Positive quadratic differential forms: topological equivalence through Newton Polyhedra

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Abstract

The purpose of this article is to establish conditions under which a positive quadratic differential form is topologically equivalent to its principal form defined by Newton Polyhedra. The problem is carried out to study the simultaneous behavior of two foliations in the plane having a common point as a singularity.

Keywords: positive quadratic differential forms, topological equivalence, Newton Polyhedra, pairs of foliations.

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1 Introduction

In this paper we deal with special classes of differential forms in dimension two. We present a geometric method for studying positive quadratic differential forms which consists of an efficient mechanism for describing the local behavior of such systems. The purpose of this article is to establish conditions under which a positive quadratic differential form is topologically equivalent to its principal form defined by Newton Polyhedra. Although our methods at first sight have some resemblance with those ones developed by Brunella & Miari for vector fields new deep difficulties arise due to the generality of our approach. To be more specific, the problem is carried out to study the simultaneous behavior of two foliations in the plane having a common point as a singularity.

We believe that the techniques and the results obtained in this work provide means to study, in a very efficient way, the phase portrait of a finitely determined singularity of a positive quadratic differential form. The reason is that the Newton polyhedra indicate a very efficient way of using the method of blowing-up.

Let M be a C^∞ orientable 2-dimensional manifold. We consider a C^r quadratic differential form on M expressed by

$$\omega = \sum_{i=1}^{n} \varphi_i \psi_i,$$

where φ_i, ψ_i are C^r-differential 1-forms on M. Note that, for each p in M, $\omega(p)$ is a quadratic form defined on the tangent space T_pM. We say that ω is positive if for each point p in M we have that $\omega^{-1}(0)$ in T_pM is either an union of two transversal lines or all T_pM. In the first case, p is called a *regular point* of ω; otherwise p is a *singular point* of ω.

Throughout this paper PQD will refer to the expression: "positive quadratic differential" and $\Omega_P(M)$ will denote the set of all PQD forms on M.

If M is orientable and ω is a PQD form then, away from the singular set, there are two foliations F_1 and F_2 globally defined. In fact, let $p \in M$ be a non singular point of ω. Let $S^1 \subset T_pM$ be the unitary circle. We say that $v_1 \in S^1$ is of type one if, given $\gamma : (-\epsilon, \epsilon) \to S^1$ an orientation preserving mapping with $\gamma(0) = v_1$ we have that $\omega(\gamma(t), \gamma(t)) < 0$ for small $t > 0$. Certainly v_1 is of type one if, and only if, $-v_1$ is of type one. So by continuity, any vector of type one generates a one dimensional foliation on M.

Quadratic differential forms have been studied by several authors including, Hartman and Wintner [11], Sotomayor and Gutierrez [16]-[22], Bruce and Fidal [1], Bruce and Tari [2], Garcia and Sotomayor [6], Guadalupe, Gutierrez and Tribuzy [7], Guinez [8]-[10], Davydov, [4]. Theses works were performed in different context, such as Differential Equation view point as Global Analysis, Geometry and Singularity Theory.

Positive quadratic differential forms can be found in control theory(as showing by Davydov in [4]) and in the study of transonic gas flows (see Kuzmin in [13]).

Two PQD forms ω_1 and ω_2 are topologically equivalent if there exists a homeomorphism h in M taking the singular points of ω_1 to the singular points of ω_2 and taking the pair of foliations defined by ω_1 at each regular point p to the pair of foliation defined by ω_2 at $h(p)$.

Here we will consider C^∞ PQD forms ω defined in the plane. Each ω can be written as

$$\omega(x,y) = a(x,y)dy^2 + b(x,y)dxdy + c(x,y)dx^2,$$

where a, b and c are real-valued functions of class C^∞, dx, dy are the canonical projections and

$$\Delta_\omega(x,y) = (b^2 - 4ac)(x,y) \geq 0$$

and

$$(b^2 - 4ac)^{-1}(0) = a^{-1}(0) \cap b^{-1}(0) \cap c^{-1}(0).$$

This work is organized as follows. We start with some preliminaries in Section 2 and illustrate the construction of a magnification associated to Newton Polyhedra. In Section 3, we proof the Desingularization Theorem that is the key of the proof of the main results of this article. In Section 4, we treat the local sector classification of pairs of planar foliations at isolated singularity and present the proof of Main Theorem. That says that the topological type of a PQD form depends only on Newton Polyhedra associated to the form. In Section 5, the topological equivalence between a PDQ form and a quasi-homogeneous component ω_j of ω_Δ is proven under stronger hypotheses, using weighted polar blowing up. Finally, in Section 6, we present some applications of the results and techniques given in this paper.

2 Newton Polyhedra associated to quadratic differential form

First, we describe how to associate Newton Polyhedra to a quadratic differential form.

Let ω be a quadratic differential form in $\mathbb{R}^2$ with an isolated and singular point at $0 \in \mathbb{R}^2$. We may expand it in its formal series as follows:

$$\omega = \sum_{(n+2,m+2)\in\mathbb{N}^2} (r_{nm}x^n y^m x^2 dy^2 + s_{nm}x^n y^m xydxdy + t_{nm}x^n y^m y^2 dx^2),$$

with $w(0) = 0$.

We introduce the following definitions:

Definition 2.1 *(i) The Support of ω is the set in $\mathbb{R}^2$ defined by*

$$S = \{(n+2, m+2) \in \mathbb{N}^2 : (r_{nm}, s_{nm}, t_{nm}) \neq 0\}.$$

(ii) The Newton Polyhedra of ω is the convex envelope Γ of the set

$$P = \bigcup_{(k,l)\in S} \{(k,l) + \mathbb{R}^2_+\},$$

where $\mathbb{R}^2_+ = [0,\infty)^2$.

(iii) The Newton Diagram of ω is the union γ of the compact faces γ_i of the Newton Polyhedra Γ.

(iv) The Principal part ω_Δ of ω is the quadratic differential form

$$\omega_\Delta = \sum_{(n,m)\in\gamma} r_{nm}x^n y^m x^2 dy^2 + s_{nm}x^n y^m xydxdy + t_{nm}x^n y^m y^2 dx^2.$$

(v) The quasi-homogeneous component of ω (resp. of ω_Δ) relative to the face γ_i, of the Newton Polyhedra, is the restriction of ω to the face γ_i.

Example 2.2 *The Newton Polyhedra of the quadratic differential form*

$$\omega(x,y) = (2x^2 + xy + y^3)dy^2 + (y^2 - xy + 5y^3)dxdy - (2x^2 + xy + y^3)dx^2$$

is illustrated in Figure 1. The principal part of ω is $\omega_\Delta = (2x^2 + xy + y^3)dy^2 + (y^2 - xy)dxdy - (x^2 + xy)dx^2$. The quasi-homogeneous part of ω are $\omega_1 = (xy + y^3)dy^2 + y^2dxdy$ and $\omega_2 = (2x^2 + xy)dy^2 + (y^2 - xy)dxdy - (x^2 + xy)dx^2$.

Definition 2.3 *A magnification of $0 \in \mathbb{R}^2$ is a pair (M, π) such that*

1. *M is a C^∞ 2-dimensional manifold; $\pi : M \to \mathbb{R}^2$ is a C^∞ map, surjective and proper.*
2. *let $Z = \pi^{-1}(0)$ be the divisor of the magnification; then Z is the finite union of one dimensional manifolds in general position on M. Moreover $\pi|_{M\setminus Z}$ is a diffeomorphism from $M \setminus Z$ to $\mathbb{R}^2 \setminus \{0\}$.*

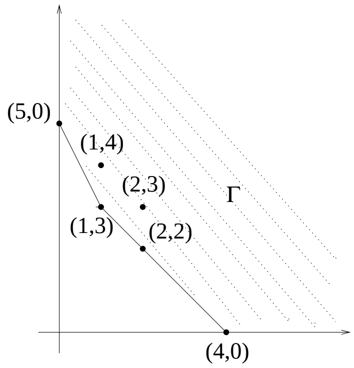

Figure 1: The Newton Polyhedra associated to ω in the Example 2.2.

Definition 2.4 *A desingularization of a PQD form ω defined in a neighborhood of $0 \in \mathbb{R}^2$, with $\omega(0) = 0$ is a PQD form $\widetilde{\omega}$ defined on M, where (M, π) is a magnification such that*

(i) the following diagram commutes

$$\begin{array}{ccc} Q(TM) & \xrightarrow{\pi^*} & Q(T\mathbb{R}^2) \\ \widetilde{\omega} \uparrow & & \uparrow \omega \\ M & \xrightarrow{\pi} & \mathbb{R}^2, \end{array}$$

where $Q(TM)$ is the fiber bundle of the quadratic forms.

(ii) for any $p \in Z$ there exist a neighborhood U of p and a function $f : U \to \mathbb{R}$ different from 0 outside Z, such that $\widetilde{\omega}|_U = f.\alpha.\beta$, where α and β are 1-forms in U. Moreover, α (resp. β) has p as either a regular point or a hyperbolic singularity or else a semi-hyperbolic singularity.

The form $\widetilde{\omega}$ will be said to be the desingularized form obtained from ω through the magnification (M, π). Moreover, we have the following:

Proposition 2.5 *Assume that (M, π) is a magnification of $0 \in \mathbb{R}^2$. Let ω_1, ω_2 be PQD forms in $\mathbb{R}^2$ with $\omega_i(0) = 0$ and $\widetilde{\omega_i}$ be the desingularized form obtained from ω_i through the magnification (M, π), $i = 1, 2$. If $\widetilde{\omega_1}$ is topologically equivalent to $\widetilde{\omega_2}$ in a neighborhood of the divisor Z then ω_1 is topologically equivalent to ω_2 in a neighborhood of $0 \in \mathbb{R}^2$.*

Now we define a *magnification (M, π) of $0 \in \mathbb{R}^2$ adapted to the Newton diagram γ of a PQD form ω on $\mathbb{R}^2$*. Let $\{(a_i, b_i)\}$ be the collection of vectors in $\mathbb{R}^2$ with mutually prime non-negative integer components, such that each vector is normal to one of the faces, say γ_i, of the Newton Diagram γ. Select a finite collection of vectors $\{e_i = (\alpha_i, \beta_i)\}_{i=1}^n$ in $\mathbb{R}^2$ with mutually prime non-negative integer components such that the following conditions are satisfied

Condition M $\{(a_i, b_i)\}_{i=1}^k \subset \{(\alpha_i, \beta_i)\}_{i=1}^n$;

Condition N $e_0 = (\alpha_0, \beta_0) = (1, 0)$ and $e_n = (\alpha_n, \beta_n) = (0, 1)$

Condition P $detP_i = \begin{pmatrix} \alpha_{i-1} & \alpha_i \\ \beta_{i-1} & \beta_i \end{pmatrix} = 1$, for all $i = 1, 2, ..., n$.

Condition Q Two consecutive vectors of this collection are not normal to two consecutive faces of Newton diagram.

From the Condition **P** given above we get that any pair of consecutive vectors $B_j = \{e_{j-1}, e_j\}$ forms a basis of $\mathbb{Z}^2$; the matrix defining the transformation that takes the base B_j to the base B_l, $j, l = 1, 2, ..., n, j \neq l$ has integer elements and its determinant is equal to ± 1. We observe that Condition **Q** is not needed to obtain a magnification; meanwhile, it will be used to obtain a simpler proof of Desingularization Theorem. Actually, Condition **Q** can always be considered, with no cost, to the other three as we shall see now. If $\det \begin{pmatrix} \alpha_{i-1} & \alpha_i \\ \beta_{i-1} & \beta_i \end{pmatrix} = 1$, for some $i = 1, 2, ..., n$, then $\det \begin{pmatrix} \alpha_{i-1} & \alpha_{i-1} + \alpha_i \\ \beta_{i-1} & \beta_{i-1} + \beta_i \end{pmatrix} = \det \begin{pmatrix} \alpha_{i-1} + \alpha_i & \alpha_i \\ \beta_{i-1} + \beta_i & \beta_i \end{pmatrix} = 1$, which means that we can always find the vector $(\alpha_{i-1} + \alpha_i, \beta_{i-1} + \beta_i) \in \mathbb{Z}^2$ belonging to the interior of the positive cone determined by the vectors $(\alpha_{i-1}, \beta_{i-1})$ and (α_i, β_i).

To build up the manifold M associated to this collection we proceed as Brunella and Miari [3]. For every pair $\{e_{i-1}, e_i\}$ there exists an associated chart (ϕ_i, U_i) on M, $\phi_i : U_i \to \mathbb{R}^2$. The transition map from (ϕ_i, U_i) to (ϕ_j, U_j) is given by $h_{ij}(x_i, y_i) = (x_i^{a_{ij}} y_i^{b_{ij}}, x_i^{c_{ij}} y_i^{d_{ij}})$, where $(x_i, y_i) = \phi_i(p)$ for $p \in U_i$ and $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ are defined by

$$\begin{pmatrix} e_{i-1} \\ e_i \end{pmatrix} = \begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix} \begin{pmatrix} e_{j-1} \\ e_j \end{pmatrix}.$$

Consider the set M obtained from gluing the n copies of $\mathbb{R}^2$ by means of the $n(n-1)$ maps h_{ij}. M is an analytic manifold.

To define the projection $\pi : M \to \mathbb{R}^2$ is enough to express $\{e_{i-1}, e_i\}$ in terms of the basis $\{(1,0),(0,1)\}$. The local representation of the projection in the chart (ϕ_i, U_i) (a local copy of $\mathbb{R}^2$) is defined by

$$h_i(x_i, y_i) = (x_i^{\alpha_{i-1}} y_i^{\alpha_i}, x_i^{\beta_{i-1}} y_i^{\beta_i}), \tag{1}$$

as

$$\begin{pmatrix} e_{i-1} \\ e_i \end{pmatrix} = \begin{pmatrix} \alpha_{i-1} & \alpha_i \\ \beta_{i-1} & \beta_i \end{pmatrix} \begin{pmatrix} e_0 \\ e_n \end{pmatrix}.$$

Now we will prove that π is a surjective map. To that end, it is enough to show that each h_i is surjective. First, consider the following lemma.

Lemma 2.6 *Let $\alpha_{i-1}, \alpha_i, \beta_{i-1}, \beta_i \in \mathbb{R}_+ = [0, \infty)$ be such that $\alpha_{i-1}\beta_i - \alpha_i\beta_{i-1} = 1$. Given $x_0, y_0 \in [0, \infty)$ there exist $x_1, y_1 \in [0, \infty)$ such that*

$$\begin{aligned} x_1^{\alpha_{i-1}} y_1^{\alpha_i} &= x_0 \\ x_1^{\beta_{i-1}} y_1^{\beta_i} &= y \end{aligned} \tag{2}$$

Proof From the system (2) we set

$$\begin{aligned} \alpha_{i-1} \log x + \beta_{i-1} \log y &= \log x_0 \\ \beta_{i-1} \log x + \beta_i, \log y &= \log y_0 \end{aligned}$$

which, by Condition **P**, has the unique solution. □

Let $i \in \{1, 2, \cdots, n\}$. The previous lemma shows that h_i takes the positive first quadrant onto itself. We shall describe below the action of h_i on the remaining quadrants. It will be convenient to introduce the following notation: $a = \alpha_{i-1}$, $b = \alpha_i$, $c = \beta_{i-1}$ and $d = \beta_i$. Recall that $ad - bc = 1$. We have the following three alternatives:

1. if a, d are even then b, c are odd. So $h_i(x, y) = (x^a y^b, x^c y^d)$ fixes the first and the third quadrant and takes the second and the fourth to the fourth and the second, respectively.

2. If a is even and d is odd (or vice-versa), b and c are odd. The map h_i fixes the first quadrant and takes the second to the fourth, the third to the second and the fourth to the third one.

3. If a, d are odd we have three situations. When b, c are even, the map h_i fixes all the quadrants. When b is even and c is odd, the map h fixes the first and the second quadrants and exchange the other two. If b is odd and c is even, the map h_i fixes the first and the fourth quadrants and interchanges the others.

This way, we conclude that the map π is surjective. Now we state the next proposition which the proof will be omitted.

Proposition 2.7 *The projection π, defined above, is an analytic map on M; moreover it is surjective and proper. The divisor $Z = \pi^{-1}(0)$ is the finite union of circles S^1 in general position. Furthermore, $\pi : M \setminus Z \to \mathbb{R}^2 \setminus \{0\}$ is a diffeomorphism.*

Lets return to the Example 2.2. The finite collection of vectors $\{(1,0),(3,1),(2,1),(5,3),$ $(3,2),(1,1),(1,2),(0,1)\}$ satisfies the conditions to define a projection in the sense of the above Proposition. The maps, representing π in local charts, are:

$$\begin{array}{rcl} h_1(x_1, y_1) & = & (x_1 y_1^3, y_1) \\ h_2(x_2, y_2) & = & (x_2^3 y_2^2, x_2 y_2) \\ & \vdots & \\ h_6(x_6, y_6) & = & (x_6 y_6, x_6 y_6^2) \\ h_7(x_7, y_7) & = & (x_7, x_7^2 y_7) \end{array}$$

3 Desingularization Theorem

Now we will show that the magnification (M, π) of $0 \in \mathbb{R}^2$ adapted to Newton diagram γ of a PQD form ω on $\mathbb{R}^2$ is indeed a desingularization which guarantees the topological equivalence between ω and ω_Δ, under non-degeneracy conditions. These conditions are established below.

Definition 3.1 *A PQD form ω, with $\omega(0) = 0$, is said to be distinguished if its associated Newton Diagram intersects the coordinate axes and if each of its quasi-homogeneous components, say ω_j relative to the face γ_i, does not have a singular point in $(\mathbb{R}\backslash 0)^2$*

If ω is distinguished then the singular points of each quasi-homogeneous forms occur at the axes $x = 0$ or $y = 0$. Moreover, in the terminology of Brunella and Miari, if ω is distinguished its Newton Polyhedra is favorable.

Lemma 3.2 *The set of all distinguished PQD forms in the plane is an open and dense subset of $\Omega_P(\mathbb{R}^2)$.*

Proof The proof is analogous to the planar vector fields case ([3]). □

This lemma guarantees that the distinguished condition is in fact a generic condition.

Theorem 3.3 (Desingularization Theorem) *Let ω be an arbitrary smooth distinguished PQD form, with $\omega(0) = 0$. Let (M, π) the magnification adapted to the Newton diagram of ω. Then ω admits a desingularization $\widetilde{\omega}$ defined on M. Moreover, up to local topological equivalence, the singularities of $\widetilde{\omega}$ are the same as those of its principal part $\widetilde{\omega_\Delta}$*

Proof Write the formal series expansion of the smooth PQD form ω as follows

$$\omega(x,y) = \sum_{(n+2,m+2)\in\mathbb{N}^2} r_{nm}x^n y^{m-2}dy^2 + s_{nm}x^{n-1}y^{m-1}dxdy + t_{nm}x^{n-2}y^m dx^2, \tag{3}$$

where, for all pair of positive integers (n,m), $s_{n0} = s_{0m} = t_{0m} = t_{1m} = r_{n0} = r_{n1} = 0$, and moreover, if (n,m) is a vertex of γ (i.e, if (n,m) is a point that belongs to two consecutive faces of the Newton diagram) then $(r_{nm}, s_{nm}, t_{nm}) \neq (0,0,0)$.

Choose a collection of vectors $\{(a_i,b_i)\}_{i=1}^n$ such that each vector has mutually prime non-negative integer components and is normal to a face γ_i of γ. Extend this collection so that the resulting collection $\{(\alpha_i,\beta_i)\}_{i=1}^k$ satisfies Conditions **M**–**P** of last section. Let (M,π) be the magnification associated to this extended collection. Given $j = 1,..,k$, let

$$\omega_j(x_j,y_j) = \frac{x_j y_j}{x_{j_x}y_{j_y} - x_{j_y}y_{j_x}} h_j^*\omega(x_j,y_j),$$

where $h_j(x_j,y_j) = (x_j^{\alpha_{j-1}}y_j^{\alpha_j}, x_j^{\beta_{j-1}}y_j^{\beta_j})$ is the corresponding local representation of π (in the chart (ϕ_j,U_j)).

We may observe that

$$\begin{aligned}\omega_j = \sum_{(n,m)\in\sigma} & x_j^{\alpha_{j-1}(n-1)+\beta_{j-1}(m-1)} y_j^{\alpha_j(n-1)+\beta_j(m-1)}.[(r_{nm}\beta_{j-1}^2 + s_{nm}\alpha_{j-1}\beta_{j-1}+ \\ & t_{nm}\alpha_{j-1}^2)y_j^2dx_j^2 + (2r_{nm}\beta_{j-1}\beta_j + s_{nm}(\alpha_{j-1}\beta_j + \alpha_j\beta_{j-1}) + 2t_{nm}\alpha_{j-1}\alpha_j)x_jy_jdx_jdy_j \\ & +(r_{nm}\beta_j^2 + s_{nm}\alpha_j\beta_j + t_{nm}\alpha_j^2)x_j^2dy_j^2]\end{aligned} \tag{4}$$

Define $t_j = min\{\alpha_j(n-1)+\beta_j(m-1)\}$, $t_0 = 0 = t_n$ and $T_j = \{(n,m) : \alpha_j(n-1)+\beta_j(m-1) = t_j\}$ for all $j = 1,2,...,n$. We have that T_j is either a face of γ or a vertex of the Polyhedra. Let $\widetilde{\omega_j}$ be the quotient of ω_j by $x_j^{t_j-1}y_j^{t_j}$. One has

$$\begin{aligned}\widetilde{\omega_j} = \sum_{(n,m)\in\sigma} & x_j^{\alpha_{j-1}(n-1)+\beta_{j-1}(m-1)-t_{j-1}} y^{\alpha_j(n-1)+\beta_j(m-1)-t_j}.[(r_{nm}\beta_{j-1}^2 + s_{nm}\alpha_{j-1}\beta_{j-1}+ \\ & t_{nm}\alpha_{j-1}^2)y_j^2dx_j^2 + (2r_{nm}\beta_{j-1}\beta_j + s_{nm}(\alpha_{j-1}\beta_j + \alpha_j\beta_{j-1}) + 2t_{nm}\alpha_{j-1}\alpha_j)x_jy_jdx_jdy_j \\ & +(r_{nm}\beta_j^2 + s_{nm}\alpha_j\beta_j + t_{nm}\alpha_j^2)x_j^2dy_j^2]\end{aligned} \tag{5}$$

Now we establish some notations. For each $j = 1,\ldots,n$ we define the expressions

$$< e_j,(n,m) >:= \alpha_j(n-1) + \beta_j(m-1) \tag{6}$$

$$A_j(x_j,y_j) := \sum_{(n,m)\in\sigma} x_j^{<e_{j-1},(n,m)>-t_{j-1}} y^{<e_j,(n,m)>-t_j}.(r_{nm}\beta_{j-1}^2 + s_{nm}\alpha_{j-1}\beta_{j-1} + t_{nm}\alpha_{j-1}^2)$$

$$B_j(x_j,y_j) := \sum_{(n,m)\in\sigma} x_j^{<e_{j-1},(n,m)>-t_{j-1}} y^{<e_j,(n,m)>-t_j}.(2r_{nm}\beta_{j-1}\beta_j + s_{nm}(\alpha_{j-1}\beta_j + \alpha_j\beta_{j-1}) +2t_{nm}\alpha_{j-1}\alpha_j)$$

$$C_j(x_j,y_j) := \sum_{(n,m)\in\sigma} x_j^{<e_{j-1},(n,m)>-t_{j-1}} y^{<e_j,(n,m)>-t_j}.(r_{nm}\beta_j^2 + s_{nm}\alpha_j\beta_j + t_{nm}\alpha_j^2)$$

So we have

$$\widetilde{\omega_j}(x_j,y_j) = C_j(x_j,y_j)x_j^2dy_j^2 + B_j(x_j,y_j)x_jy_jdx_jdy_j + A_j(x_j,y_j)y_j^2dx_j^2. \tag{7}$$

Since $b^2-4ac \geq 0$ near the origin, $B_j^2 - 4A_jC_j \geq 0$. Hence, we may write $\widetilde{\omega_j} = \varphi_j.\psi_j$, where φ_j and ψ_j are 1-forms.

It will be convenient to consider separately $\widetilde{\omega_1}$ and $\widetilde{\omega_n}$ and the general case $\widetilde{\omega_j}$.

1. $j=1$. Since γ intersects the axes, $r_{0M}, t_{N0} \neq 0$ for some positive integers M,N. In the first chart we note that the divisor is $Z=\{y_1=0\}$. We have $t_0=0$ and $T_1=\{(0,M)\}$, $\alpha_0=1, \beta_0=0, \alpha_1 \neq 0$, $\beta_1=1$. Then, $\omega_1(x,0) = r_{0M}dy_1^2 + h.o.t.$ is not identically zero. This means that ω_1 is the product $w_1 = \varphi_1.\psi_1$ of two regular 1-forms in the plane. Besides $(\varphi_1 \wedge \psi_1)(x_1,y_1) = y_1(2r_{0M} + R(x_1,y_1))$, where R vanishes in the the union of the axes.

2. $j=n$. Here we can proceed as in (1).

3. $1<j<n$. The divisor here is given by $\{x_j=0\} \cup \{y_j=0\}$. We shall study the following two different situations.

   (i) Suppose that none the vectors e_{j-1}, e_j is normal to any face of Newton Polyhedra associated to ω. Then $T_{j-1} = T_j = \{(n_j,m_j)\}$ is a vertex, so we can write

$$\begin{aligned} A_j(x_j,y_j) =&\ a_0 + R^A(x_j,y_j), \\ B_j(x_j,y_j) =&\ b_0 + R^B(x_j,y_j), \\ C_j(x_j,y_j) =&\ c_0 + R^C(x_j,y_j) \end{aligned} \tag{8}$$

   where R^A, R^B, R^C vanish in both axes.

   So the analysis on $\widetilde{\omega_j}$ can be performed via the numbers a_0, b_0, c_0.

   Now, we shall see that the sequence $\{(\alpha_j,\beta_j)\}$ can always be chosen so that $a_0^2+c_0^2>0$. In fact, suppose that

$$a_0 = r_{n_jm_j}\beta_j^2 + s_{n_jm_j}\alpha_j\beta_j + t_{n_jm_j}\alpha_j^2 = 0$$

$$c_0 = r_{n_jm_j}\beta_{j-1}^2 + s_{n_jm_j}\alpha_{j-1}\beta_{j-1} + t_{n_jm_j}\alpha_{j-1}^2 = 0,$$

   where $(r_{nm}, s_{nm}, t_{nm}) \neq (0,0,0)$. Then, we add the vector $(\alpha_{j-1}+\alpha_j, \beta_{j-1}+\beta_j)$ to the collection $\{(\alpha_j,\beta_j)\}$ (see the comments after definition of Condition **Q**). Due to the fact that the polynomial $r_{nm}v^2 + s_{nm}uv + t_{nm}u^2$ has at most two roots, for the resulting collection and new terms a_0, c_0, still denoted the same way, we will have this time that $a_0^2+c_0^2>0$.

   We shall only consider the case $a_0 \neq 0$. Then we may write $\widetilde{\omega_j} = \varphi_j.\psi_j$, where

$$\begin{aligned} \varphi_j =&\ 2A_jx_jdy_j + (B_j + \sqrt{B_j^2-4A_jC_j})y_jdx_j, \\ \psi_j =&\ 2A_jx_jdy_j + (B_j - \sqrt{B_j^2-4A_jC_j})y_jdx_j. \end{aligned}$$

   Then φ_j and ψ_j are tangent to the axes and transversal to each other, outside the axes. Moreover, the topological type of φ_j and ψ_j depends on (a_0,b_0,c_0):

   (a) If $c_0 \neq 0$, φ_j and ψ_j have hyperbolic singularities at the origin;

   (b) If $c_0=0$ and $b_0 \neq 0$, φ_j has a hyperbolic singularity at the origin and ψ_j has a semi-hyperbolic singularity at the origin;

(c) If $c_0 = 0 = b_0$, φ_j and ψ_j have semi-hyperbolic singularities at the origin; moreover, the central manifold is contained in one of the axes.

(ii) If e_j is normal to γ_l. Then $T_j = \gamma_l$ and $T_{j-1} = \{(n_j, m_j)\}$ is an endpoint of γ_l. The behavior of the foliations near the origin can be studied in the same way as item *(i)* above. To study the foliations around $y_j = 0$ when we are not in the case *(i)*, we observe that there must exist $\overline{u} \neq 0$ such that $A_j(\overline{u}, 0) = 0$ (which implies that $\widetilde{\omega_j}(\overline{u}, 0) = 0$). Using the assumptions of this theorem, we shall prove that $C_j(\overline{u}, 0) \neq 0$. In fact, otherwise, we have that the following expressions are zero

$$
\begin{aligned}
A_j(\overline{u},0).u^{t_{j-1}} &= \frac{a_{\gamma_j}(u^{\alpha_{j-1}},u^{\beta_{j-1}})}{u^{\alpha_{j-1}-\beta_{j-1}}}\beta_{j-1}^2 + \frac{b_{\gamma_j}(u^{\alpha_{j-1}},u^{\beta_{j-1}})}{u^{\alpha_{j-1}-\beta_{j-1}}}\alpha_{j-1}\beta_{j-1} + \frac{c_{\gamma_j}(u^{\alpha_{j-1}},u^{\beta_{j-1}})}{u^{\alpha_{j-1}-\beta_{j-1}}}\alpha_{j-1}^2 \\
C_j(\overline{u},0).u^{t_{j-1}} &= \frac{a_{\gamma_j}(u^{\alpha_{j-1}},u^{\beta_{j-1}})}{u^{\alpha_{j-1}-\beta_{j-1}}}\beta_j^2 + \frac{b_{\gamma_j}(u^{\alpha_{j-1}},u^{\beta_{j-1}})}{u^{\alpha_{j-1}-\beta_{j-1}}}\alpha_j\beta_j + \frac{c_{\gamma_j}(u^{\alpha_{j-1}},u^{\beta_{j-1}})}{u^{\alpha_{j-1}-\beta_{j-1}}}\alpha_j^2.
\end{aligned}
$$

where $t_{j-1} = min\{\alpha_{j-1}(n-1) + \beta_{j-1}(m-1)\}$, with $(n, m) \in \gamma_j$.

As in case *(i)* above we can add a vector (belonging to the positive cone generated by $(\alpha_{j-1}, \beta_{j-1})$ and (α_j, β_j)), say $(\bar{\alpha}, \bar{\beta})$, to the collection $\{(\alpha_i, \beta_i)\}$ such that the degree two homogeneous polynomial

$$
P_j(z,w) = \frac{a_{\gamma_j}(u^{\alpha_{j-1}}, u^{\beta_j})}{u^{\alpha_j-\beta_j}}w^2 + \frac{b_{\gamma_j}(u^{\alpha_j}, u^{\beta_j})}{u^{\alpha_j-\beta_j}}zw + \frac{c_{\gamma_j}(u^{\alpha_j}, u^{\beta_j})}{u^{\alpha_j-\beta_j}}z^2
$$

cannot vanish at $(z, w) = (\bar{\alpha}, \bar{\beta})$ unless all the coefficients of $P_j(z, w)$ are zero. By the assumptions of this Theorem $P_j(z, w)$ is not identically zero. Therefore, considering the new collection obtained by adding the vector $(\bar{\alpha}, \bar{\beta})$, we will have that $A_j(\overline{u}, 0)^2 + C_j(\overline{u}, 0)^2 > 0$.

Under these conditions, $\widetilde{\omega_j} = \varphi_j, \psi_j$, where

$$
\begin{aligned}
\varphi_j &= 2C_j y_j dx_j + (B_j + \sqrt{B_j^2 - 4A_jC_j})x_j dy_j \\
\psi_j &= 2C_j y_j dx_j + (B_j - \sqrt{B_j^2 - 4A_jC_j})x_j dy_j.
\end{aligned}
$$

Moreover, ψ_j has a semi-hyperbolic singularity at $(\overline{u}, 0)$ and φ_j is regular. In fact, since $A_j(\overline{u}, 0) = 0$ we get $(B_j^2 - 4A_jC_j)(\overline{u}, 0) = B_j^2(\overline{u}, 0) \geq 0$. But if $B_j(\overline{u}, 0) = 0$ then ψ_j and φ_j have contact outside of the divisor what contradicts the assumption $(b^2 - 4ac)^{-1}(0) = a^{-1}(0) \cap b^{-1}(0) \cap c^{-1}(0)$.

(iii) The case where e_{j-1} is normal to a face of Newton Polyhedra is similar to the case *(ii)*.

Then we can conclude that (M, π) is indeed a magnification to ω. $\square$

Remarks:

1. The form $\widetilde{\omega}$ of theorem above will be called the Newton Desingularization of ω.
2. We obtain the same result if we replace the favorable condition of the Newton Diagram by the singularity of ω_Δ is isolated.

4 Local sector classification of isolated singularities

As in case of vector fields (see [12]), we shall obtain a local sector classification of isolated singularities of the foliations induced by a smooth distinguished PQD form ω at $0 \in \mathbb{R}^2$. (see ([5]).

Let (M, π) the magnification adapted to the Newton diagram of ω and let $\widetilde{\omega}$ be the Newton desingularization of ω (See Theorem 3.3). Here we shall denote by $\mathcal{F}_1$ and $\mathcal{F}_2$ (resp. $\widetilde{\mathcal{F}_1}$ and $\widetilde{\mathcal{F}_2}$) the foliations induced by ω (resp. $\widetilde{\omega}$) in a neighborhood V (resp. $\widetilde{V}$) of 0 (resp. $Z = \pi^{-1}(0)$). We will suppose that $V = \pi(\widetilde{V})$. Given $p \in Z$ there exists a neighborhood W of p in M such that $\widetilde{\omega}|_W = \varphi_1 \cdot \varphi_2$, where each φ_i is a 1-form tangent to $\widetilde{\mathcal{F}_i}$ at W. We shall say that $\widetilde{\mathcal{F}_i}$ has p as a hyperbolic singularity (resp. saddle, node, etc) if this happen for φ_i, $i = 1, 2$. Notice that 1-forms can be seen as vector fields and so any local orientation of $\widetilde{\mathcal{F}_i}$ will refer to that induced by φ_i. Let Z be the divisor of $\widetilde{\omega}$.

Given a $\widetilde{\mathcal{F}_i}$-singularity $p \in Z$ we shall select a pair of integral curves of $\widetilde{\mathcal{F}_i}$ (*separating curves of* $\widetilde{\mathcal{F}_i}$) which will have the following properties:

(i) p will be either the ω or α-limit set of each of these curves;

(ii) π restricted to a separating curve is injective;

(iii) associated to each singularity $p \in Z$ there will be two separatrices of p contained in Z.

The referred selected separating curves are the following ones:

1. If p is a hyperbolic saddle, the two curves are the separatrices of p which do not meet $Z \setminus \{p\}$.

2. If p is a node singularity, we consider the two connected components γ_1 and γ_2 of $\overline{W_{\widetilde{\mathcal{F}_i}}(p) \setminus \{p\}}$, where $W_{\widetilde{\mathcal{F}_i}}(p)$ is its weak invariant manifold (unstable or stable, according to a orientation of $\widetilde{\mathcal{F}_i}$) at p. These curves will be called pseudo-separatrices of p.

3. If p is a saddle-node singularity (the strong manifold of p is contained in Z), the first curve is the separatrix between the two hyperbolic sectors of p. The second curve will be the connected components of $\overline{W^c(p) \setminus \{p\}}$ contained in the nodal part of p, where $W^c(p)$ denotes an arbitrarily given central manifold of p. This second curve will be called a pseudo-separatrix of p. (contained in the nodal part) and it is not uniquely determined.

Definition 4.1 *The set $S_c(i)$ of these selected curves will be said to be a family of separating curves of $\widetilde{\mathcal{F}_i}$ and their union will be denoted by S_i, $i = 1, 2$. Let $\mathcal{F}_i = \pi(\widetilde{\mathcal{F}_i})$ be the planar foliations induced by ω. The set $\pi(S_c(i))$ will be said to be a family of separating curves of $\mathcal{F}_i$.*

Definition 4.2 *Let $\widetilde{V}$ be a small neighborhood of $Z = \pi^{-1}(0)$, such that $\partial\widetilde{V}$ is a smooth curve and any separating curve of $\widetilde{\mathcal{F}_i}$ meets $\partial\widetilde{V}$ exactly once. Let U be a connected component of $\widetilde{V} \setminus (Z \cup S_i)$ then*

(i) If $\partial\overline{U} \cap S_i$ is constituted by two saddle separatrices then $\pi(\overline{U})$ is a hyperbolic sector of $\mathcal{F}_i$.

(ii) If $\partial\overline{U} \cap S_i$ is constituted by a saddle separatrix and a pseudo-separatrix then $\pi(\overline{U})$ is a parabolic sector of $\mathcal{F}_i$.

(iii) If $\partial\overline{U} \cap S_i$ is constituted by two pseudo-separatrices then $\pi(\overline{U})$ is a elliptic sector of $\mathcal{F}_i$.

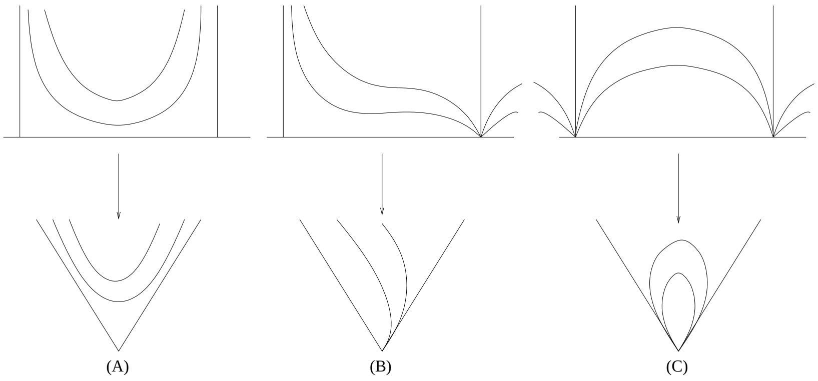

Figure 2: Sectors of a foliation in a singularity

Our main result shows that there is a topological equivalence between ω and ω_Δ. To this end, we will study the behavior of the foliations induced by $\widetilde{\omega}$.

Theorem 4.3 *Let ω be a smooth distinguished PQD form defined in a neighborhood of $0 \in \mathbb{R}^2$, with $\omega(0) = 0$. Let $\mathcal{F}$ be one of the foliations induced by ω. Then, there is a neighborhood V of 0 which can be decomposed in a finite number of elliptic, hyperbolic and parabolic sectors of $\mathcal{F}$; moreover, V and a family of separating curves of $\mathcal{F}$ can be chosen so that ∂V is smooth and any separating curve of the family meets ∂V exactly once.*

Proof Let $\widetilde{\omega}$ be the Newton Desingularization of ω. The foliations induced by $\widetilde{\omega}$ have only hyperbolic or semi-hyperbolic singularities (See Theorem 3.3 and its proof) where either both invariant manifolds belong to the divisor (in this case both foliations are singular) or one of them belongs to the divisor and the other one is transversal of it (in this case one foliation is singular and the second is regular in a neighborhood of the singularity). Then, we obtain - as Hartman's local sector classification of isolated singularities of a foliation (see [12], Chapter 8)- the conclusions of this theorem. $\square$

In the following Lemmas we shall assume that ω is a smooth distinguished PQD form defined in a neighborhood of $0 \in \mathbb{R}^2$, with $\omega(0) = 0$ and $\mathcal{F}_1$ and $\mathcal{F}_2$ are the pair of foliations induced by ω.

Lemma 4.4 *If S is a hyperbolic sector of $\mathcal{F}_1$, then*

1. *there exists at least one leaf of $\mathcal{F}_2$ that approaches the origin (i.e. a characteristic orbit) through the sector S; this leaf does not intersect the separatrices of S;*

2. *the union of the characteristic orbits of $\mathcal{F}_2$ approaching the origin through the sector S, when not reduced to a single leaf, forms a parabolic sector of $\mathcal{F}_2$. When the above union reduces to a single leaf it is a separating curve of $\mathcal{F}_2$.*

Proof Within this proof we shall refer as leaves of $\mathcal{F}_i$ to those ones which are actually leaves of $(\mathcal{F}_i, S)$. Recall that, as ω is distinguished, $\mathcal{F}_1$ and $\mathcal{F}_2$ are transversal to each other outside the origin. Denote by γ_1 and γ_2 the leaves of $\mathcal{F}_1$ that are the separatrices of the sector S. Fix a leaf γ of $\mathcal{F}_1$. Due to the transversality condition, given $i \in \{1, 2\}$, there exists a point $p_i \in \gamma$ such that the leaf of $\mathcal{F}_2$ at p_i meets transversally both γ and γ_i. See Figure 3 (A). It follows from this that if $\overline{p}$ is a point belonging to the subarc $\widehat{p_1 p_2}$ of γ, the $\mathcal{F}_2$-orbit on $\overline{p}$ either intersects γ_1 or

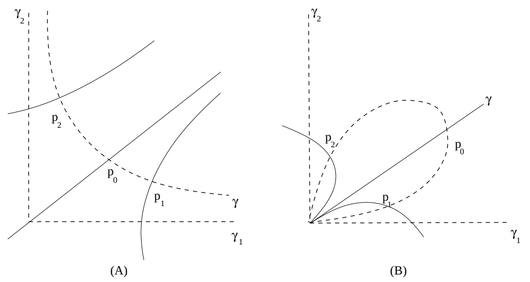

Figure 3: Hyperbolic sector and elliptic sector of $\mathcal{F}_1$

intersects γ_2 or else is a characteristic orbit. As $\mathcal{F}_1$ and $\mathcal{F}_2$ are transversal to each other, a leaf of $\mathcal{F}_2$ cannot meet simultaneously both γ_1 and γ_2 (see Figure 3(A)).

Therefore, there exists a point $p_0 \in \gamma$ such that the leaf of $\mathcal{F}_2$ approaches the origin (which proves item 1.); moreover, if there are two such points p_0 and p_0' in γ belonging to characteristic orbits such that no point of $\gamma \setminus \overset{\frown}{p_0p_0'}$ is in a characteristic orbit, then the leaves of $\mathcal{F}_2$ passing through points of $\overset{\frown}{p_0p_0'} \subset \gamma$ are characteristic orbits whose union makes up a parabolic sector of $(\mathcal{F}_2, S)$ (because the transversality condition). By the construction follows that, when there is only one leaf of $\mathcal{F}_2$ approaching the origin through the sector S, this leaf is a saddle separatrix of $\mathcal{F}_2$. $\square$

The proof of the following lemma is similar to that of Lemma 4.4 and will be omitted.

Lemma 4.5 *If S is an elliptic sector of $\mathcal{F}_1$, then there exist characteristic orbits of $\mathcal{F}_2$ that does not intersect the separatrices of S and every orbit of $(\mathcal{F}_2, S)$ is a characteristic orbit. Moreover, we can select a characteristic curve as a pseudo-separatrix (i.e. separating curve) of $\mathcal{F}_2$ in S.*

To consider parabolic sectors we shall need one more definition. Let ω, $\mathcal{F}$ and V be as in Theorem 4.3. Two sectors of $\mathcal{F}$ in V are said to be *adjacent* if they meet to each other exactly at a common separating curve. Notice that an elliptic sector can only be adjacent to either a parabolic sector or a elliptic sector. Then any finite sequence made up by adjacent sectors is obtained by concatenation of one of the following types: hyperbolic-hyperbolic, hyperbolic-parabolic-hyperbolic, hyperbolic-parabolic-elliptic, elliptic-parabolic-elliptic, elliptic- elliptic.

Now we shall study the adjacent sectors of each $\mathcal{F}_i$.

Proposition 4.6 *Let $(\mathcal{F}_1, \mathcal{F}_2)$ be the planar foliations induced by a distinguished form ω. Let V_1 and V_2 be neighborhoods, associated to $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, as in Theorem 4.3. Suppose that S_1, S_2 are two adjacent hyperbolic (resp. elliptic) sectors of $\mathcal{F}_1|_{V_1}$. Then there exists a hyperbolic (resp. elliptic) sector S of $\mathcal{F}_2|_{V_2}$ which meets $S_1 \cup S_2$ and its separating curves approaches the origin through the interior of each sector S_i, $i = 1, 2$. See Figures 4.*

Proof If S_1, S_2 are adjacent hyperbolic sectors of $\mathcal{F}_1|_{V_1}$, consider characteristic orbits γ_1 and γ_2 of $\mathcal{F}_2$ in S_1 and S_2, respectively, as in the Lemma 4.4. Denote by γ the common separating curve of S_1 and S_2. As the pair $(\mathcal{F}_1, \mathcal{F}_2)$ is transversal outside the origin, to each point $p \in \gamma$ we get a non-characteristic orbit of $\mathcal{F}_2$. On the other hand, if there exists a characteristic orbit for some $p \in \gamma$, this orbit meets some $\mathcal{F}_1$-orbit at two points, which is a contradiction with the transversality condition. Hence this orbit has a transversal intersection with the leaves of $\mathcal{F}_1$ and the region between γ_1 and γ_2 will be a hyperbolic sector of $\mathcal{F}_2$.

The same argument works in the case of a pair of adjacent elliptic sectors. $\square$

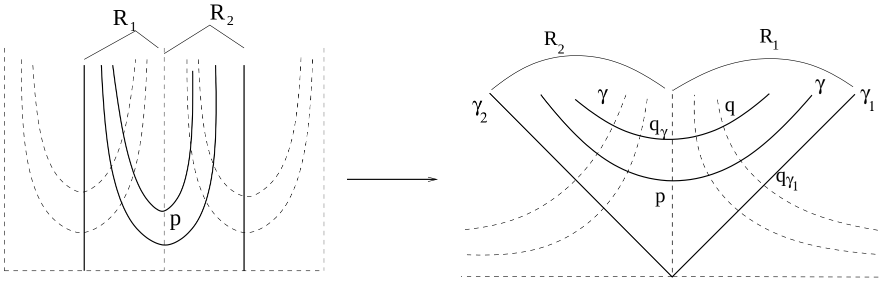

Figure 4: Adjacent hyperbolic sectors of $\mathcal{F}_1$

A PQD form possessing a pair of adjacent sectors as in previous proposition is the following one: $(xdy - ydx, (x+y)dy - (x-y)dx)$, to $y \geq 0$.

Using similar arguments of the last Proposition we can study what happens in adjacent sectors of $\mathcal{F}_1$.

Proposition 4.7 *Let $(\mathcal{F}_1, \mathcal{F}_2)$ be the planar foliations induced by a distinguished form ω. Let V_1 and V_2 be neighborhoods, associated to $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, as in Theorem 4.3. Suppose that S_0, S_1, S_2 are consecutive adjacent sectors of $\mathcal{F}_1|_{V_1}$, where S_0 and S_2 are hyperbolic sectors and S_1 is a parabolic one. Then there exists a hyperbolic sector of $\mathcal{F}_2|_{V_2}$ which meets $S_0 \cup S_1 \cup S_2$ if there is no separating curve of $\mathcal{F}_2$ contained in S_2. On the other hand, there is a hyperbolic sector adjacent to a parabolic sector of $\mathcal{F}_2$ which meets $S_0 \cup S_1 \cup S_2$. See Figure 5.*

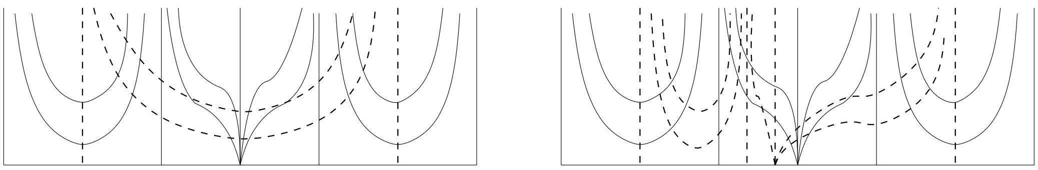

Figure 5: Sector of $\widetilde{\mathcal{F}_2}$ in $S_0 \cup S_1 \cup S_2$

Proof Notice that $\mathcal{F}_1$ has three adjacent sectors as described above if only if there is an interval of the divisor Z such that the separating curves associated to the $\widetilde{\mathcal{F}_1}$-singularities, belonging to the interval, appear orderly as follows: two saddle separatrices, a pseudo-separatrix and two saddle-separatrices. Moreover, from Lemma 4.4, we know that there is a saddle-separatrix of $\widetilde{\mathcal{F}_2}$ between each pair of consecutive saddle-separatrices of $\widetilde{\mathcal{F}_1}$. There are two alternatives to be considered:

- There is no separating curve of $\widetilde{\mathcal{F}_2}$ approaching the interval J between the saddle separatrix of $\widetilde{\mathcal{F}_2}$ in S_0 and the saddle separatrix of $\widetilde{\mathcal{F}_2}$ in S_1. In this case, we can proceed as in the previous proposition to show that there is a hyperbolic sector of $\mathcal{F}_2$ in $S_0 \cup S_1 \cup S_2$.

- Otherwise, as $(\mathcal{F}_1, \mathcal{F}_2)$ is a transversal pair, outside the origin, we conclude that there are two separating curves of $\widetilde{\mathcal{F}_2}$ in the interval J: one pseudo-separatrix and one saddle separatrix. See Figure 5. Under these conditions, there exists a parabolic and a hyperbolic sector of $\mathcal{F}_2$ in $S_0 \cup S_1 \cup S_2$. $\square$

Proposition 4.8 *Let $(\mathcal{F}_1, \mathcal{F}_2)$ be the planar foliations induced by a distinguished form ω. Let V_1 and V_2 be neighborhoods, associated to $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, as in Theorem 4.3. Suppose that S_0, S_1, S_2 are consecutive adjacent sectors of $\mathcal{F}_1|_{V_1}$, where S_0 is a hyperbolic sector, S_1 is a parabolic sector and S_2 is a elliptic sector of $\mathcal{F}_1|_{V_1}$. Then there exists a parabolic sector S of $\mathcal{F}_2|_{V_2}$ which meets $S_1 \cup S_2 \cup S_3$ provided that there is no separating curves of $\mathcal{F}_2$ on S_2. On the other hand, there is the following ordered sequence of adjacent sectors of $\mathcal{F}_2$ in $S_1 \cup S_2 \cup S_3$: hyperbolic, parabolic, elliptic sectors. See Figure 6.*

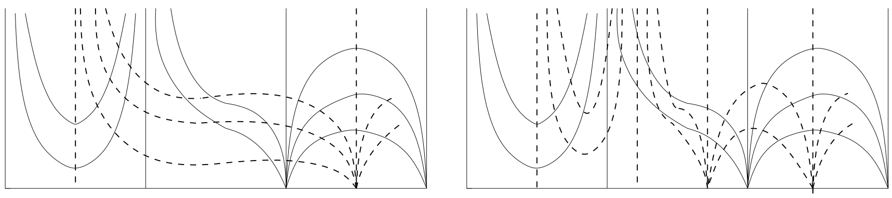

Figure 6: Sector of $\widetilde{\mathcal{F}_2}$ in $S_0 \cup S_1 \cup S_2$

Proof Under the hypotheses there is an interval of the divisor Z such that the separating curves associated to the $\widetilde{\mathcal{F}_1}$-singularities, belonging to the interval, appear orderly as follows: two saddle-separatrices and two pseudo-separatrices. From the Lemma 4.4, we conclude that there is a saddle-separatrix (resp. pseudo-separatrix) of $\widetilde{\mathcal{F}_2}$ in S_0 (resp. S_2). Then, working as in the proof of the later proposition, we observe:

- if there is no separating curves of $\widetilde{\mathcal{F}_2}$ approaching the interval between the saddle-separatrix of $\widetilde{\mathcal{F}_2}$ in S_0 and pseudo-separatrix of $\widetilde{\mathcal{F}_2}$ in S_2, there is one parabolic sector of $\mathcal{F}_2$ in $S_1 \cup S_2 \cup S_3$.

- otherwise, as the pairs are transversal outside the origin, there is a saddle-separatrix and a pseudo-separatrix of $\widetilde{\mathcal{F}_2}$ in the later interval. Then we observe one hyperbolic, one parabolic and one elliptic sector of $\mathcal{F}_2$ on $S_0 \cup S_1 \cup S_2$. □

Proposition 4.9 *Let $(\mathcal{F}_1, \mathcal{F}_2)$ be the planar foliations induced by a distinguished form ω. Let V_1 and V_2 be neighborhoods, associated to $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, as in Theorem 4.3. Suppose that S_0, S_1, S_2 are adjacent sectors of $\mathcal{F}_1|_{V_1}$ (in this ordered sequence), where S_0 and S_2 are elliptic sectors and S_1 is a parabolic sector of $\mathcal{F}_1|_{V_1}$ then there is a elliptic sector S of $\mathcal{F}_2|_{V_2}$ which meets $S_1 \cup S_2 \cup S_3$ provided that there is no separating curves of $\mathcal{F}_2$ on S_2. On the other hand, there are one elliptic and one parabolic sectors of $\mathcal{F}_2$ in the union $S_0 \cup S_1 \cup S_2$. See Figure 7*

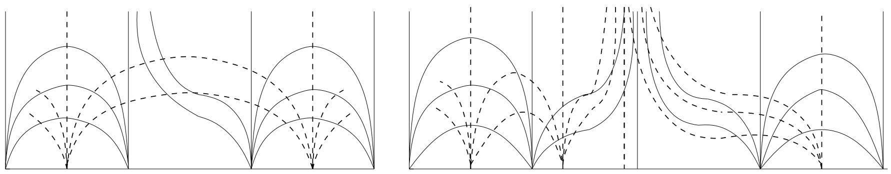

Figure 7: Sector of $\widetilde{\mathcal{F}_2}$ in $S_0 \cup S_1 \cup S_2$

The proof here uses similar arguments of the above propositions.

Now, to conclude the proof of MAIN THEOREM, we need one more result.

Consider an arbitrary distinguished form ω. Let $\widetilde{V}$ be a neighborhood of its divisor Z as described in Definition 4.2. As the induced foliations of ω are transversal out side the origin, only three non-equivalent pair of foliations can be found in an arbitrary connected component of $\widetilde{V} \setminus (Z \cap S_c)$. They are represented in Figure 8 (A) - (C). Therefore the induced foliations of $\widetilde{\omega}$ in $\widetilde{V}$ is obtained through the gluing of many finitely of these connected components.

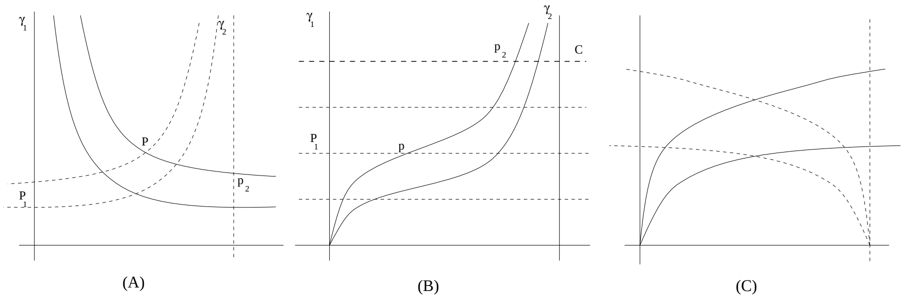

Figure 8: Sector of $\widetilde{\mathcal{F}_2}$ in $S_0 \cup S_1 \cup S_2$

We also observe that in Figure 8 (A) and (C), their separating curves belong to different foliations and both either are saddle-separatrices or are pseudo-separatrices. While in Figure 8 (B), both separating curves belong to the same foliation, but one is a saddle-separatrix while the other one is a pseudo-separatrix. The second foliation is regular in this component.

Let ω_1 and ω_2 be two distinguished PQD forms at origin. Denote by Z_i its divisor, by S_i the set of all separating curves of the induced foliations of $\widetilde{\omega_i}$ and denote by β_i the set $Z \cap S_i$, $i = 1, 2$.

Proposition 4.10 *Let ω_i, $i = 1, 2$ be two distinguished forms at $0 \in \mathbb{R}^2$ and Z_i, $i = 1, 2$ its divisors. Let $\widetilde{\mathcal{F}_j}$ (resp. $\widetilde{\mathcal{G}_j}$), $j = 1, 2$, be the foliations associated to $\widetilde{\omega_1}$ (resp. $\widetilde{\omega_2}$). The forms ω_1 and ω_2 are equivalent at a neighborhood of the origin if only if there is a homeomorphism $h : Z_1 \to Z_2$ that sends β_1 to β_2 and satisfies the following: if $p \in \beta_1$ then h restricted to a small neighborhood of p in Z_1 can be extended to a local topological equivalence between $(\widetilde{\mathcal{F}_1}, \widetilde{\mathcal{F}_2})$ around p and $(\widetilde{\mathcal{G}_1}, \widetilde{\mathcal{G}_2})$ around $h(p)$, for $j = 1, 2$.*

Proof If there exists a topological equivalence H between two forms ω_1 and ω_2 it is immediate the existence of the map h in the conditions above. Now we show the converse.

Assume that ω_1 and ω_2 are distinguished forms and h is a map in the assumptions of this proposition. We show how to build the equivalence between these two forms. First we will extend h (in an appropriate way) to the union of all separating curves of $\widetilde{\mathcal{F}_1}$ and $\widetilde{\mathcal{F}_2}$. Then we will be able to extend it to a neighborhood of Z_1.

• *The extension.* Let $\widetilde{V_i}$ be a neighborhood of Z_i, $i = 1, 2$, as in the Definition 4.2 and $\widetilde{S_i}$ (resp. $\widetilde{T_i}$) the set of all separating curves of $\widetilde{\mathcal{F}_i}$ (resp. $\widetilde{\mathcal{G}_i}$), $i = 1, 2$. We can define an equivalence relation in S_1 (resp. $\widetilde{T_1}$): given γ_1 and γ_2 two separating curves of $\widetilde{\mathcal{F}_1}$ (resp. $\widetilde{\mathcal{G}_1}$) we say that $\gamma_1 \simeq \gamma_2$ if only if there is a horizontal "flow box" of $\widetilde{\mathcal{F}_2}$ whose vertical edges are γ_1 and γ_2 (this "flow box" fails to be a true flow box because $\widetilde{\mathcal{F}_2}$ has singularities at Z_1). Analogously we can define an equivalence relation in $\widetilde{S_2}$ (resp. $\widetilde{T_2}$).

Now select one element in each equivalence class of $\widetilde{S_1}$ (resp. $\widetilde{T_1}$) and extend continuously h to the selected element homeomorphically onto its corresponding one. By definition of the equivalence relation we may use the holonomy induced by $\widetilde{\mathcal{F}_2}$ (resp. $\widetilde{\mathcal{G}_2}$) to extend h to all elements in an equivalent class.

In the same way we define an extension of h from the set of all separating curves of $\widetilde{\mathcal{F}_2}$ to the set of all separating curve of $\widetilde{\mathcal{G}_2}$. This extension h determines a correspondence between connected component of $\widetilde{V_1} \setminus (\widetilde{S_1} \cup \widetilde{S_2} \cup Z_1)$ and $\widetilde{V_2} \setminus (\widetilde{T_1} \cup \widetilde{T_2} \cup Z_2)$. In the following we shall extend $h : \widetilde{V_1} \to \widetilde{V_2}$ preserving this correspondence.

• *The equivalence in each connected component of* $\widetilde{V_1} \setminus (\widetilde{S_1} \cup \widetilde{S_2} \cup Z_1)$. Let p be a point in such a connected component $\widetilde{U_1}$. Let $\widetilde{U_2}$ be the connected component of $\widetilde{V_2} \setminus (\widetilde{T_1} \cup \widetilde{T_2} \cup Z_2)$ that corresponds to $\widetilde{U_1}$. Thanks to Propositions 4.6 - 4.9 as we have already observed after Proposition 4.9, we only need to consider two situations:

1. Suppose that p belongs to a connected component as drawn in Figure 8 (A) or (C) of ω_1. Through p there are unique leaves φ_1 and φ_2 of $\widetilde{\mathcal{F}_1}$ and $\widetilde{\mathcal{F}_2}$, respectively. These leaves meet the separating curves γ_1 and γ_1' in p_1 and p_2, respectively. From the assumptions there are points $h(q_1) = q_2 \in \gamma_2$ and $h(q_1') = q_2' \in \gamma_2'$. Through these points consider the leaves of $\widetilde{\mathcal{G}_1}$ and $\widetilde{\mathcal{G}_2}$, respectively. These leaves meet to each other at a unique point q. Then define $H(p) = q$. The application $H : \widetilde{U_1} \to \widetilde{U_2}$ is a homeomorphism which is a local equivalence between the induced foliations of $\widetilde{\omega_1}$ and $\widetilde{\omega_2}$.

2. Otherwise, p belongs to a connect component $\widetilde{U_1}$ as in Figure (C) of ω_1. Assume that $\widetilde{\mathcal{F}_2}$ is regular in this connected component. Fix one leaf C_1 of $\widetilde{\mathcal{F}_2}|_{\widetilde{U_1}}$. Notice that C_1 connects two equivalent separating curves γ_1 and γ_2 (according to the relation "$\simeq$"). Let C_2 be leaf of $\widetilde{\mathcal{G}_2}|_{\widetilde{U_2}}$ connecting $h(\gamma_1 \cap C_1)$ with $h(\gamma_2 \cap C_1)$.

Extend h to a homeomorphism between C_1 and C_2. Sliding along the leaf of $\widetilde{\mathcal{F}_2}$ through p we find a unique point $p_2 \in \gamma_1$ and sliding along of the leaf of $\widetilde{\mathcal{F}_1}$ we get a unique point p_1 in C_1. Then consider the leaf of $\widetilde{\mathcal{G}_2}$ through $h(p_2)$ and the leaf of $\widetilde{\mathcal{G}_1}$ through $h(p_1) \in C_2$. These leaves meet to each other in a unique point $q \in U_2$. The map defined by $H(p) = q$ as above is a local equivalence between the induced foliations by $\widetilde{\omega_1}|_{\widetilde{U_1}}$ and by $\widetilde{\omega_2}|_{\widetilde{U_2}}$.

In this way we have defined a topological equivalence between $\widetilde{\omega_1}|_{\widetilde{V_1} \setminus Z_1}$ and $\widetilde{\omega_2}|_{\widetilde{V_2} \setminus Z_2}$. This map induces a topological equivalence between $\omega_1|_{V_1}$ and $\omega_2|_{V_2}$. □

Definition 4.11 *A PQD form ω is called a complete positive form if each one of the directions field (or foliations) induced by ω possesses characteristic directions.*

MAIN THEOREM *Let ω be a PQD form in the plane with $\omega(0) = 0$. The form ω is equivalent to its principal part ω_Δ provided that it is distinguished and complete.*

Proof From the Desingularizarion Theorem we conclude that the pair of foliation induced by a distinguished form in the plane has finite many sectors in a neighborhood of the origin. Moreover there exists an equivalence between the divisors of ω and ω_Δ as required in the previous proposition. Then the existence of an equivalence between two pairs of induced foliations at a neighborhood of the origin is guaranteed. □

5 Weighted Polar Blowing up

As in the case of planar vector fields, we may obtain an equivalence between a PQD form and its restriction to a compact face of the Newton Diagram under suitable conditions. To do this, we need to introduce a definition.

Definition 5.1 *A function $f : \mathbb{R}^2 \to \mathbb{R}$, is called quasi-homogeneous of type $(\alpha, \beta) \in \mathbb{N}^2$ and of degree k if it satisfies*

$$f(t^\alpha x, t^\beta y) = t^k f(x, y), \forall t \in \mathbb{R}.$$

A PQD form $\omega = a(x,y)dy^2 + b(x,y)dxdy + c(x,y)dx^2$ is called quasi-homogeneous of type (α,β) and degree k if the functions a, b, c are quasi-homogeneous functions of type (α,β) and degree $k+2\alpha$, $k+\alpha+\beta$ and $k+2\beta$, respectively.

Given a PQD form, we can decompose it as $\omega = \sum_{j\geq k} \omega_j$ (sum of quasi-homogeneous forms of the degree j), where k is the first positive integer such that ω_k is not identically null. We call it the order of ω.

Consider the weighted polar coordinates

$$\begin{array}{rcl} x & = & r^{\alpha}\mathrm{Cs}\,\theta \\ y & = & r^{\beta}\mathrm{Sn}\,\theta \end{array}$$

where Cs and Sn are defined by the Cauchy problem

$$\left\{ \begin{array}{rcl} \frac{d}{d\theta}\mathrm{Cs}\,\theta & = & -\mathrm{Sn}^{2\alpha-1}\theta \\ \frac{d}{d\theta}\mathrm{Sn}\,\theta & = & \mathrm{Cs}^{2\beta-1}\theta \\ \mathrm{Cs}\,0 & = & 1 \\ \mathrm{Sn}\,0 & = & 0. \end{array} \right.$$

The functions Cs and Sn are T-periodic functions, where

$$T = \frac{2\alpha^{(1-2\alpha)/2\alpha}}{\beta^{1/2\alpha}} \int_0^1 (1-t)^{(1-2\alpha)/2\alpha} t^{(1-2\beta)/2\beta} dt,$$

and

$$\beta \mathrm{Sn}^{2\alpha}\theta + \alpha \mathrm{Cs}^{2\beta}\theta = \alpha. \tag{9}$$

We define the equivalence relation in $\mathbb{R}$: $x \sim y$ if and only if $x - y = nT$, $n \in \mathbb{Z}$ and $S^1_T = \mathbb{R}\backslash \sim$. Then, if $\omega \in \Omega_P(\mathbb{R}^2)$, with $\omega(0) = 0$, let $\widetilde{\omega} \in \Omega_P(S^1_T \times \mathbb{R})$ be define through the diagram

$$\begin{array}{ccc} Q(T(S^1_T \times \mathbb{R})) & \xrightarrow{\pi^*} & Q(T\mathbb{R}^2) \\ \widetilde{\omega} \uparrow & & \uparrow \omega \\ S^1_T \times \mathbb{R} & \xrightarrow{\pi} & \mathbb{R}^2 \end{array}$$

where $\pi : S^1_T \times \mathbb{R} \to \mathbb{R}^2$ is the weighted polar map and $Q(T(S^1_T \times \mathbb{R}))$ is the fiber bundle of the quadratic forms in $T(S^1_T \times \mathbb{R})$. The map π is a surjective and proper map. Moreover it is a diffeomorphism outside the set $\pi^{-1}(0) = S^1_T \times \{0\}$ onto its image.

Theorem 5.2 *Let ω be a PQD form with $\omega(0) = 0$. Suppose that there exists a quasi-homogeneous component ω_γ of the principal part ω_Δ, such that 0 is an isolated singularity of ω_γ. Then in a neighborhood of $0 \in \mathbb{R}^2$, ω is topologically equivalent to ω_γ provided that ω_γ is a complete positive form.*

Proof Applying the weighted polar map in ω and dividing it by $r^{k+2\alpha+2\beta-2}$, where k is the order of ω, we can write

$$\widetilde{\omega}(\theta, r) = \sum_{j\geq k} \{\alpha^2 \mathrm{Cs}^2\theta a_{\gamma_j}(\theta) + \alpha\beta \mathrm{Cs}\theta\mathrm{Sn}\theta\, b_{\gamma_j}(\theta) + \beta^2 \mathrm{Sn}^2\theta c_{\gamma_j}(\theta)\} r^{j-k} dr^2 +$$

$$\{-2\alpha \mathrm{Cs}\,\theta \mathrm{Sn}^{2\alpha-1}\theta a_{\gamma_j}(\theta) + (\alpha \mathrm{Cs}^{2\beta}\theta - \beta \mathrm{Sn}^{2\alpha}\theta) b_{\gamma_j}(\theta) + 2\beta \mathrm{Cs}^{2\beta-1}\mathrm{Sn}\,\theta c_{\gamma_j}(\theta)\} r^{j-k+1} dr d\theta$$

$$+ \{\mathrm{Sn}^{4\alpha-2}\theta a_{\gamma_j}(\theta) - \mathrm{Cs}^{2\beta-1}\theta \mathrm{Sn}^{2\alpha-1}\theta b_{\gamma_j}(\theta) + \mathrm{Cs}^{4\beta-2}\theta c_{\gamma_j}(\theta)\} r^{j-k+2} d\theta^2,$$

where $a_{\gamma_j}(\theta) = a_{\gamma_j}(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta), b_{\gamma_j}(\theta) = b_{\gamma_j}(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta)$ and $c_{\gamma_j}(\theta) = c_{\gamma_j}(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta)$.

As ω is a PQD form and we can re-write it as $\widetilde{\omega}(\theta, r) = K(\theta, r)dr^2 + M(\theta, r)rdrd\theta + N(\theta, r)r^2d\theta^2$. Moreover, it is convenient to consider this form as a product of two direction fields φ and ψ in a neighborhood of the origin, where

$$\begin{aligned} \varphi = & \ 2N(\theta, r)rd\theta + (M(\theta, r) + \sqrt{\Delta})dr, \\ \psi = & \ 2N(\theta, r)rd\theta + (M(\theta, r) - \sqrt{\Delta})dr \end{aligned}$$

and $\Delta = \Delta(\theta, r) = (M^2 - 4NK)(\theta, r) \geq 0$.

Notice that $\widetilde{\omega}(\theta, 0) = K(\theta, 0) = 0$ if and only if,

$$\alpha^2\mathrm{Cs}^2\theta a_{\gamma_k}(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta) + \alpha\beta\mathrm{Cs}\theta\mathrm{Sn}\theta b_{\gamma_k}(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta) + \beta^2\mathrm{Sn}^2\theta c_{\gamma_k}(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta) = 0.$$

But if $K(\theta, 0) = 0$ then $\Delta(\theta, 0) = M^2(\theta, 0) \neq 0$. In fact, assume for instance that $K(\theta, 0) = 0 = M(\theta, 0)$. Then we have the following equations

$$\begin{aligned} \alpha^2\mathrm{Cs}^2\theta a_{\gamma_k}(\mathrm{Cs}\theta, \mathrm{Sn}\theta) + \alpha\beta\mathrm{Cs}\theta\mathrm{Sn}\theta b_{\gamma_k}(\mathrm{Cs}\theta, \mathrm{Sn}\theta) + \beta^2\mathrm{Sn}^2\theta c_{\gamma_k}(\mathrm{Cs}\theta, \mathrm{Sn}\theta) &= 0 \\ \mathrm{Sn}^{4\alpha-2}\theta a_{\gamma_j}(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta) - \mathrm{Cs}^{2\beta-1}\theta\mathrm{Sn}^{2\alpha-1}\theta b_{\gamma_j}(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta) + \mathrm{Cs}^{4\beta-2}\theta c_{\gamma_j}(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta) &= 0. \end{aligned}$$

As each solution of the first equation is also a solution of the second one, we conclude that either

$$\alpha\mathrm{Cs}^{2\beta}\theta = \beta\mathrm{Sn}^{2\alpha}\theta$$

or there exists θ such that $a_{\gamma_j}(\mathrm{Cs}\theta, \mathrm{Sn}\theta) = b_{\gamma_j}(\mathrm{Cs}\theta, \mathrm{Sn}\theta) = c_{\gamma_j}(\mathrm{Cs}\theta, \mathrm{Sn}\theta) = 0$. Since the first equation is not possible (because of (9)) and the later can never occur (because of the hypotheses) we conclude that $M(\theta, 0) \neq 0$ provided that $K(\theta, 0) = 0$.

We also observe that given $(\theta_0, 0)$ such that $K(\theta_0, 0) = 0$ then $N(\theta_0, 0) \neq 0$. In fact, this quadratic equation has only the trivial solution: $a_{\gamma_j}(\mathrm{Cs}\theta, \mathrm{Sn}\theta) = b_{\gamma_j}(\mathrm{Cs}\theta, \mathrm{Sn}\theta) = c_{\gamma_j}(\mathrm{Cs}\theta, \mathrm{Sn}\theta) = 0$.

Then we conclude that ψ and φ, the induced vector fields of $\pi^*(\omega)$, have distinct and hyperbolic singularities on S^1_T. So applying Proposition 4.10 and conclude this proof.

□

Remark: The methods presented in this work do not permit an easy transference of the phase portrait obtained on M or S^1_T into $\mathbb{R}^2$.

6 Applications

We present here some applications from results and techniques given through out this paper.

1. Normal form for a pair of foliations in the plane

We present the usefulness of the later results to obtain models for a certain singular pairs of foliations in the plane. In [15] models to pairs of type regular/singular-exact are exhibited for pairs of type (df, dg), where f has the non-zero 1-jet at 0 and g is as arbitrary smooth real-valued function in the plane. When J^1f and J^1g are non zero, formal classification of the pairs (df, dg) are presented in [14]. Here we are interested in pairs given by level sets of smooth functions $f, g : \mathbb{R}^2, 0 \to \mathbb{R}, 0$. Observe that $\alpha.\beta$ defines a PQD form in the plane.

Proposition 6.1 *Let $\alpha = df$ and $\beta = dg$ be pairs of 1-forms in the plane, where f, g are smooth germs of functions with $j^2f(x,y) = x^2 - y^2$, $j^2g(x,y) = ax^2 + bxy + cy^2$ and $a.c.b \neq 0$. The PQD form $\omega = \alpha.\beta$ is topologically equivalent to $(2ax^2 + bxy)dx^2 + (bx^2 + 2(c-a)xy - by^2)dxdy - (bxy + 2cy^2)dy^2$ provided that $(a-c)^2 < b^2$.*

Proof From the hypotheses we have $j^1(\alpha) = xdx - ydy$ and $j^1(\beta) = (2ax+by)dx + (bx+2cy)dy$. Moreover both 1-forms have characteristic orbits provide that $(a-c)^2 - b^2 \leq 0$ and the contact between the 1-forms is characterized by the signal of the equation $(a-c)^2 + b^2$. So the conditions on a, b, c guarantee the transversality out side the origin.

Consider the quadratic form $\omega = \alpha.\beta$ and the associated Newton Polyhedra Γ of ω. As $a.b.c \neq 0$ Γ has only one face (homogeneous case) and ω_Δ is distinguished. From the Main Theorem, we conclude that ω is topological equivalent to ω_Δ, that is the product of $j^1(\alpha)$ and $j^1(\beta)$. □

2. Non-equivalent PQD forms

Consider the one-parameter family expressed by the following product

$$\begin{aligned} \alpha_\lambda = & \quad (x+\lambda)dy - (x+y)dx \\ \beta = & \quad (y-x)dy + (x+y)dx \end{aligned}$$

where $\lambda \in (-\epsilon, \epsilon) \subset \mathbb{R}$.

To each λ_0 fixed, α_λ and β are transversal out side origin and both foliations have a node singularity at origin. We also observe that α_λ is topological equivalent to α_0 to each $\lambda \neq 0$.

The PQD form $\omega_\lambda = \alpha_\lambda.\beta$ can be expressed as

$$\omega_\lambda = ((1-\lambda)xy - x^2 + \lambda y^2)dy^2 - (2x^2 + (1+\lambda)xy + (\lambda - 1)y^2)dxdy - (x^2 + 2xy + y^2)dx^2.$$

From the Main Theorem we conclude that ω_0 and ω_λ ($\lambda \neq 0$) are topologically equivalent to their principal parts. But the PQDF forms ω_0 and ω_λ are not, since they have non topologically equivalent adjacent sectors. In Figure 9 we can observe the mentioned sectors.

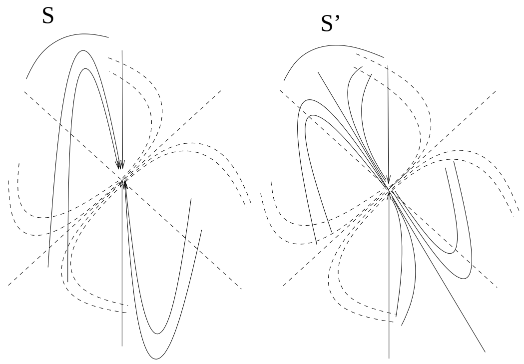

Figure 9: Non-equivalent PQD forms.

Remark: There are situations where the main result of this paper is true but Newton Diagram does not intersect the coordinate axes, as in the example before. This happens because we can substitute the condition of Newton Diagram intersecting the coordinates axes by the condition of the singularity to be an isolated point.

3. Partial Differential Equations

Many Physics, Biology, Economy phenomenons are modelled by the following partial differential equation (PDE)

$$c(x,y)u_{yy} + 2b(x,y)u_{xy} + a(x,y)u_{xx} + \alpha(x,y)u_x + \beta(x,y)u_y + d(x,y)u = f(x,y)$$

where x, y are the independent variables.

Given a PDE as above, we have associated to it characteristic curves. The characteristic curves give, for example, information about the propagation of a singularity of the PDE. Moreover, they are defined as solutions of the quadratic form

$$a(x,y)dy^2 - 2b(x,y)dxdy + c(x,y)dx^2 = 0.$$

In the case where the quadratic form is called hyperbolic $((b^2+4ac)(x,y) > 0)$, the associated characteristic curve represents a PQD form. Then our equivalence can be used in the study of the propagation of the singularities of these kind of PDE's.

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References

[1] J. W. Bruce and D. L. Fidal, *On binary differential equation and umbilics*,Proc. Roy. Soc. Edinburgh 111A (1989),147-168

[2] J. W. Bruce and F. Tari, *On binary equations*, Nonlinearity, 8 (1995), 255-271.

[3] M. Brunella and M. Miari, *Topological equivalence of a plane vector field with its principal part Defined through Newton polyedra*, J. Differential Equations 85 (1990), 338-366.

[4] A. A. Davydov, *Qualitative theory of control systems, Translations of Mathematical Monographs*, vol 141, American Mathematical Society (1994).

[5] F. Dumortier,*Singularities of vector fields on the plane*, J. Differential Equations 23 (1977), no.1, 53-106.

[6] R. Garcia and J. Sotomayor, *Lines of curvature near singular points of implicit surfaces*, Bull. Sci. Math. 117 (1993), no.3, 313-331.

[7] I. Guadalupe, C. Gutierrez and R. Tribuzy, *Lines of curvature on surfaces immersed in* $\mathbb{R}^4$, Bol. Soc. Bras. Mat. 28 (1997),nro 2, 233-251.

[8] V. Guíñez, *Positive quadratic differential forms and foliations with singularities on surfaces*, Trans. Amer. Math. Soc. 309, (1988),nro. 2, 477-502.

[9] V. Guíñez, *Locally stable singularities for PQD forms*, J. Differential Equations 110 (1994), 1-37.

[10] V. Guíñez, *Rank 2 codimension 1 singularities of PQD forms*, Nonlinearity 10 (1997), 631-654.

[11] P. Hartman and A. Wintner, *On the singularities in nets of curves defined by differential equations*, Amer. J. Math. 75, (1953), 277-297

[12] P. Hartman, *Ordinary differential equations.* Reprint second edition. Birkhuser, Boston, Mass., 1982

[13] Kuz'min, A. G. *Non-classical Equations of Mixed Type and their applications in Gas Dynamics*, Internacional Series of Numerical Mathematics, vol 109, (1992).

[14] R. D. S. Oliveira and F. Tari, *On pairs of foliations in the plane*, Discrete Contin. Dyn. Syst. 6 (2000), no. 3, 519-536.

[15] R. D. S. Oliveira, *Families of foliations in the plane*, pre-print.

[16] J. Sotomayor and C. Gutierrez, *Structural stable configurations of lines os principal curvature*, Astérisque (1982), 98-99.

[17] J. Sotomayor and C. Gutierrez, *An approximation theorem for immersions with structural satble configurations of lines os principal curvature*, Springer Lect. Notes in Math. 1007 (1983).

[18] J. Sotomayor and C. Gutierrez, *Configurations of lines of principal curvature and their bifurcations*, Aportaciones Matemáticas, Soc. Mat. Mex., Notas de Investigacioón, 1 (1985), 115-126.

[19] J. Sotomayor and C. Gutierrez, *Principal lines on surfaces immersed with constant mean curvature*, Trans. Amer. Math. Soc. 293, (1986), nro 2, 751-766.

[20] J. Sotomayor and C. Gutierrez, *Lines of principal curvature on surfaces with whitney umbrella singularities*, Tohoku Mathematical Journal 38, (1986), nro 4, 551-559.

[21] J. Sotomayor and C. Gutierrez, *Closed lines of curvature and bifurcation*, Bol. Soc. Bras. Mat. 17 (1986),no. 1, 1-19.

[22] J. Sotomayor and C. Gutierrez, *Periodics lines of curvature bifurcating from Darbouxian umbilical connections*, Springer Lec. Notes on Math. 1455 (1990).