2, Y_T^3)$ and $H(T,x) = exp\big(\int_0^T h(Y_s(x,0,0))ds\big)$, we get from (7) and (8) that

$$e^{k\lambda T} x^k.u(e^{\lambda T}\, x).dY_T^2(x,0,0) = H(T,x).x^k\, u(x)\, dy \implies dY_T^2(x,0,0) = H(T,x)\frac{u(x)}{u(e^{\lambda T} x)}.e^{-k\lambda T}.dy$$

so that

$$(9)\ \ \frac{\partial Y_T^2}{\partial y}(x,0,0) = H(T,x)\frac{u(x)}{u(e^{\lambda T} x)}.e^{-k\lambda T}\ .$$

Now, since $\frac{\partial Y_T}{\partial T} = Y(Y_T)$, we have

$$\frac{\partial^2\, Y_T^2}{\partial T \partial y}(p) = dY_2(Y_T(p)).\frac{\partial Y_T}{\partial y}(p)\ .$$

Since $Y_0(p) = p$, we have $\frac{\partial Y_0}{\partial y}(x,y,z) = (0,1,0)$, so that if we set $T = 0$ and $p = (x,0,0)$ in the above relation, we get

$$\frac{\partial^2\, Y_T^2}{\partial T \partial y}|_{(T=0,p)} = dY_2(x,0,0).(0,1,0) = \frac{\partial Y_2}{\partial y}(x,0,0) \implies$$

$$(10)\ \ \frac{\partial^2\, Y_T^2}{\partial T \partial y}|_{(T=0,0)} = \frac{\partial Y_2}{\partial y}(0)$$

Now, (9) implies that

$$\frac{\partial^2 Y_T^2}{\partial T \partial y}(x,0,0) = e^{-k\lambda T}\{\frac{\partial}{\partial T}\left[H(T,x)\frac{u(x)}{u(e^{\lambda T}x)}\right] - k\lambda\, H(T,x)\frac{u(x)}{u(e^{\lambda T}x)}\} \ .$$

Since $H(0,x)=1$, $\frac{\partial H(T,x)}{\partial T} = h(Y_T(x,0,0)).H(T,x)$ and $\frac{\partial\, u(e^{\lambda T}x)}{\partial T} = \lambda\, e^{\lambda T} x\, u'(e^{\lambda T}x)$, we get for $T=0$

$$\frac{\partial^2 Y_T^2}{\partial T \partial y}|_{(T=0,(x,0,0))} = h(x,0,0) - \lambda \frac{x.u'(x)}{u(x)} - k\lambda$$

This together with (10) gives

$$\frac{\partial Y_2}{\partial y}(0) = -k\,\lambda$$

which implies that $-k\,\lambda$ is an eigenvalue of L. □

We will use the following result, which is a consequence of the strong stable manifold Theorem (cf. [H-P-S] pg. 60):

Theorem. *Let Z be a germ at $0 \in \mathbb{C}^n$ of holomorphic vector field such that $Z(0) = 0$. Let $\{\lambda_1, ..., \lambda_n\}$ be the spectrum of $DZ(0) := L$, where $\lambda_{k+1} = ... = \lambda_n = 0$ and $0 \notin S = \{\lambda_1, ..., \lambda_k\}$, $1 \le k \le n$. Let ℓ be a straight line through $0 \in \mathbb{C}$ such that $\ell \cap S = \emptyset$ and the connected components of $\mathbb{C} \setminus \ell$ be A_1 and A_2. Let E_k be the linear subspace of $\mathbb{C}^n$ corresponding to the eigenvalues of L on A_k. Then there are germs of holomorphic invariant submanifolds of Poincaré type W_k, $k=1,2$, such that $T_0\, W_k = E_k$.*

Let $S = \{\lambda_1, \lambda_2, \lambda_3\}$ be the spectrum of $L = DY(0)$. It follows from $trace(L) = 1$, Lemma 4 and the above Theorem that one of the eigenvalues of L is zero and the other two are in the same line ℓ through $0 \in \mathbb{C}$, in such a way that $0 \in \ell$ is between them.

In fact, if the three eigenvalues were non-zero, then either L is in the Poincaré domain, or there exists a line ℓ through $0 \in \mathbb{C}$ such that two of the eigenvalues are in the same half plane of $\mathbb{C} \setminus \ell$. In the first case, we would have $F \equiv 0$ and in the second case 0 would not be a singularity of M, by the Theorem and by Lemma 4. Therefore, one of the eigenvalues is zero and we can suppose that $\lambda_3 = 0$ and $\lambda_1 \neq 0$. If $\lambda_2 = 0$, then by the Theorem, there exists an i.m.P.t., say N_1, of dimension one such that $T_0\, N_1$ is tangent to the subspace correspondent to λ_1. In this case, it follows from Lemma 4 that $-k.\lambda_1$, $k \in \mathbb{N}$, is an eigenvalue of L, so that $\lambda_2 \neq 0$. Therefore, the unique possibility is $\lambda_1\,,\, \lambda_2 \neq 0$ and $\lambda_3 = 0$. However, in this case, the Theorem implies that there is no line ℓ through $0 \in \mathbb{C}$ such that λ_1 and λ_2 are in the same component of $\mathbb{C} \setminus \ell$ ($0 \in \mathbb{C}^3$ would not be a singularity of M). Therefore, λ_1 and λ_2 are in the same line ℓ through $0 \in \mathbb{C}$ and $\lambda_3 = 0$ is between them, which proves the assertion.

In the above situation, the Theorem implies that there are invariant manifolds N_1 and N_2 of P.t., tangent at 0 to the subspaces correspondent to λ_1 and λ_2. It follows from Lemma 4 that $\lambda_2 = -k.\lambda_1$, $k \in \mathbb{N}$, and similarly $\lambda_1 = -m.\lambda_2$, $m \in \mathbb{N}$, so that $k.m = 1$ and $k = m = 1$. But this implies that $\lambda_1 = -\lambda_2$, so that $trace(L) = 0$, a contradiction. This finishes the proof of the first step.

Proof of the second step. Let $M = F^{-1}(0)$, where $F: \mathbb{C}^3 \to \mathbb{C}$ is a polynomial with an isolated singularity at $0 \in \mathbb{C}^3$, quasi-homogeneous with respect to the vector field $X = \lambda_1 x \frac{\partial}{\partial x} + \lambda_2 y \frac{\partial}{\partial y} + \lambda_3 z \frac{\partial}{\partial z}$, where $X(F) = F$ and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Q}_+$. Suppose that there exists a germ of holomorphic map $\phi: (\mathbb{C}^2, 0) \to (M, 0)$ such that $\phi^{-1}(0) = \{0\}$. We are going to prove that M is equivalent to one of the following surfaces :

(a). $M_{(p,q,r)}$, where $(p,q,r) \in \{(2,2,r), (2,3,3), (2,3,4), (2,3,5)\}$.
(b). $M_b = \{(x,y,z) \in \mathbb{C}^3 \,|\, z^2 = xy(y - x^{m+1})\}$, where $m \geq 1$.
(c). $M_c = \{(x,y,z) \in \mathbb{C}^3 \,|\, z^2 = y(y^2 + x^3)$.
(d). $M_d = \{(x,y,z) \in \mathbb{C}^3 \,|\, z^2 = x(y^2 + x^{2m+1})\}$, where $m \geq 1$.

We can suppose that $\lambda_1 \leq \lambda_2 \leq \lambda_3$.

Assertion 5. We assert that $F_{xyz} \equiv F_{yyz} \equiv F_{xzz} \equiv F_{yzz} \equiv F_{zzz} \equiv 0$. In particular, we must have

$$F(x,y,z) = A(x,y) + B(x,y)\, z + C\, z^2$$

where $X(A) = A$, $X(B) = (1 - \lambda_3)B$ and C is a constant.
Proof. The assertion follows from the following fact : if G is a germ of holomorphic function at $0 \in \mathbb{C}^3$ such that $X(G) = -a.G$, where $a > 0$, then $G \equiv 0$. In fact, this implies that $G(X_t(p)) = e^{-at}.G(p)$ for all $t \in \mathbb{R}$ and $p \in \mathbb{C}^3$, where X_t is the flow of X. Since $\lim_{t\to-\infty} X_t(p) = 0$ and $\lim_{t\to-\infty} e^{-at} = +\infty$, we must have $G(p) = 0$ for all $p \in \mathbb{C}^3$.

Let us consider $G = F_{xyz}$. As the reader can check, we have $X(F_{xyz}) = (1 - \lambda_1 - \lambda_2 - \lambda_3).F_{xyz}$, where $1 - \lambda_1 - \lambda_2 - \lambda_3 < 0$, by Theorem 1. Therefore $F_{xyz} \equiv 0$. The other cases are similar. $\square$

Now, $F_{zzz} \equiv 0$, implies that

$$F(x,y,z) = A(x,y) + B(x,y)\, z + C(x,y)\, z^2 \ ,$$

where A, B, C are polynomials. Since $F_{xzz} \equiv F_{yzz} \equiv 0$, C is a constant. The fact that $X(F) = F$ implies that $X(A) = A$ and $X(B) = (1 - \lambda_3)B$, as can be easily checked. $\square$

Now, we have two possibilities : either **(i)** $C = 0$, or **(ii)** $C \neq 0$.
Case (i). $C = 0$. In this case, we have $F = A + B\, z$. Since $F_{xyz} \equiv 0$ we must have $B(x,y) = a\, x^m + b\, y^n$, where $m, n \geq 1$ and either $a \neq 0$, or $b \neq 0$. We assert that, either $m = 1$, or $n = 1$. In fact, since $0 \in \mathbb{C}^3$ is a singularity of M, we have $A_x(0,0) = A_y(0,0) = 0$. If $m, n \geq 2$, then $F_x(0,0,z) = F_y(0,0,z) = F_z(0,0,z) = 0$, so that 0 would not be an isolated singularity of M.
Case (i.1). $C = 0$, $a = 0$ and $b \neq 0$. This implies that $n = 1$ and that $F = A(x,y) + b\, y\, z$. After the linear change of variables $z_1 = b\, z$, we can suppose that $b = 1$ and $F = A(x,y) + y\, z$. Now, since F is irreducible, y does not divide $A(x,y)$, which implies that $A(x,y) = c\, x^p + y\, A_1(x,y)$, where $c \neq 0$, $p \geq 2$ and $X(A_1) = (1 - \lambda_2)A_1$. Consider the (global) change of variables $x_1 = x$, $y_1 = y$, $z_1 = z + A_1(x,y)$. In the new variables, we have

$$F(x_1, y_1, z_1) = y_1(z_1 - A_1(x_1, y_1)) + c\, x_1^p + y_1\, A_1(x_1, y_1) = y_1\, z_1 + c\, x_1^p.$$

Finally, the change of variables $y_1 = y_2 + i\,z_2$, $z_1 = y_2 - i\,z_2$, and $x_1 = k\,x_2$, where $k^p = 1/c$, transforms F into $x_2^p + y_2^2 + z_2^2$, and so M is equivalent to $M_{(2,2,p)}$.
Case (i.2). $C = 0$, and $a, b \neq 0$. As in case **(i.1)**, we can suppose that $b = 1$. Observe that $X(B) = (1-\lambda_3)B$ and $a \neq 0$, imply that $m\,\lambda_1 = n\,\lambda_2 = 1 - \lambda_3$, so that $m \geq n$, because $\lambda_1 \leq \lambda_2$. Hence $n = 1$ and $F = A(x,y) + z(y + a\,x^m)$. After the change of variables $x_1 = x$, $y_1 = y + a\,x^m$, $z_1 = z$, we get

$$F(x_1, y_1, z_1) = y_1\,z_1 + A(x_1, y_1 - a\,x_1^m) = y_1\,z_1 + A_1(x_1, y_1)\ .$$

Observe also that $X(x_1) = \lambda_1\,x_1$, $X(y_1) = \lambda_2\,y_1$ and $X(z_1) = \lambda_3\,z_1$, so that X has the same form as before in the new coordinate system. Finally, with the same argument of case **(i.1)** we conclude that M is equivalent to $M_{(2,2,p)}$, $p \geq 2$.
Case (i.3). $C = 0$, $a \neq 0$ and $b = 0$. As the reader can check, with the same type of argument of case **(i.1)**, it is possible to prove that M is equivalent to $M_{(2,2,p)}$, $p \geq 2$.
Case (ii). $C \neq 0$. In this case, we can suppose without lost of generality that $C = 1$, so that $F = A + B\,z + z^2$. As the reader can check, the change of variables $x_1 = x$, $y_1 = y$, $z_1 = z + \frac{1}{2}B(x,y)$, transforms F into

$$F(x_1, y_1, z_1) = z_1^2 + A(x_1, y_1) - \frac{1}{4}B^2(x_1, y_1) = z_1^2 + A_1(x_1, y_1)\ .$$

Observe that $X(z^2) = z^2$, so that $X(z) = \frac{1}{2}z$ and $\lambda_3 = 1/2$. This implies that $X(B) = (1-\lambda_3)B = \frac{1}{2}B$ and so $X(z_1) = \frac{1}{2}z_1$, $X(x_1) = \lambda_1\,x_1$ and $X(y_1) = \lambda_2\,y_1$. Therefore, X has the same form as before in the new coordinate system. Moreover,

$$X(A_1) = X(A) - \frac{1}{4}X(B^2) = A - \frac{1}{4}.2BX(B) = A_1\ .$$

This implies that we can suppose without lost of generality that $F = A(x,y) + z^2$. Set $A(x,y) = \sum_{jk} a_{jk} x^j\,y^k$. We have the following facts :
(I). If $a_{jk} \neq 0$ then $j\,\lambda_1 + k\,\lambda_2 = 1$. This follows from $X(A) = A$.
(II). $2\lambda_2 \geq \lambda_1 + \lambda_2 > 1/2$. This follows from $\lambda_1 + \lambda_2 + \lambda_3 > 1$ (Theorem 1), $\lambda_1 \leq \lambda_2$ and $\lambda_3 = 1/2$.

We will consider all possibilities.
Case (ii.1). $a_{p0} \neq 0$ and $a_{0q} \neq 0$ for some p, q. In this case, since M has a singularity at 0, we must have $p, q \geq 2$. It follows from **(I)** that $\lambda_1 = 1/p$ and $\lambda_2 = 1/q$, so that $q < 4$, $q \leq p$ and $q \in \{2, 3\}$. If $q = 3$, then $\lambda_1 > 1/2 - 1/3 = 1/6$, and so $p \in \{3, 4, 5\}$. Therefore, the possible solutions for (q,p) are $(q,p) \in \{(2,p), (3,3), (3,4), (3,5)\}$. Suppose now that $a_{jk} = 0$ for all $j, k \geq 1$. In this case $F(x,y,z) = a\,x^p + b\,y^q + z^2$, so that M is equivalent to $M_{(2,q,p)}$, as the reader can check.

Let us investigate the possibily of $a_{jk} \neq 0$, for $j, k \neq 0$. It follows from **(I)** that

$$\frac{j}{p} + \frac{k}{q} = 1 \implies j\,q + k\,p = p\,q\,.$$

It is not difficult to see that if the above equation has a solution (j,k) with $j, k \geq 1$, then $mcd(q,p) > 1$, so that either $(q,p) = (2, 2m)$, or $(q,p) = (3,3)$.

(ii.1.1). $(q,p)=(2,2m)$, where $m\geq 1$. In this case we must have $j+k\,m=2m$, which implies that $k=1$ and $j=m$. Hence

$$F(x,y,z)=z^2+a\,y^2+b\,x^m\,y+c\,x^{2m}=z^2+a(y+\alpha\,x^m)^2+d\,x^{2m}\ ,$$

where $\alpha=\frac{b}{2a}$ and $d=c-\frac{b^2}{4a}$. Since M has an isolated singularity at 0, we must have $a,d\neq 0$. As the reader can check, this implies that M is equivalent to $M_{(2,2,2m)}$.

(ii.1.2). $(p,q)=(3,3)$. In this case, we have $A(x,y)=L_1(x,y).L_2(x,y).L_3(x,y)$, where L_1,L_2,L_3 are linear. We observe that the $L_{j's}$ are two by two linearly independent, because otherwise M would have a line of singularities. If we consider L_1,L_2,L_3 as three distinct points in $\mathbb{CP}(1)$, we can conclude that there exists a linear isomorphism T of $\mathbb{C}^2$ such that $L_1\circ T=a.(x+y)$, $L_2\circ T=b.(x+\delta\,y)$ and $L_3\circ T=c.(x+\delta^2\,y)$, where $a.b.c=1$ and δ is a primitive cubic root of the unity. Hence, we can suppose that $L_1.L_2.L_3=(x+y)(x+\delta\,y)(x+\delta^2\,y)=x^3+y^3$. It follows that M is equivalent to $M_{(2,3,3)}$.

Case (ii.2). $a_{p0}=0$ for all p and $a_{0q}\neq 0$ for some q. In this case, we have $\lambda_2=\frac{1}{q}$, and so $q<4$ by **(II)**. Hence $q\in\{2,3\}$. We can suppose that $a_{0q}=1$, so that $A(x,y)=y^q+x\,y\,A_1(x,y)$.

We assert that $a_{j1}\neq 0$ for some $j\geq 1$. In fact, if we had $a_{j1}=0$ for all $j\geq 1$, then $A_x(x,0)\equiv A_y(x,0)\equiv 0$ and this implies that the line $\{y=z=0\}$ is contained in the singular set of M. Let us suppose that $a=a_{m1}\neq 0$, $m\geq 1$. It follows that $m\,\lambda_1+\frac{1}{q}=1$, and so $\lambda_1=\frac{q-1}{qm}$. If we had another $a_{jk}\neq 0$, $1<k<q$, then

$$(*)\ \frac{j(q-1)}{qm}+\frac{k}{q}=1\implies j(q-1)+k\,m=q\,m\ .$$

Let us consider the two possibilities : $q=2$ and $q=3$.

(ii.2.1). $q=2$. In this case, it follows from $(*)$ that F can be written as

$$F(x,y)=z^2+y^2+a\,x^m\,y=z^2+(y+\frac{a}{2}\,x^m)^2-\frac{a^2}{4}x^{2m}\ ,$$

where $a\neq 0$, as the reader can check. Therefore, M is equivalent to $M_{(2,2,2m)}$, by the same argument of **(ii.1.1)**.

(ii.2.2). $q=3$. In this case, we have $\lambda_1=\frac{2}{3m}$. Since $\lambda_1\leq\lambda_2=\frac{1}{3}$ and $1<\lambda_1+\lambda_2+\lambda_3=\lambda_1+\frac{5}{6}$, we get that $m\in\{2,3\}$. If $m=2$, it follows from $(*)$ that A is homogeneous of degree three and the same argument of **(ii.1.2)** proves that M is equivalent to $M_{(2,3,3)}$. If $m=3$ then the possible solutions of $(*)$ are $(j,k)\in\{(0,3),(3,1)\}$, which implies that F is equivalent to

$$F(x,y)=z^2+y^3+y\,x^3=z^2+y(y^2+x^3)\ ,$$

as the reader can check. Therefore, M is equivalent to the surface M_c.

Case (ii.3). $a_{p0}\neq 0$ for some $p\geq 2$ and $a_{0q}=0$ for all q. In this case, we can suppose that $a_{p0}=1$, so that $A(x,y)=x^p+x.A_1(x,y)$. Observe that $A_1(0,y)\not\equiv 0$, because otherwise $F_x(0,y,0)=F_y(0,y,0)=F_z(0,y,0)=0$, contradicting the hypothesis. Therefore, we have $a_{1n}\neq 0$ for some $n\geq 1$. Observe that $\lambda_1=\frac{1}{p}$ and $\lambda_1+n\,\lambda_2=1$, so that $\lambda_2=\frac{p-1}{np}$. It follows from **(II)** that

$$\frac{p-1}{np}>\frac{1}{4}\implies n<4-\frac{4}{p}\implies n\in\{1,2,3\}\ .$$

We observe that, in fact we must have $n \in \{1,2\}$, as the reader can check by using that $\lambda_1 + \lambda_2 > \frac{1}{2}$ and $\lambda_1 \le \lambda_2$.

(ii.3.1). $n = 1$. Since $\frac{p-1}{p} = \lambda_2 \le \lambda_3 = \frac{1}{2}$, we get $p \le 2$, and so $p = 2$ and $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{2}$. In this case, F is homogeneous of degree 2, which implies that M is equivalent to $M_{(2,2,2)}$.

(ii.3.2). $n = 2$. In this case we have $\frac{p-1}{2p} = \lambda_2 \ge \lambda_1 = \frac{1}{p}$, so that $p \ge 3$. We divide in two subcases.

(ii.3.2.1). $n = 2$ and there exists $a_{jk} \ne 0$ for $(j,k) \notin \{(p,0),(1,2)\}$. It follows from **(I)** that

$$\begin{cases} j\,\lambda_1 + k\,\lambda_2 = 1 \\ \lambda_1 + 2\,\lambda_2 = 1 \end{cases} \implies (j-1)\lambda_1 = (2-k)\lambda_2 \implies j > 1 \text{ and } k = 1 \implies p = 2j-1\ ,$$

so that p is odd. Set $p = 2m+1$. It follows that

$$A(x,y) = x^{2m+1} + a\,x\,y^2 + b\,x^{m+1}\,y = x(x^{2m} + a\,y^2 + b\,x^m\,y) = a\,x\,(y - c_1\,x^m)(y - c_2\,x^m)\ ,$$

where $a \ne 0$ and c_1, c_2 are the roots of $a\,t^2 + b\,t + 1 = 0$. We observe that $c_1 \ne c_2$, for otherwise $0 \in \mathbb{C}^3$ would not be an isolated singularity of M. When $m = 1$, with the same argument as in **(ii.1.2)** it can be proved that M is equivalent to $M_{(2,3,3)}$. If $m > 1$ then, after a change of variables of the form $x_1 = k.x$, $y_1 = -a(y - c_1\,x^m)$, $z_1 = z$, we can put F in the form $F(x_1, y_1, z_1) = z_1^2 - x_1\,y_1(y_1 - x_1^m)$, so that M is equivalent to the surface M_b. We leave the details for the reader.

(ii.3.2.2). $n = 2$ and $a_{jk} = 0$ for all $(j,k) \notin \{(p,0),(1,2)\}$. In this case, we have $A(x,y) = x^p + a\,x\,y^2$, where $a \ne 0$ and $p \ge 3$. When p is odd, the argument is analogous to **(ii.3.2.1)** : either M is equivalent to $M_{(2,3,3)}$, or M is equivalent to the surface M_b. If $p = 2m+2$ is even then, after a linear change of variables we can put F in the form $F(x,y,z) = z^2 - x(y^2 + x^{2m+1})$, so that M is equivalent to the surface M_d.

Case (ii.4). $a_{p0} = a_{0q} = 0$ for all p, q. In this case, by using that $0 \in \mathbb{C}^3$ is an isolated singularity of M, we can conclude that there exist $m, n \ge 1$ such that $a_{m1} \ne 0$ and $a_{1n} \ne 0$. We assert that, either $m = n = 1$, or $n = 2 \le m$.

In fact, λ_1 and λ_2 satisfy the system $\lambda_1 + n\,\lambda_2 = 1$ and $m\,\lambda_1 + \lambda_2 = 1$. If $m.n > 1$, then we get $\lambda_1 = \frac{n-1}{mn-1}$ and $\lambda_2 = \frac{m-1}{mn-1}$. Since $\lambda_1 \le \lambda_2$ and $\lambda_1 + \lambda_2 > \frac{1}{2}$, we have $n \le m$ and

$$2(m+n) > m\,n + 3 \implies \frac{1}{m} + \frac{1}{n} > \frac{1}{2} \implies n \in \{1,2,3\}\ .$$

Now, $0 < \lambda_1 = \frac{n-1}{mn-1}$, so that $n \in \{2,3\}$. In fact $n = 2$, because $n = 3$ implies that $2(m+3) > 3m+3 \implies m < 3 = n$.

If $m = n = 1$, then we find $F(x,y) = z^2 + a\,x\,y$, $a \ne 0$, as the reader can check, so that M is equivalent to $M_{(2,2,2)}$. If $n = 2$ and $m \ge 2$, we have two possibilities, either $m = n$, or $m > n$. If $m = n$ then, with the same argument as in **(ii.1.2)**, we get that M is equivalent to $M_{(2,3,3)}$. If $m > n$, then M is equivalent to the surface M_b. This finishes the proof of the 2^{nd} step.

Proof of the third step. We will use two invariants : the Milnor number of M at 0 and the fundamental group of M^*. The Milnor number of M, denoted by

$\mu(M)$, is the complex dimension of $\mathcal{O}_3/ < dF >$, where $< dF >$ is the ideal of $\mathcal{O}_3$ generated by F_x, F_y, F_z. It is known that $\mu(M) = [F_x, F_y, F_z]_0$ (the intersection number of F_x, F_y, F_z at $0 \in \mathbb{C}^3$). In the case of $M_{(p,q,r)}$ we have $\mu(M_{(p,q,r)}) = [x^{p-1}, y^{q-1}, z^{r-1}]_0 = (p-1)(q-1)(r-1)$ and so

$$(11) \quad \begin{cases} \mu(M_{(2,2,r)}) = r-1 \\ \mu(M_{(2,3,3)}) = 4 \\ \mu(M_{(2,3,4)}) = 6 \\ \mu(M_{(2,3,5)}) = 8 \end{cases}$$

In the case of the surfaces in **(b)**, **(c)** and **(d)**, we have

$$(12) \quad \begin{cases} \mu(M_b) = 2(m+2) \\ \mu(M_c) = 5 \\ \mu(M_d) = 2m+3 \end{cases}$$

as the reader can check. On the other hand, we have obtained in Examples 1,2,3 and 4, maps $\phi: \mathbb{C}^2 \to M$ such that $\phi^{-1}(0) = \{0\}$. These maps are non-singular outside $0 \in \mathbb{C}^2$, so that $\psi := \phi|_{\mathbb{C}^2 \setminus \{0\}}: \mathbb{C}^2 \setminus \{0\} \to M^*$ is the universal covering of M^*. It follows that, in each case, we have $\#(\pi_1(M^*)) = dg(\psi)$, the topological degree of ψ. By using $(4')$ in Remark 3 we find (see also [Mi] pg. 65 and [Hi]) :

$$(13) \quad \begin{cases} \#(\pi_1(M^*_{(2,2,r)})) = r \\ \#(\pi_1(M^*_{(2,3,3)})) = 8 \\ \#(\pi_1(M^*_{(2,3,4)})) = 24 \\ \#(\pi_1(M^*_{(2,3,5)})) = 120 \end{cases}$$

On the other hand, it follows from the construction of the map ϕ in Examples 2,3 and 4, that

(14). There exists covering map $\psi: M^*_{(2,2,2m)} \to M^*_b$. This map is of the form $\psi(x, y, z) = (z^2, y^2, xyz)$, so that $\#(\pi_1) = dg(\phi) = dg(\psi).2m = 8m$.

(15). There exists covering map $\psi: M^*_{(2,3,4)} \to M^*_c$. This map is of the form $\psi(x, y, z) = (y, z^2, xz)$, so that $\#(\pi_1) = dg(\phi) = dg(\psi).24 = 48$.

(14). There exists covering map $\psi: M^*_{(2,2,2(2m+1))} \to M^*_d$. This map is of the form $\psi(x, y, z) = (x^2, y, xz)$, so that $\#(\pi_1) = dg(\phi) = dg(\psi).2(2m+1) = 4(2m+1)$.

As the reader can verify easily, if we take two of the above surfaces then, either they have different Milnor numbers, or different fundamental groups. This ends the proof of Theorem 2.

§4. Proof of Theorem 3.

We can suppose that $n \geq 3$. We need a Lemma :

Lemma 5. *Let M be a germ of hypersurface at $p \in \mathbb{C}^{n+1}$, $n \geq 3$. Suppose that $dim_{\mathbb{C}}(sing(M)) \leq n-3$ and that there exists a germ of holomorphic map $\phi: (\mathbb{C}^n, q) \to (M, p)$ such that $\phi^{-1}(p) = \{q\}$. Then p is not a singularity of M.*

Let us see how Lemma 5 implies the Theorem. Let U and V be connected neighborhoods of $0 \in \mathbb{C}^n$ and $0 \in \mathbb{C}^{n+1}$ be such that :

(i). F and ϕ have representatives $F\colon V \to \mathbb{C}$ and $\phi\colon U \to \mathbb{C}^{n+1}$ such that $\phi(U) \subset V$ and $F \circ \phi = 0$.
(ii). $\phi^{-1}(M_V) = U$, where $M_V = M \cap V = \{F^{-1}(0)\}$.

Lemma 5 implies that $sing(M_V)$ has at least one irreducible component with dimension $n-2$, say A. Let us prove that all components of $sing(M_V)$ have dimension $n-2$, Suppose by contradiction that $sing(M_V)$ has a component, say B, of dimension $k \leq n-3$. Let $p \in B$ be a smooth point of $sing(M)$, so that there exists a (connected) neighborhood $V_1 \subset V$ of p in $\mathbb{C}^{n+1}$ such that $B \cap V_1$ is defined by $n+1-k$ equations. It follows from **(ii)** that there exists $q \in U$ such that $\phi(q) = p$. Let U_1 be the connected component of $\phi^{-1}(V_1)$ which contains q and $\phi_1 = \phi|_{U_1}\colon U_1 \to V_1$. Since $sing(M) \cap V_1$ has dimension $k \leq n-3$, it follows from Lemma 5 that p is not a singularity of M, a contradiction. Therefore all irreducible components of $sing(M)$ have dimension $n-2$.

Given a smooth complex manifold N, we will use the following notations :
(I). $\Omega^k_N =$ the sheaf of germs of holomorphic k-forms on N.
(II). $\chi_N =$ the sheaf of germs of holomorphic vector fields on N.

We need a preliminary result in order to prove Lemma 5.

Lemma 6. *Let M be a germ of hypersurface at $0 \in \mathbb{C}^{n+1}$, $n \geq 3$. Suppose that $dim_{\mathbb{C}}(sing(M)) \leq n-2$. Then the following assertions are equivalent :*
(a). *0 is not a singular point of M.*
(b). *There exists a Stein neighborhood V of 0 and a holomorphic section $Y\colon M^*_V \to T\mathbb{C}^{n+1}|_{M^*_V}$ such that $dF_p(Y(p)) \equiv 1$ for all $p \in M^*_V$.*
(c). *There exists a Stein neighborhood V of 0 such that $H^1(M^*_V, \Omega^{n-1}_{M^*_V}) = \{0\}$.*
(d). *There exists a Stein neighborhood V of 0 such that $H^1(M^*_V, \chi_{M^*_V}) = \{0\}$.*

Proof. It is not difficult to see that **(a)** implies the other assertions. On the other hand, the interior product and the n-form η induce an isomorphism $\delta\colon \chi_{M^*_V} \to \Omega^{n-1}_{M^*_V}$ defined by $\delta(Y) = i_Y(\eta)$, where Y is a holomorphic vector field on some open subset of M^*_V. Therefore, **(c)** is equivalent to **(d)**.

(d) $\implies$ (b). Given V as in **(d)**, consider the covering $\mathcal{I} = (M_j)^n_{j=1}$ of M^*_V, where $M_j = \{p \in M_V \mid F_{x_j}(p) \neq 0\}$. Let $Z_j\colon M_j \to T\mathbb{C}^{n+1}$ be defined by $Z_j = \frac{1}{F_{x_j}} \frac{\partial}{\partial x_j}$, so that $dF(Z_j) \equiv 1$. For $i, j \in \{0, ..., n\}$, set $X_{ij} = Z_j - Z_i$. Since $dF_p(X_{ij}(p)) = 0$ for all $p \in M_{ij} := M_i \cap M_j$, the collection $\{X_{ij}\}^n_{i,j=0}$ can be considered as a cocycle in $Z^1(\mathcal{I}, \chi_{M^*_V})$, so that we can write $X_{ij} = X_j - X_i$, where X_j is a holomorphic vector field on M_j for all $j = 0, ..., n$, because $H^1(M^*_V, \chi_{M^*_V}) = \{0\}$. This implies that there exists a holomorphic section $Y\colon M^*_V \to T\mathbb{C}^{n+1}$ such that $Y|_{M_j} = Z_j - X_j$ for all $j = 0, ..., n$. Observe that $dF(Y)|_{M_j} = dF(Z_j) - dF(X_j) = 1$, which proves **(b)**.

(b) $\implies$ (a). Let $Y\colon M^*_V \to T\mathbb{C}^{n+1}$ be a holomorphic section such that $dF_p(Y(p)) = 1$ for all $p \in M^*_V$. The idea is to prove that Y can be extended to a holomorphic vector field on V, say X. Since $0 \in \overline{M^*_V}$, this implies that $dF_0(X(0)) = 1$, so that $dF_0 \neq 0$ and 0 is not a singular point of M.

We will use the following result (cf [G-R]), which is a generalization of [C] : Let N be a Stein manifold of dimension $m \geq 3$ and $A \subset N$ be an analytic subset of N such that $dim_{\mathbb{C}}(A) \leq m-3$. Then $H^1(N \setminus A, \mathcal{O}) = 0$.

In particular, since $dim_{\mathbb{C}}(sing(M)) \leq n-2 = (n+1)-3$, we have $H^1(V \setminus sing(M), \mathcal{O}) = 0$. Let us prove that the section Y can be extended to V. Since

$codim_V(sing(M)) \geq 2$, it follows from Hartogs' Theorem that it is sufficient to prove that Y can be extended to a holomorphic vector field on $V \setminus sing(M) := V^*$. Since $M \subset \mathbb{C}^{n+1}$, we can write $Y = \sum_{j=0}^n Y_j \frac{\partial}{\partial x_j}$, where $Y_j \in \mathcal{O}(M_V^*)$, so that it is sufficient to prove that $Y_0, ..., Y_n$ extend to holomorphic functions on V^*. In fact, any $f \in \mathcal{O}(M_V^*)$ can be extended to a holomorphic function on V^*. Let $\mathcal{U} = (V_j)_{j \in J}$ be a Leray covering of V^* by open sets such that, if $V_j \cap M_V^* \neq \emptyset$ then $f|_{V_j \cap M_V^*}$ can be extended to a holomorphic function , say f_j, on V_j. If $V_j \cap M_V^* = \emptyset$, set $f_j \equiv 0$. In this way we have a collection $(f_j)_{j \in J}$, where $f_j \in \mathcal{O}(V_j)$ and $f_j = f$ on $V_j \cap M_j^*$, if this set is not empty. This implies that, if $V_{ij} := V_i \cap V_j \neq \emptyset$ then $f_j - f_i = f_{ij}.F$, where $f_{ij} \in \mathcal{O}(V_{ij})$. Now, the collection $(f_{ij})_{V_{ij} \neq \emptyset}$ is an additive cocycle. Therefore, if $V_{ij} \neq \emptyset$, we can write $f_{ij} = g_j - g_i$ where $g_j \in \mathcal{O}(V_j)$, because $H^1(V^*, \mathcal{O}) = 0$. This implies that there exists $h \in \mathcal{O}(V^*)$ such that $h|_{V_j} = f_j - g_j.F$ for all $j \in J$, so that $h|_{M_V^*} = f$. This implies **(a)**. □

Let us finish the proof of Lemma 5. The idea is to prove that if $dim_{\mathbb{C}}(sing(M)) \leq n-3$ and there exists a germ of holomorphic map $\phi: (\mathbb{C}^n, 0) \to (M, 0)$ such that $\phi^{-1}(0) = \{0\}$, then $H^1(M_V^*, \Omega^{n-1}) = 0$ for some Stein neighborhood V of $0 \in \mathbb{C}^{n+1}$. At this point we will use Dolbeault's Theorem (cf. [G-H]) : if N is a complex manifold then $H^1(N, \Omega^{n-1}) \simeq H_{\overline{\partial}}^{n-1,1}(N)$.

Fix Stein neighborhoods U of $0 \in \mathbb{C}^n$ and V of $0 \in \mathbb{C}^{n+1}$ such that :
(i). F has a representative $F: V \to \mathbb{C}$ and ϕ a representative $\phi: U \to V$.
(ii). $\phi: U \setminus \phi^{-1}(sing(M)) \to M_V^*$ is a ramified covering with d sheets. We will use the notation $U \setminus \phi^{-1}(sing(M)) = U^*$.

We assert that $H_{\overline{\partial}}^{n-1,1}(U^*) = H^1(U^*, \Omega^{n-1}) = 0$. In fact, since U is Stein and $dim_{\mathbb{C}}(\phi^{-1}\ (sing(M))) \leq n-3$, we have $H^1(U^*, \mathcal{O}) = 0$. Now any holomorphic $n-1$-form on a open set $A \subset U^*$ can be written as

$$\sum_{j=1}^n a_j\, du_1 \wedge ... \wedge \hat{du_j} \wedge ... du_n \ \ , \ a_j \in \mathcal{O}(A)\,.$$

This implies that $H^1(U^*, \Omega^{n-1}) \simeq (H^1(U^*, \mathcal{O}))^n = 0$, which proves the assertion.

Now, let $\alpha \in \Omega^{n-1,1}(M_V^*)$ be C^∞ and such that $\overline{\partial}\, \alpha = 0$. We want to prove that $\alpha = \overline{\partial}\, \omega$ for some $\omega \in \Omega^{n-1,0}(M_V^*)$. Since $\phi^{-1}(M_V^*) = U^*$ and $H_{\overline{\partial}}^{n-1,1}(U^*) = 0$ we have that $\phi^*(\alpha) = \overline{\partial}\, \beta$, where β is a $(n-1,0)$-form on U^* of class C^∞. Let η be the holomorphic n-form on M_V^* defined as in §2. As we have seen, $\phi^*(\eta)$ extends to a holomorphic n-form on U which can be written as $f(u).du_1 \wedge ... \wedge du_n$, where $f \in \mathcal{O}(U)$. Then $C = \{p \in U^* |\, f(p) = 0\}$ is the set of critical points of $\phi|_{U^*}$ and $\phi(C) = CV$ is the set of critical values of ϕ, so that $\phi|_W: W \to \hat{M}$ is a covering with d-sheets, where $W = U^* \setminus \phi^{-1}(CV)$ and $\hat{M} = M_V^* \setminus CV$. Using the same method of the proof of Lemma 3, it is possible to construct a $(n-1,0)$-form of class C^∞, ω on $\hat{M}$, such that $\overline{\partial}\omega = \alpha$. At this point, we observe that if $f(0) \neq 0$ and V is small then $\hat{M} = M_V^*$, so that Lemma 5 is proved in this case. If $f(0) = 0$, we must prove that ω extends to CV. Recall that ω is defined on $\hat{M}$ by

$$(16)\ \ \omega_p = \frac{1}{d} \sum_{q \in \phi^{-1}(p)} (\phi_q)_*(\beta_q)\ \ ,$$

where $(\phi_q)_*$ denotes the map induced by the isomorphism $D\phi(q)\colon T_q(\mathbb{C}^n) \to T_p(M_V^*)$. ∎
The idea is the following. Fix a point $p \in CV$. Since $\overline{\partial}\alpha = 0$, it follows from the $\overline{\partial}$-Poincaré Lemma that there exists a $(n-1,0)$-form δ of class C^∞, defined in a neighborhood V_p of p such that $\overline{\partial}\delta = \alpha$ on V_p. Now, $\overline{\partial}(\omega - \delta) = 0$, so that $\omega - \delta$ is holomorphic. Therefore, we have only to prove that $\omega - \delta$ extends to V_p. We need a fact.

Assertion 6. There exists an analytic subset A of CV such that :
(i). $dim_{\mathbb{C}}(A) \leq n-2$.
(ii). $CV \setminus A$ is contained in M_V^* and in the smooth part of CV.
(iii). For any $p \in CV \setminus A$ and $q \in \phi^{-1}(p)$ there exist local coordinate systems $(U_q, (u,v) = (u_1, ..., u_{n-1}, v))$ and $(V_p, (x,y) = (x_1, ..., x_{n-1}, y))$ around q and p respectively, where V_p is a neighborhood of p in M_V^*, such that $u(q) = x(p) = 0 \in \mathbb{C}^{n-1}$, $v(q) = y(p) = 0 \in \mathbb{C}$, $\phi(U_q) = V_p$, $x(V_p) = \mathbb{D}^{n-1}$ and $y(V_p) = \mathbb{D}$ ($\mathbb{D} = \{z \in \mathbb{C} | \, |z| < 1\}$) and

$$\phi(u,v) = (u, v^m)$$

and $m \in \mathbb{N}$ depends only on the irreducible components of CV and $\phi^{-1}(CV)$ which contain p and q respectively.

The proof Assertion 6 will be done at the end. It implies the following :

Assertion 7. For any $p \in CV \setminus A$ there exists a coordinate system $(V_p, (x,y) = (x_1, ..., x_{n-1}, y))$ such that :
(iv). $x(p) = 0 \in \mathbb{C}^{n-1}$, $y(p) = 0 \in \mathbb{C}$, $x(V_p) = \mathbb{D}^{n-1}$ and $y(V_p) = \mathbb{D}$.
(v). There exists $0 \leq a < 1$ and a compact neighborhood K of p such that such that if $\omega = \omega_n(x,y)\, dx_1 \wedge ... \wedge dx_{n-1} + \sum_{j=1}^{n-1} \omega_j(x,y) dx_1 \wedge ... \wedge dx_{j-1} \wedge dx_{j+1} \wedge ... \wedge dy$, then

$$(17) \quad max\,\{|\omega_1(x,y)|, ..., |\omega_n(x,y)|\} \leq c.|y|^{-a} \ , \ \forall \ (x,y) \in K' := K \setminus \{y = 0\}\,,$$

where c is a positive constant.

Proof. Fix $p \in CV \setminus A$ and let $\phi^{-1}(p) = \{q_1, ..., q_r\}$. Since p is in the smooth part of CV, we can find a local coordinate system $(B, (x,y))$ around p such that $x(p) = 0 \in \mathbb{C}^{n-1}$, $y(p) = 0 \in \mathbb{C}$ and $CV \cap B = \{y = 0\}$. Given $q_j \in \phi^{-1}(p)$ fix local coordinate systems $(U_j, (u_j, v_j) = (u_{j1}, ..., u_{jn-1}, v_j))$ and $(V_j, (x_j, y_j) = (x_{j1}, ..., x_{jn-1}, y_j))$ as in Assertion 6, so that $\phi(u_j, v_j) = (u_j, v_j^{m_j})$, $m_j \in \mathbb{N}$. Observe that $m_1 + ... + m_r = d$. We can suppose without lost of generality that $V_j \subset B$ for all $j = 1, ..., r$. Let K be a compact neighborhood of p such that $K \subset \cap_j V_j$. Remark that there exists a constant $k_1 > 1$ such that $k_1^{-1}.|y(z)| \leq |y_j(z)| \leq k_1.|y(z)|$ for all $z \in K$. Now, fix a point $p_0 = (x_0, y_0) \in K \setminus \{y = 0\}$. In coordinate system V_j we have $p_0 = (x_{0j}, y_{0j})$, so that $\phi^{-1}(p_0) \cap U_j = \{(x_{0j}, \gamma^\ell.v_{0j}) := q_{0j}^\ell \,|\, \ell = 1, ..., m_j\}$, where γ is a primitive m_j^{th} root of unity and $v_{0j}^{m_j} = y_{0j}$. Let

$$\beta|_{U_j} = \beta_n(u_j, v_j)\, du_{j1} \wedge ... \wedge du_{jn-1} + \sum_{k=1}^{n-1} \beta_k(u_j, v_j)\, du_{j1} \wedge ... \wedge du_{jk-1} \wedge du_{jk+1} \wedge ... \wedge dv_j$$

where $\beta_1, ..., \beta_n \in C^\infty(U_j)$. The inverse of ϕ from a small neighborhood of p_0 to a small neighborhood of q_{0j}^ℓ can be written as $\psi_{0j}^\ell(x_j, y_j) = (x_j, \gamma^\ell.y_j^{1/m_j})$, where y_j^{1/m_j} is a branch of the m_j^{th} root of y_j. Therefore, the contribution to the sum in (16) which comes from $\phi^{-1}(p) \cap U_j$, in this neighborhood of p_0, is of the form $\frac{1}{d}\sum_{k=1}^{m_j} \theta_k$

$$\theta_k = (\psi_{0j}^k)^*(\beta) = \theta_{kn} dx_{j1} \wedge ... \wedge dx_{jn-1} + \sum_{m=1}^{n-1} \theta_{km} dx_{j1} \wedge ... \wedge dx_{j,m-1} \wedge dx_{j,m+1} \wedge ... \wedge dy_j$$

and

$$\theta_{kn} = \beta_n(x_j, \gamma^k.y_j^{1/m_j}) \text{ and } \theta_{km} = \beta_m(x_j, \gamma^k.y_j^{1/m_j}).\frac{1}{m_j}\gamma^k.y_j^{1/m_j - 1} \ , \ m = 1, ..., n-1$$

Since $\beta_1, ..., \beta_n$ are bounded in $\phi^{-1}(K) \cap U_j$ and $|y_j| \leq k_1.|y|$ on K, we get that

$$(18) \ \ |\theta_{km}(x,y)| \leq c_j.|y|^{-(1-1/m_j)} \ ,$$

where $c_j > 0$. Now, it is not difficult to prove by using (18) that $|\omega_m(x,y)| \leq c.|y|^{-a}$, for all $(x,y) \in K$, where $a = max\{1 - 1/m_j | \, j = 1, ..., r\}$ and $c > 0$ is a constant. □

Let us prove that ω extends to M_V^* by using Assertion 7. The idea is to prove first that ω can be extended to $CV \setminus A$ and then that it can be extended to $A \setminus sing(M)$. Let us prove that ω can be extended to a neighborhood V_p like in Assertion 7.

Since V_p is biholomorphic to a polydisk, there exists δ such that $\overline{\partial}\delta = \alpha$, so that $\omega - \delta$ is a holomorphic $n-1$ form. Let $\omega_1, ...\omega_n \in C^\infty(V_p \setminus \{y = 0\})$ be as before and

$$\delta = \delta_n(x,y) \, dx_1 \wedge ... \wedge dx_{n-1} + \sum_{j=1}^{n-1} \delta_j(x,y) dx_1 \wedge ... \wedge dx_{j-1} \wedge dx_{j+1} \wedge ... \wedge dy$$

where $\delta_1, ..., \delta_n \in C^\infty(V_p)$. If $0 \leq a < 1$ and K are as in Assertion 7, then it follows from (17) that

$$(17') \ \ |\omega_j(x,y) - \delta_j(x,y)| \leq c_1.|y|^{-a} \ , \ \forall \ (x,y) \in K' \ ,$$

where c_1 is a positive constant. But $\omega_j - \delta_j$ is holomorphic in $V_p \setminus \{y = 0\}$, so that $(17')$ implies that $\omega_j - \delta_j$ can be extended to K, because $a < 1$. Now, observe that $K \cup (V_p \setminus \{y = 0\})$ is a Hartogs domain whose analytic closure is V_p. Therefore $\omega_j - \delta_j$ can be extended to V_p for all $j = 1, ..., n$, so that ω can be extended to a C^∞ form on V_p. We have proved with this argument that ω can be extended to a C^∞ form on $M_V^* \setminus A$, where the codimension of $A \cap M_V^*$ in M_V^* is ≥ 2. Given $p \in A \cap M_V^*$, let δ be a C^∞ form defined in a neighborhood V_p of p in M_V^* such that $\overline{\partial}\delta = \alpha$, so that $\omega - \delta$ is holomorphic in $V_p \setminus A$. Since the codimension of $A \cap M_V^*$ in M_V^* is ≥ 2, $\omega - \delta$ can be extended to V_p by Hartogs Theorem. Therefore, ω can be extended to a C^∞ form on V_p. This implies that ω can be extended to M_V^*. It remains to prove Assertion 6.

Proof of Assertion 6. Observe first that all irreducible components of CV and of $\phi^{-1}(CV)$ have complex dimension $n-1$. Moreover, if Σ is an irreducible component of $\phi^{-1}(CV)$ then $\phi(\Sigma) := C$ is an irreducible component of CV. Denote by A_1 the singular set of CV. It is well known that $dim_{\mathbb{C}}(A_1) \leq n-2$, so that $CV \setminus A_1$ is smooth of dimension $n-1$. Let Σ and $C = \phi(\Sigma)$ be as before and denote by

ϕ_Σ the restriction $\phi|_\Sigma : \Sigma \to C$. Let $A_\Sigma = \{q \in \Sigma \setminus \phi^{-1}(A_1) | \, rank(D\phi_\Sigma(q)) \leq n-2\}$. Since $\phi(\Sigma) = C$, A_Σ is an analytic subset of Σ of dimension $\leq n-2$. Moreover, its closure $\overline{A_\Sigma}$ is an analytic subset of $U \subset \mathbb{C}^n$ of dimension $\leq n-2$. Let $A = A_1 \cup \phi(B)$, where $B = \cup_\Sigma \overline{A_\Sigma}$ (in the union Σ runs over all irreducibe components of $\phi^{-1}(CV)$). Then $dim_{\mathbb{C}}(A) \leq n-2$. Remark that if $p \in CV \setminus A$ and $q \in \phi^{-1}(p)$, then CV and $\phi^{-1}(CV)$ are smooth of dimension $n-1$ at p and q respectively and $D\phi(q)|_{T_q\phi^{-1}(CV)} : T_q\phi^{-1}(CV) \to T_p(CV)$ is an isomorphism. It follows from the implicit function Theorem that there exist local coordinate systems $(U_q, (u,v) = (u_1, ..., u_{n-1}, v))$ and $(V_p, (x,y) = (x_1, ..., x_{n-1}, y))$, where $u(q) = x(p) = 0 \in \mathbb{C}^{n-1}$, $v(q) = y(p) = 0 \in \mathbb{C}$, $\phi(U_q) = V_p$ and $\phi(u,v) = (u, f(u,v))$ where $f \in \mathcal{O}(U_q)$. We can assume also that q is a singular point of ϕ, $\phi^{-1}(CV) \cap U_q = \{v = 0\}$ and $CV \cap V_p = \{y = 0\}$. This implies that $f(u,0) \equiv 0$, so that $f(u,v) = g(u,v).v^m$ for some $m \geq 2$, where $g \not\equiv 0$. Now, the set of critical points of ϕ in U_q is defined by $f_v = 0$ and is contained in $\phi^{-1}(CV) \cap U_q = \{v = 0\}$. Since $f_v(u,v) = v^{m-1}(m.g(u,v) + v.g_v(u,v))$, we get that $g(u,0) \neq 0$ for all $(u,0) \in U_q$. Therefore by taking a smaller U_q we can suppose that U_q is simply connected and that $g \in \mathcal{O}^*(U_q)$. Let $h \in \mathcal{O}^*(U_q)$ be such that $h^m = g$ and consider the change of variables in a neighborhood of q given by $u' = u$, $v' = g(u,v).v$. In the coordinate system (u', v') we have $\phi(u', v') = (u', (v')^n)$. This proves Assertion 6 and finishes the proof of Theorem 3. $\square$

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