On Schrödinger-Boussinesq Equations

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Abstract. We study local and global well-posedness for the initial value problem associated to the one-dimensional Schrödinger–Boussinesq equations in low regularity spaces. To establish these results we make use of sharp $L^p - L^q$ estimates.

1. Introduction

In this paper we consider the initial value problem (IVP) associated to the system of nonlinear partial differential equations called the Schrödinger-Boussinesq equations, that is,

$$\left\{\begin{array}{ll} i\partial_t u + \partial_x^2 u = uv + \alpha|u|^2 u, & x \in \mathbb{R}, \quad t > 0, \\ \partial_t^2 v - \partial_x^2 v + \partial_x^4 v = \partial_x^2(\beta|v|^{p-1}v + |u|^2), & \\ u(x,0) = u_0(x), & \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & \end{array}\right. \tag{1.1}$$

where the function u is a complex valued and v is a real valued function, $p > 1$ and α and β are real parameters.

The system above appears in the study of interaction of solitons in optics (see [15], [16]). Both, the nonlinear Schrödinger equation (see, for example, [4], [6], [7], [19] and for a complete set of references [3]) and Boussinesq equation ([2], [5], [12], [13], [14], [17]) have been extensively studied but the system above have been treated more under dissipative effects and the presence of attractors ([1]).

Our purpose here is to establish local and global well-posedness results for the IVP (1.1) in the spaces $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$. Before giving our results we will begin by setting up the initial data in our problem.

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We first observe that the system (1.1) can be written as

$$\begin{cases} i\partial_t u + \partial_x^2 u = uv + \alpha |u|^2 u, & x \in \mathbb{R}, \quad t > 0, \\ \partial_t v = \partial_x n, \\ \partial_t n = \partial_x (v - \partial_x^2 v + \beta |v|^{p-1} v + |u|^2), \end{cases} \tag{1.2}$$

with initial data $u(x,0) = u_0$, $n(x,0) = n_0$, $v(x,0) = v_0$, respectively. Solutions of IVP (1.2) satisfy the following conservation laws (see [1] and references therein)

$$\begin{aligned} K(t) &= \int |u(x,t)|^2 \, dx = K(0), \\ E(t) &= \frac{1}{2} \int (|u|^2 + |\partial_x u|^2 + |\partial_x v|^2) \, dx + \frac{1}{2} \int |(-\Delta)^{-1/2} \partial_t v|^2 \, dx \\ &\quad + \int v|u|^2 dx + \frac{\beta}{p+1} \int |v|^{p+1} \, dx + \frac{\alpha}{2} \int |u|^4 \, dx = E(0). \end{aligned} \tag{1.3}$$

To make sense of the second expression above it is needed for the initial datum n_0 being a derivative of a L^2 function. So we will require in IVP (1.1) $v_1(x) = h'(x)$, that is, the derivative of some function in a suitable space.

To obtain results in low regularity spaces we will use the so called $L^p - L^q$ estimates. These type of estimates were first established by Strichartz ([18]) for solutions of the linear Schrödinger equation, i.e,

$$\begin{cases} \partial_t u = i\Delta u, \quad x \in \mathbb{R}^n, \ t \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{1.4}$$

He showed that solutions of (1.4) satisfy

$$\Big(\int_{\mathbb{R}} \int_{\mathbb{R}^n} |e^{it\Delta} u_0(x)|^{2(n+2)/n} dx dt \Big)^{n/2(n+2)} \leq c \|u_0\|_2.$$

Generalizations of this result have been obtained for several authors. (See, for instance, [6], [11]).

We proceed as follows. Instead of working with the system of nonlinear partial differential equations in (1.1) we use its equivalent integral form, that is,

$$\begin{aligned} u(t) &= U(t) u_0 - i \int_0^t U(t-s) \, (uv + \alpha |u|^2 u)(s) ds, \\ v(t) &= V_1(t) v_0 + V_2(t) v_1 + \int_0^t V_2(t-s) \, \partial_x^2 (\beta |v|^{p-1} v + |u|^2)(s) \, ds, \end{aligned} \tag{1.5}$$

where $U(t)$ is the unitary group associated to the linear Schrödinger equation and $V_1(t)$ and $V_2(t)$ are the linear operators associated to the linear Boussinesq equation to be defined in Section 3.

Then we use the $L^p - L^q$ estimates to show via the contraction mapping principle that there exists a time $T > 0$ where (1.5) has a unique solution.

The linear estimates we obtain for the Boussinesq equation allows us to obtain what we believe are the best possible results concerning local and global well-posedness. The intuition behind our affirmation is based on the recent results obtained by Kenig, Ponce and Vega in [10]. They studied ill-posedness for the initial value problem associated to the cubic Schrödinger equation, that is,

$$\begin{cases} \partial_t u = i\partial_x^2 u + |u|^2 u, & x \in \mathbb{R},\ t \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{1.6}$$

An scaling argument suggests that the best possible local well-posedness result in Sobolev spaces should be for that data in $H^s(\mathbb{R})$, $s \geq -1/2$. On the other hand, Tsutsumi [19] established the local theory for initial data in $L^2(\mathbb{R})$. These two results leave a gap in $[-1/2, 0)$. It was shown in [10] that the IVP (1.6) is in fact ill-posed for data in $H^s(\mathbb{R})$, $s < 0$. Also, there is a closed relation between the Boussinesq equation and the nonlinear Schrödinger equation as the results in [13] shown. So we do not expect to have a better regularity result than that in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ for the system (1.1) given in Theorem 2.1 below.

We also have to notice that in the L^2 case the power $p = 5$ is critical. Thus we have to analyze this problem in a different way.

The plan of the paper is as follows. In the next section we will give the statements of the main results and some remarks. We will recall the linear estimates associated to the Schrödinger as well as Boussinesq linear equations in Section 3. In Section 4, the L^2 theory will be given. The results regarding the critical case in L^2 will be treated in section 5. The H^1 theory will be established in Section 6 and finally, in Section 7 we will deal with the global results.

2. Main Results

In this section we present the statement of the main results in this paper.

Theorem 2.1. *Given $u_0, v_0 \in L^2(\mathbb{R})$ and $v_1 = h' \in H^{-1}(\mathbb{R})$, for $1 < p < 5$ there exist $T = T(p, |\alpha|, |\beta|, \|u_0\|_2, \|v_0\|_2, \|v_1\|_{H^{-1}}) > 0$ and a unique solution (u, v) of the IVP (1.1) satisfying*

$$u,\, v \in C([-T,T]; L^2(\mathbb{R})) \cap L^4([-T,T]; L^\infty(\mathbb{R})).$$

Moreover, for each $(\tilde{u}_0, \tilde{v}_0, \tilde{v}_1) \in L^2 \times L^2 \times H^{-1}$ there exists a neighborhood W of $L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times H^{-1}(\mathbb{R})$ such that the map

$$(u_0, v_0, v_1) \to (u(t), v(t)),$$

corresponding to the solution of the problem (1.1) with initial conditions (u_0, v_0, v_1), is Lipschitz.

Remark 2.2. *The same proof of this theorem shows that solutions u of 1.1 also satisfy*

$$u \in C([-T,T]; L^2(\mathbb{R})) \cap L^q([-T,T]; L^p(\mathbb{R})), \tag{2.1}$$

where (p, q) is an admissible pair (see (3.2) below).

Remark 2.3. *The solutions found in Theorem 2.1 also satisfy the smoothing effect of Kato type (see* [8], [11]*). More precisely, if* u *and* v *are solutions of (1.1) then*

$$D_x^{1/2}u, D_x^{1/2}v \in L^\infty(\mathbb{R} : L^2[0,T]).$$

See Proposition 4.2 below.

Theorem 2.4. *Given* u_0, $v_0 \in L^2(\mathbb{R})$ *and* $v_1 = h' \in H^{-1}(\mathbb{R})$*, there exists* $T > 0$, $T = T(|\alpha|, |\beta|, \|u_0\|_2, v_0, v_1)$*, such that the problem (1.1) with* $p = 5$ *has a unique solution such that*

$$u, v \in C([-T,T]; L^2(\mathbb{R})) \cap L^4([-T,T]; L^\infty(\mathbb{R})).$$

Moreover, there exists a neighborhood W *of* $(\tilde{u}_0, \tilde{v}_0, \tilde{v}_1)$ *in* $L^2 \times L^2 \times H^{-1}$ *such that the map*

$$(u_0, v_0, v_1) \to (u(t), v(t)),$$

corresponding to the solution of the problem (1.1) with initial conditions (u_0, v_0, v_1)*, is Lipschitz.*

Theorem 2.5. *Given* $u_0, v_0 \in H^1(\mathbb{R})$, $v_1 = h' \in L^2(\mathbb{R})$*, there exist*

$$T = T(p, |\alpha|, |\beta|, \|u_0\|_{H^1}, \|v_0\|_{H^1}, \|v_1\|_{L^2}) > 0$$

and a unique solution (u, v) *of IVP (1.1) such that*

$$u, v \in C([-T,T];\ H^1(\mathbb{R})).$$

Moreover, the map $(u_0, v_0, v_1) \to (u(t), v(t))$ *corresponding to the solution of the problem is locally Lipschitz.*

Remark 2.6. *The solutions in Theorem 2.5 also satisfy the following smoothing effects.*

- $L^p - L^q$ *estimates:*

$$u \in C([-T,T]; H^1(\mathbb{R})) \cap L_1^q([-T,T]; L^p(\mathbb{R})),$$
$$v \in C([-T,T]; H^1(\mathbb{R})) \cap L_1^4([-T,T]; L^\infty(\mathbb{R})),$$

  where (p, q) *is an admissible pair (see (3.2) below) and* $L_1^q([-T,T]; L^p(\mathbb{R}))$ *denotes the space whose functions and their first derivatives belong to* $L^q([-T,T]; L^p(\mathbb{R}))$*.*
- *Kato's smoothing effect:*

$$D_x^{3/2}u, D_x^{3/2}v \in L^\infty(\mathbb{R} : L^2[0,T]).$$

  See Proposition 6.2 below.

Theorem 2.7. *Let* $u_0, v_0 \in H^1(\mathbb{R})$, $v_1 = h' \in L^2(\mathbb{R})$*. The IVP (1.1) is globally well-posed in* $H^1(\mathbb{R}) \times H^1(\mathbb{R})$

a) For $\beta > 0$ *and any data.*
b) For $\beta < 0$ *and data sufficiently small.*

Remark 2.8. *In the case $\beta < 0$ we expect that solutions provided for large data blow-up in finite time. The reason to conjecture this is the fact that solutions of the Boussinesq equation blow-up in finite time, for $p > 1$, meanwhile the one-dimensional cubic Schrödinger equation is well behaved (see* [14], [17]*).*

3. Linear Estimates

In this section we will recall a series of estimates obtained for solutions of the linear problem associated to the Schrödinger equation as well as for the Boussinesq linear equation.

First we consider the (IVP) associated to the linear Schrödinger equation, that is,

$$\begin{cases} \partial_t u = i\partial_x^2 u, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), \end{cases} \tag{3.1}$$

whose solution is given by

$$u(t,x) = e^{it\Delta}u_0(x) = (e^{-it\xi^2}\hat{u}_0(\xi))^{\vee}(x).$$

Next we remind some estimates for solutions of the linear Schrödinger equation. We need the following definition.

Definition 3.1. *The pair (q,p) is an admissible pair if q, $p \geq 2$ and satisfies*

$$\frac{2}{q} + \frac{1}{p} = \frac{1}{2}. \tag{3.2}$$

Theorem 3.2. *If (q_0, p_0) and (q_1, p_1) are admissible, then we have the following estimates*

$$\|e^{it\Delta}u_0\|_{L_t^{q_0}L_x^{p_0}} \leq c\,\|u_0\|_{L^2}, \tag{3.3}$$

$$\|\int_{\mathbb{R}} e^{-is\Delta}F(\cdot,s)\,ds\|_{L^2} \leq c\,\|F\|_{L_t^{q_0'}L_x^{p_0'}}, \tag{3.4}$$

$$\|\int_{s<t} e^{i(t-s)\Delta}F(\cdot,s)\,ds\|_{L_t^{q_0}L_x^{p_0}} \leq c\,\|F\|_{L_t^{q_1'}L_x^{p_1'}}, \tag{3.5}$$

where $\frac{1}{p_i} + \frac{1}{p_i'} = \frac{1}{q_i} + \frac{1}{q_i'} = 1,\ i = 0,1$.

Solutions of (3.1) also satisfy the Kato smoothing effect

$$\sup_x \Big(\int_{\mathbb{R}} |D_x^{1/2}e^{it\Delta}f(x)|^2 dt\Big)^{1/2} \leq c\|f\|_2. \tag{3.6}$$

The proof of the estimates (3.3)–(3.5) can be found in [6]. Estimate (3.6) was proved in [11].

Next we consider the (IVP) associated to the Boussinesq linear equation:

$$\begin{cases} \partial_t^2 v - \partial_x^2 v + \partial_x^4 v = 0, \quad x \in \mathbb{R},\ t > 0, \\ v(x,0) = f(x), \\ \partial_t v(x,0) = g(x). \end{cases} \tag{3.7}$$

Using Fourier transform we obtain formally

$$v(x,t) = V_1(t)f + V_2(t)g,$$

where

$$\begin{aligned} V_1(t)f(x) &= \frac{1}{2}(e^{it\phi(\xi)}\hat{f}(\xi))^{\vee}(x) + \frac{1}{2}(e^{-it\phi(\xi)}\hat{f}(\xi))^{\vee}(x), \\ V_2(t)g(x) &= \frac{1}{2i}\Big(e^{it\phi(\xi)}\frac{\hat{g}(\xi)}{\phi(\xi)} - e^{-it\phi(\xi)}\frac{\hat{g}(\xi)}{\phi(\xi)}\Big)^{\vee}(x), \end{aligned} \tag{3.8}$$

with $\phi(\xi) = |\xi|(1+\xi^2)^{\frac{1}{2}}$.

Lemma 3.3. *For the operators $V_1(t)$ and $V_2(t)$ defined above we have the following estimates:*

$$\|V_1(t)f\|_2 \leq \|f\|_2, \tag{3.9}$$

$$\|V_2(t)\partial_x f\|_2 \leq c\|f\|_{H^{-1}}, \tag{3.10}$$

$$\|V_2(t)\partial_x^2 f\|_2 \leq c\|f\|_2. \tag{3.11}$$

The operators $V_1(t)$ and $V_2(t)$ also satisfy estimates of $L^p - L^q$ type similar to those of the solution of the linear Schrödinger equation. The proof in this case is more complicated. These estimates were obtained in [13] by using the oscillatory integrals theory developed in [11].

Lemma 3.4. *For $f \in L^2(\mathbb{R})$ we have*

$$\Big(\int_0^T \|V_1(t)f\|_\infty^4\,dt\Big)^{1/4} \leq C(1+T^{1/4})\|f\|_2, \tag{3.12}$$

$$\Big(\int_0^T \|V_2(t)\partial_x f\|_\infty^4\,dt\Big)^{1/4} \leq C(1+T^{1/4})\|f\|_{H^{-1}}, \tag{3.13}$$

$$\Big(\int_0^T \|V_2(t)\partial_x^2 f\|_\infty^4\,dt\Big)^{1/4} \leq C\|f\|_2. \tag{3.14}$$

Proof. See Lemma 2.5 in [13]. □

Next we have Kato's smoothing effect estimates satisfied by solutions of (3.7).

Lemma 3.5. *The following estimates are satisfied*

$$
\begin{aligned}
&\sup_x\Big(\int_0^T |D_x^{1/2}V_1(t)f(x)|^2\,dt\Big)^{1/2} \le c(1+T^{1/2})\|f\|_2,\\
&\sup_x\Big(\int_0^T |D_x^{1/2}V_2(t)\partial_x f(x)|^2\,dt\Big)^{1/2} \le c(1+T^{1/2})\|f\|_{H^{-1}},\\
&\sup_x\Big(\int_0^T |D_x^{1/2}V_2(t)\partial_x^2 f(x)|^2\,dt\Big)^{1/2} \le c(1+T^{1/2})\|f\|_2.
\end{aligned}
\tag{3.15}
$$

Proof. See references [11] and [13]. □

To end this section we give some estimates on the operator $\Gamma := (-\Delta)^{-1/2}\partial_t$ needed in the proof of global existence of solutions.

Lemma 3.6. *Let*

$$
\begin{aligned}
\Gamma V_1(t)f(x) &= \int_{\mathbb{R}} ie^{i(t\phi(\xi)+x\xi)}|\xi|^{-1}\phi(\xi)\hat{f}(\xi)\,d\xi,\\
\Gamma V_2(t)\partial_x f(x) &= \int_{\mathbb{R}} e^{i(t\phi(\xi)+x\xi)}\,\frac{isgn(\xi)\phi(\xi)\hat{f}(\xi)}{|\xi|(1+\xi^2)^{1/2}}\,d\xi,\\
\Gamma V_2(t)\partial_x^2 f(x) &= \int_{\mathbb{R}} e^{i(t\phi(\xi)+x\xi)}\,\frac{isgn(\xi)\phi(\xi)\,\hat{f}(\xi)}{(1+\xi^2)^{1/2}}\,d\xi.
\end{aligned}
\tag{3.16}
$$

Then we have

$$
\begin{aligned}
\|\Gamma V_1(t)f(x)\|_2 &\le c\|f\|_{1,2},\\
\|\Gamma V_2(t)\partial_x f(x)\|_2 &\le c\|f\|_2,\\
\|\Gamma V_2(t)\partial_x^2 f(x)\|_2 &\le \|f\|_{1,2}.
\end{aligned}
\tag{3.17}
$$

4. Local Theory in L^2

In this section we consider the initial value problem (1.1) with data $u_0, v_0 \in L^2(\mathbb{R})$ and $v_1 = h' \in H^{-1}(\mathbb{R})$. Our purpose is to prove Theorem 2.1. To do so we define an integral operator and a convenient metric space where this integral operator turns out to be a contraction operator. Using the contraction mapping principle we obtain the desired result. In the second part of this section we will show some smoothness properties present in solutions of (1.1).

We begin by defining the operator

$$
\Phi(u,v) = (\Phi_1(u,v), \Phi_2(u,v)),
\tag{4.1}
$$

where

$$
(4.2)\qquad \begin{cases} \Phi_1(u,v) = e^{it\Delta}u_0 - i\displaystyle\int_0^t e^{i(t-s)\Delta}(uv+\alpha|u|^2u)(s)\,ds,\\ \Phi_2(u,v) = V_1(t)v_0 + V_2(t)h' + \displaystyle\int_0^t V_2(t-s)\partial_x^2(\beta|v|^{p-1}v+|u|^2)\,ds. \end{cases}
$$

Consider the function space

$$
(4.3)\qquad E_+(T,a) = \begin{cases} (u,v) : u \in C([0,T] : L^2(\mathbb{R})) \cap L^4([0,T] : L^\infty(\mathbb{R}));\\ \qquad v \in C([0,T] : L^2(\mathbb{R})) \cap L^4([0,T] : L^\infty(\mathbb{R}));\\ \qquad \text{and } \ |||(u,v)||| \le a, \end{cases}
$$

where

$$
|||(u,v)||| = \max\Big\{ \sup_{[0,T]} \|u(t)\|_2,\ \|u\|_{L^4_T L^\infty_x},\ \sup_{[0,T]}\|v(t)\|_2,\ \|v\|_{L^4_T L^\infty_x}\Big\}.
$$

It is not difficult to show that $E_+(T,a)$ is a complete metric space.

Proposition 4.1. *There exist a and T positive, depending only on $\|u_0\|_2$, $\|v_0\|_2$, $\|v_1\|_{H^{-1}}$, and p,α,β in an appropriated manner, such that if $(u,v) \in E_+(T,a)$ then $\Phi(u,v) \subset E_+(T,a)$ and the map*

$$
\Phi : E_+(T,a) \to E_+(T,a)
$$

is a contraction.

Proof. We first estimate Φ_1. By Minkowski's inequality, group properties and Hölder's inequality it follows that

$$
(4.4)\qquad \sup_{[0,T]} \Big\| \int_0^t e^{i(t-\tau)\Delta}(uv)\,d\tau\Big\|_2 \le \sup_{[0,T]} \int_0^t \|uv\|_2\,d\tau \le cT^{3/4} \sup_{[0,T]}\|u\|_2\,\|v\|_{L^4_T L^\infty_x}.
$$

The same argument used in (4.4) produces

$$
(4.5)\qquad \sup_{[0,T]} \Big\| \int_0^t e^{i(t-\tau)\Delta}|u|^2u\,d\tau\Big\|_2 \le cT^{1/2} \sup_{[0,T]}\|u\|_2^2\,\|u\|_{L^4_T L^\infty_x}.
$$

Combining (4.4), (4.5), group properties and the definition (4.2) we get

$$
(4.6)\qquad \sup_{[0,T]} \|\Phi_1(u,v)(t)\|_2 \le c\|u_0\|_2 + cT^{3/4}\,|||(u,v)|||^2 + c|\alpha|\,T^{1/2}\,|||(u,v)|||^3.
$$

On the other hand, Minkowski's inequality, estimate (3.3), (4.4) and (4.5) give

$$
(4.7)\qquad \|\Phi_1(u,v)\|_{L^4_T L^\infty_x} \le c\|u_0\|_2 + cT^{3/4}\,|||(u,v)|||^2 + c|\alpha|\,T^{1/2}\,|||(u,v)|||^3.
$$

Now we estimate Φ_2.

From Minkowski's inequality, estimate (3.10) and Hölder's inequality it follows that

$$
\begin{aligned}
&\| \int_0^T V_2(t-\tau)\partial_x^2(\beta|v|^{p-1}v+|u|^2)\,d\tau\|_2 \\
&\leq \int_0^T |\beta|\,\||v|^{p-1}v\|_2 + \||u|^2\|_2\,d\,\tau \\
&\leq c\,|\beta| T^{(5-p)/4}\sup_{[0,T]}\|v\|_2\,\|v\|^{p-1}_{L^4_T L^\infty_x} + c\,T^{3/4}\sup_{[0,T]}\|u\|_2\,\|u\|_{L^4_T L^\infty_x}.
\end{aligned} \tag{4.8}
$$

Inequality (4.8) combined with the estimates (3.8) and (3.9) yield

$$
\begin{aligned}
&\sup_{[0,T]}\|\Phi_2(u,v)(t)\|_2 \\
&\qquad \leq c\|v_0\|_2 + c\|h\|_{H^{-1}} + c|\beta|T^{(5-p)/4}\,|||(u,v)|||^p + cT^{3/4}\,|||(u,v)|||^2.
\end{aligned} \tag{4.9}
$$

Using Minkoswki's inequality, estimate (3.14) and Hölder's inequality we obtain

$$
\begin{aligned}
&\| \int_0^T V_2(t-\tau)\partial_x^2(\beta|v|^{p-1}v+|u|^2)\,d\tau\|_{L^4_T L^\infty_x} \\
&\qquad \leq c\,(|\beta| T^{(5-p)/4}\sup_{[0,T]}\|v\|_2\,\|v\|^{p-1}_{L^4_T L^\infty_x} + C\,T^{3/4}\sup_{[0,T]}\|u\|_2\,\|u\|_{L^4_T L^\infty_x}).
\end{aligned} \tag{4.10}
$$

The last estimate combined with estimates (3.12) and (3.13) imply

$$
\begin{aligned}
\|\Phi_2(u,v)\|_{L^4_T L^\infty_x} &\leq c(1+T^{1/4})\|v_0\|_2 + c(1+T^{1/4})\|h\|_{H^{-1}} \\
&\qquad + c|\beta|\,T^{\frac{5-p}{4}}\,|||(u,v)|||^p + C\,T^{3/4}|||(u,v)|||^2.
\end{aligned} \tag{4.11}
$$

Now let $a = 4c\delta$ where $\delta = \max\{\delta_1,\ \delta_2,\ \delta_3\}$ with $\|u_0\|_2 \leq \delta_1$, $\|v_0\|_2 \leq \delta_2$ and $\|h\|_{H^{-1}} \leq \delta_3$. From (4.6), (4.7), (4.9) and (4.11) it follows

$$
\begin{aligned}
&\sup_{[0,T]}\|\Phi_1(u,v)\|_2 \leq 2c\delta\,(1 + c^3 2^3\delta T^{3/4} + 2^5 c^3|\alpha|\delta^2 T^{1/2}), \\
&\|\Phi_1(u,v)\|_{L^4_T L^\infty_x} \leq 2c\delta\,(1 + c^3 2^3\delta T^{3/4} + 2^5 c^3|\alpha|\delta^2 T^{1/2}), \\
&\sup_{[0,T]}\|\Phi_2(u,v)\|_2 \leq 2c\delta\,(1 + c^p 2^{p-1}|\beta|\delta^{p-1}T^{5-p/4} + 2c^2\delta T^{3/4}), \\
&\|\Phi_2(u,v)\|_{L^4_T L^\infty_x} \leq 2c\delta\,(1 + T^{1/4} + c^p 2^{p-1}|\beta|\delta^{p-1}T^{5-p/4} + 2c^2\delta T^{3/4}).
\end{aligned} \tag{4.12}
$$

Therefore fixing T such that

$$
\begin{aligned}
&c^3 2^3\delta T^{3/4} + 2^5 c^3|\alpha|\delta^2 T^{1/2} < 1, \\
&c^p 2^{p-1}|\beta|\delta^{p-1}T^{5-p/4} + 2c^2\delta T^{3/4} < 1, \\
&T^{1/4} + c^p 2^{p-1}|\beta|\delta^{p-1}T^{5-p/4} + 2c^2\delta T^{3/4} < 1,
\end{aligned} \tag{4.13}
$$

the first part of the proposition follows.

Using a similar argument we can also establish the next inequalities

$$\begin{aligned}
&|||\Phi_1(u,v)-\Phi_1(\tilde{u},\tilde{v})||| \leq \{cT^{3/4}(|||(u,v)|||+|||(\tilde{u},\tilde{v})|||),\\
&\qquad\qquad + c|\alpha|\,T^{1/2}(|||(u,v)|||^2+|||(\tilde{u},\tilde{v})|||^2)\}|||(u,v)-(\tilde{u},\tilde{v})|||,\\
&|||\Phi_2(u,v)-\Phi_2(\tilde{u},\tilde{v})||| \leq \{c|\beta|\,T^{5-p/4}(|||(u,v)|||^{p-1}+|||(\tilde{u},\tilde{v})|||^{p-1})\\
&\qquad\qquad + c\,T^{3/4}(|||(u,v)|||+|||(\tilde{u},\tilde{v})|||)\}|||(u,v)-(\tilde{u},\tilde{v})|||.
\end{aligned} \tag{4.14}$$

The same argument used in (4.12) and (4.13) shows then that the operator Φ is a contraction in $E_+(T,a)$. □

Proof of Theorem 2.1. The above proposition proves existence, uniqueness and local Lipschitz dependence with respect to the initial data in the space $E_+(T,a)$. Using uniqueness we can extend the result in the space $C([-T,T];L^2(\mathbb{R}))\cap L^4([-T,T];L^\infty(\mathbb{R}))$. This proves the theorem. □

Next we establish some regularity properties for solutions of IVP (1.1). More precisely, solutions of (1.1) satisfy Kato's smoothing effect estimates. That is,

Proposition 4.2. *If (u,v) is a solution of the system (1.1) with initial data $(u_0,v_0,\ v_1)$ $\in L^2\times L^2\times H^{-1}$, then*

$$D_x^{1/2}u,\ D_x^{1/2}v\in L^\infty(\mathbb{R}:L^2[0,T]). \tag{4.15}$$

Proof. From Theorem 2.1 we have that a solution of (1.1) satisfies

$$u(x,t)=e^{it\Delta}u(0)-i\int_0^t e^{i(t-s)\Delta}\partial_x^2\,(uv+\alpha|u|^2u)(s)\,ds.$$

From Theorem 3.2 (3.6) and an argument as in (4.4) it follows that

$$\begin{aligned}
&\sup_x\|D_x^{1/2}u(x,\cdot)\|_{L_t^2}\leq c\|u_0\|_2+\int_0^T\|D_x^{1/2}\,e^{i(t-s)\Delta}(uv+\alpha|u|^2u)(s)\|_{L_x^\infty L_t^2}\,ds\\
&\leq C\|u_0\|_2+C\int_0^T\|uv+\alpha|u|^2u\|_{L^2}\,ds\\
&\leq C\|u_0\|_2+cT^{3/4}\,\|u\|_{L_T^4L_x^\infty}\,\|v\|_{L_T^\infty L^2}+CT^{1/2}\,|\alpha|\,\|u\|^2_{L_T^4L_x^\infty}\,\|u\|_{L_T^\infty L^2}.
\end{aligned} \tag{4.16}$$

On the other hand,

$$v(x,t)=V_1(t)v_0+V_2(t)v_1+\int_0^t V_2(t-s)\,\partial_x^2(\beta|v|^{p-1}v+|u|^2)(s)ds.$$

Applying Lemma 3.5, Minkowski's inequality and Hölder's inequality we have

$$\begin{aligned}
&\|D_x^{1/2}v\|_{L_x^\infty L_T^2}\leq c(1+T^{1/2})\|v_0\|_2+c(1+T^{1/2})\|h\|_{H^{-1}}\\
&\qquad + c|\beta|T^{(5-p)/4}\,\|v\|^{p-1}_{L_T^4L_x^\infty}\|v\|_{L_T^\infty L_x^2}+cT^{4/3}\,\|u\|_{L_T^4L_x^\infty}\|u\|_{L_T^\infty L_x^2}.
\end{aligned} \tag{4.17}$$

This completes the proof of the proposition. □

5. Critical Case in L^2

In what follows we analyze the critical case $p = 5$ for initial data $u_0 \in L^2$, $v_0 \in L^2$ and $v_1 = h' \in H^{-1}$. We prove that in this case there is still a local solution, but the time of existence of the solution depends not only on the size of the initial data, but also on their position.

Proof. We consider the complete metric space:

$$E_+(T,b) = \begin{cases} (u,v) : u,v \in C([0,T];L^2(\mathbb{R})) \cap \in L^4([0,T];L^\infty(\mathbb{R})); \\ \qquad \sup\limits_{[0,T]} \|u(t)\|_2 \le b,\ \|u\|_{L^4_T L^\infty_x} \le b,\ \|v\|_{L^4_T L^\infty_x} \le b; \\ \qquad \sup\limits_{[0,T]} \|v(t) - \tilde{v}(t)\|_2 \le b, \end{cases} \tag{5.1}$$

where $\tilde{v}(t) = V_1(t)v_0 + V_2(t)v_1$.

As in the non critical case, we will prove that for $\|u_0\|_2 \le b$ and sufficiently small T, the operator Φ is well defined in $E_+(T,b)$ and is a contraction. The difference now is on the estimates of the operator Φ_2. Observe that from the definition of the space $E_+(T,b)$, we only have to estimate for Φ_2 the $\|\cdot\|_{L^4_T L^\infty_x}$ norm.

We begin with the simple observation that if $\lambda > 0$ then, for sufficiently small T, we have:

$$\|V_1(t)v_0\|_{L^4_T L^\infty_x} \le \lambda, \qquad \|V_2(t)v_1\|_{L^4_T L^\infty_x} \le \lambda. \tag{5.2}$$

Then by Minkowski's inequality and the argument used in (4.10) we have

$$\begin{aligned} \|\Phi_2(u,v)\|_{L^4_T L^\infty_x} &\le 2\lambda + c\,|\beta| \int_0^T \|v^5\|_2\, d\tau + c \int_0^T \||u|^2\|_2\, d\tau \\ &\le 2\lambda + c\,|\beta| \int_0^T (\|v^5 - v^4\tilde{v}\|_2 + \|v^4\tilde{v}\|_2)\, d\tau + c\,T^{3/4} \sup_{[0,T]} \|u\|_2 \|u\|_{L^4_T L^\infty_x} \\ &\le 2\lambda + c\,|\beta| (\sup_{[0,T]} \|v - \tilde{v}\|_2 + \sup_{[0,T]} \|\tilde{v}\|_2) \|v\|^4_{L^4_T L^\infty_x} + c\,T^{3/4} \sup_{[0,T]} \|u\|_2 \|u\|_{L^4_T L^\infty_x}. \end{aligned}$$

Thus

$$\|\Phi_2(u,v)\|_{L^4_T L^\infty_x} \le 2\lambda + c\,T^{3/4} b^2 + c\,|\beta|(b+M)b^4,$$

where

$$M := \sup_{[0,T]} \|\tilde{v}\|_2.$$

Setting $b = 4\lambda$ and choosing λ and T small enough such that

$$2^4 c\,T^{3/4}\lambda^2 + 2^8|\beta|(4\lambda + M)\lambda^4 \le 2\lambda \tag{5.3}$$

we obtain

$$\|\Phi_2(u,v)\|_{L^4_T L^\infty_x} \le 4\lambda = b. \tag{5.4}$$

We also have the estimate

$$\|\Phi_2(u,v)-\tilde{v}\| \le c\,|\beta|(\sup_{[0,T]}\|v-\tilde{v}\|_2 \sup_{[0,T]}\|\tilde{v}\|_2)\|v\|^4_{L^4_T L^\infty_x} + c\,T^{3/4}\sup_{[0,T]}\|u\|_2\|u\|_{L^4_T L^\infty_x}.$$

Therefore

$$\sup_{[0,T]}\|\Phi_2(u,v)-\tilde{v}\|_2 \le c\,T^{3/4}b^2 + c|\beta|(b+M)b^4 \le 2\lambda \le b. \tag{5.5}$$

From (5.3), (5.4) and the estimates for $\|\Phi_1(u,v)\|$ similar to those in Proposition 4.1 we obtain that Φ is well defined (observe that these estimates do not use the $\sup_{[0,T]}\|u\|_2$ norm.)

To see that Φ is a contraction, we use (4.10) to obtain

$$\|\Phi_2(u,v)(t)-\Phi_2(\bar{u},\bar{v})(t)\|_2 \le c(|\beta|\,b^4 + T^{3/4}\,b)(\sup_{[0,T]}\|v-\bar{v}\|_2 + \sup_{[0,T]}\|u-\bar{u}\|_2). \tag{5.6}$$

Finally, following the estimate in (4.11) we obtain

$$\|\Phi_2(u,v)-\Phi_2(\bar{u},\bar{v})\|_{L^4_T L^\infty_x} \le c(|\beta|b^4 + b\,T^{3/4})(\sup_{[0,T]}\|v-\bar{v}\|_2 + \sup_{[0,T]}\|u-\bar{u}\|_2) \tag{5.7}$$

From (5.5), (5.6) and (5.3) we see that Φ_2 is a contraction for λ and T small. As in Proposition 4.1 we see that Φ_1 is a contraction for small T, and then Φ is a contraction. Again the estimates for Φ_1 do not use the $\sup_{[0,T]}\|u(t)\|_2$ norm. This completes the proof. □

6. Local Theory in H^1

In this section we consider the problem (1.1) with initial data $u_0 \in H^1(\mathbb{R})$, $v_0 \in H^1(\mathbb{R})$, $v_1 = h' \in L^2(\mathbb{R})$. In this case, the restriction $p<5$ is not necessary since we can use the Sobolev embedding theorem in the estimates. The proof of Theorem 2.5 is analogous to the one given for the L^2 case, but now we have to estimate nonlinear terms involving an additional derivative. We begin establishing the following result.

Proposition 6.1. *Consider the operator Φ defined in section 4 and the space*

$$E_+(T,a) = \begin{cases} (u,v) : u,v \in C([0,T];H^1(\mathbb{R})); \\ \qquad |||(u,v)||| \le a. \end{cases} \tag{6.1}$$

where

$$|||(u,v)||| = \max\Big\{\sup_{[0,T]}\|u(t)\|_{H^1}, \sup_{[0,T]}\|v(t)\|_{H^1}\Big\}.$$

Then, there exist a,T positive numbers, only depending on $\|u_0\|_{H^1}$, $\|v_0\|_{H^1}$, $\|h\|_{L^2}$, and $|\alpha|,|\beta|,p$ such that the map $\Phi=(\Phi_1,\Phi_2)$ satisfies

$$\Phi : E_+(T,a) \to E_+(T,a)$$

and is a contraction.

Proof. We only estimate the terms $\partial_x\Phi_1(u,v)$ and $\partial_x\Phi_2(u,v)$ in the L^2-norm.

$$\begin{aligned}
&\|\partial_x\Phi_1(u,v)(t)\|_2 \le \|\partial_x u_0\|_2 + \int_0^T \|\partial_x(uv)\|_2 + |\alpha|\|\partial_x(u|u|^2)\|\,ds\\
&\le c\|u_0\|_{1,2} + c\int_0^T \|v\|_2^{1/2}\,\|\partial_x v\|_2^{1/2}\,\|\partial_x u\|_2\,ds\\
&+c\int_0^T (\|u\|_2^{1/2}\|\partial_x u\|_2^{1/2}\|\partial_x v\|_2 + |\alpha|\,\|\partial_x u\|_2^2\|u\|_2)\,ds\\
&\le c\|u_0\|_{1,2} + c\,T(\interleave(u,v)\interleave^2 + \interleave(u,v)\interleave^3).
\end{aligned} \tag{6.2}$$

For $\partial_x\Phi_2$ we have:

$$\begin{aligned}
\|\partial_x\Phi_2(u,v)(t)\|_2 &\le c\,(\|\partial_x v_0\|_2 + \|\partial_x h\|_{H^{-1}}) + c\int_0^t \|\partial_x(\beta|v|^{p-1}v + |u|^2)\|_2\,ds,\\
&\le c\,(\|v_0\|_{1,2} + \|h\|_2) + c\,T(\interleave(u,v)\interleave^p + \interleave(u,v)\interleave^2).
\end{aligned} \tag{6.3}$$

These estimates plus an argument similar to the one used in section 4 allow us to conclude that Φ is a contraction. □

Proof of Theorem 2.5. The proof is analogous to the proof of Theorem 2.1 so it will be omitted. □

We end this section with some regularity properties of the solutions given by Theorem 2.5.

Proposition 6.2. *If (u,v) is a solution of (1.1) with initial data in $H^1\times H^1\times L^2$ then*

$$u, v, \partial_x u, \partial_x v \quad \in L^4([-T,T]; L^\infty(\mathbb{R})), \tag{6.4}$$

$$D_x^{3/2}u,\ D_x^{3/2}v \quad \in L^\infty(\mathbb{R} : L^2[-T,T]). \tag{6.5}$$

Proof. We first prove (6.4). Using the estimate (3.3), the argument in (4.3) and Sobolev's lemma we have

$$\|\partial_x u\|_{L^4_T L^\infty_x} \le \|\partial_x u_0\|_2 + cT\{\sup_{[0,T]}\|u(t)\|_{1,2}\sup_{[0,T]}\|v(t)\|_{1,2} + |\alpha|\sup_{[0,T]}\|u(t)\|_{1,2}^3\}. \tag{6.6}$$

Using Lemma 3.4, the argument (6.3) and Sobolev's lemma we obtain

$$\|\partial_x v\|_{L^4_T L^\infty_x} \le c\|v_0\|_{1,2} + c\|h\|_2 + cT(\sup_{[0,T]}\|v(t)\|_{1,2}^p + \sup_{[0,T]}\|u(t)\|_{1,2}^2). \tag{6.7}$$

The affirmation clearly follows from (6.6) and (6.7).

The proof of (6.5) follows a similar argument as the one used in the proof of Proposition 4.2. So it will be omitted.

□

Finally, we conclude this section showing that

$$(-\Delta)^{-1/2}\partial_t v \in C([0,T];L^2(\mathbb{R})).$$

In fact, using the integral equation, Lemma 3.6 and Sobolev's lemma, it follows that

$$\sup_{[0,T]}\|(-\Delta)^{-1/2}\partial_t v\|_2 \le c\|v_0\|_{1,2} + \|h\|_2 + cT(\sup_{[0,T]}\|v(t)\|_{1,2}^p + \sup_{[0,T]}\|u(t)\|_{1,2}^2). \tag{6.8}$$

Theorem 2.5 implies the result.

This estimate will be useful to establish an *a priori* estimate for the H^1 norm.

7. Global Theory in H^1

Next we prove that under some conditions on the initial data the solutions obtained in Theorem 2.5 can be extended to any time. More precisely, we will show that for $\beta > 0$ the (IVP) (1.1) is globally well-posed for any data. Meanwhile for $\beta < 0$ global solutions will be obtained for small data. We begin establishing the conservation laws we need to show our global result.

Lemma 7.1. *The following quantities are conserved by solutions of system (1.1)*

$$K(t) = \|u(\cdot,t)\|_2^2, \tag{7.1}$$

$$E(t) = \|\partial_x u\|_2^2 + \int v|u|^2dx + \frac{1}{2}\|(-\Delta)^{-1/2}\partial_t v\|_2^2 + \frac{1}{2}\|v\|_2^2 + \frac{\beta}{p+1}\int |v|^{p+1}\,dx + \frac{1}{2}\|\partial_x v\|_2^2 + \frac{\alpha}{2}\int |u|^4\,dx. \tag{7.2}$$

Proof. To prove these identities we will proceed formally. To justify the operations we can use, for instance, Kato's quasilinear theory [9] to obtain smooth solutions for system (1.2).

To prove (7.1) we only have to multiply the first equation by $\bar{u}$, integrate over x and take the imaginary part.

To show (7.2) we argue as follows: Multiplying the first equation of the system by $\partial_t\bar{u}$, integrating the result over x and taking its real part we obtain:

$$\frac{d}{dt}\Big(\int |\partial_x u|^2\,dx + \frac{\alpha}{4}\int |u|^4\,dx\Big) + \int \partial_t(|u|^2)\,v\,dx = 0. \tag{7.3}$$

On the other hand, applying the operator $(-\Delta)^{-1/2}$ to the second equation in (1.1), multiplying by $(-\Delta)^{-1/2}\partial_t v$ the result and integrating by parts with respect to x we get

$$\frac{1}{2}\frac{d}{dt}\Big(\|(-\Delta)^{-1/2}\partial_t\, v\|_2^2 + \|\partial_x v\|_2^2 + \|v\|_2^2 + \frac{\beta}{p+1}\int |v|^{p+1}\,dx\Big) + \int |u|^2\partial_t v\,dx = 0 \tag{7.4}$$

Adding (7.3) and (7.4) we can deduce (7.2). □

Proof of Theorem 2.7

Case $\beta > 0$. From (7.2) we have

$$
\begin{aligned}
&\|\partial_x u\|_2 + \frac{1}{2}\|(-\Delta)^{-1/2}\partial_t v\|_2^2 + \frac{1}{2}\|v\|_2^2 + \frac{1}{2}\|\partial_x v\|_2^2 \\
&= E(0) - \int (v|u|^2)dx - \frac{\beta}{p+1}\int |v|^{p+1}dx - \frac{\alpha}{2}\int |u|^4\,dx \qquad (7.5)\\
&\le E(0) + |\int (v|u|^2)\,dx| + |\frac{\alpha}{2}\int |u|^4\,dx|.
\end{aligned}
$$

Using Gagliardo-Nirenberg's type inequalities we obtain

$$
|\int (v|u|^2)dx| \le \|v\|_\infty \|u\|_2^2 \le \frac{1}{4}\|v\|_2^2 + \frac{1}{4}\|\partial_x v\|_2^2 + \frac{1}{2}K^4,
$$

and

$$
\frac{|\alpha|}{2}\int |u|^4\,dx = \frac{|\alpha|}{2}\|u\|_4^4 \le \frac{1}{2}\|\partial_x u\|_2^2 + c\|u\|_2^6 = \frac{1}{2}\|\partial_x u\|_2^2 + cK^6.
$$

for some constant c.

Then we have

$$
\begin{aligned}
&\|\partial_x u\|_2 + \frac{1}{2}\|(-\Delta)^{-1/2}\partial_t v\|_2^2 + \frac{1}{2}\|v\|_2^2 + \frac{1}{2}\|\partial_x v\|_2^2 \qquad (7.6)\\
&\le E(0) + \frac{1}{4}\|v\|_2^2 + \frac{1}{4}\|\partial_x v\|_2^2 + \frac{1}{2}K^4 + \frac{1}{2}\|\partial_x u\|_2^2 + cK^6.
\end{aligned}
$$

Thus

$$
\frac{1}{2}\|\partial_x u\|_2 + \frac{1}{2}\|(-\Delta)^{-1/2}\partial_t v\|_2^2 + \frac{1}{4}\|v\|_2^2 + \frac{1}{4}\|\partial_x v\|_2^2 \le E(0) + \frac{1}{2}K^4 + cK^6.
$$

Since the last quantity is constant, we can repeat the argument of local existence of solution at time T arriving to a solution for any positive time. The same holds for negative time. $\square$

Case $\beta < 0$. As in the previous case we have

$$
\begin{aligned}
&\|\partial_x u\|_2^2 + \frac{1}{2}\|(-\Delta)^{-1/2}\partial_t v\|_2^2 + \frac{1}{2}\|v\|_2^2 + \frac{1}{2}\|\partial_x v\|_2^2 \\
&\le E(0) + \frac{1}{2}K^4 + cK^6 + \frac{|\beta|}{p+1}\int |v|^{p+1}\,dx.
\end{aligned}
$$

From the Sobolev embedding theorem we have $\|v\|_{p+1} \le c\|v\|_{H^1}$. Using this inequality we obtain

$$
\begin{aligned}
\frac{1}{4}\|v\|_{H^1} &\le \frac{1}{2}\|\partial_x u\|_2 + \frac{1}{2}\|(-\Delta)^{-1/2}\partial_t v\|_2^2 + \frac{1}{4}\|v\|_2^2 + \frac{1}{4}\|\partial_x v\|_2^2 \\
&\le E(0) + \frac{1}{2}K^4 + cK^6 + \frac{|\beta|}{p+1}\int |v|^{p+1}dx \qquad (7.7)\\
&\le M + c\|v\|_{H^1}^{p+1}.
\end{aligned}
$$

Thus

$$
\|v\|_{H^1} - c\|v\|_{H^1}^{p+1} \le M,
$$

where $M = \frac{1}{4}(E(0) + \frac{1}{2}K^4 + cK^6)$ is a constant.

Since the function $x - cx^{p+1}$ is not negative for small x, we have that M is not negative if $\|v(\cdot,0)\|_{H^1}$ is sufficiently small.

Also, the function $x - cx^{p+1}$ has maximum value at $x = x_p = (\frac{1}{c(p+1)})^{\frac{1}{p}}$. We denote this maximum by y_p.

Then, if $\|u(\cdot,0)\|_{H^1}$ is small enough, we have: $0 \le M \le \frac{y_p}{2}$. In this case, the values of x for which $x - cx^{p+1} \le M$ are contained in two disjoint intervals $[0, x_M]$ and $[\bar{x}_M, \infty)$. Then, if $\|v(0)\|_{H^1} \le x_M$ we have, by continuity:

$$\|v(t)\|_{H^1} \le x_M$$

for each $t \in [-T, T]$.

Finally, we have

$$\frac{1}{2}\|\partial_x u\|_2 + \frac{1}{2}\|(-\Delta)^{-1/2}\partial_t v\|_2^2 \le M + \frac{1}{4}\|v\|_{H^1} + c\|v\|_{H^1}^{p+1}.$$

Since this last quantity is bounded, we can proceed as in the previous case to extend the solution. □

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