Unique Continuation Principles for some Equations of Benjamin-Ono Type.

Rafael José Iorio, Jr.
Instituto Nacional de Matemática Pura e Aplicada (IMPA).
Estrada Dona Castorina 110
Jardim Botânico, Rio de Janeiro, RJ, Brazil
22460-320

1 Introduction.

By equations of Benjamin-Ono type we mean either

$$\partial_t u(t) + \sigma L u(t) + F(u(t)) = 0, \tag{1}$$

or

$$\partial_t (u(t) + \sigma L u(t)) + F(u(t)) = 0, \tag{2}$$

where L is a linear (possibly unbounded) operator, F is (in general) a nonlinear function of its argument and σ denotes the Hilbert transform

$$(\sigma f)(x) = pv \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{y-x} dy. \tag{3}$$

In what follows we will be mainly interested in determining conditions under which the solutions of the Cauchy problems associated to (1) and (2) are identically zero. Moreover we will concentrate on the case of the Generalized Benjamin-Ono equation (GBO), namely (1) with $L = \partial_x^2$ and $F(v) = \partial_x(\Phi(v))$ where Φ maps an appropriate function space into $\mathbb{R}$.

To explain what we have in mind, consider the following problem

$$\begin{cases} \partial_t u(t) + \sigma u(t) = 0 \\ u(0) = \phi \in L^2(\mathbb{R}). \end{cases} \tag{4}$$

Recall that if $f \in L^2(\mathbb{R})$ then

$$(\sigma f)^\wedge(\xi) = isg(\xi) \widehat{f}(\xi), \ \xi \text{ a.e.}, \tag{5}$$

where

$$sgn(\xi) = \begin{cases} -1 \ if \ \xi < 0 \\ 0 \ if \ \xi = 0 \\ 1 \ if \ \xi > 0 \end{cases} \tag{6}$$

and $\widehat{g}$ denotes the Fourier transform of the tempered distribution g. In case $g \in L^1(\mathbb{R})$, our definition is

$$\widehat{g}(\xi) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} g(x) \exp(-i\xi x)\, dx. \tag{7}$$

In particular σ is an unitary operator in $L^2(\mathbb{R})$ and

$$\sigma^2 = -1. \tag{8}$$

For a proof of these facts see ([3]) or ([4]).

It is easy to verify that the unique solution of (4) is given by the following unitary group

$$u(t) = \exp(-\sigma t)\, \phi = (\cos t - \sigma \sin t)\, \phi,\ t \in \mathbb{R}. \tag{9}$$

Assume that there exists $R > 0$ and $\tau \neq 0$ such that, $\sin \tau \neq 0$, and $supp(\phi)$, $supp(u(\tau))$ are contained in $[-R, R]$. We claim that under these conditions we have $u \equiv 0$. Indeed, combining these assumptions with (9) we conclude that

$$supp(\sigma\phi) \subset [-R, R]. \tag{10}$$

Therefore,

$$supp(P_{\pm}\phi) \subset [-R, R] \tag{11}$$

where

$$P_{\pm} = \frac{1}{2}(1 \mp i\sigma) \tag{12}$$

are the projections onto the Hardy spaces

$$\mathfrak{H}^2_{\pm}(\mathbb{R}) = \left\{\varphi \in L^2(\mathbb{R})\, |\chi_{\mp}(\xi)\, \widehat{\varphi}(\xi) = 0\, a.e.\right\} \tag{13}$$

and $\chi_{\mp}$ denote the characteristic functions of $(-\infty, 0]$ and $[0, \infty)$ respectively. But then

$$(P_{\pm}\phi)^{\wedge}(z) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int_{-R}^{R} (P_{\pm}\phi)(x) \exp(-izx)\, dx,\ z \in \mathbb{C} \tag{14}$$

are entire functions. In particular they are continuous. In view of (12) and (13) we have

$$(P_{+}\phi)^{\wedge}(\xi) = 0,\ \forall\ \xi < 0 \text{ and } (P_{-}\phi)^{\wedge}(\xi) = 0,\ \forall\ \xi > 0. \tag{15}$$

Hence $(P_{\pm}\phi)^{\wedge} = 0$ and it follows that $\phi = 0$. This proves our claim.

Remark 1 *Note that the preceding arguments show that if $\phi \in L^2(\mathbb{R})$ has compact support and is such that $\sigma\phi$ also has compact support, then $\phi = 0$.*

Remark 2 *In Section 3 we will present a more fruitful way of obtaining this result.*

These ideas can be extended to nonlinear equations to a certain extent. Let $F : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ be such that $F(0) = 0$ and

$$\|F(v) - F(w)\|_0 \leq \alpha\left(\|v\|_0, \|w\|_0\right) \|v - w\|_0 \tag{16}$$

where $\alpha : [0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$ is a continuous function, nondecreasing with respect to each of its arguments and $\|\cdot\|_0$ denotes the L^2 norm. Combining Banach's fixed point theorem and Gronwall's inequality, it is possible to show that the problem

$$\begin{cases} \partial_t u(t) + \sigma u(t) + F(u(t)) = 0 \\ u(0) = \phi. \end{cases} \tag{17}$$

is globally well posed.[1]

Theorem 3 *There exists a unique $u \in C\left(\mathbb{R}, L^2(\mathbb{R})\right)$ such that $u(0) = \phi$ and the equation in (17) is satisfied with the time derivative computed in the L^2 norm, that is,*

$$\lim_{h \to 0} \left\| \frac{u(t+h) - u(t)}{h} + \sigma u(t) + F(u(t)) \right\|_0 = 0. \tag{18}$$

Now we are in position to prove a unique continuation principle for (17).

Theorem 4 *Let u be as in Theorem 3. Assume that*
(a) there exists a $t_0 \in \mathbb{R}$ and a positive R such that

$$\operatorname{supp}(u(t)) \subset [-R, R] \tag{19}$$

for all t in some neighborhood of t_0.
(b) F is local, that is, if $v \in L^2(\mathbb{R})$ has compact support then so does $F(v)$.
Then $u \equiv 0$.

Proof. Our assumptions imply that

$$supp\left(\frac{u(t_0 + h) - u(t_0)}{h}\right) \subset [-R, R] \tag{20}$$

for all sufficiently small h. In view of (18) there exists a sequence (h_n) such that

$$\frac{u(t_0 + h_n) - u(t_0)}{h_n} \longrightarrow -\left(\sigma u(t_0) + F(u(t_0))\right) \tag{21}$$

almost everywhere. Thus $\sigma u(t_0) + F(u(t_0))$ must have compact support. Since F is local and $u(t_0)$ has compact support, it follows that $\sigma(u(t_0))$ also has compact support. Remark (1) implies that $u(t_0) = 0$. Uniqueness implies the theorem. ■

[1] In order to keep this paper reasonably short, here, and in what follows, we shall not dwell on the details of such applications. They are rather standard. See for instance ([5]), sections 5.1 and 6.1 of ([8]) or section Chapter 14 of ([16]).

As can be seen from the preceding remarks, there is a huge gap between the linear and the nonlinear results. At the moment it seems that this gap cannot be bridged unless we introduce more detailed structure than that allowed in Theorem 4. In what follows we will present some results in this direction. This article is organized as follows. In section 2 we sketch the well-posedness theory associated to the generalized Benjamin-Ono equation. Section 3 presents two closely related unique continuation principles for GBO. Finally, in Section 4, we make a few comments on the existing literature, and indicate some open problems.

Before proceeding it is convenient to establish notations that will be used throughout this article. Let $s \in \mathbb{R}$ and denote by $H^s(\mathbb{R})$ the L^2 type Sobolev spaces, that is, the set of all tempered distributions f such that

$$\|f\|_s^2 = \int_{\mathbb{R}} \left(1+\xi^2\right)^s \left|\widehat{f}(\xi)\right|^2 d\xi < \infty. \tag{22}$$

As is well known, they are Hilbert spaces with respect to the inner product

$$(f\,|g)_s = \int_{\mathbb{R}} \left(1+\xi^2\right)^s \widehat{f}(\xi)\,\overline{\widehat{g}(\xi)}dx. \tag{23}$$

Define the usual weighted L^2 spaces by the formula

$$L_r^2(\mathbb{R}) = \left(H^r(\mathbb{R})\right)^\wedge,\ r \in \mathbb{R}. \tag{24}$$

The norm and inner product in these spaces are given, respectively, by

$$\|f\|_{L_r^2}^2 = \int_{\mathbb{R}} \left(1+x^2\right)^r |f(x)|^2 dx \tag{25}$$

and

$$(f\,|g)_{L_r^2} = \int_R \left(1+x^2\right)^r f(x)\,\overline{g(x)}dx. \tag{26}$$

We will also have occasion to use the following weighted Sobolev spaces

$$\mathfrak{F}_{s,r} = \mathfrak{F}_{s,r}(\mathbb{R}) = H^s(\mathbb{R}) \cap L_r^2(\mathbb{R}). \tag{27}$$

They are all Hilbert spaces when provided with the inner product

$$(f\,|g)_{s,r} = (f\,|g)_s + (f\,|g)_{L_r^2}. \tag{28}$$

The corresponding norm will be denoted by $\|\cdot\|_{s,r}$. Since in what follows we will be interested only in real valued solutions, it is convenient to introduce

$$h^s(\mathbb{R}) = \operatorname{Re} H^s(\mathbb{R}). \tag{29}$$

and

$$\mathfrak{f}_{s,r} = h^s(\mathbb{R}) \cap \mathfrak{F}_{s,r}. \tag{30}$$

Constants whose precise values are of no consequence to our arguments will be denoted simply by C. If X and Y are Banach spaces, we denote the set of all bounded linear operators from Y into X by $\mathcal{B}(Y, X)$ If $Y = X$ we write simply $\mathcal{B}(Y)$. Finally, $[A, B]$ denotes the commutator of the linear operators A and B.

2 The Generalized Benjamin-Ono Equation: Local Well-Posedness.

In this section we will sketch the well-posedness theory associated to the real valued solutions of the Cauchy problem associated to the generalized Burgers-Benjamin-Ono equation (GBBO), namely[2]

$$\left\{ \begin{array}{c} \partial_t u(t) + \sigma \partial_x^2 u(t) + \partial_x \Phi(u(t)) = \mu \partial_x^2 u(t) \\ u(0) = \phi \in h^s(\mathbb{R}) \end{array} \right. \tag{31}$$

where $s > \frac{3}{2}$ is fixed, $\mu \geq 0$, and Φ: $h^s(\mathbb{R}) \longrightarrow h^s(\mathbb{R})$. If $\mu = 0$ we refer to the pde in (31) simply as the generalized Benjamin-Ono equation (GBO). Examples of such Φ 's are

$$\Phi(w) = \frac{w^{p+1}}{p+1}, \tag{32}$$

where $p \geq 1$ is an integer and

$$\Phi(w) = \cos(w). \tag{33}$$

We begin with the linear equation. If $r \in \mathbb{R}$ and $\phi \in H^r(\mathbb{R})$, a straightforward application of the Fourier transform shows that the unique solution to

$$\left\{ \begin{array}{c} \partial_t v(t) + \sigma \partial_x^2 v(t) = \mu \partial_x^2 v(t) \\ u(0) = \phi \in H^r(\mathbb{R}) \end{array} \right. \tag{34}$$

is given by

$$v_\mu(t) = E_\mu(t)\phi = \exp(-Q_\mu t)\phi = \left[F_\mu(t,\cdot)\widehat{\phi}(\cdot)\right]^\vee, \tag{35}$$

where

$$-Q_\mu = \mu \partial_x^2 - 2\sigma \partial_x^2, \tag{36}$$

and

$$F_\mu(t,\xi) = \exp\left[(-\mu + 2ih(\xi))\xi^2 t\right]. \tag{37}$$

Combining the Fourier transform with the parabolic character of the pde in (34) we obtain the following Sobolev space version of the smoothing properties of the heat equation.

Theorem 5 *Let $\mu \in (0,\infty)$, $\lambda \geq 0$ and $\phi \in H^s(\mathbb{R})$. Then there exists a constant $K_\lambda \geq 0$, depending only on λ, such that*

$$\|E_\mu(t)\phi\|_r \leq K_\lambda \left[1 + \left(\frac{1}{2\mu t}\right)^\lambda\right]^{\frac{1}{2}} \|\phi\|_s. \tag{38}$$

Moreover the map $t \in (0,\infty) \longmapsto E_\mu(t)\phi$ is continuous in the topology of $H^{r+\lambda}(\mathbb{R})$.

[2]To simplify the notation, from now on we will write $\partial_x(\Phi(v)) = \partial_x \Phi(v)$.

Proof. The proof is contained in any of the following references: ([5]), ([7]), and Theorem 4.17 and Exercise 7.96 of ([8]). ■

It is now easy to show that (31) is equivalent to the integral equation

$$u(t) = E_\mu(t)\,\phi - \int_0^t E_\mu(t-t')\,\partial_x\Phi(u(t'))\,dt' \tag{39}$$

and a standard application of Banach's fixed point theorem and Gronwall's inequality yield the next theorem. For details see ([5]), ([7]) and Theorems 8.1 and 8.2 of ([8]).

Theorem 6 *Let $\mu > 0$, $\phi \in h^s(\mathbb{R})$, $s > \frac{3}{2}$. Assume*[3] *that $\partial_x\Phi$ is Lipchitz in the following sense: there exists a continuous function $L : [0,\infty)\times[0,\infty) \longrightarrow [0,\infty)$, nondecreasing with respect to each of its arguments, such that*

$$\|\partial_x\Phi(v) - \partial_x\Phi(w)\|_{s-1} \le L(\|v\|_s, \|w\|_s)\,\|v-w\|_s\,. \tag{40}$$

Then there exists a unique $u = u_\mu \in C([0,T]; h^s(\mathbb{R}))$ such that $u_\mu(0) = \phi$ and the pde in (31) is satisfied with the time derivative computed with respect to the norm of $h^{s-2}(\mathbb{R})$. Moreover,

$$u_\mu \in C((0,T]; h^\infty(\mathbb{R})) \tag{41}$$

for all $\mu > 0$, where

$$h^\infty(\mathbb{R}) = \bigcap_{r\ge 0} h^r(\mathbb{R})\,, \tag{42}$$

provided with its usual Frechet space structure.

In general, the existence time determined in the preceding theorem tends to zero as μ tends to zero so that to establish existence of solutions with $\mu = 0$, we must be able to extend all the u_μ 's to an interval which is independent of μ.

Lemma 7 *Let $\phi \in h^s(\mathbb{R})$, as before and assume that*

$$|(w\,|\partial_x\Phi(w))_s| \le g_1\left(\|w\|_s^2\right), \tag{43}$$

where $g_1 : [0,\infty) \longrightarrow [0,\infty)$ is continuous. Let ρ denote the maximal solution of

$$\begin{cases} \partial_t\rho(t) = 2g_1(\rho(t)) \\ \rho(0) = \|\phi\|_s^2\,. \end{cases} \tag{44}$$

Then

$$\|u_\mu\|_s^2 \le \rho(t) \tag{45}$$

for all $\mu > 0$, whenever both sides are defined. Since ρ does not depend on μ it follows that all the u_μ 's can be extend to any closed interval $[0,T]$ contained in the maximal interval of existence of ρ.

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[3] Note that $\partial_x\Phi(0) = 0$.

Proof. In view of (41) we have

$$\begin{aligned}
&\partial_t \|u_\mu(t)\|_s^2 = 2\left(u_\mu(t)\,|\partial_t u_\mu(t)\right)_s \\
&= 2\left(u_\mu(t)\,\middle|\mu\partial_x^2 u_\mu(t) - \sigma\partial_x^2 u_\mu(t) - \partial_x \Phi(u_\mu(t))\right)_s \\
&= -2\mu\|\partial_x u_\mu(t)\|_s^2 - 2\left(u_\mu(t)\,\middle|\sigma\partial_x^2 u_\mu(t)\right)_s - 2\left(u_\mu(t)\,|\partial_x \Phi(u_\mu(t))\right)_s .
\end{aligned} \tag{46}$$

Since the first term on the last member of (46) is nonpositive and the second is zero because $\sigma\partial_x^2$ is antisymmetric and we are dealing with real-valued solutions, we obtain

$$\partial_t \|u_\mu(t)\|_s^2 \leq \left|\left(u_\mu(t)\,|\partial_x \Phi(u_\mu(t))\right)_s\right| \leq g_1\left(\|u_\mu(t)\|_s^2\right), \tag{47}$$

and the lemma follows at once. ■

Remark 8 *Sufficient conditions for the validity of (43) can be found in Appendix A of [11]*

A long, but by now standard argument, implies existence and uniqueness for $\mu = 0$. See ([5]), ([7]), Section 8.1 of ([8]) and ([12]).)

Theorem 9 *Assume that Φ satisfies the conditions of Theorem 6 and of Lemma 7. Let $\phi \in h^s(\mathbb{R})$ and $[0,T]$, $T > 0$, be any closed interval contained in the maximal interval of existence of (44). Then there exists a unique $u = u_0 \in C([0,T], h^s(\mathbb{R}))$ such that $u_0(0) = \phi$ and*

$$\partial_t u_0(t) + \sigma\partial_x^2 u_0(t) + \partial_x \Phi(u_0(t)) = 0, \tag{48}$$

with the time derivative computed with respect to the norm of $h^{s-2}(\mathbb{R})$.

Remark 10 *It is possible to show that the solution of GBBO depends continuously on the initial data in the following sense. Let $\mu \geq 0$ be fixed $\phi_n \in h^s(\mathbb{R})$, $n = 1, 2, ..., \infty$ be such that $\phi_n \xrightarrow{h^s} \phi_\infty$. Let $u_n \in C([0,T_n]; h^s(\mathbb{R}))$ denote the corresponding solutions. Then if $T' \in (0, T_\infty)$, the solutions u_n, $n < \infty$ can be extended to $[0,T']$ for all n sufficiently large and*

$$\lim_{n\longrightarrow\infty} \sup_{[0,T']} \|u_n(t) - u_\infty(t)\|_s = 0.$$

If $\mu > 0$ this statement follows directly from the analysis of the integral equation (39). In the case of GBO, one must resort to the Bona-Smith approximation procedure. (See ([1]) and Chapters 6 and 8 of ([8]).) Alternatively one can use Kato's theory of quasilinear equations to get continuous dependence for each fixed $\mu \geq 0$. (See ([13]).)

Now we turn to the $\mathfrak{F}_{2,2}$ theory. Assume that $\mu > 0$ for the moment. Multiplying (39) by x^2, noting that $\partial_x \Phi(u_\mu) \in C\left((0,T]; h^{s-1}(\mathbb{R})\right)$ and using the exponential decay of $F_\mu(t,\xi)$ it follows that

$$x^2 u_\mu \in C\left((0,T]; L^2(\mathbb{R})\right), \tag{49}$$

for fixed $\mu > 0$[4]. Let

$$\omega(x) = 1 + x^2 \tag{50}$$

and note that

$$\begin{aligned} \partial_t \left\| u_\mu(t) \right\|_{0,2}^2 &= 2 \left(\omega u_\mu(t) \left| \omega \partial_t u_\mu(t) \right. \right)_0 \\ &= 2 \left(\omega u_\mu(t) \left| \omega \left(\mu \partial_x^2 u(t) - \sigma \partial_x^2 u_\mu(t) - \partial_x \Phi(u(t)) \right) \right. \right)_0 . \end{aligned} \tag{51}$$

The first two inner products in the last member of (51) can be rewritten as follows

$$\begin{aligned} &\left(\omega u_\mu(t) \left| \omega \mu \partial_x^2 u(t) \right. \right) \\ &= \mu \left(\omega u_\mu(t) \left| \left[\omega, \partial_x^2 \right] u(t) \right. \right)_0 - \mu \left\| \partial_x \omega u(t) \right\|_0^2 \\ &\leq \mu \left(\omega u_\mu(t) \left| \left[\omega, \partial_x^2 \right] u(t) \right. \right)_0 \end{aligned} \tag{52}$$

and

$$\left(\omega u_\mu(t) \left| \omega \sigma \partial_x^2 u_\mu(t) \right. \right)_0 = \left(\omega u_\mu(t) \left| \left[\omega, \sigma \partial_x^2 \right] u_\mu(t) \right. \right)_0 , \tag{53}$$

because $\sigma \partial_x^2$ is antisymmetric and we are dealing with real-valued functions so that, $\left(\omega u_\mu(t) \left| \sigma \partial_x^2 \omega u_\mu(t) \right. \right)_0 = 0$. But

$$\left[\omega, \sigma \partial_x^2 \right] \psi = \left[\omega, \sigma \right] \partial_x^2 \psi + \sigma \left[\omega, \partial_x^2 \right] \psi \tag{54}$$

and

$$\begin{aligned} \left[\omega, \sigma \right] \partial_x^2 \psi(x) &= pv \frac{1}{\pi} \int_{\mathbb{R}} \frac{x^2 - y^2}{y - x} \partial_y^2 \psi(y) \, dy \\ &= -\frac{1}{\pi} \int_R (x + y) \partial_y^2 \psi(y) \, dy = 0 \end{aligned} \tag{55}$$

where the last equality was obtained integrating by parts.[5]. Therefore (53) becomes

$$\left(\omega u_\mu(t) \left| \left[\omega, \sigma \partial_x^2 \right] u_\mu(t) \right. \right)_0 = \left(\omega u_\mu(t) \left| \sigma \left[\omega, \partial_x^2 \right] u_\mu(t) \right. \right)_0 . \tag{56}$$

Now,

$$\left[\omega, \partial_x^2 \right] \psi = -2\psi - 2x \partial_x \psi . \tag{57}$$

Integration by parts shows that if $\psi \in \mathfrak{f}_{2,2}$ then $x \partial_x \psi \in L^2(\mathbb{R})$[6] and

$$\left\| x \partial_x \psi \right\|_0^2 \leq C \left\| \psi \right\|_{2,2}^2 . \tag{58}$$

Combining these results we obtain

$$\partial_t \left\| u_\mu(t) \right\|_{0,2}^2 \leq C \left\| u_\mu(t) \right\|_{2,2}^2 + 2 \left| \left(\omega u_\mu(t) \left| \omega \partial_x \Phi(u_\mu(t)) \right. \right)_0 \right| . \tag{59}$$

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[4] For details see [6] where the case $\Phi(v) = \dfrac{v^2}{2}$ is considered.

[5] Strictly speaking, the computation in (55) holds only if ψ is sufficiently smooth and falls off sufficiently fast. However, an easy limiting argument settles the account for $\psi \in \mathfrak{f}_{2,2}$.

[6] This is a special case of Theorem 16

Lemma 11 *Let Φ satisfy the conditions of Theorem 9 with $s = 2$ and let $\phi \in \mathfrak{f}_{2,2}$. Assume that*
(a) $\partial_x \Phi(\cdot)$ maps $\mathfrak{f}_{2,2}$ into $L^2_2(\mathbb{R})$ and
(b) there exists a continuous function $g_2 : [0, \infty) \longrightarrow [0, \infty)$ such that

$$\left|(\omega\psi \left|\omega\partial_x \Phi(\psi)\right.)_0\right| \leq g_2\left(\|\psi\|_{2,2}\right). \tag{60}$$

Let $\widetilde{\rho}$ be the maximal solution of

$$\begin{cases} \partial_t \widetilde{\rho}(t) = C\widetilde{\rho}(t) + 2\left(g_1(\widetilde{\rho}(t)) + g_2(\widetilde{\rho}(t))\right) \\ \widetilde{\rho}(0) = \|\phi\|^2_{2,2}. \end{cases} \tag{61}$$

Then

$$\|u_\mu(t)\|^2_{2,2} \leq \widetilde{\rho}(t) \tag{62}$$

whenever both sides are defined.

Remark 12 *Assumptions (a) and (b) of Lemma 11 are satisfied by the maps defined in (32) and (33).*

Once again, the standard limiting argument implies existence in the case $\mu = 0$. Uniqueness is trivial in view of the h^s theory. For details see ([5]), ([6]) and Chapter 8 of ([8]).

Theorem 13 *Let $\phi \in \mathfrak{f}_{2,2}$. Then there exist a $T > 0$ and a unique $u = u_0 \in C([0, T]; \mathfrak{f}_{2,2})$ such that $u(0) = \phi$ and*

$$\partial_t u_0(t) + \sigma\partial_x^2 u_0(t) + \partial_x \Phi(u_0(t)) = 0 \tag{63}$$

with the time derivative computed with respect to the norm of $h^s(\mathbb{R})$.

Remark 14 *Note that the previous theorem says that the solution remains in $\mathfrak{f}_{2,2}$ while it exists. This is also true for u_μ, $\mu > 0$, as we have noted in (49).*

3 Unique Continuation for GBBO, $\mu \geq 0$.

Consider (4) once again. Suppose that $x\phi \in L^2(\mathbb{R})$ and that there exists a τ such that $\sin\tau \neq 0$ and $xu(\tau)$ also belongs to $L^2(\mathbb{R})$. Then (9) implies $x\sigma\phi \in L^2(\mathbb{R})$. But then

$$\begin{aligned} & x(\sigma\phi)(x) \\ & = pv\frac{1}{\pi}\int_{\mathbb{R}} \frac{x-y}{y-x}\phi(y)\,dy + (\sigma(x\phi))(x) \\ & = -\int_{\mathbb{R}} \phi(y)\,dy + \sigma(x\phi)(x). \end{aligned} \tag{64}$$

Since $\sigma(x\phi) \in L^2(\mathbb{R})$ we conclude that

$$(2\pi)^{\frac{1}{2}} \widehat{\phi}(0) = \int_{\mathbb{R}} \phi(y)\, dy = 0. \tag{65}$$

Conversely, if (65) holds then (64) implies that $x\sigma\phi \in L^2(\mathbb{R})$. A similar computation, using the formula

$$x^k - y^k = -\sum_{j=0}^{k-1} x^j y^{k-1-j} \tag{66}$$

yields the following result.

Proposition 15 *Let k be a positive integer and assume that $\phi \in L_k^2(\mathbb{R})$. Then $\sigma\phi \in L_k^2(\mathbb{R})$ if and only if*

$$\widehat{\phi}(0) = \partial_\xi \widehat{\phi}(0) = ... = \partial_\xi^{k-1} \widehat{\phi}(0) = 0. \tag{67}$$

In particular if the above assumption is satisfied for all k then $\widehat{\phi}$ has a zero of infinite order at $\xi = 0$.

Now, if ϕ and $u(\tau)$ have compact support where τ is such that $\sin\tau \neq 0$, then the assumptions of the previous proposition are satisfied for all positive integers k. This implies $\phi = 0$ so $u(t) = \exp(-\sigma t)\phi = 0$ for all values of t.

We will now apply these ideas to GBO. Before proceeding, it is worthwhile mentioning a technical result that will be used several times in the sequel. For a proof see Appendix A of ([5]) or H. Triebels' book ([17]).

Theorem 16 *Let $r \in \mathbb{N}$ and $\phi \in \mathcal{F}_{r,r}$. Then $x^j \partial_x^k \phi \in L^2(\mathbb{R})$ for all $j, k \in \mathbb{N}$ such that $0 \leq j + k \leq r$ and there exists a constant C depending only on r such that*

$$\left\| x^j \partial_x^k \phi \right\|_0 \leq C \left\| \phi \right\|_{r,r}. \tag{68}$$

First we consider the linear equation. Combining the Fourier transform, Leibniz's rule, Theorem 16 and the formulas

$$\partial_\xi F_\mu(t,\xi) = (-2t\xi)(\mu - 2ih(\xi)) F_\mu(t,\xi) \tag{69}$$

$$\partial_\xi^2 F_\mu(t,\xi) = -2t(\mu - 2ih(\xi)) F_\mu(t,\xi) + (-2t\xi)^2 (\mu - 2ih(\xi))^2 F_\mu(t,\xi). \tag{70}$$

$$\begin{aligned} &\partial_\xi^3 F_\mu(t,\xi) = 8it\delta + 3(-2t)^2 \xi^2 (\mu - 2ih(\xi))^2 F_\mu(t,\xi) \\ &+ (-2t)^3 \xi^3 (\mu - 2ih(\xi))^3 F_\mu(t,\xi) \end{aligned} \tag{71}$$

$$\partial_\xi^j F_\mu(t,\xi) = 8it\delta^{(j)} + \sum_{k=1}^{j} a_k t^k \xi^{m(k)} (\mu - 2ih(\xi))^k F_\mu(t,\xi)\ , j \geq 4, \tag{72}$$

we obtain the behavior of the solutions to the linear equation in the weighted L^2 spaces under consideration.

Theorem 17 *Let $r \in \mathbb{N}$ and $\phi \in \mathfrak{f}_{r,r}(\mathbb{R})$. Then for each fixed $\mu \geq 0$ we have (a) $v_\mu \in C([0,\infty);\mathfrak{f}_{r,r})$ if $r = 0,1,2$ and satisfies*

$$\|v_\mu(t)\|_{r,r} \leq \Theta_\mu(t)\|\phi\|_{r,r}, \tag{73}$$

where $\Theta_\mu(t)$ is a polynomial of degree r with positive coefficients depending only on r and μ.
(b) if $r \geq 3$, the following statements are equivalent

$$v_\mu \in C([0,\infty);\mathfrak{f}_{r,r}), \tag{74}$$

$$\exists\, t_1 < t_2 \text{ such that } v_\mu(t_j) \in \mathfrak{f}_{r,r}, j = 1,2, \tag{75}$$

$$\partial_\xi^j \widehat{\phi}(0) = 0,\ j = 0,1,...,r-3, \tag{76}$$

$$\partial_\xi^j \widehat{v}_\mu(t,0) = 0,\ j = 0,1,...,r-3,\ \forall t \in [0,\infty). \tag{77}$$

In this case an estimate of the form (73) also holds.

Proof. See ([5]), ([7]) and ([9]). We just note that if $r \geq 3$ we obtain

$$(x^r v_\mu(t))^\wedge(\xi) = (x^r E_\mu(t)\phi)^\wedge(\xi) = 8it\sum_{j=0}^{r-3} c_j \delta^{(j)}\phi^{(j)} + (R(t))^\wedge(\xi),\ r \geq 3 \tag{78}$$

where δ denotes Dirac's delta function, $a_k \in \mathbb{R}$ and $m(k) \in \{1,2,...,k\}$, and $R \in C\left([0,\infty);L^2(\mathbb{R})\right)$. ■

The next corollary is an immediate consequence of Theorem 17.

Corollary 18 *If ϕ and $v_\mu(\tau)$ have compact support for some $\tau > 0$ then $v_\mu \equiv 0$.*

At this point it is convenient to introduce the following (non-standard) definition.

Definition 19 *Let*

$$\mathbb{R}_+^n = \{\theta = (\theta_1,\theta_2,...,\theta_n)\,|\theta_\ell \geq 0, \ell = 1,2...,n\}. \tag{79}$$

If $n = 1$ we write $\mathbb{R}_+^1 = \mathbb{R}_+$. A function $R : \mathbb{R}_+^n \longrightarrow \mathcal{S}'(\mathbb{R})$ is called a regular map if there exists a finite number of (Banach) subspaces $X_j \subset \mathcal{S}'(\mathbb{R})$, $j = 1,2,...,N$ and maps $R_j \in C\left(\mathbb{R}_+^n, X_j\right)$ such that

$$R = \sum_{j=1}^{N} R_j \tag{80}$$

and $\widehat{R}_j(\theta) = (R_j(\theta))^\wedge$ is a measurable function for all $j = 1,2,...,N$. In particular, $\widehat{R}(\theta) = (R(\theta))^\wedge$ is a measurable function. We will write $R \in reg(N, X_1, X_2, ..., X_N.)$

Remark 20 *In the sequel, the letter R, and symbols like $\widetilde{R}$, $R^{\#}$, will be reserved to denote regular maps, whose precise form is not relevant to our arguments. Thus, in different formulas, R, $\widetilde{R}$, $R^{\#}$... may represent different regular maps, but in all instances they have the properties described in Definition 19.*

Assume that[7] $\phi \in \mathfrak{f}_{3,3} \hookrightarrow \mathfrak{f}_{2,2}$, and let $u = u_\mu \in C([0,T];\mathfrak{f}_{22})$, $\mu \geq 0$ fixed, be the solution constructed in Theorem 13. Multiplying the integral equation (39)[8] by x^3 and using (78) with $r = 3$ we obtain

$$\left(x^3 u(t)\right)^\wedge(\xi) = 8it\widehat{\phi}(0)\,\delta - \int_0^t \left(x^3 E_\mu(t-t')\,\partial_x \Phi(u(t'))\right)^\wedge dt'. \tag{81}$$

To proceed we must examine the behavior of the nonlinear term in (81). Let $\tau > 0$ be fixed and $\psi \in \mathfrak{f}_{2,2}$ A formal computation combining (69)-(71) and Leibniz's formula implies

$$\begin{aligned}&\left(x^3 E_\mu(\tau)\,\partial_x \Phi(\psi)\right)^\wedge(\xi) = -i\partial_\xi^3\left(F_\mu(\tau,\xi)\,(\partial_x\Phi(\psi))^\wedge(\xi)\right) = \qquad (82)\\ &= -i\left[\underbrace{8i\tau\delta\,(\partial_x\Phi(\psi))^\wedge(0) + Q_1(\tau,\xi)}_{j=3} + \sum_{j=0}^{2} c_j\left(\partial_\xi^j F(\tau,\xi)\right)\left(\partial_\xi^{3-j}(\partial_x\Phi(\psi))^\wedge(\xi)\right)\right]\end{aligned}$$

where

$$Q_1(\tau,\xi) = \left(\partial_\xi^3 F_\mu(\tau,\xi) - 8i\tau\delta\right)(\partial_x\Phi(\psi))^\wedge(\xi). \tag{83}$$

Since

$$(\partial_x\Phi(\psi))^\wedge(0) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\int_{\mathbb{R}}(\partial_x\Phi(\psi))(x)\,dx = 0 \tag{84}$$

we conclude that

$$\begin{aligned}&\left(x^3 E_\mu(\tau)\,\partial_x\Phi(\psi)\right)^\wedge(\xi) = -i\partial_\xi^3\left(F_\mu(\tau,\xi)\,(\partial_x\Phi(\psi))^\wedge(\xi)\right) = \qquad (85)\\ &= -i\left[Q_1(\tau,\xi) + \sum_{j=0}^{2} c_j\left(\partial_\xi^j F(\tau,\xi)\right)\left(\partial_\xi^{3-j}(\partial_x\Phi(\psi))^\wedge(\xi)\right)\right].\end{aligned}$$

Theorem 21 *Let $\phi \in \mathfrak{f}_{3,3}$ and u be as above. Assume that*
(a) $\partial_x\Phi(\cdot)$ maps $\mathfrak{f}_{2,2}$ into $L_1^2(R)$,
(b) $(t,t') \in \mathbb{R}_+^2 \longmapsto x^3 E_\mu(t-t')\,\partial_x\Phi(u(t'))$ is a regular map,
(c) there exists a $t_1 > 0$ such that $x^3 u(t_1) \in L^2(\mathbb{R})$.
Then $\widehat{\phi}(0) = 0$.

Proof. Combining (81), (85) with our assumptions we obtain

$$\left(x^3 u(t_1)\right)^\wedge(\xi) = 8it_1\widehat{\phi}(0)\,\delta(\xi) + \left(\widetilde{R}(t)\right)^\wedge(\xi), \tag{86}$$

where $\widetilde{R}$ is a regular map. Hence the left hand side of (86) belongs to $L^2(\mathbb{R})$ if and only if $\widehat{\phi}(0) = 0$. ■

[7]The symbol $Y \hookrightarrow X$ means that the inclusion$Y \subseteq X$ is continuous and dense.
[8]If $\mu = 0$ we regard (39) as an identity in $H^{s-1}(\mathbb{R})$.

Remark 22 *Assumptions (a) and (b) in Theorem 21 are rather implicit. They are satisfied by the* Φ *'s defined in (32). (See ([9]) where the case* $p = 1$ *is considered. The case* $p > 1$ *is easier and can be dealt with much in the same way.)*

Remark 23 *Note that if* $\widehat{\phi}(0) = 0$ *it follows that* $(u_\mu(t))^\wedge(0) = 0$ *for all values* t *in the interval of existence. Indeed, taking the Fourier transform of (39) (which makes sense even for* $\mu = 0$*, because of assumption (a) in Theorem 21) we obtain*

$$(u_\mu(t))^\wedge = F_\mu(t,\cdot)\,\widehat{\phi} - \int_0^t F(t-t',\cdot)\,(\partial_x \Phi(u_\mu(t')))^\wedge(\cdot)\,dt'.$$

Since the Fourier transform of the nonlinearity is automatically equal to zero at $\xi = 0$*, we are done.*

In view of the preceding remark it is natural to seek solutions to GBO in

$$\widetilde{\mathfrak{f}}_{3,3} = \left\{ \psi \in \mathfrak{f}_{3,3} \,\middle|\, \widehat{\psi}(0) = 0 \right\}. \tag{87}$$

Once again, Banach's fixed point theorem, Gronwall's inequality and the standard limiting process can be applied to obtain existence and uniqueness. (See ([5]), ([7]) and Chapter 8 of ([8]), where $\Phi(v) = \dfrac{v^2}{2}$ is considered. The proof in the general case is the same.)

Theorem 24 *Let* $\phi \in \widetilde{\mathfrak{f}}_{3,3}$ *and* Φ *be as in Theorem 21 Then the solution* u_μ *constructed in Theorem 13,* $\mu \geq 0$*, belongs to* $C\left([0,T];\widetilde{\mathfrak{f}}_{3,3}\right)$.

Our next task is to establish the unique continuation principle for GBBO with $\mu \geq 0$.[9]

Theorem 25 . *Let* $\phi \in \mathfrak{f}_{4,4} \hookrightarrow \mathfrak{f}_{2,2}$ *and* $u_\mu \in C([0,T];\mathfrak{f}_{2,2})$ *be the solution constructed in Theorem 13,* $\mu \geq 0$*. Assume that* Φ *satisfies the conditions of Theorem 21 and*
(a) $\partial_x \Phi(\cdot)$ *maps* $\mathfrak{f}_{3,3}$ *into* $L_2^2(\mathbb{R})$ *;*
(b) $\Phi(v) \geq 0$ *for all* $v \in L_2^2(\mathbb{R})$ *and if* $\Phi(v) = 0$ *then* $v = 0$*;*
(c) $(t,t') \longrightarrow \left(x^4 E_\mu(t-t')\,\partial_x \Phi(u(t'))\right)^\wedge - 8i(t-t')(1+c_3)\,\delta\partial_\xi\left(\partial_x \Phi(u(t'))^\wedge\right)(0)$*, where* c_3 *is defined in (90), is a regular map.*
Suppose also that one of the following conditions hold
(d) There exists $0 < t_1 < t_2$ *such that* $u_\mu(t_j) \in \mathfrak{f}_{4,4}$*,* $j = 1,2$*;*
(e) There exists $t_1 > 0$ *such that* $u_\mu(t_1) \in \mathfrak{f}_{4,4}$ *and*

$$\int_{\mathbb{R}} x\phi(x)\,dx \geq 0. \tag{88}$$

Then $u(t) = 0$ *for all* $t \in [0,T]$.

[9] Conclusion (d), with $\Phi(v) = \frac{v^2}{2}$, was obtained in [9].

Proof. Write $u = u_\mu$ for simplicity's sake. Since $\mathfrak{f}_{4,4} \hookrightarrow \mathfrak{f}_{3,3}$ Theorem 21 implies that $\widehat{\phi}(0) = 0$, so that $u_\mu \in C([0,T];\mathfrak{f}_{3,3})$ in view of Theorem 24. Multiply the integral equation (39) by x^4 and take the Fourier transform to get

$$\left(x^4 u(t)\right)^\wedge(\xi) = \left(x^4 E_\mu(t)\phi\right)^\wedge(\xi) - \int_0^t \left(x^4 E_\mu(t-t')\,\partial_x \Phi(u(t'))\right) dt'. \quad (89)$$

Combining Leibniz's rule with formulas (69)-(72) and Theorem 16 we obtain

$$\begin{aligned}\left(x^4 E_\mu(t)\phi\right)^\wedge(\xi) &= \partial_\xi^4\left(F_\mu(t,\xi)\,\widehat{\phi}(\xi)\right) \\ &= \sum_{j=0}^{4} c_j \partial_\xi^j F_\mu(t,\xi)\,\partial_\xi^{4-j}\widehat{\phi}(\xi) = 8it\delta'\widehat{\phi} + c_3 8it\delta\widehat{\phi}' + \widehat{R}(t,\xi),\end{aligned} \quad (90)$$

where $R \in C\left([0,T];L^2(\mathbb{R})\right)$. For all appropriate test functions φ we have

$$\begin{aligned}\langle\psi\delta',\varphi\rangle &= \langle\delta',\psi\varphi\rangle = -\left\langle\delta,(\psi\varphi)'\right\rangle \\ &= -\left(\psi'\varphi + \psi\varphi'\right)(0) \\ &= -\psi'(0)\,\varphi(0) - \psi(0)\,\varphi'(0) \\ &= -\langle\psi'(0)\,\delta,\varphi\rangle - \langle\psi(0)\,\delta,\varphi'\rangle = \\ &= -\langle\psi'(0)\,\delta,\varphi\rangle + \langle\psi(0)\,\delta',\varphi\rangle.\end{aligned} \quad (91)$$

Therefore,

$$\psi\delta' = -\psi'(0)\,\delta + \psi(0)\,\delta'. \quad (92)$$

Since $\widehat{\phi}(0) = 0$ we get

$$\left(x^4 E_\mu(t)\phi\right)^\wedge(\xi) = 8it\delta(1+c_3)\,\widehat{\phi}'(0) + \widehat{R}(t). \quad (93)$$

Next we turn to the nonlinear term in (89). Since $(\partial_x\Phi(u(t')))^\wedge(0) = 0$ for all t', (92), (93) (with ϕ replaced by $\partial_x\Phi(u(t'))$), and assumption (b), imply that

$$\begin{aligned}&\left(x^4 E_\mu(t-t')\,\partial_x\Phi(u(t'))\right)^\wedge(\xi) = \partial_\xi^4\left(F_\mu(t,\xi)\left(\partial_x\Phi(u(t'))\right)^\wedge(\xi)\right) \\ &= 8i(t-t')\,\delta'\left(\partial_x\Phi(u(t'))^\wedge\right)(\xi) + c_3 8i(t-t')\,\delta\partial_\xi\left(\partial_x\Phi(u(t'))\right)^\wedge(\xi) + \widehat{R^\#}(t,\xi) \\ &= 8i(t-t')(1+c_3)\,\delta\partial_\xi\left(\partial_x\Phi(u(t'))^\wedge\right)(0) + \widehat{R^\#}(t,t',\xi),\end{aligned} \quad (94)$$

where $R^\#$ is a regular map. Therefore

$$\left(x^4 u(t)\right)^\wedge(\xi) = 8i\delta(1+c_3)\left(t\widehat{\phi}'(0) - \int_0^t (t-t')\,\partial_\xi\left(\partial_x\Phi(u(t'))^\wedge\right)(0)\,dt'\right) + \widetilde{R}(t,\xi) \quad (95)$$

where $\widetilde{R}$ is a regular map. Note that

$$\begin{aligned}\partial_t \int_{\mathbb{R}} xu(t,x)\,dx \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (96)\\ = \int_{\mathbb{R}} x\left(\mu\partial_x^2 u(t) - \sigma\partial_x^2 u(t) - \partial_x \Phi(u(t))\right) dx = \int_{\mathbb{R}} \Phi(u(t))\,dx,\end{aligned}$$

and

$$\partial_\xi \left(\partial_x \Phi(u(t))^\wedge\right)(0) = i(2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} x\partial_x \Phi(u(t))\,dx = -i(2\pi)^{\frac{1}{2}} \int_{\mathbb{R}} \Phi(u(t))\,dx. \tag{97}$$

These formulas imply that (95) can be rewritten as

$$\begin{aligned}&\left(x^4 u(t)\right)^\wedge(\xi) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (98)\\ &= K\delta\left(t\int_{\mathbb{R}} x\phi(x)\,dx + \int_0^t (t-t')\,\partial_t\left(\int_{\mathbb{R}} xu(t',x)\,dx\right)dt'\right) + \widetilde{R}(t,\xi)\end{aligned}$$

where

$$K = -8(2\pi)^{-\frac{1}{2}}(1+c_3). \tag{99}$$

Integration by parts shows that

$$t\int_{\mathbb{R}} x\phi(x)\,dx + \int_0^t (t-t')\,\partial_t\left(\int_{\mathbb{R}} xu(t',x)\,dx\right)dt' = \int_0^t \left(\int_{\mathbb{R}} xu(t',x)\,dx\right)dt'. \tag{100}$$

Thus

$$\left(x^4 u(t)\right)^\wedge(\xi) = K\delta\left(\int_0^t\left(\int_{\mathbb{R}} xu(t',x)\,dx\right)dt'\right) + \widetilde{R}(t,\xi). \tag{101}$$

Since $x^4 u(t_1) \in L^2(\mathbb{R})$, we must have

$$\int_0^{t_1}\left(\int_{\mathbb{R}} xu(t',x)\,dx\right)dt' = 0, \tag{102}$$

so that the function

$$F(t) = \int_0^t\left(\int_{\mathbb{R}} xu(t',x)\,dx\right)dt' \tag{103}$$

satisfies $F(0) = F(t_1) = 0$. Thus, there exists a $\tau_1 \in (0,t_1)$ such that

$$F'(\tau_1) = \int_{\mathbb{R}} xu(\tau_1,x)\,dx = 0. \tag{104}$$

We are now in position to prove the conclusions of the theorem.
Proof of (d).
Since $x^4 u(t_j) \in L^2(\mathbb{R})$, we can repeat the above arguments with ϕ replaced by

$u(t_1)$ and $u(t_1)$ replaced by $u(t_2)$ to conclude that there exists a $\tau_2 \in (0, \tau_1)$, such that

$$F'(\tau_2) = \int_{\mathbb{R}} xu(\tau_2, x)\, dx = 0. \tag{105}$$

Equation (96) implies that for t, $\tau \in [0, T]$ we have

$$\int_{\mathbb{R}} xu(t, x)\, dx = \int_{\mathbb{R}} xu(\tau, x)\, dx + \int_{\tau}^{t} \left(\int_{\mathbb{R}} \Phi(u(t'))\, dx \right) dt'. \tag{106}$$

Applying (106) with $t = \tau_2$ and $\tau = \tau_1$ we conclude that

$$\int_{\tau_1}^{\tau_2} \left(\int_{\mathbb{R}} \Phi(u(t'))\, dx \right) dt' = 0. \tag{107}$$

Hence

$$\int_{\mathbb{R}} \Phi(u(t'))\, dx = 0 \ \forall\, t' \in [\tau_1, \tau_2], \tag{108}$$

so that $\Phi(u(t')) = 0$, $t' \in [\tau_1, \tau_2]$. Hence

$$u(t') = 0 \ \forall\, t' \in [\tau_1, \tau_2]. \tag{109}$$

uniqueness implies (d).

Proof of (e). Taking $\tau = 0$ in (106) our assumptions imply that $F(t) \geq 0$ for all $t \in [0, T]$ and

$$F'(t) = \int_{\mathbb{R}} xu(t, x)\, dx \geq 0 \ \forall\, t \in [0, T]. \tag{110}$$

Thus F is nondecreasing.. Since $F(0) = F(t_1) = 0$ we must have $F(t) = 0$ for all $t \in [0, T]$. The result then follows from (106) and our assumptions on Φ. ■

4 Final Remarks.

First, it deserves remark that there are unique continuation principles for evolution equations that are not of BO type. Saut and Scheurer ([15]) proved that if u is a sufficiently smooth solution of the Korteweg-de Vries equation (KdV)

$$\partial_t u(t) + \partial_x^3 u(t) + u(t)\, \partial_x u(t) = 0 \tag{111}$$

such that $\mathrm{supp}(u(t)) \subset (a, b)$ for all values of t in some interval (τ_1, τ_2) then $u \equiv 0$. Their arguments employ a Carleman type estimate. Bourgain ([2]) reobtained this unique continuation principle using complex variable techniques. Stronger results were obtained in Zhang ([18]) and Kenig Ponce and Vega in ([14]). Both works assume there are $t_1 < t_2$ such that $u(t_j)$, $j = 1, 2$, are supported in a half line. Moreover that ([18]) deals with KdV using the inverse scattering transform while in ([14]) the authors consider the generalized KdV equation, make use of Carlemann type estimates to obtain their results. It should also be noted that ([2]) and ([14]) deal with equations which are much more general than KdV.

The results of ([18]) and ([14] together with those obtained above for the linear GBBO suggest that Theorem 25 should be true assuming only that ϕ and $u(t_1)$ belong to $\mathfrak{f}_{4,4}$ for some $t_1 > 0$. Although we believe this to be true, we have not been able to prove it so far. It is also natural to inquire about unique continuation principles for solutions supported in half-lines. In view of the methods used above this question leads naturally to the study of GBBO in asymmetric weighed L^2 spaces. These questions will be dealt with somewhere else. Another natural question, posed by Otared Kavian, is the following. Suppose that $\phi, \psi \in \mathfrak{f}_{2,2}$ let $u, v \in C([0,T]; \mathfrak{f}_{2,2})$ be the corresponding solutions. Assume $\phi - \psi \in \mathfrak{f}_{4,4}$ and that there are two times $0 < t_1 < t_2$ satisfying $u(t_j) - v(t_j) \in \mathfrak{f}_{4,4}$. Is it true that $u \equiv v$?

Finally we wish to remark that Henrik Kalisch ([10]) obtained a result, similar to part (e) of Theorem 25, for the BBM-BO equation, namely

$$\partial_t u + \partial_x u - \sigma \partial_x \partial_t u + u \partial_x u = 0. \tag{112}$$

He assumes that $u \in C\left([0,T]; h^2(\mathbb{R}) \cap L_3^2(\mathbb{R})\right)$. However his proof can be modified to yield a unique continuation principle where ϕ and $u(t_1)$ belong to $h^2(\mathbb{R}) \cap L_3^2(\mathbb{R})$ and the condition

$$\int \phi(x)\, dx \geq 0 \tag{113}$$

is satisfied.

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