Journal of Mathematical Analysis and Applications
(accepted for publication)

MERIT FUNCTIONS AND ERROR BOUNDS FOR GENERALIZED VARIATIONAL INEQUALITIES*

M. V. Solodov†

September 19, 2001

ABSTRACT

We consider the generalized variational inequality and construct certain merit functions associated with this problem. In particular, those merit functions are everywhere nonnegative and their zero-sets are precisely solutions of the variational inequality. We further use those functions to obtain error bounds, i.e., upper estimates for the distance to solutions of the problem.

Key words. Variational inequality, merit function, error bound.
AMS subject classifications. 90C30, 90C33.

---

* Research of the author is partially supported by CNPq Grant 300734/95-6, by PRONEX–Optimization, and by FAPERJ.
† Instituto de Matemática Pura e Aplicada, Estrada Dona Castorina 110, Jardim Botânico, Rio de Janeiro, RJ 22460-320, Brazil.
Email : `solodov@impa.br` .

1 Introduction

Let $\mathcal{H}$ be a real Hilbert space. Given two mappings $F, g : \mathcal{H} \to \mathcal{H}$ and a function $f : \mathcal{H} \to (-\infty, +\infty]$, we consider the generalized variational inequality problem, GVIP(F, g, f) for short, which is to find a point $x \in \mathcal{H}$ such that

$$\langle F(x), y - g(x) \rangle + f(y) - f(g(x)) \geq 0 \quad \text{for all } y \in \mathcal{H}\,, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathcal{H}$. We assume that f is proper convex, with its effective domain $\mathrm{dom}(f) = \{x \in \mathcal{H} \mid f(x) < +\infty\}$ being closed. When f is the indicator function of a closed convex set $C \subset \mathcal{H}$ (i.e., $f(x) = 0$ if $x \in C$, $f(x) = +\infty$ otherwise) then GVIP(F, g, f) reduces to the problem considered, for example, in [22]. If $g(x) = x \,\forall\, x \in \mathcal{H}$ then GVIP(F, g, f) is the same as studied, for example, in [23]. Furthermore, it is not difficult to see that if both $g(x) = x$ and f is the indicator function of a closed convex set C, then GVIP(F, g, f) reduces to the classical variational inequality VIP(F, C) [2, 5]: find x such that

$$x \in C\,, \quad \langle F(x), y - x \rangle \geq 0 \quad \text{for all } y \in C\,.$$

We note that there exist also some other formulations of generalized variational inequalities, for example [10, 11, 20]. These formulations will not be considered in the present paper.

Many fruitful approaches to both theoretical and numerical treatment of VIP(F, C) make use of constructing a function $M : \mathcal{H} \to (-\infty, +\infty]$ such that

$$M(x) \geq 0 \;\; \forall\, x \in D \supset C \quad \text{and} \quad M(x) = 0\,,\; x \in D \;\Leftrightarrow\; x \text{ solves VIP}(F, C).$$

The set D is usually either the whole space or the set C itself, and $M(\cdot)$ with such properties is commonly referred to as a *merit function*. We refer the reader to [4, 12, 16] for surveys, further references, and some specific examples of merit functions for classical variational inequalities VIP(F, C). One of the many useful applications of merit functions is in deriving the so-called *error bounds*, i.e., upper estimates on the distance to the solution set S of VIP(F, C):

$$\mathrm{dist}(x, S) \leq \gamma M(x)^{\lambda} \quad \forall\, x \in D' \subset D\,,$$

where γ, $\lambda > 0$ are independent of x (but possibly dependent on D'). The bound is global if $D' = D = \mathcal{H}$, and it is local if the set D' is some neighbourhood of the solution set S. We refer the reader to [12] for a recent survey.

To our knowledge, merit functions have not been studied beyond the classical formulation VIP(F, C), except for norms of certain equation reformulations (more on this in Section 2). In this paper, we exhibit three merit functions for GVIP(F, g, f), and show how they can be used to obtain error bounds.

Our notation is standard. By $\arg\min_{z\in Z} t(z)$ we mean the set of minimizers of the function $t : \mathcal{H} \to (-\infty, +\infty]$ over the set $Z \subset \mathcal{H}$. The subdifferential of a convex function f at $x \in \mathcal{H}$ is denoted by $\partial f(x) = \{u \in \mathcal{H} \mid f(y) \geq f(x) + \langle u, y - x\rangle \ \forall y \in \mathcal{H}\}$. For $t : \mathcal{H} \to \mathcal{H}$ and $Z \subset H$, we denote $t^{-1}(Z) := \{x \in \mathcal{H} \mid t(x) \in Z\}$.

2 Natural residual

For VIP(F, C), it is well-known that $x \in \mathcal{H}$ is a solution if, and only if,

$$0 = x - P_C\left(x - \alpha F(x)\right) ,$$

where $P_C(\cdot)$ is the orthogonal projector onto C and $\alpha > 0$ is arbitrary. Hence, the norm of the right-hand side in this equation can serve as a merit function for VIP(F, C), which is commonly called *natural residual.*

We next derive a similar characterization for GVIP(F, g, f). We note that analogous results are quite well-known also for other extensions of VIP(F, C), e.g., [11, 21]. So this characterization is, in a sense, as expected. However, it provides some building blocks for deriving certain gap functions in the subsequent sections. And we note that our error bound for GVIP(F, g, f) based on the natural residual is new.

Recall that the *proximal map* [9, 15], $p_f^\alpha : \mathcal{H} \to \mathrm{dom}(f)$, is given by

$$p_f^\alpha(z) := \arg\min_{y\in\mathcal{H}} \left\{ f(y) + \frac{1}{2\alpha}\|y - z\|^2 \right\} , \quad z \in \mathcal{H}\,, \alpha > 0\,.$$

Note that the objective function above is proper strongly convex. Since $\mathrm{dom}(f)$ is closed, $p_f^\alpha(\cdot)$ is well-defined and single-valued. Define

$$R_\alpha(x) := g(x) - p_f^\alpha(g(x) - \alpha F(x))\,, \quad x \in \mathcal{H}\,, \alpha > 0\,.$$

We next show that $R_\alpha(\cdot)$ plays the role of natural residual for GVIP(F, g, f).

Theorem 1 *Let $\alpha > 0$ be arbitrary. An element $x \in \mathcal{H}$ solves GVIP(F, g, f) if, and only if, $R_\alpha(x) = 0$.*

Proof. We have that $R_\alpha(x) = 0$ is equivalent to

$$g(x) = \arg\min_{y\in\mathcal{H}} \left\{ f(y) + \frac{1}{2\alpha}\|y - (g(x) - \alpha F(x))\|^2 \right\} .$$

By optimality conditions (which are necessary and sufficient, by convexity), the latter is equivalent to

$$0 \in \partial f(g(x)) + \frac{1}{\alpha}(g(x) - (g(x) - \alpha F(x))) = \partial f(g(x)) + F(x)$$

or

$$-F(x) \in \partial f(g(x)),$$

which in turn is equivalent, by the definition of the subgradient, to

$$f(y) \geq f(g(x)) - \langle F(x), y - g(x) \rangle \quad \forall\, y \in \mathcal{H},$$

which means that x solves GVIP(F, g, f). ∎

We next exhibit an error bound in terms of $R_\alpha(\cdot)$. Let $\bar{x}$ be a solution of GVIP(F, g, f). We say that F is g-strongly monotone with respect to $\bar{x}$ with modulus $\mu > 0$ if

$$\langle F(\bar{x}) - F(y), g(\bar{x}) - g(y) \rangle \geq \mu \|\bar{x} - y\|^2 \quad \forall\, y \in \mathcal{H}. \tag{2}$$

First note that in this case the solution is in fact unique. Indeed, if $\hat{x}$ is another solution of GVIP(F, g, f) then taking $x = \bar{x}$, $y = g(\hat{x})$ and $x = \hat{x}$, $y = g(\bar{x})$ in (1), we have that

$$\begin{aligned}
\langle F(\bar{x}), g(\hat{x}) - g(\bar{x}) \rangle + f(g(\hat{x})) - f(g(\bar{x})) \geq 0, \\
\langle F(\hat{x}), g(\bar{x}) - g(\hat{x}) \rangle + f(g(\bar{x})) - f(g(\hat{x})) \geq 0.
\end{aligned}$$

Adding the two inequalities and using (2), we obtain

$$0 \leq \langle F(\bar{x}) - F(\hat{x}), g(\hat{x}) - g(\bar{x}) \rangle \leq -\mu \|\bar{x} - \hat{x}\|^2,$$

which implies that necessarily $\bar{x} = \hat{x}$.

Finally, we say that F is Lipschitz-continuous with respect to $\bar{x}$ with modulus $L > 0$ if

$$\|F(\bar{x}) - F(x)\| \leq L \|\bar{x} - x\| \quad \forall\, x \in \mathcal{H}.$$

An analogous definition will be used for g.

Theorem 2 *Suppose that F is g-strongly monotone with modulus $\mu > 0$ with respect to solution $\bar{x}$ of GVIP(F, g, f), and that F, g are Lipschitz-continuous with modulus $L > 0$ with respect to $\bar{x}$. Then*

$$\|x - \bar{x}\| \leq \frac{L}{\mu} \left(1 + \frac{1}{\alpha}\right) \|R_\alpha(x)\| \quad \forall\, x \in \mathcal{H},\ \alpha > 0.$$

Proof. Fix any $x \in \mathcal{H}$ and $\alpha > 0$. By the definition of p_f^α, we have that $p_f^\alpha(g(x) - \alpha F(x))$ satisfies

$$0 \in \partial f(p_f^\alpha(g(x) - \alpha F(x))) + \frac{1}{\alpha}(p_f^\alpha(g(x) - \alpha F(x)) - (g(x) - \alpha F(x))).$$

Hence,

$$-F(x) + \frac{1}{\alpha}(g(x) - p_f^\alpha(g(x) - \alpha F(x))) \in \partial f(p_f^\alpha(g(x) - \alpha F(x)))\,.$$

It follows that for all $y \in \mathcal{H}$,

$$f(y) - f(p_f^\alpha(g(x) - \alpha F(x))) + \langle F(x) - \frac{1}{\alpha}(g(x) - p_f^\alpha(g(x) - \alpha F(x))), y - p_f^\alpha(g(x) - \alpha F(x))\rangle \geq 0\,. \tag{3}$$

Taking $y = g(\bar{x})$ in the inequality above, $y = p_f^\alpha(g(x) - \alpha F(x))$ in (1), and adding the two inequalities, we obtain that

$$\langle F(\bar{x}) - F(x) + \frac{1}{\alpha}(g(x) - p_f^\alpha(g(x) - \alpha F(x))), p_f^\alpha(g(x) - \alpha F(x)) - g(\bar{x})\rangle \geq 0\,. \tag{4}$$

By (2), we have that

$$\begin{aligned}
\mu\|x - \bar{x}\|^2 &\leq \langle F(\bar{x}) - F(x), g(\bar{x}) - g(x)\rangle \\
&= \langle F(\bar{x}) - F(x), g(\bar{x}) - p_f^\alpha(g(x) - \alpha F(x))\rangle \\
&\quad + \langle F(\bar{x}) - F(x), p_f^\alpha(g(x) - \alpha F(x)) - g(x)\rangle \\
&\leq \frac{1}{\alpha}\langle g(x) - p_f^\alpha(g(x) - \alpha F(x)), p_f^\alpha(g(x) - \alpha F(x)) - g(\bar{x})\rangle \\
&\quad + \langle F(\bar{x}) - F(x), p_f^\alpha(g(x) - \alpha F(x)) - g(x)\rangle \\
&\leq -\frac{1}{\alpha}\|R_\alpha(x)\|^2 + \frac{1}{\alpha}\|R_\alpha(x)\|\|g(\bar{x}) - g(x)\| \\
&\quad + \|F(\bar{x}) - F(x)\|\|R_\alpha(x)\| \\
&\leq L(1 + \frac{1}{\alpha})\|x - \bar{x}\|\|R_\alpha(x)\|\,,
\end{aligned}$$

where the second inequality is by (4), the third is by the Cauchy-Schwarz inequality, and the last is by the Lipschitz-continuity assumptions. The assertion of the theorem now follows. ∎

3 Regularized-gap function

The following *regularized-gap function* was introduced (in the finite-dimensional setting) for VIP(F, C) in [1, 3]:

$$\langle F(x), x - P_C(x - \alpha F(x))\rangle - \frac{1}{2\alpha}\|x - P_C(x - \alpha F(x))\|^2\,, \quad x \in \mathcal{H}\,,\ \alpha > 0\,.$$

In the special case when C is the nonnegative orthant, it is equivalent to the restricted *implicit Lagrangian* [7, 16]. This function has a number of interesting properties. For example, it is nonnegative on the set C and its set of zeroes in C coincides with solutions of VIP(F,C). Furthermore, it has better smoothness properties when compared to the natural residual.

We next construct the regularized-gap function for GVIP(F,g,f). To this end, define $G_\alpha : \mathcal{H} \to [0,+\infty]$ by

$$G_\alpha(x) := \max_{y\in\mathcal{H}} \left\{ \langle F(x), g(x)-y\rangle + f(g(x)) - f(y) - \frac{1}{2\alpha}\|g(x)-y\|^2 \right\}, \quad x\in\mathcal{H}, \ \alpha>0. \tag{5}$$

We next derive an explicit formula for G_α, and in particular we show that this function is nonnegative everywhere on $\mathcal{H}$.

Lemma 3 *It holds that*

$$\begin{aligned} G_\alpha(x) &= \langle F(x), g(x) - p_f^\alpha(g(x)-\alpha F(x))\rangle + f(g(x)) - f(p_f^\alpha(g(x)-\alpha F(x))) \\ &\quad -\frac{1}{2\alpha}\|g(x) - p_f^\alpha(g(x)-\alpha F(x))\|^2, \quad x\in\mathcal{H}. \end{aligned} \tag{6}$$

In particular, $G_\alpha(x)\geq 0$ for all $x\in\mathcal{H}$.

Proof. Denote by $t(y)$ the function being maximized in (5), with $x\in\mathcal{H}$ fixed. If $x\in g^{-1}(\mathrm{dom}(f))$ then $t(g(x))=0$, and hence $0=t(g(x))\leq \max_{y\in\mathcal{H}} t(y) = G_\alpha(x)$. If $x\notin g^{-1}(\mathrm{dom}(f))$ then for any $y\in\mathrm{dom}(f)$ it holds that $t(y)=+\infty$, and hence $+\infty = \max_{y\in\mathcal{H}} t(y) = G_\alpha(x)$.

First note that if $x\notin g^{-1}(\mathrm{dom}(f))$ then formula (6) is correct, because $f(g(x)) = +\infty$ while the other terms are all finite (recall that $p_f^\alpha(z)\in\mathrm{dom}(f)$ for any $z\in\mathcal{H}$).

Consider now any $x\in g^{-1}(\mathrm{dom}(f))$. Let z be the (unique, by strong concavity of $t(y)$) element at which the maximum is realized in (5). Then z is uniquely characterized by the optimality condition

$$0\in F(x) + \partial f(z) + \frac{1}{\alpha}(z-g(x)).$$

But this inclusion also uniquely characterizes the solution of the problem

$$\min_{z\in\mathcal{H}} \left\{ f(z) + \frac{1}{2\alpha}\|z-(g(x)-\alpha F(x))\|^2 \right\},$$

which by definition is $p_f^\alpha(g(x)-\alpha F(x))$. We conclude that $z = p_f^\alpha(g(x)-\alpha F(x))$, from which (6) follows by evaluating the maximum in (5). ∎

The next result shows that the set of zeroes of G_α is precisely the solution set of GVIP(F,g,f). Furthermore, it establishes the order of growth of G_α when compared to the natural residual R_α.

Theorem 4 *It holds that*

$$G_\alpha(x) \geq \frac{1}{2\alpha}\|R_\alpha(x)\|^2 \quad \forall\, x \in \mathcal{H}\,,\ \alpha > 0\,.$$

In particular, $G_\alpha(x) = 0$ *if, and only if,* x *solves GVIP*(F, g, f).

Proof. Fix any $x \in \mathcal{H}$ and $\alpha > 0$. Taking $y = g(x)$ in (3), we have that

$$f(g(x)) - f(p_f^\alpha(g(x) - \alpha F(x))) + \langle F(x), R_\alpha(x)\rangle \geq \frac{1}{\alpha}\|R_\alpha(x)\|^2\,.$$

Using this relation together with (6), we obtain

$$\begin{aligned} G_\alpha(x) &= \langle F(x), R_\alpha(x)\rangle + f(g(x)) - f(p_f^\alpha(g(x) - \alpha F(x))) - \frac{1}{2\alpha}\|R_\alpha(x)\|^2 \\ &\geq \frac{1}{\alpha}\|R_\alpha(x)\|^2 - \frac{1}{2\alpha}\|R_\alpha(x)\|^2 \\ &= \frac{1}{2\alpha}\|R_\alpha(x)\|^2\,. \end{aligned}$$

We now prove the last assertion of the theorem. If $G_\alpha(x) = 0$ then $R_\alpha(x) = 0$, by the first assertion. Hence, by Theorem 1, x solves GVIP(F, g, f). Conversely, suppose x is a solution. Again by Theorem 1, $g(x) = p_f^\alpha(g(x) - \alpha F(x))$. In particular, it holds that $g(x) \in \mathrm{dom}(f)$. Now (6) implies that $G_\alpha(x) = 0$. ∎

As was already mentioned, one advantage of the regularized-gap function over the natural residual is the more atractive differentiability properties in certain situations. For example, for VIP(F, C) it holds that $G_\alpha(\cdot)$ is differentiable whenever $F(\cdot)$ is differentiable [3], while the same is not true for $R_\alpha(\cdot)$. Another advantage is exhibited by the following error bound result, where, unlike in Theorem 2, we do not need Lipschitz-continuity of F.

Theorem 5 *Suppose that* F *is* g*-strongly monotone with modulus* $\mu > 0$ *with respect to solution* $\bar{x}$ *of GVIP*(F, g, f)*, and that* g *is Lipschitz-continuous with modulus* $L > 0$ *with respect to* $\bar{x}$*. Then*

$$\|x - \bar{x}\| \leq \sqrt{\frac{2\alpha}{2\alpha\mu - L^2}}\sqrt{G_\alpha(x)} \quad \forall\, x \in \mathcal{H}\,,\ \alpha > \frac{L^2}{2\mu}\,.$$

Proof. Again, let $t(y)$ denote the function being maximized in (5), with $x \in \mathcal{H}$ fixed.

It holds that

$$\begin{aligned} G_\alpha(x) &\geq t(g(\bar{x})) \\ &= \langle F(x), g(x) - g(\bar{x})\rangle + f(g(x)) - f(g(\bar{x})) - \frac{1}{2\alpha}\|g(x) - g(\bar{x})\|^2 \\ &\geq \langle F(\bar{x}), g(x) - g(\bar{x})\rangle + \mu\|x - \bar{x}\|^2 \\ &\quad + f(g(x)) - f(g(\bar{x})) - \frac{1}{2\alpha}\|g(x) - g(\bar{x})\|^2 \,, \end{aligned} \tag{7}$$

where the second inequality follows from (2). Taking $y = g(x)$ in (1), we have

$$\langle F(\bar{x}), g(x) - g(\bar{x})\rangle + f(g(x)) - f(g(\bar{x})) \geq 0 \,.$$

Combining the latter relation with (7) and using the Lipschitz-continuity assumption, we obtain

$$\begin{aligned} G_\alpha(x) &\geq \mu\|x - \bar{x}\|^2 - \frac{1}{2\alpha}\|g(x) - g(\bar{x})\|^2 \\ &\geq \left(\mu - \frac{L^2}{2\alpha}\right)\|x - \bar{x}\|^2 \,, \end{aligned}$$

which concludes the proof. ∎

4 D-gap function

The *D-gap function* can be thought of as a difference of two regularized-gap functions with distinct parameters. It was introduced, via a rather different construction, in [7] for VIP(F, C) with C being the nonnegative orthant (i.e., for the nonlinear complementarity problem, where it is known as the *implicit Lagrangian* [16]). It was subsequently extended for a general closed convex set C (in finite-dimensional space) [13, 19]. For properties and applications, see also [17, 8, 18, 14, 6].

In the setting of GVIP(F, g, f), the expression

$$G_\alpha(x) - G_\beta(x) \,, \quad x \in \mathcal{H} \,,\ \alpha > \beta > 0 \,,$$

will not be well-defined for $x \notin g^{-1}(\mathrm{dom}(f))$, as both quantities are not finite. Nevertheless, we shall define the D-gap function by taking a formal difference of the expressions (6) for two parameters α and β. As we shall show, this would result in an appropriate merit function for GVIP(F, g, f). Furthermore, there is an advantage that, unlike the regularized-gap function, the resulting function will be not only nonnegative but also everywhere finite. To this end, taking a formal difference of expressions given by (6) with parameters α and β, we define $H_{\alpha,\beta} : \mathcal{H} \to [0, +\infty)$ by

$$\begin{aligned} H_{\alpha,\beta}(x) &= \langle F(x), p_f^\beta(g(x) - \beta F(x)) - p_f^\alpha(g(x) - \alpha F(x))\rangle \\ &\quad + f(p_f^\beta(g(x) - \beta F(x))) - f(p_f^\alpha(g(x) - \alpha F(x))) \\ &\quad + \frac{1}{2\beta}\|g(x) - p_f^\beta(g(x) - \beta F(x))\|^2 - \frac{1}{2\alpha}\|g(x) - p_f^\alpha(g(x) - \alpha F(x))\|^2 \\ &= \langle F(x), R_\alpha(x) - R_\beta(x)\rangle + f(p_f^\beta(g(x) - \beta F(x))) - f(p_f^\alpha(g(x) - \alpha F(x))) \\ &\quad + \frac{1}{2\beta}\|R_\beta(x)\|^2 - \frac{1}{2\alpha}\|R_\alpha(x)\|^2, \quad x \in \mathcal{H},\ \alpha > \beta > 0. \end{aligned} \tag{8}$$

Note that since $p_f^\alpha(z) \in \mathrm{dom}(f)$ for all $z \in \mathcal{H}$ and all $\alpha > 0$, it follows that $H_{\alpha,\beta}$ given by (8) is indeed everywhere finite. We next show that the D-gap function is of the order of the natural residual. Hence, it is nonnegative and its zeroes are solutions of GVIP(F, g, f).

Theorem 6 *It holds that*

$$\frac{1}{2}\left(\frac{1}{\beta} - \frac{1}{\alpha}\right)\|R_\alpha(x)\|^2 \ge H_{\alpha,\beta}(x) \ge \frac{1}{2}\left(\frac{1}{\beta} - \frac{1}{\alpha}\right)\|R_\beta(x)\|^2 \quad \forall\, x \in \mathcal{H},\ \alpha > \beta > 0.$$

In particular, $x \in \mathcal{H}$ solves GVIP(F, g, f) if, and only if, $H_{\alpha,\beta}(x) = 0$ with $\alpha > \beta > 0$.

Proof. Fix any $x \in \mathcal{H}$ and $\alpha > \beta > 0$. Using (3) with $y = p_f^\beta(g(x) - \beta F(x))$, we have that

$$\begin{aligned} \frac{1}{\alpha}\langle R_\alpha(x), R_\alpha(x) - R_\beta(x)\rangle &\le f(p_f^\beta(g(x) - \beta F(x))) - f(p_f^\alpha(g(x) - \alpha F(x))) \\ &\quad + \langle F(x), R_\alpha(x) - R_\beta(x)\rangle. \end{aligned}$$

Combining this relation with (8), we obtain

$$\begin{aligned} H_{\alpha,\beta}(x) &\ge \frac{1}{\alpha}\langle R_\alpha(x), R_\alpha(x) - R_\beta(x)\rangle + \frac{1}{2\beta}\|R_\beta(x)\|^2 - \frac{1}{2\alpha}\|R_\alpha(x)\|^2 \\ &= \frac{1}{2}\left(\frac{1}{\beta} - \frac{1}{\alpha}\right)\|R_\beta(x)\|^2 + \frac{1}{\alpha}\langle R_\alpha(x), R_\alpha(x) - R_\beta(x)\rangle \\ &\quad - \frac{1}{2\alpha}\|R_\alpha(x) - R_\beta(x)\|^2 - \frac{1}{\alpha}\langle R_\beta(x), R_\alpha(x) - R_\beta(x)\rangle \\ &= \frac{1}{2}\left(\frac{1}{\beta} - \frac{1}{\alpha}\right)\|R_\beta(x)\|^2 + \frac{1}{2\alpha}\|R_\alpha(x) - R_\beta(x)\|^2, \end{aligned}$$

which implies the right-most inequality in the assertion.

On the other hand,

$$-F(x) + \frac{1}{\beta} R_\beta(x) \in \partial f(p_f^\beta(g(x) - \beta F(x)))$$

implies that

$$\begin{aligned}
\frac{1}{\beta}\langle R_\beta(x), R_\alpha(x) - R_\beta(x)\rangle \quad &\geq \quad f(p_f^\beta(g(x) - \beta F(x))) - f(p_f^\alpha(g(x) - \alpha F(x))) \\
&\quad + \langle F(x), R_\alpha(x) - R_\beta(x)\rangle \, .
\end{aligned}$$

Similarly to the analysis above, we then obtain

$$\begin{aligned}
H_{\alpha,\beta}(x) \quad &\leq \quad \frac{1}{\beta}\langle R_\beta(x), R_\alpha(x) - R_\beta(x)\rangle + \frac{1}{2\beta}\|R_\beta(x)\|^2 - \frac{1}{2\alpha}\|R_\alpha(x)\|^2 \\
&= \quad \frac{1}{2}\left(\frac{1}{\beta} - \frac{1}{\alpha}\right)\|R_\alpha(x)\|^2 - \frac{1}{2\beta}\|R_\alpha(x) - R_\beta(x)\|^2 \, ,
\end{aligned}$$

which implies the left-most inequality in the assertion.

The last assertion now follows from Theorem 1. ■

As a consequence of Theorems 6 and 2, we obtain the following error bound. Note that compared to Theorem 5 for the regularized-gap function, in Theorem 7 regularization parameters α and β do not depend on the problem data.

Theorem 7 *Suppose that F is g-strongly monotone with modulus $\mu > 0$ with respect to solution $\bar{x}$ of GVIP(F, g, f), and that F, g are Lipschitz-continuous with modulus $L > 0$ with respect to $\bar{x}$. Then*

$$\|x - \bar{x}\| \leq \frac{L}{\mu}\left(1 + \frac{1}{\beta}\right)\sqrt{\frac{2\alpha\beta}{\alpha - \beta}}\sqrt{H_{\alpha,\beta}(x)} \quad \forall\, x \in \mathcal{H}\, ,\ \alpha > \beta > 0\, .$$

Finally, we note the following advantage of the D-gap function compared to the natural residual: as a difference of two regularized-gap functions, it inherits smoothness properties of the latter mentioned in Section 3.

References

[1] G. Auchmuty. Variational principles for variational inequalities. *Numerical Functional Analysis and Optimization*, 10:863–874, 1989.

[2] R.W. Cottle, F. Giannessi, and J.-L. Lions. *Variational Inequalities and Complementarity Problems: Theory and Applications.* Wiley, New York, 1980.

[3] M. Fukushima. Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems. *Mathematical Programming*, 53:99–110, 1992.

[4] M. Fukushima. Merit functions for variational inequality and complementarity problems. In G. Di Pillo and F. Giannessi, editors, *Nonlinear Optimization and Applications*, pages 155–170. Plenum Publishing Corporation, New York, New York, 1996.

[5] P.T. Harker and J.-S. Pang. Finite-dimensional variational inequality problems: A survey of theory, algorithms and applications. *Mathematical Programming*, 48:161–220, 1990.

[6] C. Kanzow and M. Fukushima. Solving box constrained variational inequality problems by using the natural residual with D-gap function globalization. *Operations Research Letters*, 23:45–51, 1998.

[7] O. L. Mangasarian and M. V. Solodov. Nonlinear complementarity as unconstrained and constrained minimization. *Mathematical Programming*, 62:277–297, 1993.

[8] O. L. Mangasarian and M. V. Solodov. A linearly convergent derivative-free descent method for strongly monotone complementarity problems. *Computational Optimization and Applications*, 14:5–16, 1999.

[9] J.-J. Moreau. Proximité et dualité dans un espace Hilbertien. *Bulletin de la Société Mathématique de France*, 93:273–299, 1965.

[10] M.A. Noor. General nonlinear variational inequalities. *Journal of Mathematical Analysis and Applications*, 126:78–84, 1987.

[11] M.A. Noor. An iterative algorithm for variational inequalities. *Journal of Mathematical Analysis and Applications*, 158:448–455, 1991.

[12] J.-S. Pang. Error bounds in mathematical programming. *Mathematical Programming*, 79:299–332, 1997.

[13] J.-M. Peng. Equivalence of variational inequality problems to unconstrained optimization. *Mathematical Programming*, 78:347–356, 1997.

[14] J.-M. Peng and M. Fukushima. A hybrid Newton method for solving the variational inequality problem via the D-gap function. *Mathematical Programming*, 86:367–386, 1999.

[15] R.T. Rockafellar. Monotone operators and the proximal point algorithm. *SIAM Journal on Control and Optimization*, 14:877–898, 1976.

[16] M. V. Solodov. Implicit Lagrangian. In C. Floudas and P. Pardalos, editors, *Encyclopedia of Optimization*. Kluwer Academic Publishers, 2001.

[17] M. V. Solodov and P. Tseng. Some methods based on the the D-gap function for solving monotone variational inequalities. *Computational Optimization and Applications*, 17:255–277, 2000.

[18] N. Yamashita and M. Fukushima. On stationary points of the implicit Lagrangian for nonlinear complementarity problems. *Journal of Optimization Theory and Applications*, 84:653–663, 1995.

[19] N. Yamashita, K. Taji, and M. Fukushima. Unconstrained optimization reformulations of variational inequality problems. *Journal of Optimization Theory and Applications*, 92:439–456, 1997.

[20] L.C. Zeng. Iterative algorithms for finding approximate solutions to completely generalized strongly nonlinear quasi-variational inequalities. *Journal of Mathematical Analysis and Applications*, 201:180–194, 1996.

[21] L.C. Zeng. On a general projection algorithm for variational inequalities. *Journal of Optimization Theory and Applications*, 97:229–235, 1998.

[22] Y.-B. Zhao. The iterative methods for monotone generalized variational inequalities. *Optimization*, 42:285–307, 1997.

[23] D.L. Zhu and P. Marcotte. Co-coercivity and its role in the convergence of iterative schemes for solving variational inequalities. *SIAM Journal on Optimization*, 6:714–726, 1996.