EXCEPTIONAL FAMILIES OF FOLIATIONS AND THE POINCARÉ PROBLEM

A. Lins Neto*

Abstract. A 1-parameter family of foliations $(\mathcal{F}_\alpha)_{\alpha \in X}$ on a compact complex surface M is called exceptional and elliptic if it satisfies the following properties : (a). The family has singularities of fixed analytic type; (b). The set $E = \{\alpha \in X \mid \mathcal{F}_\alpha$ has a first integral$\}$ is countable and non-discrete; (c). There is $\alpha \in E$ such that the generic fibre of the first integral is elliptic. In this paper we show that, if a surface M admits an exceptional and elliptic family of foliations, then M is algebraic and biholomorphically equivalent to a torus, to a K3 surface, or to CP(2) (Theorem 3). In the case of CP(2) we classify all possible equireducible and exceptional families such that the singularities of the generic foliations in the family are non-degenerate (Theorem 2). This classification is connected to the Poincaré problem of deciding if an algebraic foliation on CP(2) has a first integral (cf. [P-1]).

§1 Introduction

Around 1891, Poincaré asked the following question (cf. [P-1]): "Is it possible to decide if an algebraic differential equation in two variables is algebraically integrable ?" (in the sense that it has a rational first integral). In [P-2] he starts, by observing that it is sufficient to bound the degree of a possible algebraic solution. In fact, in [P-2] and [P-3] he tries to bound this degree, in terms of the degree of the equation and some local invariants associated to the singularities. He supposes that all the singularities of the equation are non-degenerate and that the equation has a first integral around each singularity of the type $u^p/v^q = cte$, where $p \in \mathbb{N}$ and $q \in \mathbb{Z} \setminus \{0\}$ are relatively primes and depend only on the singularity. When $q > 0$ the first integral is meromorphic and he calls the singularity "dicritical" or node" ("noed"). When $q < 0$, the first integral is holomorphic and he calls the singularity a "saddle" ("col"). In [P-2] he solves the problem in the particular case where in all the saddles we have $p = 1$ and $q = -1$.

In a previous paper (cf. [LN]) we have given some examples of one parameter families of foliations in CP(2), of any degree $d \geq 2$, which show that the Poincaré problem of bounding the degree of an algebraic solution in terms of d and of local data involving the analytic type of its singularities, does not have solution. These examples, in degree $d \geq 5$, provide also a negative answer for the analogous Painlevé problem of bounding the genus of the generic level of a pencil which gives origin to a degree d foliation. The main purpose of this paper is to classify these families in special cases. In order to state properly our results, we give some definitions which synthetize some properties of the families of [LN].

First of all, let us recall the definition of the tangent line bundle associated to a foliation on a complex compact surface. A holomorphic singular foliation $\mathcal{F}$ on a compact complex surface M, with isolated singularities, can be defined by local holomorphic vector fields or 1-forms. More precisely, let $\mathcal{U} = (U_j)_{j \in J}$ be an open covering of M. In each U_j, the foliation is defined by a holomorphic vector field X_j with isolated singularities. If $U_i \cap U_j \neq \emptyset$, we require that X_i and X_j are multiple in $U_i \cap U_j$, that is there exists $f_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ such that $X_i = f_{ij} X_j$. This means

*This research was partially supported by Pronex.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TeX that the local integral curves of X_i in U_i and of X_j in U_j glue together, up to reparametrization, in the intersection $U_i \cap U_j$. The collection $(f_{ij})_{U_i \cap U_j \neq \emptyset}$ is a multiplicative cocycle and therefore defines a line bundle on M, which is called the cotangent bundle of F. The class of this bundle in $H^1(M, O^*)$ is denoted by T^*_F. The tangent bundle of F is the dual, T_F, of T^*_F. When $M = CP(2)$, the tangent bundle of a foliation F is related with the degree d of F by $T_F = (1 - d)H$, where H is the line bundle associated to a line in $CP(2)$ (cf. [Br]). Given a line bundle L on M, we will use the notation

$$F(M, L) = \{H \,;\, H \text{ is a foliation on } M \text{ such that } T_H = L\}.$$

It is well known that, if $F(M, L)$ is not empty, then it has a natural structure of holomorphic manifold (cf. [G-M]). A holomorphic family of foliations on M, is a holomorphic map $t \in X \mapsto F_t \in F(M, L)$, for some line bundle L and some complex manifold X. We will use the notation $(F_t)_{t\in X}$ for such a family. Given two foliations F and G on M such that $T_F = T_G$, defined by collections of holomorphic vector fields $(X_j)_{j\in J}$ and $(Y_j)_{j\in J}$, respectively, associated to the same covering $(U_j)_{j\in J}$ of M and the same cocycle $(f_{ij})_{U_i\cap U_j\neq\emptyset}$, we define the pencil of foliations generated by F and G as the family $(F_\alpha)_{\alpha\in\overline{\mathbb{C}}}$, where $F_\infty = G$ and F_α is defined by the collection of vector fields $(X_j + \alpha.Y_j)_{j\in J}$, if $\alpha \in \mathbb{C}$.

We say that F has a first integral, if there exists a non-constant map $f: M \to S$, where S is a Riemann surface, such that any level of f, $f^{-1}(c)$, $c \in S$, is an union of leaves and singularities of F. In this case, we will say also that f is tangent to F. We will suppose that the generic level curve of f is irreducible. It is well known that, the genus of two different generic levels of f are the same. This genus will be denoted by $g(f)$. For the basic definitions of the theory of foliations such as leaf, holonomy, etc..., we recomend [C-LN]. Given a family of foliations $P = (F_t)_{t\in X}$, we will use the notation

$$E(P) = \{t \in X \,|\, F_t \text{ has a first integral}\}.$$

1.1 Definition. Let M be compact complex surface and X be a Riemann surface. We say that a family of holomorphic foliations $P = (F_t)_{t\in X}$ is exceptional if :

(a). There exists a discrete subset $F \subset X$, such that if $t_1, t_2 \in X \setminus F$, then for any singularity p of F_{t_1}, there is a singularity q of F_{t_2}, such that the germs of F_{t_1} at p and of F_{t_2} at q are analitically equivalent. In this case, we will say also that the family has singularities of fixed analytic types. In the case where all singulatities of F_t ($t \notin F$) are non-degenerate we will say that the family is non-degenerate. Recall that the singularity p of F_t is non-degenerate if $\det(DX(p)) \neq 0$, for some (and so for any) holomorphic vector field X which represents F_t in a neighborhood of p.

(b). The set $E(P)$ is an infinite, countable and non-discrete subset of X.

We will say that the family is weakly exceptional, if $E(P)$ is at most countable and contains at least two different points.

Given $t \in E(P)$, let us denote by $f_t: M \to \overline{X}_t$ a rational first integral of F_t, whose generic level curve $f_t^{-1}(c)$ is irreducible. We say that the exceptional family $(F_t)_{t\in X}$ has unbounded genus, if for any $k > 0$, the set $\{t \in E(P);\, g(f_t) \le k\}$ is finite.

We can resume the results of [LN] in the following :

Theorem.[LN]. For any $d \ge 2$ there exists a non-degenerate exceptional pencil $P^d = (F^d_t)_{t\in\overline{\mathbb{C}}}$ on $CP(2)$ of degree d. Given $t \in E(P^d)$, let $f_t: CP(2) \to \overline{\mathbb{C}}$ be a first integral, whose generic levels are irreducible, and denote by $d(t)$ the degree of a level of f_t. Then, for any $k > 0$, the set $\{t \in E(P^d);\, d(t) \le k\}$ is finite. In particular, we can find in the family foliations with rational first integrals of arbitrarily large degrees. Moreover, if $d \ge 5$ then the family has unbounded genus.

1.2 Remark. We would like to observe that the families contructed in [LN] have the following additional properties :

(I). For any fixed family $(\mathcal{F}_t^d)_{t\in\overline{\mathbb{C}}}$, the blowing-up process used to reduce the singularities of $\mathcal{F}_t^d$ (in the sense of Seidemberg [Se]) is the same for all $t \in \overline{\mathbb{C}}$. A 1-parameter family of foliations which satisfies this property will be called *equireducible*.

(II). For $d = 2, 3, 4$ and $t \in E(P_d)$, if f_t is as before, then $g(f_t) = 1$. An exceptional family which satisfies this property will be called *elliptic*.

Moreover, in the families of degrees 2, 3 and 4 of [LN], the generic level (after normalization) of a first integral is biholomorphic to the torus $\mathbb{C}/<1, e^{2\pi i/3}>$. Here, $<1, b>$ denotes the lattice of $\mathbb{C}$ generated by 1 and $b \notin \mathbb{R}$. In §2.4 we will describe an exceptional pencil of degree three on $\mathbb{CP}(2)$ such that the generic level (after normalization) of a first integral is biholomorphic to the torus $\mathbb{C}/<1, i>$, $i = \sqrt{-1}$.

When the family $(\mathcal{F}_t)_{t\in S}$ is equireducible, then after the blowing-up process we obtain a rational surface M and a bimeromorphism $\pi\colon M \to \mathbb{CP}(2)$ such that for all $t \in S$, all singularities of the strict transform $\tilde{\mathcal{F}}_t$ of $\mathcal{F}_t$ by π are reduced in the sense of Seidemberg.

1.3 Definition. Let M be a compact connected complex surface, S a Riemann surface, and $f\colon M \to S$ be an elliptic fibration, that is a holomorphic map such that the generic level $f^{-1}(c)$ is irreducible. We say that a foliation $\mathcal{F}$ in M is *turbulent* with respect to f, if $\mathcal{F}$ is transverse to some level curve of f.

The main facts about turbulent foliations, that will be used here, are the following : let $\mathcal{F}$ be a foliation on a surface M, turbulent with respect to some elliptic fibration $f\colon M \to S$. Then the set $A = \{c \in S;\ \mathcal{F}$ is not transverse to $f^{-1}(c)\}$ is finite. Moreover, if $V = f^{-1}(S \setminus A)$, then $g := f|_V\colon V \to S \setminus A$ is a fibre bundle locally holomorphically trivial. In particular, if $c_1, c_2 \in S \setminus A$, then the fibres $f^{-1}(c_1)$ and $f^{-1}(c_2)$ are biholomorphic. In this case, we will say that the fibration f is *isotrivial* (cf. [Br-1] and [Br-2]). Note that the leaves of the restricted foliation $\mathcal{F}|_V$ are transverse to the fibres of $f|_V$, so that we can use the theory of foliations transverse to the fibres of a fibration (cf. [Eh] and [C-LN]).

1.4 Remark. Let $(\mathcal{F}_t^d)_{t\in\overline{\mathbb{C}}}$ be one of the families in [LN], of degree $d \in \{2, 3, 4\}$. Since it is equirreducible, there exists a rational surface M_d and a bimeromorphism $\pi_d\colon M_d \to \mathbb{CP}(2)$ (a composition of blowing-ups), which reduce the singularities of all foliations $\mathcal{F}_t^d$ simultaneously. Denote by $\tilde{\mathcal{F}}_t^d$ the strict transform of the foliation $\mathcal{F}_t^d$ by π_d. Then, in each case ($d = 2, 3, 4$), for any $t_o \in E$, if F_{t_o} is the rational first integral of $\mathcal{F}_{t_o}^d$ as in (c) of Definition 1.1, then $f_{t_o} := F_{t_o} \circ \pi_d\colon M_d \to \overline{\mathbb{C}}$ extends to an elliptic fibration. Moreover, if $t \neq t_o$, then the foliation $\tilde{\mathcal{F}}_t^d$ is turbulent with respect to f_{t_o}.

We need one more definition.

1.5 Definition. Let V and W be compact complex surfaces and $(\mathcal{F}_t)_{t\in T}$, $(\mathcal{G}_s)_{s\in S}$ be holomorphic families of foliations on V and W respectively, where T and S are Riemann surfaces. We say that $(\mathcal{F}_t)_{t\in T}$ *immerges* (resp. *immerges bimeromorphicaly*) in $(\mathcal{G}_s)_{s\in S}$, if there exists a map $\Phi = (\phi_1, \phi_2)\colon T \times V \to S \times W$ such that :

(a). ϕ_1 depends only on $t \in T$ and $\phi_1\colon T \to S$ is holomorphic.

(b). For each $t \in T$, if $f_t\colon V \to W$ is defined by $f_t(p) = \phi_2(t, p)$, then f_t is a biholomorphism (resp. bimeromorphism).

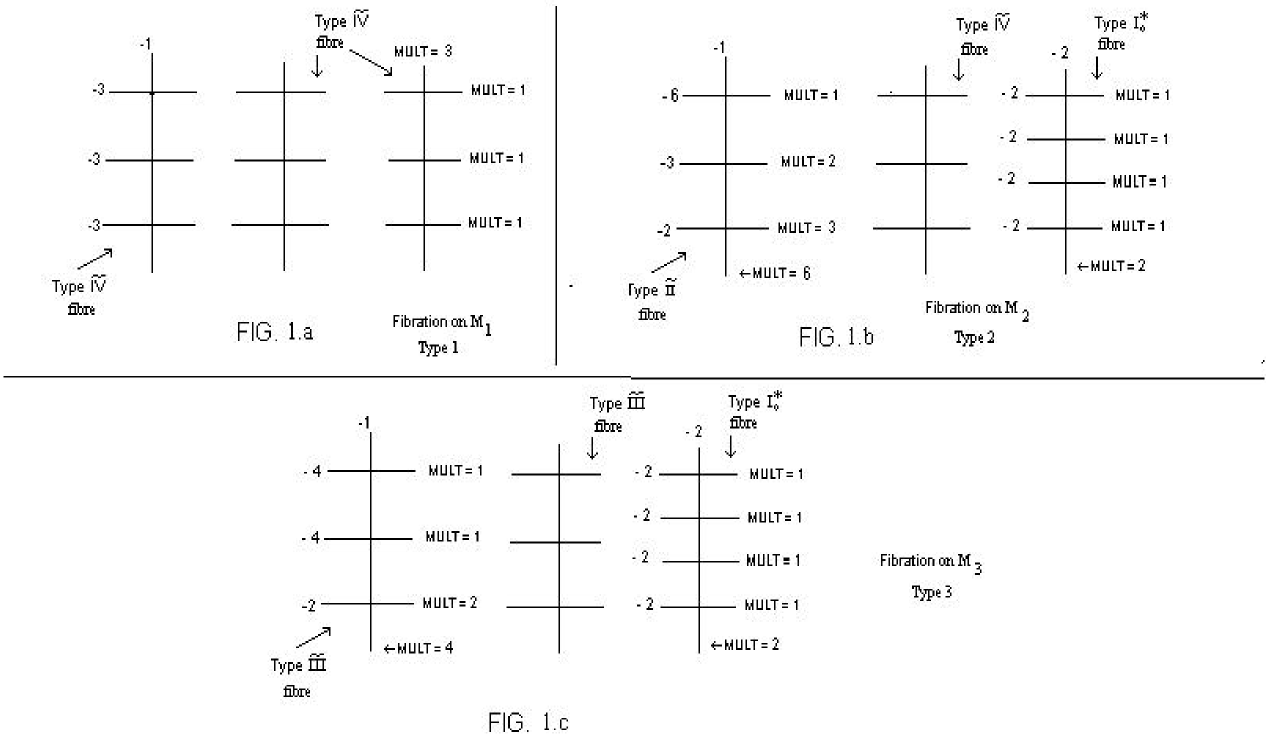
(c). For each $t \in T$, we have $f_t^*(\mathcal{G}_{\phi_1(t)}) = \mathcal{F}_t$.

If ϕ_1 is a biholomorphism, we will say that the families are *equivalent* (resp. *bimeromorphicaly equivalent*).

We now state our first result :

Theorem 1. *There are exactly three holomorphic pencils of foliations, say $P^j = (\mathcal{G}_\alpha^j)_{\alpha\in\overline{\mathbb{C}}}$, on three rational surfaces, say M_j, $j = 1, 2, 3$, such that any elliptic, equireducible, exceptional family of*

foliations on CP(2) immerges bimeromorphicaly in one of them. These pencils satisfy the following properties :
(a). For any $\alpha \in E(P^j)$, the first integral is an isotrivial elliptic fibration $f^j_\alpha : M_j \to \overline{\mathbb{C}}$, with three singular fibres. The generic fibre of of f^j_α is biholomorphic to $\mathbb{C}/\Gamma_j$, where $\Gamma_j = \langle 1; e^{2\pi i/3} \rangle$ for $j = 1; 2$ and $\Gamma_3 = \langle 1; i \rangle$ ($i = \sqrt{-1}$).
(b). For any $s_o \in E(P^j)$, if $s \neq s_o$, then the foliation G^j_s is turbulent with respect to $f^j_{s_o}$.
(c). If $s_1; s_2 \in E(P^j)$, then there exist biholomorphisms $\Phi : M_j \to M_j$ and $\phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $\phi \circ f^j_{s_1} = f^j_{s_2} \circ \Phi$.
(d). $E(P^j) = \mathbb{Q}.\Gamma_j \cup \{\infty\}$, $j = 1; 2; 3$, where $\mathbb{Q}.\Gamma_j = \{x.y \mid x \in \mathbb{Q}$ and $y \in \Gamma_j\}$. In particular, $E(P^j)$ is countable and dense in $\overline{\mathbb{C}}$.



We will call the family $(G^j_s)_{s \in \overline{\mathbb{C}}}$ the family of type j, $j = 1; 2; 3$. In the sections 2.2, 2.3 and 2.4, we will give examples of equirreducible exceptional families of foliations on $CP(2)$, such that after the resolution we get these families. In the figures 1.a, 1.b and 1.c, we sketch the typical fibrations f^j_s, $j = 1; 2; 3$, $s \in E_j$. The singular fibres which appear in the fibrations are the following :
Fibres of the type $\tilde{IV}$. This fibre is composed of four rational irreducible components. Three of these components have multiplicity one and self-intersection -3, as in figure 1.a. The reason for the notation is that it is obtained from the Kodayra fibre of type IV (cf. [K] and [BPV]) by doing one blowing-up at the intersection of the three rational components, as it is sketched in the figure 2.a.
Fibre of the type $\tilde{II}$. This fibre is composed of four rational irreducible components. The multiplicities and self-intersections of the components are sketched in figure 1.b. The reason for the notation is that it is obtained from the Kodayra fibre of type II (cf. [K]) by doing the blowing-up process sketched in the figure 2.c.
Fibres of the type $\tilde{III}$. This fibre is composed of four rational irreducible components. The multiplicities and self-intersections of the components are sketched in figure 1.c. The reason for the

notation is that it is obtained from the Kodayra fibre of type II (cf. [K]) by the doing blowing-up process sketched in the figure 2.b.

Fibres of the type I_0^*. This fibre appears in the fibrations of types 2 and 3.

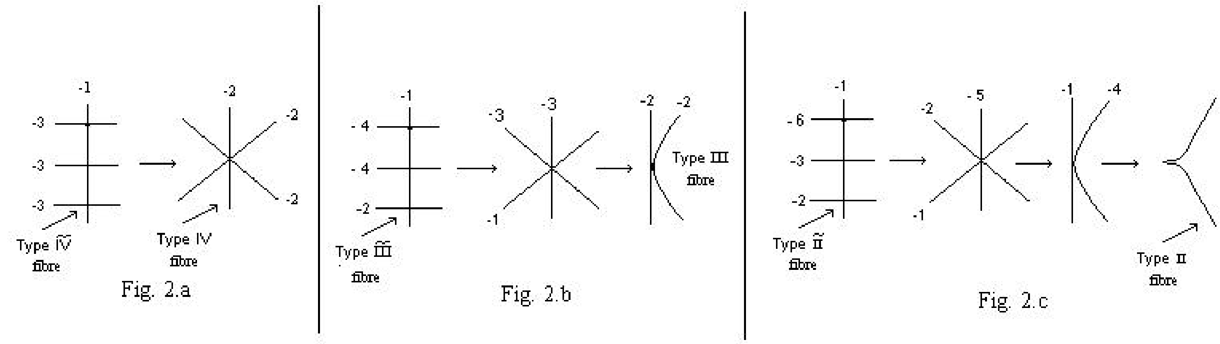


Fig. 2.a Fig. 2.b Fig. 2.c

As a consequence of Theorem 1, we will prove the following :

Theorem 2. There are exactly four elliptic non-degenerate exceptional pencils on $CP(2)$ such that any elliptic, exceptional, equireducible and non-degenerate family of foliations in $CP(2)$ immerges in one of them.

In §2 we will describe the families stated in Theorems 1 and 2. The prototypes of the families of Theorem 2 are given below.

1.6 Example. Each pencil, say $(F_\alpha)_{\alpha\in\overline{\mathbf{C}}}$, of Theorem 2 is defined in an appropriate affine coordinate system $(x;y)\in\mathbf{C}^2$ by polynomial vector fields X and Y, in such a way that X defines F_0, Y defines F_∞ and $X+\alpha.Y$ defines F_α. There are two pencils of degree three, one of degree two and one of degree four.

(1.6.1). The pencil of degree two. In this case, the vector fields X and Y are the following $X(x;y)=(4x-9x^2+y^2)\frac{\partial}{\partial x}+(6y-12xy)\frac{\partial}{\partial y}$ and $Y(x;y)=(2y-4xy)\frac{\partial}{\partial x}+3(x^2-y^2)\frac{\partial}{\partial y}$.

(1.6.2). The pencil of degree four. In this case, the vector fields X and Y are the following $X(x;y)=x(x^3-1)\frac{\partial}{\partial x}+y(y^3-1)\frac{\partial}{\partial y}$ and $Y(x;y)=y^2(x^3-1)\frac{\partial}{\partial x}+x^2(y^3-1)\frac{\partial}{\partial y}$.

(1.6.3). The first pencil of degree three. In this case, the vector fields X and Y are the following $X(x;y)=(-x+2y^2-4x^2y+x^4)\frac{\partial}{\partial x}+y(-2-3xy+x^3)\frac{\partial}{\partial y}$ and $Y(x;y)=(2y-x^2+xy^2)\frac{\partial}{\partial x}+(3xy-x^3+2y^3)\frac{\partial}{\partial y}$.

(1.6.4). The second pencil of degree three. In this case, the vector fields X and Y are the following $X(x;y)=(-4x+x^3+3xy^2)\frac{\partial}{\partial x}+2y(y^2-1)\frac{\partial}{\partial y}$ and $Y(x;y)=(x^2y-y^3)\frac{\partial}{\partial x}+2x(y^2-1)\frac{\partial}{\partial y}$.

The proofs of Theorems 1 and 2, will be based in the following :

Theorem 3. Let M be a complex compact surface and F, G, be two foliations on M such that $T_F=T_G$ and $P=(F_\alpha)_{\alpha\in\overline{\mathbf{C}}}$ be the pencil generated by F and G. Suppose that :

(i). $F\neq G$.

(ii). The singularities of F are reduced in the sense of Seidemberg.

(iii). F and G have holomorphic first integrals, say $f: M\to S_1$ and $g: M\to S_2$, respectively, where f is an elliptic fibration.

Then :

(a). The pencil $(F_\alpha)_{\alpha\in\overline{\mathbf{C}}}$ is a non-degenerate and elliptic weakly exceptional family.

(b). For any foliation H on M, such that $T_H=T_F$, there exists $\alpha\in\overline{\mathbf{C}}$ such that $H=F_\alpha$. In particular $F(M;T_F)=\{F_\alpha|\alpha\in\overline{\mathbf{C}}\}$.

(c). If $K_M \not\equiv 0$, then M is a rational surface. In this case, the pencil is exceptional and bimeromorphically equivalent to one of the families of types 1,2 or 3. Moreover, we have $E(P) = \lambda.\mathbb{Q}.\Gamma_j \cup \{\infty\}$, where $\lambda \in \mathbb{C}^*$ and $j \in \{1, 2, 3\}$. In particular, $E(P)$ is countable and dense in $\overline{\mathbb{C}}$.
(d). If $K_M = 0$ then, either M is a complex algebraic torus, or M is an algebraic $K3$ surface. Moreover, the family is exceptional if, and only if, $E(P)$ contains at least three elements.

As a consequence of (c) of Theorem 3, we have the following :

1.7 Corollary. Let $P = (\mathcal{F}_\alpha)_{\alpha\in\overline{\mathbb{C}}}$ be a pencil of foliations bimeromorphically equivalent to the pencil of type j, where $j \in \{1, 2, 3\}$. Let Γ_j be as before. If $\infty, \alpha_1, \alpha_2 \in E(P)$, where $\alpha_1, \alpha_2 \in \mathbb{Q}.\Gamma_j$ and $\alpha_1 \neq \alpha_2$, then $E(P) = \mathbb{Q}.\Gamma_j \cup \{\infty\}$.

In §2.1 we will describe two exceptional pencils of foliations, the first one in a complex 2-torus and the second in a Kummer surface (which is a special type of $K3$ surface). In §2.2, 2.3 and 2.4, we will describe, without details, the resolutions of the pencils in the examples 1.6.1,...,1.6.4. We will see also that they satisfy the hypothesis of Theorem 3. Theorem 3 will be proved in §3.2, Theorem 1 in §3.3 and Theorem 2 in §3.4. Before finishing this section, we would like to make some remarks and state some problems.

1.8 Remark. We would like to observe that the fact that $E(P) = \lambda.\mathbb{Q}.\Gamma_j \cup \{\infty\}$ in assertion (c) of Theorem 3, can be proved by using a result of [McQ] (see also [Br-2] pg. 110), once we know that the generic fibre of a first integral is biholomorphic to $\mathbb{C}/\Gamma_j$. This result says that if $kod(\mathcal{F}) = 0$, which is the case, then it is possible to find a ramified covering $\pi: N \to M$ and a birational morphism $p: N \to K$ such that $p_*(\pi^*(\mathcal{F}))$ is defined by a global holomorphic vector field on K, say X. Once we know some of the informations given in the proof of Theorem 3, it is possible to prove that $p_*(\pi^*(\mathcal{G}))$ is also defined by a global holomorhic vector field, say Y, in such a way that $p_*(\pi^*(\mathcal{F}_\alpha))$ is defined by $X + \alpha.Y$. These facts imply that K is a torus (see §2.1). In this paper we give a different proof, more adapted for our situation.

1.9 Remark. In the proof of our results we use strongly that the families are equireducibles. A natural question is if Theorems 1 and 2 are true for exceptional families, not necessarily equireducibles a priori. We would like to pose the following :

Problem 1. Let $(\mathcal{F}_n)_{n\geq 1}$ be a sequence of foliations on $\mathbb{CP}(2)$ with the following properties :

(i). All $\mathcal{F}_n$ have the same degree, say d.

(ii). For all $n \geq 1$, the singularities of $\mathcal{F}_n$ are non-degenerate. Moreover, for any singularity p of $\mathcal{F}_n$, there is a singularity q of $\mathcal{F}_1$, such that the germs of $\mathcal{F}_n$ at p and of $\mathcal{F}_1$ at q are analitically equivalent.

(iii). For all $n \geq 1$, $\mathcal{F}_n$ has a meromorphic first integral $f_n: \mathbb{CP}(2) \dashrightarrow \overline{\mathbb{C}}$, such that $g(f_n) = 1$, the general level curve $f_n^{-1}(c)$ is irreducible and $\lim_{n\to\infty}(deg(f_n)) = +\infty$.

Is it possible to immerge the sequence $(\mathcal{F}_n)_{n\geq 1}$ in one of the families of Theorem 2 ? In other words, is there a sequence of automorphisms of $\mathbb{CP}(2)$, say $(\varphi_n)_{n\geq 1}$, such that $\varphi_n^*(\mathcal{F}_n)$ is in one of these families, for all $n \geq 1$?

1.10 Remark. In our results we deal only with elliptic families of foliations. A natural question is the following:

Problem 2. Is it possible to classify all equireducible non-degenerate exceptional families of foliations on $\mathbb{CP}(2)$?

We would like to observe that the exceptional families in $\mathbb{CP}(2)$ of [LN], with unbounded genus, are obtained from the elliptic families by pulling back the elliptic families with fixed endomorphisms of $\mathbb{CP}(2)$ of topological degree ≥ 2. Since the endomorphims used in this construction are more or less arbitrary (generic), we can not expect to obtain a finite list of models, like in Theorem 2, for the general case.

§2 Description of the models

In this section we will describe some examples of non-degenerate, exceptional, elliptic families of foliations, including the four families in $\mathbb{CP}(2)$, one of degree two, two of degree three and one of degree four, which give origin to the three exceptional families of the statement of Theorem 1. Three of these families were already described in [LN], so that we will only give an idea of their construction and properties.

§2.1 Examples in a complex 2-torus and in a Kummer surface.

Let $M = T_1 \times T_2$, where $T_j = \mathbb{C}/\Gamma_j$ is an elliptic curve, such that Γ_j is the lattice in $\mathbb{C}$ generated by 1 and $a_j \notin \mathbb{R}$, $j = 1, 2$. We will take coordinates $(x, y) \in M$, where $x \in \mathbb{C}/\Gamma_1$ and $y \in \mathbb{C}/\Gamma_2$. Let $\mathcal{F}$ and $\mathcal{G}$ be the foliations generated by the non-vanishing vector fields $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$, respectively. If $\mathcal{P} = (\mathcal{F}_\alpha)_{\alpha \in \overline{\mathbb{C}}}$ is the pencil generated by $\mathcal{F}$ and $\mathcal{G}$, then $\mathcal{F}_\alpha$ is defined by the vector field $X_\alpha = X + \alpha.Y$, for every $\alpha \in \mathbb{C}$. This pencil is weakly exceptional in all cases, but it is not exceptional, in general. In fact, the set $E(\mathcal{P})$ contains at least two points, $\alpha = 0$ and $\alpha = \infty$. On the other hand, as the reader can check, the following assertions are equivalent :

(a). $\alpha \in E(\mathcal{P}) \setminus \{0, \infty\}$.

(b). If $\Gamma_1(\alpha)$ is the lattice $< \alpha, \alpha.a_1 > = \{\alpha(m + n.a_1) \mid m, n \in \mathbb{Z}\}$ and D is a fundamental domain of Γ_2, then $\Gamma_1(\alpha) \cap D$ is finite.

(c). There exists $k \in \mathbb{N} \setminus \{0\}$ such that $k.\Gamma_1(\alpha) \subset \Gamma_2$.

Assertion (c) implies that :

(d). $E(\mathcal{P}) \setminus \{0, \infty\} \neq \emptyset$ if, and only if, there exists $h \in PSL(2, \mathbb{Q})$ such that $a_1 = h(a_2)$.

In this case, we can write $a_1 = \frac{k + \ell.a_2}{m + n.a_2}$, where $k, \ell, m, n \in \mathbb{Z}$ and $kn - m\ell \neq 0$. It is easy to see that

$$E(\mathcal{P}) \supset \{\alpha \in \mathbb{C} \mid \text{there exists } p \in \mathbb{Z} \text{ such that } p.\alpha = m + n.a_2 \text{ and } p.\alpha.a_1 = k + \ell.a_2\} .$$

Under assumption (d), this last set is infinite and countable, so that the pencil is exceptional. In particular, if $T_1 = T_2$, the pencil is exceptional. On the other hand, if $a_1 \notin \{h(a_2) \mid h \in PSL(2, \mathbb{Q})\}$, then $E(\mathcal{P}) = \{0, \infty\}$, and so the pencil is not exceptional.

Given the torus M as above, it can be defined the Kummer surface $Km(M)$. This surface is defined as follows : let $I : M \to M$ be the involution , which in representation $\mathbb{C}/\Gamma_1 \times \mathbb{C}/\Gamma_2$ is of the form $I(x, y) = (-x, -y)$. This involution has sixteen fixed points, say $p_1, \ldots, p_{16}$, so that $M_1 = M/ < I >$, is a singular surface with sixteen singularities, say $q_1, \ldots, q_{16}$. When we resolve these singularities, we obtain the Kummer surface $Km(M)$, which contains sixteen rational curves with self-intersection -2, say $C_1, \ldots, C_{16}$, where C_j corresponds to q_j, $j = 1, \ldots, 16$ (for the details see [BPV] pg. 170). Note that $Km(M) \setminus (\cup_j C_j)$ is naturally biholomorphic to $M_1 \setminus \{p_1, \ldots, p_{16}\}$ and the quotient map by the involution, induces a covering map of degree two, say $P : M \setminus \{p_1, \ldots, p_{16}\} \to Km(M) \setminus (\cup_j C_j)$. On the other hand, $I_*(X) = -X$ and $I_*(Y) = -Y$, so that $I_*(X + \alpha.Y) = -(X + \alpha.Y)$ and the foliation $\mathcal{F}_\alpha$ is invariant by the involution. This implies that there exists a foliation $\mathcal{G}_\alpha$ on $Km(M) \setminus (\cup_j C_j)$ such that $P^*(\mathcal{G}_\alpha) = \mathcal{F}_\alpha$. Since the curves C_j are -2-curves, this foliation extends to a foliation on $Km(M)$, which we denote also by $\mathcal{G}_\alpha$. This defines a pencil of foliations $\mathcal{Q} := (\mathcal{G}_\alpha)_{\alpha \in \overline{\mathbb{C}}}$. Note that $E(\mathcal{Q}) = E(\mathcal{P})$, so that the pencil $\mathcal{Q}$ is always weakly exceptional and it is exceptional if, and only if, $\mathcal{P}$ is exceptional. We observe that if $\alpha \in E(\mathcal{Q})$, then the first integral of $\mathcal{G}_\alpha$ is a fibration $f_a : Km(M) \to \overline{\mathbb{C}}$ which has four critical fibres of type I_0^*. This last fact will be proved in §3.2.

§2.2 The type 1 exceptional family.

The exceptional family of type 1 can be obtained by the resolution of the singularities of an equirreducible family of degree four in $CP(2)$, which will be denoted by $\mathcal{P}^4 = (\mathcal{F}^4_\alpha)_{\alpha\in\overline{\mathbb{C}}}$. This family is characterized by the fact that the lines in $CP(2)$ defined in homogeneous coordinates by the equation

$$(1)\quad (y^3 - x^3)(z^3 - y^3)(x^3 - z^3) = 0 .$$

are invariant for the foliation $\mathcal{F}^4_\alpha$, for all $\alpha \in \overline{\mathbb{C}}$. If $j = e^{2\pi i/3}$, then these lines, in the affine coordinate system $\{z = 1\}$, are given by $\ell_1 := \{x = 1\}$, $\ell_2 := \{x = j\}$, $\ell_3 := \{x = j^2\}$, $\ell_4 := \{y = 1\}$, $\ell_5 := \{y = j\}$, $\{\ell_6 := y = j^2\}$, $\ell_7 := \{y = x\}$, $\ell_8 := \{y = j.x\}$ and $\ell_9 := \{y = j^2.x\}$. They intersect in twelve points, that we will denoted by $p_1, ..., p_{12}$. These sets of lines and points define a configuration of lines and points $\mathcal{C} := (\mathcal{L}, \mathcal{P})$, where $\mathcal{L} := \{\ell_1, ..., \ell_9\}$ and $\mathcal{P} := \{p_1, ..., p_{12}\}$. We observe that each line $\ell_j \in \mathcal{L}$, contains four points in the set $\mathcal{P}$, and each point $p_i \in \mathcal{P}$ is contained in three lines of $\mathcal{L}$.

In the above affine coordinate system, the foliation $\mathcal{F}_\alpha$ is defined by the vector field $X + \alpha.Y$, where $X(x, y) = x(x^3 - 1)\frac{\partial}{\partial x} + y(y^3 - 1)\frac{\partial}{\partial y}$ and $Y(x, y) = y^2(x^3 - 1)\frac{\partial}{\partial x} + x^2(y^3 - 1)\frac{\partial}{\partial y}$. This pencil is described in §2.2 of [LN], so that we will only resume its main properties. Before the description, let us fix a notation.

2.2.1 Notation. Let $\mathcal{F}$ be a foliation on a surface M. We say that a singularity P of $\mathcal{F}$ is of the type $p : q$, where $p, q \in \mathbb{Z}^*$ and $gcd(p, q) = 1$, if in suitable holomorphic coordinate system (x, y) around P with $x(P) = y(P) = 0$, the foliation $\mathcal{F}$ is represented by the vector field $X(x, y) = px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y}$. Note that this vector field has the first integral x^q/y^p. For this reason, we will say also that a singularity of type $1 : 1$ is a radial singularity. In the notation $p : q$, we will identify $p : q \equiv q : p \equiv -p : -q \equiv -q : -p$.

If $\alpha \notin F = \{1, j, j^2, \infty\}$, then $\mathcal{F}^4_\alpha$ has twelve radial singularities at the points of $\mathcal{P}$ and nine singularities of the type $1 : -3$, say $q_1(\alpha), ..., q_9(\alpha)$, where $q_k(\alpha) \in \ell_k$. We observe that, for each $k = 1, ..., 9$, the map $\alpha \in \overline{\mathbb{C}} \mapsto q_k(\alpha) \in \ell_j$, is a regular parametrization of ℓ_k. When $\alpha \in \{1, j, j^2, \infty\}$, then the point $q_k(\alpha)$ coincides with some point in $\mathcal{P}$, and so the foliation $\mathcal{F}^4_\alpha$ has a degenerate singularity at this point. In [LN] it is proved that the pencil is elliptic and exceptional. Moreover, $\{0, 1, \infty\} \subset \mathcal{E}(\mathcal{P}^4)$, so that $\mathcal{E}(\mathcal{P}^4) = \mathbb{Q}. < 1, j > \cup \{\infty\}$, by the Corollary of Theorem 3. In fact, in §2.2 of [LN] it is proved that, for $\alpha \in \{0, 1, \infty\}$, $\mathcal{F}^4_\alpha$ has the following first integrals

$$f_0(x, y) = \frac{x^3(y^3 - 1)}{y^3(x^3 - 1)} \;;\; f_1(x, y) = \frac{(x - j^2)(y - j)(y - j^2x)}{(x - j)(y - j^2)(y - jx)} \;;\; f_\infty(x, y) = \frac{y^3 - 1}{x^3 - 1}$$

The process of reduction of the singularities for $\mathcal{F}^4_\alpha$ involves twelve blowing-ups at the points of $\mathcal{P}$. Let us denote by M_1 the surface obtained from $CP(2)$ by doing one blowing-up at each point of $\mathcal{P}$, and by $\pi : M_1 \to CP(2)$ the composition of these blowing-ups. The family of type 1 is defined as $\mathcal{Q}^1 = (\mathcal{G}^1_\alpha)_{\alpha\in\overline{\mathbb{C}}}$, where $\mathcal{G}^1_\alpha = \pi^*(\mathcal{F}^4_\alpha)$. We observe that $\mathcal{E}(\mathcal{Q}^1) = \mathcal{E}(\mathcal{P}^4)$ and for any $\alpha \in \mathcal{E}(\mathcal{P}^4)$ the fibration f_α tangent to $\mathcal{G}^1_\alpha$ is like in fig. 1.a. Moreover, the strict transforms $\tilde{\ell}_1, ..., \tilde{\ell}_9$ of the lines $\ell_1, ..., \ell_9$, are the unique curves in M_1 that are invariant for all foliations in the family $\mathcal{Q}^1$. Each curve $\tilde{\ell}_k$ contains an unique singularity of $\mathcal{G}^1_\alpha$, say $\tilde{q}_k(\alpha)$, such that $\pi(\tilde{q}_k(\alpha)) = q_k(\alpha)$. This singularity is of the type $1 : -3$.

§2.3 The type 2 exceptional family.

In this section we will describe two non-degenerate families of foliations on $CP(2)$ which give origin two the type 2 exceptional family. The first one is a family of degree three, which is obtained from the family of §2.2 by using that the differential equations which define it, are invariant with respect to the change of variables $S(x,y) = (y,x)$. In §2.3 of [LN], it is proved that there exists another family of foliations, say $\mathcal{P}^3 = (\mathcal{F}^3_\alpha)_{\alpha\in\overline{\mathbb{C}}}$, such that for every $\alpha \in \overline{\mathbb{C}}$ we have $\mathcal{F}^4_\alpha = T^*(\mathcal{F}^3_\alpha)$ where $T\colon \mathbb{C}P(2) \to \mathbb{C}P(2)$ is the rational map which in the coordinate system (x,y) of §2.2 is expressed as $T(x,y) = (u,v) = (x+y, x.y)$. The foliation $\mathcal{F}^3_\alpha$ is defined in the affine coordinate system (u,v) by the vector field $X + \alpha.Y$, where the expresssions of X and Y are given in the example 1.6.3 (in terms of x and y). The main facts about the pencil $\mathcal{P}^3$ are the following :

2.3.1. $E(\mathcal{P}^3) = E(\mathcal{P}^4)$.

2.3.2 **Invariant curves.** There are five curves in $\mathbb{C}P(2)$ which are invariant for all foliations in the family. These curves are the images by T of the lines in the configuration $\mathcal{L}$:

(I). The lines $(x = j^k)$ and $(y = j^k)$ are sent by T into the line $(v - j^k u + j^{2k} = 0)$, $k = 0,1,2$. This implies that the foliation $\mathcal{F}^3_\alpha$ has three invariant lines; $\ell_k := (v - j^k u + j^{2k} = 0)$, $k = 0,1,2$.

(II). The line $(y = x)$ is sent by T into the conic $C_1 := (v = \frac{1}{4}u^2)$.

(III). The lines $(y = j\,x)$ and $(y = j^2 x)$ are sent by T into the conic $C_2 := (v = u^2)$.

In figure 3 we sketch this configuration of curves. Denote by Γ^3 the union of these curves.

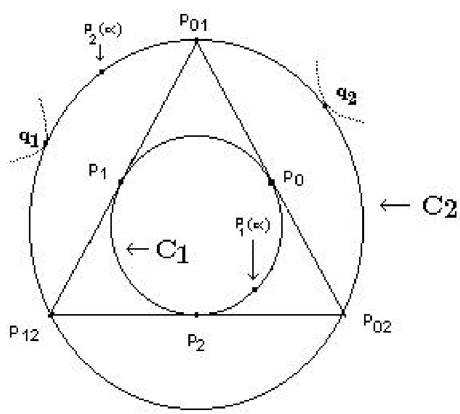


Fig. 3

2.3.3 **Singularities.** Observe that the singular points of Γ^3 are singularities of all foliations in the pencil $\mathcal{P}^3$. The conics C_1 and C_2 are tangent at the points $q_1 = [0:0:1]$ and $q_2 = [0:1:0]$, the lines ℓ_k, $k = 0,1,2$, intersect at the points $p_{01} = [-j^2 : j : 1] \in \ell_0 \cap \ell_1$, $p_{02} = [-j : j^2 : 1] \in \ell_0 \cap \ell_2$ and $p_{12} = [-1 : 1 : 1] \in \ell_1 \cap \ell_2$ and the lines are tangent to the conic C_1 at the points $p_0 = [2:1:1] \in \ell_0 \cap C_1$, $p_1 = [2j : j^2 : 1] \in \ell_1 \cap C_1$ and $p_2 = [2j^2 : j : 1] \in \ell_2 \cap C_1$. Observe also that $p_{01}, p_{02}, p_{12} \in C_2$. In Proposition 7 of [LN], it is proved that, if $\alpha \notin \{0,1,j,j^2,\infty\} := F$, then the singularities of $\mathcal{F}^3_\alpha$ are non-degenerate of the following types :

(IV). The points p_{01}, p_{02} and p_{12} are radial singularities.

(V). The points p_1, p_2, p_3, q_1 and q_2 are of the type $2:1$.

(VI). Each one of the five curves contains another singularity, say $P_1(\alpha) \in C_1$, $P_2(\alpha) \in C_2$ and $Q_k(\alpha) \in \ell_k$, $k = 0,1,2$. They are of the following types : $P_1(\alpha)$ is of the type $1 : -6$, the others are of the type $1 : -3$.

The reduction of the singularities of the elements of the family is done with a total of thirteen blowing-ups, as follows : one blowing-up at each of the three radial singularities and two blowing-ups at each of the five singularities of the type $2:1$. Denote by M_2 the rational surface obtained from $\mathbb{C}P(2)$ by this blowing-up process, by $\pi\colon M_2 \to \mathbb{C}P(2)$ the blowing-up map and let $\mathcal{G}^2_\alpha := \pi^*(\mathcal{F}^3_\alpha)$. The pencil $\mathcal{Q}^2 := (\mathcal{G}^2_\alpha)_{\alpha\in\overline{\mathbb{C}}}$ will be called the type 2 family. In §2.3 of [LN] it is proved that this pencil satisfies properties (a) and (b) of Theorem 1. Property (c) will be proved in §3.2. The typical elliptic fibration which appears in this case is sketched in fig. 1.b. This fibration appears, for instance, as a first integral of the foliation $G_1^2 = \pi^*(F_1^3)$. The foliation F_1^3 has the following rational first integral :

$$R(u,v) = \frac{(u^2 - 4v)(v - u^2)^2}{(u^3 - 3uv - 2)^2}.$$

The reader can check that $g = R \circ \pi : M_2 \to \overline{\mathbb{C}}$ is an elliptic fibration with three critical levels, namely $\{g = 0\}$, $\{g = 1\}$ and $\{g = \infty\}$, as sketched in figure 1.b.

There is another non-degenerate family of foliations on CP(2) which gives origin to the type 2 family. This family is obtained from the family $(F^3_\alpha)_{\alpha \in \overline{\mathbb{C}}}$ by a Cremona transformation as ilustrated in Fig. 4 (see also Lemma 3.4.14).

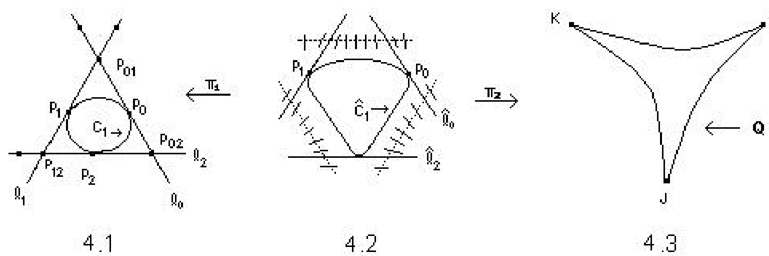


Fig 4

In this figure, we denote by π_1 the blowing-up at the three points p_{01}, p_{02} and p_{12} (see figure 3). After this blowing-up process, we obtain three divisors, not invariant for the strict transform $\pi_1^*(F^3_\alpha)$, because p_{01}, p_{02} and p_{12} are radial singularities ($\alpha \notin \{0, 1, j, j^2, \infty\}$). Moreover, the strict transforms of ℓ_0, ℓ_1 and ℓ_2, say $\hat{\ell}_0$ $\hat{\ell}_1$ and $\hat{\ell}_2$, have self-intersection -1, so that, we can blow-down these three curves. The map indicated by π_2 in Fig. 4, is the blowing-up associated to this blowing-down process. The curve indicated by $\hat{C}_1$ is the strict transform of the curve C_1. This curve is sent by π_2 in the curve Q of Fig. 4.3, which is a quartic with three cuspidal points, which we denote by J, K and L. We call π the bimeromorphism $\pi_2 \circ (\pi_1)^{-1}$. This type of blowing-up-blowing-down process is known in the literature as a "Cremona transformation". It is well known that the manifold, obtained after a Cremona transformation in CP(2), is again CP(2). The curve C_2 is transformed by π in a straight line, say R, which meets Q in two tangent points, which we denote by M and N. The pencil $P^2 := (F^2_\alpha)_{\alpha \in \overline{\mathbb{C}}}$ is defined by $F^2_\alpha = \pi_*(F^3_\alpha)$. The main facts about the pencil P^3 are the following (see §2.4 of [LN]):

2.3.4. Any foliation F^2_α in the pencil has degree two. Moreover, $E(P^2) = E(P^3)$.

2.3.5 Invariant curves. The algebraic invariant curves for all foliations in the pencil are the quartic Q and line R.

2.3.6 Singularities. For $\alpha \notin \{0, 1, j, j^2, \infty\}$ the singularities of F^2_α are non-degenerate of the following types :

(VII). The cuspidal points of Q are of the type $3 : 2$.

(VIII). The tangency points M and N between Q and R are of the type $2 : 1$.

(IX). The quartic Q contains a singularity $P_1(\alpha)$ of the type $1 : -6$.

(X). The line R contains a singularity $P_2(\alpha)$ of the type $1 : -3$.

Finally, we would like to observe that it is possible to find an affine coordinate system $(\mathbb{C}^2, (x, y))$ in CP(2) such that F^2_α is defined by $X + \alpha.Y$, where $X(x, y) = (4x - 9x^2 + y^2)\frac{\partial}{\partial x} + (6y - 12xy)\frac{\partial}{\partial y}$

and $Y(x,y) = (2y - 4xy)\frac{\partial}{\partial x} + 3(x^2 - y^2)\frac{\partial}{\partial y}$. In this coordinate system, the line R is the line at infinity, the quartic Q is given by $F(x,y) = 0$, where $F(x,y) = 4y^2(1-3x) - 4x^3 + (3x^2+y^2)^2$, $P_1(\alpha) = (\frac{4(1+\alpha^2)}{(3+\alpha^2)^2}, \frac{-8\alpha}{(3+\alpha^2)^2})$ and $P_2(\alpha) = [1:\alpha:0]$. Moreover, the foliations $\mathcal{F}^2_\infty$, $\mathcal{F}^2_1$ and $\mathcal{F}^2_{-1}$ have the following first integrals :

$$g_\infty(x,y) = \frac{F(x,y)}{(2x-1)^3}\ ;\ g_1(x,y) = \frac{F(x,y)}{(y-x)^3} \text{ and } g_{-1}(x,y) = \frac{F(x,y)}{(y+x)^3}$$

respectively, as the reader can check. This implies that $E(\mathcal{P}^2) = \mathbb{Q}/<1,j> \cup \{\infty\}$.

§2.4 The type 3 family.

In this section we show an example of an exceptional non-degenerate family, for which, the elliptic fibration which appear after the reduction of singularities has elliptic fibres biholomorphic to $\mathbb{C}/<1,i>$, where $i = \sqrt{-1}$. This family is obtained as the set of foliations of degree three which leave invariant all curves of the configuration sketched in figure 5. The five curves in this figure, in some affine coordinate system $(\mathbb{C}^2,(x,y))$, are :

(a). The circle $C_1 := \{(x-1)^2 + y^2 = 1\}$.
(b). The circle $C_{-1} := \{(x+1)^2 + y^2 = 1\}$.
(c). The line $L_1 := \{y = 1\}$.
(d). The circle $L_{-1} := \{y = -1\}$.
(e). The line at infinity in this affine system, denoted by L_∞.

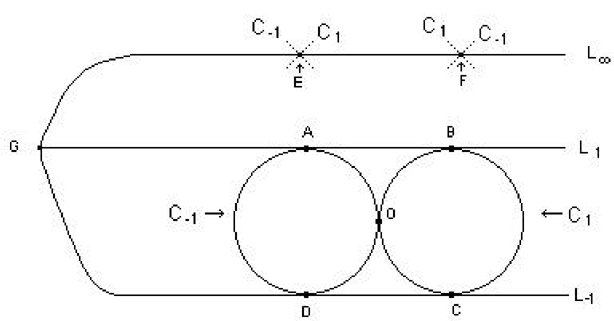


Fig. 5

The two circles are tangent at origin, $O = (0,0)$. The line L_1 is tangent to the circle C_{-1} at the point $A = (-1,1)$ and to the circle C_1 at the point $B = (1,1)$. The line L_{-1} is tangent to the circle C_{-1} at the point $D = (-1,-1)$ and to the circle C_1 at the point $C = (1,-1)$. The two circles intersect in two more points, $E = [1:i:0]$ and $F = [1:-i:0]$, which belong to L_∞. Finaly, the three lines intersect at the point $G = [1:0:0] \in L_\infty$. We will denote by $\mathcal{C}^{3,1}$ the union of these curves.

The reader can check that any foliation of degree three which leaves invariant the five curves in (a), (b), (c), (d) and (e), is defined the polynomial vector field $X + \alpha.Y$, where $X(x,y) = (-4x + x^3 + 3xy^2)\frac{\partial}{\partial x} + 2y(y^2-1)\frac{\partial}{\partial y}$ and $Y(x,y) = (x^2y - y^3)\frac{\partial}{\partial x} + 2x(y^2-1)\frac{\partial}{\partial y}$. The pencil defined in this way will be denoted by $\mathcal{P}^{3,1} = (\mathcal{F}^{3,1}_\alpha)_{\alpha\in\overline{\mathbb{C}}}$. Next we will see that this family is equirreducible and that after the desingularisation process we obtain a pencil of foliations which satisfies the hypothesis of Theorem 3.

2.4.1 Lemma. If $\alpha \notin \{1,-1,i,-i,\infty\}$, then $\mathcal{F}^{3,1}_\alpha$ has 13 non degenerated singularities :

(I). The points E, F and G are radial singularities.
(II). The points A, B, C, D and O are singularities of the type 2 : 1.

Each irreducible component of $\Gamma^{3:1}$ contains a singularity outside $\mathrm{sing}(\Gamma^{3:1})$. They are the following :
(III). The points $P_{-1}(\alpha) := (\alpha, -1) \in L_{-1}$, $P_1(\alpha) := (-\alpha, 1) \in L_1$, $Q_{-1}(\alpha) := (\frac{-2}{1+\alpha^2}, \frac{2\alpha}{1+\alpha^2}) \in C_{-1}$ and $Q_1(\alpha) := (\frac{2}{1+\alpha^2}, \frac{-2\alpha}{1+\alpha^2}) \in C_1$. These singularities are of the type 1 : −4.
(IV). The point $P_\infty(\alpha) := [\alpha : 1 : 0] \in L_\infty$. This singularity is of the type 1 : −2.

Proof. The fact that $\mathrm{sing}(\mathcal{F}^{3:1}_\alpha)$ has thirteen points as described in (I),...,(IV), can be proved as follows : by solving the system of algebraic equations given by $X(x,y) + \alpha.Y(x,y) = 0$ we find the finite singularities, which are A, B, C, D, O, $P_{-1}(\alpha)$, $P_1(\alpha)$, $Q_{-1}(\alpha)$ and $Q_1(\alpha)$. The four singularities at the line L_∞ can be found by solving the homogeneous equation of degree four $y[A_3(x,y) + \alpha.C_3(x,y)] - x[B_3(x,y) + \alpha.D_3(x,y)] = 0$, where $A_3\frac{\partial}{\partial x} + B_3\frac{\partial}{\partial y}$ and $C_3\frac{\partial}{\partial x} + D_3\frac{\partial}{\partial y}$ are the homogeneous parts of degree three of X and Y, respectively (see [LN 1]). As the reader can check, this equation gives $y(x^2 + y^2)(x - \alpha.y) = 0$. The solution of this equation gives the points E, F, G and $P_\infty(\alpha)$. The fact that $\alpha \notin \{1, -1, i, -i, \infty\}$ implies that these thirteen points are distinct.

Let us prove (I) and (II). Observe first that, since $\mathcal{F}^{3:1}_\alpha$ is of degree three and has $13 = 3^2 + 3 + 1$ singularities, then these singularities are non-degenerate (see [LN] or 3.1.6). The following result implies (I) and (II) (see §2.3 of [LN] for the proof):

2.4.2 Lemma. Let Z be a holomorphic vector field defined in a neighborhood of $0 \in \mathbb{C}^2$. Suppose that :
(a). 0 is a non-degenerate singularity of Z and the quotient of the eigenvalues of of $DZ(0)$ are rational and positive, say p/q, where $p, q \in \mathbb{N}$ are relatively primes.
(b). Either $p, q \geq 2$ or Z has at least two distinct local analytic separatrices through 0.

Then there exists a holomorphic coordinate system $(W, (u, v))$ with $0 \in W$, $u(0) = v(0) = 0$, in which Z can be written as

$$Z(u, v) = k(q.u\frac{\partial}{\partial u} + p.v\frac{\partial}{\partial v}),$$

where $k \in \mathbb{C}^*$. In particular, $\frac{u^p}{v^q}$ is a meromorphic first integral of Z in a neighborhood of 0.

Let us consider a point $P \in \{A, B, C, D, O\}$. The curve $\Gamma^{3:1}$ has two smoth branches through P with an ordinary tangency at P. It follows from Lemma 2.4.2 that there exists a holomorphic coordinate system (u, v) in a neighborhood U of P, such that $u(P) = v(P) = 0$ and $\mathcal{F}^{3:1}_\alpha$ is represented on U by the vector field $Z(u, v) = q.u\frac{\partial}{\partial u} + p.v\frac{\partial}{\partial v})$, where $1 \leq p < q$ and $\gcd(p, q) = 1$. Since $\frac{u^p}{v^q}$ is a first integral of $\mathcal{F}^{3:1}_\alpha$ and the invariant branches of $\Gamma^{3:1}$ have an ordinary tangency at P, then $p = 1$ and $q = 2$, so that P is of the type 2 : 1. In the case $P \in \{E, F, G\}$, the argument is similar and uses that the curve $\Gamma^{3:1}$ has three smooth branches throgh P, two by two transverse. We leave the details for the reader.

In the proof of (III) and (IV) we use the desingularization process for the foliation $\mathcal{F}^{3:1}_\alpha$. This process involves thirteen blowing-ups : one blowing-up at each radial singularity and two blowing-ups at each singularity of the type 2 : 1 (see §3.4). In the figure 6 we sketch the resolution process for a singularity of the type 2 : 1.

Note that the divisor which appears after the first blowing-up is invariant for the new foliation, whereas the second divisor is not. Let M_3 be the rational surface obtained from $\mathbb{C}P(2)$ after this blowing-up process, $\pi : M_3 \to \mathbb{C}P(2)$ be the blowing-up map, $\mathcal{G}^3_\alpha$ be the strict transform of $\mathcal{F}^{3:1}_a$ by π and $\tilde{L}_{-1}$, $\tilde{L}_1$, $\tilde{L}_\infty$, $\tilde{C}_{-1}$ and $\tilde{C}_1$ be the strict transforms of the rational curves L_{-1}, L_1, L_∞, C_{-1} and C_1, respectively. Denote by D_P the invariant divisor which appears after the two blowing-ups at $P \in \{A, B, C, D, O\}$. Note that the ten curves $\tilde{L}_{-1}$, $\tilde{L}_1$, $\tilde{L}_\infty$, $\tilde{C}_{-1}$, $\tilde{C}_1$ and D_P, $P \in \{A, B, C, D, O\}$, are disjoint, smooth, rational and invariant for G^3_α.

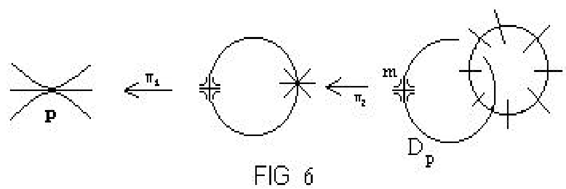


FIG 6

Each one of these ten curves contain one singularity of G^3_α : the five singularities $\pi^{-1}(P_k(\alpha))$, $k = 1, -1, \infty$, $\pi^{-1}(Q_m(\alpha))$, $m = 1, -1$, and one in each of the divisors D_P, $P \in \{A, B, C, D, O\}$ (the singularity m in Fig. 6). Denote by $M_P(\alpha)$ the singularity of G^3_α in the divisor D_P, $P \in \{A, B, C, D, O\}$. For the singularities $\pi^{-1}(P_k(\alpha))$, $k = 1, -1, \infty$, and $\pi^{-1}(Q_m(\alpha))$, $m = 1, -1$, we keep the same notation of before : $P_k(\alpha)$, $k = 1, -1, \infty$, and $Q_m(\alpha)$, $m = 1, -1$. Observe that $sing(G^3_\alpha)$ consists exactly of these ten singularities. The analytic type of these singularities can be obtained by using Camacho-Sad index Theorem (see [C-S] and 3.1.9) and a Lemma of linearization of Mattei-Moussu [M-M]. Let C be one of the ten rational invariant curves and q be the singularity of G^3_α on C. Let Z be a holomorphic vector field which represents G^3_α in a neighborhood of q and λ_n and λ_t be the eigenvalues of $DZ(q)$, where λ_t is the eigenvalue in the tangent direction to C and λ_n in the normal direction. According to Camacho-Sad Theorem, $\frac{\lambda_n}{\lambda_t} = C^2$, that is the self-intersection number of C. On the other hand, since $C \setminus \{q\}$ is a leaf of G^3_α biholomorphic to $\mathbb{C}$, the holonomy of G^3_α in a transverse section to C is the identity. It follows from [M-M], that the vector field Z is linearizable at q. Moreover, if $\frac{\lambda_n}{\lambda_t} = C^2 = -n$, then there exists a coordinate system $(U, (z, w))$ in a neighborhood of q such that $z(q) = w(q) = 0$, $U \cap C = (w = 0)$ and $Z(z, w) = k(z\frac{\partial}{\partial z} - nw\frac{\partial}{\partial w})$, where $k \in \mathbb{C}^*$. In particular, $z^n.w = cte$, is a local first integral of G^3_α. In particular, q is a singularity of the type $1 : -n$. It follows that :

(i). $P_\infty(\alpha)$, $M_A(\alpha)$, $M_B(\alpha)$, $M_C(\alpha)$, $M_D(\alpha)$ and $M_O(\alpha)$ are of the type $1 : -2$, because the curves $\tilde{L}_\infty$, D_A, D_B, D_C, D_D and D_O, have self-intersection -2 in M_3.

(ii). $P_{-1}(\alpha)$, $P_1(\alpha)$, $Q_{-1}(\alpha)$ and $Q_1(\alpha)$ are of the type $1 : -4$, because the curves $\tilde{L}_{-1}$, $\tilde{L}_1$, , $\tilde{C}_{-1}$ and $\tilde{C}_1$ have self intersection -4.

In the proof of (i) and (ii), we can use the following fact : let S be a smooth curve on a surface N and $\pi : \tilde{N} \to N$ be a blowing-up at a point $p \in S$. If, $\tilde{S}$ is the strict transform of S by π, then $\tilde{S}^2 = S^2 - 1$. So, for instance, $\tilde{L}_\infty$ has self-intersection -2 because $L^2_\infty = 1$ and the process involves three blowing-ups at points of L_∞. Another way is to calculate explicitly the quotient of the eigenvalues at the singularities $P_j(\alpha)$ and $Q_k(\alpha)$, by using the expression of $X + \alpha.Y$. We leave the details for the reader. ∎

Let $\pi : M_3 \to CP(2)$ be as in the proof of Lemma 2.4.1. The pencil of foliations $Q^3 = (G^3_\alpha = \pi^*(F^{3:1}_\alpha))_{\alpha \in \overline{\mathbb{C}}}$ will be called the family of type 3.

2.4.3 Corollary. The family of type 3 satisfies the hypothesis (ii) and (iii) of Theorem 3. Moreover, $F^{3:1}_\infty$, $F^{3:1}_1$ and $F^{3:1}_{-1}$ have the following first integrals :

$$f_\infty(x, y) = \frac{C_1(x, y).C_{-1}(x, y)}{4L_\infty(y).L_{-1}(y)} \; ; \; f_1(x, y) = \frac{L_{-1}(y).C_1(x, y)}{L_1(y).C_{-1}(x, y)} \; ; \; f_{-1}(x, y) = \frac{L_1(y).C_1(x, y)}{L_{-1}(y).C_{-1}(x, y)}$$

respectively, where $C_1(x, y) = x^2 + y^2 - 2x$, $C_{-1}(x, y) = x^2 + y^2 + 2x$, $L_1(y) = y - 1$ and $L_{-1}(y) = y + 1$. Moreover, $f_\infty \circ \pi$ is an elliptic fibration. In particular, $E(P^{3:1}) = E(Q^3) = \mathbb{Q}.\langle 1, i \rangle \cup \{\infty\}$.

Proof. Lemma 2.4.1 implies that it satisfies the hypothesis (iii). The fact that f_α is a first integral of $\mathcal{F}^{3,1}_\alpha$, for $\alpha \in \{1, -1, \infty\}$, can be proved by checking that $(X + \alpha.Y)(f_\alpha) = 0$ in each case. We leave the details for the reader. Note that, since all singularities of the foliation $\mathcal{G}^3_\alpha$ are reduced, then for any $\alpha \in E(Q^3)$, we can suppose that the first integral of $\mathcal{G}^3_\alpha$ is a fibration. Let us prove that $h := f_\infty \circ \pi$ is an elliptic fibration. Consider the generic level curve $\{f_\infty = c\}$, which in homogeneous coordinates, can be written as $F_c(x, y, z) := (x^2 + y^2 + 2xz)(x^2 + y^2 - 2xz) - 4cz^2(y^2 - z^2) = 0$. An easy calculation, shows that, if $c \notin \{0, 1, \infty\}$, then the curve F_c is irreducible and that its singular set consists of two nodal singularities at the points $[1 : i : 0]$ and $[1 : -i : 0]$, so that it is elliptic, because $g(F_c) = \frac{(4-1)(4-2)}{2} - 2 = 1$, by the genus formula. Since in the resolution process we have done one blowing-up at the each one of the points $[1 : i : 0]$ and $[1 : -i : 0]$, the level curves of $h = f_\infty \circ \pi$ are all disjoint, so that h is a fibration. $\square$

§3. Proofs

§3.1. Basic facts. In this section we state some facts that will be used in the proofs of Theorems 1, 2 and 3. Let $\mathcal{F}$ a foliation on the surface M defined in a covering $(U_j)_{j\in J}$ of M by a collection of holomorphic vector fields, say $(X_j)_{j\in J}$. Suppose that each U_j is a domain of a holomorphic chart $(x_j, y_j)\colon U_j \to \mathbb{C}^2$ and consider the 2-form $\theta_j := dx_j \wedge dy_j$ and the 1-form $\omega_j = i_{X_j}\theta_j$. Note that the differential equation $\omega_j = 0$ also defines $\mathcal{F}$ on U_j. If $U_i \cap U_j \neq \emptyset$, then $\omega_i = g_{ij}\,\omega_j$, where $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$. The cocycle $(g_{ij})_{U_i\cap U_j\neq\emptyset}$ defines a line bundle on M, called the normal bundle of $\mathcal{F}$. The class of this bundle in $H^1(M, \mathcal{O}^*)$ is denoted by $N_\mathcal{F}$. The conormal bundle of $\mathcal{F}$ is the dual, $N^*_\mathcal{F}$, of $N_\mathcal{F}$. Given another foliation $\mathcal{G}$ on M such that $T_\mathcal{G} = T_\mathcal{F}$, defined by collections of vector fields $(Y_j)_{j\in J}$, consider the pencil generated by $\mathcal{F}$ and $\mathcal{G}$, that is, the family $(\mathcal{F})_{\alpha\in\overline{\mathbb{C}}}$ of foliations, where $\mathcal{F}_\alpha$ is defined by the collection $(X_j + \alpha.Y_j)_{j\in J}$, if $\alpha \in \mathbb{C}$, or $(Y_j)_{j\in J}$, if $\alpha = \infty$. Note that $T_{\mathcal{F}_\alpha} = T_\mathcal{F}$ for all $\alpha \in \overline{\mathbb{C}}$.

3.1.1 Remark. Even if the singularities of $\mathcal{F}$ and $\mathcal{G}$ are isolated, for some values of $\alpha \neq 0, \infty$, the singularities of $\mathcal{F}_\alpha$ could not be isolated. Since we are considering always foliations with isolated singularities, when $\alpha \in \overline{\mathbb{C}}$ is such that $\mathrm{sing}(\mathcal{F}_\alpha)$ is not isolated, that is contains a curve with divisor $(f_j)_{j\in J}$, $f_j \in \mathcal{O}(U_j)$, then we redefine $\mathcal{F}_\alpha$ as the foliation given by the collection of vector fields $(f_j^{-1}[X_j + \alpha.Y_j])_{j\in J}$. Note that, in this case, $T_{\mathcal{F}_\alpha} - T_\mathcal{F}$ is an effective divisor, that is

$$T_{\mathcal{F}_\alpha} - T_\mathcal{F} = \sum_{k=1} n_k.C_k\,,$$

where $n_k \in \mathbb{N}$ and $\cup_k C_k$ is the curve defined in U_j by $\{f_j = 0\}$. We will consider the set

$$B(\mathcal{F}, \mathcal{G}) = \{\alpha \in \overline{\mathbb{C}} \mid T_{\mathcal{F}_\alpha} = T_\mathcal{F} = T_\mathcal{G}\}\,.$$

Observe that the set $\overline{\mathbb{C}} \setminus B(\mathcal{F}, \mathcal{G})$ is always finite.

In the sequel, we will recall some known facts about foliations on surfaces that will be used in the proof of the above result. The proofs and definitions of some concepts involved can be found [Br], [Br-1], [Br-2], [BPV] and [S]. Let M be a compact surface and $\mathcal{F}$ be a foliation on M with isolated singularities. Suppose that $\mathcal{F}$ is defined by a collection of holomorphic vector fields $(X_j)_{j\in J}$, or 1-forms $(\omega_j)_{j\in J}$, associated to a covering $(U_j)_{j\in J}$ of M, as before.

3.1.2 Seidenberg's Theorem. (cf. [Se] or [M-M]) In order to state Seidenberg's Theorem, we recall the concept of reduced singularity. Let p be an isolated singularity of a foliation $\mathcal{F}$ on a surface M and X be a holomorphic vector field which represents $\mathcal{F}$ in a neighborhood of p. Let

λ_1 and λ_2 be the eigenvalues of $DX(p)$. We say that p is a reduced singularity of $\mathcal{F}$, if one of the following condictions holds :
(I). p is a non-degenerate singularity, that is $\lambda_1, \lambda_2 \neq 0$, and the charachteristic values, λ_2/λ_1 and λ_1/λ_2, are not rational positive.
(II). $\lambda_1 = 0$ and $\lambda_2 \neq 0$, or vice-versa. In this case, we say that p is a saddle-node for $\mathcal{F}$.

These condictions do not depend on the vector field X.

Theorem. ([Se] or [M-M]). For any foliation $\mathcal{F}$, with isolated singularities, on a surface M, there exists a surface N and bimeromorphism $\pi : N \to M$, which is a sequence of blowing-ups, such that all singularities of the strict transform foliation, $\pi^*(\mathcal{F})$, are reduced.

In the sequel, we resume other results that will be used, involving the bundles associated to the foliation $\mathcal{F}$.

3.1.3. If Y is a meromorphic non-vanishing vector field on M tangent to $\mathcal{F}$ (that is $\omega_j(Y) = 0$, $\forall j \in J$) then

$$T_{\mathcal{F}} = (Y)_0 - (Y)_\infty ;$$

where $(Y)_0$ and $(Y)_\infty$ denote the divisors of zeroes and poles of Y respectively. Analogously, if ω is a meromorphic non-vanishing 1-form on M such that $\omega(Y_j) \equiv 0$, $\forall j \in J$, then

$$N^*_{\mathcal{F}} = (\omega)_0 - (\omega)_\infty ;$$

where $(\omega)_0$ and $(\omega)_\infty$ denote the divisors of zeroes and poles of Y respectively.

The relation between $N_{\mathcal{F}}$ and $T_{\mathcal{F}}$ is the following :

3.1.4. $K_M = N^*_{\mathcal{F}} + T^*_{\mathcal{F}}$, where K_M denotes the canonical bundle of M.

In the case of a foliation $\mathcal{F}$ of degree d on $\mathbb{CP}(2)$ we have the following :

3.1.5. $T^*_{\mathcal{F}} = (d-1)H$, $N_{\mathcal{F}} = (d+2)H$ and $K_{\mathbb{CP}(2)} = -3H$, where H denotes the divisor associated to a line.

Given two line bundles L_1 and L_2 on M, we will use the notation $L_1.L_2$ for the number $\int_M c_1(L_1) \wedge c_1(L_2)$, where $c_1(L_j) \in H^2_{DR}(M)$ is the first Chern class of L_j, $j = 1, 2$. When $L_1 = L_2$ we will use the notation $L_1.L_1 = L_1^2$.

If we denote by $\mu(\mathcal{F})$ the number of singularities of $\mathcal{F}$ counted with multiplicities, then :

3.1.6. $\mu(\mathcal{F}) = c_2(T^*_{\mathcal{F}} + TM) = c_2(M) + T^*_{\mathcal{F}}.c_1(M) + (T^*_{\mathcal{F}})^2 = c_2(M) + T_{\mathcal{F}}.K_M + T^2_{\mathcal{F}}$. In particular, if $M = \mathbb{CP}(2)$ and $\mathcal{F}$ has degree d the $\mu(\mathcal{F}) = d^2 + d + 1$. Moreover, the singularities are non-degenerate if, and only if, $\mathcal{F}$ has $d^2 + d + 1$ singularities.

Now, let C be a curve on M. We say that C is not invariant for $\mathcal{F}$, if $C \cap U_j$ is not a solution of $\omega_j = 0$ for any $j \in J$ such that $C \cap U_j \neq \emptyset$, where ω_j defines $\mathcal{F}$ on U_j. We say that C is invariant for $\mathcal{F}$, if $C \cap U_j$ is a solution of $\omega_j = 0$ for any $j \in J$ such that $C \cap U_j \neq \emptyset$. Given a reduced curve C, which is not invariant for $\mathcal{F}$, and $p \in C$, the order of tangency between $\mathcal{F}$ and C at p is

$$\operatorname{tang}(\mathcal{F}, C, p) := \dim_{\mathbb{C}} \frac{\mathcal{O}_p}{< f, X(f) >} = [f, X(f)]_p ;$$

where $f = 0$ is a reduced equation of C, X is a holomorphic vector field which defines $\mathcal{F}$ in a neighborhood of p and $[f, X(f)]_p$ denotes the intersection number of f and $X(f)$ at p. Observe that, since f is reduced and not invariant for X, then f and $X(f)$ have no common components at p, so that $0 \leq \operatorname{tang}(\mathcal{F}, C, p) < +\infty$. Moreover, $\operatorname{tang}(\mathcal{F}, C, p) = 0$ if, and only if, the leaf of $\mathcal{F}$ through p is transverse to C at p. This implies that

$$0 \leq \operatorname{tang}(\mathcal{F}, C) := \sum_{p \in C} \operatorname{tang}(\mathcal{F}, C, p) < +\infty .$$

3.1.7. Let C be a reduced curve on M, not invariant for $\mathcal{F}$. Then :

$$N_{\mathcal{F}}.C = \chi(C) + tang(\mathcal{F}, C) \text{ and } T_{\mathcal{F}}.C = C^2 - tang(\mathcal{F}, C);$$

where $\chi(C) = -K_M.C - C^2$ is the virtual Euler characteristic of C (cf. [Br-1]). We observe that, if C is a smooth curve, then $\chi(C)$ coincides with the topological Euler characteristic of C. On the other hand, if C is not smooth, then $\chi(C)$ is the Euler characteristic of a smoothing of C (cf. [BPV]).

In order to compute $N_{\mathcal{F}}.C$ and $T_{\mathcal{F}}.C$ when C is invariant for $\mathcal{F}$, we have to introduce another local index involving $\mathcal{F}$ and a point $p \in C$. This index is denoted by $Z(\mathcal{F}, C, p)$ in [Br-1] and [Br-2]. When C is smooth at p, $Z(\mathcal{F}, C, p)$ is the Poincaré-Hopf index of the "restricted" foliation at p, which is defined as follows. Let $p \in C$ be smooth point of C and X be a holomorphic vector field which defines $\mathcal{F}$ in a neighborhood of p. Since C is smooth at p and C is invariant for X, there exists a holomorphic coordinate system $(U, (x, y))$ in a neighborhood of p such that $C \cap U = (y = 0)$, $x(p) = y(p) = 0$ and $X|_{U \cap C} = x^k.u(x)\frac{\partial}{\partial x}$, where $u(0) \neq 0$. In this case, $Z(\mathcal{F}, C, p) = k \geq 0$. This index can be defined also when C is not smooth at p, but since we will use it only in the smooth case, we refer the general definition for [Br-1] or [Br-2]. Given a reduced curve C, define

$$Z(\mathcal{F}, C) = \sum_{p \in C} Z(\mathcal{F}, C, p).$$

We have the following :

3.1.8. Let C be a reduced curve on M, invariant for $\mathcal{F}$. Then :

$$N_{\mathcal{F}}.C = C^2 + Z(\mathcal{F}, C) \text{ and } T_{\mathcal{F}}.C = \chi(C) - Z(\mathcal{F}, C).$$

When C is an invariant reduced curve for $\mathcal{F}$ and $p \in C \cap sing(\mathcal{F})$, it is defined the so called Camacho-Sad index of p with respect to C. In the case where C is smooth at p and p is a non-degenerate singularity of $\mathcal{F}$, this index can be expressed in terms of the eigenvalues of $DX(p)$, where X is a holomorphic vector field which represents $\mathcal{F}$ in a neighborhood of p. If λ_t is the eigenvalues of $DX(p)$ relative to the eigendirection tangent to C at p and λ_n is the other eigenvalue, then the Camacho-Sad index of $\mathcal{F}$ at p with respect to C is $I(\mathcal{F}, C, p) = \frac{\lambda_n}{\lambda_t}$. In the case where p is not a singularity of $\mathcal{F}$ we have $I(\mathcal{F}, C, p) = 0$. In the general case, the definition can be found in [Br-2] or [S]. Set $I(\mathcal{F}, C) = \sum_{p \in C} I(\mathcal{F}, C, p)$. The main fact about this index is that

3.1.9 Camacho-Sad Theorem. We have $I(\mathcal{F}, C) = C^2$, the self-intersection number of C.

Another ingredient that will be used is the divisor of tangency between two foliations. Let $\mathcal{F}$ and $\mathcal{G}$ be two foliations on the surface M. Let $(U)_{j \in J}$ be a covering of M by open sets and let $\mathcal{F}|_{U_j}$ be defined by the vector field X_j and $\mathcal{G}_{U_j}$ by the 1-form ω_j, where $X_i = f_{ij} X_j$ and $\omega_i = g_{ij}\omega_j$ on $U_i \cap U_j \neq \emptyset$. Set $f_j = i_{X_j}(\omega_j) \in \mathcal{O}(U_j)$. Then the foliations are tangent along the curve $\Delta_j = (f_j = 0) \subset U_j$. Moreover, since $f_i = f_{ij}.g_{ij}.f_j$, on $U_i \cap U_j \neq \emptyset$, the curves Δ_i and Δ_j glue togheter on $U_i \cap U_j$, and this gives origin to a divisor on M, which we denote by $\Delta(\mathcal{F}, \mathcal{G})$. We have

3.1.10. $[\Delta(\mathcal{F}, \mathcal{G})] = T^*_{\mathcal{F}} + N_{\mathcal{G}} = T^*_{\mathcal{G}} + N_{\mathcal{F}}$.

In the particular case of $CP(2)$, we get from (3.1.5) that if $\mathcal{F}$ and $\mathcal{G}$ are foliations on $CP(2)$ of degrees k and ℓ respectively, then $[\Delta(\mathcal{F}, \mathcal{G})] = (k + \ell - 1)H$.

Finally, we will see how the line bundles above change when we do a blowing-up at a point $p \in M$. Let us denote by $\hat{M}$ the surface obtained from M by performing this blowing-up, by $\pi : \hat{M} \to M$

the blowing-up map, by D the excepcional divisor $\pi^{-1}(p)$ and by $\hat{\mathcal{F}}$ the strict transform of the foliation $\mathcal{F}$ by π. Let ω be a holomorphic 1-form which defines $\mathcal{F}$ in a neighborhood of p. If p is a singularity of ω, then D is in the divisor of zeroes of the 1-form $\pi^*(\omega)$ with some multiplicity, say m(p). By using the definitions, it is possible to prove that (cf. [Br-1]):

3.1.11. $N^*_{\hat{\mathcal{F}}} = \pi^*(N^*_{\mathcal{F}}) + m(p)[D]$ and $T_{\hat{\mathcal{F}}} = \pi^*(T_{\mathcal{F}}) + (m(p) - 1)[D]$.

§3.2. **Proof of Theorem 3.** Let M be a complex surface and $\mathcal{F}$ and $\mathcal{G}$ be holomorphic foliations on M such that $T_{\mathcal{F}} = T_{\mathcal{G}}$. We will denote by T the class of $T_{\mathcal{F}} = T_{\mathcal{G}}$ in $H^1(M, \mathcal{O}^*)$ and by $\mathcal{F}(M, T)$ the set

$$\{\mathcal{H}\,;\ \mathcal{H} \text{ is a foliation on } M \text{ such that } T_{\mathcal{H}} = T\}.$$

Suppose that $\mathcal{F}$ has a holomorphic first integral $f : M \to S$, where S is some compact Riemann surface. Denote by $g(f)$ the genus of the regular level curves, $f^{-1}(c)$, of f.

3.2.1 Lemma. Let M, $\mathcal{F}$, $\mathcal{G}$, f and $g(f)$ be as above. Then :

(a). If $g(f) = 0$ then $\mathcal{F} \equiv \mathcal{G}$. In particular $\mathcal{F}(M, T) = \{\mathcal{F}\}$.

(b). If $g(f) = 1$ and $\mathcal{F} \not\equiv \mathcal{G}$, then $\mathcal{G}$ is turbulent with respect to f.

(c). If $g(f) \geq 2$, then for any regular fibre $F = f^{-1}(c)$ of f, which is not invariant for $\mathcal{G}$, we have $tang(\mathcal{G}, F) > 0$.

In particular, if $\mathcal{F} \not\equiv \mathcal{G}$, then $\mathcal{G}$ is transverse to some regular fibre of f if, and only if, $g(f) = 1$. Moreover, in this case, $\mathcal{G}$ is turbulent with respect to f.

Proof. Let $F = f^{-1}(c)$ be a regular fibre of f. Since F is invariant for $\mathcal{F}$ and $\mathcal{F}$ has no singular points on F, we get from 3.1.8 that

$$T.F = T_{\mathcal{F}}.F = \chi(F) - Z(\mathcal{F}, F) = \chi(F).$$

On the other hand, if F is not invariant for $\mathcal{G}$, we get from 3.1.7 that

$$T.F = T_{\mathcal{G}}.F = F^2 - tang(\mathcal{G}, F) = -tang(\mathcal{G}, F);$$

so that, $\chi(F) = -tang(\mathcal{G}, F) \leq 0$. In particular, if the fibres of f are rational curves, then $\chi(F) = 2 > 0$ and F is invariant for $\mathcal{G}$. In this case, all regular fibres of f are invariant for $\mathcal{G}$, which implies that, $\mathcal{G} \equiv \mathcal{F}$. On the other hand, $\mathcal{G}$ is transverse to F if, and only if, $tang(\mathcal{G}, F) = 0 = \chi(F)$. This implies (b) and (c). ∎

Now, let us suppose $T_{\mathcal{F}} = T_{\mathcal{G}}$, but $\mathcal{F} \not\equiv \mathcal{G}$, that the singularities of $\mathcal{F}$ are reduced in the sense of Seidemberg and that $\mathcal{F}$ is tangent to an elliptic fibration $f : M \to S$. Let C be a smooth irreducible component of a critical fibre $F = f^{-1}(c)$ of f.

3.2.2 Lemma. In the above situation, we have :

(a). If $F = m.C$, for $m \geq 2$, then $g(C) = 1$ and, either C is a leaf of $\mathcal{G}$, or $\mathcal{G}$ is transverse to C.

(b). If C is rational then $C^2 < 0$. Moreover, if C is not invariant for $\mathcal{G}$, then $Z(\mathcal{F}, C) \geq 3$.

(c). $\mathcal{G}$ is turbulent with respect to f and $sing(\mathcal{G}) \subset f^{-1}(A)$, where

$$A = \{c \in S;\ c \text{ is a critical value of } f \text{ such that } f^{-1}(c) \text{ is not smooth}\}.$$

Proof. Lemma 3.2.1 implies that $\mathcal{G}$ is turbulent with respect to f. Suppose that the critical fibre $F = m.C$, $m \geq 2$, so that F, as a subset of M, is smooth. It follows from Kodaira's classification of critical fibers in [K], that C is an elliptic curve and the fibre F is of the type ${}_mI_0$, $m \geq 2$. In particular, F is a multiple fibre. Since C is smooth, given $p \in C$, there exist holomorphic coordinate systems $(U, (x, y))$ in M and (V, z) in S, such that $p \in U$, $C \cap U = (x = 0)$, $f(p) \in V$,

$z(f(p)) = 0 \in \mathbb{C}$ and $z \circ f(x,y) = x^m$. This implies that $\mathcal{F}|_U$ is defined by $dx = 0$. In particular $Z(\mathcal{F}, C) = 0$. Therefore, if $F = m.C$ ($m \geq 1$), we have

$$T.C = T_{\mathcal{F}}.C = X(C) - Z(\mathcal{F}, C) = 0 .$$

If C is invariant for $\mathcal{G}$, we have

$$0 = T.C = T_{\mathcal{G}}.C = X(C) - Z(\mathcal{G}, C) = -Z(\mathcal{G}, C) \implies Z(\mathcal{G}, C) = 0 .$$

On the other hand, if C is not invariant for $\mathcal{G}$, then

$$T.C = T_{\mathcal{G}}.C = C^2 - tang(\mathcal{G}, C) = -tang(\mathcal{G}, C) = 0 .$$

Therefore, if $F = m.C$, where C is smooth and $m \geq 1$ then, either $\mathcal{G}$ is transverse to C, or C is invariant for $\mathcal{G}$ and $Z(\mathcal{G}, C) = 0$. This implies (a) and (c). Let us prove (b). First of all, observe that $sing(\mathcal{F}) \cap C \neq \emptyset$. In fact, if $sing(\mathcal{F}) \cap C$ was empty, then Reeb's stability Theorem would imply that there exists a neighborhood V of C, saturated for $\mathcal{F}$, such that all leaves of $\mathcal{F}$ in V are rational curves (cf. [C-LN]), which is not possible, because f is an elliptic fibration. Let $p_1, ..., p_k$ be the singularities of $\mathcal{F}$ on C. For each $j = 1, ..., k$ let I_j be the Camacho-Sad index of $\mathcal{F}$ with respect to C at p_j. It follows from Camacho-Sad Theorem that $C^2 = \sum_{j=1}^{k} I_j$. On the other hand, for each $j \in \{1, ..., k\}$, p_j is a reduced singularity of $\mathcal{F}$ and f is tangent to $\mathcal{F}$. This implies that there exist holomorphic coordinate systems $(U, (x, y))$ in M and (V, z) in S, such that $p_j \in U$, $x(p_j) = y(p_j) = 0$, $C \cap U = (y = 0)$, $f(p_j) \in V$, $z(f(p_j)) = 0 \in \mathbb{C}$ and $z \circ f|_U(x, y) = x^{m_j}.y^{n_j}$, $m_j, n_j > 0$, so that $\mathcal{F}$ is represented in U by the vector field $X(x, y) = n_j x \frac{\partial}{\partial x} - m_j y \frac{\partial}{\partial y}$. Hence $I_j = -\frac{m_j}{n_j} < 0$. It follows that $C^2 < 0$.

Now, let us suppose that C is not invariant for $\mathcal{G}$. In this case, it follows from 3.1.7 that

$$T.C = C^2 - tang(\mathcal{G}, C) < 0 .$$

On the other hand, since C is invariant for $\mathcal{F}$, it follows from 3.1.8 that

$$T.C = X(C) - Z(\mathcal{F}, C) = 2 - Z(\mathcal{F}, C) \implies 2 - Z(\mathcal{F}, C) < 0 \implies Z(\mathcal{F}, C) \geq 3 .$$

□

Before stating the next result we need a definition.

3.2.3 Definition. Let $\mathcal{F}$ be a holomorphic foliation on a surface M. We say that a smooth rational curve $C \subset M$ is contractible for $\mathcal{F}$ if :

(a). $C^2 = -1$ and C is invariant for $\mathcal{F}$.

(b). When we blow down C, thus obtaining a surface N and a blowing-down map $\pi: M \to N$, where $\pi(C) = p \in N$, then, either p is not a singularity for the transformed foliation $\pi_*(\mathcal{F})$, or it is a reduced singularity for $\pi_*(\mathcal{F})$.

3.2.4 Remark. If C is contractible for $\mathcal{F}$ as in definition 3.2.3, then we have three possibilities (cf. [Br-2]):

1^{st}) . p is a non-singular point for $\pi_*(\mathcal{F})$. In this case, $\mathcal{F}$ has just one non-degenerate singularity, say q, on C, such that $I(\mathcal{F}, C, q) = -1$. We have also that $Z(\mathcal{F}, C, q) = Z(\mathcal{F}, C) = 1$.

2^{nd}) . p is a non-degenerate singularity of $\pi_*(\mathcal{F})$. In this case, $\mathcal{F}$ has two non-degenerate singularities and $Z(\mathcal{F}, C) = 2$. If the characteristic numbers of $\pi_*(\mathcal{F})$ at p are λ and λ^{-1}, where

$\lambda, \lambda^{-1} \notin \mathbb{Q}_+$ (because p is a reduced singularity), then the Camacho-Sad index of the singularities with respect to C at the two singularities are $\frac{\lambda}{1-\lambda}$ and $\frac{1}{\lambda-1}$.
3^{rd}) . p is a saddle node of $\pi_*(F)$. In this case F has two singularities on C, one saddle-node, say q_1, and the other non-degenerate, say q_2. Moreover, $I(F; C; q_1) = 0$, $I(F; C; q_2) = -1$, $Z(F; C; q_1) = Z(F; C; q_2) = 1$ and $Z(F; C) = 2$. We observe that this case does not occur if F (or $\pi_*(F)$) is tangent to a fibration.

Let F, G and $f : M \to S$ be as in Lemma 3.2.2 , and $(F_\alpha)_{\alpha \in \overline{\mathbb{C}}}$ be the pencil of foliations on M generated by F and G, where $F_0 = F$ and $F_\infty = G$. Set $T_F = T_G = T$ and let $B = B(F; G)$ be as in Remark 3.1.1. Recall that $\overline{\mathbb{C}} \setminus B$ is finite.

3.2.5 Lemma. Let $F \neq G$, $f : M \to S$, $(F_\alpha)_{\alpha \in \overline{\mathbb{C}}}$ and B be as before. Suppose that F_{α_0} has a contractible curve C, for some $\alpha_0 \in B \setminus \{0\}$. Let $\pi : M \to N$ be the blowing-down map obtained by contracting C, where $\pi(C) = p \in N$. Then :
(a). C is invariant for F_α and $1 \le Z(F_a; C) = Z(F; C) \le 2$, for all $\alpha \in B$.
(b). Suppose that F_{α_1} is tangent to some fibration $f_1 : M \to S_1$, for some $\alpha_1 \in B$. Then $f_1 \circ \pi^{-1} : N \to S_1$ is a fibration.
(c). C is contractible for $F_0 = F$. In particular, all singularities of $\pi_*(F)$ are reduced and $f \circ \pi^{-1} : N \to S$ is a fibration.
(d). There exists $\epsilon > 0$ such that if $|\alpha| < \epsilon$ then p is a reduced singularity for $\pi_*(F_\alpha)$ and $T_{\pi_*(F_\alpha)} = T_{\pi_*(F)}$.
(e). If $B' = B(\pi_*(F); \pi_*(G))$, then $\overline{\mathbb{C}} \setminus B'$ is finite.

Proof. By definition we have $T_{F_\alpha} = T$ for all $\alpha \in B$. Since C is rational and invariant for F_{α_o}, it follows from (a) and (b) of Lemma 3.2.2 that $1 \le Z(F; C) \le 2$. Note that (b) of Lemma 3.2.2 also implies that C is invariant for F_α, for all $\alpha \in B \setminus \{0\}$, because $F_\alpha \neq F$ if $\alpha \neq 0$. This implies that C is also invariant for F, so that it is contained in a critical fibre of f. Moreover, if $\alpha \in B$ then

$$Z(F_\alpha; C) = \chi(C) - T.C = 2 - T.C\ ;$$

so that $Z(F_\alpha; C)$ does not depends on $\alpha \in B$. Hence $1 \le Z(F; C) = Z(F_\alpha; C) \le 2$, which proves (a). Let us prove (b). Since C is invariant for F_{α_1}, which is tangent to the fibration f_1, we must have that $f_1|_C$ is constant, say $f_1(C) = a \in S_1$. Now, π^{-1} is a biholomorphism outside C, so that $f_1 \circ \pi^{-1}$ is holomorphic outside p and hence in N, by Hartog's Theorem, so that it is also a fibration.

Let us prove (c). Since $1 \le Z(F; C) \le 2$, F has one or two singularities on C. Given $q \in \mathrm{sing}(F) \cap C$, denote by $I(F; q)$ the Camacho-Sad index of F at q with respect to C. Since the singularities of F are reduced, as we have seen in the proof of Lemma 3.2.2, they are non-degenerate and if $q \in \mathrm{sing}(F) \cap C$ then $I(F; q) \in \mathbb{Q}_-$ and $Z(F; C; q) = 1$. We have two possibilities :
(i). $Z(F; C) = 1$. In this case, if q is the singularity of F on C, then $I(F; q) = C^2 = -1$, and p is not a singular point of $\pi_*(F)$.
(ii). $Z(F; C) = 2$. In this case, if q_1 and q_2 are the singular points of F on C, then $I(F; q_1) + I(F; q_2) = -1$. Set $I(F; q_1) = -\lambda < 0$, so that $I(F; q_2) = \lambda - 1 < 0$. In this case, the point p will be a non-degenerate singularity of $\pi_*(F)$ with negative characteristic numbers $\frac{\lambda}{\lambda-1}$ and $\frac{\lambda-1}{\lambda}$, so that it is reduced for $\pi_*(F)$. This implies (c).

Let us prove (d). We have seen above that F has one or two non-degenerate singularities on C. For each one of these singularities the Camacho-Sad index of F with respect to C is negative. It follows from the facts that non-degenerate singularities are stable by small perturbations and the characteristic values vary continuously with parameters (cf. [Ar]), that:

(iii). There exists $\epsilon > 0$ such that, if $|\alpha| < \epsilon$ then $\alpha \in B$ and F_α has the same number of singularities as $F = F_0$ on C, all of them non-degenerate with Camacho-Sad indexes with respect to C negative.

Let ω_α be a 1-form representing $\pi_*(F_\alpha)$ in a neighborhood of p. If $|\alpha| < \epsilon$, we have two possibilities, according to (i) or (ii) :

Case (i). Since F_α has just one singularity on C, say q, we must have $I(F_\alpha; C; q) = -1$. In this case, p is a regular point of $\pi_*(F_\alpha)$, so that $\omega_\alpha(p) \neq 0$ and the multiplicity of C in the divisor of zeroes of $\pi^*(\omega_\alpha)$ is zero. It follows from $T_F = T_{F_\alpha}$ and from 3.1.11 that

$$T_F = \pi^*(T_{\pi_*(F_a)}) - [C] = \pi^*(T_{\pi_*(F)}) - [C] \implies T_{\pi_*(F_a)} = T_{\pi_*(F)}$$

Case (ii). In this case, if $q_1(\alpha)$ and $q_2(\alpha)$ are the singularities of F_α on C and $I(F_\alpha; C; q_1(\alpha)) = \lambda(\alpha) < 0$, then the charachteristic numbers of $\pi_*(F_\alpha)$ at p are $\mu(\alpha) := \frac{\lambda(\alpha)}{1-\lambda(\alpha)}; \mu(\alpha)^{-1} < 0$, so that p is a reduced singularity of $\pi_*(F_\alpha)$. Moreover, the multiplicity of C in the divisor of zeroes of $\pi^*(\omega_\alpha)$ is one (see [Br-2]). It follows from $T_F = T_{F_\alpha}$ and from 3.1.11 that

$$T_F = \pi^*(T_{\pi_*(F_a)}) = \pi^*(T_{\pi_*(F)}) \implies T_{\pi_*(F_a)} = T_{\pi_*(F)}$$

Note that this implies (e), because the pencil generate by $\pi_*(F)$ and $\pi_*(G)$ coincides, up to reparametrization, with the pencil generate by $\pi_*(F)$ and $\pi_*(F_\alpha)$, if $\alpha \neq 0$. ∎

3.2.6 Corollary. Let $F \neq G$ be foliations on a surface M such that $T_F = T_G$. Suppose that all singularities of F and G are reduced and that F and G are tangent to fibrations, say $f: M \to S$ and $g: M \to S_1$, where f is elliptic. Then there exist a complex surface N and a bimeromorphism $\phi: N \to M$ such that :

(a). $f \circ \phi: N \to S$ and $g \circ \phi: N \to S_1$ are fibrations.

(b). All the singularities of $\phi^*(F)$ are reduced and $\phi^*(F)$ has no contractible curves.

(c). $T_{\phi^*(F)} = T_{\phi^*(G)}$.

Proof. Note that in the proof that $T_{\pi_*(F)} = T_{\pi_*(F_\alpha)}$ in (d) of Lemma 3.2.5, we have used only that the singularities of F_α on C are reduced. Therefore, the proof of the Corollary can be done by induction. We leave the details for the reader. ∎

Consider now two foliations F and G on a complex compact surface M, such that $F \neq G$, $T_F = T_G = T$, all singularities of F and G are reduced, F and G are tangent to fibrations $f: M \to S$ and $g: M \to S_1$, respectively, where f is elliptic. It follows from Lemma 3.2.1 that G is turbulent with respect to f, so that f is isotrivial. On the other hand, the Corollary 3.2.6 implies that there exists a bimeromorphism $\phi: N \to M$ such that $\phi^*(F)$ is reduced, has no contractible curve, $f \circ \phi$ and $g \circ \phi$ are fibrations and $T_{\phi^*(F)} = T_{\phi^*(G)}$. Hence, in this situation, after applying Corollary 3.2.6, we can suppose that :

(I). All singularities of F are reduced and F has no contractible curves.

(II). f is isotrivial.

(III). $T_F = T_G = T$.

3.2.7 Lemma. In the above situation, any critical fibre of f is of one of the following types : ${}_mI_0$ $(m \geq 2)$, I_0^*, II^*, III^* or IV^*.

Proof. The idea is to use Kodaira's classification of the critical fibres of an elliptic fibration. In [K], Kodaira classifies the possible fibres of an elliptic fibration h, which satisfies the following hypothesis : if C is a smooth rational curve contained in a critical fibre, then $C^2 \neq -1$. Although F has no contractible curve, the fibration f could have some. More precisely, it could happen that there are -1 rational smooth curves contained in some critical fibres of f, but when we blow-down one of these curves the singularity of F which appears is not reduced. However, after a finite number of blowing-downs, we can obtain a new surface N, a bimeromorphism $\phi\colon M \to N$ (a composition of blowing-downs) and a fibration $f_1 = f \circ \phi^{-1}\colon N \to S$, such that f_1 has no contractible fibres. According to [K] or [BPV], the critical fibres of f_1 could be of the following types : ${}_mI_0$ ($m \geq 2$), I_0^*, II, III, IV, II^*, III^*, IV^*, ${}_mI_b$ or I_b^* (cf. pages 564 and 604 of [K], or page 159 of [BPV]). The fibres of the types ${}_mI_b$ and I_b^* can not occur in isotrivial fibrations, so that the critical fibres of f_1 could be of the types : ${}_mI_0$ ($m \geq 2$), I_0^*, II, III, IV, II^*, III^* or IV^*. The fibre of type I_0^* is sketched in figures 1.b and 1.c, and the fibres II, III and IV are sketched in figure 2.

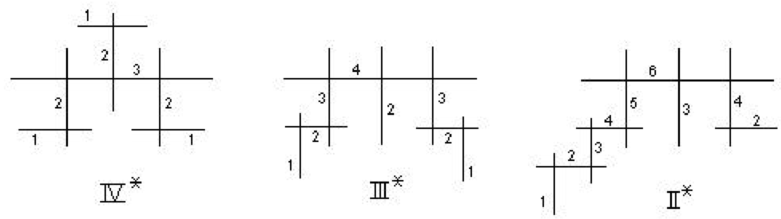


FIG. 7

In figure 7 we sketch the fibres of types II^*, III^* and IV^*. In that figure, the lines represent smooth rational components of the fibre and the numbers the multiplicity of the component. The self intersection of each component is -2. Moreover, if two components C_1 and C_2, of multiplicities m_1 and m_2, respectively, intersect in a point p, then there are coordinate systems $(U,(x,y))$ in N and (V,z) in S such that $x(p) = y(p) = 0$, $C_1 \cap U = (x = 0)$, $C_2 \cap U = (y = 0)$, $f_1(U) \subset V$, and $z \circ f_1(x,y) = x^{m_1}.y^{m_2}$. This implies that $\phi_*(F)$ is represented in U by the vector field $X(x,y) = m_2 x\frac{\partial}{\partial x} - m_1 y\frac{\partial}{\partial y}$, so that p is a non-degenerate, reduced singularity of F and $Z(F,C_1,p) = Z(F,C_2,p) = 1$. On the other hand, the singularities which appear in the fibres of types II, III or IV, are not reduced for F, but if we perform some blowing ups in such a way that the fibres become of types $\tilde{II}$, $\tilde{III}$ or $\tilde{IV}$, respectively, then the singularities of the new foliation become reduced and non-degenerate for the transformed foliation. This last foliation has no contractible curve, and so it coincides with F. Therefore, the critical levels of f are of one of the following types : ${}_mI_0$ ($m \geq 2$), I_0^*, $\tilde{II}$, $\tilde{III}$, $\tilde{IV}$, II^*, III^* or IV^*.

Let us prove that f has no critical fibres of the types II^*, III^* or IV^*. Suppose by contradiction that there is a critical fibre, say $F_c = f^{-1}(c)$, of one of these types. Observe first that F_c has only one component, say C_0, such that $Z(F,C_0) = 3$ (see figure 7). If C is another component of F_c, then $Z(F,C) \leq 2$. It follows from Lemma 3.2.2 that the unique component of F_c that could be not invariant for G is C_0. Here we use that $T_G = T_F$, $F \neq G$ and g is a fibration tangent to G. Let $C_0, C_1, \ldots, C_k$ be the components of F_c, where C_0 is as above. Since $C_1,\ldots,C_k$ are invariant for G, the function g must be constant in each C_j, $j = 1,\ldots,k$. Set $b_j = g(C_j)$, $j = 1,\ldots,k$. Since $G \neq F$, almost all regular fibres of g are not invariant for F. Let $G_b = g^{-1}(b)$ be a regular fibre of g, not invariant for F, such that $b \neq b_j$, $j = 1,\ldots,k$. Since the map $h = f|_{G_b}\colon G_b \to S$ is holomorphic and non constant, it is surjective, so that $h(p) = c$ for some $p \in G_b$. This implies that $F_c \cap G_b \neq \emptyset$ and the leaf G_b of G cuts F_c at the point p. Since $b \neq b_j$, $j = 1,\ldots,k$, we must have that $p \in C_0$. We have found a leaf G_b of G, which is not invariant for F, such that $p \in C_0 \cap G_b \neq \emptyset$ and $p \notin C_1 \cup \ldots \cup C_k$. Therefore C_0 is not invariant for G. If we apply 3.1.7 and 3.1.8 to G, F

and C_0 we get

$$T.C_0 = T_G.C_0 = C_0^2 - \operatorname{tang}(G;C_0) = -2 - \operatorname{tang}(G;C_0)$$

and

$$-2 - \operatorname{tang}(G;C_0) = T.C_0 = T_F.C_0 = \chi(C_0) - Z(F;C_0) = 2 - 3 = -1 \implies \operatorname{tang}(G;C_0) = -1\,;$$

which is an absurd. This proves that f has no fibres of types II^*, III^* or IV^*. ∎

Let F be a fixed foliation , tangent to an elliptic fibration f as in Lemma 3.2.7. We will use the following notations :

(A). A_0^m for the set of ${}_mI_0$ fibres, $A_0 = \cup_m A_0^m$.
(B). A_0^* for the set of I_0^* fibres.
(C). A_2 for the set of fibres of type II, A_3 for the set of fibres of type III and A_4 for the set of fibres of type IV.

Denote by $F_1,\dots,F_r$ the fibres of f in $A_0^* \cup A_2 \cup A_3 \cup A_4$.
(D). Given $F_j \in A_0^* \cup A_2 \cup A_3 \cup A_4$, let $C_{j,i}$, $j = 0,1,\dots,k_j$, be the rational irreducible components of F_j. By convention, $C_{j,0}$ will be fibre which contains more than two singularities of F. Denote by $m_{j,i}$ the multiplicity of f in the component $C_{j,i}$ (see figure 1). The divisor of F_j can be written as

$$F_j = \sum_{i=0}^{k_j} m_{j,i} C_{j,i}\,;$$

Observe that $F_j^2 = 0$ (see [K]). Moreover, if $F_j \in A_0^*$ then $Z(F;C_{j,0}) = 4$ and $C_{j,0}^2 = -2$, whereas in the other cases we have $Z(F;C_{j,0}) = 3$ and $C_{j,0}^2 = -1$.

From now on, we will consider the following situation : F, G will be two foliations on M such that $T_F = T_G$ and $(F_\alpha)_{\alpha\in\overline{\mathbb{C}}}$ will be the pencil of foliations generated by F and G, where $F_0 = F$ and $F_\infty = G$. Denote by $B := B(F;G)$ the set $\{\alpha \in \overline{\mathbb{C}} \mid T_{F_\alpha} = T_F\}$ and by $\Delta = \Delta(F;G)$, the divisor of tangency between F and G (see Remark 6 and 3.1.10). Suppose that :
(I). F and G are tangent to fibrations $f: M \to S$ and $g: M \to S_1$.
(II). f is an elliptic fibration such that any critical fibre is of one the types $m.I_0$ $(m \geq 2)$, I_0^*, II, III or IV.
(III). $F \neq G$.

3.2.8 Remark. In the above situation, the surface M is algebraic, because its algebraic dimension is two (cf. [BPV] pg. 127). In fact, if $f: M \to S$ and $g: M \to S_1$ are as in (I) and $\phi: S \to \overline{\mathbb{C}}$, $\phi_1: S_1 \to \overline{\mathbb{C}}$ are non-constant holomorphic functions, then we can define two meromorphic functions $f_1, g_1: M \to \overline{\mathbb{C}}$ by $f_1 = \phi \circ f$ and $g_1 = \phi_1 \circ g$. These functions are algebraically independent, because $F \neq G$,

In the Lemma below, we keep the notations of (A), (B), (C) and (D).

3.2.9 Lemma. In the situation considered, we have $B := B(F;G) = \overline{\mathbb{C}}$ and :
(a). If $F_c = f^{-1}(c)$ is a regular level or a critical fibre of type ${}_mI_0$, then, for any $\alpha \in \overline{\mathbb{C}}$, $\alpha \neq 0$, F_c is not invariant for F_α and $\operatorname{tang}(F_\alpha;F_c) = 0$, so that F_α is transverse to F_c.
(b). If $F_j \in A_0^* \cup A_2 \cup A_3 \cup A_4$ then the curves $C_{j,1},\dots,C_{j,k_j}$ are invariant for F_α, for any $\alpha \in \overline{\mathbb{C}}$. On the other hand, if $\alpha \neq 0$, then $C_{j,0}$ is not invariant for F_α and $\operatorname{tang}(F_\alpha;C_{j,0}) = 0$, so that F_a is transverse to $C_{j,0}$.
(c). For all $\alpha \in \overline{\mathbb{C}}$, the singularities of F_α are reduced and

$$\operatorname{sing}(F_a) \subset \cup_{j=1}^{r} \left(\cup_{i>0} C_{j,i} \right) :$$

Moreover, for each $j \in \{1,\dots,r\}$ and $i > 0$, $\mathcal{F}_\alpha$ contains exactly one singularity on the $C_{j,i}$, denoted by $q_{j,i}(\alpha)$, such that :
(c.1). The map $\alpha \in \overline{\mathbb{C}} \mapsto q_{j,i}(\alpha) \in C_{j,i}$ is a regular parametrization of $C_{j,i}$.
(c.2). If $C_{j,i}^2 = -m < 0$, then the singularity $q_{j,i}(\alpha)$ is of the type $1 : -m$ and $I(\mathcal{F}_\alpha, C_{j,i}, q_{j,i}(\alpha)) = -m$ (see 3.1.9).
(d). If $\alpha \in \overline{\mathbb{C}}$ is such that $\mathcal{F}_\alpha$ is tangent to a fibration $f_\alpha : M \to S_\alpha$, then f_α is elliptic. Moreover, if $A_0^* \cup A_2 \cup A_3 \cup A_4 \neq \emptyset$, then $S_\alpha = \overline{\mathbb{C}}$.
(e). The divisor of tangencies is

$$\Delta = \sum_j \sum_{i \neq 0} C_{j,i} .$$

Proof. Let us prove first that $\mathcal{G}$ is transverse to any fibre $F_c = f^{-1}(c) \notin A_0^* \cup A_2 \cup A_3 \cup A_4$. In this case, we have $F_c = m.C$, where $m \geq 1$ and C is smooth and elliptic. If $m = 1$, then F_c is a regular fibre of f, whereas it is of the type ${}_mI_0$, if $m \geq 2$. According to (a) of Lemma 3.2.2, either C is a leaf of $\mathcal{G}$, or $\mathcal{G}$ is transverse to C. Suppose by contradiction that C is a leaf of $\mathcal{G}$. The idea is to prove that this implies that $\mathcal{G} \equiv \mathcal{F}$. Since g is tangent to $\mathcal{G}$, $g|_C$ is constant, say $g(C) = b$. In fact, we must have $g^{-1}(b) = C$, because the generic levels of g are irreducible. Let D and D' be a small neighborhoods of $c \in S$ and $b \in S_1$, respectively. Set $V = f^{-1}(D) \cap g^{-1}(D')$. Note that, if c_1 is near c in S, then $F_{c_1} = f^{-1}(c_1) \subset V$. On the other hand, $g|_{F_{c_1}} : F_{c_1} \to S_1$ is a holomorphic map, and so it is, either surjective, or constant. Since $g(F_{c_1}) \subset D'$, $g|_{F_{c_1}}$ is constant. This implies that f and g have the same fibres in a neighborhood of C, and so $\mathcal{F} = \mathcal{G}$.

Now, fix a fibre $F_j \in A_0^* \cup A_2 \cup A_3 \cup A_4$ and let $C_{j,i}$, $i \in \{0, 1, \dots, k_j\}$, the irreducible components of F_j, as in (D). Note that $Z(\mathcal{F}, C_{j,i}) = 1$ if $i > 0$ (see figure 1). It follows from (b) of Lemma 3.2.2 that $C_{j,i}$ is invariant for $\mathcal{G}$, if $i > 0$. Let us prove that $\mathcal{G}$ is transverse $C_{j,0}$. First of all, observe that $C_{j,0}$ is not invariant for $\mathcal{G}$. The proof of this fact is similar to the argument in the proof of Lemma 3.2.7 that almost all levels of g must cut the singular fibre. These intersections must be on $C_{j,0}$, because the other components are invariant for $\mathcal{G}$. It follows from 3.1.7 that

$$T.C_{j,0} = T_{\mathcal{G}}.C_{j,0} = C_{j,0}^2 - \operatorname{tang}(\mathcal{G}, C_{j,0}) .$$

On the other hand, since $C_{j,0}$ is invariant for $\mathcal{F}$, we get from 3.1.8

$$T.C_{j,0} = 2 - Z(\mathcal{F}, C_{j,0}) \implies \operatorname{tang}(\mathcal{G}, C_{j,0}) = Z(\mathcal{F}, C_{j,0}) + C_{j,0}^2 - 2 .$$

Since $Z(\mathcal{F}, C_{j,0}) = 4$, $C_{j,0}^2 = -2$ if F_j is of the type I_0^* and $Z(\mathcal{F}, C_{j,0}) = 3$, $C_{j,0}^2 = -1$ in the other cases, we get that $\operatorname{tang}(\mathcal{G}, C_{j,0}) = 0$ in all the cases. This implies that $\mathcal{G}$ is transverse to $C_{j,0}$.

Let $W := M \setminus \left(\cup_j (\cup_{i>0} C_{j,i})\right)$. The above facts and the definition of pencil of foliations, imply that :
(i). If $\alpha \neq \beta$ and $p \in W$, then $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ are transverse in a neighborhood of p.

The fact that $C_{j,i}$, $i > 1$, is invariant for both foliations, $\mathcal{F}$ and $\mathcal{G}$, implies that :
(ii). The curve $C_{j,i}$, $i > 0$, is invariant for $\mathcal{F}_\alpha$, if $\alpha \in B$. Moreover, $Z(\mathcal{F}_\alpha, C_{j,i}) = 1$ for all $\alpha \in B$, $j = 1, \dots, r$ and $i > 0$. In particular, $\mathcal{F}_\alpha$ has just one singularity on $C_{j,i}$, if $i > 0$ and $\alpha \in B$.

Let us prove the last relation. Since $C_{j,i}$ is invariant for both foliations and $T_{\mathcal{F}} = T_{\mathcal{F}_\alpha} = T$, we get from 3.1.8, that

$$1 = 2 - Z(\mathcal{F}, C_{j,i}) = T.C_{j,i} = 2 - Z(\mathcal{F}_\alpha, C_{j,i}) \implies Z(\mathcal{F}_\alpha, C_{j,i}) = 1 .$$

Let us denote the singularity of $\mathcal{F}_\alpha$ on $C_{j,i}$ $(i > 0)$ by $q_{j,i}(\alpha)$, $\alpha \in B$.

(iii). Suppose that for a fixed pair (j,i), $i > 0$, and for some $\alpha \in B$, the singularity $q := q_{j,i}(\alpha)$ is non-degenerate. Let $C_{j,i}^2 = -m < 0$. Then q is a singularity of the type $1 : -m$ for F_α and $I(F_\alpha, C_{j,i}, q) = -m$. In particular, there exists a coordinate system $(U,(x,y))$ around q, such that $x(q) = y(q) = 0$, $U \cap C_{j,i} = (y = 0)$ and F_α is represented on U by the vector field $X_\alpha = x\frac{\partial}{\partial x} - my\frac{\partial}{\partial y}$.

In fact, since q is the unique singularity of F_α on $C_{j,i}$, the Camacho-Sad index of F_α at q with respect to $C_{j,i}$ is $-m = C_{j,i}^2$. Therefore, if Y is a vector field representing F_α is a neighborhood of q and λ_1, λ_2 are the eigenvalues of $DY(q)$, where λ_1 corresponds to the direction tangent to $C_{j,i}$, then $\lambda_2/\lambda_1 = -m$. On the other hand, since the curve $C_{j,i}$ is rational, we have that the leaf $C_{j,i} \setminus \{q\}$ of F_α, is homeomorphic to $\mathbb{C}$, so that its holonomy is trivial. It follows from a Lemma of Mattei and Moussu (cf. [M-M]), that the foliation is linearizible at q, that is represented by a linear vector field in some coordinate system in neighborhood of q. Since $\lambda_2/\lambda_1 = -m$, we can choose the coordinate system $(U,(x,y))$ in such a way that the linear vector field is given by $x\frac{\partial}{\partial x} - my\frac{\partial}{\partial y}$ and that $U \cap C_{j,i} = (y = 0)$. This proves (iii).

Let us prove that $B = \overline{\mathbb{C}}$ and that the singularity of F_α on $C_{j,i}$, $i > 0$, is non-degenerate for every $\alpha \in \overline{\mathbb{C}}$. First of all, observe that if $\alpha \in \overline{\mathbb{C}} \setminus B$, then there exist $q \in M$ and holomorphic vector fields X and Y representing F and G, respectively, in a neighborhood U of q, such that $sing(X + \alpha.Y)$ contains a holomorphic curve through q (see Remark 3.1.1). Denote by P the set of such points. In order to prove that $B = \overline{\mathbb{C}}$, it is sufficient to verify that $P = \emptyset$. Note that (i) implies that $P \subset \cup_j(\cup_{i>0}C_{j,i})$. Moreover, if $P \neq \emptyset$, then P contains at least a curve. Since P is an analytic subset of M, it follows that if $C_{j,i} \cap P \neq \emptyset$ then $C_{j,i} \subset P$. Suppose by contradiction that $C_{j,i} \subset P$ for some $j \in \{1,\dots,r\}$ and some $i > 0$. Let $q_0 := q_{j,i}(0)$ and X and Y be vector fields representing F and G, respectively, in a neighborhood U of q_0 such that F_α is represented by $X_\alpha := X + \alpha.Y$ on U. If we take U small, we can suppose that there exists a coordinate system $(U,(x,y))$ such that $U \cap C_{j,i} = (y = 0)$, $x(q_0) = y(q_0) = 0$ and $X = x\frac{\partial}{\partial x} - my\frac{\partial}{\partial y}$, $C_{j,i}^2 = -m$. Since $C_{j,i}$ is invariant for G and $q_0 \notin sing(G)$, the vector field $Y|_{C_{j,i}}$ can be written as $Y(x,0) = (b + x^k.u(x))\frac{\partial}{\partial x}$, where $b \neq 0$ and $k \geq 1$. Hence, $X_\alpha|_{C_{j,i}}$ can be written as

$$(*)\quad X_\alpha(x,0) = \left(x + \alpha(b + x^k.u(x))\right)\frac{\partial}{\partial x} .$$

Since $C_{j,i} \subset P$, it follows that $x + \alpha(b + x^k.u(x)) \equiv 0$ on $U \cap C_{j,i}$, for some fixed $\alpha \in \mathbb{C}$. But this is impossible, so that $P = \emptyset$. It follows from (ii) that $Z(F_\alpha, C_{j,i}) = 1$ for all $i > 0$ and all $\alpha \in \overline{\mathbb{C}}$. In particular, F_α has just one singularity on $C_{j,i}$, $q_{j,i}(\alpha)$, and if X_α is a vector field representing F_α in a neighborhood of $q_{j,i}(\alpha)$, then the eigenvalue of $DX_\alpha(q_{j,i}(\alpha))$ relative to the eigendirection tangent to $C_{j,i}$, say $\lambda_1(\alpha)$, is non-zero. Moreover, if $\lambda_2(\alpha)$ is the other eigenvalue, then $\lambda_2(\alpha)/\lambda_1(\alpha) = I(F_\alpha, C_{j,i}, q_{j,i}(\alpha)) = C_{j,i}^2 \neq 0$ (see 3.1.9). This implies that $\lambda_2(\alpha) \neq 0$, so that $q_{j,i}(\alpha)$ is a non-degenerate singularity of F_α.

We have already proved assertion (a), (b) and (c.2) of the statement of the Lemma. Let us prove (c.1). Observe first that, for a fixed $C_{j,i}$, $i > 0$, the map $\alpha \in \overline{\mathbb{C}} \mapsto q_{j,i}(\alpha) \in C_{j,i}$ is holomorphic. This follows from the general theory of differential equations (see [Ar]). In order to prove that it is a regular parametrization of $C_{j,i}$, it is sufficient to verify that it has no critical point. We will prove that $\alpha = 0$ is not a critical point of the map $q_{j,i}(\alpha)$ and leave the general case for the reader. Represent F_α in a neighborhood U of $q_{j,i}(0)$ by a vector field X_α such that $X_\alpha|_{U \cap C_{j,i}}$ has an expression as in (*). In the coordinate system (x,y) considered, we have that $q_{j,i}(0) = (0,0)$ and that, for $|\alpha|$ small, $q_{j,i}(\alpha) := (x(\alpha),0)$, where $x(\alpha)$ is the solution of the equation $\phi(x,\alpha) := x + \alpha(b + x^k.u(x)) = 0$. Since $\frac{\partial \phi}{\partial x}(0,0) = 1$ and $\frac{\partial \phi}{\partial \alpha}(0,0) = b \neq 0$, we get that $x'(0) = -b \neq 0$, so that $\alpha = 0$ is not a critical point of $q_{j,i}(\alpha)$.

Let us prove assertion (d). Suppose that F_α is tangent to a fibration $f_\alpha\colon M \to S_\alpha$, where $\alpha \neq 0$. Let $G_b = f_\alpha^{-1}(b)$ be a generic fibre of f_α. We have seen that $G_b \cap F_j \subset C_{j,0}$, for any $j = 1, \dots, r$. It follows from (a) and (b) that F is transverse to G_b, so that G_b is an elliptic curve, by Lemma 3.2.1. Let us suppose that $A_0^* \cup A_2 \cup A_3 \cup A_4 \neq \emptyset$ and prove that $S_\alpha \simeq \overline{\mathbb{C}}$. Let $F_j \in A_0^* \cup A_2 \cup A_3 \cup A_4$. Since F_α is transverse to $C_{j,0}$ it follows that $f_\alpha|_{C_{j,0}}\colon C_{j,0} \to S_\alpha$ is a non-constant holomorphic map. This implies that $S_\alpha \simeq \overline{\mathbb{C}}$, because $C_{j,0}$ is a rational curve.

Finally, let us prove (e). It follows from (a) and (b) that, as a set, Γ is contained in $\cup_j(\cup_{i>0} C_{j,i})$. This implies that, as a divisor we must have

$$\Gamma = \sum_j \Big(\sum_{i>0} n_{j,i} C_{j,i} \Big);$$

where $n_{j,i} \in \mathbb{N}$. Since $C_{j,i}.C_{k,\ell} = 0$ if, either $j \neq k$, or $j = k$ and $0 \neq i \neq \ell \neq 0$, we have

$$\Gamma.C_{j,i} = n_{j,i} C_{j,i}^2 \,, \text{ for } i \neq 0.$$

By using 3.1.10 and 3.1.8, we have $[\Gamma] = T_G^* + N_F$ and $T_G^*.C_{j,i} = Z(G, C_{j,i}) - \chi(C_{j,i}) = -1$, $N_F.C_{j,i} = C_{j,i}^2 + Z(F, C_{j,i}) = C_{j,i}^2 + 1$, so that

$$n_{j,i} C_{j,i}^2 = \Gamma.C_{j,i} = C_{j,i}^2 \implies n_{j,i} = 1;$$

because $C_{j,i}^2 \neq 0$. □

3.2.10 Corollary. In the situation considered, let

$$F(M, T_F) = \{H;\ H \text{ is a foliation on } M \text{ such that } T_H = T_F\}.$$

Then $F(M, T_F) = \{F_\alpha;\ \alpha \in \overline{\mathbb{C}}\}$, where $(F_\alpha)_{\alpha \in \overline{\mathbb{C}}}$ is the pencil generated by F and G. In particular, $\dim(F(M, T_F)) = 1$ and if $(H_s)_{s \in S}$ is a holomorphic family of foliations on $F(M, T_F)$, then there exists a holomorphic map $\varphi\colon S \to \overline{\mathbb{C}}$ such that $H_s = F_{\varphi(s)}$ for all $s \in S$.

Proof. Let $H \in F(M, T_F)$ and fix $\alpha \in \overline{\mathbb{C}}$ such that $H \neq F_\alpha$. Since $T_H = T_{F_\alpha}$, we have $T_H^* = T_{F_\alpha}^*$ and $N_H = N_{F_\alpha}$, which implies that $[\Gamma(H, F_\alpha)] = [\Gamma(G, F)] = [\Gamma]$ (as an element of $H^1(M, \mathcal{O}^*)$). Let us prove that, as a curve, we have also

$$\Gamma(H, F_\alpha) = \Gamma = \cup_{j=1}^{r} \Big(\cup_{i=1}^{k_j} C_{j,i} \Big):$$

Write

$$\Gamma(H, F_\alpha) = \sum_{i=1} m_i.S_i + \sum_{j=1} k_j.V_j;$$

where $S_1, \dots, S_s$ are the components of $\Gamma(H, F_\alpha)$ which are not contained in fibres of f, $V_1, \dots, V_v$ are the components contained in fibres of f and $m_1, \dots, m_s, k_1, \dots, k_v$ are non negative integers. Since $[\Gamma(H, F_\alpha)] = [\Gamma]$, we must have $\Gamma(H, F_\alpha).C = \Gamma.C$ for any curve C on M. Let $F = f^{-1}(c)$ be a regular fibre of f. We have, $F.S_i > 0$ for all $i \in \{1, \dots, s\}$ and $F.V_j = 0$ for all $j \in \{1, \dots, v\}$. This implies that $F.\Gamma(H, F_\alpha) = \sum_{i=1}^{s} m_i F.S_i \geq 0$. On the other hand, it follows from (e) of Lemma 3.2.9 that $\Gamma.F = 0$, and so

$$\sum_{i=1} m_i F.S_i = 0 \implies m_i = 0 \text{ for all } i = 1, \dots, s:$$

Now, let $G = g^{-1}(b)$ be a regular fibre of g. It follows from Lemma 3.2.9 that $G.F_j = G.C_{j,0} > 0$, and $G.C_{j,i} = 0$, for any $j = 1,...,r$ and $i > 0$, and that $G.F > 0$ if F is, either a regular fibre of f, or a fibre of type $m.I_0$. Therefore (e) of Lemma 3.2.9, implies that

$$0 = G.\Gamma = G.\Gamma(H;F_\alpha) = \sum_{j=1} k_j G.V_j \geq 0 \implies k_j = 0 \text{ if } V_j = C_{t,0} \text{ for some } t = 1,...,r\,.$$

Hence $|\Gamma(H;F_\alpha)| \subset |\Gamma|$. Finally, if we take $C = C_{t,i}$ for some $t = 1,...,r$ and $i > 0$, we obtain $\Gamma(H;F_\alpha).C = \Gamma.C = C^2 \neq 0$, which shows that $\Gamma(H;F_\alpha) = \Gamma$. This fact implies that, if $p \notin \Gamma$ is fixed, and H and F_α have the same tangent line at p, then $H = F_\alpha$. On the other hand, given $p \notin \Gamma$, there exists $\alpha \in \overline{\mathbb{C}}$ such that H and F_α have the same direction, so that $H = F_\alpha$. This implies that $F(M;T_F) = \{F_\alpha;\ \alpha \in \overline{\mathbb{C}}\}$. The remaining conclusions are a consequence of this fact, as the reader can check. ■

3.2.11 Corollary. If F_α is tangent to a fibration $f_\alpha\colon M \to S_\alpha$, then :
(a). f_α is an elliptic fibration and S_α is either a rational, or an elliptic curve.
(b). Any critical fibre of f_α is of one of the types ${}_mI_0$, I_0^*, II, III or IV.

Proof. We have seen in (d) of Lemma 3.2.9 that f_α is an elliptic fibration. Let F be a generic level of f. Since $h := f_\alpha|_F\colon F \to S_\alpha$ is holomorphic and non-constant and F is an elliptic curve, it follows that S_α is, either a rational, or an elliptic curve (cf. [G-H]). According to Lemma 3.2.7, in order to prove assertion (b), it is sufficient to verify that F_α has no contractible curve. Suppose by contradiction that F_α has a contractible curve, say C. Since $C^2 = -1$, we must have $C \subset M \setminus \left(\cup_j (\cup_{i>0} C_{j,i})\right)$, because $C_{j,i}^2 \leq -2$, if $i > 0$. This implies that F is transverse to C and it follows from 3.1.7 that $T_F.C = C^2 = -1$. On the other hand, since C is contractible for F_α, we must have that, either $Z(F_\alpha;C) = 1$, or $Z(F_\alpha;C) = 2$, so that $T_F.C = \chi(C) - Z(F_\alpha;C) \geq 0$, because $T_F = T_{F_\alpha}$, which is a contradiction. ■

Let A_0^m, $A_0 = \cup_m A_0^m$, A_0^*, A_2, A_3 and A_4 be as (A), (B) and (C). We will use the following notations : $a_0 = \#A_0$, $a_0^* = \#A_0^*$, $a_j = \#A_j$, for $j = 2,3,4$, and $a = a_0 + a_0^* + a_2 + a_3 + a_4$.
(E). If $A_0 \neq \emptyset$, we will use the notation $G_1,...,G_{a_0}$ for the fibres in A_0. Note that each G_i is an elliptic fibre with multiplicity, say $m_i \geq 2$, so that we can write $G_i = m_i.C_i$, where C_i is an (irreducible) elliptic curve.

Recall that $f\colon M \to S$ is an elliptic fibration, where S is, either rational, or elliptic. We will consider both cases.

3.2.12 Lemma. Suppose that S is an elliptic curve. Then M is a complex algebraic torus and the foliations F and G can be defined by global non-vanishing holomorphic vector fields on M. Moreover, the pencil generated by F and G is a weakly exceptional family of foliations.

Proof. Let us prove first that $f\colon M \to S$ has no critical fibres, so that it is a fibre bundle. Since S is an elliptic curve, we must have $A_0^* \cup A_2 \cup A_3 \cup A_4 = \emptyset$, by (d) of Lemma 3.2.9. This implies that F and G are everywhere transverse. Let G be a generic fibre of g and $h := f|_G\colon G \to S$. Then h is a holomorphic non-constant map, so that it has no critical point, by Riemann-Hurwitz formula. On the other hand, if f had some critical fibre, say $G_j \in A_0$, then the points in $G_j \cap G$ would be critical points of h. Therefore $A_0 = \emptyset$ and f has no critical fibre. In particular, $f\colon M \to S$ is a fibre bundle. Since G is transverse to F, this bundle is a principal bundle with transiction maps locally constant. It follows from BIa) of page 146 of [BPV], that M is a complex 2-torus and the foliations F and G are defined by global vector fields. This implies that the pencil is a weakly exceptional family of foliations. We leave the details for the reader. ■

From now on, in this section, we will suppose that $f\colon M \to \overline{\mathbb{C}}$.

3.2.13 Lemma. In the above hypothesis, we have the following :

(a). $N^*_F = (a-2)[F] - \sum_{j=1}^{r} \Lambda_j - \sum_{i=1}^{a_0}[C_i]$, where F denotes any fixed fibre of f, $\Lambda_j = \sum_{i=0}^{k_j}[C_{j;i}]$ and the $C_{j;i}$ are as in (D).
(b). $K_M = [\Delta] + 2N^*_F = 2(a-2)[F] - \sum_{j=1}^{r} \Gamma_j - 2\sum_{i=1}^{a_0}[C_i]$, where $\Gamma_j = 2[C_{j;0}] + \sum_{i=1}^{k_j}[C_{j;i}]$. In particular,

$$K_M^2 = c_1^2(M) = \sum_{j=1}^{r} \Gamma_j^2 = -3a_2 - 2a_3 - a_4 .$$

(c). $6a_0^* + 10a_2 + 9a_3 + 8a_4 + 12\sum_{i=1}^{a_0}(1 - \frac{1}{m_i}) = 24$.
(d). $C_2(M) = 6a_0^* + 5a_2 + 5a_3 + 5a_4$.

Proof. Let us prove (a). After composing f with a Moebius transformation, we can suppose that the fibre $C_\infty := f^{-1}(\infty)$ is a regular level of f, so that we can consider f as a meromorphic function on M with pole divisor $(f)_\infty = [C_\infty]$. In this case, the foliation F is tangent to the meromorphic 1-form df, so that $N^*_F = (df)_0 - (df)_\infty$ (see 3.1.3). Note that $(df)_\infty = 2[C_\infty]$. Since C_∞ is a regular fibre of f, we have that $[C_\infty] = [F]$, where F is any fixed fibre of f. On the other hand, $df(p) = 0$ if, and only if, p belongs to a multiple component C of a critical fibre of f. Moreover, if the multiplicity of f at C is $m \geq 2$, then C will be a component of order $m-1$ of the divisor of zeroes of df, $(df)_0$. If $F_j \in A_0^* \cup A_2 \cup A_3 \cup A_4$, with the notation of (D), we have

$$[F_j] = \sum_{i=0}^{k_j} m_{j;i}[C_{j;i}] = [F];$$

so that,

$$[(df)_0] = \sum_{j=1}^{r}(\sum_{i=0}^{k_j}(m_{j;i}-1)[C_{j;i}]) + \sum_{i=0}^{a_0}(m_i - 1)[C_i] =$$

$$= \sum_{j=1}^{r}(\sum_{i=0}^{k_j} m_{j;i}[C_{j;i}]) + \sum_{i=0}^{a_0} m_i[C_i] - \sum_{j=1}^{r}(\sum_{i=0}^{k_j}[C_{j;i}]) - \sum_{i=0}^{a_0}[C_i] = a[F] - \sum_{j=1}^{r}\Lambda_j - \sum_{i=0}^{a_0}[C_i].$$

Hence :

$$N^*_F = (a-2)[F] - \sum_{j=1}^{r}\Lambda_j - \sum_{i=0}^{a_0}[C_i];$$

which proves (a). Since $[\Delta] = [\Delta(F;G)] = T^*_G + N_F = T^*_F + N_F$ and $K_M = T^*_F + N^*_F$, we get $K_M = [\Delta] + 2N^*_F$ (see 3.1.2). Therefore, (e) of Lemma 3.2.9 implies that

$$K_M = 2(a-2)[F] - \sum_{j=1}^{r}(2[C_{j;0}] + \sum_{i=1}^{k_j}[C_{j;i}]) - 2\sum_{i=0}^{a_0}[C_i] = 2(a-2)[F] - \sum_{j=1}^{r}\Gamma_j - 2\sum_{i=0}^{a_0}[C_i].$$

In particular, $K_M^2 = \sum_{j=1}^{r}\Gamma_j^2$, as the reader can check. Hence, (b) follows from : $\Gamma_j^2 = 0$, if $F_j \in A_0^*$, $\Gamma_j^2 = -3$ if $F_j \in A_2$, $\Gamma_j^2 = -2$ if $F_j \in A_3$ and $\Gamma_j^2 = -1$ if $F_j \in A_4$. For instance, if $F_j \in A_2$ we have $\Gamma_j = 2[C_{j;0}] + [C_{j;1}] + [C_{j;2}] + [C_{j;3}]$, where $C_{j;0}^2 = -1$, $C_{j;1}^2 = -6$, $C_{j;2}^2 = -3$ and $C_{j;3}^2 = -2$ (see figure 1.b), so that $\Gamma_j^2 = 4C_{j;0}^2 + C_{j;1}^2 + C_{j;2}^2 + C_{j;3}^2 + 4\sum_{n=1}^{3} C_{j;n}.C_{j;0} = -4 - 6 - 3 - 2 + 4.3 = -3$. The other identities can be checked in the same way.

In order to prove (c) we will use the other fibration $g: M \to \overline{\mathbb{C}}$. Let G be a regular fibre of g and consider $h := f|_G: G \to \overline{\mathbb{C}}$. It follows from Riemann-Hurwitz formula and the fact that g is an elliptic fibration that

$$0 = \chi(G) = d.\chi(\overline{\mathbb{C}}) - \sum_{p \in G} (m_p - 1) = 2d - \sum_{p \in G} (m_p - 1) \implies 2d = \sum_{p \in G} (m_p - 1)$$

where m_p is the ramification number of h at the point $p \in G$ and d is the topological degree of h. We observe the following facts :

(i). If F is a regular fibre of f, then $d = F.G$.

(ii). The critical points of h are contained in the intersection of G with the critical fibres of f.

(iii). If $F_j \in A_0^* \cup A_2 \cup A_3 \cup A_4$ then $G \cap F_j = G \cap C_{j;0}$ and G intersects $C_{j;0}$ transversely (Lemma 3.2.9). This implies that $\#(G \cap F_j) = \frac{d}{m_{j;0}}$, where $m_{j;0}$ is the multiplicity of $C_{j;0}$. Moreover, if $p \in G \cap F_j$ then the ramification number of h at p is $m_{j;0}$.

(iv). If $G_i \in A_0$ then G intersects G_i transversely at $\frac{d}{m_i}$ points (Lemma 3.2.9). Moreover, if $p \in G \cap G_i$ then the ramification number of h at p is m_i.

The above facts imply that (see figure 1 for the multiplicities $m_{j;0}$)

$$2d = (2-1)\frac{d}{2}a_0^* + (6-1)\frac{d}{6}a_2 + (4-1)\frac{d}{4}a_3 + (3-1)\frac{d}{3}a_4 + \sum_{i=1}^{k} (m_i - 1)\frac{d}{m_i} ;$$

and the above equality implies (c), as the reader can check.

It remains to prove (d). We use here the following well known result (cf. [BPV]) :

"Let $f: M \to S$ be a fibration, where S is a compact Riemann surface and M is a compact complex surface. Then

$$c_2(M) = \chi(S).\chi(F_g) + \sum_{c \in S} (\chi(F_c) - \chi(F_g)) ;$$

where in the above sum F_g denotes a generic fibre of f and $\chi(F_c)$ denotes the topological Euler characteristic of the curve $(f^{-1}(c))_{red}$."

In the above statement, $(f^{-1}(c))_{red}$ denotes the curve $f^{-1}(c)$ reduced, that is if $f^{-1}(c) = \sum_j m_j C_j$, then $(f^{-1}(c))_{red} = \sum_j C_j$. In our case, $f: M \to \overline{\mathbb{C}}$, $\chi(S) = 2$, $\chi(F_g) = 0$ and $\chi(F_c) = 0$ if $F_c \in A_0$, so that

$$c_2(M) = \sum_{F_c \in A_0^* \cup A_2 \cup A_3 \cup A_4} \chi(F_c) .$$

On the other hand, we have $\chi(F_c) = 6$ if $F_c \in A_0^*$ and $\chi(F_c) = 5$ if $F_c \in A_2 \cup A_3 \cup A_4$. Therefore,

$$c_2(M) = 6a_0^* + 5a_2 + 5a_3 + 5a_4 .$$

$\square$

3.2.14 Remark. In the table below we give all the non-negative integer possible solutions of the equation in (c) of Lemma 3.2.13 :

Sol.	1	2	3	4	5	6	7	8	9	10	11	12	13
a_0	0	0	0	0	1	1	1	1	1	1	2	3	4
a_0^*	0	1	1	4	3	1	1	1	0	0	2	1	0
a_2	0	1	0	0	0	1	0	0	1	0	0	0	0
a_3	0	0	2	0	0	0	1	0	0	0	0	0	0
a_4	3	1	0	0	0	0	0	1	1	2	0	0	0
m_i	$-$	$-$	$-$	$-$	2	3	4	6	2	3	2; 2	2; 2; 2	2; 2; 2; 2
a	3	3	3	4	4	3	3	3	3	3	4	4	4
$C_2(M)$	15	16	16	24									
K_M^2	-3	-4	-4	0									

Table of solutions

In the botton of the table we give the possible values of $C_2(M)$ and K_M^2 for the four first solutions. Note that the solutions 1, 2 and 3 correspond to the families of types 1, 2 and 3, respectively, whereas the solution 4 coresponds to the second example in §2.1.

3.2.15 Lemma. The only solutions of the equation in (c) of Lemma 3.2.13, which come from foliations $\mathcal{F}$ and $\mathcal{G}$ as before, are the solutions 1, 2, 3 and 4.

Proof. First of all, let us consider the monodromy of the fibration f. By using it, we will prove that there are no fibrations corresponding to solutions $5; 6; 7; 8; 9; 10$ and 12. Let $c_1; \ldots; c_a \in \overline{\mathbb{C}}$ be the critical values of f, c be a regular value and $F = f^{-1}(c)$. Recall that the monodromy is a homomorphism $\rho\colon \pi_1(\overline{\mathbb{C}} \setminus \{c_1; \ldots; c_a\}; c) \to \mathrm{Aut}(\mathbb{Z}^2) \simeq \mathrm{Aut}(H_1(F; \mathbb{Z}))$ (cf. [BPV]). Note that $\pi_1(\overline{\mathbb{C}} \setminus \{c_1; \ldots; c_a\}; c)$ is generated by a curves, $\gamma_1; \ldots; \gamma_a$, as in figure 8, with the relation $\gamma_1 * \ldots * \gamma_a = 1$. Let $G = \rho(\pi_1(\overline{\mathbb{C}} \setminus \{c_1; \ldots; c_a\}; c))$. If we use the notation $\rho(\gamma_j) := T_j$, then $G = < T_1; \ldots; T_a >$, where $T_1 \circ \ldots \circ T_a = id$. The monodromy of the Kodaira fibres, along curves as in figure 8, is well known (cf. [BPV]). We observe that the monodromy of the fibres $\tilde{II}$, $\tilde{III}$ and $\tilde{IV}$, coincides with that of the fibres II, III and IV, respectively, of Kodaira's classification.

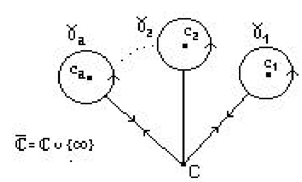


Fig 8

The monodromy T_j, $j = 1; \ldots; a$, can be of the one of the following types :

(i). $T_j = id$, if $f^{-1}(c_j)$ is of the type ${}_mI_0$, $m \geq 2$.

(ii). $T_j = -id$, if $f^{-1}(c_j)$ is of the type I_0^*.

(iii). T_j is conjugated to the matrix $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, if $f^{-1}(c_j)$ is of the type $\tilde{II}$. In particular, the order of T_j is 6.

(iv). T_j is conjugated to the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, if $f^{-1}(c_j)$ is of the type $\tilde{III}$. In particular, the order of T_j is 4.

(v). T_j is conjugated to the matrix $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$, if $f^{-1}(c_j)$ is of the type $\tilde{IV}$. In particular, the order of T_j is 3.

Let us prove that the solutions 6,7,8,9 and 10, cannot occur. In these cases we have $a = 3$ and one of the critical fibres, say $f^{-1}(c_3)$, is of the type ${}_mI_0$, so that $G = < T_1, T_2 >$, where $T_1 \circ T_2 = id$. This implies that G is abelian, so that we can suppose that T_1 and T_2 are given by same matrixes as in (ii),...,(v). As the reader can check, in all these cases, we have $T_1 \circ T_2 \neq id$, which is a contradiction. Therefore, these cases cannot happen. On the other hand, for the solutions 5 and 12, the fibres can be only of the types ${}_2I_0$ and I_0^*, so that $G = \{id, -id\}$. In these cases, as the reader can check, we have $T_1 \circ T_2 \circ T_3 \circ T_4 = -id \neq id$ and so these cases cannot occur also.

It remains to prove that the solutions 11 and 13 do not occur. Let us prove first that $K_M = 0$ in the case of solutions 4, 11 and 13. Note that, for a fibre F_j of type I_0^*, we have $[F_j] = \Gamma_j$. For the solutions 11 and 13, we have $a = 4$ and $a_0 \in \{2, 4\}$, so that $A_0 \neq \emptyset$. Moreover, if $G_i \in A_0$, then $m_i = 2$ and $G_i = 2.C_i$, so that (b) of Lemma 3.2.13 implies that :

$$K_M = 4[F] - \sum_{j=1}^{a_0^*} \Gamma_j - \sum_{i=0}^{a_0} 2.[C_i] = (4 - a_0^* - a_0)[F] = 0 .$$

The above fact, implies that there exists a holomorphic non-vanishing 2-form on M, say Θ (cf. [BPV]). We will prove that this leads to a contradiction. The idea is to prove that if $G_i = 2.C_1$ and W is a small neighborhood of C_i, then any holomorphic 2-form on W must vanish along C_i, which contradicts the fact that Θ does not vanishes. Let $G_i = f^{-1}(c_i)$, fix a small disk D around c_i, such that c_i is the unique critical value of f on D, and set $W = f^{-1}(D)$. We know that f is isotrivial, so that we can suppose that its generic fibre is biholomorphic to $\mathbb{C}/ < 1, b >$ for some $b \notin \mathbb{R}$. We will use the following fact (see [BPV] pages 151 and 155) :
(*). We can choose the representation $\mathbb{C}/ < 1, b >$ above and D small enough, in such a way that W is biholomorphic to $(\mathbb{C} \times D)/\equiv$, where D is the unit disk on $\mathbb{C}$ and $\equiv$ is the equivalence relation on $\mathbb{C} \times D$ defined by the action generated by $T_1, T_2 : \mathbb{C} \times D \to \mathbb{C} \times D$, where $T_1(z, w) = (z + 1/2, -w)$ and $T_2(z, w) = (z + b, w)$. In this representation of W, we have $C_i = \{w = 0\}$.

Let $\pi : \mathbb{C} \times D \to (\mathbb{C} \times D)/\equiv$ be the projection of the equivalence relation and set $\Theta_1 = \pi^*(\Theta)$. Note that π is a covering map with two sheets, so that Θ_1 do not vanishes on $\mathbb{C} \times D$. Let $\Theta_1 = \phi(z, w).dz \wedge dw$, where ϕ is holomorphic. Note that $T_j^*(\Theta_1) = \Theta_1$, for $j = 1, 2$. This implies that

$$\phi(z + 1/2, -w) = -\phi(z, w) \text{ and } \phi(z + b, w) = \phi(z, w) \implies \phi \text{ does not depend on } z ;$$

so that $\phi(z, w) = \psi(w)$, where $\psi(-w) = -\psi(w)$. But, this implies that $\psi(0) = 0$ and that $\Theta|_{C_i} \equiv 0$, which is a contradiction. ∎

3.2.16 Corollary. In the situation of Lemma 3.2.15, let $E = \{\alpha \mid F_a$ has a first integral$\}$ and for $\alpha \in E$ let $f_\alpha : M \to S_\alpha$ be a fibration tangent to F_α. Then $S_\alpha = \overline{\mathbb{C}}$ and the critical fibres of f_α are of the same type as the critical fibres of f.

Proof. Observe first that $A_0^* \cup A_2 \cup A_3 \cup A_4 \neq \emptyset$, so that $S_\alpha = \overline{\mathbb{C}}$. Moreover, it follows from the Corollary 3.2.11 that the critical fibres of f_α can be only of the types mI_0, I_0^*, II^*, III^* or IV^*. Let $a_0(\alpha)$, $a_0^*(\alpha)$, $a_2(\alpha)$, $a_3(\alpha)$ and $a_4(\alpha)$ be the number of such fibres, respectively. It is enough to prove that these numbers are the same as a_0,...,a_4. Note that Lemma 3.2.15 implies that they must be as in the solutions 1, 2, 3 or 4 in the table of solutions. On the other hand, the Chern class $C_2(M)$ in the same table, shows that the unique possibility that they are different is in the case of solutions 2 and 3. At this point we can use the fact that the curves $C_{j,i}$, $j = 1, 2, 3$, $i > 0$, are invariant for both foliations, so that they must be contained in the critical levels of f_α. Since a fibre of type III^* contains one curve C with $C^2 = -4$ and the critical fibres I_0^*, II^* and IV^* do not contain any component like that, we conclude that the critical fibres of the fibrations are of the same type. ∎

3.2.17 Corollary. If $K_M = 0$, then M is an algebraic K_3 surface. Moreover, if E is as before, then for any $\alpha \in E$, the first integral is a fibration with four I_0^* fibres.

Proof. We have already proved that M is algebraic. Let us prove that M is minimal, that is, does not contain a smooth rational curve with self-intersection -1. Suppose by contradiction that M contains a smoth rational -1-curve, say C. Since the curves that are invariant for both foliations, $\mathcal{F}$ and $\mathcal{G}$, are contained in $\cup_j(\cup_{i>0} C_{j,i})$ and for all curves in this set we have $C_{j,i} \leq -2$, we get that C is not invariant for one of the foliations, say $\mathcal{F}$. In this case, we get from $K_M = N^*_{\mathcal{F}} + T^*_{\mathcal{F}}$ and from 3.1.7 that :

$$T^*_{\mathcal{F}}.C = 1 + tang(\mathcal{F}, C) \text{ and } N^*_{\mathcal{F}}.C = -2 - tang(\mathcal{F}, C) \implies 1 + tang(\mathcal{F}, C) = 2 + tang(\mathcal{F}, C)$$

which is a contradiction. Therefore, M is minimal. On the other hand, in the table of solutions we see that the unique possibility for $K_M = 0$ is the 4^{th} solution, so that $C_2(M) = 24$ and for any $\alpha \in E$ the fibration f_α has four critical fibres, all of the type I_0^*. The fact that $K_M = 0$ implies that $kod(M) = 0$, so that M is biholomorphic to a $K3$ surface, by Enriques-Kodaira classification of surfaces (see table 10, pg. 188 of [BPV]). ∎

In order to finish the proof of Theorem 3, it remains to prove that the pencil $(\mathcal{F}_\alpha)_{\alpha \in \overline{\mathbb{C}}}$ is weakly exceptional and assertions (c) and (d). Here, we will use the global holonomy groups of the foliations in the pencil with respect to the fibration g. Let $c_1, ..., c_a$ be the critical levels of g, where $a = 3$ in the case of solutions 1, 2, 3, and $a = 4$ in the case of solution 4. Set $F_j = g^{-1}(c_j)$. It follows from (e) of Lemma 3.2.9 that if $\alpha \neq \infty$ then $\mathcal{F}_\alpha$ is transverse to $\mathcal{G} = \mathcal{F}_\infty$ outside $\cup_j(\cup_{i>0} C_{j,i})$ and so, a fortiori, in the set $W = M \setminus \cup_j F_j$. Note that $g|_W : W \to V$ is a fibre bundle, where $V = \overline{\mathbb{C}} \setminus \{c_1, ..., c_a\}$. Therefore, if $F := f^{-1}(c)$, $c \in V$, then we can define a global holonomy representation

$$H_\alpha : \pi_1(V, c) \to Aut(F);$$

where $Aut(F)$ denotes the set of automorphisms of the fibre F (cf. [Eh] and [C-LN]). We denote by G_α the holonomy group of $\mathcal{F}_\alpha$, that is the image $H_\alpha(\pi_1(V, c)) \subset Aut(F)$. Note that $\pi_1(V, c)$ is generated by a closed curves $\gamma_1, ..., \gamma_a$, sketched in figure 8, where $\gamma_1 * ... * \gamma_a = 1$. We denote by $f_{k,\alpha}$ the holonomy map $H_\alpha(\gamma_k)$, $k = 1, ..., a$. Hence, we have $G_\alpha = < f_{1,\alpha}, ..., f_{a-1,\alpha} >$. Fix a holomorphic universal covering $\pi : \mathbb{C} \to F$ of F, with automorphism group $Aut(\pi) = < h_1, h_b >$, where $H_\lambda(z) = z + \lambda$, $\lambda \in \{1, b\}$, $b \notin \mathbb{R}$, so that $F \simeq \mathbb{C}/\Gamma$, where $\Gamma = < 1, b >$. Given $\alpha \in \mathbb{C}$ and $k \in \{1, ..., a\}$, we will consider a covering of $f_{k,\alpha}$ in $\mathbb{C}$ by π, that is a map $\phi_{k,\alpha} \in Aut(\mathbb{C})$ such that $\pi \circ \phi_{k,\alpha} = f_{k,\alpha} \circ \pi$. Let us see how $\phi_{k,\alpha}$ looks like, according to the type of the fibre F_k :

(1) F_k is of the type I_0^*. In this case $\phi_{k,\alpha}(z) = -z + b_k(\alpha)$, where $b_k(\alpha) \in \mathbb{C}$. In particular, $f_{k,\alpha}$ has order two.

(2) F_k is of the type II. In this case $\phi_{k,\alpha}(z) = \omega^{-1}.z + b_k(\alpha)$, where $\omega = e^{2\pi i/6}$ and $b_k(\alpha) \in \mathbb{C}$. In particular $f_{k,\alpha}$ has order six.

(3) F_k is of the type III. In this case $\phi_{k,\alpha}(z) = -i.z + b_k(\alpha)$, where $i = \sqrt{-1}$ and $b_k(\alpha) \in \mathbb{C}$. In particular $f_{k,\alpha}$ has order four.

(4) F_k is of the type IV. In this case $\phi_{k,\alpha}(z) = \omega^{-2}.z + b_k(\alpha)$, where $b_k(\alpha) \in \mathbb{C}$. In particular $f_{k,\alpha}$ has order three.

The proof of (4) is done in Proposition 4 of [LN]. The idea, in the general case, is that the fibre F_k contains at least one component, say $C_{k,1}$, with multiplicity one (see figure 1). This component, contains a singularity $q_{k,1}(\alpha) := q(\alpha)$ with a separatrix, say $S(\alpha)$, transverse to $C_{k,1}$ and with holonomy conjugated to $z \mapsto e^{-2\pi i/m}.z$, where $m = -C^2_{k,1}$ (see (c.2) of Lemma 3.2.9).

It follows that $f_{k,\alpha}$ must have a fixed point, say $z_k(\alpha)$, and it is conjugated in a neighborhood of it to the linear map $z \mapsto e^{-2\pi i/m}.z$. This fixed point corresponds to some intersection of the leaf of F_α which contains $S(\alpha)$ with F (see the proof of Proposition 4 in [LN]). As the reader can check, this implies that $f_{k,\alpha}$ has period m, so that $\phi_{k,\alpha}$ must be like in (1), (2), (3) or (4). Remark also that (2), (3) and (4) imply that in the cases (2) and (4) the lattice Γ must be $< 1, \omega := e^{\frac{2\pi i}{3}} >$, whereas in the case (3) it is $< 1, i >$.

For each $k = 1, \dots, a$, consider the function $f_k\colon C \times F \to F$ defined by $f_k(\alpha, q) = f_{k,\alpha}(q)$. It follows from the theorem of holomorphic dependence of the solutions with respect to parameters and initial conditions, that f_k is holomorphic, for all $k = 1, \dots, a$. In particular, this implies that in all the cases, the map $\alpha \in C \mapsto b_k(\alpha)\ md(\Gamma) \in C/\Gamma$ is holomorphic. Therefore, we can choose $b_k(\alpha)$ in such a way that $\alpha \mapsto b_k(\alpha) \in C$ is holomorphic. In particular, if we write $\phi_{k,\alpha}(z) = \lambda_k.z + b_k(\alpha)$, the point $z_k(\alpha) = \frac{b_k(\alpha)}{1-\lambda_k}$ is a fixed point of $\phi_{k,\alpha}$. Hence, by conjugating the group G_α with the automorphism corresponding to the translation $\sigma_\alpha(z) = z + z_1(\alpha)$, we can suppose that $\phi_{k,\alpha}(z) = \lambda_k.z + \mu_k(\alpha)$, where $\mu_1(\alpha) = 0$ and $\mu_k(\alpha) = b_k(\alpha) - \frac{1-\lambda_k}{1-\lambda_1} b_1(\alpha)$, $k = 2, \dots, a$, so that $\alpha \mapsto \mu_k(\alpha)$ is holomorphic. Since G_α is generated by $f_{1,\alpha}, \dots, f_{a-1,\alpha}$, we get that, if g has three critical fibres, then G_α is conjugated to a group, whose universal covering is of the form $G_\alpha = \{\lambda.z + d.\mu_2(\alpha) \mid \lambda \in \Lambda$ and $d \in \Gamma\}$, where :

(I). In the case of solution 1 we have $\Lambda = \{1, \omega^2, \omega^4\}$ and $\Gamma = \mathbb{Z} \oplus \omega.\mathbb{Z}$ (cf. Proposition 5 of [LN]).

(II). In the case of solution 2 we have $\Lambda = \{\omega^j \mid j = 0, \dots, 5\}$ and $\Gamma = \mathbb{Z} \oplus \omega.\mathbb{Z}$

(III). In the case of solution 3 we have $\Lambda = \{1, i, -1, -i\}$ and $\Gamma = \mathbb{Z} \oplus i.\mathbb{Z}$.

On the other hand, in the case of solution 4, we have :

(IV). $G_\alpha = \{\lambda.z + m.\mu_2(\alpha) + n.\mu_3(\alpha) \mid \lambda \in \{1, -1\}$ and $m, n \in \mathbb{Z}\}$, where $\Gamma = < 1, b >$, $b \notin \mathbb{R}$.

The proof of (I) can be found in Proposition 5 of [LN]. The proof of (II), (III) and (IV) is analogous and is left for reader. Another result that we will use, whose proof is analogous to the proof of Proposition 5 and of its Corollary in [LN], is the following :

3.2.18 Lemma. For $\alpha \neq \infty$, the following assertions are equivalent :

(i). F_α has a first integral.

(ii). G_α is finite.

(iii). G_α has a finite orbit.

(iv). F_α has an algebraic leaf which is not contained in the critical levels of g.

Moreover, in the cases of solutions 1, 2 and 3, the above assertions are equivalent to :

(v). If $\alpha \neq 0$, then there exists $n \in \mathbb{N}$ such that $n.\mu_2(\alpha) \in \Gamma$.

Another important fact is the following :

3.2.19 Lemma. For any $k \in \{2, \dots, a\}$ we have $\mu_k(\alpha) = a_k.\alpha + d_k$, where $a_k, d_k \in C$. Moreover, if g has three critical fibres then $a_2 \neq 0$, whereas if g has four critical fibres then, either $a_2 \neq 0$, or $a_3 \neq 0$. In particular, we have the following :

(a). The pencil P is always weakly exceptional.

(b). If g has three critical fibres then $E(P) = \lambda.\mathbb{Q}.\Gamma \cup \{\infty\}$, where $\lambda = a_2^{-1}$.

(c). If g has four critical fibres and $E(P)$ contains at least three distinct points, then the family is exceptional.

Proof. Recall that the fibration $g|_W\colon W \to V$ is locally holomorphically trivial and that the leaves of $F_0 = F$ are transverse to the fibres of g in W. In particular, for every $c \in V$, there exists a neighborhood V_c of c in V with the following properties :

(i). V_c is biholomorphic to a disk and $W_c = g^{-1}(V_c)$ is biholomorphic to $V_c \times C/\Gamma$, by a biholomorphism $\psi_c\colon W_c \to V_c \times C/\Gamma$.

(ii). $g \circ \psi_c^{-1} : V_c \times \mathbb{C}/\Gamma \to V_c$ is the first projection. In particular, the sets of the form $\{x\} \times \mathbb{C}/\Gamma$, $x \in V_c$, correspond to the leaves of $\mathcal{G}$ in W_c.
(iii). The leaves of $\psi_c^*(\mathcal{F})$ are of the form $V_c \times \{y\}$, $y \in \mathbb{C}/\Gamma$.

Consider an universal covering $\pi: id \times \pi_2 : V_c \times \mathbb{C} \to V_c \times \mathbb{C}/\Gamma$, where $\pi_2 : \mathbb{C} \to \mathbb{C}/\Gamma$ is an universal covering with automorphism group $Aut(\pi_2) = < T_1, T_2 >$, $T_1(y) = y + 1$, $T_2(y) = y + b$, $\Gamma = < 1, b >$. Here, y is a fixed affine coordinate system in $\mathbb{C}$. For simplicity, we will denote by $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ the vector fields on W_c defined by $\psi_c^*(\pi_*(\frac{\partial}{\partial x} \times 0))$ and $\psi_c^*(\pi_*(0 \times \frac{\partial}{\partial y}))$, respectively, where x is some coordinate system in V_c. We assert that, if V_c is sufficiently small then there exists a coordinate system z on V_c such that $\mathcal{F}_\alpha$ is represented on W_c by the vector field

$$X_{c,\alpha}(z, y) = \frac{\partial}{\partial z} + \alpha . \frac{\partial}{\partial y} \ ,$$

for every $\alpha \in \mathbb{C}$.

In fact, fix a coordinate system x in V_c and let us represent $\mathcal{F}|_{W_c}$ and $\mathcal{G}|_{W_c}$ by the vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, respectively, as in (ii) and (iii). Recall that two foliations of the pencil coincide if, and only if, they have the same tangent space at some point $p \in W_c$ ((e) of Lemma 3.2.9). In particular, if $\alpha \in \mathbb{C}$, then the tangent space of $\mathcal{F}_\alpha$ in any point $p \in W_c$ is not "vertical", so that this tangent space is generated by a holomorphic vector field of the form $Z_\alpha(p) = \frac{\partial}{\partial x}(p) + A(p, \alpha)\frac{\partial}{\partial y}(p)$. It follows from (e) of Lemma 3.2.9 that, for any fixed $p \in W_c$ the function $\alpha \in \mathbb{C} \to A(p, \alpha)$ is injective. This implies that $A(p, \alpha)$, as a function of α, is affine. Since $A(p, 0) = 0$, we must have $A(p, \alpha) = a(p).\alpha$, where $a : W_c \to \mathbb{C}^*$ is holomorphic. Now, the fibres $g^{-1}(x)$, $x \in V_c$, are compact and contained in W_c, which implies that a is constant in these fibres. It follows that $a(x, y) = b(x)$ for some holomorphic function $b : V_c \to \mathbb{C}^*$. Hence $\mathcal{F}_\alpha$ can be represented on W_c by the vector field $X_\alpha(x, y) = \frac{1}{b(x)} . \frac{\partial}{\partial x} + \alpha . \frac{\partial}{\partial y}$. This implies that there exists a coodinate system z around $c \in V_c$ such that $\frac{1}{b(x)} . \frac{\partial}{\partial x} = \frac{\partial}{\partial z}$, which proves the assertion.

It follows that there exist coverings $(V_j)_{j \in J}$ of V and $(W_j := g^{-1}(V_j))_{j \in J}$, of V and W, by open sets, and a collection $(\psi_j)_{j \in J}$ of biholomorphisms $\phi_j : W_j \to V_j \times \mathbb{C}/\Gamma$, such that for each $j \in J$, V_j, W_j and ψ_j satisfy (i), (ii), (iii) and :
(iv). For each $j \in J$, there exist coordinate systems x_j on V_j and y_j on the universal covering $\mathbb{C} \to \mathbb{C}/\Gamma$, such that $\mathcal{F}_\alpha|_{W_j}$ is represented by the vector field $X_\alpha^j = \frac{\partial}{\partial x_j} + \alpha . \frac{\partial}{\partial y_j}$, for every $\alpha \in \mathbb{C}$. In particular, if we fix two points $z_0, z_1 \in V_j$, then the holonomy map $h_{z_0,z_1} : g^{-1}(z_0) \to g^{-1}(z_1)$ can be written as

$$(*) \quad h_{z_0,z_1}(y_j) = y_j + \alpha(z_1 - z_0)$$

The last assertion can be proved by integrating the differential equation $\frac{dy}{dx} = \alpha$ between z_0 and z_1. On the other hand, (ii) and (iii) imply that :
(v). If $i \neq j \in J$ are such that $V_{i,j} := V_i \cap V_j \neq \emptyset$, then $V_{i,j}$ is diffeomorphic to a disk and the change of chart $\psi_{ij} = \psi_j \circ \psi_i^{-1} : V_{i,j} \times \mathbb{C}/\Gamma \to V_{i,j} \times \mathbb{C}/\Gamma$ is of the form $\psi_{ij}(x_i, y_i) = (h_{ij}(x_i), g_{i,j}(y_i))$, where $g_{i,j} \in Aut(\mathbb{C}/\Gamma)$. In particular, we have

$$(**) \quad g_{i,j}(y_i) = \lambda_{ij} . y_i + \mu_{ij} \ \text{,where } \lambda_{ij} \in \mathbb{C}^* \text{ and } \mu_{ij} \in \mathbb{C} .$$

Note that the holonomy of $\mathcal{F}_\alpha$, $h_{\gamma,\alpha}$, with respect to a path $\gamma : [0, 1] \to V$, is a composition of finite sequence of maps as in $(*)$ and $(**)$. This implies that for every $y \in g^{-1}(\gamma(0))$ the map $\alpha \in \mathbb{C} \mapsto h_{\gamma,\alpha}(y) \in g^{-1}(\gamma(1))$ is affine (in the universal covering). In particular, the maps $\phi_{k,\alpha}$ defined in (1),...,(4), are of the form $\phi_{k,\alpha}(z) = \lambda_k.z + b_k(\alpha)$, where $b_k(\alpha) = A_k.\alpha + B_k$, where $A_k \in \mathbb{C}^*$ and $B_k \in \mathbb{C}$. On the other hand, we have $\mu_k(\alpha) = b_k(\alpha) - \frac{1-\lambda_k}{1-\lambda_1} b_1(\alpha)$, so that $\mu_k(\alpha) = a_k.\alpha + d_k$, where $a_k = A_k - \frac{1-\lambda_k}{1-\lambda_1} A_1$. Note that, although $A_1, A_k \neq 0$, we could have $a_k = 0$, for some $k > 1$.

Suppose for a moment that we have proved that in the case of three critical fibres we have $a_2 \neq 0$. In this case, it follows from Lemma 3.2.18 that, if $a_2.\alpha + d_2 \neq 0$ then

$$\alpha \in E(P) \setminus \{\infty\} \iff \exists\, n \in \mathbb{N}^* \text{ such that } n(a_2.\alpha + d_2) \in \Gamma \iff a_2.\alpha + d_2 \in \mathbb{Q}.\Gamma\ .$$

In particular, if $d_2 \neq 0$, then $d_2 \in \mathbb{Q}.\Gamma$, because $0 \in E(P)$. This implies that $E(P) = a_2^{-1}.\mathbb{Q}.\Gamma$, as the reader can check, which proves (b) of Lemma 3.2.19. On the other hand, suppose that we have proved that in the case of four critical fibres then, either $a_2 \neq 0$, or $a_3 \neq 0$. In this case, we have $E(P) \setminus \{\infty\} = \{\alpha \in \mathbb{C} \mid G_\alpha \text{ has a finite orbit}\}$. It follows from (IV) that the orbit of 0 by G_α is $\{m\mu_2(\alpha) + n\mu_3(\alpha) \mid m, n \in \mathbb{Z}\}$. Hence

$$E(P) \setminus \{\infty\} = \{\alpha \in \mathbb{C} \mid \mu_2(\alpha), \mu_3(\alpha) \in \mathbb{Q}.\Gamma\}\ ;$$

as the reader can check. Since, either $a_2 \neq 0$, or $a_3 \neq 0$, we conclude that $E(P)$ is countable, so that the pencil is weakly exceptional. Note that, $\mu_j(0) = d_j$, $j = 2, 3$, and $0 \in E(P)$, so that $d_2, d_3 \in \mathbb{Q}.\Gamma$. Therefore

$$E(P) \setminus \{\infty\} = \{\alpha \in \mathbb{C} \mid a_2.\alpha, a_3.\alpha \in \mathbb{Q}.\Gamma\}\ .$$

In particular, if there exists $\alpha_0 \in E(P) \setminus \{0, \infty\}$, then $a_2.\alpha_0, a_3.\alpha_0 \in \mathbb{Q}.\Gamma$, so that for every $x \in \mathbb{Q}$ we have that $a_2.(x\alpha_0), a_3.(x\alpha_0) \in \mathbb{Q}.\Gamma$ and $x.\alpha_0 \in E(P)$. Hence the family is exceptional.

Let us finish the proof of the Lemma. Note that if, either $a_2 = 0$ in the case of three critical levels, or $a_2 = a_3 = 0$ in the case of four critical levels, then the group G_α does not depends on $\alpha \in \mathbb{C}$, so that it is finite and $\mathcal{F}_\alpha$ has a first integral for all $\alpha \in \mathbb{C}$, say $f_\alpha: M \to \overline{\mathbb{C}}$. Let us prove that this is impossible in our case. Suppose by contradiction that $G_\alpha = G_0$ for all $\alpha \in \mathbb{C}$ and let $m = \#(G_\alpha) = \#(G_0)$. Note that the integer m is also the number of points of a generic orbit of G_α, that is the number of points in which a generic fibre of f_α cuts a generic fibre of $g = f_\infty$. It follows that $m = [f_\alpha^{-1}(c)].[g^{-1}(d)]$, the intersection number of these fibres. Fix a regular fibre $F_0 = f_0^{-1}(c_0)$ of f_0. Since $\mathcal{G}$ is transverse to F_0, there exists a neighborhood V_0 of c_0, biholomorphic to a disk, such that $W_0 := f_0^{-1}(V_0)$ is biholomorphic to $V_0 \times \mathbb{C}/\Gamma_1$, where Γ_1 is the lattice associated to the generic fibres of f_0. We can choose coordinates (x, y) on W_0 such that the sets $\{x = ct\}$ are leaves of $\mathcal{F}$ and the sets $\{y = ct\} \cap W_0$ are leaves of $\mathcal{G}$ and biholomorphic to disks. This defines a tubular neighborhood $\pi: W_0 \to F_0$, where $\pi^{-1}(y)$ is a leaf of $\mathcal{G}|_{W_0}$ for all $y \in F_0$. The idea is to prove that there exists $\epsilon > 0$ such that, if $0 < |\alpha| < \epsilon$, then W_0 contains some generic fibre, say F_α, of f_α. This is not possible, because in this case $f_0|_{F_\alpha}$ must be constant, so that F_α coincides with some fibre of f_0, which implies that $\mathcal{F}_\alpha = \mathcal{F}_0$ for $\alpha \neq 0$.

Fix a point $p_0 = (x_0, y_0) \in F_0$ and let F_α be the leaf of $\mathcal{F}_\alpha$ through p_0. Given $p \in F_0$, denote by L_p the leaf of $\mathcal{G}$ through p. Note that $\pi^{-1}(p) \subset L_p$, by the definition of π. If $g(p)$ is a regular value of g, then $L_p = g^{-1}(g(p))$ and $L_p \cap F_0$ contains $m = \#(G_0)$ points, say $p = p_1, \ldots, p_m$, where $p_i \neq p_j$ if $i \neq j$. Fix m paths in F_0, say $\gamma_1, \ldots, \gamma_m: [0, 1] \to F_0$, joining p_0 to $p_1, \ldots, p_m$, respectively. Since the pencil is a holomorphic family, there exists $\epsilon_p > 0$ such that if $|\alpha| < \epsilon_p$ then, for all $j = 1, \ldots, m$, the path γ_j can be lifted in the leaf F_α to a path $\gamma_{j,\alpha}: [0, 1] \to F_\alpha$ such that $\gamma_{j,\alpha}(0) = p_0$, $\gamma_{j,\alpha}[0, 1] \subset W_0$ and $\pi \circ \gamma_{j,\alpha} = \gamma_j$. This fact, whose proof we leave for the reader, follows from the general theory of

foliations. This defines m holomorphic functions, say $p_1,\ldots,p_m\colon D_p \to L_p$, where $p_j(\alpha) = \gamma_{j,\alpha}(1)$ and $D_p = \{|\alpha| < \epsilon_p\}$. Note that for every $\alpha \in D_p$ we have $p_1(\alpha),\ldots,p_m(\alpha) \in L_p \cap F_\alpha$ and $p_i(\alpha) \neq p_j(\alpha)$ if $i \neq j$. Since $L_p.F_\alpha = m$, then $L_p \cap F_\alpha = \{p_1(\alpha),\ldots,p_m(\alpha)\} \subset W_0$, for every $\alpha \in D_p$. In particular, $F_\alpha \cap \pi^{-1}(p) = \{p_1(\alpha)\}$, because $\pi^{-1}(p) \subset L_p$. The same type of argument can be done in the case where $g(p)$ is a critical value of g. In this case, the closure of L_p, is an irreducible component of the fibre $g^{-1}(g(p))$, say C. If the multiplicity of g along C is ℓ, then $F_0 \cap C$ contains m/ℓ different points, and it can be proved that :
(vi). For every $p \in F_0$, there exist $\epsilon_p > 0$ and a holomorphic map $P_p\colon D_p \to \pi^{-1}(p)$, such that $\pi^{-1}(p) \cap F_\alpha = \{P_p(\alpha)\}$ for every $\alpha \in D_p := \{\alpha \mid |\alpha| < \epsilon_p\}$.

Another fact that follows from the general theory of foliations is the following :
(vii). Given $p \in F_0$, there exist $0 < \delta_p \le \epsilon_p$ and neighborhoods U_p and Σ_p of p in F_0 and $\pi^{-1}(p)$, respectively, such that if $|\alpha| < \delta_p$ and $q \in \Sigma_p$ then the leaf of $F_\alpha|_{\pi^{-1}(U_p)}$ through q, say $X_q(\alpha)$, is such that $\pi|_{X_q(\alpha)}\colon X_q(\alpha) \to U_p$ is a biholomorphism. Moreover, if we choose δ_p small enough, then we can suppose that $P_p(\alpha) \in \Sigma_p$; $\forall \alpha \in D_{\delta_p}$.

Note that (vi) and (vii) imply that if $|\alpha| < \delta_p$ then $F_\alpha \cap \pi^{-1}(U_p) = X_{P_p(\alpha)}(\alpha)$ and $X_{P_p(\alpha)}(\alpha)$ cuts every fibre $\pi^{-1}(s)$, $s \in U_p$, in exactly one point. Let $U_{p_1} = U_1,\ldots,U_{p_r} = U_r$ be be a finite covering of F_0 by open sets as above and set $\delta = \min\{\delta_{p_1},\ldots,\delta_{p_r}\}$. As the reader can check, if $|\alpha| < \delta$ then F_α is entirely contained in W_0, which proves the Lemma. □

In order to finish the proof of Theorem 3, it remains to prove that in the case of three critical fibres then the pencil is equivalent to one of the families of types 1, 2 or 3, of §2.2, §2.3 and §2.4, respectively. Note that this fact implies also that M is a rational surface.

We will consider the following situation : let M_1 and M_2 be two compact complex surfaces and $(F^1_\alpha)_{\alpha\in\overline{\mathbb{C}}}$ and $(F^2_\alpha)_{\alpha\in\overline{\mathbb{C}}}$ be pencils of foliations on M_1 and M_2, generated by foliations F^1, G^1 and F^2, G^2 on M_1 and M_2, respectively. Suppose that :
(I). The foliations F^j and G^j are tangent to fibrations $f_j, g_j\colon M_j \to \overline{\mathbb{C}}$, respectively, $j = 1,2$, where $f_j \neq g_j$.
(II). f_j is an elliptic fibration with three critical fibres, as in one of the solutions 1, 2 or 3, in the table of solutions, $j = 1,2$. In particular, g_j has also three critical fibres, of the same type of the critical fibres of f_j, $j = 1,2$ (Corollary 3.2.16).
(III). The critical fibres of f_1, g_1 and f_2, g_2 are of the same type.

In this situation, let us call F^j_i the critical fibres of g_j, where $i \in \{1,2,3\}$, $j = 1,2$ and the indexes are choosen in such a way that F^1_i is of the same type as F^2_i, $i = 1,2,3$. After composition of g_j with a Moebius transformation, we can suppose that $F^j_1 = g_j^{-1}(0)$, $F^j_2 = g_j^{-1}(1)$ and $F^j_3 = g_j^{-1}(\infty)$, $j = 1,2$. Fix generators γ_1 and γ_2 of $\pi_1(V,c)$ as in figure 8, where $V := \overline{\mathbb{C}} \setminus \{0,1,\infty\}$. Set $W_j := g_j^{-1}(V)$, so that $g_j|_{W_j}\colon W_j \to V$ is a holomorphic fibre bundle, $j = 1,2$. Denote by G^j_α the global holonomy group of F^j_α calculated in the fibre $F_j := g_j^{-1}(c)$, $j = 1,2$. We have seen that, given $\alpha \in \mathbb{C}$, we can choose an universal covering $\pi_{j,\alpha}\colon \mathbb{C} \to F_j$ such that the generators of the global holonomy group of F^j_α, corresponding to γ_1 and γ_2, say $h^j_{1,\alpha}$ and $h^j_{2,\alpha}$, can be written (in the respective universal covering) as :

$$h^j_{1,\alpha}(y) = \lambda_1.y \text{ and } h^j_{2,\alpha}(y) = \lambda_2.y + a^j_2.\alpha + d^j_2 .$$

where $a^j_2 \neq 0$, $j = 1,2$. Recall that both fibres F_1 and F_2 are biholomorphic elliptic curves of the form $\mathbb{C}/\Gamma$, where $\Gamma = <1, \omega = e^{2\pi i/6}>$, in the case of solutions 1 and 2, and $\Gamma = <1, i>$ in the case of solution 3. In all cases, the exponents λ_k are roots of unity and $\lambda_k \neq 1$, $k = 1,2$.

3.2.20 **Lemma.** In the above situation, let $\alpha, \beta \in \mathbb{C}$ be such that $a^1_2.\alpha + d^1_2 = a^2_2.\beta + d^2_2$. Then there exists a biholomorphism $\Phi\colon M_1 \to M_2$ such that :

(a). $\Phi^*(F^2_\beta) = F^1_\alpha$.
(b). $g_2 \circ \Phi = g_1$. In particular, $\Phi^*(F^2_1) = F^1_1$.

Proof. Since α and β are fixed we will use the notations $F^1_\alpha = F_1$, $F^2_\beta = F_2$, $h^1_{k,\alpha} = h_{1,k}$ and $h^2_{k,\beta} = h_{2,k}$, $k = 1,2$. Recall that the universal coverings $\pi_{1,\alpha}\colon \mathbb{C} \to F_1$ and $\pi_{2,\beta}\colon \mathbb{C} \to F_2$ where constructed by composing two fixed universal coverings $\pi_j\colon \mathbb{C} \to F_j$, with two translations in $\mathbb{C}$, say σ_j, $j = 1,2$, where $\sigma_1(0)$ is the fixed point of $h_{1,1}$ and $\sigma_2(0)$ is the fixed point of $h_{1,2}$. The coverings π_j where chosen in such a way that $Aut(\pi_j) = \{z \mapsto z + \lambda \mid \lambda \in \Gamma\}$, $j = 1,2$, so that the map $\varphi\colon F_1 \to F_2$ defined by $\varphi(q) = \pi_2 \circ \sigma_2 \circ \sigma_1^{-1} \circ \pi_1^{-1}(q)$ is a well defined biholomorphism. This map is a conjugation between G^1_α and G^2_β. More precisely, φ satisfies $h_{2,k} \circ \varphi = \varphi \circ h_{1,k}$, $k = 1,2$. Following a standard construction (see [C-LN]) it is possible to extend φ to a biholomorphism $\Psi\colon W_1 \to W_2$ such that :
(i). Ψ sends leaves of $F_1|_{W_1}$ onto leaves of $F_2|_{W_2}$.
(ii). $g_2 \circ \Psi = g_1$ on W_1, so that $\Psi(g_1^{-1}(q)) = g_2^{-1}(q)$ for every $q \in V$.

The proof of the Lemma is then reduced to show that Ψ and Ψ^{-1} can be extended holomorphically to the critical levels of g_1 and g_2, respectively. We will only prove that Ψ can be extended to the critical levels of g_1. Consider for instance the levels $F^j_1 = g_j^{-1}(0)$, $j = 1,2$, and let us prove that Ψ can be extended to a holomorphic map $\Phi_1\colon (W_1 \cup F^1_1) \to M_2$. Note that, if Ψ can be extended to Φ_1 as above, then $\Phi_1(F^1_1) \subset F^2_1$, because in this case we must have $g_2 \circ \Phi_1 = g_1$, by (ii). To fix the ideas, we will suppose that F^1_1 and F^2_1 are of the type $\tilde{II}$. In the other cases, the proof is similar and will be left for the reader. In this case, we have the decomposition

$$(*)\quad F^j_1 = 6C^j_0 + C^j_1 + 2C^j_2 + 3C^j_3\,,$$

where $[C^j_0]^2 = -1$, $[C^j_1]^2 = -6$, $[C^j_2]^2 = -3$ and $[C^j_3]^2 = -2$, $j = 1,2$ (see figure 1.b). Note that the curve C^j_0 is the one for which the foliation F_j is transverse, $j = 1,2$. Set $C^{j*}_k = C^j_k \setminus C^j_0$, $j = 1,2$, $k = 1,2,3$. We assert that Ψ can be extended to a holomorphic map $\Psi_k\colon (W_1 \cup C^{1*}_k) \to M_2$ such that $\Psi_k(C^{1*}_k) = C^{2*}_k$.

Fix $k \in \{1,2,3\}$. Recall that C^j_k contains a unique singularity of F_j, say q_j, $j = 1,2$, of the type $1 : -m_k$, $m_k = -(C^j_k)^2$. Note that $q_j \in C^{j*}_k$, by (c.1) of Lemma 3.2.9.

Assertion. For $j = 1,2$, there exists a coordinate system $(U_j, \varphi_j = (x_j, y_j))$ such that $x_j(q_j) = y_j(q_j) = 0$, $C^j_k \cap U_j = \{y_j = 0\}$ and :
(iii). F_j is represented on U_j by the linear vector field $X_j(x_j, y_j) = x_j \frac{\partial}{\partial x_j} - m_k.y_j \frac{\partial}{\partial y_j}$.
(iv). The foliation $G^j|_{U_j}$ is represented by $dy_j = 0$ and $g_j(x_j, y_j) = y_j^k$.

Proof. We have seen in Lemma 3.2.9 that there exists a coordinate system $(U,(u,v))$ around q_j such that $u(q_j) = v(q_j) = 0$, $C^j_k \cap U = \{v = 0\}$ and $F_j|_U$ is represented by the vector field $u\frac{\partial}{\partial u} - m.v\frac{\partial}{\partial v}$, so that $\psi(u,v) = u^m.v$ is a first integral of $F_j|_U$, where $m = m_k$. The proof will be based in the following remark : consider a change of coordinates around $u = v = 0$ of the form $x = u.A(u,v)$, $y = v.B(u,v)$, where $A(0,0).B(0,0) \neq 0$ and $(A(u,v))^m.B(u,v) \equiv cte \neq 0$. Note that, after this change of variables, the first integral becomes $\varphi(x,y) = cte.x^m.y$, so that $x^m.y$ is a first integral of F_j near q_j. In this case, the vector field $x\frac{\partial}{\partial x} - m.y\frac{\partial}{\partial y}$ represents F_j in a neighborhood of q_j. Recall that C^j_k is invariant for G^j, this foliation has no singularities near q_j and it is tranverse to F_j outside C^j_k, in a neighborhood of q_j. It follows that G^j has a holomorphic first integral, near q_j, of the form $v.D(u,v)$, where $D(0,0) \neq 0$. Consider the change of variables defined in a neighborhood of $(0,0)$ by $z = u.C(u,v)$, $w = v.D(u,v)$, where $C(u,v)$ is a holomorphic m^{th} root of $(D(u,v))^{-1}$ near $(0,0)$. After this change of variables, $z^m.w$ and w are first integrals of F_j and G^j, respectively, in neighborhood U_1 of q_j. Now, since g_j is also a first integral of G^j, the funcion $g_j|_{U_1}$ depends only on w, so that $g_j(z,w) = w^\ell.h(w)$ on a neighborhood U_2 of q_j, where $h(0) \neq 0$ and ℓ is the multiplicity of g_j along C_k^j. As the reader can see in (*), this multiplicity was chosen in such a way that $\ell = k$, so that $g_j(z,w) = w^k.h(w)$. Let $B(w)$ be a k^{th} root of $h(w)$ and $A(w)$ be a m^{th} root of $(B(w))^{-1}$ and consider the change of variables $x = z.A(w)$, $y = w.B(w)$, in a neighborhood U of $(0,0)$. After this change of variables, $x^m.y$ and y are first integrals of F_j and G_j in U. Moreover $g_j(x,y) = y^k$. ∎

Let us prove that Φ extends to a holomorphic map $\Phi_k: W_1 \cup C_k^{1*} \to M_2$ such that $\Phi_k(C_k^{1*}) \subset C_k^{2*}$. We prove first that Φ can be extended to a neighborhood of q_1 in C_k^{1*}, in such a way that the extension sends q_1 to q_2. For $j = 1,2$, consider coordinate system $(U_j,(x_j,y_j))$ around q_j, and a vector field X_j, as in the Assertion. We can suppose that $x_j(U_j) = y_j(U_j) = D$, where $D = \{y \in \mathbb{C} \mid |y| < r\}$, so that, $g_j(U_j) = \{z \in \mathbb{C} \mid |z| < r^k\} = D_1$. Let $S_j = \{(0,y_j) \mid y_j \in D\}$ be the local separatrix of X_j transverse to C_k^j and set $S_j^* = S_j \setminus \{(0,0)\}$. Note that $S_j^* \subset W_j$, $j = 1,2$. We assert that $\Phi(S_1^*) = S_2^*$.

In fact, suppose first that the curve γ_1, used to define $h_{j,1}$, $j = 1,2$, is contained in D_1 and that $\gamma_1(t) = y_0^k.e^{2\pi i t}$, $t \in [0,1]$, $y_0^k = c$. We recall that $h_{j,1}(z) = \omega^{-1}.z$ in a certain universal covering $\mathbb{C} \to \mathbb{C}/<1,\omega>$ of $F_j = g_j^{-1}(c)$, where $\omega = e^{2\pi i/6}$. This implies that $h_{j,1}$ has one fixed point, one orbit of period two and one orbit of period three. The other orbits are generic and are of period six. On the other hand, $F_j \cap S_j = \{(0,\lambda^n.y_0) \mid n = 0,\dots,k-1\}$, where $\lambda = e^{2\pi i/k}$. Moreover, the lifting of γ_1 on S_j through g_j with initial point $(0,\lambda^n.y_0)$ is $(0,\lambda^n.y_0.e^{2\pi i t/k})$, $t \in [0,1]$. It follows that $h_{j,1}(0,\lambda^n.y_0) = (0,\lambda^{n+1}.y_0)$. This implies that the orbit of period k of $h_{j,1}$ is $O(y_0) := \{(0,\lambda^n.y_0) \mid n = 0,\dots,k-1\}$. Since $\Phi|_{F_1}: F_1 \to F_2$ is a conjugation between $h_{1,1}$ and $h_{2,1}$, we must have $\Phi(O(y_0)) = O(y_0)$. It follows from (i) that Φ must send the leaf of F_1 which contains S_1^* onto the leaf of F_2 which contains S_2^*. By analytic continuation and the fact that $g_2 \circ \Phi = g_1$ we get that $\Phi(S_1^*) = S_2^*$, as the reader can check. In the general case, that is when $\gamma_1[0,1] \not\subset D_1$, we can suppose that $\gamma_1 = \delta * \gamma * \delta^{-1}$, where $\gamma(t) = y_0.e^{2\pi i t}$ and δ is a curve in $\overline{\mathbb{C}}$ joining c to $c_1 \in D_1$, $c_1 = y_0^k$ (figure 8). The lifting of the curve δ on the leaves of F_j, $j = 1,2$, produces a holonomy map $h_{j,\delta}: g_j^{-1}(c) \to g_j^{-1}(c_1)$ which conjugates the holonomy map of the curve γ on $g_j^{-1}(c_1)$, say h_j, to $h_{j,1}$, that is $h_j = h_{j,\delta} \circ h_{j,1} \circ h_{j,\delta}^{-1}$. It follows from (i), (ii) and analytic continuation that $\Phi(g_1^{-1}(c_1)) = g_2^{-1}(c_1)$ and that $\phi_1 := \Phi|_{g_1^{-1}(c_1)}$ satisfies $h_2 \circ \phi_1 = \phi_1 \circ h_1$. Hence, the general case can be reduced to the first one.

The facts that $g_2 \circ \Phi = g_1$ and $\Phi(S_1^*) = S_2^*$, imply that $\Phi(0,y_1) = (0,\lambda^n.y_1)$, for some $n \in \mathbb{Z}$, as the reader can check. After the change of variables $y = \lambda^{-n}.y_1$ we get $\Phi(0,y) = (0,y)$. Let $A \subset U_1$ be a neighborhood of S_1^* such that $\Phi(A) \subset U_2$. Since $\Phi(0,y) = (0,y)$ and $g_2 \circ \Phi = g_1$, we get that $\Phi(x_1,y) = (\phi_y(x_1),y)$ for all $(x_1,y) \in A$. In particular, if we denote by $L_j(y)$ the germ at $(0,y)$ of the set $\{(x_j,y) \mid x_j \in \mathbb{C}\}$, then we get that $\Phi(L_1(y)) = L_2(y)$. We will consider ϕ_y as a map from $L_1(y)$ to $L_2(y)$. Let $X_{j,T}$ be the flows of X_j, $j = 1,2$, so that $X_{j,T}(x_1,y) = (e^T.x_1, e^{-mT}.y)$. Note that $X_T^j(L_j(y)) = L_j(e^{-mT}.y)$. This fact together with $\Phi(L_1(y)) = L_2(y)$ and (i) imply that $\Phi \circ X_{1,T}(x_1,y) = X_{2,T} \circ \Phi(x_1,y)$, for all $(T,x_1,y) \in \mathbb{C} \times U_1$ such that both members of the equality are defined. In particular, if we set $T = -\frac{2\pi i}{m}$ then we get $\Phi(e^{-2\pi i/m}.x_1,y) = (e^{-2\pi i/m}.\phi_y(x_1),y)$, so that $\phi_y(e^{-2\pi i/m}.x_1) = e^{-2\pi i/m}.\phi_y(x_1)$. Hence, ϕ_y conjugates the holonomies of the separatrices S_1 and S_2 for the vector fields X_1 and X_2 in $L_1(y)$ and $L_2(y)$, respectively. Now, the fact that Φ extends as a biholomorphism from a neighborhood of q_1 to a neighborhood of q_2, follows from a Lemma of Mattei-Moussu in [M-M]. The main facts used in the proof of the Lemma of Mattei-Moussu are that ϕ_y conjugates the two holonomies, the flows preserve the "horizontal" fibrations $L_j(y)$ and the quotient of the eigenvalues are equal and negative (in our case $-1/m$).

The extension of Φ to C_k^{1*}, can be done by using Hartogs' Theorem. Let $C \subset C_k^{1*}$ be the maximal connected open set of C_k^{1*} such that Ψ can be extended to C. Note that, if there exists q in the boundary of C in C_k^{1*}, then there exists an open neighborhood U of q, where $U \simeq D \times D$, such that $U \cap C_k^{1*} \simeq D \times \{0\}$ and Ψ is holomorphic on $H = (D \times D^*) \cup (C \cap U)$. According to Hartogs' Theorem, the holomorphic closure of H is U. Observe that $\Psi|_H$ must be $1-1$, because it is $1-1$ on $D \times D^*$ and non-constant on C. Hence, $\Psi_H : H \to M_2$ is an embedding and this impies that it can be extended holomorphically to U. It follows that $C = C_k^{1*}$ and this proves that Ψ extends to C_k^*.

We have proved that Ψ extends to a biholomorphism $\Psi_0 : W_1 \cup (\cup_{k=1}^3 C_k^{1*}) \to W_2 \cup (\cup_{k=1}^3 C_k^{2*})$ in such a way that $\Psi_0(C_k^{1*}) = C_k^{2*}$, $k = 1, 2, 3$. It remains to prove that Ψ_0 extends to the component C_0^1 in such a way that $\Psi_0(C_0^1) = C_0^2$. For this extension, we can use, for example, that the curves C_0^j are -1 rational curves. These curves can be blow-down to points $p_1 \in \tilde{M}_1$ and $p_2 \in \tilde{M}_2$, so that we have blowing-downs maps $\pi_j : M_j \to \tilde{M}_j$, where $\pi_j^{-1}(p_j) = C_0^j$, $j = 1, 2$. The map $\tilde{\Psi}_0 = \pi_2 \circ \Psi_0 \circ \pi_1^{-1}$ is a biholomorphism of a punctured neighborhood of p_1 to a punctured neighborhood of p_2, so that it can be extended to p_1 in such a way that $\tilde{\Psi}_0(p_1) = p_2$. This implies that Ψ_0 extends biholomorphically to C_0^1 in such a way that $\Psi_0(C_0^1) = C_0^2$.

There are small differences in the proof when the fibres F_1^j are not of the type II. The first one is the following : in order to prove that Ψ sends the separatrix S_1 to the separatrix S_2, we have used that the maps $h_{j,1}$ have three special orbits : one fixed point, one of period two and one of period three. Each of these orbits correspond to one of the components C_k^j, $k = 1, 2, 3$, of F_1^j, $j = 1, 2$. For instance, if F_1^j is of the type III, then $h_{j,1}(z) = -i.z$, so that it has also three special orbits, but this time two of them are fixed and the third has period two. According to figure 1.c, we can write the decomposition of F_1^j as

$$F_1^j = 4C_0^j + C_1^j + C_2^j + 2C_3^j .$$

The component C_0^j is transverse to F_j, whereas the other three are invariant for F_j, $j = 1, 2$. For each $k = 1, 2, 3$, the component C_k^j contains a singularity, say q_k^j, and there is a local separatrix for F_j, say S_k^j, such that $q_k^j \in S_k^j$. The separatrix S_3^j corresponds to the orbit of period two of $h_{j,1}$, whereas S_1^j and S_2^j correspond to the two fixed points. By using an argument similar to the proof that $\Psi(S_1^*) = S_2^*$, we can conclude that, in the case we are considering, we have $\Psi(S_3^{1*}) = S_3^{2*}$. However, the same argument implies only that, either $\Psi(S_1^{1*}) = S_1^{2*}$ and $\Psi(S_2^{1*}) = S_2^{2*}$, or $\Psi(S_1^{1*}) = S_2^{2*}$ and $\Psi(S_2^{1*}) = S_1^{2*}$. The rest of the proof is similar and at the end we will get that in the first case we will have $\Psi(C_1^1) = C_1^2$ and $\Psi(C_2^1) = C_2^2$, whereas in the second case we will have $\Psi(C_1^1) = C_2^2$ and $\Psi(C_2^1) = C_1^2$. The proof of the extension of Φ to $\cup_{k>0} C_k^1$ is similar for the other types of fibres. The second difference is in the proof of the extension of Ψ to the component C_0^1 in the case where F_1^j is of the type I_0^*. In this case, the components C_0^j are -2 rational curves and not -1 curves. However, we can contract them, thus obtaining two singular surfaces, each one with one singularity, say p_j. Since these singularities are normal, it can be proved that the map Ψ_0 can be extended to a biholomorphism, exactly as in the -1 case. We leave the details for the reader. ∎

3.2.21 Corollary. Let $(F_\alpha^1)_{\alpha \in \overline{\mathbb{C}}}$ and $(F_\alpha^2)_{\alpha \in \overline{\mathbb{C}}}$ be pencils of foliations on surfaces M_1 and M_2, respectively, which satisfy (I), (II) and (III) before Lemma 3.2.20. Then there exist a biholomorphism $\Phi : M_1 \to M_2$ and $a, d \in \mathbb{C}$, $a \neq 0$, such that $\Phi^*(F_1^2) = F_1^1$ and $\Phi^*(F_\beta^2) = F_{(a.\beta+d)}^1$ for every $\beta \in \mathbb{C}$.

Proof. Let $a_2^j \neq 0$, d_2^j, $j = 1, 2$, be as in Lemma 3.2.20. Choose $\alpha_0, \beta_0 \in \mathbb{C}$ such that $a_2^1.\alpha_0 + d_2^1 = a_2^2.\beta_0 + d_2^2$. As we have seen in Lemma 3.2.20, we have $\Phi^*(F_1^2) = F_1^1$ and $\Phi^*(F_{\beta_0}^2) = F_{\alpha_0}^1$. After changing the variables as $\alpha' = \alpha - \alpha_0$ and $\beta' = \beta - \beta_0$, we can suppose that $\Phi^*(F^2_0) = F^1_0$. Let $(U^2_j)_{j\in J}$ be a covering of M_2 by open sets and $(X^2_j)_{j\in J}$ and $(Y^2_j)_{j\in J}$ be collections of holomorphic vector fields, such that X^2_j, Y^2_j and $X^2_j + \beta.Y^2_j$ define F^2_0, F^2_1 and F^2_β on U^2_j, respectively, for every $j \in J$ and $\beta \in \mathbb{C}$. Note that there exists a multiplicative cocycle $(f^2_{ij})_{U^2_{ij}\neq\emptyset}$ such that $X^2_i + \beta.Y^2_i = f^2_{ij}(X^2_j + \beta.Y^2_j)$ on $U^2_{ij} := U^2_i \cap U^2_j$. Consider the covering $(U^1_j := \Phi^{-1}(U^2_j))_{j\in J}$ of M_1 and the collections of vector fields $(X^1_j := \Phi^*(X^2_j))_{j\in J}$ and $(Y^1_j := \Phi^*(Y^2_j))_{j\in J}$. Since $\Phi^*(F^2_0) = F^1_0$ and $\Phi^*(F^2_1) = F^1_1$, the vector fields X^1_j and Y^1_j represent F^1_0 and F^1_1 on U^1_j, respectively, $j \in J$. Set $f^1_{ij} = f^2_{ij} \circ \Phi^{-1}$ for (i,j) such that $U^1_i \cap U^1_j \neq \emptyset$. Since $X^1_i = f^1_{ij}.X^1_j$ and $Y^1_i = f^1_{ij}.Y^1_j$, it follows that there exists $\lambda \in \mathbb{C}^*$ such that F^1_α is represented by $X^1_j + \alpha.\lambda.Y^1_j$ on U^1_j, for all $j \in J$. On the other hand, the fact that $X^1_j + \alpha.\lambda.Y^1_j = \Phi^*(X^2_j + \alpha.\lambda.Y^2_j)$, for all $j \in J$, implies that $\Phi^*(F^2_{\lambda.\alpha}) = F^1_{\lambda.\alpha}$ for all $\alpha \in \mathbb{C}$. ∎

The result below is a consequence Corollary 3.2.21 and of the description of the families of §2.2, 2.3 and 2.4.

3.2.22 Corollary. Let $(F_\alpha)_{\alpha\in\overline{\mathbb{C}}}$ be a pencil of foliations on a surface M, satisfying the hypothesis of Theorem 3, where $K_M \neq 0$. Then it is holomorphically equivalent to one of the families of types 1, 2 or 3, described in §2.2, 2.3 or 2.4. In particular, M is a rational surface.

Another interesting fact, is the following :

3.2.23 Corollary. Let $(F_\alpha)_{\alpha\in\overline{\mathbb{C}}}$ be a pencil of foliations on a surface M, satisfying the hypothesis of Theorem 3, where $K_M \neq 0$. Given $\alpha, \beta \in E(P)$ and fibrations f_α and f_β, tangent to F_α and F_β, respectively, then there exist biholomorphisms $\Phi\colon M \to M$ and $\varphi\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $f_\alpha \circ \Phi = \varphi \circ f_\beta$.

We leave the proof for the reader.

§3.3. Proof of Theorem 1. Let $P = (F_s)_{s\in X}$ be an equirreducible, elliptic and exceptional family of foliations on $\mathbb{CP}(2)$, where X is a Riemann surface. According to the definition, the set $E(P)$ is countable, infinite and has an accumulation point, say $s_0 \in X$. Since the family is equirreducible, there exists a rational surface M_1 and a a bimeromorphism $\pi_1\colon M_1 \to \mathbb{CP}(2)$ such that the family $(G_s := \pi_1^*(F_s))_{s\in X}$ satisfies

(i). $T_{G_{s_1}} = T_{G_{s_2}}$ for all $s_1, s_2 \in X$.

(ii). For all $s \in X$ the singularities of G_s are reduced in the sense of Seidemberg.

It follows from Lemma 3.2.5 that there exist a neighborhood V of s_0 and a bimeromorphism $\pi\colon M_1 \to M$, which consists of a sequence of blowing-downs, such that the family $Q := (H_s := \pi_*(G_s))_{s\in X}$ satisfies

(iii). For all $s \in V$, H_s has no contractible fibres and the singularities of H_s are reduced.

(iv). $T_{H_{s_1}} = T_{H_{s_2}}$ for all $s_1, s_2 \in V$.

Let $F(M) = \{H \mid H$ is a foliation on M sush that $T_H = T_{H_{s_0}}\}$. Note that $E(Q) = E(P)$, so that the family Q is exceptional. We assert that there exists $s_1 \in E(H_s) \cap V$ such that $H_{s_1} \neq H_{s_0}$. In fact, let $(t_n)_{n\geq 1}$ be a sequence in $E(H_s) \cap V$ such that $\lim_{n\to\infty} t_n = s_0$ and $t_n \neq s_0$ for all $n \geq 1$. Note that $s \in V \mapsto H_s \in F(M)$ is a holomorphic map, so that, if $H_{t_n} = H_{s_0}$ for all $n \geq 1$, then the map $s \mapsto H_s$ would be constant. On the other hand, since $E(H_s)$ is countable, there exists $s \in V$ such that H_s has no first integral, that is $H_s \neq H_{s_0}$. This implies that the map $s \mapsto H_s$ is not constant. Therefore, there exists $n \geq 1$ such that $H_{t_n} \neq H_{s_0}$.

Let $(K_\alpha)_{\alpha\in\overline{\mathbb{C}}}$ be the pencil generated by $K_0 = H_{s_0}$ and $K_1 = H_{s_1}$. It follows from Corollary 3.2.10 that $F(M;T) = \{K_a \mid \alpha \in \overline{\mathbb{C}}\}$, where $T = T_{H_{s_0}}$. This implies that $H_s \in F(M;T)$ for all $s \in X$ and that there exists a holomorphic map $\varphi\colon X \to \overline{\mathbb{C}}$ such that $H_s = K_{\varphi(s)}$ for all $s \in X$. In particular, if $\Phi\colon M \to \mathbb{CP}(2)$ is the bimeromorphism defined by $\Phi = \pi_1 \circ \pi^{-1}$ then $\Phi^*(F_s) = K_{\varphi(s)}$

for all $s \in X$. Now, Corollary 3.2.22 implies that the pencil $(K_\alpha)_{\alpha\in\overline{\mathbb{C}}}$ is equivalent to one of the families of types 1, 2 or 3. Assertion (c) of Theorem 1 follows from the Corollary 3.2.23. This ends the proof of Theorem 1.

§3.4. **Proof of Theorem 2.** Let $(\mathcal{F}_\alpha)_{\alpha\in X}$ be an equirreducible, non-degenerate, elliptic and exceptional family of foliations on $\mathbb{C}P(2)$. According to Theorem 1, the family immerges bimeromorphically in one of the pencils of types 1, 2 or 3, described in §2. In particular, we can suppose that $X = \overline{\mathbb{C}}$ and the family is the pencil generated by two foliations on $\mathbb{C}P(2)$, say $\mathcal{F}_0$ and $\mathcal{F}_\infty$, of the same degree d. We can suppose also that $\mathcal{F}_0$ and $\mathcal{F}_\infty$ have rational first integrals and that their singularities are non-degenerate. Let $\Phi\colon M_j \to \mathbb{C}P(2)$ be a bimeromorphism such that $\Phi^*(\mathcal{F}_\alpha) = \mathcal{G}^j_\alpha$, where $\mathcal{P}^j := (\mathcal{G}^j_\alpha)_{\alpha\in\overline{\mathbb{C}}}$ is the family of type j, $j \in \{1,2,3\}$. The proof will be done in three steps :
1st step. We will prove that $d \in \{2,3,4\}$.
2nd step. We will prove that we can suppose that the bimeromorphism Φ consists of a sequence of blowing-ups.
3rd step. We will prove that there exists an automorphism Ψ of $\mathbb{C}P(2)$ such that $(\Psi^*(\mathcal{F}_\alpha))_{\alpha\in\overline{\mathbb{C}}}$ is one of the four families in $\mathbb{C}P(2)$ described in §2.

Proof of the 1st step. This part follows from a Theorem of M. Brunella :

Theorem ([Br-3]). Let $\mathcal{F}$ be a foliation on $\mathbb{C}P(2)$ of degree d, whose singularities are reduced in the sense of Seidemberg. Suppose that there exists a non-constant entire map $f\colon \mathbb{C} \to \mathbb{C}P(2)$ such that $f(\mathbb{C})$ is the union of non-algebraic leaf and some singularities of $\mathcal{F}$. Then $d \le 4$.

Since the family is bimeromorphically equivalent to the family of type $k \in \{1,2,3\}$, $(\mathcal{G}^k_\alpha)_{\alpha\in\overline{\mathbb{C}}}$, it is sufficient to prove that there exists $\beta \in \overline{\mathbb{C}}$ such that $\mathcal{G}^k_\beta$ has a non-algebraic leaf bimeromorphic to $\mathbb{C}$. This fact is proved for the families of types 1 and 2 in Proposition 6 of [LN]. In fact, in this proposition we prove the following : let L be a generic leaf of $\mathcal{G}^k_\beta$, where $k \in \{1,2\}$. Then there exists a holomorphic covering $\pi\colon L \to \mathbb{C}/\Gamma$, where $\Gamma = \langle 1; \omega\rangle$. When $\beta \notin E(\mathcal{P}^j)$ then the generic leaves of $\mathcal{G}^j_\beta$ are not algebraic, so that they must be biholomorphic to $\mathbb{C}$ or $\mathbb{C}^*$. In [LN] it is proved that they are biholomorphic to $\mathbb{C}$, but for our purposes it is sufficient that they are not algebraic and covered by $\mathbb{C}$. An analogous result can be proved for the family of type 3 : let L be a generic leaf of $\mathcal{G}^3_\beta$. Then there exists a holomorphic covering $\pi\colon L \to \mathbb{C}/\Gamma$, where $\Gamma = \langle 1; i\rangle$. In particular, if $\beta \notin E(\mathcal{P}^3)$ then the generic laves of $\mathcal{G}^3_\beta$ are covered by $\mathbb{C}$ and non-algebraic. Since the proof is analogous in this case, we leave it for the reader. From this fact, we get that $0 \le d \le 4$. Since foliations of degrees 0 or 1 can not have elliptic first integrals, we conclude that $2 \le d \le 4$. In the proof of the 3rd step we will need the following result :

3.4.1 Lemma. Let $\mathcal{F}$ be a foliation of degree d on $\mathbb{C}P(2)$ and ℓ be a straight line of $\mathbb{C}P(2)$. Then ℓ is invariant for $\mathcal{F}$, if one of the conditions below is verified :
(a). $d = 2$ and ℓ contains, either two singularities of $\mathcal{F}$, where one of then is radial (of the type 1 : 1), or three singularities of $\mathcal{F}$.
(b). $d = 3$ and ℓ contains two radial singularities of $\mathcal{F}$.
(c). $d = 4$ and ℓ contains three singularities of $\mathcal{F}$, where two of them are radial.

Proof. The proof is based in the following fact : Let m be a radial singularity of $\mathcal{F}$ and C be a curve such that $m \in C$, the multiplicity of C at m is ν and all irreducible components of C are non-invariant for $\mathcal{F}$. Then $\mathrm{tang}(\mathcal{F}; C; m) \ge \nu(\nu+1)$. In particular, if $\nu = 1$ then $\mathrm{tang}(\mathcal{F}; C; m) \ge 2$.

In fact, we can suppose that $\mathcal{F}$ is represented in a neighborhood of m by a vector field of the form $X = R + \sum_{j\ge 2} X_j$, where $R = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and X_j is homogeneous of degree j, in some coordinate system such that $x(m) = y(m) = 0$. On the other hand, C has a local equation of the form $f = 0$, where $f = f_\nu + \sum_{j>\nu} f_j$, where f_j is homogeneous of degree j and $f_\nu \neq 0$. It follows that the Taylor series of $X(f)$ at m is of the form :

$$X(f) = \nu . f_\nu + \sum_{j>\nu} g_j \implies X(f) - \nu . f = \sum_{j>\nu} (g_j - f_j) \implies [f; X(f)]_m \geq \nu(\nu+1) .$$

Since $\text{tang}(F; C; m) = [f; X(f)]_m$, we get the result.

Now, if ℓ was non-invariant for F, then we would get from 3.1.7 and 3.1.3 that $d - 1 = T^*_F . \ell = -\ell^2 + \text{tang}(F; \ell)$, so that $\text{tang}(F; \ell) = d$. Since $\text{tang}(F; \ell) = \sum_{p \in \ell} \text{tang}(F; \ell; p)$, then any one of the conditions in (a), (b) or (c), implies that $\text{tang}(F; \ell) > d$, a contradiction. ∎

Proof of the 2nd step. Since the singularities of F_0 are non-degenerate and F_0 has a rational first integral, it follows that for any singularity p_0 of F_0, there exists a local coordinate system $(U; (x; y))$ around p_0 such that $x(p_0) = y(p_0) = 0$ and $f(x; y) = x^p / y^q$ is a first integral of $F_0|_U$. In this case, $F_0|_U$ is represented by the differential equation $\omega = 0$, where

$$\text{(R)} \quad \omega = p\, y\, dx - q\, x\, dy\, ; \; p \in \mathbb{N}; \; q \in \mathbb{Z}^* \text{ and } \gcd(p; q) = 1 .$$

In particular, the singularity is of the type $p : q$. When $q < 0$, the first integral is holomorphic and the singularity is reduced in the sense of Seidemberg, whereas when $q > 0$ the first integral is meromorphic and the singularity is not reduced. According to the Corollary 3.2.6, the resolution process for the family can be done as follows :

1. Reduce all singularities as in (R) with $q > 0$. This is done by a sequence of blowing-ups, say $\Phi_1 : M \to \mathbb{C}P(2)$. After this sequence of blowing-ups we consider the family of foliations on M, $(H_\alpha := \Phi_1^*(F_\alpha))_{\alpha \in \overline{\mathbb{C}}}$.
2. If H_0 has no contractible curve, then all elements of the family $(H_\alpha)_{\alpha \in \overline{\mathbb{C}}}$ have only reduced singularities, $M = M_j$ and the family coincides with the family $(G^j_\alpha)_{\alpha \in \overline{\mathbb{C}}}$, for some $j \in \{1; 2; 3\}$, up to a biholomorphism Φ_2. If there is some contractible curve for H_0, then this curve is contractible for all elements of the family (Lemma 3.2.5). After a sequence of blowing-downs which at each step contracts a -1-curve, contractible for all foliations in the pencil, we obtain a bimeromorphism $\Phi_2 : M \to M'$, and we get a pencil $(H'_\alpha := (\Phi_2)_*(H_\alpha))_{\alpha \in \overline{\mathbb{C}}}$. By Lemma 3.2.20, this family is biholomorphically equivalent to one of the families of types 1, 2 or 3. Therefore, we can suppose that $M' = M_j$ and that $(H'_\alpha = G^j_\alpha)_{\alpha \in \overline{\mathbb{C}}}$, for some $j \in \{1; 2; 3\}$.

We have concluded that $\Phi = \Phi_1 \circ \Phi_2^{-1}$, where Φ_1 is a sequence of blowing-ups and Φ_2 is, either a biholomorphism, or a sequence of blowing-downs. Therefore, in order to conclude the 2nd step, it is enough to prove that after the sequence of blowing-ups Φ_1, the generic foliation H_α has no contractible curve, so that Φ_2 is an biholomorphism. To do this, we will describe the resolution process of a singularity like in (R) with $p; q > 0$.

3.4.2 Remark. Note that the singularities of the foliations of types 1, 2 or 3 can be only of the following types : $1 : -2$, $1 : -3$, $1 : -4$ or $1 : -6$.

3.4.3 The resolution process of a singularity of the type $p : q$, $1 \leq q \leq p$, $\gcd(p; q) = 1$. Let F_0 be a foliation on a surface N_0 and $m_0 \in N_0$ be a singularity of type $p : q$. Denote by $\pi_1; \ldots; \pi_r$ the minimal sequence of blowing-ups necessary for the resolution of m_0. The sequence is defined inductively in such a way that $\pi_1 : N_1 \to N_0$ is the blowing-up at m_0 and $\pi_{n+1} : N_{n+1} \to N_n$ is the blowing-up at some point $m_n \in N_n$, $n = 1; \ldots; r-1$. The composition $\pi_1 \circ \ldots \circ \pi_n$ will be denoted by Π_n. Note that $\Pi_n^{-1}(m_0)$ is the union of n exceptional divisors, say $D^n_1; \ldots; D^n_n$. These divisors are ordered inductively in such a way that $D^1_1 = \pi_1^{-1}(m_0)$, $D^n_n = \pi_n^{-1}(m_{n-1})$ and $D^n_1; \ldots; D^n_{n-1}$ are the strict transforms by π_n of $D^{n-1}_1; \ldots; D^{n-1}_{n-1}$ respectively. In all steps of the resolution, the point m_n belongs to D^n_n and Π_n is a biholomorphism between $N_n \setminus (\cup_{i=1}^n D^n_i)$ and $N_0 \setminus \{m_0\}$. The foliation induced by the form $\omega = pydx - qxdy$ in a neighborhood of m_0 will be denoted by $\mathcal{F}_0^0$ and the strict transform of $\Pi_n^*(\mathcal{F}_0^0)$ by $\mathcal{F}_0^n$. Note that $\mathcal{F}_0^n = \pi_n^*(\mathcal{F}_0^{n-1})$ for all $n = 1, \ldots, r$. Let ω_n be a holomorphic 1-form representing $\mathcal{F}_0^n$ in a neighborhood of m_n. The form ω_n, in our case, can always be written as in (R) in some coordinate system around m_n, so that it is of type $p_n : q_n$, $p_n, q_n > 0$, $\gcd(p_n, q_n) = 1$. On the other hand, the divisor D_{n+1}^{n+1} is contained in the divisor of zeroes of $\pi_n^*(\omega_n)$ with some multiplicity, say $\mu_n \geq 1$ (see 3.1.11). Let us see how the foliation $\mathcal{F}_0^{n+1}$ looks like in a neighborhood of the divisor D_{n+1}^{n+1}. If we suppose that $1 \leq q_n \leq p_n$, then we have two possibilities :

(I). $p_n = q_n = 1$. In this case m_n is a radial singularity of ω_n, $\mu_n = 2$ and $\mathcal{F}_0^{n+1}$ is transverse to D_{n+1}^{n+1}. This is the last step of the resolution of m_0, so that $r = n + 1$.

(II). $1 \leq q_n < p_n$. In this case, the divisor D_{n+1}^{n+1} is invariant for $\mathcal{F}_0^{n+1}$, $\mu_n = 1$ and D_{n+1}^{n+1} contains two singularities, one of type $p_n : q_n - p_n$ and the other of type $q_n : p_n - q_n$. Since $q_n < p_n$, the singularity of type $q_n : p_n - q_n$ is non-reduced, so that we need more blowing-ups. The point m_{n+1} will be this singularity. The singularity of type $p_n : q_n - p_n$ is reduced and in any other step of the resolution Π_r, it will appear a singularity of the same type. From this process, we get the following conclusions :

3.4.4 Remark. (a). m_{r-1} is of the type $1 : 1$ and $\mathcal{F}_0^r$ is transverse to D_r^r, the last divisor which appears in the resolution.

(b). If m_n is of the type $p_n : q_n$, then m_{n-1} is, either of the type $[p_n, p_n + q_n]$, or of the type $q_n : p_n + q_n$. In particular, m_{r-2} is of the type $2 : 1$, the divisor D_{r-1}^r has self-intersection -2 and contains an unique singularity of $\mathcal{F}_0^r$, say P, which is of the type $-2 : 1$. The Camacho-Sad index of this singularity with respect to D_{r-1}^r is $I(\mathcal{F}_0^r, D_{r-1}^r, P) = -2$.

(c). If $r \geq 3$ then the singularity m_{r-3} is, either of the type $3 : 2$, or of the type $3 : 1$. Moreover :

(c.1). If m_{r-3} is of the type $3 : 1$ then the divisor D_{r-2}^r cuts D_{r-1}^r, but does not cut D_r^r.

(c.2). If m_{r-3} is of the type $3 : 2$, then D_{r-2}^r has self-intersection -3 and contains an unique singularity of $\mathcal{F}_0^r$, say Q, such that $I(\mathcal{F}_0^r, D_{r-2}^r, Q) = -3$. In this case, D_r^r cuts both divisors D_{r-2}^r and D_{r-1}^r.

We leave the details of the proof of the above Remark for the reader. In figures 9.a and 9.b we sketch the divisors which appear in the resolution of the types $p : 1$, $p > 1$, and $p : q$, $1 < q < p$, respectively. Note that the last divisor which appears, D_r^r, is always transverse to $\mathcal{F}_0^r$. Moreover, $[D_r^r]^2 = -1$, $[D_{r-1}^r]^2 = -2$ and $[D_k^r]^2 \leq -2$ if $k < r$, in both cases. In the case $p : 1$ we have $r = p$, $[D_p^p]^2 = -1$ and $[D_k^p]^2 = -2$ for all $k < p$ (fig. 9.a).

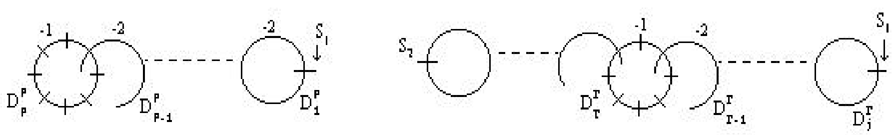


Fig. 9.a Fig. 9.b

Observe that there are separatrices cutting the invariant divisors of the extremities of the resolution, denoted by S_1 in figure 9.a and S_1, S_2 in figure 9.b. The function x^p/y^q is a first integral of the form $\omega = p\,y\,dx - q\,x\,dy$, and these separatrices correspond to the non-generic levels of the pencil $y^q - c.x^p = 0$, which are the axis $\{x = 0\}$ in the case $p : 1$ and the two axis $\{x = 0\}$ and $\{y = 0\}$ in the case $p : q$, $1 < q < p$.

Consider the pencil $(\mathcal{F}_\alpha)_{\alpha \in \overline{\mathbb{C}}}$ in $\mathbb{C}P(2)$, as in the hypothesis of Theorem 2. Since it is non-degenerate and equirreducible, we can suppose that that each non-reduced singularity, say m_0, of $\mathcal{F}_0$ is of the type $p : q = p(m_0) : q(m_0)$ and is also a non-reduced singularity of $\mathcal{F}_\infty$ of the same type and with the same resolution process. From now on, we will assume that all non-reduced singularities of $\mathcal{F}_0$ and $\mathcal{F}_\infty$, were reduced, but we will fix the singularity m_0 and we will keep the notations for the foliations and exceptional divisors obtained along the resolution of this singularity. In this way, π_1 coincides with σ_r in a neighborhood N_r of $D_1^r \cup \dots \cup D_r^r$. We will denote by $\mathcal{F}_\alpha^n$ the strict transform of $\mathcal{F}_\alpha$ in a neighborhood W of m_0 by $\sigma_n : N_n \to W$ and by Δ_n the divisor of tangency between $\mathcal{F}_0^n$ and $\mathcal{F}_\infty^n$.

3.4.5 Lemma. The following properties are true :

(a). $D_r^r \cap \Delta_r$ is a discrete subset of D_r^r. In particular, $\mathcal{F}_0^r$ and $\mathcal{F}_\infty^r$ are transverse in almost all points of D_r^r.

(b). For all $n \in \{0, \dots, r\}$ the divisor Δ_n is invariant for both foliations, $\mathcal{F}_0^n$ and $\mathcal{F}_\infty^n$.

(c). The divisor of tangency $\Delta(\mathcal{F}_0, \mathcal{F}_\infty) \subset \mathbb{CP}(2)$ is invariant for all foliations in the pencil $(\mathcal{F}_\alpha)_{\alpha \in \overline{\mathbb{C}}}$.

(d). $\pi_2(D_r^r)$ is a smooth rational -1-curve in M_j. In particular, π_2 is a biholomorphism in a neighborhood of D_r^r.

Proof. Let us prove (a). Suppose by contradiction that $D_r^r \cap \Delta_r$ is not discrete. In this case, since $D_r^r \cap \Delta_r$ is an analytic set, we must have $D_r^r \subset \Delta_r$ and that $\mathcal{F}_0^r$ is tangent to $\mathcal{F}_\infty^r$ along D_r^r. Recall that, at each step, π_2 contracts only curves that are invariant for all foliations in the pencil at the correspondent step. Since D_r^r is not invariant for $\mathcal{F}_0^r$, it can not be contracted to a point by π_2. It follows that $C := \pi_2(D_r^r) \subset M_j$ is a curve and that π_2 is a biholomorphism in a neighborhood of almost all points of D_r^r. Since $\mathcal{F}_0^r$ and $\mathcal{F}_\infty^r$ are tangent along D_r^r, it follows that $\mathcal{G}_0^j = (\pi_2)_*(\mathcal{F}_0^r)$ and $\mathcal{G}_\infty^j = (\pi_2)_*(\mathcal{F}_\infty^r)$ are tangent along C, so that $C \subset \Delta(\mathcal{G}_0^j, \mathcal{G}_\infty^j) := \Delta$. On the other hand, we have seen in Lemma 3.2.9 that Δ is invariant for all foliations in the pencil $(\mathcal{G}_\alpha^j)_{\alpha \in \overline{\mathbb{C}}}$. This is a contradiction, because C can not be invariant for $\mathcal{G}_0^j$.

Since Δ is invariant for all foliations in the pencil $(\mathcal{G}_\alpha^j)_{\alpha \in \overline{\mathbb{C}}}$ and π_2, at each step, contracts only curves that are invariant, we get that $\Delta_r = \pi_2^{-1}(\Delta) \cap \pi_1^{-1}(W)$ (as a set) and that Δ_r is invariant for both foliations $\mathcal{F}_0^r$ and $\mathcal{F}_\infty^r$. It follows by induction, from the process of resolution of m_0, that Δ_n is invariant for all foliations of the pencil $(\mathcal{F}_\alpha^n)_{\alpha \in \overline{\mathbb{C}}}$. Applying this argument for all non-reduced singularities of $\mathcal{F}_0$, we get that $\Delta(\mathcal{F}_0, \mathcal{F}_\infty)$ is invariant for all foliations in the pencil $(\mathcal{F}_\alpha)_{\alpha \in \overline{\mathbb{C}}}$. This proves (b) and (c).

Let us prove (d). Observe first that there exists $\beta \in \mathbb{C}$ such that the curve D_r^r is invariant for $\mathcal{F}_\beta^r$. In fact, fix a point $m \in D_r^r \setminus \Delta^r$. Since $\mathcal{F}_0^r$ and $\mathcal{F}_\infty^r$ are transverse at m, there exists $\beta \in \mathbb{C}$ such that the leaf of $\mathcal{F}_\beta^r$ through m is tangent to D_r^r at m. Since $\mathcal{F}_0^r$ is transverse to D_r^r, we get from 3.1.7 that $T_{\mathcal{F}_0^r} . D_r^r = (D_r^r)^2 = -1$. This is true for all $\alpha \in \overline{\mathbb{C}}$ such that $T_{\mathcal{F}_\alpha^r} = T_{\mathcal{F}_0^r}$, so that if $T_{\mathcal{F}_\alpha^r} = T_{\mathcal{F}_0^r}$ and D_r^r is not invariant for $\mathcal{F}_\alpha^r$, then $\mathcal{F}_\alpha^r$ is transverse to D_r^r. Since $\mathcal{F}_\beta^r$ is tangent to D_r^r at m, we conclude that, either D_r^r is invariant for $\mathcal{F}_\beta^r$, or $T_{\mathcal{F}_\beta^r} \neq T_{\mathcal{F}_0^r}$. Suppose that $T_{\mathcal{F}_\beta^r} \neq T_{\mathcal{F}_0^r}$. We have seen in Remark 3.1.1 that $T_{\mathcal{F}_\beta^r} - T_{\mathcal{F}_0^r}$ is an effective divisor, in this case, so that $T_{\mathcal{F}_\beta^r} = T_{\mathcal{F}_0^r} + \sum_{k=1}^n n_k.[C_k]$, where $n_k \geq 1$ and C_k is a divisor associated to some irreducible curve on N_r, $k = 1, \dots, n$. Note that each curve C_k is contained in Δ_r, so that $D_r^r \neq C_k$, for all k. If D_r^r was not invariant for $\mathcal{F}_\beta^r$ then we would get from 3.1.7 that

$$-1 - \operatorname{tang}(\mathcal{F}_\beta^r, D_r^r) = T_{\mathcal{F}_\beta^r} . D_r^r = T_{\mathcal{F}_0^r} . D_r^r + \sum_{k=1}^{n} n_k.(C_k . D_r^r) \geq -1 \implies$$

$$\implies \operatorname{tang}(\mathcal{F}_\beta^r, D_r^r) \leq 0 \implies \operatorname{tang}(\mathcal{F}_\beta^r, D_r^r) = 0$$

and this would imply that $\mathcal{F}_\beta^r$ would be transverse to D_r^r, a contradiction. Now, since D_r^r is invariant for $\mathcal{F}_\beta^r$, but not for $\mathcal{F}_0^r$, it follows that $C := \pi_2(D_r^r)$ must be invariant for $\mathcal{G}_\beta^j$, but not for

G_0^j, so that $C \not\subset \Gamma$. This implies that C is smooth. This last assertion, follows from Lemma 3.2.9. We have seen in Lemma 3.2.9 that $\Gamma = \sum_{k=1}^{3}(\sum_{i>0} C_{k,i})$, where each $C_{k,i}$ is a rational curve containing just one reduced singularity of G^j, say $q_{k,i}$. Since C is connected, the set $L := C \setminus \Gamma$ is a leaf of G^j and $C \setminus L$ is an union of a certain number of singularities $q_{k,i}$ as above. These singularities are reduced, so that C is smooth. We leave the details for the reader. Moreover, the Camacho-Sad index of a singularity $q_{k,i}$ with respect to $C_{k,i}$ is $I(G^j, C_{k,i}, q_{k,i}) = C_{k,i}^2 \in \{-2, -3, -4, -6\}$. This implies that, if $q_{k,i} \in C$ then $I(G^j, C, q_{k,i}) \in \{-1/2, -1/3, -1/4, -1/6\}$ (see 3.1.9). The fact that C is a rational curve implies that C can not be a leaf of G^j, so that it contains at least one singularity $q_{k,i}$. It follows from Camacho-Sad Theorem that $C^2 = I(G^j, C) < 0$. Since C^2 must be integer, we get that $C^2 \leq -1$. On the other hand, Π_2 is a sequence of blowing-downs and $C = \Pi_2(D^r_r)$ is smooth, so that $C^2 \geq (D^r_r)^2 = -1$. This implies that $C^2 = -1$. We conclude Π_2 can not contract any curve cutting D^r_r, for otherwise $C^2 > -1$. This implies that Π_2 is a biholomorphism in a neighborhood of D^r_r. ∎

3.4.6 Lemma. If m_0 is a non-reduced singularity of type $p : q$, then $p : q \in \{1 : 1, 2 : 1, 3 : 2\}$. Moreover, Π_2 is a biholomorphism in a neighborhood of $\Pi_1^{-1}(m_0)$.

Proof. Let us suppose that $1 \leq q < p$. Consider the resolution of m_0, sketched in one of the figures 9.a or 9.b. In any case, the divisor D^r_r cuts the divisor D^r_{r-1} and if $2 \leq q < p$ then D^r_r cuts another divisor, which we will call $D^r_{k_1}$, $k_1 < r - 1$. We have also that $(D^r_{r-1})^2 = -2$. Let $D^r_{r-1} = D^r_{j_1}, D^r_{j_2}, ..., D^r_{j_s}$ be the maximal chain of divisors contained in the resolution of m_0, such that $D^r_{j_i} \cap D^r_{j_{i+1}} \neq \emptyset$, for $1 \leq i \leq s - 1$, and $D^r_{j_i} \neq D^r_r$ for all $i = 1, ..., s$. If $2 \leq q < p$, then consider also the analogous chain $D^r_{k_1}, ..., D^r_{k_t}$ such that $D^r_{k_i} \neq D^r_r$ and $D^r_{k_i} \cap D^r_{k_{i+1}} \neq \emptyset$ for $1 \leq i \leq t - 1$, where $s + t = r - 1$. By convention we will set $t = 0$ if $1 = q < p$. Set also $J = D_{j_1} \cup ... \cup D^r_{j_s}$ and $K = D_{k_1} \cup ... \cup D^r_{k_t}$ (if $t > 0$). Since Π_2 is holomorphic, only contracts invariant curves and $J \subset \Gamma_r$, we must have that $\Pi_2(J)$ is connected and $\Pi_2(J) \subset \Gamma = \Gamma(G_0, G_1)$. Hence $\Pi_2(J)$ must be contained in some connected component of Γ. Since the connected components of Γ are the curves $C_{\ell,i}$, $1 \leq \ell \leq 3$, $i > 0$, which are also irreducible components, we get that $\Pi_2(J) \subset C_{\ell,i}$ for some $\ell = 1, 2, 3$ and $i > 0$. Since $D^r_r \cap D^r_{r-1} \neq \emptyset$, the curve $D^r_{r-1} = D^r_{j_1}$ can not be contracted by Π_2, by (d) of Lemma 3.4.5. This implies that $\Pi_2(J) = C_{\ell,i}$. We assert that $s = 1$, $\Pi_2(D^r_{r-1}) = C_{\ell,i}$ and that Π_2 is a biholomorphism in a neighborhood of D^r_{r-1}.

In fact, suppose by contradiction that $s > 1$. This implies that all divisors $D^r_{j_2}, ..., D^r_{j_s}$ are contracted by Π_2. Let us follow the process of contractions of these curves in Π_2, step by step. In each step only -1-curves can be contracted, so that the first curve to be contracted in the chain J must cut some curve that was contracted before, because $(D^r_{j_i})^2 \leq -2$ for all $i = 1, ..., s$. This curve can only be D^r_s, because this curve is the unique one in J which cuts the closure of some leaf outside the chain : the leaf containing the separatrix S_1. For simplicity we will use the same notation for the curves that was not contracted after some step. Just after contracting the -1-curve that contains S_1, the divisor D^r_s becomes a -1-curve containg one or two reduced singularities and the divisors $D^r_{j_1}, ..., D^r_{j_{s-1}}$ remain with same self-intersection. After the contraction of $D^r_{j_s}$, the unique divisor that can be contracted is $D^r_{j_{s-1}}$, because the others don't change the self-intersection. Proceding in this argument, we see that the last divisor to be contracted in J is $D^r_{j_2}$ and before its contraction it cuts D^r_{r-1} transversely in just one point, which is a reduced singularity of the transformed foliation. This implies that, after the contraction of $D^r_{j_2}$, the self-intersection of D^r_{r-1} increases of $+1$, so that D^r_{r-1} becomes a -1-curve. But this implies that after this step, D^r_{r-1} can be contracted, which is a contradiction. Therefore, we conclude that $s = 1$. This implies already that if $1 = q < p$ then $p : q = 2 : 1$. Moreover, Π_2 does not contract any invariant curve that meets D^r_{r-1}. This implies that Π_2 is a biholomorphism in a neighborhood of $D^r_r \cup D^r_{r-1}$. Set $\Pi_2(D^r_{r-1}) = C_{\ell,i} := C_1$.

Suppose now that $t > 0$ and $K \neq \emptyset$. Observe that, in this case, $D^r_{k_1} = D^r_{r-2}$ and D^r_{r-2} has self-intersection -3. This fact follows from (c) of Remark 3.4.4 and the fact that $s = 1$. By an argument analogous to the above one, we get that $\Phi_2(K) = C_{k,i}$, an irreducible component of Δ. Moreover, $D^r_{k_1}$ is not contracted by Φ_2 and, if $t > 1$ then, all divisors $D^r_{k_2}, ..., D^r_{k_t}$ are contracted by Φ_2. Following the contractions step by step, as before, we get that these divisors are contracted in the order $D^r_{k_t}, D^r_{k_{t-1}}, ..., D^r_{k_2}$. When we contract $D^r_{k_2}$, then the self-intersection of $D^r_{k_1}$ increases by one, so that it becomes -2. We conclude that $\Phi_2(K) = \Phi_2(D^r_{k_1}) = C_{k,i}$, $C^2_{k,i} = -2$ and $C_{k,i}$ contains just one singularity of G^j_0, say Q, such that $I(G_0, C_{k,i}, Q) = -2$. Let us prove that this is impossible. Set $\Phi_2(D^r_{k_1}) = C_2$.

We have seen that there exists $\beta \in \mathbb{C}$ such that D^r_r is invariant for F^r_β. This implies that $C = \Phi_2(D^r_r)$ is invariant for G^j_β. Hence G^j_β has an invariant set which consists of a chain of three smooth rational curves $L = C_1 \cup C \cup C_2$ and $sing(G^j_\beta) \cap L = \{P, Q\}$, where $P = C \cap C_1$ and $Q = C \cap C_2$ are reduced singularities, so that $Z(G^j_\beta, C) = 2$. Since G^j_0 is transverse to C, we get that $T_{G^j_0}.C = C^2 - tang(G^j_0, C) = -1$. On the other hand, the fact that $T_{G^j_0} = T_{G^j_\beta}$ and 3.1.8 imply that $-1 = T_{G^j_\beta}.C = X(C) - Z(G^j_\beta, C) = 2 - 2 = 0$, a contradiction. This contradiction implies that $t = 1$ and that there is no -1-curve contracted by Φ_2 meeting C_2. Therefore, $p : q = 3 : 2$ and Φ_2 is a biholomorphism in a neighborhood of $C_1 \cup C \cup C_2$. □

3.4.7 Corollary. Let $m_1, ..., m_k$ be the non-reduced singularities of the pencil $(F_\alpha)_{\alpha \in \overline{\mathbb{C}}}$. Then m_i is of the type $p_i : q_i$ for the generic foliation of the pencil, where $p_i : q_i \in \{1 : 1, 2 : 1, 3 : 2\}$. Moreover, Φ_2 is a biholomorphism.

Proof. The first part follows directly from Lemma 3.4.6. It follows also from Lemma 3.4.6 that, Φ_2 is a biholomorphism in a neighborhood of $\Phi_1^{-1}\{m_1, ..., m_k\}$. This implies that, if Φ_2 contracts some -1-curve, say D, then $D \cap \Phi_1^{-1}\{m_1, ..., m_k\} = \emptyset$. Since Φ_1 is a biholomorphism outside $\Phi_1^{-1}\{m_1, ..., m_k\}$, we obtain that $\Phi_1(D)$ is a smooth -1-curve in CP(2), which is not possible. □

The next result will be used in the proof of the 3rd step.

3.4.8 Lemma. Let m_0 be a non-reduced singularity of F_0 of type $p : q \in \{1 : 1, 2 : 1, 3 : 2\}$. Let $\{f = 0\}$ be an equation of the germ of Δ_0 at m_0 and ν_0 be the multiplicity of f at m_0. Then there exists a local coordinate system (x, y) at m_0 where F_0 is represented by a linear vector field and

(a). If $p : q = 1 : 1$ then $\nu_0 = 3$ and $f(x, y) = x.y(y - x).u(x, y)$, where $u(0, 0) \neq 0$.
(b). If $p : q = 2 : 1$ then $\nu_0 = 2$ and $f(x, y) = y(y - x^2).u(x, y)$, where $u(0, 0) \neq 0$.
(c). If $p : q = 3 : 2$ then $\nu_0 = 2$ and $f(x, y) = (y^2 - x^3).u(x, y)$, where $u(0, 0) \neq 0$.

In particular, if $sing(\Delta_0)$ denotes the singular set of Δ_0 then, $sing(\Delta_0)$ coincides with the set of non-reduced singularities of F_α, for a generic $\alpha \in \overline{\mathbb{C}}$.

Proof. Keeping the notation of Lemma 3.4.5, denote by Δ'_n the strict transform of Δ_n by π_n. Note that $\pi^*_{n+1}(\Delta_n) = \Delta'_n + \nu_n.D^{n+1}_{n+1}$, where ν_n is the multiplicity of Δ_n at m_n. On the other hand, it follows from 3.1.11 that

$$T^*_{F^{n+1}_0} = \pi^*_n(T^*_{F^n_0}) - (\mu_n - 1).D^{n+1}_{n+1} \text{ and } N_{F^{n+1}_1} = \pi^*_n(N_{F^n_1}) - \mu_n.D^{n+1}_{n+1} \implies$$

$$(13) \quad \Delta_{n+1} = \pi^*_n(\Delta_n) - (2\mu_n - 1)D^{n+1}_{n+1} = \Delta'_n + (\nu_n - 2\mu_n + 1)D^{n+1}_{n+1}$$

Recall that $\mu_{r-1} = 2$, whereas $\mu_n = 1$ if $1 \leq n < r - 1$. If $n = r - 1$ then $\pi^*_r(\Delta_{r-1}) = \Delta'_{r-1}$, because F^r_0 and F^r_1 are not tangent along D^r_r. This implies that $\nu_{r-1} = 2.2 - 1 = 3$. On the other hand, after the resolution Φ, all components of Δ are smooth rational curves with multiplicity one (Lemma 3.2.9). Since the resolution Π_r coincides with Φ in a neighborhood W of $\Phi^{-1}(m_0)$, we get that $W \cap \Gamma = W \cap \Gamma_r$ and all the components of this curve must have multiplicity one. Let $(x;y)$ be a local coordinate system where F_0 is represented by the vector field $X = qx\frac{\partial}{\partial x} + py\frac{\partial}{\partial y}$. Note that $g(x;y) = y^q/x^p$ is a local first integral of X and that the germ of the components of Γ_0 at m_0 are level curves of g.

Consider the case $p : q = 1 : 1$. In this case, $r = 1$ and $\pi_1^*(\Gamma_0) = \Gamma_1^0$, so that $\nu_0 = 3$. Since the components of Γ_0 have multiplicity one, it follows that Γ_0 has three branches passing through m_0, which are level curves of $g = y/x$. Hence, after a linear change of variables we can suppose that f is like in (a). When $p : q = 2 : 1$ or $p : q = 3 : 2$, we have $r = 2$ or $r = 3$, respectively, and after the first blowing-up we get that $\Gamma_1 = \Gamma_0^0 + (\nu_0 - 1).D_1^1$. Since D_1^r is the strict transform of D_1^1 at the final step of the resolution and D_1^r has multiplicity one in Γ, in both cases, we get that $\nu_0 = 2$. In particular, $\Gamma_1 = \Gamma_0^0 + D_1^1$. If $p : q = 2 : 1$, then m_1 is a singularity of type $1 : 1$ for F_0^1, so that $\nu_1 = 3$. Hence, the multiplicity of Γ_0^0 at m_1 is two and its germ consists of two curves meeting transversely at m_1. This implies that the germ of Γ_0 at m_0 consists of two tangent curves meeting at m_0, so that after a linear change of coordinates, we can suppose that f is like in (b). If $p : q = 3 : 2$ then, after the first blowing-up, the singularity m_1 is of the type $2 : 1$ and, by the previous argument, the germ of Γ_1 at m_1 contains two tangent branches, where one of them is D_1^1. When we blow-down the other branch, we obtain a cuspidal curve like in (c).

Let us prove the last assertion. Let N be the set of non-reduced singularities of F_α. It follows from (a), (b) and (c) that $\mathrm{sing}(\Gamma_0) \subset N$. On the other hand, if $m \in \mathrm{sing}(\Gamma_0)$ then m must be a singularity of any F_α in the pencil. This singularity must be non-reduced, for otherwise after the resolution process the set Γ would have singularities, which is not the case. ∎

Proof of the 3rd step. Since Φ_2 is a biholomorphism, we can suppose that the resolution of the pencil is a sequence of blowing-ups $\Phi\colon M_j \to CP(2)$ and $\Phi^*(F_\alpha) = G_\alpha^j$, for all $\alpha \in \overline{C}$. We have seen that the divisors of tangencies $\Gamma(F_0;F_1) := \Gamma_0$ and $\Gamma(G_0^j;G_1^j) := \Gamma$ are invariant for all foliations in the pencils $P = (F_\alpha)_{\alpha\in\overline{C}}$ and $Q^j = (G_\alpha^j)_{\alpha\in\overline{C}}$, respectively. Let $\Gamma_0 = \sum_{i=1}^{\ell} n_i.B_i$, $n_i > 0$, and $\Gamma = \sum_{k=1}^{3}(\sum_{i>0} C_{k;i})$ be the decompositions of these divisors in irreducible components (see Lemma 3.2.9). Note that, if we consider Γ and Γ_0 as sets, then $\Phi(\Gamma) = \Gamma_0$. This implies the following facts :

(i). For any $(k;i)$, $k = 1;2;3$, $i > 0$, either $\Phi(C_{k;i}) = B_r$, for some $r \in \{1;\ldots;\ell\}$, or Φ contracts $C_{k;i}$ and $\Phi(C_{k;i})$ is a point. Moreover, if $\Phi(C_{k;i}) = B_r$, then $r \in \{1;\ldots;\ell\}$ is unique and $n_r = 1$. This is a consequence of the fact that Φ is a biholomorphism outside the set of curves that it contracts. It follows that $\Gamma_0 = \sum_{i=1}^{\ell} B_i$. Since r is unique, we will use the notation $C_{k;i} := C_r$.

(ii). If $\Phi(C_r) = B_r$, then B_r contains an unique singularity $q_r(\alpha)$ such that the map $\alpha \in \overline{C} \mapsto q_r(\alpha)$ is a regular parametrization of B_r. In fact, if $C_r = C_{k;i}$, then we have seen that $C_{k;i}$ contains an unique singularity $q_{k;i}(\alpha)$ such that the map $\alpha \in \overline{C} \mapsto q_{k;i}(\alpha) \in C_{k;i}$ is a regular parametrization of $C_{k;i}$. If we set $q_r(\alpha) = \Phi(q_{k;i}(\alpha))$, then $\alpha \mapsto q_r(\alpha)$ is a regular parametrization of B_r. We will say that $\alpha \in \overline{C}$ is generic, if for all $r \in \{1;\ldots;\ell\}$ the point $q_r(\alpha) \in B_r \setminus \mathrm{sing}(\Gamma_0)$.

(iii). Φ contracts only curves that are contained in Γ and $\mathrm{sing}(\Gamma_0)$ coincides with the set of non-reduced singularities of F_0.

(iv). If $\alpha \in \overline{C}$ is generic then, for all $r \in \{1;\ldots;\ell\}$, $q_r(\alpha)$ is a non-degenerate singularity of the type $1 : C_r^2 \in \{1 : -2; 1 : -3; 1 : -4; 1 : -6\}$. This follows from (iii) and the fact that Φ is a biholomorphism in a neighborhood of $q_{k;i}(\alpha)$, if $\Phi(q_{k;i}(\alpha)) = q_r(\alpha)$ and α is generic.

If $\alpha \in \overline{C}$ is generic, then all the singularities of F_α are non-degenerate. Moreover, it follows from Lemma 3.4.6 and (iv) that they are of one of the types : $1 : 1$, $2 : 1$, $3 : 2$, $1 : -2$, $1 : -3$, $1 : -4$, $1 : -6$. Will use the notations r_1, r_2, r_3, s_2, s_3, s_4 and s_6 for the number of the singularities of the types $1 : 1$, $2 : 1$, $3 : 1$, $1 : -2$, $1 : -3$, $1 : -4$ and $1 : -6$, respectively, of the generic foliations of the pencil $(F_\alpha)_{\alpha\in\overline{C}}$. Similarly, we will use the notations s_2^1, s_3^1, s_4^1 and s_6^1 for the number of singularities of the types $1:-2$, $1:-3$, $1:-4$ and $1:-6$, respectively, of the foliations in the pencil $(\mathcal{G}_\alpha^j)_{\alpha\in\overline{\mathbb{C}}}$.

3.4.9 Lemma. The numbers $d, \ell, r_1, ..., s_6$ and $s_2^1, ..., s_6^1$ satisfy the following relations :
(a). $2d+1 = \sum_{i=1}^{\ell} dg(B_i) = dg(\Delta_0)$.
(b). $s_2+s_3+s_4+s_6 = \ell$.
(c). $s_2+r_2+r_3 = s_2^1$, $s_3+r_3 = s_3^1$, $s_4 = s_4^1$ and $s_6 = s_6^1$.
(d). $r_1+r_2+r_3+s_2+s_3+s_4+s_6 = d^2+d+1$.
(e). $4r_1 + \frac{9}{2}r_2 + \frac{25}{6}r_3 - \frac{1}{2}s_2 - \frac{4}{3}s_3 - \frac{9}{4}s_4 - \frac{25}{6}s_6 = (d+2)^2$.

Proof. Relation (a) follows from $[\Delta_0] = T^*_{\mathcal{F}_\alpha} + N_{\mathcal{F}_\alpha} = (2d+1)H$, where H is the divisor associated to a hyperplane in $\mathbb{C}P(2)$ (see 3.1.10 and 3.1.5). Relation (b) follows from (ii) and (iv). We get (c) from the process of resolution of the singularities of the types $2:1$ and $3:2$. Each singularity of the type $3:2$ gives origin, after the resolution, to two singularities, one of the type $1:-2$ and the other of the type $1:-3$. On the other hand, each singularity of the type $2:1$ gives origin, after the resolution, to just one singularity of the type $1:-2$. This implies that $s_2+r_2+r_3 = s_2^1$ and $s_3+r_3 = s_3^1$. Since these resolutions do not create any singularity of one of the types $1:-4$ or $1:-6$, we get the other relations in (c). Relation (d) follows from 3.1.6. Finally, relation (e) is a consequence of Baum-Bott Theorem (cf. [B-B] and [Br-2]). We will state this result in the particular case in which all singularities of the foliation are non-degenerate. Given a foliation $\mathcal{H}$ on a compact surface M, with non-degenerate singularities, say $p_1, ..., p_n$, define

$$BB(\mathcal{H}, p_j) = \frac{(tr(DX(p_j)))^2}{det(DX(p_j))}$$

where X is a holomorphic vector field which represents $\mathcal{H}$ in a neighborhood of p_j, $j = 1, ..., n$.

Theorem (Baum-Bott). In the above situation, we have that $\sum_{j=1}^{n} BB(\mathcal{H}, p_j) = N_{\mathcal{H}}^2$. In particular, if $M = \mathbb{C}P(2)$ and $\mathcal{H}$ has degree d, then $\sum_{j=1}^{n} BB(\mathcal{H}, p_j) = (d+2)^2$.

In the case of a singularity p_j of the type $p:q$ we have that $BB(\mathcal{H}, p_j) = \frac{(p+q)^2}{p.q}$. If we apply this result in the case of a generic foliation in the pencil $(\mathcal{F}_\alpha)_{\alpha\in\overline{\mathbb{C}}}$ then we get (e). $\square$

Next, we will consider all possible cases for the pencil $(\mathcal{G}_\alpha^j)_{\alpha\in\overline{\mathbb{C}}}$. The strategy in any case, will be to prove that the divisor of tangencies Δ_0 of the pencil $\mathcal{P}$ coincides with the divisor of tangencies of one of the pencils of §2.2, 2.3 or 2.4, modulo an automorphism $\mathbb{C}P(2)$. This implies the Theorem, because if the divisor of tangencies of two pencils coincide then the pencils are equivalent, as the reader can check.

3.4.10 The pencil is bimeromorphically equivalent to the family of type 1 (j=1). Let us prove that the pencil $(\mathcal{F})_{\alpha\in\overline{\mathbb{C}}}$ is equivalent to the pencil $(\mathcal{F}^4)_{\alpha\in\overline{\mathbb{C}}}$ of §2.2. In this case, all the members of the pencil $(\mathcal{G}_\alpha^1)_{\alpha\in\overline{\mathbb{C}}}$ have nine singularities, all of them of the type $1:-3$ (see fig. 1.a). Hence, $s_2^1 = s_4^1 = s_6^1 = 0$ and $s_3^1 = 9$. It follows from (c) of Lemma 3.4.9 that $s_2 = s_4 = s_6 = r_2 = r_3 = 0$ and $s_3 = 9$, so that $\ell = 9$, by (b). On the other hand, (a) implies that $2d+1 = \sum_{i=1}^{9} dg(B_i) \geq 9$, and so $d \geq 4$. Therefore, $d = 4$, by the 1st step, and $dg(B_i) = 1$ for all $i = 1, ..., 9$. In particular, Δ_0 contains nine straight lines, all of them with multiplicity one. It follows from (d) that $r_1 + s_3 = d^2+d+1 = 21$, and so $r_1 = 12$. Let $P := \{m_1, ..., m_{12}\}$ be the set of singularities of the type $1:1$ and $L := \{B_1, ..., B_9\}$. The idea is to consider the configuration of lines and points (L, P) and prove that it satisfies the following properties :
(I). Each line of L contains four points of P.
(II). Each point of P belongs to three lines of L.
(III). If three points of P are not in the same line of L, then the points are not aligned.

The rest of the proof is based in Proposition 1 of [LN]. Proposition 1 of [LN] says that, if a configuration as above satisfies (I), (II) and (III), then there exists an automorphism T of $\mathbb{C}P(2)$ such that the lines in $T(P)$ are the lines defined by $(Y^3 - X^3)(Z^3 - Y^3)(X^3 - Z^3) = 0$, in homogeneous coordinates. On the other hand, the divisor of tangencies of the pencil $(F^4_\alpha)_{\alpha\in\overline{\mathbb{C}}}$ is also $(Y^3 - X^3)(Z^3 - Y^3)(X^3 - Z^3) = 0$, so that this pencil is equivalent to $(F_\alpha)_{\alpha\in\overline{\mathbb{C}}}$.

Let us prove (I), (II) and (III). Assertion (I) follows from 3.1.8 : if F_α is a generic foliation in the pencil and $B_i \in L$, then

$$d - 1 = T^*_{F_\alpha}.B_i = X(B_i) - Z(F_\alpha; B_i) \implies Z(F_\alpha; B_i) = 5\ ;$$

so that B_i contains five singularities of F_α. Since only one of these singularities is of the type $1 : -3$, the other four must be of the type $1 : 1$. Assertion (II) follows from Lemma 3.4.8 : the multiplicity of $\Gamma_0 = \sum_{i=1}^{9} B_i$ at m_j is three, for all $j = 1, ..., 12$. Hence, each m_j belongs to the intersection of three lines of L. Finally, assertion (III) follows from Lemma 3.4.1 : if $m_{i_1}, m_{i_2}, m_{i_3}$ belong to the same line, say B, then B must be invariant for any F_α such that m_{i_1}, m_{i_2} and m_{i_3} are radial singularities. Hence $B \in L$. This ends the proof of this case. ∎

3.4.11 The pencil is bimeromorphically equivalent to the family of type 2 (j=2). We will prove in this case that, either $d = 2$ and P is equivalent to the pencil $(F^2_\alpha)_{\alpha\in\overline{\mathbb{C}}}$ of §2.3, or $d = 3$ and P is equivalent to the pencil $(F^3_\alpha)_{\alpha\in\overline{\mathbb{C}}}$ of §2.3. Note that for a foliation G^2_α we have $s^1_2 = 5$, $s^1_3 = 4$, $s^1_4 = 0$ and $s^1_6 = 1$. From (c) of Lemma 3.4.9 we get the following relations : $s_2 + r_2 + r_3 = 5$, $s_3 + r_3 = 4$, $s_4 = 0$ and $s_6 = 1$. In particular, $s_3 = 4 - r_3$ and $s_2 = 5 - r_2 - r_3$. If we substitute these relations in (d) and (e), we obtain that $r_1 - r_3 = d^2 + d - 9$ and $4r_1 + 5r_2 + 6r_3 = d^2 + 4d + 16$, which implies that $5(r_1 + r_2 + r_3) = 2d^2 + 5d + 7$, and so 5 divides $2d^2 + 5d + 7$. As the reader can check, if $d \in \{2, 3, 4\}$, this is possible only for $d \in \{2, 3\}$. Moreover, if $d = 2$ then we get that $r_1 + r_2 + r_3 = 5$ and $\ell = s_2 + s_3 + s_6 = 2$, whereas if $d = 3$ then we get $r_1 + r_2 + r_3 = 8$ and $\ell = s_2 + s_3 + s_6 = 5$.

3.4.12 The case $d = 2$. In this case, $dg(\Gamma_0) = 5$. We assert that $r_1 = 0$. In fact, suppose by contradiction that for a generic $\alpha \in \overline{\mathbb{C}}$ the foliation F_α has a radial singularity, say m. It follows from (a) of Lemma 3.4.1 that for any other singularity, say q, of F_α, the straight line $L(m, q)$, which joins m to q is invariant for F_α. On the other hand, since $\ell = 2$, Γ_0 contains exactly two irreducible components, say B_1 and B_2. For each $j = 1, 2$, the component B_j does not change with the parameter and contains an unique singularity $q_j(\alpha)$ such that $\alpha \mapsto q_j(\alpha) \in B_j$ is a regular parametrization of B_j. Since $L(m, q_j(\alpha))$ is invariant for F_α, we have two possibilities : either the line $L(m, q_j(\alpha))$ does not change with parameter, or it changes. In the first case, we must have $B_j \equiv L(m, q_j(\alpha))$, whereas in the second, the foliation F_α has an algebraic invariant curve outside Γ_0. We assert that the second possibility can not happen. In fact, if $L(m, q_j(\alpha))$ is an algebraic invariant curve outside Γ_0, then $\Pi^{-1}(L(m, q_j(\alpha)))$ is an algebraic invariant curve for G^1_α, outside Γ. It follows from (iv) of Lemma 3.2.18 that G^1_α has a first integral, so that $\alpha \in E(Q^2)$. But this implies that $E(Q^2) = \overline{\mathbb{C}}$, a contradiction. From this, we get that B_1 and B_2 are straight lines, and so $dg(\Gamma_0) = 2$, which is a contradiction. This proves that $r_1 = 0$.

It follows from $r_1 = 0$ and Lemma 3.4.9 that : $r_2 = 2$, $s_2 = 0$, $r_3 = 3$ and $s_3 = s_6 = 1$. Since $dg(\Gamma_0) = 5$, we have two possibilities for the components B_1 and B_2 of Γ_0 : if $dg(B_1) \le dg(B_2)$ then, either $dg(B_1) = 1$ and $dg(B_2) = 4$, or $dg(B_1) = 2$ and $dg(B_2) = 3$. Let us exclude the second possibility. Suppose by contradiction that $dg(B_1) = 2$. This implies that B_1 is a smooth conic, so that it contains four singularities of F_α, for a generic $\alpha \in \overline{\mathbb{C}}$, by 3.1.8. One of these singularities is $q_1(\alpha)$, which is of one the types $1 : -3$ or $1 : -6$. The other three, say m_1, m_2, m_3, are of one the types $2 : 1$ or $3 : 2$. Let us apply Camacho-Sad Theorem : we have $I(F_\alpha, B_1, q_1(\alpha)) \in \{-3, -6\}$ and $I(F_\alpha, B_1, m_j) \in \{2, 1/2, 2/3, 3/2\}$, because the tangent direction of B_1 at each m_j corresponds to a local separatrix of this singularity. Since $4 = B_1^2 = \sum_{q\in B_1} I(\mathcal{F}_\alpha; B_1; q)$, we get that $\sum_{j=1}^{3} I(\mathcal{F}_\alpha; B_1; m_j) = 4 - I(\mathcal{F}_\alpha; B_1; q_1(\alpha)) \in \{1; -2\}$. On the other hand, $\sum_{j=1}^{3} I(\mathcal{F}_\alpha; B_1; m_j) \geq 3/2$, which is a contradiction. Therefore, $dg(B_1) = 1$ and $dg(B_2) = 4$.

Let us analyse the singularities of $\mathcal{F}_\alpha$ in the straight line B_1, by using Camacho-Sad Theorem. Observe first that B_1 contains three singularities, by 3.1.8. One of these singularities is $q_1(\alpha)$. Call m_1 and m_2 the other two. We assert that, for a generic α, $q_1(\alpha)$ is of the type $1 : -3$ and m_1, m_2 are of the type $2 : 1$. In fact, consider the Camacho-Sad indexes $I_\alpha := I(\mathcal{F}_\alpha; B_1; q_1(\alpha))$ and $I_j := I(\mathcal{F}_\alpha; B_1; m_j)$. We have that $I_\alpha \in \{-3; -6\}$, $I_j \in \{2; 1/2; 2/3; 3/2\}$ and $I_\alpha + I_1 + I_2 = B_1^2 = 1$, so that $I_1 + I_2 = 1 - I_\alpha$. Since $I_1 + I_2 \leq 4$, we get that $I_\alpha \geq -3$, so that $I_\alpha = -3$ and $I_1 = I_2 = 2$, as the reader can check. This implies that $q_1(\alpha)$ is of the type $1 : -3$ and m_1 and m_2 are of the type $2 : 1$. Moreover, $\mathcal{F}_\alpha$ has four singularities outside B_1, one of the type $1 : -6$ and three of the type $3 : 2$. The curve B_2 must contain these singularities and also the points in $B_2 \cap B_1$, which are also singularities of $\mathcal{F}_\alpha$. Since $q_1(\alpha)$ changes with the parameters, for a generic α, B_2 does not contain $q_1(\alpha)$. This implies that $B_2 \cap B_1 \subset \{m_1; m_2\}$. On the other hand, (b) Lemma 3.4.8 implies that the germ of Δ_0 at m_j contains two smooth tangent branches. Hence, B_2 is a quartic tangent to B_1 at m_1 and m_2. Let m_3, m_4 and m_5 be the non-reduced singularities of $\mathcal{F}_0$, outside B_1. These singularities are of the type $3 : 2$ and must be contained in B_2. It follows from (c) of Lemma 3.4.8 that these points are cuspidal singularities of B_2. Therefore, B_2 is a quartic with three cuspidal singularities and tangent to B_1 at m_1 and m_2. Note that three different points in the set $\{m_1; \dots; m_5\}$, are not aligned, for otherwise the line containing them would be a component of Δ_0, which can not happen.

Choose a homogeneous coordinate system $[x : y : z]$ such that B_1 is the line $z = 0$ and m_3, m_4 and m_5 are the points, $[0 : 0 : 1]$, $[1/2 : 1/2 : 1]$ and $[1/2 : -1/2 : 1]$, respectively. As the reader can check, in the affine coordinate system $z = 1$, the quartic B_2 is then given by $4y^2(1 - 3x) - 4x^3 + (3x^2 + y^2)^2 = 0$. This finishes the proof in this case, because the divisor of tangencies of the pencil $(\mathcal{F}_\alpha^2)_{\alpha\in\overline{\mathbb{C}}}$ is also given by these curves (see §2.3). $\square$

3.4.13 The case $d = 3$. We will consider the following situation : let $\mathcal{F}$ be a foliation on $\mathbb{CP}(2)$ of degree three with three non-aligned radial singularities, say $m_1; m_2; m_3$. Let ℓ_{ij} be the straight line joining m_i and m_j, $1 \leq i < j \leq 3$. Consider the Cremona transformation $\Psi : \mathbb{CP}(2) \to \mathbb{CP}(2)$ defined by blowing-up at the points $m_1; m_2; m_3$ and blowing-down the strict transforms of the lines ℓ_{ij}, $1 \leq i < j \leq 3$, as in figure 4. Set $\mathcal{G} = \Psi_*(\mathcal{F})$. We have the following result :

3.4.14 Lemma. The foliation $\mathcal{G}$ has degree two. Moreover, the singularities of $\mathcal{G}$ are non-degenerate if, and only if, the singularities of $\mathcal{F}$ are non-degenerate.

Proof. Note first that the lines ℓ_{ij} are invariant for $\mathcal{F}$ ((b) of Lemma 3.4.1). Since m_1, m_2 and m_3 are not aligned, we can choose a homogeneous coordinate system $[x : y : z]$ such that $m_1 = [0 : 0 : 1]$, $m_2 = [0 : 1 : 0]$ and $m_3 = [1 : 0 : 0]$, so that $\ell_{12} = \{x = 0\}$, $\ell_{13} = \{y = 0\}$ and $\ell_{23} = \{z = 0\}$. In this coordinate system, we have $\Psi[x : y : z] = [y.z : x.z : x.y]$. Since the lines $\{x = 0\}$, $\{y = 0\}$ and $\{z = 0\}$ are invariant and $[0 : 0 : 1]$ is a radial singularity of $\mathcal{F}$, this foliation can be represented in the affine coordinate system $\{z = 1\}$, by a polynomial vector field X of the form

$$X(x; y) = x(1 + \alpha x + \beta y + P_2(x; y))\frac{\partial}{\partial x} + y(1 + \gamma x + \delta y + Q_2(x; y))\frac{\partial}{\partial y} \ ;$$

where $\alpha; \beta; \gamma; \delta \in \mathbb{C}$ and $P_2; Q_2$ are homogeneous polynomials of degree two. The fact that $[0 : 1 : 0]$ and $[1 : 0 : 0]$ are radial singularities of $\mathcal{F}$, is equivalent to $P_2(0; 1) = Q_2(1; 0) = 0$ and

$P_2(1,0).Q_2(0,1) \neq 0$, as the reader can check. Hence, we can suppose that

$$X(x,y) = x(1+\alpha x+\beta y+Ax^2+Bxy)\frac{\partial}{\partial x} + y(1+\gamma x+\delta y+Cxy+Dy^2)\frac{\partial}{\partial y}\ ;$$

where $A.D \neq 0$. Now, in this coordinate system, we have $\Psi(x,y) = (1/x, 1/y) = (u,v)$, so that, if $Y(u,v) = -u.v.\Psi_*(X)$, then

$$Y(u,v) = (Bu + Av + \alpha uv + \beta u^2 + u^2 v)\frac{\partial}{\partial u} + (Du + Cv + \gamma v^2 + \delta uv + uv^2)\frac{\partial}{\partial v}$$

and Y represents $\mathcal{G}$ in the affine coordinate system $(u,u) = [u : v : 1]$. This implies that $\mathcal{G}$ has degree two, because the homogeneous part of degree three of Y is $u.v(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v})$ (see [LN 1]). Note that the point $n_1 := [0 : 0 : 1]$ is a singularity of $\mathcal{G}$. Similarly, the points $n_2 := [0 : 1 : 0]$ and $n_3 := [1 : 0 : 0]$ are singularities of $\mathcal{G}$. On the other hand, if the singularities of $\mathcal{F}$ are non-degenerate, then each line ℓ_{ij} contains four singularities, so that there are nine singularities in $\cup_{ij}\ell_{ij}$ and $4 = 13 - 9$ singularities of $\mathcal{F}$ in $\mathbb{CP}(2) \setminus \cup_{ij}\ell_{ij}$, because the total number of singularities is $13 = 3^2 + 3 + 1$ (see 3.1.6). Since Ψ is a biholomorphism outside $\cup_{ij}\ell_{ij}$, $\mathcal{G}$ must have four non-degenerate singularities in $\Psi(\mathbb{CP}(2)\setminus\cup_{ij}\ell_{ij}) \subset \mathbb{CP}(2)\setminus\{n_1, n_2, n_3\}$. Hence, $\mathcal{G}$ has seven singularities, so that they must be non-degenerate, because $7 = 2^2 + 2 + 1$. We leave the proof of the converse for the reader. ∎

The idea of the proof is the following : we will prove that, for a generic $\alpha \in \overline{\mathbb{C}}$, $\mathcal{F}_\alpha$ has three radial singularities, say m_1, m_2 and m_3, which are not aligned. If Ψ is as in Lemma 3.4.14, then the pencil $(\mathcal{H}_\alpha := \Phi_*(\mathcal{F}_\alpha))_{\alpha\in\overline{\mathbb{C}}}$ satisfies the hypothesis of the case of degree two. Therefore, we can suppose that $\mathcal{H}_\alpha = \mathcal{F}^2_\alpha$, for every $\alpha \in \overline{\mathbb{C}}$. The result then follows from the fact that the pencil $(\mathcal{F}^3_\alpha)_{\alpha\in\overline{\mathbb{C}}}$ is obtained from the pencil $(\mathcal{F}^2_\alpha)_{\alpha\in\overline{\mathbb{C}}}$ by a Cremona transformation, as was showed in §2.3 (see also §2.3 of [LN]). Let us prove the existence of the radial singularities m_1, m_2, m_3.

We have seen before that $dg(\Delta_0) = 7$, $r_1 + r_2 + r_3 = 8$, $s_4 = 0$, $s_6 = 1$, $s_2 + s_3 = 4$ and $\ell = s_2 + s_3 + s_6 = 5$. In particular, since Δ_0 has five irreducible components, at least three of them, say B_1, B_2 and B_3, are straight lines. Observe that $r_1 \geq 3$. This follows from $s_2 + r_2 + r_3 = s^1_2 = 5$ ((c) of Lemma 3.4.9) and $r_1 + r_2 + r_3 = 8$, so that $r_1 = s_2 + 3$ and $r_1 \geq 3$. We assert that $r_1 = 3$. In fact, suppose by contradiction that $r_1 > 3$ and let $m_1, ..., m_4$ be four radial singularities of $\mathcal{F}_\alpha$. Let us prove that at least three of them are not aligned. Suppose by contradiction that they are aligned. Note that the line which contains these singularities is invariant for all foliations in the pencil, and so we can suppose that $m_1, ..., m_4 \in B_1$. Since $Z(\mathcal{F}_\alpha, B_1) = 4$, by 3.1.8, we get that $sing(\mathcal{F}_\alpha) \cap B_1 = \{m_1, ..., m_4\}$. But, this is impossible, by Camacho-Sad Theorem, because $I(\mathcal{F}_\alpha, B_1, m_j) = 1$, $j = 1, ..., 4$, and $B^2_1 = 1$. Hence, three of the singularities are not aligned. In this case, by the previous argument, the pencil $(\mathcal{F}_\alpha)_{\alpha\in\overline{\mathbb{C}}}$ is equivalent to the pencil $(\mathcal{F}^3_\alpha)_{\alpha\in\overline{\mathbb{C}}}$. Since the generic foliations in this pencil have three radial singularities, we get $r_1 = 3$.

Now, $r_1 = 3$ and the system of equations in Lemma 3.4.9 gives, $r_2 = 5$, $r_3 = 0$, $s_2 = s_4 = 0$, $s_3 = 4$ and $s_6 = 1$. We leave this computation for the reader. Let us prove that the radial singularities, m_1, m_2 and m_3, are not aligned. Suppose by contradiction that they are aligned. Since the line that contains them is contained in Δ_0 (Lemma 3.4.1), we can suppose that $m_1, m_2, m_3 \in B_1$. On the other hand, $I_j := I(\mathcal{F}_\alpha, B_1, m_j) = 1$, for a generic α, so that by Camacho-Sad Theorem, we must have $I(\mathcal{F}_\alpha, B_1, q_1(\alpha)) = 1 - \sum_{j=1}^3 I_j = -2$. This implies that $q_1(\alpha)$ is of the type $1 : -2$, and so $s_2 > 0$, a contradiction with $s_2 = 0$. Hence m_1, m_2 and m_3 are not aligned. This finishes the proof of this case. ∎

3.4.15 The pencil is bimeromorphically equivalent to the family of type 3 (j=3). We will prove that the pencil $(\mathcal{F}_\alpha)_{\alpha\in\overline{\mathbb{C}}}$ is equivalent to the pencil $(\mathcal{F}^{3:1}_\alpha)_{\alpha\in\overline{\mathbb{C}}}$ of §2.4. First of all, let us prove that $d = 3$, $r_1 = 3$, $r_2 = 5$, $r_3 = s_3 = s_6 = 0$, $s_2 = 1$ and $s_4 = 4$, in this case. For the pencil of type 3, we have $s_2^1 = 6$, $s_3^1 = s_6^1 = 0$ and $s_4^1 = 4$ (see fig. 1.c). It follows from (c) of Lemma 3.4.9 that $s_2 + r_2 = 6$, $s_3 = s_6 = r_3 = 0$ and $s_4 = 4$. If we substitute these values in (d) and (e) of Lemma 3.4.9, we get $r_1 = d^2 + d - 9$ and $4r_1 + \frac{9}{2}r_2 - \frac{1}{2}s_2 = d^2 + 4d + 13$, so that $9r_2 - s_2 = -6d^2 + 98$. This last relation, together with $r_2 + s_2 = 6$, gives $5s_2 = 3d^2 - 22 > 0$, and so $3 \leq d \leq 4$. Since 5 divides $3d^2 - 22$, we get that $d = 3$ and $s_2 = 1$. This implies that $r_1 = 3$, $r_2 = 5$, $r_3 = s_3 = s_6 = 0$ and $s_4 = 4$, as the reader can check. In particular, there is no pencil of degree two bimeromorphically equivalent to the pencil of type 3. Moreover, since $\ell = s_2 + s_4 = 5$, Γ_0 has five irreducible components. Let us denote by m_1, m_2, m_3 the three radial singularities, by $m_4, \ldots, m_8$ the five singularities of the type $2:1$ and by $B_1, \ldots, B_5$ the five irreducible components of Γ_0. Set $P = \{m_1, \ldots, m_8\}$ and $L = \{B_1, \ldots, B_5\}$. We choose the order $B_1, \ldots, B_5$ in such a way that $dg(B_j) \leq dg(B_{j+1})$, $1 \leq j \leq 4$. Recall that for a generic $\alpha \in \overline{\mathbb{C}}$ and for each $j = 1, \ldots, 5$, B_j contains a reduced singularity $q_j(\alpha)$, such that $\alpha \mapsto q_j(\alpha) \in B_j$ is a regular parametrization of B_j. We will see before, that we can suppose that $q_1(\alpha)$ is of the type $1 : -2$ and that $q_j(\alpha)$ is of the type $1 : -4$ for $j \geq 2$. We assert that the configuration of points and curves (P, L) satisfies the following properties :

(I). B_1, B_2, B_3 are straight lines and B_4, B_5 are conics. Moreover, each line contains four singularities and each conic contains six singularities of F_α, for a generic $\alpha \in \overline{\mathbb{C}}$.

(II). $m_1, m_2, m_3 \in B_1$ and $\emptyset \neq B_1 \cap B_2 \cap B_3 \subset \{m_1, m_2, m_3\}$, so that we can suppose that $B_1 \cap B_2 \cap B_3 = \{m_1\}$. In particular, $sing(F_\alpha) \cap B_1 = \{q_1(\alpha), m_1, m_2, m_3\}$.

(III). Besides m_1, B_2 (resp. B_3) contains two singularities of the type $2:1$, so that we can suppose that $sing(F_\alpha) \cap B_2 = \{q_2(\alpha), m_1, m_4, m_5\}$ (resp. $sing(F_\alpha) \cap B_3 = \{q_3(\alpha), m_1, m_6, m_7\}$).

(IV). The lines B_2 and B_3 are tangent to the conics B_4 and B_5. Moreover, we can order the points $m_4, \ldots, m_7$ in such a way that $B_2 \cap B_4 = \{m_4\}$, $B_2 \cap B_5 = \{m_5\}$, $B_3 \cap B_4 = \{m_6\}$ and $B_3 \cap B_5 = \{m_7\}$.

(V). $B_4 \cap B_5 = \{m_2, m_3, m_8\}$, where m_8 is a point of tangency and B_4, B_5 are transverse at m_2, m_3. In particular, $sing(F_\alpha) \cap B_4 = \{q_4(\alpha), m_2, m_3, m_4, m_5, m_8\}$ and $sing(F_\alpha) \cap B_5 = \{q_4(\alpha), m_2, m_3, m_6, m_7, m_8\}$.

Observe first that $dg(B_1) = dg(B_2) = dg(B_3) = 1$ and that, either $dg(B_4) = dg(B_5) = 2$, or $dg(B_4) = 1$ and $dg(B_5) = 3$. This follows from $dg(\Gamma_0) = 7$, $\Gamma_0 = \sum_{j=1}^{5} B_j$ and $dg(B_j) \leq dg(B_{j+1})$, as the reader can check. Note also that m_1, m_2 and m_3, are aligned, for otherwise the pencil would be bimeromorphically equivalent to an elliptic, non-degenerate, exceptional pencil of degree two (by Lemma 3.4.14), which is not possible. The straight line that contains m_1, m_2, m_3 is invariant for every foliation F_α, so that it is contained in Γ_0, by Lemma 3.4.1, and we can suppose that this line is B_1. By 3.1.8, each line contains four singularities of F_α, for a generic α. On the other hand, Camacho-Sad Theorem implies that $q_1(\alpha)$ is of the type $1 : -2$: since $I(F_\alpha, B_1, m_j) = 1$, $j = 1, 2, 3$, we get that $1 = I(F_\alpha, B_1, q_1(\alpha)) + 3$, so that $I(F_\alpha, B_1, q_1(\alpha)) = -2$ and $q_1(\alpha)$ is of the type $1 : -2$. Since $s_2 = 1$ and $s_4 = 4$, we get that $q_j(\alpha)$ is of the type $1 : -4$, $j = 2, 3, 4, 5$. Let us prove that $dg(B_4) = dg(B_5) = 2$.

Suppose by contradiction that $dg(B_4) = 1$ and $dg(B_5) = 3$. Consider a straight line B_j, $j \geq 2$, and set $sing(F_\alpha) \cap B_j = \{q_j(\alpha), m_{k_1}, m_{k_2}, m_{k_3}\}$. Observe that $I(F_\alpha, B_2, q_j(\alpha)) = -4$ and $I_i := I(F_\alpha, B_2, m_{k_i}) \in \{1, 2, 1/2\}$, $i = 1, 2, 3$. If we choose k_1, k_2, k_3 in such a way that $I_1 \leq I_2 \leq I_3$, then we get $I_1 = 1$ and $I_2 = I_3 = 2$, as the reader can check by using Camacho-Sad Theorem. Hence, m_{k_2} and m_{k_3} are of the type $2:1$. It follows from (b) of Lemma 3.4.8 that the curve B_5, which is the unique component of degree > 1 of Γ_0, must be tangent to B_j at the points m_{k_2} and m_{k_3}. This implies that $B_j . B_5 \geq 4$. But, $B_j . B_5 = 3$, because $dg(B_j) = 1$ and $dg(B_5) = 3$. This contradiction implies that $dg(B_4) = dg(B_5) = 2$. Note that we have proved also that B_j, $j = 2, 3$, contains one singularity of the type $1:1$ and two of the type $2:1$.

Now, we have two possibilities, either $B_1 \cap B_2 \cap B_3 \neq \emptyset$, or $B_1 \cap B_2 \cap B_3 = \emptyset$. Suppose by contradiction that $B_1 \cap B_2 \cap B_3 = \emptyset$. In this case, $B_2 \cap B_3$ is one of the points m_j, $4 \leq j \leq 8$, because $m_1, m_2, m_3 \in B_1$. This implies that B_2 and B_3 meet transversely at m_j and this contradicts the fact that the germ of Γ_0 consists of two tangent branches ((b) of Lemma 3.4.8). Hence, $B_1 \cap B_2 \cap B_3$ consists of one radial singularity, and so we can suppose that $B_1 \cap B_2 \cap B_3 = \{m_1\}$. Note that $m_1 \notin B_4 \cup B_5$, because the germ of Γ_0 at m_1 contains exactly three different branches ((a) of Lemma 3.4.8), and these branches are contained in $B_1 \cup B_2 \cup B_3$.

We can chose the order m_j, $4 \leq j \leq 8$, in such a way that $sing(F_\alpha) \cap B_2 = \{q_2(\alpha), m_1, m_4, m_5\}$ and $sing(F_\alpha) \cap B_3 = \{q_3(\alpha), m_1, m_6, m_7\}$. Since $m_1 \notin B_4 \cup B_5$ and $sing(\Gamma_0) = \{m_1, ..., m_8\}$, we get that $B_4 \cap B_2 \subset \{m_4, m_5\}$. Note that the germ of Γ_0 at m_4 and m_5 contains two tangent branches at each one of these points, because they are of the type 2 : 1. This implies that $B_4 \cap B_2$ contains just one of these points, because otherwise we would have $B_2.B_4 \geq 4$, whereas $B_2.B_4 = 2$. Hence, we can suppose that $B_2 \cap B_4 = \{m_4\}$ and B_4 is tangent to B_2 at m_4. Analogously, we can suppose that $B_3 \cap B_4 = \{m_6\}$ and B_4 is tangent to B_3 at m_6. This implies that B_4 is a conic tangent to the two lines B_2 and B_3 at m_4 and m_6, respectively. Similarly, B_5 is a conic tangent to the lines B_2 and B_3 at the points m_5 and m_7, respectively. Note that $m_8 \in B_4 \cap B_5$. Since m_8 is of the type 2 : 1, B_4 and B_5 are tangent at m_8, by (b) of Lemma 3.4.8. On the other hand, $B_4.B_5 = 4$ and $[B_4, B_5]_{m_8} = 2$, so that $B_4 \cap B_5$ must contain two other points, which are m_2 and m_3, where B_4 and B_5 meet transversely, because m_2 and m_3 are of the type 1 : 1. From this, we get that $sing(F_\alpha) \cap B_4 = \{q_4(\alpha), m_2, m_3, m_4, m_6, m_8\}$ and $sing(F_\alpha) \cap B_5 = \{q_5(\alpha), m_2, m_3, m_5, m_7, m_8\}$. This finishes the proof of (I),...,(V).

Now, consider a homogeneous coordinate system $[x : y : z]$ in $\mathbb{C}P(2)$ such that $B_1 = \{z = 0\}$, $m_2 = [1 : i : 0]$ and $m_3 = [1 : -i : 0]$. This implies that, in the affine coordinate system $\{z = 1\}$, B_1 is the line at infinity and that for $j = 4, 5$, B_j has an equation of the form $f_j(x, y) = P_j(x, y) + x^2 + y^2$, where P_j is of degree one, $j = 4, 5$. Note that in this coordinate system, the lines B_2 and B_3 are parallel, because they meet at $m_1 \in B_1$. After a translation in the plane (x, y), we can suppose that the tangency point between B_4 and B_5 is $(0, 0)$, so that $P_1(0, 0) = P_2(0, 0) = 0$ and $dP_1(0, 0) \wedge dP_2(0, 0) = 0$. Observe that $dP_j(0, 0) \neq 0$, $j = 1, 2$. Hence, after a linear change of variables of the form $(x, y) \mapsto (a.x + b.y, -b.x + a.y)$, with $a^2 + b^2 = 1$, we can suppose that $f_j(x, y) = -2a_j.x + x^2 + y^2$, where $a_j \neq 0$, $j = 1, 2$, and $a_1 \neq a_2$. Since the lines B_2 and B_3 are parallel, but not parallel to the direction $\{x = 0\}$, we can suppose that they have equations of the form $y = a.x + A_j$, where $a \in \mathbb{C}$ and $0 \neq A_1 \neq A_2 \neq 0$, $j = 1, 2$. The fact that they are tangent to B_4 and B_5 implies the following relations :

$$(a.A_j - a_i)^2 = A_j^2(1 + a^2)\ ;\ i, j = 1, 2 \quad \Rightarrow \quad (a.A_j - a_1)^2 = (a.A_j - a_2)^2\ ;\ j = 1, 2 \quad \Rightarrow$$

$a_1 + a_2 = 2a.A_1 = 2a.A_2$. Since $0 \neq A_1 \neq A_2 \neq 0$, we get that $a = 0$, $a_1 = -a_2$ and $A_1^2 = A_2^2 = a_1^2$. After a linear change of variables of the form $(x, y) \mapsto (\lambda.x, \lambda.y)$, $\lambda \neq 0$, we can suppose that $a_1 = -1$ and $a_2 = 1$, so that $f_1(x, y) = (x + 1)^2 + y^2 - 1$ and $f_2(x, y) = (x - 1)^2 + y^2 - 1$. This implies that $A_1^2 = A_2^2 = 1$ and we can suppose that $B_2 = \{y = 1\}$ and $B_3 = \{y = -1\}$. In these coordinates Γ_0 coincides with the divisor of tangencies of the pencil $(F_\alpha^{3:1})_{\alpha \in \overline{\mathbb{C}}}$, of §2.4. This finishes the proof of Theorem 2. □

References

[Ar] V. Arnold : "Chapitres Supplémentaires de la Théorie des Équations Différentielles Ordinaires" ; Éditions MIR, 1980.

[BPV] W. Barth, C. Peters, A. Van de Ven : "Complex Compact Surfaces"; Springer Verlag (1984).

[Br] M. Brunella : "Birational geometry of foliations" ; text book for a course in the First Latin American Congress of Mathematics, IMPA (2000).

[Br-1] M. Brunella : "Feuilletages Holomorphes sur les Surfaces Complexes Compactes" ; Ann. Scient. Éc. Norm. Sup. , 4e série, t. 30, 1997, p. 569-594.

[Br-2] M. Brunella : "Birational geometry of foliations" ; text book for a course in the First Latin American Congress of Mathematics, IMPA (2000).

[Br-3] M. Brunella : "Courbes entières et feuilletages holomorphes" ; L'Enseignement Mathématique, tm. 45, fasc. 1-2, pg. 196-216 (1999).

[C-LN] C. Camacho e A. Lins Neto : "Geometric theory of foliations"; Birkhauser 1985.

[C-LN-S] C. Camacho, A. Lins Neto e P. Sad: "Topological invariants and equidesingularization for holomorphic vector fields"; J. of Diff. Geometry, vol. 20, nº 1 (1984), pg. 143–174.

[C-S] C. Camacho e P. Sad: "Invariant varieties through singularities of holomorphic vector fields"; Ann. of Math. 115 (1982), pg. 579-595.

[Eh] C. Ehresmann, G. Reeb: "Les connexions infinitésimales dans un space fibré différentiable" ; Coloque de topologie, Bruxelles, june 1950, p. 29-55.

[G-H] Griffiths-Harris : "Principles of Algebraic Geometry"; John-Wiley and Sons, 1994.

[G-M] X. Gomez-Mont : "The Transverse Dynamics of a Holomorphic Flow"; Annals of Mathematics 127 (1988) pg. 49-92.

[K] K. Kodaira : "On Compact Analytic Surfaces: II" ; Annals of Mathematics, vol. 77, n. 3 (1963).

[LN] A Lins Neto: "Some examples for the Poincaré and Painlevé problems". Ann. Scient. Éc. Norm. Sup., 4e série, t. 35, 2002, p. 231-266.

[LN 1] A. Lins Neto: "Algebraic solutions of polynomial differential equations e foliations in dimension two"; Lect. Notes em Math. nº 1345, pg. 192-231.

[McQ] M. McQuillan : "Noncommutative Mori theory", preprint IHES (2000).

[M-M] J.F. Mattei e R. Moussu: "Holonomie et intégrales premières"; Ann. Sc. Ec. Norm. Sup. 13 (1980), pg. 469-523.

[Re] G. Reeb : "Sur certains propriétés topologiques des variétés feuilletés"; Actualités Sci. Ind., 1183, Herman, Paris, 1952.

[P-1] H. Poincaré : "Sur l'intégration algébrique des équations diff."; Comptes rendus de l'academie des sciences, t. 112, pp. 761-764.

[P-2] H. Poincaré : "Sur l'intégration algébrique des équations diff. du 1er ordre et du 1er degré"; Rendiconti del circolo matemàtico di Palermo, t. 5, pp. 161-191 (1891).

[P-3] H. Poincaré : "Sur l'intégration algébrique des équations diff. du 1er ordre et du 1er degré"; Rendiconti del circolo matemàtico di Palermo, t. 11, pp. 193-239 (1897).

[S] T. Suwa : "Indices of vector fields and residues of holomorphic singular foliations"; Hermann (1998).

[Se] A. Seidenberg: "Reduction of singularities of the differential equation Ady = Bdx"; Amer. J. de Math. 90 (1968), pg. 248-269.

A. Lins Neto
Instituto de Matemtica Pura e Aplicada
Estrada Dona Castorina, 110
Horto, Rio de Janeiro, Brasil
E-Mail - alcides@impa.br