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COMPLEMENTARITY CONSTRAINT QUALIFICATION VIA THE THEORY OF 2-REGULARITY *

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Abstract. We exhibit certain second-order regularity properties of parametric complementarity constraints, which are notorious for being irregular in the classical sense. Our approach leads to a constraint qualification in terms of 2-regularity of the mapping corresponding to the subset of constraints which must be satisfied as equalities around the given feasible point, while no qualification is required for the rest of constraints. Under this 2-regularity assumption, we derive constructive sufficient conditions for tangent directions to feasible sets defined by complementarity constraints. A special form of primal-dual optimality conditions is also obtained. We further show that our 2-regularity condition always holds under the piecewise Mangasarian-Fromovitz constraint qualification, but not vice versa. Relations with other constraints qualifications and optimality conditions are also discussed. It is shown that our approach can be useful when alternative ones are not applicable.

Key words. equilibrium constraints, tangent cone, constraint qualification, 2-regularity, optimality conditions.

AMS subject classifications. 90C30, 90C33.

1. Introduction.

This paper is devoted to the analysis of local structure of a set defined by parametric complementarity constraints, such as

$$D := \{(x, y) \in \Re^n \times \Re^m \mid g(x, y) \geq 0\,, y \geq 0\,, \langle g(x, y), y\rangle = 0\}\,, \tag{1.1}$$

where $g : \Re^n \times \Re^m \to \Re^m$, and $\langle \cdot, \cdot \rangle$ denotes the usual inner product in an appropriate space. Geometry of sets having this (or similar) structure is important in connection with mathematical programs with complementarity constraints, where D appears as the feasible region:

$$\begin{array}{ll} \text{minimize} & f(x, y) \\ \text{subject to} & (x, y) \in D\,. \end{array} \tag{1.2}$$

Mathematical programs with equilibrium constraints (MPEC), of which (1.2) is a special case, is a relatively new active field of research. We refer the reader to the monograph [10] for further references. It is known [4], and not difficult to see, that for D given by (1.1) the classical Mangasarian-Fromovitz constraint qualification (MFCQ) [13] does not hold at any feasible point $(x, y) \in D$. Arguably, this is the single most important reason that makes treatment of this problem considerably more difficult when compared to regular (i.e., satisfying some standard constraint qualifications) nonlinear programs. Specifically, the non-satisfaction of classical constraint qualifications makes a simple constructive description of the tangent cone to the set D (and its dual) difficult. This rules out standard stationarity/optimality conditions. The latter, in turn, makes it problematic to come up with reliable computational algorithms for solving MPEC.

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To deal with these issues, a number of approaches have been proposed, among which are [11, 10, 18, 15, 14, 20]. In this paper, we study complementarity constraints from the point of view of the theory of 2-regularity of twice differentiable mappings, which has not been used in this context previously. Our analysis clarifies the role that 2-regularity can play for MPEC, as well as provides an alternative view of some issues related to MPEC constraint qualifications and optimality conditions. We also exhibit some situations when other approaches do not apply but our results appear useful to characterize optimality.

Given any $(x^*, y^*) \in D$, define the three index sets

$$\begin{aligned} I_0 &:= \{i \mid y_i^* = 0 = g_i(x^*, y^*)\}\,, \\ I_g &:= \{i \mid y_i^* > 0 = g_i(x^*, y^*)\}\,, \\ I_y &:= \{i \mid y_i^* = 0 < g_i(x^*, y^*)\}\,, \end{aligned}$$

with I_0 being the *degenerate* set. Note that these index sets depend on the point under consideration. Since this point will be fixed throughout our analysis, we shall omit this dependency in our notation. Note also that locally, the constraints $g_i(x, y) \geq 0\,, i \in I_y$ are never active, and we can further deal with constraints $y_i = 0\,, i \in I_y$ explicitly by eliminating the variables. Thus, to simplify the notation, we can assume that

$$I_y = \emptyset\,.$$

It is further easy to see that there exists a neighborhood $\mathcal{V}$ of $(x^*, y^*) \in D$ such that $D \cap \mathcal{V} = D^* \cap \mathcal{V}$, where

$$D^* := \left\{ (x, y) \;\middle|\; \begin{array}{l} y_i \geq 0\,, g_i(x, y) \geq 0\,, y_i g_i(x, y) = 0\,,\ i \in I_0 \\ g_i(x, y) = 0\,,\ i \in I_g \end{array} \right\}. \tag{1.3}$$

The sets D and D^* are therefore locally equivalent, and we can work with the representation given by (1.3). Let us define the mapping $F : \Re^n \times \Re^m \to \Re^{|I_0|+|I_g|}$ associated with equality constraints in D^*:

$$F_i(x, y) := \left\{ \begin{array}{ll} y_i g_i(x, y)\,, & i \in I_0 \\ g_i(x, y)\,, & i \in I_g \end{array} \right. .$$

In what follows, we show that certain (second-order) regularity properties of this mapping F are sufficient to completely characterize tangent directions to the set D, and to derive corresponding optimality conditions for MPEC.

The rest of the paper is organized as follows. In Section 2 we review the theory of 2-regularity (for smooth equality constraints) and show, by means of a simple example, that it is relevant in the context of MPEC. Section 3 contains our necessary and sufficient conditions for tangency, and a demonstration that this description is always valid under the piecewise MFCQ, and does go beyond. A special form of primal-dual necessary optimality conditions is derived in Section 4. This section also contains a comparison with some alternative approaches to MPEC constraint qualifications and optimality conditions.

We next briefly describe our notation. All vectors will be column-vectors. When writing a pair (x, y), we mean a column-vector comprised by two vectors x and y. For a vector v of arbitrary (finite) dimension, v_i will denote its i-th component, and for a matrix M, M_i will denote its i-th row. By M^T we shall denote the transposed matrix of M. For a differentiable scalar function $\varphi : \Re^n \times \Re^m \to \Re$, $\nabla\varphi(z)$ stands for the row-vector of its partial derivatives at $z = (x, y) \in \Re^n \times \Re^m$ with respect to all the variables, while $\nabla_x \varphi(z)$ will denote the vector of partial derivatives with respect to $x \in \Re^n$ (similarly for y). If φ is twice differentiable, $\nabla^2 \varphi(z)$ denotes its Hessian matrix, and $\nabla^2_x \varphi(z)$ - its Hessian matrix with respect to x. For a differentiable vector-function $F : \Re^l \to \Re^k$, $F'(z)$ will denote its Jacobian, i.e., the $k \times l$ matrix whose i-th row is $\nabla F_i(z)$. If F is twice differentiable, then $F''(z)[h]$ denotes the $k \times l$ matrix whose i-th row is $\nabla^2 F_i(z)h$, $i = 1, \ldots, k$. Furthermore, $F''(z)[h, p] := (F''(z)[h])p$, and $F''(z)[h]^2 := F''(z)[h, h]$. For a linear operator $A : S_1 \to S_2$, $\operatorname{Ker} A = \{w \in S_1 \mid Aw = 0\}$ is its null space, and $\operatorname{Im} A = \{v \in S_2 \mid v = Aw \text{ for some } w \in S_1\}$ is its image space. Given a subspace S of an arbitrary space, we shall denote its orthogonal complement in this space by $S^\perp$. If K is a cone, then $K^* = \{p \mid \langle p, h\rangle \geq 0 \ \forall h \in K\}$ is the (positive) dual cone of K. For a set K, $\operatorname{cl} K$ will stand for its closure. Finally, for a (finite) index set I its cardinality will be denoted by $|I|$.

2. Elements of the theory of 2-regularity. In this section, we describe some results from the theory of 2-regularity for smooth nonlinear mappings [12, 19, 1, 9, 5, 6, 7]. Our emphasis will be only on those constructions that will be used later in the paper. We therefore shall limit our presentation to equality constraints only, to the finite-dimensional setting, and to the case of mappings differentiable at least twice. After introducing the necessary objects, we provide an example which illustrates certain 2-regularity properties of MPEC.

Let $F : \Re^l \to \Re^k$ be differentiable at a point $z^* \in C$, where

$$C := \{z \in \Re^l \mid F(z) = 0\}.$$

We are interested in the tangent cone to the set C at the point $z^* \in C$, denoted $T_C(z^*)$, which is the set of all vectors $h \in \Re^l$ for which there exists a mapping $r : \Re_+ \to \Re^l$ such that

$$z^* + th + r(t) \in C \quad \forall\, t \in \Re_+, \quad \|r(t)\| = o(t).$$

We note that in the context of constructive constraint qualifications there is no distinction between this definition of the tangent cone and the more general Bouligand tangent cone. In the sequel, we shall use the given definition, as it is more convenient for our development (and leads to the same results as when using the Bouligand cone anyway). It is well known that

$$T_C(z^*) \subset H_1 := \operatorname{Ker} F'(z^*),$$

which is the (first-order) necessary condition of tangency. A sufficient condition is given by the classical Lyusternik theorem: the equality

$$T_C(z^*) = H_1$$

holds if F is (first-order) *regular*, i.e.,

$$\operatorname{Im} F'(z^*) = \Re^k.$$

In the *irregular* case when $\operatorname{Im} F'(z^*) \neq \Re^k$, cone $T_C(z^*)$ can be smaller than H_1. In that case, a more accurate representation is needed.

Suppose now that F is twice differentiable at z^*. Then it can be verified that

$$T_C(z^*) \subset H_2 := H_1 \cap \{h \mid F''(z^*)[h]^2 \in \operatorname{Im} F'(z^*)\},$$

which is the second-order necessary condition of tangency. Let P be the orthogonal projector onto $(\operatorname{Im} F'(z^*))^\perp$ in $\Re^k$. With this notation, we can equivalently write

$$H_2 = \operatorname{Ker} F'(z^*) \cap \{h \mid PF''(z^*)[h]^2 = 0\}\,. \tag{2.1}$$

DEFINITION 2.1. *Under the assumptions above, the mapping F is called 2-regular at the point z^* with respect to an element $h \in \Re^l$ if*

$$\operatorname{Im}(F'(z^*) + PF''(z^*)[h]) = \Re^k\,.$$

Obviously, if F is (first-order) regular in the classical sense, i.e., $\operatorname{Im} F'(z^*) = \Re^k$, then it is 2-regular with respect to *every* h. The following generalization of the classical Lyusternik theorem can be found in [12, 19, 1, 9, 5], and more general results under weaker smoothness assumptions, with applications, in [6, 7].

THEOREM 2.2. *Let $F : \Re^l \to \Re^k$ be twice differentiable at a point $z^* \in \Re^l$ such that $F(z^*) = 0$. Assume further that F is 2-regular at z^* with respect to an element $h \in H_2$. Then there exists a mapping $r : \Re_+ \to \Re^l$ such that*

$$F(z^* + th + r(t)) = 0 \quad \forall\, t \in \Re_+\,, \quad \textit{where } \|r(t)\| = o(t)\,.$$

In other words, 2-regularity of F on some h satisfying the second-order necessary conditions of tangency ($h \in H_2$) guarantees that this h is indeed a tangent direction. In particular, if F is 2-regular with respect to every $h \in H_2 \setminus \{0\}$, then

$$T_C(z^*) = H_2\,.$$

In fact, due to the closedness of the tangent cone, for the above equality to hold it is enough to require that F be 2-regular with respect to every h is some dense subset of H_2 (i.e., $h \in K$ such that $\operatorname{cl} K = H_2$).

Note that this representation of the tangent cone subsumes the classical regular case. Indeed, in that case $\operatorname{Im} F'(z^*) = \Re^k$, hence $(\operatorname{Im} F'(z^*))^\perp = \{0\}$, $P = 0$, and thus $\{h \mid PF''(z^*)[h]^2 = 0\} = \Re^l$, and so we have that $T_C(z^*) = \operatorname{Ker} F'(z^*) = H_1 = H_2$. Of course, the description of tangent directions via constructions of 2-regularity is useful precisely in the absence of classical regularity. It is therefore natural to see what, if anything, the approach of 2-regularity has to offer in the context of MPEC. To quickly convince the reader that 2-regularity is relevant here, we next consider a simple one-dimensional problem and show that sufficient conditions for tangency can be interpreted as 2-regularity of a certain subset of constraints.

EXAMPLE 2.1. Let $n = m = 1$ and consider the set D given by

$$y \geq 0\,, \quad g(x,y) \geq 0\,, \quad 0 = F(x,y) := yg(x,y)\,. \tag{2.2}$$

Obviously, the cause of difficulties in MPEC is the degenerate set I_0, so the case of interest is when $I_0 \neq \emptyset$. Here, this means that $I_0 = \{1\}$ while the other index sets introduced in Section 1 are empty. Therefore we shall consider a point $(x^*, y^*) \in \Re^2$ such that

$$y^* = 0\,, \quad g(x^*, y^*) = 0\,.$$

As is easy to see, in this case

$$F'(x^*, y^*) = (0,0)\,.$$

Hence, the last constraint in (2.2) is not regular (even by itself, separately from the other constraints defining D!). Note further that

$$F''(x^*,y^*) = \begin{pmatrix} 0 & \nabla_x g(x^*,y^*) \\ \nabla_x g(x^*,y^*) & 2\nabla_y g(x^*,y^*) \end{pmatrix}.$$

As is easy to see, the necessary conditions for some $(u,v) \in \Re \times \Re$ to be a tangent direction to D at (x^*,y^*) are the following:

$$\begin{gathered} v \geq 0\,, \quad \nabla_x g(x^*,y^*)u + \nabla_y g(x^*,y^*)v \geq 0\,, \\ v(\nabla_x g(x^*,y^*)u + \nabla_y g(x^*,y^*)v) = 0\,. \end{gathered} \tag{2.3}$$

When these conditions are sufficient? Consider the two possible cases.

Case 1: $v = 0$. It is clear that the point $(x^* + tu, y^* + tv) = (x^* + tu, 0)$ satisfies the first and third constraints in (2.2) for any $t > 0$. Suppose that $\nabla_x g(x^*,y^*) \neq 0$. Then, by (2.3), if $u \neq 0$ it holds that $\nabla_x g(x^*,y^*)u > 0$. By differentiability of g we then obtain for all $t > 0$ small enough that

$$g(x^* + tu, 0) = t\nabla_x g(x^*,y^*)u + o(t) \geq 0\,.$$

Hence, $(x^* + tu, 0) \in D$ for all $t > 0$ small, which means that $h = (u,0) \in T_D(z^*)$. Observe now that

$$F'(x^*,y^*) + PF''(x^*,y^*)[h] = (0, \nabla_x g(x^*,y^*)u)\,,$$

and condition $\nabla_x g(x^*,y^*) \neq 0$ means precisely 2-regularity of F with respect to $h = (u,0)$, where $u \neq 0$.

Case 2: $v > 0$. By (2.3), $\langle \nabla g(x^*,y^*), (u,v) \rangle = 0$. Suppose that $\nabla g(x^*,y^*) \neq 0$. Then g is regular in the classical sense, and the standard Lyusternik theorem implies that $h = (u,v)$ is a tangent direction to the set $\{(x,y) \mid g(x,y) = 0\}$: there exists some mapping $r(\cdot) = (r^1(\cdot), r^2(\cdot))$ such that

$$g(z^* + th + r(t)) = 0, \quad \|r(t)\| = o(t)\,.$$

Furthermore, for any $t > 0$ sufficiently small

$$y^* + tv + r^1(t) = tv + o(t) \geq 0\,.$$

Hence, the point $z^* + th + r(t)$ satisfies all constraints in (2.2). Therefore, $h \in T_D(z^*)$. Again observe that

$$\begin{aligned} F'(x^*,y^*) + PF''(x^*,y^*)[h] &= (\nabla_x g(x^*,y^*)v, \nabla_x g(x^*,y^*)u + 2\nabla_y g(x^*,y^*)v) \\ &= (\nabla_x g(x^*,y^*)v, \nabla_y g(x^*,y^*)v)\,. \end{aligned}$$

As is now easy to see, condition $\nabla g(x^*,y^*) \neq 0$ is precisely 2-regularity of F with respect to $h = (u,v)$, where $v \neq 0$. [Note that if $\nabla_x g(x^*,y^*) = 0$ then necessarily $\nabla_y g(x^*,y^*) = 0$ (by (2.3), because $v > 0$). Hence, under our assumption, case $v > 0$ can occur only if $\nabla_x g(x^*,y^*) \neq 0$.]

We have demonstrated that in both cases the assumption which makes necessary conditions of tangency for some direction h be sufficient can be interpreted as 2-regularity of the mapping F with respect to this h.

In the next section we show that the intuition derived from this simple one-dimensional case is essentially valid in general. Specifically, 2-regularity of constraints that have to be satisfied as equalities (in a neighborhood of the point being considered), yields a precise description of the tangent cone to complementarity constraints at this point.

3. 2-regularity for MPEC. We start with recalling some useful objects from the MPEC theory [11, 10, 18, 15]. The standard *necessary* condition of tangency in MPEC is

$$T_D(z^*) \subset L\,, \tag{3.1}$$

where L is the "linearized cone":

$$L := \left\{ (u,v) \in \Re^n \times \Re^m \;\middle|\; \begin{array}{l} \langle \nabla g_i(z^*), (u,v)\rangle \geq 0\,, v_i \geq 0\,, i \in I_0 \\ v_i \langle \nabla g_i(z^*), (u,v)\rangle = 0\,, i \in I_0 \\ \langle \nabla g_i(z^*), (u,v)\rangle = 0\,, i \in I_g \end{array} \right\}. \tag{3.2}$$

Linearization here is understood differently from the usual notion in nonlinear programming. Indeed, L is not a polyhedral cone, except when the degenerate set I_0 is empty. Instead, L consists of a finite union of polyhedral cones. It is known [11] that (3.1) holds as equality when $g(\cdot,\cdot)$ is an affine function, or when $z^* \in D$ is (strongly) regular in the sense of [17].

It is also clear that $T_D(z^*) = L$ under some *piecewise* constraint qualification. Let (R,Q) be the family of partitions of the degenerate set I_0, i.e., $R \cup Q = I_0$, $R \cap Q = \emptyset$. Associated with each partition (R,Q), define *branch* of the feasible set

$$D_{RQ} := \left\{ (x,y) \;\middle|\; \begin{array}{l} g_i(x,y) \geq 0\,, y_i = 0\,, i \in R \\ g_i(x,y) = 0\,, y_i \geq 0\,, i \in Q \cup I_g \end{array} \right\}. \tag{3.3}$$

It is easy to see that

$$\bigcup_{(R,Q)} (D_{RQ} \cap \mathcal{V}) = D \cap \mathcal{V}\,, \quad T_D(z^*) = \bigcup_{(R,Q)} T_{D_{RQ}}(z^*)\,, \tag{3.4}$$

where $\mathcal{V}$ is a neighbourhood of z^*. If one assumes that, say, the MFCQ is satisfied at z^* for constraints defining each of the sets D_{RQ}, then the corresponding branch of L given by

$$L_{RQ} := \left\{ (u,v) \;\middle|\; \begin{array}{l} v_i = 0\,, \langle \nabla g_i(z^*), (u,v)\rangle \geq 0\,, i \in R \\ \langle \nabla g_i(z^*), (u,v)\rangle = 0\,, i \in Q \cup I_g \\ v_i \geq 0\,, i \in Q \end{array} \right\} \tag{3.5}$$

will represent necessary *and* sufficient conditions of tangency for this D_{RQ}. Putting together all the pieces, one then obtains that $T_D(z^*) = L$.

In general however, (3.1) does not hold as equality, and so further analysis is needed to obtain a precise description of the tangent directions. We next show how this can be done with the help of 2-regularity outlined in Section 2.

Recall the mapping $F : \Re^n \times \Re^m \to \Re^{|I_0|+|I_g|}$ associated with equality constraints in D^* (which is locally equivalent to D):

$$F_i(x,y) := \left\{ \begin{array}{ll} y_i g_i(x,y)\,, & i \in I_0 \\ g_i(x,y)\,, & i \in I_g \end{array} \right. . \tag{3.6}$$

With this notation,

$$D^* := \left\{ (x,y) \;\middle|\; \begin{array}{l} y_i \geq 0\,, g_i(x,y) \geq 0\,,\ i \in I_0 \\ F(x,y) = 0 \end{array} \right\}.$$

Note that the i-th row of $F'(x^*,y^*)$ is

$$F'(x^*,y^*)_i = \nabla F_i(x^*,y^*) = \begin{cases} 0\,, & i \in I_0 \\ \nabla g_i(x^*,y^*)\,, & i \in I_g \end{cases}\,.$$

Observe that F cannot be regular, unless the degenerate set I_0 is empty. On the other hand, it is quite natural to study F from the point of view of 2-regularity. Note however that F represents only part of the constraints. Indeed, application of 2-regularity (and arguably, of any general regularity concept) to MPEC cannot be straightforward. This can already be seen from the one-dimensional example considered in Section 2. The current general theory for irregular problems does not permit mixed equality and inequality constraints, except in some very special cases. Within the general theory available, if one is to insist on mixed constraints, then all irregularity has to be induced by equalities with inequalities being regular [2, 9], or vice versa [8]. Yet it can be verified that for MPEC even these two extreme cases cannot be applied. Fortunately, and thanks to the special complementarity structure of D, it appears possible to develop a special approach for MPEC, different from the general theory. We next show that whenever tangent directions can be characterized via 2-regularity for the subset of constraints in D^* corresponding to $F(x,y)=0$, then these directions are automatically tangent for the complete set of constraints, without any further assumptions involving these constraints.

We start with computing some objects described in Section 2. Let A_g be the $|I_g| \times (n+m)$ matrix with rows $\nabla g_i(z^*)\,, i \in I_g$. With this notation,

$$\operatorname{Ker} F'(z^*) = \operatorname{Ker} A_g\,.$$

Furthermore,

$$\operatorname{Im} F'(z^*) = \{(0,w) \in \Re^{|I_0|} \times \Re^{|I_g|} \mid w \in \operatorname{Im} A_g\}\,,$$

$$(\operatorname{Im} F'(z^*))^\perp = \{(p,q) \in \Re^{|I_0|} \times \Re^{|I_g|} \mid q \in (\operatorname{Im} A_g)^\perp\}\,.$$

Now, let P be the orthogonal projector onto $(\operatorname{Im} F'(z^*))^\perp$. Then

$$P(x,y) = (x, P_g y)\,,$$

where P_g is the orthogonal projector onto $(\operatorname{Im} A_g)^\perp$. Differentiating F twice, it can be verified that for any $(u,v) \in \Re^n \times \Re^m$

$$F''(z^*)[(u,v)] = \begin{pmatrix} B_0[(u,v)] \\ B_g[(u,v)] \end{pmatrix}\,,$$

where $B_0[(u,v)]$ is the $|I_0| \times (n+m)$ matrix with rows

$$B_0[(u,v)]_i = v_i \nabla g_i(z^*) + \langle \nabla g_i(z^*), (u,v)\rangle e^i\,, \quad i \in I_0\,, \tag{3.7}$$

with $e^i \in \Re^{n+m}$ being the vector of zeroes except for the $(n+i)$-th component which is equal to one; and $B_g[(u,v)]$ is the $|I_g| \times (n+m)$ matrix with rows

$$B_g[(u,v)]_i = \nabla^2 g_i(z^*)(u,v)\,, \quad i \in I_g\,. \tag{3.8}$$

Observe further that

$$(B_0[(u,v)]^2)_i = 2 v_i \langle \nabla g_i(z^*), (u,v)\rangle\,. \tag{3.9}$$

Taking into account this information, we have that

$$PF''(z^*)[(u,v)]^2 = 0 \quad \Leftrightarrow \quad B_0[(u,v)]^2 = 0\,,\ P_gB_g[(u,v)]^2 = 0\,,$$

which in turn is equivalent to

$$v_i\langle \nabla g_i(z^*), (u,v)\rangle = 0\,, i \in I_0\,, \quad P_gB_g[(u,v)]^2 = 0\,.$$

We can now state necessary conditions for tangent directions to equality constraints in D^*, in terms of cone H_2 defined by (2.1):

$$\begin{array}{l} \langle \nabla g_i(z^*), (u,v)\rangle = 0\,, i \in I_g\,, \\ v_i\langle \nabla g_i(z^*), (u,v)\rangle = 0\,, i \in I_0\,, \\ P_gB_g[(u,v)]^2 = 0\,. \end{array}$$

Putting these together with standard necessary conditions for inequality constraints in D^*, we obtain

$$T_D(z^*) \subset H\,,$$

where

$$H := \left\{ (u,v) \;\middle|\; \begin{array}{l} \langle \nabla g_i(z^*), (u,v)\rangle \geq 0\,, v_i \geq 0\,, i \in I_0 \\ v_i\langle \nabla g_i(z^*), (u,v)\rangle = 0\,, i \in I_0 \\ \langle \nabla g_i(z^*), (u,v)\rangle = 0\,, i \in I_g \\ P_gB_g[(u,v)]^2 = 0 \end{array} \right\}. \tag{3.10}$$

Note that the above is not a standard set of necessary conditions: we have a mix of second-order conditions for equality constraints with first-order conditions for inequality constraints. In general, H is not piecewise polyhedral, although it can be such under some natural assumptions discussed below. But perhaps more interesting is the difficult case when H is not piecewise polyhedral. Even in that case, as we shall see, (3.10) gives a precise (i.e., $T_D(z^*) = H$) constructive description of tangency whenever the mapping F is 2-regular. This can further be used for deriving a special form of primal-dual optimality conditions in the situation when other techniques are not applicable.

We are now ready to state our description of the tangent cone to complementarity constraints.

THEOREM 3.1. *Let g be twice differentiable at $z^* \in D$, where D is given by* (1.1). *Then the following statements hold.*

1. If $h \in T_D(z^*)$ then $h \in H$.

2. If F defined by (3.6) *is 2-regular at z^* with respect to $h \in H$, then $h \in T_D(z^*)$.*

Proof. In view of the preceding discussion, the first assertion of the theorem requires no further justification.

Suppose now that F is 2-regular with respect to some $h = (u,v) \in H$. By Theorem 2.2, it then follows that h is tangent to the set $\{z \mid F(z) = 0\}$:

$$F(z^* + th + r(t)) = 0\,, \quad \|r(t)\| = o(t)\,,$$

where $r(\cdot) = (r^1(\cdot), r^2(\cdot))$ is the mapping from Theorem 2.2. By the definition of F, this is equivalent to

$$\begin{aligned} (y_i^* + tv_i + r_i^2(t))g_i(x^* + tu + r^1(t), y^* + tv + r^2(t)) &= 0\,, i \in I_0\,, \\ g_i(x^* + tv + r^1(t), y^* + tv + r^2(t)) &= 0\,, i \in I_g\,. \end{aligned} \tag{3.11}$$

To prove that $h \in T_D(z^*)$, it remains to show that $z^*+th+r(t)$ satisfies the remaining inequality constraints (for $t>0$ sufficiently small). According to prior calculations,

$$F'(z^*)+PF''(z^*)[(u,v)] = \begin{pmatrix} B_0[(u,v)] \\ A_g + P_g B_g[(u,v)] \end{pmatrix},$$

and the assumption of 2-regularity means that the matrix in the right-hand side has full row rank. In particular, $B_0[(u,v)]$ must have full row rank. Hence (see (3.7)),

$$v_i \nabla g_i(z^*) + \langle \nabla g_i(z^*), (u,v)\rangle e^i\,,\ i \in I_0\,,\ \text{are linearly independent.} \tag{3.12}$$

Define

$$I_0^0 := \{i \in I_0 \mid v_i = 0\}\,, \quad I_0^1 := I_0 \setminus I_0^0 = \{i \in I_0 \mid v_i > 0\}\,.$$

It immediately follows from (3.10) that

$$\langle \nabla g_i(z^*), (u,v)\rangle = 0\,, i \in I_0^1\,.$$

Note that the linear independence of vectors in (3.12) corresponding to $i \in I_0^0$ means that $\langle \nabla g_i(z^*), (u,v)\rangle \neq 0\,, i \in I_0^0$. Taking further into account (3.10), this is equivalent to

$$\langle \nabla g_i(z^*), (u,v)\rangle > 0\,, i \in I_0^0\,. \tag{3.13}$$

Using (3.13), differentiability of g implies that for all $t>0$ sufficiently small

$$g_i(x^*+tv+r^1(t), y^*+tv+r^2(t)) = t\langle \nabla g_i(z^*), (u,v)\rangle + o(t) > 0\,, i \in I_0^0\,.$$

The latter relation, together with (3.11), yields

$$y_i^* + tv_i + r_i^2(t) = r_i^2(t) = 0\,, i \in I_0^0\,.$$

Similarly, for all $t>0$ small enough,

$$y_i^* + tv_i + r_i^2(t) = tv_i + r_i^2(t) > 0\,, i \in I_0^1\,,$$

and so (3.11) implies that

$$g_i(x^*+tv+r^1(t), y^*+tv+r^2(t)) = 0\,, i \in I_0^1\,.$$

In particular, we have established that for all $t>0$ sufficiently small, $(x,y)=z^*+th+r(t)$ satisfies the inequality constraints $y_i \geq 0\,, g_i(x,y) \geq 0\,, i \in I_0 = I_0^0 \cup I_0^1$, which completes the proof that $h \in T_D(z^*)$. □

COROLLARY 3.2. *The following statements hold.*

1. *If there exists some set $K \subset H$ such that F is 2-regular with respect to every $h \in K$ and $\operatorname{cl} K = H$, then $T_D(z^*) = H$.*
2. *If*

$$\nabla g_i(z^*)\,, i \in I_g\,,\quad \text{are linearly independent}, \tag{3.14}$$

then $H = L$. In particular, under the assumption of the previous item, $T_D(z^*) = L$.

Proof. The first assertion follows from Theorem 3.1 and closedness of the tangent cone. For the second assertion, just notice that (3.14) implies that $(\operatorname{Im} A_g)^\perp = \{0\}$, and so $P_g = 0$. Hence, $H = L$. □

Therefore, whenever there exists a dense subset of H with respect to which F is 2-regular, H gives a precise description of the tangent cone. If also (3.14) holds, necessary conditions for tangency take a simpler piecewise polyhedral form.

We next show that the piecewise MFCQ is a sufficient condition for 2-regularity of F on a certain dense subset of H, as well as for (3.14). A branch D_{RQ}, defined by (3.3), satisfies MFCQ if

$$\begin{array}{l} \nabla g_i(z^*)\,, i \in Q \cup I_g\,,\ e^i\,, i \in R\,, \text{ are linearly independent, and} \\ \exists\,(u,v) \in \Re^n \times \Re^m \text{ such that} \\ v_i = 0\,, \langle \nabla g_i(z^*), (u,v)\rangle > 0\,, i \in R\,, \\ \langle \nabla g_i(z^*), (u,v)\rangle = 0\,, i \in Q \cup I_g\,, \\ v_i > 0\,, i \in Q\,, \end{array} \tag{3.15}$$

where $e^i = (0, 0, \ldots, 1, \ldots, 0) \in \Re^{n+m}$ with one in the $(n+i)$-th position. Under the piecewise MFCQ, for each branch D_{RQ} the (classical) linearized cone L_{RQ}, defined by (3.5), represents all the tangent directions. Note that, as discussed above, the fact that a piecewise constraint qualification implies the equality $T_D(z^*) = L$ is fairly obvious. The significance of the following result lies not in establishing this equality, but in providing a new interpretation for piecewise MFCQ in terms of 2-regularity of a subset of constraints of the original non-decomposed problem. Since piecewise MFCQ is arguably the most natural condition guaranteeing that $T_D(z^*) = L$, this goes to show that the introduced 2-regularity condition is also natural for MPEC. Furthermore, piecewise MFCQ is merely a sufficient condition for 2-regularity, as would be demonstrated by an example.

THEOREM 3.3. *Suppose that piecewise MFCQ holds at $z^* \in D$. Then F is 2-regular with respect to every $h = (u,v) \in \Re^n \times \Re^m$ satisfying* (3.15) *for some pair (R,Q), and it holds that $T_D(z^*) = H = L$.*

Proof. Consider an arbitrary branch D_{RQ} of the feasible set D, and an arbitrary $h = (u,v)$ satisfying (3.15). We have to show that the rows of matrix

$$F'(z^*) + PF''(z^*)[(u,v)] = \begin{pmatrix} B_0[(u,v)] \\ A_g + P_g B_g[(u,v)] \end{pmatrix} = \begin{pmatrix} B_0[(u,v)] \\ A_g \end{pmatrix}$$

are linearly independent. By (3.7) and (3.15), we have that

$$B_0[(u,v)]_i = \begin{cases} \langle \nabla g_i(z^*), (u,v)\rangle e^i\,, & i \in R \\ v_i \nabla g_i(z^*)\,, & i \in Q \end{cases}.$$

By (3.15), it further holds that

$$\begin{array}{l} \langle \nabla g_i(z^*), (u,v)\rangle > 0\,, i \in R\,, \\ v_i > 0\,, i \in Q\,. \end{array}$$

The linear independence condition in (3.15) implies now the desired linear independence of the rows of $F'(z^*) + PF''(z^*)[(u,v)]$.

Taking into account that MFCQ subsumes (3.14), the last assertion follows immediately from Corollary 3.2. □

The following example demonstrates that 2-regularity of F is a weaker condition than piecewise MFCQ, i.e., the former may be satisfied when the latter is not. Furthermore, it shows that our results can yield a relatively simple description of the tangent cone even in the difficult situation when piecewise constraint qualifications fail.

EXAMPLE 3.1. Let $n = 1$, $m = 2$ and consider the set D given by (1.1), where

$$g_1(x, y) = x + y_1, \quad g_2(x, y) = x^2 - (y_2 - a)^2,$$

with $a > 0$ being any given number. Consider the point $z^* = (x^*, y^*)$ with $x^* = 0$, $y_1^* = 0$, $y_2^* = a$. We have that $z^* \in D$, and $I_0 = \{1\}$, $I_g = \{2\}$,

$$H = \left\{ (u, v) \;\middle|\; \begin{array}{l} v_1 \geq 0,\, u + v_1 \geq 0 \\ v_1(u + v_1) = 0 \\ u^2 - v_2^2 = 0 \end{array} \right\}.$$

Since $\nabla g_2(z^*) = 0$, it is clear from (3.15) that the piecewise MFCQ does not hold at z^*. By direct computations, we have that

$$F'(z^*) + PF''(z^*)[(u,v)] = \left(\begin{array}{ccc} v_1 & u + 2v_1 & 0 \\ 2u & 0 & 2v_2 \end{array} \right).$$

It can be easily seen that F is 2-regular with respect to any element in $H \setminus \{0\}$, and hence, $T_D(z^*) = H$.

It is further interesting to note that even though $H \neq L$, the structure of H is no more complex than that of L. Indeed, observe that $u^2 - v_2^2 = 0 \;\Leftrightarrow\; u = v_2$ or $u = -v_2$. Hence, cone H is piecewise polyhedral, just like L. However, $H = T_D(z^*) \neq L$. This shows that 2-regularity can give a constructive description of the tangent cone when $T_D(z^*) \neq L$, even without going beyond the situation where the tangent cone is piecewise polyhedral.

We conclude this example by noting that it remains valid if we perturb g by any function δ such that $\delta(z) = o(\|z - z^*\|^2)$. In particular, this would not affect 2-regularity properties of F or constraints defining cone H (this is because the first and second derivatives at z^* would not change). After such a perturbation, it would still hold that $H = T_D(z^*) \neq L$.

4. Optimality conditions. In this section we show that 2-regularity can be used to derive optimality conditions, including a special form of primal-dual conditions, in situations when alternative approaches may not be applicable. Our results are meant as a complement (rather than a substitute) to other types of optimality conditions for MPEC, some of which we shall discuss below.

We start with the primal form of optimality conditions, which can be derived in a standard way using the relations

$$\mathrm{cl}\, K \subset T_D(x^*) \subset H\,,$$

where

$$K := \{h \in H \mid F \text{ is 2-regular with respect to } h\}\,.$$

THEOREM 4.1. *Let f be once, and g be twice differentiable at a point $z^* \in D$. If z^* is a local solution of* (1.2)*, then*

$$\langle \nabla f(z^*),\, h\rangle \geq 0 \quad \forall\, h \in \mathrm{cl}\, K\,. \tag{4.1}$$

If it holds that

$$\langle \nabla f(z^*),\, h\rangle > 0 \quad \forall\, h \in H \setminus \{0\}\,, \tag{4.2}$$

then z^* is an isolated local solution of (1.2).

Note that when $\operatorname{cl} K = H = L$ (e.g., under the assumptions of Corollary 3.2) then we recover B-stationarity as defined in [18], which is generally considered one of the natural primal stationarity concepts for MPEC (see also Example 4.1).

To obtain a primal-dual form of optimality conditions, one way is to compute the dual cone of K. In general, this is a difficult problem, as this cone need not be even piecewise polyhedral. However, we are able to derive special primal-dual optimality conditions using a different technique. This is possible under the assumption that there exists some $\bar{h} \in K$ for which inequality in (4.1) holds as equality (a "critical" direction). This assumption is justifiable in this context, and it is quite common in the literature, e.g., [3, 16]. If $\langle \nabla f(z^*), \bar{h} \rangle > 0$ for a given $\bar{h} \in K$, then it is clear that the same property holds for all h in some neighborhood $\mathcal{U}$ of $\bar{h}$. Hence, on $\mathcal{U}$ there is no contradiction with necessary optimality conditions, and z^* is still a candidate for being a solution. There is no more local information to be extracted using this $\bar{h}$. On the other hand, if $\langle \nabla f(z^*), \bar{h} \rangle = 0$, a further investigation is needed, and indeed more local information relevant to optimality of z^* can be obtained. Note that in the following result, 2-regularity of F is assumed with respect to this $\bar{h}$ only, and in particular, the knowledge of the entire tangent cone is not necessary. We first state our optimality conditions using the same objects as in 2-regularity constructions in Section 3, which allows a more compact form. After the proof, we shall re-write them in terms of the original problem data, and give comparisons with some other results in the literature.

THEOREM 4.2. *Let f be once, and g be twice differentiable at a point $z^* \in D$, which is a local minimizer for problem* (1.2). *Assume further that*

$$\exists\, \bar{h} \in K \text{ such that } \langle \nabla f(z^*),\, \bar{h} \rangle = 0\,. \tag{4.3}$$

Then there exist unique $\bar{\lambda} \in \Re^{|I_0|}$ and $\bar{\mu}^1$, $\bar{\mu}^2 \in \Re^{|I_g|}$ (all depending on $\bar{h}$) such that

$$\nabla f(z^*) + (B_0[\bar{h}])^T \bar{\lambda} + A_g^T \bar{\mu}^1 + (B_g[\bar{h}])^T \bar{\mu}^2 = 0\,, \tag{4.4}$$

$$\bar{\mu}^1 \in \operatorname{Im} A_g, \quad \bar{\mu}^2 \in (\operatorname{Im} A_g)^\perp\,, \tag{4.5}$$

where the index sets I_0 and I_g are as defined earlier.

Moreover, the multiplier $\bar{\mu}^2$ satisfies the following additional condition: for every h such that

$$h \in \operatorname{Ker} A_g\,, \quad B_0[\bar{h},\, h] = 0\,, \quad B_g[\bar{h},\, h] \in \operatorname{Im} A_g\,, \tag{4.6}$$

it holds that

$$\langle \bar{\mu}^2,\, B_g[h]^2 \rangle \geq 0\,. \tag{4.7}$$

Proof. It can be easily seen from the Definition 2.1 that if 2-regularity of F holds for some $\bar{h} = (\bar{u}, \bar{v}) \in H$, then it holds for all h sufficiently close to $\bar{h}$. In other words, there exists a neighbourhood $\mathcal{U}$ of $\bar{h}$ in $\Re^n \times \Re^m$ such that

$$H \cap \mathcal{U} \subset K\,.$$

Hence, by Theorem 4.1, we have that

$$\langle \nabla f(z^*),\, h \rangle \geq 0 \quad \forall\, h \in H \cap \mathcal{U}\,.$$

Observe that the latter relation and (4.3) imply that $\bar{h}$ is a local solution of the optimization problem

$$\min_{h \in H} \langle \nabla f(z^*),\, h \rangle . \tag{4.8}$$

The feasible set H of problem (4.8), given by (3.10), can be equivalently written in the following form (recall definitions of B_0 and A_g):

$$H = \left\{ h = (u,\, v) \;\middle|\; \begin{array}{l} v_i \geq 0,\ \langle \nabla g_i(z^*),\, h \rangle \geq 0,\ i \in I_0 \\ \frac{1}{2} B_0[h]^2 = 0 \\ A_g h + \frac{1}{2} P_g B_g[h]^2 = 0 \end{array} \right\} . \tag{4.9}$$

Here we also took into account the following obvious observation: for any h the elements $A_g h$ and $\frac{1}{2} P_g B_g [h]^2$ belong to orthogonal subspaces in $\Re^{|I_g|}$, and hence, they both are equal to zero if and only if their sum is equal to zero.

Next we show that in a neighborhood of $\bar{h}$ the set H is completely defined by equality constraints only, so in our local considerations we can omit the inequality constraints in (4.9). Indeed, recall that since F is 2-regular with respect to $\bar{h}$, we have by (3.13) that

$$\langle \nabla g_i(z^*),\, \bar{h} \rangle > 0, \quad i \in I_0^0, \tag{4.10}$$

where

$$I_0^0 = \{ i \in I_0 \mid \bar{v}_i = 0 \}. \tag{4.11}$$

Let h satisfy the equality $B_0[h]^2 = 0$. By (3.9), we then obtain that

$$v_i \langle \nabla g_i(z^*),\, h \rangle = 0, \quad i \in I_0 .$$

This relation and (4.10) imply that if $h = (u, v)$ is sufficiently close to $\bar{h}$, then it must hold that

$$\begin{array}{ll} v_i = 0,\ \langle \nabla g_i(z^*),\, h \rangle > 0, & i \in I_0^0, \\ v_i > 0,\ \langle \nabla g_i(z^*),\, h \rangle = 0, & i \in I_0^1, \end{array} \tag{4.12}$$

where

$$I_0^1 = I_0 \setminus I_0^0 = \{ i \in I_0 \mid \bar{v}_i > 0 \}. \tag{4.13}$$

We have therefore established that there exists a neighborhood $\mathcal{V}$ of $\bar{h}$ such that whenever $h \in \mathcal{V}$ satisfies the equality constraints in (4.9), it also satisfies the inequality constraints (by (4.12)). In other words, H is completely defined by its equality constraints (locally, in $\mathcal{V}$). Writing these equality constraints as $\Phi(h) = 0$, where

$$\Phi(h) = \begin{pmatrix} \frac{1}{2} B_0[h]^2 \\ A_g h + \frac{1}{2} P_g B_g [h]^2 \end{pmatrix} ,$$

we conclude that problem (4.8) is locally equivalent to

$$\min_{\Phi(h)=0} \langle \nabla f(z^*), h \rangle . \tag{4.14}$$

Furthermore,

$$\Phi'(\bar{h}) = \left(\begin{array}{c} B_0[\bar{h}] \\ A_g + P_g B_g[\bar{h}] \end{array} \right), \tag{4.15}$$

and, as we have verified in the proof of Theorem 3.1, the assumption of 2-regularity of F with respect to $\bar{h}$ means that the matrix in the right-hand side has full row rank. Thus Φ is regular at $\bar{h}$ in the classical sense. This means that classical first and second order optimality conditions for regular equality-constrained problems are applicable for problem (4.14) at its local solution $\bar{h}$. To this end, define the standard Lagrangian for (4.14):

$$\begin{aligned} L(h,\, \lambda,\, \mu) &= \langle \nabla f(z^*),\, h\rangle + \langle (\lambda,\, \mu),\, \Phi(h)\rangle \\ &= \langle \nabla f(z^*),\, h\rangle + \langle \lambda,\, \tfrac{1}{2} B_0[h]^2\rangle + \langle \mu,\, A_g h + \tfrac{1}{2} P_g B_g[h]^2\rangle\,. \end{aligned}$$

Representing μ as $\mu = \mu^1 + \mu^2$, where $\mu^1 \in \operatorname{Im} A_g$ and $\mu^2 \in (\operatorname{Im} A_g)^\perp$, the Lagrangian can be further re-written as

$$L(h,\, \lambda,\, \mu) = \langle \nabla f(z^*),\, h\rangle + \frac{1}{2}\langle \lambda,\, B_0[h]^2\rangle + \langle \mu^1,\, A_g h\rangle + \frac{1}{2}\langle \mu^2,\, B_g[h]^2\rangle\,. \tag{4.16}$$

Now, by the classical first-order necessary optimality conditions, there exists a (unique, by necessity) pair $(\bar{\lambda},\, \bar{\mu}) \in \Re^{|I_0|} \times \Re^{|I_g|}$ such that

$$\nabla_h L(\bar{h},\, \bar{\lambda},\, \bar{\mu}) = 0\,,$$

and, by the second-order necessary optimality conditions, it further holds that

$$\langle \nabla_h^2 L(\bar{h},\, \bar{\lambda},\, \bar{\mu}) h,\, h\rangle \geq 0 \quad \forall\, h \in \operatorname{Ker} \Phi'(\bar{h})\,.$$

Computing all the derivatives involved (by taking into account (4.15) and (4.16)), we obtain the asserted results. ∎

Using the definition of A_g, (3.7), (3.8) and (4.12) (for $h = \bar{h}$), we can re-write our optimality conditions explicitly in terms of the problem data. Specifically, for (4.4) and the second relation in (4.5), we obtain

$$\nabla f(z^*) + \sum_{i\in I_0^0} \tilde{\lambda}_i e^i + \sum_{i\in I_0^1} \tilde{\lambda}_i \nabla g_i(z^*) + \sum_{i\in I_g} (\bar{\mu}_i^1 \nabla g_i(z^*) + \bar{\mu}_i^2 \nabla^2 g_i(z^*)\bar{h}) = 0\,, \tag{4.17}$$

and

$$\sum_{i\in I_g} \bar{\mu}_i^2 \nabla g_i(z^*) = 0\,,$$

where e^i is as defined earlier, the index sets I_0^0 and I_0^1 are defined in (4.11) and (4.13), respectively, and

$$\tilde{\lambda}_i = \left\{ \begin{array}{ll} \bar{\lambda}_i \langle \nabla g_i(z^*),\, \bar{h}\rangle, & i \in I_0^0 \\ \bar{\lambda}_i \bar{v}_i, & i \in I_0^1 \end{array} \right. .$$

Similarly, we can re-write the first two conditions on $h = (u, v)$ in (4.6) as follows:

$$\langle \nabla g_i(z^*),\, h\rangle = 0, \quad i \in I_g\,,$$

and

$$v_i = 0, \quad i \in I_0^0, \quad \langle \nabla g_i(z^*),\, h\rangle = 0, \quad i \in I_0^1,$$

respectively. Finally, taking into account (3.9), condition (4.7) takes the form

$$\sum_{i \in I_g} \bar{\mu}_i^2 \langle \nabla^2 g_i(z^*) h,\, h\rangle \geq 0\,.$$

Note that if the linear independence condition (3.14) is satisfied, then necessarily $\bar{\mu}^2 = 0$, and our optimality conditions take a simpler form. We next comment on some other approaches to constraint qualifications and optimality conditions for MPEC.

First, note that Theorem 4.2 subsumes optimality conditions for the *nondegenerate* case of $I_0 = \emptyset$, but in a way different from conditions based on disjunctive programming [11, 10]. Observe that if $I_0 = \emptyset$ then F is comprised by components $g_i\,, i \in I_g$. If further $\nabla g_i(z^*)\,, i \in I_g$ are linearly independent (condition (3.14)), then F is regular at z^* in the classical sense, and hence, 2-regular with respect to every h. We can then choose $\bar{h} = 0$ in (4.3). With this choice, primal-dual characterization of optimality given by Theorem 4.2 reduces to $\nabla f(z^*) = \sum_{i \in I_g} \bar{\mu}_i^1 \nabla g_i(z^*)$, which is precisely the Karush-Kuhn-Tucker conditions in the situation where $I_0 = \emptyset$ and $I_y = \emptyset$ (recall that we set $I_y = \emptyset$ merely to simplify notation; if $I_y \neq \emptyset$ then the gradients of $y_i = 0\,, i \in I_y$ enter into play in the obvious way).

In [15], some conditions are obtained which ensure that the tangent cone and/or its dual can be computed as corresponding cones of certain standard nonlinear programs associated with MPEC. In particular, these cones are therefore polyhedral. Since this need not be the case in our development, it is clear that the two approaches are principally different. The motivation of [15] is to identify situations where MPEC optimality can be characterized without an excessive combinatorial burden of disjunctive programming. Note that our constraint qualification does not require decomposition of the feasible region for its verification. In this sense, it also contributes to the same goal.

We next comment on some results in [18], and in particular show that our approach can be useful to verify optimality when [18] may not be applicable, and vice versa. We start with some general comments, and then give a number of examples.

For the feasible set D given by (1.1), all constraint qualifications considered in [18] are equivalent to the following linear independence constraint qualification (LICQ):

$$\nabla g_i(z^*),\, i \in I_0 \cup I_g,\, e^i,\, i \in I_0, \quad \text{are linearly independent.} \tag{4.18}$$

Under this assumption, the following primal-dual necessary conditions hold:

$$\nabla f(z^*) = \sum_{i \in I_0 \cup I_g} \bar{\lambda}_i \nabla g_i(z^*) + \sum_{i \in I_0} \bar{\mu}_i e^i, \qquad \bar{\lambda}_i \geq 0\,, \bar{\mu}_i \geq 0\,, i \in I_0\,. \tag{4.19}$$

To compare this result with ours, first note that LICQ (4.18) implies piecewise MFCQ, which is stronger than 2-regularity (recall Example 3.1). On the other hand, for our primal-dual conditions we need a tangent direction $\bar{h}$ which is critical (condition (4.3)). It is therefore clear that the two constraint qualifications are not directly comparable. In other words, one can be satisfied when the other is not, and vice versa. Secondly, note that primal-dual conditions (4.17) and (4.19) are also essentially different. If

(3.14) holds (which is certainly implied by LICQ), as discussed above, we have that $\bar{\mu}^2 = 0$. Observe now that, since $I_0^0 \cup I_0^1 = I_0$, the right-hand side of (4.17) involves twice fewer gradients of constraints from the index set I_0 than (4.19). I.e., in (4.17) the multipliers corresponding to half of constraints in I_0 are claimed to be zero. In this respect, (4.17) is sharper than (4.19). On the other hand, (4.19) contains additional conditions on the signs of the multipliers in I_0. Hence, the two optimality conditions are essentially different, none of them being stronger than the other.

We next give some examples. Note that in the first example below, LICQ (4.18) holds, which is a much stronger condition than is needed to apply our results. Thus this example is certainly not the most suitable for showing the utility of our approach. We include it mainly because it is extensively used in the literature, e.g., [18]. Examples 4.2 and 4.3 better illustrate the differences and our contribution.

EXAMPLE 4.1. Let $n = m = 1$ and

$$f(x, y) = \alpha x + \beta y\,, \quad g(x, y) = x\,,$$

where α, $\beta \in \Re$ are parameters. In the terminology of [18], the point $z^* = (x^*, y^*) = 0$ is a B-stationary point (equivalent here to strongly stationary) if and only if $\alpha \geq 0$, $\beta \geq 0$. This is a primal stationarity condition. Concerning primal-dual stationarity, z^* is a weak stationary point for any α and β; it is a C-stationary point if and only if either $\alpha \geq 0$, $\beta \geq 0$ or $\alpha \leq 0$, $\beta \leq 0$; and it is an M-stationary point if and only if $\alpha \geq 0$, $\beta \geq 0$ or $\alpha\beta = 0$.

As is easy to see, $I_0 = \{1\}$, $F'(z^*) = 0$, $H = \{(u, v) \mid u \geq 0, v \geq 0, uv = 0\}$ and F is 2-regular at z^* with respect to all $(u, v) \in H \backslash \{0\}$ because $F'(z^*) + PF''(z^*)[(u, v)] = (v, u)$.

We start with illustrating our primal optimality conditions. If $\alpha > 0$, $\beta > 0$ then (4.2) holds, and the second assertion of Theorem 4.1 guarantees that z^* is an isolated local minimizer. The first assertion of Theorem 4.1 shows that z^* cannot be a minimizer unless $\alpha \geq 0$, $\beta \geq 0$. The latter is consistent with B-stationarity because in this example (3.14) holds, and $T_D(z^*) = \operatorname{cl} K = H = L$ (recall the comment after Theorem 4.1). Example 4.2 below shows that Theorem 4.1 is in fact useful beyond B-stationarity in the sense of [18].

Consider now primal-dual conditions.

Let $\alpha = 0$ and $\beta > 0$, in which case z^* is a minimizer. Note that $\bar{h} = (\bar{u}, \bar{v}) = (1, 0)$ satisfies (4.3). We have that $I_0^0 = \{1\}$, $I_0^1 = \emptyset$, $\tilde{\lambda}_1 = \bar{\lambda}_1 \langle \nabla g(z^*), \bar{h} \rangle = \bar{\lambda}_1$ and (4.17) holds with $\tilde{\lambda}_1 = -\beta$. All the stationarity conditions from [18] are also satisfied here.

Let $\alpha = 1$ and $\beta = -1$, in which case z^* is not a minimizer. In (4.3) we can take $\bar{h} = (\bar{u}, \bar{v}) = (1, 1)$, so that $I_0^1 = \{1\}$, $I_0^0 = \emptyset$ and $\tilde{\lambda}_1 = \bar{\lambda}_1 \bar{v}_1 = \bar{\lambda}_1$. There exists no $\tilde{\lambda}_1$ such that $(1, -1) + \tilde{\lambda}_1(1, 0) = 0$, and so (4.17) does not hold. Therefore this non-optimal z^* is not stationary in our sense. Note that it is weakly stationary, but not C- or M-stationary.

In the case $\alpha = 0$ and $\beta < 0$, the point z^* is not a minimizer. But all the stationarity concepts of [18] hold, and it can be seen that Theorem 4.2 also applies. In this case, none of those stationarity concepts is able to detect non-optimality of z^*.

The following examples are intended to show that the presented optimality conditions can be useful in situations when other approaches (e.g., [15, 18, 6]) do not apply.

EXAMPLE 4.2. Let g, x^* and y^* be as defined in Example 3.1, and the objective function be given by $f(x, y) = \alpha x + \beta_1 y_1 + \beta_2 y_2 + o(\|(x, y)\|)$, where α, β_1, β_2 are parameters. Recall that not only (4.18), but even much weaker condition (3.14) does not hold in this example. In particular, constraint qualifications used in [15, 18] do not hold, and the corresponding results do not apply.

According to Example 3.1, the cone H consists of four rays spanned by (1, 0, 1), (1, 0, −1), (−1, 1, 1) and (−1, 1, −1), and the mapping F is 2-regular at z^* with respect to every element in $H \setminus \{0\}$. From the primal necessary optimality conditions stated in Theorem 4.1 it can be derived that z^* cannot be a local minimizer unless

$$0 \le \alpha \le \beta_1, \quad |\beta_2| \le \min\{a,\, |\beta_1 - a|\}.$$

Sufficient condition (4.2) is satisfied if, and only if, both of these inequalities hold strictly.

Next, for $\bar{h} = (1,\, 0,\, \pm 1)$, equality in (4.3) takes place if, and only if, $\alpha \pm \beta_2 = 0$, and (4.4), (4.5) hold with $\bar{\lambda} = -\beta_1$, $\bar{\mu}^1 = 0$, $\bar{\mu}^2 = \pm\beta_2/2$. For $\bar{h} = (-1,\, 1,\, \pm 1)$, equality in (4.3) takes place if, and only if, $-\alpha + \beta_1 \pm \beta_2 = 0$, and (4.4), (4.5) hold with the same multipliers.

"Second-order" condition (4.7) does not provide any additional information in this example, as for every $\bar{h} \in H \setminus \{0\}$, the subspace comprised by elements h satisfying (4.6) coincides with span$\{\bar{h}\}$, and hence $B_g[h]^2 = 0$ for such h.

We next demonstrate the utility of the "second-order" condition (4.7). Note that this example also highlights the differences of the present paper as compared to optimality conditions in [6], where a general theory of 2-regularity is developed for $C^{1,1}$ mappings, and optimality conditions for MPEC are obtained after reformulating the constraints as $C^{1,1}$ equations.

EXAMPLE 4.3. Let $n = m = 2$,

$$f(x,\, y) = x_1 + y_1 - y_2, \quad g_1(x,\, y) = x_1 + y_1, \quad g_2(x,\, y) = x_1^2 + x_2^2 - (y_2 - a)^2,$$

where $a > 0$ is a parameter. Consider the point $z^* = (x^*,\, y^*) \in D$ with $x^* = 0$, $y_1^* = 0$, $y_2^* = a$. It can be verified that this z^* is not a minimizer, but the stationarity conditions given in [6, Section 6] hold. Results from [15, 18] are not applicable (because $\nabla g_2(z^*) = 0$ and thus constraint qualifications do not hold). We next show that Theorem 4.2 correctly classifies z^* as non-optimal.

We have that $I_0 = \{1\}$, $I_g = \{2\}$,

$$H = \left\{ (u,\, v) \;\middle|\; \begin{array}{l} v_1 \ge 0,\, u_1 + v_1 \ge 0 \\ v_1(u_1 + v_1) = 0 \\ u_1^2 + u_2^2 - v_2^2 = 0 \end{array} \right\}.$$

It is easy to see that F is 2-regular with respect to all elements $h = (u,\, v) \in H$ satisfying $u_1 \ne 0$.

Take, e.g., $\bar{h} = (1,\, 0,\, 0,\, 1) \in H$. Then equality in (4.3) is satisfied, and (4.4), (4.5) hold with $\bar{\lambda} = -1$, $\bar{\mu}^1 = 0$, $\bar{\mu}^2 = -1$. Hence, z^* is still a candidate for being optimal. A different analysis based on [6] leads to the same conclusion up to this point, and no further analysis is possible based on that reference.

However, using Theorem 4.2, we see that linear equalities (4.6) hold for elements $h = (u,\, v)$ such that $v_1 = 0$, $u_1 - v_2 = 0$. For such h, it holds that

$$\bar{\mu}^2 B_g[h]^2 = -u_2^2 < 0,$$

for any $u_2 \ne 0$. Hence, condition (4.7) is violated here, and z^* cannot be a local minimizer.

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