PERIODIC ORBITS FOR EXACT MAGNETIC FLOWS ON SURFACES

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Abstract. We show that any exact magnetic flow on a closed surface has periodic orbits in all energy levels. Moreover, we give homological and homotopic properties of these periodic orbits in terms of the Mañé's critical values of the corresponding Lagrangian. We also prove that if M is not the 2-torus the energy level k is of contact type if and only if $k > c_0$, where c_0 is Mañé's strict critical value. When M is the 2-torus we give examples for which the energy level c_0 is of contact type.

1. Introduction

Let M be a closed n-dimensional manifold endowed with a C^∞ Riemannian metric g, and let $\pi : TM \to M$ be the canonical projection. Let ω_0 be the symplectic form on TM obtained by pulling back the canonical symplectic form of T^*M via the Riemannian metric. Let Ω be a closed 2-form on M and consider the new symplectic form ω_1 defined as:

$$\omega_1 \stackrel{\text{def}}{=} \omega_0 + \pi^*\Omega.$$

The 2-form ω_1 is a symplectic form and defines what is called a *twisted symplectic structure.*

Let $E : TM \to \mathbb{R}$ be given by

$$E(x, v) = \frac{1}{2} g_x(v, v).$$

The magnetic flow of the pair (g, Ω) is the Hamiltonian flow of E with respect to ω_1. The magnetic flow models the motion of a particle of unit mass and charge under the effect of a magnetic field, whose Lorentz force $Y : TM \to TM$ is the bundle map defined by:

$$\Omega_x(u, v) = g_x(Y_x(u), v),$$

for all $x \in M$ and all u and v in T_xM. In other words, the curve

$$t \mapsto (\gamma(t), \dot\gamma(t)) \in TM$$

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is an orbit of the Hamiltonian flow if and only if

$$\frac{D\dot{\gamma}}{dt} = Y_{\gamma}(\dot{\gamma}), \tag{1}$$

where D stands for the covariant derivative of g. The magnetic flow of the pair $(g, 0)$ is the geodesic flow of the Riemannian metric g. A curve γ that satisfies (1) will be called a *magnetic geodesic*.

1.1. **Existence of periodic orbits.** We will study the problem of existence of periodic orbits on prescribed energy levels for these flows. When $\Omega = -d\theta$ is exact, the magnetic flow can also be obtained as the Euler-Lagrange flow of the Lagrangian

$$L(x, v) = \frac{1}{2}g_x(v, v) + \theta_x(v).$$

Recall that the action of the Lagrangian L over an absolutely continuous curve $\gamma : [a, b] \to M$ is defined by

$$A_L(\gamma) = \int_a^b L(\gamma(t), \dot{\gamma}(t))\, dt.$$

A closed magnetic geodesic with energy k can be seen as a critical point of the functional

$$\gamma \mapsto A_{L+k}(\gamma),$$

where γ is an absolutely continuous closed curve defined on *any* closed interval $[a, b]$.

However, in the case of magnetic monopoles (i.e. when Ω is not exact), we cannot define an action functional as above but the differential of A_{L+k} is well defined for any magnetic field Ω. Thus, the general problem of existence of periodic orbits of magnetic flows is equivalent to the existence of singularities of an appropriate 1-form on the space of absolutely continuous closed curves with arbitrary period.

For the case of the geodesic flow, that is, when the magnetic field vanishes, the classical Morse theory ensures the existence of at least one closed geodesic for any closed Riemannian manifold (see [17]). An important point here is that the associated action functional, given by the total kinetic energy of the curve, is bounded from below and satisfies the so called Palais-Smale condition.

However, for non-vanishing magnetic fields, the associated action functional (which is multi-valued for magnetic monopoles) is no longer bounded from below and also may not satisfy the Palais-Smale condition [7]. Hence, a priori, we cannot use the classical methods of Morse theory as in the Riemannian case.

This problem was first considered by S. Novikov [26, 27, 33, 34, 35] and V. Arnold [1, 2] whose works begin with two essentially distinct approaches.

The first one introduces the so called Morse-Novikov theory, developed by Novikov and I. Taimanov (see [27, 33, 34, 35]). In his work, Taimanov uses variational principles for multi-valued functionals defined not in the space of closed curves, but in the space of films (see [33, 34, 35]) on surfaces. In this space, the functional becomes bounded from below and the Palais-Smale condition is replaced by the property that the minimal point can be taken in a compact subset consisting of films whose boundaries are polygons with a sufficiently great number of segments (given by local solutions of the Euler-Lagrange equation). At the basis of this property is the method of throwing out cycles [35]. With these techniques, he shows the existence of simple closed magnetic geodesic homologous to zero for sufficiently strong exact magnetic fields on surfaces and for strong non-exact magnetic fields taking both positive and negative values.

In higher dimensions, some partial results have been obtained by Taimanov and A. Bahri [3] using an approximation of the Lagrangian functional by auxiliary functionals satisfying the Palais-Smale condition.

The other approach uses methods from symplectic geometry and is closely related to the Weinstein conjecture which states that every contact hypersurface in a symplectic manifold (with trivial first cohomology group) carries a closed characteristic. However, the essential difference here is that the energy levels may fail to have contact type turning the problem more delicate.

For the case of surfaces, V. Ginzburg [9, 10] proved the existence of periodic orbits for sufficiently strong nondegenerate magnetic fields (corresponding to low energy levels with fixed intensity). His proof is based on the fact that sufficiently strong magnetic flows can be viewed (after a reparametrization) as a C^1 perturbation of the flow given by the vertical vector field on the unit sphere bundle of M. A survey of these and other results can be found in [11, 12].

In the higher-dimensional setting, recent results were obtained by Ginzburg and E. Kerman [13, 15] for magnetic fields given by a symplectic form satisfying a compatibility condition with the metric (e.g. Kähler forms). In this case, the non-degeneracy of the magnetic field implies essentially that the limit (reparametrized) dynamics of the magnetic flow defines a free S^1-action.

On the other hand, for sufficiently high energies the magnetic flow can be regarded as a small perturbation of the underlying geodesic flow and from this observation various existence results follow, see [10]. For exact magnetic flows in any dimensions a result of H. Hofer and C. Viterbo [14] implies the existence of periodic orbits for every energy level greater than $\max_{x\in M} \frac{1}{2}|\theta_x|^2$. This result is sharpened in [7, Theorem 27] with the introduction of the critical value of the universal covering (cf. Subsection 3.1 below).

Thus, the search for periodic orbits was divided into the three realms of high, low and intermediate energy levels, where, in the last case, we do not have, in general, information about the existence of such orbits. In fact, for magnetic monopoles we cannot expect to find periodic orbits in all energy levels as it is shown by the example given by a hyperbolic surface of constant curvature -1 and the magnetic field given by the area form. In this example, the magnetic geodesics are the curves with constant geodesic curvature and consequently, the magnetic flow restricted to the unit sphere bundle coincides with the horocycle flow and hence is minimal.

These results suggest that there exists an essential change in the dynamics of the magnetic flow in the transition from the higher to the lower energy levels. This change can be expressed for example in terms of periodic orbits by the appearance of *contractible* (or *homologous to zero*) closed orbits with low energy. In fact, it was proved by L. Macarini [18], extending previous results of L. Polterovich [31], that for every nontrivial weakly exact magnetic field there exist non-trivial contractible closed orbits of the magnetic flow in a sequence of arbitrarily small energy levels. Recall that a closed 2-form Ω on M is weakly exact if $\Omega|_{\pi_2(M)} = 0$. Recently, E. Kerman [16] proved the same result for magnetic fields given by symplectic forms.

1.2. **Results.** The present note arises as an attempt to relate the results of I. Taimanov [33, 34, 35] about existence of closed orbits of magnetic flows with Mañé's critical values [20] and Mather's theory of minimizing measures [23, 24]. Mañé's critical values single out those energies at which various decisive changes in the behaviour of the flow take place.

Let M be a closed oriented surface and let L be the Lagrangian:

$$L(x,v) = \frac{1}{2} g_x(v,v) + \theta_x(v),$$

where g is a smooth Riemannian metric and θ is a smooth 1-form. The Euler-Lagrange flow of L is an exact magnetic flow whose magnetic field is given by the 2-form $-d\theta$. Recall that the energy in this case is simply given by $E(x,v) = \frac{1}{2} g_x(v,v)$.

For a probability measure μ, we define the action of the probability μ, as the value $A(\mu) = \int L d\mu$. For a homological position $h \in H_1(M,\mathbb{R})$, we let $\beta(h) := \inf_{\rho(\mu)=h} A(\mu)$, where μ is assumed to be invariant under the magnetic flow and with homology $\rho(\mu)$. Minimizing measures always exist and if h is an extremal point for β then there exists an ergodic minimizing measure with homology h (β is convex and superlinear).

Theorem A. *Any exact magnetic flow on M possesses closed orbits in all energy levels. Moreover if M is not the 2-torus, an energy level k is of contact type if and only if $k > -\beta(0)$.*

When M is the 2-torus we give an example (cf. Section 5) for which the energy level $-\beta(0)$ is of contact type.

The next result clarifies the relationship between Taimanov's work and Mañé's critical value.

Recall that Mañé's *strict* critical value is defined as [20, 30]:

$$c_0(L) := -\beta(0) = \inf\{k :\ A_{L+k}(\gamma) \geq 0 \text{ for any absolutely continuous}$$
$$\text{closed curve } \gamma \text{ homologous to zero}\}.$$

Define now:

$$c_0^s(L) := \inf\{k :\ A_{L+k}(\gamma) \geq 0 \text{ for any absolutely continuous}$$
$$\textbf{simple} \text{ closed curve } \gamma \text{ homologous to zero}\}.$$

Obviously $c_0^s(L) \leq c_0(L)$.

Theorem B. $c_0(L) = c_0^s(L)$ *if and only if there exists an ergodic minimizing measure with zero homology .*

Corollary. *Suppose that the graph of Mather's beta function* $\beta : H_1(M, \mathbb{R}) \to \mathbb{R}$ *exhibits an extremal point at* $h = 0$*. Then* $c_0(L) = c_0^s(L)$ *.*

In Section 5 we give an example without ergodic minimizing measures with zero homology. By Theorem B this example has $c_0^s(L) < c_0(L)$.

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2. Preliminaries

We say that a a homology $h \in H_1(M, \mathbb{R})$ is *rational* if there exists $\lambda > 0$ such that $\lambda\, h \in i_* H(M, \mathbb{Z})$, where $i_* : H_1(M, \mathbb{Z}) \to H_1(M, \mathbb{R})$ is the natural map. The following proposition is attributed to A. Haefliger; a sketch of proof can be found in [22] and a detailed proof is given in the appendix.

Proposition 2.1. *Let* μ *be a minimizing measure such that* $\rho(\mu)$ *is rational. Then the support of* μ *is a union of closed orbits of* L.

The next theorem can be found in [19].

Theorem 2.2 (Tonelli's theorem for closed curves)**.** *Let* L *be a convex superlinear Lagrangian on a closed manifold* M*. Take* $h \in H_1(M, \mathbb{Z})$*. For any* $a > 0$*, there exists a closed orbit* $\gamma : [0, a] \to M$ *of* L *with homology* h *such that*

$$A_L(\gamma) \leq A_L(\tau),$$

for any absolutely continuous closed curve $\tau : [0, a] \to M$ *with homology class* h.

Let C_ℓ^0 be the set of continuous functions $f : TM \to \mathbb{R}$ having linear growth, i.e.

$$\sup_{(x,v)\in TM} \frac{|f(x, v)|}{|v|} < +\infty.$$

Let $\mathcal{M}_\ell$ be the set of Borel probabilities μ on TM such that

$$\int_{TM} |v|\; d\mu < +\infty,$$

endowed with the topology such that $\lim_n \mu_n = \mu$ if and only if

$$\lim_n \int f\; d\mu_n = \int f\; d\mu$$

for all $f \in C_\ell^0$.

If $\gamma : [0, T] \to M$ is a closed absolutely continuous curve, let $\mu_\gamma \in \mathcal{M}_\ell$ be defined by

$$\int f\; d\mu_\gamma = \frac{1}{T} \int_0^T f\big(\gamma(t), \dot\gamma(t)\big)\; dt$$

for all $f \in C^0_\ell$. Observe that $\mu_\gamma \in \mathcal{M}_\ell$ because if γ is absolutely continuous then $\int |\dot\gamma(t)|\,dt < +\infty$. Let $\mathcal{C}(M)$ be the set of such μ_γ's and let $\overline{\mathcal{C}(M)}$ be its closure in $\mathcal{M}_\ell$. Observe that the set $\overline{\mathcal{C}(M)}$ is convex.

Let $\mathcal{M}(L)$ be the set of all invariant probability measures in $\mathcal{M}_\ell$. Observe that for $\mu \in \mathcal{M}(L)$, all its ergodic components are in $\mathcal{M}_\ell$. By Birkhoff's theorem and the convexity of $\overline{\mathcal{C}(M)}$, we have that $\mathcal{M}(L) \subset \overline{\mathcal{C}(M)}$.

The proof of the next theorem due to Mañé can be found in [5, 21].

Theorem 2.3. *Let L be a convex superlinear Lagrangian on a closed manifold M. We have:*

$$c(L) = -\min\{A_L(\mu) :\ \mu \in \overline{\mathcal{C}(M)}\},$$

and any measure $\mu \in \overline{\mathcal{C}(M)}$ that achieves the minimum must belong to $\mathcal{M}(L)$.

We recall that [21]

$$c(L) := \inf\{k :\ A_{L+k}(\gamma) \geq 0 \text{ for any absolutely continuous closed curve } \gamma\}.$$

2.1. **A criterion for contact type.** Let N be a closed manifold and let X be a non-vanishing vector field on N. The following proposition is proved by D. McDuff in [25] and is based on Sullivan's structural cycles [32].

Proposition 2.4. *Suppose that X does not admit a global cross section and let Θ be a smooth 1-form on N. The following are equivalent:*

1. *$\int \Theta(X)\,d\mu \neq 0$ for every invariant probability measure μ with zero homology.*
2. *There exists a smooth closed 1-form φ such that $\Theta(X) + \varphi(X)$ never vanishes.*

Let Θ be the pull back of the canonical 1-form of T^*M under the Legendre transform. Let X be the Euler-Lagrange vector field associated with the Euler-Lagrange flow. A regular energy level $E^{-1}(k)$ is said to be of contact type if there exists a smooth closed 1-form φ on $E^{-1}(k)$ such that for all $(x,v) \in E^{-1}(k)$, $\Theta_{(x,v)}(X) + \varphi_{(x,v)}(X)$ is not zero. Equivalently, we could say that $E^{-1}(k)$ is of contact type if there exists a smooth 1-form α on $E^{-1}(k)$ such that $\alpha(X)$ is never zero and $d\alpha = j^*\omega_1$ where $j : E^{-1}(k) \to TM$ is the inclusion map and ω_1 is the twisted symplectic form defined in the introduction. Proposition 2.4 therefore gives a criterion for contact type in terms of invariant measures with zero homology. Note that the implication $2 \implies 1$ is fairly straightforward and hence the interesting part of the criterion is $1 \implies 2$.

3. Proof of Theorem A

We now state Taimanov's main result in [33, 34, 35] for exact magnetic flows in a form that is particularly suited for our purposes.

Theorem 3.1. *Let $\tau : [0,T] \to M$ be a piecewise differentiable closed curve such that*

1. *τ is simple and homologous to zero;*
2. *τ has energy k;*
3. $A_{L+k}(\tau) < 0$.

Then there exists a closed magnetic geodesic γ such that:
1. *γ is simple and homologous to zero;*
2. *γ has energy k;*
3. $A_{L+k}(\gamma) < 0$.

We need a preliminary lemma.

Lemma 3.2. *Let $\tau : [0, \ell] \to \widetilde{M}$ be an absolutely continuous curve parametrized by arc length. The reparametrization of τ that minimizes A_{L+k} has constant speed $\sqrt{2k}$.*

Proof. Suppose the reparametrization has speed $v(t)$ at $\tau(t)$. Then the action of $L+k$ along the reparametrization is

$$\int_0^\ell \frac{v(t)^2}{2} + k + \theta(v(t)\dot{\tau}(t))\,\frac{dt}{v(t)} = \int_0^\ell \frac{v(t)}{2} + \frac{k}{v(t)}\,dt + \int_\sigma \theta.$$

Since the last integral is independent of the reparametrization and the function

$$v \mapsto \frac{v}{2} + \frac{k}{v}$$

has a unique minimum at $v = \sqrt{2k}$, the action is minimized when $v(t) \equiv \sqrt{2k}$. □

Given a cover $\widehat{M} \mapsto M$ of M, let $\widehat{L}$ be the lift of L to $\widehat{M}$. The next lemma is the key result that allows us to relate Taimanov's results with the critical value c_0.

Lemma 3.3. *Let $k < c_0(L)$, then there exist a finite cover $\widehat{M} \mapsto M$ and a piecewise differentiable closed curve $\tau : [0, T] \to \widehat{M}$ such that:*
1. *τ is simple and homologous to zero;*
2. *τ has energy k;*
3. $A_{\widehat{L}+k}(\tau) < 0$.

Proof. Let M_0 be the abelian cover of M and let L_0 be the lift of L to M_0. The group of deck transformations can be identified with $H_1(M, \mathbb{Z}) = \mathbb{Z}^{b_1}$ where b_1 is the first Betti number of M. If $k < c_0(L)$, then we can find an absolutely continuous closed curve $\alpha : [0, T] \to M_0$ such that $A_{L_0+k}(\alpha) < 0$. By Tonelli's theorem (cf. Theorem 2.2) we can assume that α is a closed magnetic geodesic (and therefore it has constant energy) with a finite number of self intersections. From α we can "extract" a simple closed curve β with negative $(L_0 + k)$-action as we now explain. Let the operation $*$ denote concatenation of paths. Since α is an immersion we can decompose it as $\alpha_1 * \alpha_2 * \cdots * \alpha_n$ where each α_i is an embedding. Some of these $\alpha_i' s$ may be simple loops. If for some i, α_i is a simple loop and has negative $(L_0 + k)$-action then we set $\beta := \alpha_i$. If all the $\alpha_i' s$ which are loops have non-negative $(L_0 + k)$-action then we simply remove them and we are left with a new closed curve, let us say, α^1 which has negative $(L_0 + k)$-action since clearly

$$A_{L_0+k}(\alpha^1) \leq \sum_i A_{L_0+k}(\alpha_i) = A_{L_0+k}(\alpha) < 0.$$

We now repeat the process with the curve α^1 and remove simple loops with non-negative (L_0+k)-action. Since α was an immersion this process will stop after a finite number of steps and we will be left with a simple piecewise smooth closed curve β with negative (L_0+k)-action as desired.

Since β is a closed curve in M_0 its projection to M is a curve homologous to zero. The only problem is that the projection may not be simple, however this is easily fixed by passing to a suitable finite cover of M such that β lies in the interior of a fundamental domain. It can be constructed as follows. Let $f_1,\dots,f_{b_1}$ be generators of $\mathbb{Z}^{b_1}$. Given positive integers $n_1,\dots,n_{b_1}$, let $H(n_1,\dots,n_{b_1})$ be the subgroup of $\mathbb{Z}^{b_1}$ generated by $f_1^{n_1},\dots,f_{b_1}^{n_{b_1}}$. Let $M_{n_1,\dots,n_{b_1}}$ be the finite covering of M whose fundamental group is given by the kernel of

$$\pi_1(M)\mapsto\mathbb{Z}^{b_1}\mapsto\mathbb{Z}^{b_1}/H(n_1,\dots,n_{b_1}).$$

Then $M_{n_1,\dots,n_{b_1}}$ is the quotient of the abelian cover M_0 by the subgroup of deck transformations $H(n_1,\dots,n_{b_1})\subset H_1(M,\mathbb{Z})$. It is clear now that we can find sufficiently large positive integers $n_1,\dots,n_{b_1}$ such that the projection of β to $M_{n_1,\dots,n_{b_1}}$ is a simple curve τ. If necessary we now reparametrize τ so that it has energy k. On account of Lemma 3.2 this reparametrization can only decrease the $L+k$-action and hence τ has all the desired properties.

□

We now split the proof of Theorem A into three cases:

- $k>c_0(L)$. It was proved in [6] that in this case the magnetic flow in the energy level k can be seen as a reparametrization of the geodesic flow of a suitable Finsler metric. But it is well known that the geodesic flow of a Finsler metric on a closed manifold always has a closed orbit. Also this shows that the energy levels are of contact type.
- $k=c_0(L)$. In this case there exists a minimizing measure with zero homology in the energy level k (recall that $c_0(L)=-\beta(0)$). By Proposition 2.1 the support of such a minimizing measure is foliated by closed orbits.
- $k<c_0(L)$. In this case Theorem 3.1 and Lemma 3.3 show that there exists a closed magnetic geodesic with energy k homologous to zero and negative $L+k$-action.

Finally we show that for $k\le c_0(L)$ the energy level cannot be of contact type. This will conclude the proof of the theorem. Without loss of generality we can assume that $k>0$.

We will use the following:

Lemma 3.4. *Let Θ be the pull back of the canonical 1-form of T^*M under the Legendre transform. Let X be the Euler-Lagrange vector field associated with the Euler-Lagrange flow. We have:*

$$\Theta(X)|_{E^{-1}(k)}=L+k.$$

Proof. Let $\mathcal{L}$ be the Legendre transform. Since the projection of $X(x,v)$ to M is v, we see that

$$\begin{aligned}\Theta_{can}(d\mathcal{L}(X(x,v))) &= \mathcal{L}(x,v)(d\pi_{T^*M}(d\mathcal{L}(X(x,v))))\\ &= \frac{\partial L}{\partial v}(x,v)v\\ &= L(x,v)+E(x,v)\\ &= L(x,v)+k\end{aligned}$$

on $E^{-1}(k)$. ∎

Our previous discussion shows that for any $k \leq c_0(L)$ there exists an invariant probability measure μ with energy k and zero homology in M for which

$$\int (L+k)\, d\mu \leq 0.$$

We recall that given an invariant measure μ supported on a regular energy level $E^{-1}(k)$ one can consider two homologies associated with μ. The homology $\rho(\mu) \in H_1(M,\mathbb{R})$ (which is the one that we have been considering so far) is defined by

$$\langle \rho(\mu), [\omega]\rangle = \int \omega\, d\mu,$$

for any $[\omega] \in H_1(M,\mathbb{R})$. We can also consider the homology $S(\mu)$ given by

$$\langle S(\mu), [\varphi]\rangle = \int \varphi(X)\, d\mu,$$

for any $[\varphi] \in H_1(E^{-1}(k),\mathbb{R})$, where X is the Euler-Lagrange field restricted to $E^{-1}(k)$. Let $\pi : E^{-1}(k) \mapsto M$ be the canonical projection. Since $d\pi(X(x,v)) = v$ it follows that the two homologies are related by $\pi_*(S(\mu)) = \rho(\mu)$, where

$$\pi_* : H_1(E^{-1}(k),\mathbb{R}) \to H_1(M,\mathbb{R}).$$

It follows from the Gysin exact sequence of the circle bundle $\pi : E^{-1}(k) \mapsto M$ that π_* is an isomorphism, provided that M is not diffeomorphic to a 2-torus (see for example [29, Lemma 1.45]). Hence any invariant measure with zero homology ρ in M will also have zero homology S in $E^{-1}(k)$. Using Lemma 3.4 we conclude that for any $k \leq c_0(L)$ there exists an invariant probability measure μ with $S(\mu) = 0$ for which

$$\int \Theta(X)\, d\mu \leq 0.$$

Let μ_ℓ be the Liouville measure in $E^{-1}(k)$. Note that μ_ℓ has $S(\mu_\ell) = 0$ and is invariant under the flip $v \mapsto -v$ since it coincides with the Liouville measure of the geodesic flow [28]. Therefore

$$\int \theta\, d\mu_\ell = 0,$$

which yields

$$\int \Theta(X)\, d\mu_\ell = 2k > 0$$

since $L(x,v) + k = 2k + \theta_x(v)$ for $(x,v) \in E^{-1}(k)$. Therefore we can always find an appropriate convex combination ν of μ and μ_ℓ such that $S(\nu) = 0$ and

$$\int \Theta(X)\, d\nu = 0.$$

By Proposition 2.4 the energy level $E^{-1}(k)$ cannot be of contact type.

$\square$

3.1. **Remarks.** We can also introduce the critical value of the universal covering as:

$$c_u(L) := \inf\{k :\ A_{L+k}(\gamma) \geq 0 \text{ for any absolutely continuous}$$
$$\text{closed curve } \gamma \text{ homotopic to zero}\}.$$

Obviously $c_u(L) \leq c_0(L)$ but the inequality can be strict (see below). Theorem 27 in [7] says that for any $k > c_u(L)$ there are closed magnetic geodesics in any non-trivial homotopy class.

We remark that we cannot expect, in general, the periodic orbits with energies $k < c_0^s(L)$ to be contractible. In fact, there exists an example of a magnetic Lagrangian given by G.P. Paternain & M. Paternain [30] with an energy level $k < c_0^s(L)$ restricted to which the magnetic flow is Anosov and hence without contractible closed orbits. For this example, $c_u(L) < c_0^s(L) \leq c_0(L)$. As we mentioned in the introduction we know that there exists a sequence of contractible periodic orbits of arbitrarily small energy. This naturally raises the following:

Question: Given $k < c_u(L)$ is there a contractible periodic orbit with energy k?

4. Proof of Theorem B

Suppose first that $c_0^s = c_0$. By the definition of c_0^s there exists a sequence of absolutely continuous simple closed curves $\gamma_n : [0, T_n] \to M$ homologous to zero and such that

$$\lim_{n\to\infty} \frac{1}{T_n} A_{L+c_0}(\gamma_n) = 0.$$

Let $\nu_n \in \mathcal{C}(M)$ be the measure defined by

$$\int f\, d\nu_n = \frac{1}{T_n} \int_0^{T_n} f(\gamma_n(t), \dot\gamma_n(t))\, dt.$$

Thus

$$A_{L+c_0}(\nu_n) \to 0.$$

It follows from Theorem 2.3, that a subsequence of ν_n (which we still denote in the sequel by ν_n) converges to a minimizing measure μ with $\rho(\mu) = 0$. Since μ is minimizing then its support is contained in the energy level c_0 (cf. [8]). It follows that

$$\int \theta \, d\mu = -2c_0. \tag{2}$$

Since the curves γ_n are simple and homologous to zero, there exists an embedded surface D_n such that $\partial D_n = \gamma_n$. By Stokes theorem

$$\int_{\gamma_n} \theta = \int_{D_n} d\theta.$$

Since the surfaces are embedded it follows that there exists a positive number K so that

$$\left| \int_{\gamma_n} \theta \right| \leq K,$$

for all n. Suppose that $\sup_n T_n = \infty$. It follows from the last inequality and (2) that $c_0 = 0$. In this case it can be easily checked (cf. [30]) that $d\theta = 0$ (there is no effective magnetic field) and the minimizing measures with zero homology are just Dirac measures supported on points on the zero section of TM. Obviously these Dirac measures are ergodic and have zero homology. Henceforth we will restrict our attention to the case when $a := \sup_n T_n < \infty$.

Let $\delta_n : [0, a] \to M$ be the curve given by:

$$\delta_n(t) := \begin{cases} \gamma_n(t), & t \in [0, T_n], \\ \gamma_n(T_n), & t \in [T_n, a]. \end{cases}$$

Observe that $A_L(\gamma_n) = A_L(\delta_n)$.

By Theorem 2.2 there exists a closed orbit $\gamma : [0, a] \to M$ with zero homology such that

$$A_L(\gamma) \leq A_L(\delta_n) = A_L(\gamma_n)$$

for all n. Since $T_n \leq a$ for all n and

$$\lim_{n \to \infty} \frac{1}{T_n} A_{L+c_0}(\gamma_n) = 0,$$

it follows that

$$A_{L+c_0}(\gamma) \leq 0;$$

which implies that the measure μ supported on γ is a minimizing measure with zero homology. Since it is supported on a single closed orbit, then it is ergodic.

Suppose now that there exists an ergodic minimizing measure μ with zero homology. By Proposition 2.1 the support of μ is a union of closed orbits of L. Since μ is ergodic it must be supported in a single closed orbit γ. But since μ is minimizing:

$$A_{L+c_0}(\gamma) = 0,$$

and therefore for any $k < c_0$

$$A_{L+k}(\gamma) < 0,$$

which implies that $c_0^s = c_0$ because γ is a simple closed curve homologous to zero: recall that by the Graph Theorem [19], the support of μ is a Lipschitz graph.

$\square$

5. Examples

5.1. **Example of a magnetic Lagrangian on $\mathbb{T}^2$ for which the energy level c_0 is of contact type.** Let us consider the 2-torus $\mathbb{T}^2$ and let $\langle\cdot,\cdot\rangle$ be the flat metric. Consider a smooth vector field Z on $\mathbb{T}^2$ such that Z has a simple closed orbit γ homotopic to zero and with speed one with respect to the flat metric.

Take a C^∞ function $\psi : \mathbb{T}^2 \to \mathbb{R}$ such that $\psi(x) \geq 0$ and $\psi(x) = 0$ iff $x \in \gamma$. Set $\theta_x(v) := \langle Z(x), v\rangle$ and $\varphi(x) := |Z(x)|^2 + 2\psi(x)$. Our Lagrangian will be:

$$L(x,v) = \frac{1}{2}\varphi(x)|v|^2 - \theta_x(v).$$

An easy computation shows that

$$L(x,v) + \frac{1}{2} = \frac{1}{2}\varphi(x)\left|v - \frac{Z(x)}{\varphi(x)}\right|^2 + \frac{\psi(x)}{\varphi(x)}.$$

It follows that $L(x,v) + 1/2 \geq 0$ with equality iff $x \in \gamma$ and $v = Z(x)$ and therefore γ is a closed magnetic geodesic and the probability measure associated with γ is a minimizing measure with zero homology. In particular, it follows that $c_0(L) = 1/2$.

We claim that the energy level $1/2$ is of contact type. On account of Lemma 3.4 and Proposition 2.4 it suffices to show that for any invariant probability measure μ supported in $E^{-1}(1/2)$ and with homology $S(\mu) = 0$ in $E^{-1}(1/2)$ we have

$$\int (L + 1/2)\, d\mu > 0.$$

Suppose there exists such a μ for which

$$\int (L + 1/2)\, d\mu = 0.$$

It follows that μ has to be supported on the closed magnetic geodesic γ. But the curve $t \mapsto (\gamma(t), \dot\gamma(t))$ in $E^{-1}(1/2)$ is not homologous to zero, in fact it is homotopic to the fibre of the bundle $E^{-1}(1/2) \mapsto \mathbb{T}^2$. Therefore there is no μ with $S(\mu) = 0$ for which

$$\int (L + 1/2)\, d\mu = 0$$

and thus the energy level $E^{-1}(1/2)$ is of contact type.

Note that since being of contact type is an open condition we also obtain energy levels of contact type for $k < c_0$.

5.2. **Example for which $c_0 > c_0^s$.** Let us consider the 2-torus $\mathbb{T}^2$ equipped with the flat metric. Consider on $\mathbb{T}^2$ a vector field X with norm one and such that its orbits form a Reeb foliation. By this we mean that X has only two closed orbits γ_1 and γ_2 in *opposite* homology classes and any other orbit approaches asymptotically γ_1 in forward time and γ_2 in backward time.

The vector field X gives rise to a magnetic field

$$\theta_x(v) = \langle X(x), v\rangle.$$

Since X has unit norm we have:

$$L(x, v) + \frac{1}{2} = \frac{1}{2}|v - X(x)|^2,$$

from which it easily follows that $c_0 = 1/2$ and that the only minimizing measure with zero homology is the one supported on γ_1 and γ_2. It follows that there is no ergodic minimizing measure with zero homology and thus by Theorem B, $c_0 > c_0^s$.

In this example, the energy levels $E^{-1}(k)$ for $k \in (c_0^s, c_0)$ contain closed orbits homologous to zero and negative $(L + k)$-action, but none of them will be simple curves.

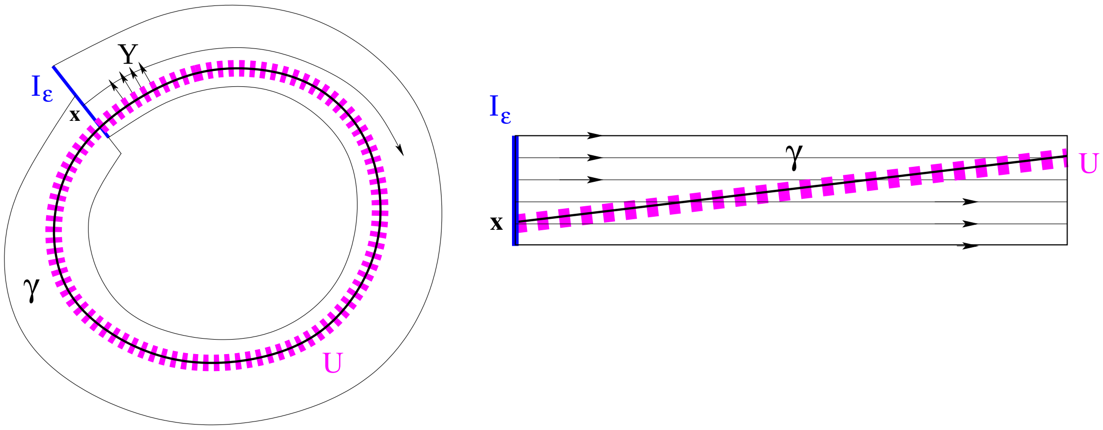
6. Appendix: proof of Proposition 2.1

Proof. Assume first that the surface M is orientable. By Mather's Lipschitz graph theorem, $\mathcal{L} = \pi(\mathrm{supp}(\mu))$ is a Lipschitz lamination on M, the Lagrangian flow φ_t on $\mathrm{supp}(\mu)$ projects to a flow ϕ_t on $\mathcal{L}$ and μ projects into a invariant measure $\nu = \pi_*(\mu)$ for ϕ_t. Let $z \in \mathcal{L}$ suppose that z is not in the projection of a closed orbit in $\mathrm{supp}(\mu)$. Let I_ε be an open segment centered at z such that I_ε is transverse to $\mathcal{L}$ and $\mathrm{diam}(I_\varepsilon) = \varepsilon > 0$. Since $z \in \mathrm{supp}(\nu) = \mathcal{L}$ then $\nu(\phi_{[0,1]}(I_\varepsilon)) > 0$. By Poincaré recurrence, the open segment I_ε returns to itself. We consider its first return:

$$S(\varepsilon) := \inf\{\, t > 0 \mid \phi_t(I_\varepsilon) \cap I_\varepsilon \neq \emptyset \,\}.$$

Since z is not in a periodic orbit, then $\lim_{\varepsilon\to 0} S(\varepsilon) = +\infty$. Let $\lambda > 0$ be such that $\lambda\, \rho(\mu) \in i_* H_1(M, \mathbb{Z})$. Assume that $\varepsilon > 0$ is small enough so that

$$S(\varepsilon) > 1 \qquad \text{and} \qquad S(\varepsilon) > \lambda.$$



There is a closed differentiable curve γ_ε which is transverse to $\mathcal{L}$ which intersects I_ε only once. Let $X = \frac{d}{dt}\phi_t$ be the vector field of the projected flow ϕ_t. Then X is smooth in the flow direction and transversally Lipschitz. Let Y be a smooth vector field defined on a small neighbourhood of γ_ε such that

$$\|Y\| \equiv 1, \quad \langle Y, X\rangle > 0 \quad \text{and} \quad Y \text{ is transversal to } \gamma_\varepsilon. \tag{3}$$

Denote by ψ_s the flow of Y. Let $0 < \delta \ll \varepsilon$ and $U := \psi_{]-\delta,\delta[}(\gamma_\varepsilon)$. Let $f :]\delta,\delta[\to [0,+\infty[$ be a C^∞ function such that $f(s) = 0$ if $|s| \geq \frac{\delta}{2}$ and $\int f(s)\,ds = 1$. Define a 1-form η^ε_x by $\eta^\varepsilon(\psi_s(x), v) = f(s)\,\langle Y, v\rangle$, where $x \in \gamma_\varepsilon$. In local coordinates given by a flow box for ψ_s around γ_ε, the form η^ε can be written as $\eta^\varepsilon = f(s)\,ds$. Hence η^ε is closed form. Moreover, since Y is transversal to γ_ε, for any closed curve ζ, the integral $\oint_\zeta \eta^\varepsilon$ is the oriented intersection number of ζ with γ_ε up to a constant sign. Choose the orientation of γ_ε such that that sign is always positive. Then the cohomology class of η^ε is the Poincaré dual of the homology class of γ_ε, in particular, it is an integer class:

$$[\eta^\varepsilon] \in H^1(M,\mathbb{Z}).$$

We see η^ε as a function in TM and consider its Birkhoff limits:

$$\begin{aligned} \widetilde{\eta^\varepsilon}(x,v) &:= \limsup_{T\to+\infty} \frac{1}{T}\int_0^T \eta^\varepsilon(\varphi_t(x,v))\,dt \\ &= \limsup_{T\to+\infty} \frac{1}{T}\int_0^T \eta^\varepsilon\left(\tfrac{d}{dt}\phi_t(x)\right)\,dt. \end{aligned} \tag{4}$$

For $x \in I_\varepsilon$, let $0 < \tau(x) \leq +\infty$ be its first return time to I_ε. Let $J_{\varepsilon,U} := \phi_{[-\frac{1}{2},-\frac{1}{4}]}(U) \cap I_\varepsilon$. If $x \in J_{\varepsilon,U}$, then the segment $\phi_{[0,\tau(x)]}(x)$ intersects exactly once the curve γ_ε. Moreover, its endpoints are not in U. Let $\rho : [0,\tau(x)] \to M$ be a smooth curve, homotopic with fixed endpoints to $[0,\tau(x)] \ni t \mapsto \phi_t(x)$, and such that $\frac{d}{dt}\rho(t) = Y(\rho(t))$ at the points $\rho(t) \in U$. Since η^ε is closed, we have that

$$\int_0^{\tau(x)} \eta^\varepsilon\left(\tfrac{d}{dt}\phi_t(x)\right)\,dt = \int_\rho \eta^\varepsilon = \int_{-\delta}^{\delta} f(s)\,\|Y(\rho)\|^2\,ds = 1. \tag{5}$$

Moreover, by the definition of $S(\varepsilon)$, $\tau(x) \geq S(\varepsilon)$. For $x \in \mathcal{L}$ let $N(x,T)$ be the number of returns of x to I_ε in the orbit segment $\phi_{[0,T]}(x)$. Then

$$T \geq S(\varepsilon)\,[N(x,T) - 2] \tag{6}$$

Decomposing the integral in (4) into return times, since $\eta^\varepsilon \equiv 0$ outside U, we have that

$$\int_0^T \eta^\varepsilon\left(\tfrac{d}{dt}\phi_t(x)\right)\,dt \leq N(x,T).$$

Let χ_ε be the characteristic function for $\phi_{[0,1]}(J_{\varepsilon,U})$. Each time an orbit segment reaches $J_{\varepsilon,U}$, it crosses U in the following time interval of length 1. Then

$$\int_0^T \chi_\varepsilon(\phi_t(x))\,dt \leq M(x,T) + 2,$$

where $M(x,T)$ is the number of times at which the orbit segment $\phi_{[0,T]}(x)$ crosses U. Since $\langle Y, X\rangle > 0$ then $\eta^\varepsilon\left(\frac{d}{dt}\phi_t(x)\right) \geq 0$. From (5), we get that

$$-2 + \int_0^T \chi_\varepsilon(\phi_t(x))\, dt \leq \int_0^T \eta^\varepsilon\left(\tfrac{d}{dt}\phi_t(x)\right)\, dt \leq N(x,T).$$

Hence, using (6),

$$-\frac{2}{T} + \frac{1}{T}\int_0^T \chi_e(\phi_t(x))\, dt \leq \frac{1}{T}\int_0^T \eta^\varepsilon\left(\tfrac{d}{dt}\phi_t(x)\right)\, dt \leq \frac{1}{S(\varepsilon)}\left[1 + \frac{2}{N(x,T)-2}\right].$$

By Poincaré recurrence we have that $\lim_{T\to+\infty} N(x,T) = +\infty$ for ν-almost every $x \in \mathcal{L}$. Taking the lim sup, integrating with respect to ν, and using Birkhoff's theorem, we get that

$$0 < \kappa := \nu\left(\phi_{[0,1]}(J_{\varepsilon,U})\right) \leq \int \eta^\varepsilon\, d\mu \leq \frac{1}{S(\varepsilon)}.$$

Recall that $\lambda > 0$ is such that $\lambda\,\rho(\mu) \in i_* H_1(M,\mathbb{Z})$ and that $S(\varepsilon) > \lambda$. Then

$$\mathbb{Z} \ni \langle[\eta^\varepsilon], \lambda\,\rho(\mu)\rangle = \lambda \int \eta^\varepsilon\, d\mu \in \left(\kappa\,\lambda, \tfrac{\lambda}{S(\varepsilon)}\right) \subset\ (0,1).$$

This is a contradiction. Hence z is in the projection of a periodic orbit in $\operatorname{supp}(\mu)$.

Now, if M is non-orientable, we can lift the lamination $\mathcal{L}$ and the flow ϕ_t to the double cover $\hat{M}$ of M. The asymptotic cycle of the lift $\hat{\nu}$ of ν is still rational. Then $\operatorname{supp}(\hat{\nu})$ is a union of closed curves, and hence $\operatorname{supp}(\mu)$ is a union of periodic orbits. ☐

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