ON THE DENSITY OF ALGEBRAIC FOLIATIONS WITHOUT ALGEBRAIC INVARIANT SETS

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Abstract. Let X be a complex projective variety of dimension greater than or equal to 2, and let $k \gg 0$ be an integer. We prove that a generic global section of the twisted tangent sheaf $\Theta_X(k)$ gives rise to a foliation of X without any proper algebraic invariant subvarieties of non-zero dimension. We also extend this result to fields of m-vectors over X.

1. Introduction

The study of holomorphic foliations over projective varieties can be traced back to the work of G. Darboux and H. Poincaré in the 19th century. In the papers [6] and [24] they studied differential equations over the complex projective plane and posed several questions concerning projective algebraic curves invariant under holomorphic foliations, many of which are still actively pursued.

In the late 1970s, J. P. Jouanolou reworked and extended the work of Darboux [6] in the algebraic geometric framework provided by Grothendieck. One of the key results of Jouanolou's celebrated monograph [13, théorème 1.1, p. 158] states that a very generic holomorphic foliation of the projective plane, of degree at least 2, does not have any invariant algebraic curves. Recall that a property P holds for a *very generic point* of a variety V if the set of points on which it fails is contained in a countable union of hypersurfaces of V. Jouanolou's result has been extended in various ways; see [16], [25], [17], [18] and [20].

In this paper we prove a generalization of Jouanolou's result for one dimensional foliations over any smooth projective variety. Our result is related to a problem posed by V. I. Arnold in [2, §10, pp. 6-7], and it also leads to a simpler proof of [18, theorem 2, p. 533]. Throughout the paper X denotes a smooth complex projective variety of dimension $d \geq 2$

Theorem 1.1. *Let $k \gg 0$ be an integer, and let f be a very generic global section of the twisted tangent sheaf $\Theta_X(k)$. The foliation of X determined by f has no proper invariant algebraic subvarieties of non-zero dimension.*

It should be noted that a similar result does not hold for foliations of codimension 1 when the underlying variety has dimension greater than 2. This follows from the fact that, for $n \geq 3$, the space of foliations of any degree over $\mathbb{P}^n$ has a logarithmic

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component, in the sense of [5, Theorem 2, p. 580]. The result also fails for more general compact holomorphic manifolds, even in dimension 2 [14, théorème 2.3.1, p. 171], and for varieties over fields of positive characteristic [23].

On the other hand, it is possible to generalize theorem 1.1 to fields of m-vectors, or Pfaff systems as they are sometimes called, and in section 6 we prove the following theorem.

Theorem 1.2. *Let $k \gg 0$ be an integer, and let f be a very generic section of $\bigwedge^m \Theta_X(k)$.*

(1) *If $1 \leq m \leq d-1$, then f has no proper invariant algebraic subvarieties of non-zero dimension.*
(2) *If $2 \leq m \leq d-2$, then f has no singular points.*

As an application of theorem 1.1 we prove in section 7 the following *dynamical* characterization of ampleness when X is a surface.

Theorem 1.3. *A line bundle $\mathcal{L}$ on a smooth projective surface X is ample if, and only if $\mathcal{L}^2 > 0$, and there exists a positive integer k such that a section of $\Theta_X \otimes \mathcal{L}^{\otimes k}$ induces a foliation of X without invariant algebraic curves.*

2. Fields of m-vectors

In this section we collect some facts about fields of m-vectors and foliations that are used to prove the main theorems.

Throughout the paper X denotes a smooth complex projective variety of dimension $d \geq 2$, and $i : X \to \mathbb{P}^n$ its embedding in projective space. Let Θ_X be the tangent sheaf of X and let $\mathcal{L}$ be a line bundle over X. A *field of m-vectors* of X is an $\mathcal{O}_X$-homomorphism $f : \Omega_X^m \to \mathcal{L}$.

A field of 1-vectors f determines a *singular foliation of dimension one* $\mathcal{F}$ of X. The same foliation is also completely defined by the kernel of f. Throughout the paper, such an $\mathcal{F}$ is simply called a *foliation* of X. The bundle $\mathcal{L}$ is sometimes called the *cotangent bundle* of the foliation $\mathcal{F}$.

Lemma 2.1. *The field of m-vectors $f : \Omega_X^m \to \mathcal{L}$ can also be defined by*

(1) *a global section of $\bigwedge^m \Theta_X \otimes \mathcal{L}$;*
(2) *the $\mathcal{O}_X$-homomorphism $f^\vee : \mathcal{L}^\vee \to \bigwedge^m \Theta_X$;*

where $\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$.

We swap between these definitions, whenever needed, without further comment. Moreover, we do not always distinguish between a foliation $\mathcal{F}$ and the map or section f that is used to define it.

A *singularity* of the field of m-vectors f is a point $x \in X$ such that f is not surjective at x. The set of all singularities of f is denoted by $\mathrm{Sing}(f)$.

A projective algebraic subvariety Y of X is *invariant* under f if there exists a map $\Omega_Y^m \to \mathcal{L}|_Y$ such that the diagram

$$\begin{array}{ccc} \Omega_X^m|_Y & \xrightarrow{f|_Y} & \mathcal{L}|_Y \\ \downarrow & \nearrow & \\ \Omega_Y^m & & \end{array}$$

is commutative. In particular, if $\dim(Y) < m$ then $\Omega_Y^m = 0$, and consequently $Y \subseteq \mathrm{Sing}(f)$.

If Y is a smooth subvariety, we also say that f is *extended* from the field of m-vectors g of Y defined by the map $\Omega_Y^m \to \mathcal{L}|_Y$.

Theorem 2.2 (Extension theorem)**.** *Let $k \gg 0$ be an integer. Then every field of m-vectors $f : \Omega_X^m \to \mathcal{O}_X(k)$ extends to a field $\Omega_{\mathbb{P}^n}^m \to \mathcal{O}(k)$.*

Proof. We must show that the horizontal dotted line in the diagram

$$\begin{array}{ccc} i^*\Omega_{\mathbb{P}^n}^m & \dashrightarrow & \mathcal{O}_X(k) \\ \downarrow & \nearrow & \\ \Omega_X^m & & \end{array}$$

is the pullback of a map $\Omega_{\mathbb{P}^n}^m \to \mathcal{O}(k)$ under i. But, by adjointness,

$$\mathrm{Hom}(i^*(\Omega_{\mathbb{P}^n}^m), \mathcal{O}_X(k)) \cong \mathrm{Hom}(\Omega_{\mathbb{P}^n}^m, i_*(\mathcal{O}_X(k)).$$

Hence, it is enough to show that the map

$$\mathrm{Hom}(\Omega_{\mathbb{P}^n}^m, \mathcal{O}(k)) \to \mathrm{Hom}(\Omega_{\mathbb{P}^n}^m, i_*(\mathcal{O}_X(k)) \tag{2.1}$$

is onto.

Let $\mathcal{I}$ be the kernel of the canonical map $\mathcal{O}_{\mathbb{P}^n} \to i_*(\mathcal{O}_X)$. The long exact sequence of homomorphisms of

$$0 \to \mathcal{I}(k) \to \mathcal{O}(k) \to i_*(\mathcal{O}(k)) \to 0,$$

gives

$$\mathrm{Hom}(\Omega_{\mathbb{P}^n}^m, \mathcal{O}(k)) \to \mathrm{Hom}(\Omega_{\mathbb{P}^n}^m, i_*(\mathcal{O}_X(k)) \to \mathrm{Ext}^1(\Omega_{\mathbb{P}^n}^m, \mathcal{I}(k)).$$

By [12, proposition III.6.7, p. 234] and [12, proposition III.6.3, p. 234]

$$\mathrm{Ext}^1(\Omega_{\mathbb{P}^n}^m, \mathcal{I}(k)) \cong \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^n}, \textstyle\bigwedge^m \Theta_{\mathbb{P}^n} \otimes \mathcal{I}(k)) \cong \mathrm{H}^1(\mathbb{P}^n, \textstyle\bigwedge^m \Theta_{\mathbb{P}^n} \otimes \mathcal{I}(k))$$

which is zero by Serre's theorem, [12, proposition III.5.2, p. 228]. Therefore the map of equation (2.1) is surjective, as we wanted to prove. □

We finish the section with a theorem that will play a key rôle in the proofs of the main theorems. Throughout the remainder of the section, $k \gg 0$ will be a positive integer and $\Sigma = \mathbb{P}(\mathrm{H}^0(X, \bigwedge^m \Theta_X(k)))$. The class of $f \in \mathrm{H}^0(X, \bigwedge^m \Theta_X(k))$ in Σ will be denoted by $[f]$. Define the subset of $\Sigma \times X$ by

$$\mathcal{Y} = \{([f], x) : [f] \in \Sigma \ \text{ and } \ x \in \mathrm{Sing}(f)\}.$$

Theorem 2.3. *$\mathcal{Y}$ is an irreducible subvariety of $\Sigma \times X$ of dimension*

$$d + \dim(\Sigma) - \binom{d}{m}.$$

Proof. It is clear that $\mathcal{Y}$ is a closed set, we must show that it is irreducible.

Since $k \gg 0$ it follows from Serre's theorem that $\bigwedge^m \Theta_X(k)$ is generated by its global sections. Denote by $\mathbb{T}$ the trivial bundle with fiber $\mathrm{H}^0(X, \bigwedge^m \Theta_X(k))$ and by $\pi : \mathbb{PT} \to X$ the standard projection. There exists a surjective map of vector bundles $u : \mathbb{T} \to \bigwedge^m \mathrm{T}_X(k)$ which takes $(x, \theta) \in \mathbb{T}$ to the vector $\theta(x) \in \bigwedge^m \mathrm{T}_x X(k)$.

Moreover, since u is surjective, $\ker(u)$ is also a vector bundle, and we have an exact sequence

$$\begin{array}{ccccc} \pi^*(\ker u) & \longrightarrow & \pi^*(\mathbb{T}) & \xrightarrow{\pi^*(u)} & \pi^*(\bigwedge^m \mathrm{T}_X(k)) \\ & & \uparrow{\scriptstyle j} & & \\ & & \mathcal{O}_{\mathbb{T}}(-1). & & \end{array}$$

Now $([f],x) \in \mathcal{Y}$ if and only if $\pi^*(u)j(f,x) = 0$. By [9, B.5.6, p.434], the zero scheme of $\pi^*(u)j$, which is $\mathcal{Y}$, is isomorphic to $\mathbb{P}(\ker(u))$. But $\mathbb{P}(\ker(u))$ is irreducible of dimension

$$\dim(X) + (\dim(\mathrm{H}^0(X, \bigwedge^m \Theta_X(k)))) - \mathrm{rank}(\bigwedge^m \Theta_X(k))) - 1 = d + \dim(\Sigma) - \binom{d}{m}$$

so the same holds for $\mathcal{Y}$. □

Corollary 2.4. *Let $2 \leq m \leq d-2$ be an integer and let f be a generic section of $\bigwedge^m \Theta_X(k)$. Then:*

(1) $\mathrm{Sing}(f) = \emptyset$*;*
(2) *any subscheme of Y invariant under f must have dimension at least m.*

Proof. Let $p : \Sigma \times X \to \Sigma$ be the projection on the first component of the product. Since $2 \leq m \leq d-2$, it follows from theorem 2.3 that $\dim(\mathcal{Y}) < \dim(\Sigma)$. Hence, $p(\mathcal{Y}) \subsetneq \Sigma$. But, by the definition of $\mathcal{Y}$ we have that every $f \in \Sigma \setminus p(\mathcal{Y}) \neq \emptyset$ has an empty singular set. Now (2) follows from (1) and the fact that every closed subscheme of X, invariant under a field of m-vectors and of dimension smaller than m must be contained in $\mathrm{Sing}(f)$. □

3. Foliations

In this section we collect some results on foliations, that is fields of 1-vectors, that will be used in the proof of theorem 1.1.

Let $\mathcal{E}$ be a coherent $\mathcal{O}_X$-submodule of Ω^1_X. We denote by $\mathcal{E}^a$ the sheaf whose stalk at $x \in X$ is $(\mathcal{E}^a)_x = \{\theta \in \Theta_{X,x} : \alpha(\theta) = 0 \text{ for all } \alpha \in \mathcal{E}\}$. There is a dual notion for coherent submodules of the tangent sheaf.

Let $\mathcal{F}$ be a foliation of X defined by a homomorphism $f : \Omega^1_X \to \mathcal{L}$ and let $\phi : Y \to X$ be a finite map of smooth irreducible projective varieties. Denote by $\mathcal{G}$ the image of the composition

$$\phi^*(\ker(f)) \to \phi^*(\Omega^1_X) \to \Omega^1_Y.$$

The *pullback* $\phi^\dagger(\mathcal{F})$ of $\mathcal{F}$ under ϕ is $\phi^\dagger(\mathcal{F}) = (\mathcal{G})^{aa}$. Note that we pass to $\mathcal{G}^{aa}$ in order to remove what one might call the 'apparent singularities' of $\mathcal{G}$. For more details see [26, p. 183ff].

Lemma 3.1. *Let $k \gg 0$ be an integer and let $\pi : X \to \mathbb{P}^d$ be a finite projection. If $\mathcal{F}$ is the foliation of $\mathbb{P}^d$ defined by the map $f : \Omega^1_{\mathbb{P}^d} \to \mathcal{O}(k)$, then the pullback $\pi^\dagger(\mathcal{F})$ is defined by a map $\Omega^1_X \to \mathcal{O}_X(k)$*

Proof. Since π is finite, there exists an exact sequence

$$0 \to \pi^*(\Omega^1_{\mathbb{P}^d}) \to \Omega^1_X \to \Omega^1_{X/\mathbb{P}^d},$$

whose rightmost term is a torsion $\mathcal{O}_X$-module.

Denote by $\mathcal{K}$ the kernel of f. Then, we have a commutative diagram

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
\pi^*\mathcal{K} & \longrightarrow & \pi^*\Omega^1_{\mathbb{P}^d} & \longrightarrow & \pi^*\mathcal{O}(k) \\
\downarrow & & \downarrow & & \\
\mathcal{G} & \longrightarrow & \Omega^1_X & & \\
 & & \downarrow & & \\
 & & \Omega^1_{X/\mathbb{P}^d} & &
\end{array}$$

where $\mathcal{G}$ is the image of the composition

$$\pi^*\mathcal{K} \to \pi^*\Omega^1_{\mathbb{P}^d} \to \Omega^1_X.$$

In particular the vertical map $\pi^*\mathcal{K} \to \mathcal{G}$ is surjective. Moreover, since π is finite, it follows that $\pi^*\mathcal{O}(k) = \mathcal{O}_X(k)$.

Dualizing this diagram, and taking into account the remarks above, we obtain

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \uparrow & & \\
\pi^*\mathcal{K}^\vee & \longleftarrow & \pi^*(\Omega_{\mathbb{P}^d})^\vee & \longleftarrow & \mathcal{O}_X(-k) \\
\uparrow{\scriptstyle \beta} & & \uparrow{\scriptstyle \alpha} & & \\
\mathcal{G}^\vee & \xleftarrow{\;u\;} & \Theta_X & & \\
\uparrow & & \uparrow & & \\
0 & & 0 & &
\end{array}$$

Since α is an isomorphism, every map $\mathcal{O}_X(-k) \to (\pi^*(\Omega^1_{\mathbb{P}^d}))^\vee$ gives rise to a map $v : \mathcal{O}_X(-k) \to \Theta_X$. Moreover, since α and β are injective, it follows that

$$0 \to \mathcal{O}(-k) \xrightarrow{v} \Theta_X \xrightarrow{u} \mathcal{G}^\vee$$

is exact. Thus $\ker(u) = \mathcal{G}^a = \mathrm{im}(v)$, so that $\pi^\dagger(\mathcal{F}) = \mathcal{G}^{aa}$ is the kernel of a map $\Omega^1_X \to \mathcal{O}_X(k)$ which is the dual of v. $\square$

Next, we need some results about the singularities of a foliation. Let $\mathcal{F}$ be a foliation of X defined by the $\mathcal{O}_X$-homomorphism $f : \Omega^1_X \to \mathcal{L}$. If p is a singularity of $\mathcal{F}$ then, taking a local system of coordinates $\alpha : (\mathbb{C}^d, 0) \to X$, which maps 0 to p, we have the following commutative diagram.

$$\begin{array}{ccc}
\Omega^1_{X,p} & \xrightarrow{\quad f \quad} & \mathcal{L}_p \\
\downarrow{\scriptstyle \alpha^*} & & \downarrow{\scriptstyle \alpha^*} \\
\Omega^1_{\mathbb{C}^d,0} & \xrightarrow{\alpha^* f \alpha^{-1*}} & \mathcal{O}_{\mathbb{C}^d,0}
\end{array}$$

In this way we can identify the foliation $\mathcal{F}$ on a neighbourhood of p with a germ of section of $\mathrm{Hom}(\Omega^1_{\mathbb{C}^d}, \mathcal{O}_{\mathbb{C}^d})$; that is, with a germ of holomorphic vector field Z defined on a neighbourhood of 0 in $\mathbb{C}^n$.

The *algebraic multiplicity* of Z at 0, or equivalently of $\mathcal{F}$ at p, denoted by $m = m(Z, 0) = m(\mathcal{F}, p)$, is the total degree of the first non-zero jet of Z. In other words,

$$Z = \sum_{i=m}^{+\infty} Z_i \,,$$

where Z_i is a homogeneous vector field of degree i, and $Z_m \neq 0$. We say that $0 \in \mathbb{C}^n$ is a *dicritical* singularity of Z if $m \geq 0$ and Z_m is a multiple of the radial vector field.

When Z is non-dicritical, the jet Z_m satisfies

$$[Z_m, R] = (1 - m) Z_m,$$

hence Z_m and R form an involutive system of vector fields. This system defines a two-dimensional foliation of $\mathbb{C}^d$, whose leaves are two-dimensional cones with vertex at the origin. In other words, the leaves of this foliation are invariant under the action of $\mathbb{C}^*$ on $\mathbb{C}^d \setminus \{0\}$. Therefore, passing to the quotient variety, we have that Z_m induces a foliation on $\mathbb{P}^{d-1}$. This foliation coincides with the restriction of the saturated foliation induced by $\pi^* Z$ on the exceptional divisor. In general this foliation is not saturated even when Z is saturated, because the singular set of $\pi^* Z$ may have codimension 2 on the underlying variety, while its restriction to the exceptional divisor has codimension one.

Lemma 3.2. *Let $\mathcal{F}$ be foliation of X and let $p \in X$ be a non-dicritical singularity of $\mathcal{F}$. If (W, p) is an irreducible germ of subvariety invariant under $\mathcal{F}$, then the restriction of $\widetilde{W}$, the strict transform of W, to the exceptional divisor is invariant under $\widetilde{\mathcal{F}}$.*

Proof. By hypothesis p is a non-dicritical singularity and therefore the exceptional divisor E is invariant under $\widetilde{\mathcal{F}}$. Since $\widetilde{W}$ is invariant under $\widetilde{\mathcal{F}}$ then its intersection with E is also invariant under $\widetilde{\mathcal{F}}$ and the lemma follows. □

We end the section with some results that will play a major rôle in the proof of the main theorem.

Lemma 3.3. *Let $\mathcal{F}$ be a foliation of X. If W is an irreducible closed subvariety of X invariant under $\mathcal{F}$ then the singular locus of W is invariant under $\mathcal{F}$. In particular, if the singularities of W are isolated then*

$$\mathrm{Sing}(W) \subset \mathrm{Sing}(\mathcal{F}).$$

Proof. We may assume, without loss of generality, that W is a subvariety of X whose singular locus is not contained in $\mathrm{Sing}(\mathcal{F})$. Thus, in order to prove the global statement it is enough to show that $\mathrm{Sing}(W)$ is invariant under $\mathcal{F}$ in the neighbourhood of every point which is not a singularity of $\mathcal{F}$.

Let $p \in \mathrm{Sing}(W) \setminus \mathrm{Sing}(\mathcal{F})$, and let U be a neighbourhood of p over which $\mathcal{F}_{|U}$ is described by a nowhere vanishing holomorphic vector field Z. Let $V \subset U$ be a neighborhood of p where the local flow of Z is defined. In other words, there exists a holomorphic map $\Phi : (\mathbb{C}, 0) \times V \to U$ such that for every t, $\Phi(t, \cdot)$ is biholomorphic onto its image and

$$\frac{d}{dt}\Phi(t,z) = Z(\Phi(t,z)) \text{ and } \Phi(0,z) = z. \tag{3.1}$$

Since W is invariant under Z it follows that $\Phi(t, W \cap V) \subset W \cap U$, for every $t \in (\mathbb{C}, 0)$. Moreover, $\Phi(t,\cdot)$ is a biholomorphism, so it must preserve the singular set of W. In other words,

$$\Phi(t, \operatorname{Sing}(W) \cap V) \subset \operatorname{Sing}(W) \cap U,$$

for every $t \in (\mathbb{C}, 0)$. This proves that $\operatorname{Sing}(W)$ is invariant under $\mathcal{F}$.

When p is an isolated singularity of W we have that $\Phi(t,p) = p$, for every $t \in (\mathbb{C}, 0)$. Together with (3.1) this implies that $Z(p) = 0$, which shows that every isolated singularity of W must be a singularity of $\mathcal{F}$. □

Proposition 3.4. *Let $\mathcal{G}$ be a foliation $\mathbb{P}^n$. If an irreducible smooth closed algebraic subvariety V of $\mathbb{P}^n$ is invariant under $\mathcal{G}$ then* $\operatorname{Sing}(\mathcal{G}) \cap V \neq \emptyset$.

Proof. Suppose, by contradiction, that $\mathcal{G}$ does not have singularities on V. Then, by [15, théorème 2, p. 223], any polynomial on the Chern classes of the normal bundle $\mathcal{N}_V$ must vanish in dimension greater than $r-1$, where $r = \dim(V)$. Thus, to prove that $\mathcal{G}$ has a singularity on V it is enough to show that the rth power of the first Chern class of $\mathcal{N}_V$ is nonzero.

However, $\det(\mathcal{N}_V)$ is an ample line bundle by [11, Proposition 2.1, p. 87], so it induces an embedding j of V into $\mathbb{P}^n$. Let ω be the 2-form of V which is the pull-back of the Fubini-Study 2-form of $\mathbb{P}^n$ under j, see [10, p. 31]. The first Chern class of $\mathcal{N}_V$ is the image of ω in $\mathrm{H}^2(V, \mathbb{Q})$. Since the isomorphism

$$\mathrm{H}^0(V, \mathbb{Q}) \cong \mathrm{H}^{2d}(V, \mathbb{Q})$$

of the Hard Lefschetz Theorem is induced by ω, we conclude that $c_1(\mathcal{N}_V)^r \neq 0$, which completes the proof. □

Theorem 3.5. *Let $\theta \in \Theta_X(k)$ for some integer $k \gg 0$. If W is a closed subscheme of X invariant under θ then $W \cap \operatorname{Sing}(\theta) \neq \emptyset$.*

Proof. Since $k \gg 0$, it follows that θ extends to $\mathbb{P}^n$. Thus we may assume that $X = \mathbb{P}^n$. If W is contained in the singular set of θ, there is nothing to do. Otherwise, W is invariant under the saturation $\mathcal{F}$ of θ in $\mathbb{P}^n$. But this implies that each irreducible component of the reduced scheme W_{red} is also invariant under $\mathcal{F}$. Therefore, we may assume that W is an irreducible closed subvariety of $\mathbb{P}^n$.

We proceed by induction on $\dim(W)$. If $\dim(W) = 0$ then W is an invariant point, hence a singularity of $\mathcal{F}$. Suppose that the result holds for all invariant closed subvarieties of dimension less than r. If W is an invariant subvariety of dimension r, then either W is smooth, or it has a singularity set V of dimension smaller than r. In the first case, the result follows from proposition 3.4. If W is singular, then its singularity set V is invariant under $\mathcal{F}$ by lemma 3.3. Thus,

$$\emptyset \neq V \cap \operatorname{Sing}(\mathcal{F}) \subseteq W \cap \operatorname{Sing}(\mathcal{F}),$$

by the induction hypothesis; and the theorem is proved. □

4. Some geometry

Throughout this section $k \gg 0$ and m will be positive integers and Σ will denote the projective space $\mathbb{P}(\bigwedge^m \mathrm{H}^0(X, \Theta_X(k)))$. The class of $f \in \bigwedge^m \mathrm{H}^0(X, \Theta_X(k))$ in Σ will be denoted by $[f]$. It follows from Serre's theorem that $\bigwedge^m \Theta_X(k)$ is generated by its global sections. Denote by $\mathbb{T}$ the trivial bundle with fibre $\bigwedge^m \mathrm{H}^0(X, \Theta_X(k))$. There exists a surjective map of vector bundles $u : \mathbb{T} \to \bigwedge^m \mathrm{T}_X(k)$ which takes $(x, \theta) \in \mathbb{T}$ to the m-vector $\theta(x) \in \bigwedge^m \mathrm{T}_x X$.

If $\pi : \mathbb{P}\mathbb{T} \to X$ is the standard projection, there exists a diagram

(4.1)
$$\begin{array}{ccc} \pi^*(\mathbb{T}) & \xrightarrow{\pi^*(u)} & \pi^*(\bigwedge^m \mathrm{T}_X(k)) \\ {\scriptstyle j}\uparrow & \nearrow {\scriptstyle v} & \\ \mathcal{O}_{\mathbb{T}}(-1). & & \end{array}$$

Now v gives rise to a map

(4.2)
$$\Omega^m_{\Sigma \times X/\Sigma} \to \mathcal{O}_{\mathbb{T}}(k+1),$$

which plays the rôle of a universal field of m-vectors over X. Note that $\mathbb{P}(\mathbb{T}) = X \times \Sigma$. Let S be a scheme and consider the diagram

$$\begin{array}{ccc} \mathcal{X} = \Sigma \times X \times S & \xrightarrow{q_1} & \Sigma \times S \\ \downarrow {\scriptstyle q_3} & & \downarrow \\ \Sigma \times X & \longrightarrow & \Sigma \end{array}$$

where q_1 and q_3 are the canonical projections. Then it follows from (4.2) by base change that

$$g : \Omega^m_{\mathcal{X}/\Sigma \times S} \to q_3^*(\mathcal{O}_{\mathbb{T}}(k+1)).$$

Now, let $V \subset X \times S$ be a flat family over S. The pull-back $\widetilde{V} \subset \mathcal{X}$ of V under the cnonical projection $q_2 : \mathcal{X} \to X \times S$ is a flat family over $T = \Sigma \times S$. Moreover, for a given $t = ([f], s) \in \Sigma \times S$ the scheme V_s is invariant under the field of m-vectors $f : \Omega^1_X \to \mathcal{O}_X(k)$ if and only if the map θ defined by

(4.3)
$$\begin{array}{ccc} 0 & & \\ \downarrow & & \\ \mathcal{K} & \searrow {\scriptstyle \theta} & \\ \downarrow & & \\ (\Omega^m_{\mathcal{X}/T})_{|\widetilde{V}} & \xrightarrow{g|\widetilde{V}} & q_3^*(\mathcal{O}_{\mathbb{T}}(k+1))_{|\widetilde{V}} \\ \downarrow & & \\ \Omega^m_{\widetilde{V}/T} & & \\ \downarrow & & \\ 0 & & \end{array}$$

is zero at t. We want to show that the set

$$\mathcal{Z} = \{([f], s) \in T : V_s \text{ is invariant under } \ f\} = \{t \in T : \theta_t = 0\}$$

is closed in $\Sigma \times S$. But first we need a technical lemma.

Proposition 4.1. *Let $p : \mathcal{X} \to T$ be a proper morphism. Assume that $\mathcal{F}$ is a p-flat coherent $\mathcal{O}_{\mathcal{X}}$-module such that $\mathrm{R}^i p_* \mathcal{F} = 0$, for all $i > 0$. If $\mathcal{G}$ is a quasi-coherent $\mathcal{O}_T$-module and $\sigma : p^*\mathcal{G} \to \mathcal{F}$ is a homomorphism of $\mathcal{O}_{\mathcal{X}}$-modules, then the zero scheme of σ is closed in T.*

Proof. The result follows immediately from [1, proposition 2.3, p. 16] if we prove that $p_*\mathcal{F}$ is locally free, and that its formation commutes with base change.

However, p is proper and $\mathcal{F}$ is $\mathcal{O}_{\mathcal{X}}$-coherent, so that $p_*(\mathcal{F})$ is $\mathcal{O}_T$-coherent by [7, théorème 3.2.1, p. 460]. Since $\mathcal{F}$ is p-flat it follows that $p_*(\mathcal{F})$ is also flat over $\mathcal{O}_T$. Thus it is locally free, as required. The fact that taking the direct image commutes with base change follows from [22, p. 19]. □

Proposition 4.2. *The set $\mathcal{Z}$ is closed in $T = \Sigma \times S$.*

Proof. Let $\mathcal{L}$ be a very ample sheaf over $\widetilde{V}$, the pullback of V under $q_2 : \mathcal{X} \to T$. Given an integer $r \gg 0$ it follows by Serre's theorem that, for some positive integer N, there exists a surjective map $\alpha : \mathcal{O}_{\widetilde{V}}^N \to \mathcal{K} \otimes \mathcal{L}^r$. Denote by σ the composition

$$\mathcal{O}_{\widetilde{V}}^N \xrightarrow{\alpha} \mathcal{K} \otimes \mathcal{L}^r \to (\Omega^m_{\mathcal{X}/T})_{|\widetilde{V}} \otimes \mathcal{L}^r.$$

Since α is surjective, $\theta_t = 0$, for some $t \in T$ if and only if $\sigma_t = 0$. But this implies that $\mathcal{Z}$ is the scheme of zeroes of σ. We must show that this scheme is closed in T.

In order to do this we apply proposition 4.1 with $\mathcal{F} = q_3^*(\mathcal{O}_{\mathbb{T}}(k+1))_{|\widetilde{V}} \otimes \mathcal{L}^r$ and $\mathcal{G} = \mathcal{O}_T^{\oplus N}$. Denoting by p the composition of the embedding of $\widetilde{V}$ in $\mathcal{X}$ with the projection q_1, we have that $p^*\mathcal{G} = \mathcal{O}_{\widetilde{V}}^{\oplus N}$, and σ is a map $p^*\mathcal{G} \to \mathcal{F}$. Moreover, $\mathrm{R}^i p_* \mathcal{F} = 0$ for $i > 0$ by [12, theorem III.8.8, p. 252]. Since $\mathcal{Z}$ is the scheme of zeroes of σ, the result follows from proposition 4.1. □

5. Proof of theorem 1.1

Throughout this section $k \gg 0$ is a positive integer and $\Sigma = \mathbb{P}(\mathrm{H}^0(X, \Theta_X(k)))$. The class of $f \in \mathrm{H}^0(X, \Theta_X(k))$ in Σ will be denoted by $[f]$. Let $\chi \in \mathbb{Q}[t]$. Define two subsets of $\Sigma \times X$ by

$$\mathcal{Y} = \{([f], x) : [f] \in \Sigma \ \text{ and } \ x \in \mathrm{Sing}(f)\},$$

as in section 2, and

$$\mathcal{X} = \{([f], x) : x \text{ is in a subscheme, invariant under } f\text{, with Hilbert polynomial } \chi\}.$$

We will write $\mathcal{X}_\chi$ if we need to call attention to the corresponding Hilbert polynomial. Let p_1 and p_2 denote the projections of $\Sigma \times X$ on the first and second coordinates, respectively.

Let $\mathbf{Hilb}_\chi(X)$ be the Hilbert scheme of X with respect to the Hilbert polynomial χ. Denote by V_s the closed subscheme of X that corresponds to $s \in \mathbf{Hilb}_\chi(X)$.

Lemma 5.1. *$\mathcal{X}$ is a closed subset of $\Sigma \times X$.*

Proof. Let

$$\begin{array}{ccc} \Sigma \times X \times \mathbf{Hilb}_\chi(X) & \xrightarrow{q_1} & \Sigma \times \mathbf{Hilb}_\chi(X) \\ \downarrow q_2 & \searrow q_3 & \\ X \times \mathbf{Hilb}_\chi(X) & & \Sigma \times X \end{array}$$

be the canonical projections. Since $k \gg 0$, it follows from theorem 4.2 that

$$\mathcal{Z}_\chi = \{([f], s) : \ V_s \text{ is invariant under } \ f\}$$

is a closed subset of $\Sigma \times \mathbf{Hilb}_\chi(X)$. Let $\mathcal{C}$ be the universal family in $X \times \mathbf{Hilb}_\chi(X)$. It follows that

$$\mathcal{X} = q_3(q_1^{-1}(\mathcal{Z}_\chi) \cap q_2^{-1}(\mathcal{C}))$$

is closed in $\Sigma \times X$, as we wished to prove. □

Lemma 5.2. *Let $\pi : X \to \mathbb{P}^d$ be a finite projection and let $x \in X$. Suppose that $\mathcal{G}$ is a foliation of $\mathbb{P}^d$ and that $\pi(x)$ is an isolated singular point of $\mathcal{G}$ that does not belong to the branch locus of π. Then x is a singular point of $\pi^\dagger(\mathcal{G})$ and if no invariant proper subvariety of dimension $r > 0$ of $\mathcal{G}$ passes through $\pi(x)$, then no subvariety of dimension $r > 0$, invariant under $\pi^\dagger(\mathcal{G})$, passes through x.*

Proof. Suppose, by contradiction, that $Y \subseteq X$ is an irreducible algebraic closed subvariety of dimension $r > 0$ that passes through x and is invariant under $\pi^\dagger\mathcal{G}$. Since π is a proper map, $\pi(Y)$ is an algebraic subvariety of dimension r of $\mathbb{P}^d$. Moreover, $\pi(Y)$ is not invariant under $\mathcal{G}$ by hypothesis.

Since $\pi(x)$ does not belong to the branch locus of π, there exist neighbourhoods U of x and V of $\pi(x)$ such that $\pi|_U : U \to V$ is a biholomorphism. But, $\pi(Y) \cap V$ is not invariant under $\mathcal{G}_{|V}$. Since $\pi_{|U}$ is a biholomorphism, this contradicts the fact that $Y \cap U$ is invariant under $\pi^\dagger(\mathcal{G})_{|U}$. □

Corollary 5.3. *Let $\pi : X \to \mathbb{P}^d$ be a finite projection and let $x \in X$. Then, for all $k \geq 3$ there exists $f \in \Theta_X(k)$ such that the corresponding foliation $\mathcal{F}$ of X has a singularity at x which is not contained in any algebraic curve invariant under f.*

Proof. The result follows from lemma 5.2 and [17, Theorem 2, p. 653]. □

From now on we denote by p the restriction to $\mathcal{X}$ of the projection $p_1 : \Sigma \times X \to \Sigma$.

Lemma 5.4. *Suppose that*

1. *$p^{-1}([f]) \cap \mathrm{Sing}(f) \neq \emptyset$ for every $[f] \in \Sigma$, and that*
2. *there exists a foliation $\mathcal{F}$ of $\mathbb{P}^d$, of degree $k - 1$, and a singularity x of $\mathcal{F}$ which is not contained in any closed subscheme of $\mathbb{P}^d$ invariant under $\mathcal{F}$.*

Then $p(\mathcal{X})$ is a proper closed subset of Σ.

Proof. Since p is a proper map, it is enough to show that $p(\mathcal{X}) \subsetneq \Sigma$. We will assume, by contradiction, that $p(\mathcal{X}) = \Sigma$. Thus, for every $[f] \in \Sigma$ the subvariety $p^{-1}([f])$ is invariant under f. Moreover, hypothesis (1) implies that

$$p(\mathcal{X} \cap \mathcal{Y}) = \Sigma.$$

Since $\dim(\mathcal{Y}) = \dim(\Sigma)$, it follows that $\dim(\mathcal{X} \cap \mathcal{Y}) = \dim(\mathcal{Y})$. However, $\mathcal{Y}$ is irreducible by lemma 2.3, therefore $\mathcal{X} \cap \mathcal{Y} = \mathcal{Y}$. This means that given $f \in \Sigma$ and $x \in \mathrm{Sing}(f)$, there exists a closed subscheme V_x of X, invariant under f, with Hilbert polynomial χ, and such that $x \in V_x$.

Now, let $\pi : X \to \mathbb{P}^d$ be a finite projection. After a linear change of coordinates, we may assume that the singularity x of $\mathcal{F}$ does not belong to the branch locus of π. Thus, it follows from lemma 5.2 that the pullback of $\mathcal{F}$ under π has a singular point which is not contained in any invariant closed subscheme of X with Hilbert polynomial χ. This contradicts the statement at the end of the previous paragraph, and completes the proof of the lemma. □

We are ready to prove theorem 1.1

Proof of theorem 1.1 For every $(d, r) \in \mathbb{N}^2$ with $d \geq 2$ and $1 \leq r \leq d-1$, consider the following statement:

$A(d, r)$: if X is a d-dimensional smooth projective variety and $k \gg 0$ is an integer, then a very generic section of $\Theta_X(k)$ does not have any invariant closed r-dimensional subschemes.

Note, first of all, that since Σ is irreducible, and since there are only countably many Hilbert polynomials, then $A(d, r)$ follows if we prove that

$$p(\mathfrak{X}_\chi) \not\subset \Sigma = \mathbb{P}(\mathrm{H}^0(X, \Theta_X(k))),$$

for all $\chi \in \mathbb{Q}[t]$ of degree r. Thus we may assume, from now on, that $\chi \in \mathbb{Q}[t]$ is such a polynomial.

We begin by proving that $A(d, 1)$ holds for all $d \geq 2$. Suppose that χ has degree one. It follows from theorem 3.5 that

$$p^{-1}([f]) \cap \mathrm{Sing}([f]) \neq \emptyset$$

for every $[f] \in \Sigma$. Thus, by corollary 5.3 and lemma 5.4, $p(\mathfrak{X}_\chi)$ is a proper closed subset of Σ. As we noted above this is enough to prove $A(d, 1)$.

In order to prove the theorem by induction it is enough to show that $A(d, r)$ follows from $A(d-1, r-1)$, for every $r \geq 2$. But, just as in the case $r = 1$ dealt with above, $A(d, r)$ follows from corollary 3.5 and lemma 5.4 if we prove the following statement

> there exists a foliation $\mathcal{F}$ of $\mathbb{P}^d$, of degree $k-1$, and a singularity x of $\mathcal{F}$ which is not contained in any closed r-dimensional subscheme of $\mathbb{P}^d$ invariant under $\mathcal{F}$.

The case $d = 2$ is covered by $A(d, 1)$. We show that for $d \geq 3$, the statement follows from $A(d-1, r-1)$ applied to $X = \mathbb{P}^{d-1}$. If f is a very generic section of $\Theta_{\mathbb{P}^{d-1}}(k)$ then $A(d-1, r-1)$ implies that the only proper closed subschemes invariant under f are its singularities. Let Z be an homogeneous vector field on $\mathbb{C}^d$ such that $\omega(f) = i_Z\omega$. Note that any two such Z must differ by a multiple of the radial vector field. The vector field Z induces a one dimensional foliation on $\mathbb{C}^d$. Its singular set is contained in union of the lines through the origin whose directions correspond to the singular points of f on $\mathbb{P}^{d-1}$. Moreover, since f is very generic the origin is a non-dicritical singular point of Z.

If there exists a germ of r-dimensional subvariety at the origin that is invariant under Z then, by lemma 3.2, f admits a proper invariant algebraic set of dimension $r-1$. But this is impossible by the choice of f. Therefore the origin is a singularity of Z which is not contained in any invariant r-dimensional germ of subvariety. Extending this foliation to $\mathbb{P}^d$ we have proved the statement above, and the proof of the theorem is complete.

Remark. In the above proof, the integer k depends on the variety X. However, when X is a projective space (of any dimension) we may take k to be any integer greater than or equal to 2. Hence, the choice of k does not interfere with the induction step, since we use $A(d-1, r-1)$ only to prove $A(d, r)$ for $\mathbb{P}^d$.

6. Proof of theorem 1.2

Let $k \gg 0$ and m be positive integers. As in section 2, denote by Σ the projective space $\mathbb{P}(\mathrm{H}^0(X, \bigwedge^p \Theta_X(k)))$. For $i = 1, \ldots, m$, let $k_i \gg 0$, be positive integers which add up to k, and write

$$\Psi : \bigoplus_{i=1}^{m} H^0(X, \Theta_X(k_i)) \to H^0(X, \bigwedge{}^m \Theta_X(k))$$

for the natural map.

Lemma 6.1. *For $i = 1, \ldots, m$, let $k_i \gg 0$, be positive integers which add up to k. If f is a generic element of $\bigoplus_{i=1}^{m} H^0(X, \Theta_X(k_i))$ then*

$$\dim \mathrm{Sing}(\Psi(f)) = m - 1.$$

Proof. For $m = 1$ the result follows from Theorem 2.3. Suppose that the result holds for $m - 1$ and let $g = (g_1, \ldots, g_{m-1})$ be an element of $\bigoplus_{i=1}^{m-1} H^0(X, \Theta_X(k_i))$ such that

$$\dim \mathrm{Sing}(\Psi(g)) = m - 2.$$

Denote by U the complement of $\mathrm{Sing}(g)$ in X. Consider the trivial bundle $\mathbb{T}_U$ over U with fibre $\mathrm{H}^0(U, \Theta_U(k_m))$. Since the codimension of $\mathrm{Sing}(g)$ is at least 2, it follows that $H^0(X, \Theta_X(k_m)) \cong H^0(U, \Theta_U(k_m))$. Thus $\mathbb{T}_U$ is the restriction to U of the bundle $\mathbb{T}$ defined at the beginning of section 4. Once again we have a map of vector bundles $u : \mathbb{T}_U \to \bigwedge^m TU(k)$, of constant rank, which takes $(x, \theta) \in \mathbb{T}_U$ to the m-vector $(g \wedge \theta)(x) \in \bigwedge^m T_x U$. Thus $\ker(u)$ has dimension

$$\dim X + (h^0(U, \Theta_U(k)) - \mathrm{rank}(\mathrm{Im}(u))).$$

But $\mathrm{rank}(\mathrm{Im}(u)) = \dim X - (m-1)$, so that $\dim \ker(u) = h^0(U, \Theta_U(k)) + (m-1)$. However, as in the proof of theorem 2.3, we have that $\mathbb{P}(\ker(u))$ is isomorphic to the set of $([\theta], x)$ such that x is a singularity of $g \wedge \theta$. Hence, for a generic θ,

$$\dim \mathrm{Sing}(g \wedge \theta) = m - 1\,.$$

□

We may now prove theorem 1.2.

Proof of theorem 1.2. If $m = 1$ then the theorem has already been proved, so we may assume that $m > 1$.

Since $\mathbb{Q}[t]$ is a countable set, it is enough to prove that, for a given $\chi \in \mathbb{Q}[t]$, the generic field of m-vectors does not have have any invariant subvariety with Hilbert polynomial χ.

But the set of m-vectors which do not admit an invariant closed subvariety with Hilbert polynomial χ is open in Σ by proposition 4.2. Thus, the result follows if we prove that this open set is non-empty. If $m > \deg(\chi)$ this is a consequence of of corollary 2.4. So we may assume that $m < \deg(\chi)$.

For $1 \le i \le m$ choose integers $k_i \gg 0$ which add up to k. It follows from theorem 1.1 that there exist sections g_i of $\Theta_X(k_i)$ which do not have any proper invariant closed algebraic subvarieties apart from their singularities. Write $g = (g_1, \dots, g_m)$. By lemma 6.1 the singularity set of $\Psi(g)$ has dimension $m - 1 < d$. But, if a closed subvariety Y of X, invariant under $\Psi(g)$, goes through a non-singular point of $\Psi(g)$ then it must be invariant under each g_i. This contradicts the fact that $\dim(Y) = \deg(\chi) > m$, and the proof of (1) is complete. (2) follows from corollary 2.4.

7. Foliations on Surfaces

The birational theory of foliations on surfaces will play a decisive rôle in this section. For details see [19, 21], and specially the last three chapters of [4]. Throughout this section S denotes a smooth complex projective surface and $\mathcal{F}$ a foliation $f : \Omega^1_S \to \mathcal{L}$ over S.

Let C be a curve on S and p a point of C. Denote by $\mathcal{O}_p$ the local algebra of S at p. If the the curve has local equation $f = 0$ and the foliation is described by a vector-field v in a neighbourhood of p, let

$$tang(\mathcal{F}, C, p) = \dim_{\mathbb{C}} \frac{\mathcal{O}_p}{(f, v(f))}.$$

Note that this number is 0 except at the finite number of points where f is not transverse to C. Define the *tangency number* between $\mathcal{F}$ and C by

$$tang(\mathcal{F}, C) = \sum_{p \in C} tang(\mathcal{F}, C, p).$$

Lemma 7.1. *If $\mathcal{F}$ is a foliation without algebraic invariant curves, then $\mathcal{L} \cdot C \geq 0$ for every irreducible curve C.*

Proof. By Miyaoka's theorem [4, theorem 7.1, p. 89] $\mathcal{L}$ is pseudo-effective. Thus, by [8, Theorem 1.12, p. 108], $\mathcal{L}$ can be decomposed in the form $P + N$, where P is a semi-positive $\mathbb{Q}$-divisor and $P \cdot C = 0$ for each irreducible component of the support of N. The result follows if we show that the support of N is empty.

Assume, by contradiction, that the support of N is nonempty. Then, by [8, Theorem 1.12(c), p. 108] it contains an irreducible component E such that $\mathcal{L} \cdot E < 0$ and $E^2 < 0$. Since $\mathcal{F}$ does not have any invariant algebraic curves, E cannot be invariant under $\mathcal{F}$. Then, by [4, proposition 2.2, p.23],

$$\mathcal{L} \cdot E = tang(\mathcal{F}, E) - E^2,$$

so that $\mathcal{L} \cdot E > 0$, contradicting the choice of E. □

Proof of Theorem 1.3 Let $\mathcal{F}$ be a foliation of S with no algebraic invariant curves and suppose that $\mathcal{L} \cdot C = 0$ for some irreducible curve C on S. Since $\mathcal{L}^2 > 0$, by hypothesis, it follows from the Hodge index theorem [3, corollary 2.4, p.18] that $C^2 < 0$. However, [4, proposition 2.2, p.23] implies that $C^2 = tang(\mathcal{F}, C) \geq 0$, and we conclude that there exists no such C. Hence, it follows from lemma 7.1 that $\mathcal{L} \cdot C > 0$ for every irreducible curve C. Thus $\mathcal{L}$ is ample by the Nakai-Moishezon criterion [12, theorem 1.10,p. 365].

In order to prove the converse, assume that $\mathcal{L}$ is an ample line-bundle. Hence, $\mathcal{L}^{\otimes n}$ is very ample for some positive integer n. In other words, there exists an embedding of S in a projective space so that $\mathcal{L}^{\otimes n} \cong \mathcal{O}_S(1)$. Hence, by the main theorem, $\mathcal{L}^{\otimes kn}$ is the cotangent bundle of a foliation without invariant algebraic curves whenever $k \gg 0$, and the proof is complete.

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