A QUASILINEAR PARABOLIC SYSTEM FOR THREE-PHASE CAPILLARY FLOW IN POROUS MEDIA

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Abstract. A parabolic system is proposed to describe three-phase capillary flows in porous media. The assumptions imposed on the phase interaction mean that the capillarity matrix is triangular. The unique solvability of an associated non-degenerate parabolic system is proved for the Cauchy problem in a class of x-periodic solutions.

1. Introduction

We consider a quasi-linear parabolic problem

$$u_t + f(u)_x = (D(u)u_x)_x, \quad u|_{t=0} = u_0(x), \quad x \in \mathbb{R}, \tag{1.1}$$

related to three-phase capillary flows in porous media, e.g., the flow of gas, water and oil in a reservoir. Here,

$$u = (u_1, u_2), \quad f = (f_1, f_2), \quad D = (D_{ij}), \quad i, j \in \{1, 2\},$$

where, say, u_1 and u_2 are the gas and water saturations, respectively, while $1 - u_1 - u_2$ is the oil saturation. The meaning of the flux vector f and the capillary matrix D is explained below. We study unique global solvability. The main assumptions are

$$D_{21} = 0, \quad D_{22} = D_{22}(u_2) > 0, \quad D_{11}(u_1, u_2) > 0. \tag{1.2}$$

Global existence for equation (1.1) in the most general setting is an open problem. Global well-posedness is proved in [7] if $D_{11} = D_{22}$ and $D_{12} = D_{21} = 0$. The restriction (1.2) is also present in the theory of H.Amann [2] but therein it is also assumed that $f_2 = f_2(u_2)$. In our case the flux vector f may be rather general.

Due to the physical motivation, it is required that

$$0 \leq u_i, \quad u_1 + u_2 \leq 1. \tag{1.3}$$

We prove a global existence theorem for system (1.1) with the verification of (1.3) with the use of the principle of positively invariant regions of Chueh, Conley and Smoller [5]. A new feature in the present paper is concerned with the constitutive law of the capillary pressures $P_1(u)$ and $P_2(u)$. These functions are defined by

$$P_1 = p_1 - p_3, \quad P_2 = p_2 - p_3, \tag{1.4}$$

where p_i is the pressure in the i-th phase $i \in \{1, 2, 3\}$. It is shown below how the functions P_i define the capillary matrix D. Very little is known about the functions P_i both theoretically and experimentally. An approximation of the real

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three-phase flows by two artificial two-phase flows is proposed in [12]. In [9], some engineer formulas are given by interpolation into the triangle

$$0 < u_i, \quad u_1 + u_2 < 1, \quad i \in \{1,2\}$$

of equalities (1.4) which are assumed known at the boundaries $u_1 = 0$, $u_2 = 0$, and $u_1 + u_2 = 1$, corresponding to two-phase flows. A class of capillary pressures P_1 and P_2, efficient from the computational point of view, is considered in [4]. Therein, it is assumed the existence of a function $p_c(u_1, u_2)$ such that

$$\nabla_u p_c = f_1(u)\nabla_u P_1 + f_2(u)\nabla_u P_2,$$

where f_i are the fractional flow functions described in Section 2.

It is proved in Section 2 that equalities (1.2) are equivalent to (2.11) below, a linear system of differential equations for P_1 and P_2. This system is solved in the case of linear phase mobilities

$$\lambda_i = k_i u_i, \quad k_1 \neq k_3, \quad i \in \{1,2,3\}.$$

In this case, a class of solutions of (2.11) is given by the formulas

$$P_1 = \alpha\xi^m - \beta k_0 \frac{u_2^2}{2}, \quad P_2 = \alpha\varphi(\xi; k_1, k_3, m) + \frac{\beta}{2} u_2^2 \left(\frac{1}{k_3} - \frac{1}{k_2}\right) + \frac{\beta}{k_2} u_2, \tag{1.5}$$

where $\xi = u_1/(1 - u_2)$ and the function φ is defined from the equation

$$\frac{\partial\varphi}{\partial\xi} = \frac{mk_1}{k_1 - k_3} \frac{(k_1 - k_3)\xi^m}{(k_1 - k_3)\xi + k_3},$$

$$m > 1, \quad k_0 = \frac{k_3 - k_1}{k_1 k_3}, \quad \alpha = \text{const}, \quad \beta = \text{const} .$$

Whatever capillary pressures in (1.4) underly a three-phase model, the governing equations (1.1) should preserve the saturation bounds (1.3), as time grows, provided that the initial saturations obey it. From the analysis in [5], not every matrix D fits this invariance requirement. Below (see (2.17)) we formulate conditions for D which guarantee (1.3). As will become clear, physical capillarity matrices (together with their perturbations by multiples of the identity matrix) satisfy the corresponding conditions. In particular, the material functions P_i given by (1.5) allow the verification of (1.3).

The three-phase flow equations without capillarity effect were addressed in a number of papers. It is well-known that the corresponding system may fail to be hyperbolic. Indeed, an example is given in [3] in which the system is elliptic in a region inside the triangle of saturations and hyperbolic elsewhere in the triangle. The asymptotic behavior of measure-valued solutions was studied in [6]. Additional informations, including references, on the subject can be found in [1, 13].

We study the Cauchy problem (1.1) in a class of x-periodic functions with period 2:

$$u(t,x) = u(t, x \pm 2), \quad x \in \mathbb{R}.$$

We denote

$$\Omega = (-1,1), \quad Q = (0,T) \times \Omega, \quad \mathbb{R}^2_+ = (0,\infty) \times \mathbb{R}.$$

We assume that there exist nonnegative functions $\lambda_i(u)$, satisfying $\lambda_i(u) > 0$ if $u_i > 0$, $i \in \{1,2,3\}$, such that

$$f_i = \frac{\lambda_i}{\lambda}, \quad \lambda = \sum_1^3 \lambda_i; \quad \lambda_j|_{u_j=0} = 0, \quad j \in \{1,2\}; \quad \lambda_3|_{u_1+u_2=1} = 0. \tag{1.6}$$

In the model for three-phase flow in porous media, λ_i represents the mobility of the i-th phase. It is also assumed that the functions D_{ij} obey the inequalities

$$D_{ii} \geq \nu = \text{const} > 0 \tag{1.7}$$

and satisfy (2.17). The main result is the following.

Theorem 1.1. *Let the functions $f_i(u)$, $D_{ij}(u)$ and their first derivatives be Hölder continuous with the Hölder exponent $\beta \in (0,1)$. Let the hypotheses (1.2), (1.6), (1.7), (2.17) about f and D be satisfied and $u_0 \in H^{2+\beta}(\mathbb{R})$, periodic with period 2 assuming values in Δ. Then the Cauchy problem (1.1) has a unique solution $u(t,x) \in H^{2+\beta,1+\frac{\beta}{2}}(\mathbb{R}^2_+)$, periodic in x with period 2 and taking values in Δ.*

The remaining of this paper is organized as follows. In section 2 we describe the basic equations of three-phase flow in porous media, state our assumptions on the capillarity matrix and give examples. In section 3 we obtain apriori estimates which allow the solution of the question of global existence. In section 4 we complete the proof of the Theorem 1.1. In section 5 we provide an appendix with details about the construction of the examples given in section 2.

2. Basic equations of three-phase flow

We recall here some basic facts about multiphase flow in a porous medium (cf., e.g., [1]). We consider a one-dimensional horizontal flow of three fluid phases in a porous medium. These phases are gas, water, and oil, with saturations u_1, u_2, and u_3, respectively. The balance of masses is governed by the mass conservation equations

$$\frac{\partial}{\partial t}(m u_i \rho_i) + \frac{\partial}{\partial x}(\rho_i v_i) = 0, \tag{2.1}$$

where m denotes the porosity of the porous medium, ρ_i is the density, and v_i is the seepage velocity of the i-th phase. The functions u_i satisfy the volume-balance equation

$$u_1 + u_2 + u_3 = 1. \tag{2.2}$$

The theory of multiphase flows in porous media is based on the following form of Darcy's law

$$v_i = -k\lambda_i p_{ix}, \quad \lambda_i = \lambda_i(u_1, u_2), \tag{2.3}$$

where k stands for the absolute permeability, λ_i is the mobility of the i-th phase, and p_i is the pressure of the i-th phase.

The capillary pressures are defined as the presure differences (cf.,e.g.,[8, 1]), and we assume here that they are functions of the saturations u_1, u_2, that is,

$$P_1(u_1, u_2) = p_1 - p_3, \quad P_2(u_1, u_2) = p_2 - p_3, \tag{2.4}$$

with

$$\frac{\partial P_1}{\partial u_1} \geq 0, \quad \frac{\partial P_2}{\partial u_2} \geq 0. \tag{2.5}$$

Denote

$$\lambda = \sum_1^3 \lambda_i, \quad f_i = \frac{\lambda_i}{\lambda}, \qquad i = 1, 2, 3. \tag{2.6}$$

Assume, for simplicity, $k = m = \rho_i \equiv 1$, $i = 1, 2, 3$. For

$$v = \sum_1^3 v_i,$$

we find from (2.1) and (2.2) that $v_x = 0$, so v depends on t only. Thus, we also assume for simplicity that $v \equiv 1$ as well.

Eliminating the pressure derivative p_{3x}, we have from (2.3)

$$v_1 = f_1(1 + \lambda_2 P_{2x} - (\lambda_2 + \lambda_3)P_{1x}), \quad v_2 = f_2(1 + \lambda_1 P_{1x} - (\lambda_1 + \lambda_3)P_{2x}).$$

When we substitute these velocities into the first two equations in (2.1) we obtain the system in (1.1), where u and f are two-dimensional vectors with the components u_1, u_2 and f_1, f_2 respectively, and the 2×2-matrix D is given by

$$\begin{cases} D_{11} = \frac{\lambda_1(\lambda_2+\lambda_3)}{\lambda}\frac{\partial P_1}{\partial u_1} - \frac{\lambda_1\lambda_2}{\lambda}\frac{\partial P_2}{\partial u_1}, \; D_{12} = -\frac{\lambda_1\lambda_2}{\lambda}\frac{\partial P_2}{\partial u_2} + \frac{\lambda_1(\lambda_2+\lambda_3)}{\lambda}\frac{\partial P_1}{\partial u_2}, \\ \\ D_{21} = \frac{\lambda_2(\lambda_1+\lambda_3)}{\lambda}\frac{\partial P_2}{\partial u_1} - \frac{\lambda_1\lambda_2}{\lambda}\frac{\partial P_1}{\partial u_1}, \quad D_{22} = -\frac{\lambda_1\lambda_2}{\lambda}\frac{\partial P_1}{\partial u_2} + \frac{\lambda_2(\lambda_1+\lambda_3)}{\lambda}\frac{\partial P_2}{\partial u_2}. \end{cases} \tag{2.7}$$

Since u is a saturation vector, we should impose the restriction

$$u \in \Delta = \{u : u \in \mathbb{R}^2, \quad 0 \le u_i \le 1, \quad u_1 + u_2 \le 1\}. \tag{2.8}$$

Observe, that the triangle Δ can be defined as an intersection

$$\Delta = \cap_1^3 \{G_i(u) \le 0\}, \quad G_1 = -u_1, \quad G_2 = -u_2, \quad G_3 = u_1 + u_2 - 1.$$

Now, we formulate some hypotheses about λ_i and P_i. As mentioned above we require

$$\lambda_i > 0, \text{ if } u_i > 0, \quad \lambda_i|_{G_i=0} = 0, \quad i \in \{1, 2, 3\}, \tag{2.9}$$

and, concerning the capillarity matrix, we assume

$$D_{21} = 0, \quad D_{11} \ge 0, \quad D_{22} = D_{22}(u_2) \ge 0 \quad \text{in} \quad \Delta. \tag{2.10}$$

Conditions (2.9) are discussed in [1]. The hypothesis (2.10) means, in particular, that we are restricted to the case when the first phase (gas) is not responsible for the amount of diffusion in the equation for the second phase (water). The restrictions $D_{21} = 0$ and $D_{22} = D_{22}(u_2)$ are equivalent to

$$A\frac{\partial P_1}{\partial u_1} = \frac{\partial P_2}{\partial u_1}, \quad \frac{\partial P_2}{\partial u_2} = A\frac{\partial P_1}{\partial u_2} + \frac{\lambda D_{22}}{\lambda_2(\lambda_1 + \lambda_3)}, \quad A = \frac{\lambda_1}{\lambda_1 + \lambda_3}. \tag{2.11}$$

In particular, equations (2.11) are valid if

$$\lambda_i = k u_i, \quad P_1 = \alpha\xi, \quad P_2 = \frac{\alpha\xi^2}{2} + \frac{\beta u_2}{k}, \quad \xi \equiv \frac{u_1}{1 - u_2}, \tag{2.12}$$

where k, α, β are constants. The corresponding capillarity matrix is

$$\begin{cases} D_{11}^* = k\alpha\xi(1-\xi), \quad D_{12}^* = k\alpha\xi^2(1-\xi) - \beta u_1 u_2, \\ \\ D_{21}^* = 0, \quad D_{22}^* = \beta u_2(1 - u_2). \end{cases} \tag{2.13}$$

Hence, the example (2.12) verifies the hypotheses (2.9) and (2.10). When

$$\lambda_i = k_i u_i, \quad k_1 = k_3, \quad P_1 = \alpha\xi, \quad P_2 = \frac{\alpha\xi^2}{2} + \frac{\beta}{2}(\frac{1}{k_3} - \frac{1}{k_2})u_2^2 + \frac{u_2}{k_2} \tag{2.14}$$

or when

$$\lambda_i = k_i u_i, \quad k_1 \neq k_3, \quad P_1 = \alpha\xi - \frac{\beta k_0}{2} u_2^2, \quad k_0 = \frac{k_3 - k_1}{k_1 k_3},$$

(2.15)

$$P_2 = \frac{\beta}{2}\left(\frac{1}{k_3} - \frac{1}{k_2}\right)u_2^2 + \frac{\beta}{k_2}u_2 + \frac{\alpha k_1 \xi}{k_1 - k_3} - \frac{\alpha k_1 k_3}{(k_1 - k_3)^2}\ln\left((k_1 - k_3)\xi + k_3\right),$$

equations (2.11) are satisfied and the hypotheses (2.9) and (2.10) also hold, with D_{21} and D_{22} given by (2.13). Later on we will also use the example (2.12)-(2.13) for other hypotheses. The derivation of formulas (2.12), (2.14), (2.15), and (1.5) is given in the Appendix.

If $u \in \Delta$, the variable ξ takes values in the interval $[0, 1]$ and ξ is a continuous function of u everywhere in Δ except the point $(u_1, u_2) = (0, 1)$. From this example we see that equations (1.1) will, in general, constitute a degenerate parabolic system with bounded coefficients.The set of degeneration in the example given above coincides with the boundary of Δ.

Since the system without capillarity effects may even fail to be hyperbolic, as mentioned before, there is little hope of constructing a satisfactory well-posedness theory for the corresponding degenerate parabolic system, accounting for capillarity effects. So, instead of (2.10), we impose the following stronger condition:
There exists $\nu > 0$ such that

$$D_{11} \geq \nu, \quad D_{22} = D_{22}(u_2) \geq \nu. \tag{2.16}$$

Finally, in order to get the invariance of the triangle of saturations, which is the physically meaningful region, we further assume

$$D^\top \cdot \nabla_u G_i = \mu_i \nabla_u G_i \quad \text{if} \quad G_i(u) = 0, \tag{2.17}$$

i.e. $\nabla_u G_i$ is an eigenvector of the matrix $D^\top$, transpose of D, for $u \in \partial\Delta$ satisfying $G_i(u) = 0$.

Remark 2.1. We observe that the physical capillarity matrix given by (2.7) immediately satisfies (2.17). We also notice that (2.7) together with (2.5) implies $D_{11} \geq 0$ and $D_{22} \geq 0$, whenever we have $D_{21} = 0$.

An example of D satisfying (2.16), (2.17) is provided by the following perturbation of D^* defined above. Consider

$$D_{11}^{*\nu} = \nu + \frac{k\alpha u_1(1 - u_1 - u_2)}{(1 + \nu - u_2)^2}, \quad D_{12}^{*\nu} = \frac{k\alpha u_1^2(1 - u_1 - u_2)}{(1 + \nu - u_2)^2} - \beta u_1 u_2,$$
$$D_{21}^{*\nu} = 0, \quad D_{22}^{*\nu} = \nu + \beta u_2(1 - u_2).$$

Clearly, $D^{*\nu}$ satisfies the properties (2.16). Equations (2.17) are equivalent to the relations

$$D_{12}^{*\nu} = 0, \quad D_{21}^{*\nu} = 0, \quad D_{11}^{*\nu} - D_{12}^{*\nu} - D_{22}^{*\nu} = 0, \tag{2.18}$$

at the sets $G_1 = 0$, $G_2 = 0$, and $G_3 = 0$, respectively. The relationships (2.18) are valid by the choice of $D^{*\nu}$. We remark that D^* obeys the restriction (2.17) as well. It will become clear further on that this property of D^* jointly with (2.9) yields the invariance property $u \in \Delta$ for any smooth solution u of system (1.1), (2.6) , with $D = D^*$.

3. A priori estimates

In this section we prove some apriori estimates for the solutions of (1.1), (2.6), (2.7), subjected to hypotheses (2.9), (2.16), (2.17). We first consider the perturbed problem

$$u_t + f(u)_x = (D(u)u_x)_x + \varepsilon h(u), \quad u|_{t=0} = u_0(x), \tag{3.1}$$

for f, D as above, given by (2.6), (2.7), satisfying (2.9), (2.16), (2.17), and $\varepsilon > 0$ arbitrary. We prove the apriori estimates for (3.1), and the same apriori estimates for (1.1) will follow by a simple continuity argument letting $\varepsilon \to 0$. Here u_0 is a periodic function with the period 2. From now on, it is assumed that any x-periodic function has period 2. We denote the solution of (3.1) also by u without reference to ε until the study of the convergence problem when $\varepsilon \to 0$. The function h is assumed to satisfy the inequalities

$$\nabla_u G_i(u) \cdot h(u) < 0, \tag{3.2}$$

in a neighbourhood of $\partial\Delta \cap \{G_i = 0\}$, $i \in \{1, 2, 3\}$. Such a choice of h is possible since $\nabla_u G_i(u) \neq 0$ and Δ is a convex set. For instance, we may choose $h(u) = u_* - u$, where u_* is any point in the interior of Δ.

Lemma 3.1. *Any smooth periodical solution u of problem* (3.1) *satisfies the inclusion*

$$u(t, x) \in \Delta, \quad \forall (t, x) \in Q = (0, T) \times \Omega, \quad \Omega = \{x : -1 < x < 1\}. \tag{3.3}$$

Proof. We apply the method of positively invariant regions [5] (see also [11]). According to this method, the inclusion (3.3) results from the following three conditions.

(i) $\nabla_u G_i(u)$ is an eigenvector of the matrices ${f'_u}^\top$ and $D^\top$ for any u such that $G_i(u) = 0$;

(ii) if $D^\top \nabla_u G_i(u) = \mu_i(u) \nabla_u G_i(u)$, with $\mu_i \neq 0$, then $G_i''(u)(\eta, \eta) \geq 0$ whenever $\eta \cdot \nabla_u G_i(u) = 0$;

(iii) $h(u) \cdot \nabla_u G_i(u) < 0$ for any $u \in \partial\Delta \cap \{G_i = 0\}$, $i \in \{1, 2, 3\}$.

Due to the choice of G_i and h, the conditions (ii) and (iii) are satisfied. Let us verify the condition (i) for f'_u. We only consider the case of the function G_3, since two other cases require less calculations. The condition (i) is equivalent to the equality

$$\frac{\partial f_1}{\partial u_1} + \frac{\partial f_2}{\partial u_1} = \frac{\partial f_1}{\partial u_2} + \frac{\partial f_2}{\partial u_2}.$$

When $\lambda_3 = 0$ this reads

$$\frac{\partial \lambda_3}{\partial u_1} - \frac{\partial \lambda_3}{\partial u_2} = 0.$$

Since $\lambda_3(u_1, u_2) = 0$ on the line $u_1 + u_2 = 1$, by hypothesis (2.9), the above equation is clearly verified. Condition (i) for D is given by hypothesis (2.17). ☐

Lemma 3.2. *There is a constant c such that*

$$\|u_{1x}\|_{L^2(Q)} + \|u_{2x}\|_{L^2(Q)} \leq c.$$

Proof. Due to equation $(3.1)_2$, ...

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_2^2\, dx + \int_\Omega D_{22}u_{2x}^2\, dx = \int_\Omega f_2 u_{2x} + \varepsilon h_2 u_2\, dx.$$

Hence, one can apply Lemma 3.1, the Cauchy inequality, and the condition $D_{22} \geq \nu$ to conclude that u_{2x} is bounded in $L^2(Q)$. From equation $(3.1)_1$ we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_1^2\, dx + \int_\Omega D_{11}u_{1x}^2\, dx = \int_\Omega f_1 u_{1x} - D_{12}u_{1x}u_{2x} + \varepsilon h_1 u_1\, dx.$$

By the same argument we arrive at the estimate of u_{1x} in $L^2(Q)$. □

Lemma 3.3. *There are constants $c > 0$ and $\alpha \in (0,1)$ such that*

$$\|u_2\|_{H^{\alpha,\alpha/2}(\bar{Q})} \leq c.$$

Proof. Let $\tilde{\Omega}$ be a bounded open interval containing $\bar{\Omega}$, the closure of Ω. We take a test function ζ with values between 0 and 1, which is different from zero only for $x \in K_\rho$, the ball of radius ρ centered at $x_0 \in \tilde{\Omega}$. We multiply equation $(3.1)_2$ by

$$\zeta^2 \max\{u_2 - k, 0\} \equiv \zeta^2 u_2^{(k)}, \quad k \in \mathbb{R},$$

and integrate over the cylinder $(s,t) \times K_\rho$. We have

$$\frac{1}{2}\int_{K_\rho} \zeta^2 |u_2^{(k)}|^2\, dx\Big|_s^t + \nu\int_s^t\int_{K_\rho} \zeta^2 |u_{2x}^{(k)}|^2\, dxdt \leq$$

$$\int_s^t\int_{K_\rho} \zeta\zeta_t|u_2^{(k)}|^2 - 2D_{22}u_{2x}^{(k)}u_2^{(k)}\zeta\zeta_x + 2f_2\zeta\zeta_x u_2^{(k)} + f_2\zeta^2 u_{2x}^{(k)} + h_2 u_2^{(k)}\zeta^2 \equiv I.$$

By the Young inequality

$$I \leq \int_s^t\int_{K_\rho} \frac{\nu}{2}\zeta^2|u_{2x}^{(k)}|^2 + |u_2^{(k)}|^2(|\zeta\zeta_t| + c|\zeta_x|^2 + c\zeta^2) + c\zeta^2\chi_{A_{k,\rho}(t)}\, dxd\tau,$$

where $A_{k,\rho}(t)$ is the intersection of the support of $u_2^{(k)}$ with K_ρ, and χ_A is the characteristic function of the set A. These two inequalities imply that the function u_2 belongs to the class $\mathcal{B}_2(Q, M, \gamma, r, \delta, k)$ defined in [7] and hence, by theorem II.7.1 of [7], $u_2 \in H^{\alpha,\alpha/2}(\bar{Q})$ for some $\alpha \in (0,1)$.

□

Lemma 3.4. *There is a constant $c > 0$ such that*

$$\sup_{0\leq t\leq T}\int_\Omega u_{2x}^2\, dx + \int_Q u_{2xx}^2 + u_{2x}^4 + u_{2t}^2\, dxdt \leq c.$$

Proof. Let $\zeta(t,x)$ be a test function as above. We multiply equation $(3.1)_2$ by $(u_{2x}\zeta^2)_x$ and integrate over $(0,t)\times K_\rho$ to arrive at the inequality

$$\frac{1}{2}\int_{K_\rho} u_{2x}^2\zeta^2\,dx\Big|_0^t + \nu\int_0^t\int_{K_\rho} u_{2xx}^2\zeta^2\,dxdt \le \int_0^t\int_{K_\rho} u_{2x}^2\zeta\zeta_t - \frac{\partial}{\partial u_2}D_{22}u_{2x}^2u_{2xx}\zeta^2$$
$$-2\frac{\partial}{\partial u_2}D_{22}u_{2x}^3\zeta\zeta_x - 2D_{22}u_{2xx}u_{2x}\zeta\zeta_x + \frac{\partial f_2}{\partial u_1}u_{1x}u_{2xx}\zeta^2 + \frac{\partial f_2}{\partial u_2}u_{2x}u_{2xx}\zeta^2$$
$$+2\frac{\partial f_2}{\partial u_1}u_{1x}u_{2x}\zeta\zeta_x + 2\frac{\partial f_2}{\partial u_2}u_{2x}^2\zeta\zeta_x + \varepsilon h_2(u_{2x}\zeta^2)_x\,dxdt \equiv J.$$

By Lemmas 3.1 and 3.2, and by the Young inequality,

$$J \le \frac{\nu}{2}\int_0^t\int_{K_\rho} u_{2xx}^2\zeta^2 + c_*u_{2x}^4\zeta^2 + c_*\,dxdt.$$

We estimate the last integral, using the inequality (see lemma II.5.4 of [7])

$$\int_{K_\rho} v_x^4\zeta^2\,dx \le 16\mathrm{osc}^2\{v,K_\rho\}\int_{K_\rho} 2v_{xx}^2\zeta^2 + v_x^2\zeta_x^2\,dx. \tag{3.4}$$

By Lemma 3.3,

$$\mathrm{osc}^2\{u_2,K_\rho\} \le c\rho^\alpha.$$

Now, the assertion of the lemma follows if we take ρ such that $32c_*\rho^\alpha < \nu/4$. □

Lemma 3.5. *There are constants $c>0$ and $\alpha\in(0,1)$ such that $\|u_1\|_{H^{\alpha,\alpha/2}(\bar Q)} \le c$.*

Proof. We consider equation $(3.1)_1$ as a quasi-linear equation with respect to u_1:

$$u_{1t} - a(t,x,u_1,u_{1x})_x + b(t,x,u_1,u_{1x}) = 0, \tag{3.5}$$

with

$$a(t,x,u_1,p) = D_{11}(u_1,u_2(t,x))p,$$
$$b(t,x,u_1,p) = \frac{\partial f_1}{\partial u_1}(u_1,u_2(t,x))p + \frac{\partial f_1}{\partial u_2}(u_1,u_2(t,x))u_{2x}(t,x)$$
$$-\Big(\frac{\partial D_{12}}{\partial u_1}(u_1,u_2(t,x))p + \frac{\partial D_{12}}{\partial u_2}(u_1,u_2(t,x))u_{2x}(t,x)\Big)u_{2x}(t,x)$$
$$-D_{12}(u_1,u_2(t,x))u_{2xx}(t,x) - \varepsilon h_1(u_1,u_2(t,x)).$$

By the above lemmas, we have

$$ap \ge \nu p^2, \quad |a| \le \mu p, \quad \mu = \mathrm{const} > 0,$$
$$|b(t,x,u_1,p)| \le p^2 + \varphi(t,x), \quad (\int_0^T(\int_\Omega \varphi^q\,dx)^{r/q})\,dt)^{1/r} \equiv \|\varphi\|_{q,r,Q} \le \mu,$$

when $q=2, r=2$. Morever, the constants $q=2$ and $r=2$ satisfy the restrictions

$$\frac{1}{r} + \frac{1}{2q} = 1-\kappa_1, \quad q\in[1,\infty], \quad r\in[\frac{1}{1-\kappa_1},\frac{2}{1-2\kappa_1}], \quad 0<\kappa_1<\frac{1}{2}, \tag{3.6}$$

when $\kappa_1 = 1/4$. Thus, the functions a and b verify the conditions of [7], Chapter 5, implying the Hölder continuity of u_1 which is a solution of the quasilinear parabolic equation (3.5). □

Lemma 3.6. *There is a constant $c > 0$ such that*

$$\sup_{0\le t\le T}\int_\Omega u_{1x}^2\,dx+\int_Q u_{1xx}^2+u_{1x}^4+u_{1t}^2\,dxdt\le c.$$

Proof. By the same argument as in Lema 3.4,

$$\frac{1}{2}\int_{K_\rho} u_{1x}^2\zeta^2\,dx\Big|_0^t+\nu\int_0^t\int_{K_\rho} u_{1xx}^2\zeta^2\,dxdt\le$$

$$\int_0^t\int_{K_\rho}\frac{\nu}{2}u_{1xx}^2\zeta^2+c(u_{1x}^2+u_{2x}^2+u_{2x}^4+u_{1x}^4\zeta^2+u_{2xx}^2)\,dxdt.$$

Applying inequality (3.4) for $v = u_1$, we arrive at the conclusion of the lemma. □

Lemma 3.7. *There are constants $c > 0$ and $\alpha \in (0, 1)$ such that $\|u_{2x}\|_{H^{\alpha,\alpha/2}(\bar Q)} \le c$.*

Proof. Consider equation $(3.1)_2$ as an equation for u_2 in the domain $Q_2 = (0, T) \times (-2, 2)$:

$$u_{2t}-\frac{\partial}{\partial x}a(t,x,u_2,u_{2x})+b(t,x,u_2,u_{2x})=0,$$

with

$$a(t,x,u_2,p)=D_{22}(u_2)p,$$

$$b(t,x,u_2,p)=\frac{\partial f_2}{\partial u_1}(u_1(t,x),u_2)u_{1x}(t,x)+\frac{\partial f_2}{\partial u_2}(u_1(t,x),u_2)p-\varepsilon h_2(u_1(t,x),u_2).$$

By the above lemmas, we have

$$\nu\le a_p\le\mu,\quad |a|+|a_u|\le\mu(p+1),\quad a_x=0,|b|\le\mu p^2+\varphi(t,x),\quad \|\varphi\|_{2q,2r,Q_2}\le\mu,$$

if $q = r = 2$. Clearly, these constants q and r satisfy the restrictions (3.6) with $\kappa_1 = 1/4$. These properties of the functions a and b imply the estimate of the lemma for the subdomain $Q \subseteq Q_2$ according to the theory of quasilinear parabolic equations (cf., theorem V.3.1 of [7]).

□

Lemma 3.8. *There are constants $c > 0$ and $\alpha \in (0, 1)$ such that*

$$\|u_{1x}\|_{H^{\alpha,\alpha/2}(\bar Q)}\le c.$$

Proof. Let $\zeta(x)$ be a smooth function such that $\zeta(x) = 1$ if $|x| < 1$ and $\zeta(x) = 0$ if $|x| \ge 3/2$. Then the function $w = \zeta u_2$ solves the initial boundary-value problem

$$w_t-D_{22}(u_1,u_2)w_{xx}=F,\quad w|_{|x|=2}=0,\quad w|_{t=0}=\zeta u_{02},\tag{3.7}$$

$$F\equiv -D_{22}\zeta_{xx}-D_{22}\zeta_x u_{2x}-(\frac{\partial f_2}{\partial u_1}u_{1x}+\frac{\partial f_2}{\partial u_2}u_{2x})\zeta+\frac{\partial D_{22}}{\partial u_2}u_{2x}^2\zeta+\varepsilon h_2\zeta.$$

By the above estimates, there exists a constant c such that

$$\|F\|_{L^4(Q_2)}\le c,\quad Q_2=(0,T)\times(-2,2).$$

Hence, (see, e.g., [7]) u_{2xx} verifies the estimate $\|u_{2xx}\|_{L^4(Q)} \le c$ and the function b from (3.5) satisfies the inequality

$$|b(t,x,u_1,p)|\le p^2+\varphi(t,x),\quad \|\varphi\|_{4,4,Q}\le\mu.$$

Now, the estimate of the lemma can be obtained by the same argument as in Lemma 3.7. □

Following [7], we denote by $|v|_Q^{(\alpha)}, |v_0|_\Omega^{(\alpha)}$ the norms of v, v_0 in $H^{\alpha,\alpha/2}(Q)$ and $H^\alpha(\Omega)$, respectively.

Lemma 3.9. *If $u_0 \in H^{2+\beta}(\bar\Omega)$, $0<\beta<1$, then there exists a constant c such that for any solution u from $H^{2+\beta,1+\beta/2}(\bar Q)$ of equation (3.1) the estimate $|u|_Q^{(2+\beta)} \le c$ holds, where $c>0$ depends on T, $|u_0|_\Omega^{(2+\beta)}$, the C-norms of the functions $f(u)$, $D(u)$, and their first derivatives.*

Proof. We know from the lemmas above that there are constants $c>0$ and $\alpha \in (0,1)$ such that $|u_1, u_{1x}|_Q^{(\alpha)} \le c$. If $\gamma = \min\{\alpha,\beta\}$, it follows from the linear equation (3.7) that $|u_2|_Q^{(2+\gamma)} \le c$ (see, e.g., [7]). Let ζ be as in the proof of Lemma 3.8. The function $z = u_1\zeta(x)$ solves the linear problem

$$z_t - D_{11}(u)z_{xx} = G, \quad z|_{|x|=2} = 0, \quad z|_{t=0} = \zeta(x)u_0, \tag{3.8}$$

$$G = D_{11}(u_1\zeta_x)_x - \frac{\partial f_1}{\partial x}\zeta + \frac{\partial D_{11}}{\partial x}u_{1x}\zeta + (D_{12}u_{2x})_x\zeta + \varepsilon h_1\zeta.$$

Hence, by the same argument $|u_1|_Q^{(2+\gamma)} \le c$. To increase γ up to β, one should return to problem (3.7), which now ensures that $|u_2|_Q^{(2+\beta)} \le c$, and then pass to problem (3.8) to obtain that $|u_1|_Q^{(2+\beta)} \le c$.

□

4. Existence and uniqueness.

To prove the solvability of problem (3.1), we apply a fixed-point argument in the form of the Leray-Schauder principle as in [7]. Let B_γ be a Banach space of x-periodic vector functions $\mathbf{u}(t,x) \in \mathbb{R}^2$, such that

$$\|\mathbf{u}\|_{B_\gamma} := |\mathbf{u}|_Q^{(\gamma)} + |\mathbf{u}_x|_Q^{(\gamma)} < \infty, \qquad 0<\gamma<1.$$

The choice of γ will be made later. Given $\mathbf{a} \equiv (a_1,a_2) \in B_\gamma$ and $\lambda \in [0,1]$, we define $\mathbf{u} = (u,v)$ as a solution to the linear problem

$$v_t + \lambda[\frac{\partial f_2(\mathbf{a})}{\partial x} - (D_{22}(a_2)v_x)_x - \varepsilon h_2(\mathbf{a})] = (1-\lambda)\nu v_{xx},$$

$$u_t + \lambda[\frac{\partial f_1(\mathbf{a})}{\partial x} - (D_{11}(\mathbf{a})u_x)_x - (D_{12}(\mathbf{a})v_x)_x - \varepsilon h_1(\mathbf{a})] = (1-\lambda)\nu u_{xx},$$

$$\mathbf{u}|_{t=0} = (u_{01}(x), u_{02}(x)).$$

Due to uniqueness of the linear Cauchy problem (cf.,e.g., [7]), u is an x-periodic vector function. Thus, the operator $\mathbf{a} \mapsto \mathbf{u} \equiv A_\lambda(\mathbf{a})$ is well-defined, and its fixed points are solutions to problem (3.1) when $\lambda = 1$. By repeating the arguments of the above lemmas, one arrives at the apriori estimates for the fixed points $\mathbf{u}_\lambda$ of the operator A_λ:

$$\mathbf{u}_\lambda \in \Delta, \quad |\mathbf{u}_\lambda, \mathbf{u}_{\lambda x}|_Q^{(\alpha)} \le M, \quad |\mathbf{u}_\lambda|_Q^{(2+\beta)} \le M_1, \tag{4.1}$$

where the constants M, M_1, and $\alpha \in (0,1)$ are independent of λ. Setting $\gamma = \alpha$, we restrict A_λ to the set

$$\mathcal{U} = \{\mathbf{u} \in B_\alpha : \mathbf{u} \in \Delta', \quad |\mathbf{u}, \mathbf{u}_x|_Q^{(\alpha)} \le M', \quad \mathbf{u}|_{t=0} = \mathbf{u}_0(x)\}$$

where $\mathrm{int}(\Delta') \supset \bar{\Delta}$ and $M' > M$. Clearly, $\mathcal{U}$ is a bounded convex set in B_α, and all the fixed-points $\mathbf{u}_\alpha$ of A_α are strictly inside of $\mathcal{U}$.

As in [7], one can prove that the other conditions of the Leray-Schauder theorem are also verified. Namely,

(i) The map $(\mathbf{a}, \lambda) \mapsto A_\lambda(\mathbf{a})$ from $\mathcal{U} \times [0,1]$ to B_α is compact;

(ii) The family of maps $\mathbf{a} \mapsto A_\lambda(\mathbf{a})$, $\lambda \in [0,1]$, is equicontinuous on $\mathcal{U}$;

(iii) The family of maps $\lambda \mapsto A_\lambda(\mathbf{a})$, $\mathbf{a} \in \mathcal{U}$, is equicontinuous on $[0,1]$;

(iv) The operator A_0 has a unique fixed point inside of $\mathcal{U}$, and the mapping $\mathbf{a} \mapsto \mathbf{a} - A_0(\mathbf{a})$ has an inverse near this fixed point.

Hence, the Cauchy problem (3.1) has at least one x-periodic solution in the Hölder space $H^{2+\beta,1+\beta/2}(\bar{Q})$. Uniqueness can be established in the same manner as in [7]. Thus, we have proved the following.

Theorem 4.1. *Let the functions $f_i(u)$, $D_{ij}(u)$, their first derivatives, and the function $h(u)$ be the Hölder continuous with the Hölder exponent $\beta \in (0,1)$. Let the hypotheses* (2.9), (2.16), (2.17) *hold, and assume that the x-periodic (with the period 2) initial data $u_0(x)$ satisfies $u_0 \in H^{2+\beta}(\bar{\Omega})$, $u_0(x) \in \Delta$. Then the Cauchy problem* (3.1) *has a unique x-periodic solution $u(t,x) \in H^{2+\beta,1+\beta/2}(\bar{Q})$.*

Proof of Theorem 1.1. The assertion of Theorem 1.1 about existence follows as a consequence of Theorem 4.1. Indeed, since the estimate of Lemma 3.9 does not depend on ε, there is a sequence $\varepsilon_k \downarrow 0$ such that the corresponding sequence $u_k(t,x)$ of solutions of problem (3.1) converges to a function $u(t,x) \in H^{2+\beta,1+\beta/2}(\bar{Q})$ in the $|\cdot|_Q^{(2+\gamma)}$-norm for any $\gamma < \beta$. Clearly, u solves problem (1.1). Uniqueness can be proved in a straightforward manner as in Theorem 4.1. □

5. Appendix

Here we consider the linear system (2.11) and describe a class of solutions which includes (2.12), (2.14), (2.15), and (1.5) when

$$\lambda_i = k_i u_i, \quad D_{22} = \beta u_2(1-u_2).$$

First, we explain how the variable $\xi = \frac{u_1}{1-u_2}$ appears. Eliminating the function P_2 from (2.11) by cross-differentiation, we get

$$\frac{u_1}{1-u_2}\frac{\partial P_1}{\partial u_1} - \frac{\partial P_1}{\partial u_2} = \frac{D_{22}(u_2)k_0}{1-u_2}. \tag{5.1}$$

Clearly, this equation has a particular solution $P_1^*(u_2)$ depending only on the variable u_2. On the other hand, one can verify that any smooth function $\Phi(\xi)$ solves the homogeneous equation (5.1). Thus, the function $\Phi(\xi) + P_1^*(u_2)$ is a solution of (5.1).

Now, we look for a solution of (2.11) having the form

$$P_i = q_i(\xi) + p_i(u_2). \tag{5.2}$$

Since $u_3 = 1 - u_1 - u_2$, we calculate

$$\frac{\partial \xi}{\partial u_1} = \frac{1}{1-u_2}, \quad \frac{\partial \xi}{\partial u_2} = \frac{\xi}{1-u_2}, \quad A = \frac{k_1 \xi}{(k_1 - k_3)\xi + k_3},$$
$$\frac{\partial P_1}{\partial u_1} = \frac{q_1'}{1-u_2}, \quad \frac{\partial P_1}{\partial u_2} = \frac{\xi q_1'}{1-u_2} + p_1',$$
$$\frac{\partial P_2}{\partial u_1} = \frac{q_2'}{1-u_2}, \quad \frac{\partial P_2}{\partial u_2} = \frac{\xi q_2'}{1-u_2} + p_2',$$
$$\frac{\lambda D_{22}}{\lambda_2(\lambda_1+\lambda_3)} = \frac{\beta(1-u_2)}{k_2} + \frac{\beta u_2}{(k_1-k_3)\xi + k_3}.$$

With these formulas at hand, system (2.11) reads

$$\xi A(\xi) q_1'(\xi) = q_2'(\xi), \quad p_2'(u_2) = \frac{k_1 \xi p_1'(u_2) + \beta u_2}{(k_1-k_3)\xi + k_3} + \frac{\beta(1-u_2)}{k_2}. \tag{5.3}$$

If one choose

$$p_1'(u_2) = -k_0 \beta u_2, \tag{5.4}$$

equation $(5.3)_2$ reads

$$p_2'(u_2) = \beta u_2 (\frac{1}{k_3} - \frac{1}{k_2}) + \frac{\beta}{k_2}. \tag{5.5}$$

Let us take

$$q_1(\xi) = \alpha \xi^m, \quad m \geq 1.$$

Then

$$q_2'(\xi) = \alpha m \xi^{m-1} A(\xi). \tag{5.6}$$

Now, to obtain a class of solutions of type (5.2), we just integrate equations (5.4),(5.5), and (5.6).

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