ON THE INVERSE PROBLEM FOR SCATTERING OF ELECTROMAGNETIC RADIATION BY A PERIODIC STRUCTURE

L. O. CASTELLANO PÉREZ AND JORGE P. ZUBELLI

ABSTRACT. We consider a smooth perturbation $\delta\epsilon(x, y, z)$ of a constant background permittivity $\epsilon = \epsilon_0$ that varies periodically with x, does not depend on y, and is supported on a finite-length interval in z. We investigate the theoretical and numerical determination of such perturbation from (several) fixed frequency y-invariant electromagnetic waves.

By varying the direction and frequency of the probing radiation a scattering matrix is defined. By using an invariant-imbedding technique we derive an operator Riccati equation for such scattering matrix. We obtain a theoretical uniqueness result for the problem of determining the perturbation from the scattering matrix.

We also investigate a numerical method for performing such reconstruction using multi-frequency information of the truncated scattering matrix. This relies on ideas of regularization and recursive linearization. Numerical experiments are presented validating such approach.

Date: October 8, 2002.
Key words and phrases. Inverse Scattering, Periodic Structures, Diffraction.
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Acknowledgments: J.P.Z. was supported by FAPERJ through grant number E-26/172.009, by CNPq through grant number 30.1003 and 200258, J.P.Z. also acknowledges the participation at the MGSS which was supported through NSF grant DMS97-09320. This work was concluded at MSRI which is supported in part by NSF grant DMS-9810361.

1. Introduction

This work concerns the scattering of electromagnetic waves by a periodic structure along one of the coordinate axis that is also invariant along an orthogonal direction. Such waves will be used to probe into the medium in such a way that the scattered fields will carry information about the internal periodic structure. Our goal being that of imaging this internal structure. This is motivated by technological applications in thin films and similar technologies [1].

Our main theoretical results are the uniqueness for the corresponding inverse scattering problem with data given as the scattering matrix for several frequencies, and a Riccati-type differential equation for the scattering matrix. We also present a practical algorithm for the solution of a regularized version of the inverse problem.

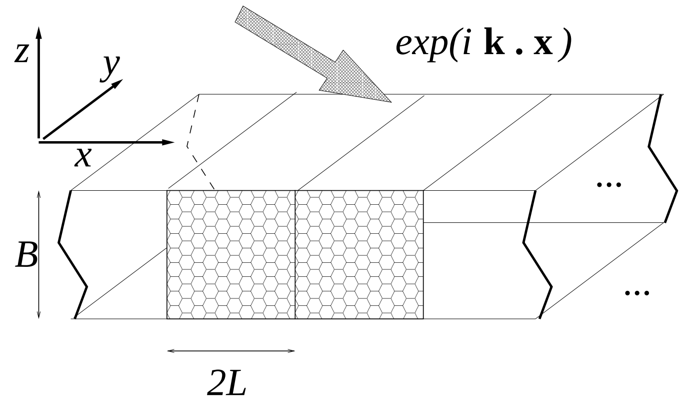

Figure 1. The underlying geometry

In a fairly general context, electromagnetic waves are characterized by two vector fields, the electric field $\mathcal{E}$ and the magnetic intensity $\mathcal{H}$. Such fields depend on the spatial coordinate $\mathbf{x} = (x, y, z)$ and the time t, and satisfy Maxwell's equations

$$\nabla \times\ \mathcal{E} + \mu_0 \frac{\partial \mathcal{H}}{\partial t} = 0\ , \tag{1}$$

$$\nabla \times\ \mathcal{H} - \epsilon(\mathbf{x}) \frac{\partial \mathcal{E}}{\partial t} = J\ , \tag{2}$$

where $\epsilon = \epsilon(\mathbf{x})$ and μ_0 are the electric permittivity and magnetic permeability, respectively. Here, $J = J(\mathbf{x})$ is the current density vector, which satisfies Ohm's law

$$J = \sigma \mathcal{E}\ , \tag{3}$$

where $\sigma = \sigma(\mathbf{x})$ is the conductivity. In this work we focus at (stationary) time-harmonic fields, which correspond to

$$\mathcal{E}(\mathbf{x}, t) = \Re\{E(\mathbf{x}) e^{-i\omega t}\}\,,$$

$$\mathcal{H}(\mathbf{x}, t) = \Re\{H(\mathbf{x}) e^{-i\omega t}\}\,,$$

where ω is the frequency and $\Re$ denotes the real part. Since our structure is invariant along the y direction, we look for solutions that are independent of such direction. Under such assumptions, Maxwell equations uncouple into two sets of equations. One for H_1, E_2 and H_3, the other for E_1, H_2 and E_3. This uncoupling determines two independent polarizations: transversal electric (TE) and transversal magnetic (TM), respectively.

In this work we shall concentrate on the the TE case. In this case, the equation becomes the well known Helmholtz equation:

$$\left(\Delta + \omega^2 \mu_0 \epsilon + i\omega\mu_0\sigma\right) E_2 = 0\,,$$

where now Δ denotes the Laplacian in the variables x and z. Let us denote by ϵ_0 the free-space permittivity and introduce the wave number $k = \omega\sqrt{\mu_0\epsilon_0}$. Thus, the equation takes the form

$$\left(\Delta + k^2 n(\mathbf{x})\right) u = 0\ , \tag{4}$$

where $u = E_2$ and

$$n(\mathbf{x}) \stackrel{\text{def}}{=} \frac{1}{\epsilon_0}\left(\epsilon(\mathbf{x}) + i\frac{\sigma(\mathbf{x})}{\omega}\right)\,.$$

We shall write $n(\mathbf{x}) = n_0(1 + q(\mathbf{x}))$ where n_0 is a constant and known background. After an elementary rescaling of the variables we assume that $n_0 = 1$.

The plan of the present work is the following:

In the rest of the present introduction we discuss some preliminary results concerning the direct scattering problem for Equation (4) in the periodic context.

Section 2 defines the scattering matrix. It will be the input to our inverse problem of finding the scatterer q from the the scattering matrix.

Section 4 states and proves a uniqueness result for the inverse problem assuming we know the scattering matrix for an interval of frequencies. This result, although very reassuring, relies on analytical continuation and information of the full scattering matrix. It calls for a practical algorithm.

If we truncate the scatterer $q(\mathbf{x})$ by multiplying it by the characteristic function of the interval $[0, z]$ where $\mathbf{x}_3 = z$, the scattering matrix becomes a function of z. Section 3 presents a matrix Riccati equation that is satisfied by the scattering matrix as a function of such variable z. This result is of theoretical, as well as of practical interest. It will be the starting point for the development of the numerical algorithm we propose in Section 5. In such section we also present numerical experiments validating this approach.

We also include two appendices where we collect a number of technical results that were used throughout the paper concerning the periodic scattering problem for the Helmholtz equation.

1.1. **The Direct Scattering Problem.** Let us assume, as alluded above, that the function determining the medium is given by $q(x, z)$ where q is $2L$-periodic with respect to x. Because of the invariance with respect to y our geometry becomes bi-dimensional. We shall also assume that our perturbation $q(\mathbf{x})$ has compact support in the z variable. Most of the action will take place on the following sets, which are depicted in Figure 2:

$$\begin{array}{lcll} \Omega & = & \{(x,z) \in \mathbb{R}^2 & \mid \quad 0 < z < B\}, \\ \Omega^+ & = & \{(x,z) \in \mathbb{R}^2 & \mid \quad z > B\}, \\ \Omega^- & = & \{(x,z) \in \mathbb{R}^2 & \mid \quad z < 0\}, \\ \Gamma_a & = & \{(x,z) \in \mathbb{R}^2 & \mid \quad z = a\,, a \in \mathbb{R}.\} \end{array} \tag{5}$$

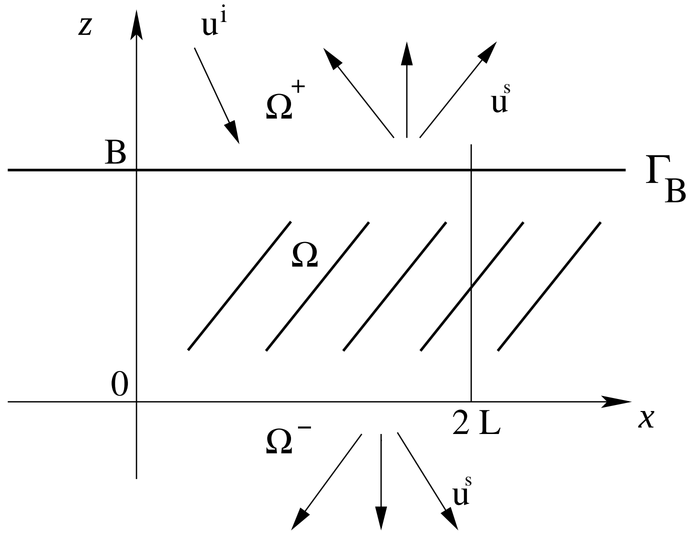

FIGURE 2. The scattering geometry and the fundamental domain.

By the *direct scattering problem* we mean the following scenario: An incident time-harmonic plane wave propagating from the region Ω^+ is scattered by q whose support is Ω. See Figure 2. The problem consists in determining the scattered field u^s.

More precisely, we assume that the incident wave has the form

$$u^i(x,z) = \exp(ik(xd_1 + zd_3)) \ , \tag{6}$$

where $(d_1, d_3) = (\sin(\theta), -\cos(\theta))$ is the propagation direction, and $\theta \in (-\pi/2, \pi/2)$ is the angle of incidence. In the absence of a scatterer ($q = 0$) the total field u would be the same as u^i. However, if $q \neq 0$ we have a scattered field which would be defined by

$$u^s(\mathbf{x}) \stackrel{\text{def}}{=} u(\mathbf{x}) - u^i(\mathbf{x})\,, \tag{7}$$

where u is such that

$$u \in \mathcal{V} \stackrel{\text{def}}{=} \left\{ u \in C^1\left(\mathbb{R}^2\right) \quad \mid \quad u \text{ is } C^2 \text{ in } \Omega,\, \Omega^+, \text{ and } \Omega^- \right\}$$

satisfies

$$\Delta u + k^2(1+q)u = 0 \text{ in } \Omega\,, \tag{8}$$

$$\Delta u + k^2 u = 0 \text{ in } \Omega^\pm\,, \tag{9}$$

and both u and $\partial u / \partial z$ are continuous across the boundaries Γ_B and Γ_0.

Since the scattering perturbation q (in general) does not vanish as $x \to \infty$, we cannot request the scattered field to satisfy the classical Sommerfeld radiation condition. We have to impose substitutive conditions. We shall now recall such conditions, which date back to Rayleigh's studies of diffraction by periodic structures [2, 3].

We start off by remarking that the incident field u^i satisfies:

$$u^i(x+2Lm, z) = u^i(x,z)e^{i\,2Lm\,k\sin\theta}, \quad \forall z \in \mathbb{R}, \quad \forall m \in \mathbb{Z}\,.$$

Because of the periodicity of q w.r.t. x, it follows that if u is a solution of (8) and (9), then

$$u_L(x,z) \stackrel{\text{def}}{=} u(x+2Lm, z)e^{-i\,2Lm\,k\sin\theta}, \quad m \in \mathbb{Z}\,,$$

also satisfies the same equations. Furthermore, the scattered field u^s_L has the same asymptotic behavior of u^s as $|z| \to \infty$. Therefore, if we set $\alpha \stackrel{\text{def}}{=} k\sin\theta$ it is natural to request u to satisfy the α *quasi-periodicity* condition

$$u(x+2Lm, z) = u(x,z)e^{i\,2Lm\alpha} \quad \forall m \in \mathbb{Z}\,, \quad \forall z \in \mathbb{R}. \tag{10}$$

Let us take $p \stackrel{\text{def}}{=} \pi/L$ and $\alpha_m \stackrel{\text{def}}{=} \alpha + pm,\ m \in \mathbb{Z}$. Denote by

$$\omega_m(k) \stackrel{\text{def}}{=} \begin{cases} \left(k^2 - \alpha_m^2\right)^{1/2}, & |\alpha_m| \le k \\ i\left(\alpha_m^2 - k^2\right)^{1/2}, & |\alpha_m| > k. \end{cases} \tag{11}$$

Assume that $\omega_m(k) \neq 0$ for all $m \in \mathbb{Z}$. Then, it is not hard to verify that any α quasi-periodic solution of the free Helmholtz equation (9) in the region $\Omega^\pm$ has the form

$$u(x,z) = \sum_{m\in\mathbb{Z}} \left(A^\pm_m e^{i\alpha_m x}\, e^{i\omega_m z} + B^\pm_m e^{i\alpha_m x}\, e^{-i\omega_m z}\right)\,.$$

In this equation each term for which $|\alpha_m| \le k$ corresponds to a propagating plane wave with normal vector $(\alpha_m,\, \omega_m)/k$ or $(\alpha_m,\, -\omega_m)/k$. However, if $|\alpha_m| > k$, the corresponding terms in the series propagate along the x direction but decay (or explode) as $|z| \to \infty$. The exploding modes are physically unacceptable. As for the decaying ones, they are called *evanescent waves* or *surface waves*. If, for some m, $\omega_m(k) = 0$, we are in a physically anomalous situation, which will be excluded from our analysis.

In order to properly define the scattering problem we shall request that the scattered field u^s satisfy the *Rayleigh expansion*

$$u^s(x,z) = \sum_{m\in\mathbb{Z}} u^\pm_m\, e^{i\alpha_m x}\, e^{\pm i\omega_m z}, \quad (x,z) \in \Omega^\pm. \tag{12}$$

Recall that $q(x,z)$ is $2L$-periodic in x, supported on the interior region Ω bounded by Γ_0 and Γ_B. Furthermore, we shall assume henceforth that

$$q \in C^1(\overline{\Omega})\ , \Re q > -1\ , \Im q \ge 0\,. \tag{13}$$

We are now ready to formalize the definition of the *direct scattering* problem:

Definition 1. The Direct Scattering Problem *Given a perturbation q satisfying (13) and an incident field u^i of the form (6), we shall say that a function $u \in \mathcal{V}$ is a solution of the direct* *scattering problem if, and only if, u satisfies the equations (8),(9), the quasi-periodicity condition (10), the scattered field u^s defined by (7) satisfies the Rayleigh expansion (12) in Ω^+ and Ω^-, and u and $\partial u/\partial z$ are continuous across the boundary of Ω.*

It turns out that even under the hypothesis spelled out above a general result ensuring the existence and uniqueness of scattered solutions is not available. In fact, one can find in [4] several nontrivial examples of non-uniqueness. This illustrates the need for further investigation of the direct problem posed above. We shall content ourselves, however, by quoting a result which can be found (in a bit more general context) in [5, 6] where the scattering problem for a more general structure is analyzed. See also [4, 7] where similar conditions are requested.

Theorem 1 ([5, 6])**.** *Let q be a function defined in $\mathbb{R}^2$ satisfying $q(x+2L,z)=q(x,z)$ for all z, $q \in C^1(\overline{\Omega})$, and $q|_{\Omega^+\cup\Omega^-}=0$. Assume further that one of the following conditions is satisfied:*

- *i) There exists $(x_0,z_0)\in\Omega$ such that $\Im q(x_0,z_0)>0$.*
- *ii) The function q takes real values only, there exists $\beta\in\mathbb{R}$ such that $(z+\beta)\,q(x,z)\le 0$ if $(x,z)\in\Gamma_B$, $(z+\beta)\,q(x,z)\ge 0$ if $(x,z)\in\Gamma_0$, and $(z+\beta)\,\frac{\partial q}{\partial z}\ge 0$ in Ω, and furthermore there exists at least one point in Ω where the strict inequality is satisfied.*

Then, there exists at most one solution to the direct scattering problem in the sense of Definition 1.

2. The Direct and Inverse Problems

In this section we shall define the scattering matrix for the periodic problem under consideration. Intuitively, the scattering matrix relates incoming fields from far away with outgoing fields produced by the x-periodic scatterer $q(x,z)$. The scattering matrix will play an instrumental role in our approach to the inverse problem.

We shall assume from now on that we are dealing with a scatterer q for which the uniqueness of the direct scattering problem of Definition 1 holds for arbitrary incident fields u^i satisfying the Helmholtz equation $\Delta u^i + k^2 u^i = 0$. Sufficient conditions for such uniqueness to hold were spelled out in Theorem 1.

In order to discuss the scattering matrices we start by introducing some notation and terminology. Let us define for $B\in\mathbb{R}$ arbitrary

$$\Omega_B^{\pm} \stackrel{\text{def}}{=} \left\{ (x,z)\in\mathbb{R}^2 \quad | \quad 0<x<2L,\ \pm z > \pm B \right\} .$$

Definition 2. *An α quasi-periodic solution of the unperturbed Helmholtz equation $\Delta\varphi + k^2\varphi = 0$ in the domain $\Omega_B^{\pm}$ will be called a* radiation field *in such domain if it satisfies the Rayleigh expansion (12) for large enough $|z|$ and*

$$\sup_{\pm z > \pm B} \left(\int_0^{2L} |\varphi(x,z)|^2 dx \right) < \infty$$

The integral condition in the above definition is physically plausible and to be expected for the solutions of the unperturbed Helmholtz equation in $\Omega_B^{\pm}$. Indeed, at the light of the Rayleigh expansion (12), it is clear that if φ satisfies such condition for a given B_0 then it satisfies it for every $B > B_0$ (respec. $B < B_0$) in the Ω_B^+ case (respec. Ω_B^-). Note that, due to the Rayleigh expansion,

$$\int_0^{2L} |\varphi(x,z)|^2 dx \;=\; 2L \sum_{m\in\mathbb{Z}} |u_m^{\pm}|^2 \, e^{\mp 2\Im\omega_m z} \qquad \pm\, z > \pm B.$$

So, the integral condition implies that

$$\sum_{m\in\mathbb{Z}} |u_m^{\pm}|^2 \, e^{\mp 2\Im\omega_m B} \;<\infty.$$

The energy condition above makes it natural to introduce certain sequence spaces as follows:

We fix $k\in\mathbb{R}$ and $\theta\in(-\pi/2,\,\pi/2)$ and define the (k,θ)-dependent spaces

$$X_z \;=\; \left\{ (d_m)_{m\in\mathbb{Z}} \subset \mathbb{C} \quad\Big|\quad \sum_{m\in\mathbb{Z}} |d_m|^2 \left|e^{i\omega_m z}\right|^2 \;<\; \infty \right\},$$

$$Y_z \;=\; \left\{ (c_m)_{m\in\mathbb{Z}} \subset \mathbb{C} \quad\Big|\quad \sum_{m\in\mathbb{Z}} |c_m|^2 \left|e^{-i\omega_m z}\right|^2 \;<\; \infty \right\}.$$

We denote by $E_{k,\theta,z}$ the infinite diagonal matrix

$$E_{k,\theta,z} \;=\; \operatorname{diag}\left\{ \cdots, e^{i\omega_{-1} z}, e^{i\omega_0 z}, e^{i\omega_1 z}, \cdots \right\}.$$

Recall from (5) that Γ_z denotes the points in $\mathbb{R}^2$ of the form (x,z) with a fixed z. Since we are dealing with quasi-periodic functions defined on this set we shall identify it with $(0,2L]$. Let $F_{\rm qp} \,:\, L^2(\Gamma_z) \longrightarrow l^2(\mathbb{Z})$ denote the Fourier transform that sends a square summable function in Γ_z to its Fourier series with respect to the orthogonal system $\{\exp(i\alpha_m\cdot)\}_{m\in\mathbb{Z}}$. Thus, we can write

$$f(x) = \sum_{m\in\mathbb{Z}} c_m \, e^{i\alpha_m x}$$

in the form $f = F_{\rm qp}^{-1}\, c$, or if necessary to emphasize the x-dependence:

$$f(x) = F_{x,{\rm qp}}^{-1}\, c\,.$$

The next lemma, whose proof will be omitted because of its simplicity, gives analytic expressions for radiation fields.

Lemma 1. *Let $k>0$, $B\in\mathbb{R}$, and $\varphi\,:\,\Omega_B^+ \longrightarrow \mathbb{C}$ an α quasi-periodic function w.r.t x. Then, φ is a radiation field on Ω_B^+ if, and only if, there exists a sequence $d\in X_B$ such that*

$$\varphi(x,z) \;=\; \sum_{m\in\mathbb{Z}} d_m \, e^{i\omega_m z} e^{i\alpha_m x} \;=\; F_{x,{\rm qp}}^{-1} E_z\, d\,, \tag{14}$$

for all $(x,z)\in\Omega_B^+$. Analogously, $\psi\,:\,\Omega_B^- \longrightarrow \mathbb{C}$ is a radiation field on Ω_B^- if, and only if, there exists a sequence $c\in Y_B$ such that

$$\psi(x,z) \;=\; \sum_{m\in\mathbb{Z}} c_m \, e^{-i\omega_m z} e^{i\alpha_m x} \;=\; F_{x,{\rm qp}}^{-1} E_z^{-1}\, c\,, \tag{15}$$

for all $(x,z)\in\Omega_B^-$.

It is easy to prove that the series defining φ converges uniformly in $\overline{\Omega_{B_1}^+}$ for all $B_1 > B$. Analogously, one can show that the series defining ψ converges uniformly in $\overline{\Omega_{B_1}^-}$ for all $B_1 < B$. We note that conceptually φ is generated by sources located in Ω_B^-, whereas ψ by sources located in Ω_B^+.

We shall denote by Ω_{B_1,B_2} the fundamental region in $\mathbb{R}^2$ bounded by $z = B_1$ and $z = B_2 > B_1$. For convenience, we take $\Omega_{\mathbb{R}} \stackrel{\text{def}}{=} \{0 < x < 2L,\ z \in \mathbb{R}\}$ and $\Omega_B \stackrel{\text{def}}{=} \Omega_{0,B}$. Also, if $D \subset \mathbb{R}^2$, we shall denote by q_D the product $q_D = q\,\chi_D$, where χ_D is the characteristic function of D.

From now on we look for solutions of the scattering problem with a more general incident field instead of a single plane wave in the upper region.

Definition 3. *An incident field (or incoming field) to the region Ω_{B_1,B_2} is a superposition in this region of a radiation field in $\Omega_{B_1}^+$ and another one in $\Omega_{B_2}^-$. The corresponding scattered field in $\Omega_{B_1}^-$ (respec. $\Omega_{B_2}^+$) will be called the radiation field in the region $\Omega_{B_1}^-$ (respec. $\Omega_{B_2}^+$)*

As a direct consequence of Lemma 1 we have the following:

Lemma 2. *i) Any $(c, d) \in Y_{B_2} \times X_{B_1}$ defines an incoming field $u^i : \Omega_{B_1,B_2} \longrightarrow \mathbb{C}$ via*

$$u^i(x,z) \;=\; \varphi_1^i(x,z) \;+\; \varphi_2^i(x,z)\,, \quad (x,z) \in \Omega_{B_1,B_2}\,, \tag{16}$$

where

$$\varphi_1^i(x,z) \;=\; F_{x,\mathrm{qp}}^{-1} E_z^{-1}\, c\,, \quad (x,z) \in \Omega_{B_2}^-\,, \tag{17}$$

and

$$\varphi_2^i(x,z) \;=\; F_{x,\mathrm{qp}}^{-1} E_z\, d\,, \quad (x,z) \in \Omega_{B_1}^+\,. \tag{18}$$

ii) If $u^s : \Omega_{\mathbb{R}} \longrightarrow \mathbb{C}$ is the scattered field due to a scatterer located in Ω_{B_1,B_2}, then there exist two sequences $\tilde{d} \in X_{B_2}$ and $\tilde{c} \in Y_{B_1}$ such that

$$\begin{aligned} u^s(x,z) &= F_{x,\mathrm{qp}}^{-1} E_z\, \tilde{d}\,, \quad (x,z) \in \Omega_{B_2}^+\,, & (19)\\ u^s(x,z) &= F_{x,\mathrm{qp}}^{-1} E_z^{-1}\, \tilde{c}\,, \quad (x,z) \in \Omega_{B_1}^-\,. & (20) \end{aligned}$$

Since the support of our scatterer q_{Ω_B} splits $\mathbb{R}^2$ into two disconnected sets, it is possible to envisage several scattering problems. We shall concentrate our analysis in the case where the sources that originate the incoming field and detectors that measure the scattered field are located on the same half-space (in the upper half-space, to be more specific). This is due to the fact that in the inverse problem considered in the next section the sources and the detectors are located on Ω_B^+. Nevertheless, for technical reasons which will become clear in the sequel, we also have to consider the situation where the sources and detector are located on either sides of the scatterer.

2.1. **The Scattering Matrix.** The key object in our analysis is the so-called *scattering matrix*. Broadly speaking, it relates the Fourier coefficients of an incident wave with the coefficients of the scattered wave. Our next goal is to make this definition rigorous according to the two cases discussed in the previous paragraphs.

For convenience, we shall assume that our scatterer has its support in Ω_{B_0} for some fixed $B_0 > 0$. In the first case, according to Lemma 2 any sequence $c \in Y_{B_0}$ defines an incident field via (17). The corresponding scattered field can be represented in the form (19) in $\Omega_{B_0}^+$ for some sequence $d \in X_{B_0}$. This leads to the following:

Definition 4. *Assume that the scatterer q is supported in Ω_{B_0} and the sources and detectors are located in $\Omega^+_{B_0}$. The linear transformation $S(k,\theta;q) : Y_{B_0} \longrightarrow X_{B_0}$ such that*

$$d = S(k,\theta;q)\, c\,, \quad c \in Y_{B_0}\,,$$

is called the scattering matrix *for q.*

Given $B \geq 0$, we shall denote by S_B the scattering matrix $S(k,\theta;q)$ associated to the truncated scatterer q_{Ω_B}. In other words,

$$S_B \stackrel{\text{def}}{=} S\left(k,\theta;q_{\Omega_B}\right)\,.$$

Although the scattering matrix depends on the incidence angle θ and the frequency k, we shall omit such dependence unless needed. Furthermore, if $B=0$, then $q_{\Omega_B} = 0$ and thus $S_0 = 0$. If $B \geq B_0$, then $S_B = S(k,\theta;q)$.

Let us now consider the case where sources and detectors are located in both sides of the scatterer supported in the slab $\Omega_{B,B+h}$ of size h. Using Lemma 2 with $B_1 = B$ and $B_2 = B+h$, we introduce the scattering matrix associated to the scatterer $q^h_{\Omega_B} \stackrel{\text{def}}{=} q_{\Omega_{B,B+h}}$ according to the following:

Definition 5. *Under the assumptions of Lemma 2, the scattering matrix S^h_B corresponding to scattering by $q^h_{\Omega_B}$ in the strip of size h is the linear map*

$$S^h_B : Y_{B+h} \times X_B \longrightarrow X_{B+h} \times Y_B$$

that associates each pair (c,d) defining the incident field, as in (16), to the pair $(\tilde{d},\tilde{c})$ corresponding to the scattered field (19) and (20) and such that

$$\begin{bmatrix} \tilde{d} \\ \tilde{c} \end{bmatrix} = S^h_B \begin{bmatrix} c \\ d \end{bmatrix}.$$

Here, we can partition the matrix S^h_B into

$$\tilde{d} = S_{dc}\, c + S_{dd}\, d\,, \tag{21}$$

$$\tilde{c} = S_{cc}\, c + S_{cd}\, d\,. \tag{22}$$

We remark that since we are assuming that the (direct) scattering problem has a unique solution, the scattering matrix is well defined. Furthermore, it is not hard to see from the integral equation formulation of the problem that the above defined scattering matrices act as bounded operators on their corresponding domains of definition. To prove that we proceed as follows:

Recall that we are considering the incident field of the form

$$u^i(x,z) = \sum_{m\in\mathbb{Z}} c_m\, e^{-i\omega_m z} e^{i\alpha_m x}, \quad (x,z) \in \Omega_B,$$

where $c \in Y_B$, i.e., $\|c\|^2_{Y_B} \stackrel{\text{def}}{=} \sum_{m\in\mathbb{Z}} |c_m|^2\, e^{2\Im\omega_m B} < \infty$, and the corresponding scattered field in Ω^+_B may be represented as

$$u^s(x,z) = \sum_{m\in\mathbb{Z}} d_m\, e^{i\omega_m z} e^{i\alpha_m x}, \quad (x,z) \in \Omega^+_B,$$

for some $d \in X_B$, i.e., $\|d\|_{X_B}^2 \stackrel{\text{def}}{=} \sum_{m\in\mathbb{Z}} |d_m|^2 \, e^{-2\Im\omega_m B} < \infty$. The scattering matrix associated to q_{Ω_B} is the linear transformation $S_B : Y_B \to X_B$ such that $d = S_B\, c$.

Due to the fact that the direct problem is well-posed, there exists a constant $C > 0$ such that $\|u\|_2 \le C\, \|u^i\|_2$ (see the proof of Lemma 9), where u is the solution of the Lippmann-Schwinger equation (88) $u = u^i - Tu$, $(x,z) \in \overline{\Omega}$. On the other hand, by Theorem 5

$$u^s(x,z) = -k^2 \int_\Omega G(x,z;\xi,\eta) q(\xi,\eta) u(\xi,\eta) d\xi d\eta\,, \quad (x,z) \in \Omega^+\,.$$

Using the expression of the Green's function and the fact that $z > B$, we have

$$u^s(x,z) = \sum_{m\in\mathbb{Z}} \left(\frac{ik^2}{4L\omega_m} \int_\Omega e^{-i\omega_m \eta} e^{-i\alpha_m \xi} q(\xi,\eta) u(\xi,\eta) d\xi d\eta \right) e^{i\omega_m z} e^{i\alpha_m x}\,.$$

Therefore,

$$d_m \;=\; \frac{ik^2}{4L\omega_m} \int_\Omega e^{-i\omega_m \eta} e^{-i\alpha_m \xi} q(\xi,\eta) u(\xi,\eta) d\xi d\eta\,.$$

From this expression, we obtain

$$\begin{aligned} |d_m|^2 &= \tfrac{k^4}{16L^2|\omega_m|^2} \left| \int_\Omega e^{-i\omega_m\eta} e^{-i\alpha_m\xi} q(\xi,\eta) u(\xi,\eta) d\xi d\eta \right|^2 \\ &\le \tfrac{k^4}{16L^2|\omega_m|^2} \int_\Omega \left| e^{-i\omega_m \eta} \right|^2 |q(\xi,\eta)|^2 d\xi d\eta \int_\Omega |u(\xi,\eta)|^2 d\xi d\eta \\ &\le \tfrac{k^4}{16L^2} \tfrac{2L}{|\omega_m|^2} \|q\|_\infty^2 \, B e^{2\Im\omega_m B}\, C^2 \|u^i\|_2^2\,. \end{aligned}$$

Note that

$$\begin{aligned} \|u^i\|_2^2 &= \int_0^B \int_0^{2L} |u^i(x,z)|^2\, dx\, dz \;=\; \int_0^B 2L \sum_{m\in\mathbb{Z}} |c_m|^2 \left| e^{-i\omega_m z} \right|^2 dz \\ &= 2L \sum_{m\in\mathbb{Z}} |c_m|^2 \int_0^B e^{2\Im\omega_m z}\, dz \\ &\le 2LB \sum_{m\in\mathbb{Z}} |c_m|^2 e^{2\Im\omega_m B} \\ &= 2LB \|c\|_{Y_B}^2\,. \end{aligned}$$

Then,

$$\begin{aligned} \|S_B\, c\|_{X_B}^2 &= \|d\|_{X_B}^2 = \\ \sum_{m\in\mathbb{Z}} |d_m|^2\, e^{-2\Im\omega_m B} &\le \frac{k^4}{8L} \|q\|_\infty^2 \, B\, C^2\, 2LB \|c\|_{Y_B}^2 \sum_{m\in\mathbb{Z}} |\omega_m|^{-2} \\ &= \frac{k^4}{4} B^2 \, \|q\|_\infty^2 \, C^2\, \mathsf{C}_\omega^2 \|c\|_{Y_B}^2\,. \end{aligned}$$

The proof of the boundedness of the full matrix S_B^h is analogous, we only should define an appropriate norm in the spaces $Y_{B+h} \times X_B$ and $X_{B+h} \times Y_B$. More precisely, we set

$$\|(c,d)\|^2 \;=\; \|c\|_{Y_{B+h}}^2 + \|d\|_{X_B}^2 \text{ and } \|(\tilde{d},\tilde{c})\|^2 \;=\; \|\tilde{d}\|_{X_{B+h}}^2 + \|\tilde{c}\|_{Y_B}^2\,.$$

2.2. **Multiple Scattering.** Our goal is to derive a Riccati-type equation for the scattering matrix. This will be done following the general methodology of invariant imbedding [8]. We start by considering the effect of several scatterers with disjoint support.

Given an incident field, we know how to compute the scattered field resulting from each scatterer. Our first task is to compute the resulting field from the totality of scatterers. These results follow the ideas of the work of V. Twersky [9, 10] on multiple scattering. For our purposes we need to consider only two scatterers as shown in Figure 3, that occupy domains Ω_1 and Ω_2 in such way that $\Omega_1 \cap \Omega_2 = \emptyset$ and $\overline{\Omega_1} \cup \overline{\Omega_2} = \overline{\Omega}$.

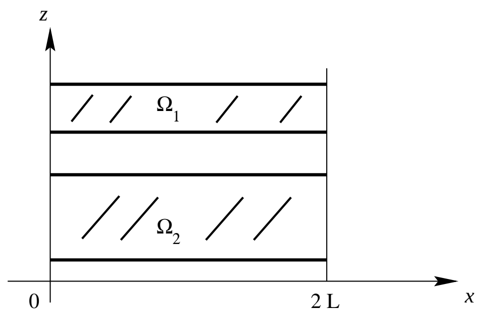

FIGURE 3. Multiple scattering geometry

Let us now define, for $j = 1, 2$, the operators $T_j : C_{\mathrm{qp}}(\overline{\Omega_j}) \longrightarrow C_{\mathrm{qp}}(\Omega_{\mathbb{R}})$ as

$$(T_j u)(x, z) = k^2 \int_{\Omega_j} G(x, z; \xi, \eta) q(\xi, \eta) u(\xi, \eta) d\xi d\eta \ ,$$

where the sub-index qp emphasizes the quasi-periodicity (in x) of the functions involved. The Lippmann-Schwinger equation for the domain Ω takes the form

$$(I + T_1)u \ = \ u^i - T_2 u \, , \tag{23}$$

or

$$(I + T_2)u \ = \ u^i - T_1 u \, , \tag{24}$$

We now define $u_j^s : \Omega_{\mathbb{R}} \longrightarrow \mathbb{C}, j = 1, 2$, via

$$u_1^s \stackrel{\text{def}}{=} -T_1 u \, , \quad u_2^s \stackrel{\text{def}}{=} -T_2 u \, , \tag{25}$$

and see that (23) can be interpreted as the Lippmann-Schwinger equation for the total field u in the domain Ω_1, with an incident field given by

$$u_1^i \stackrel{\text{def}}{=} u^i \, + \, u_2^s \, , \quad \mathbf{x} \in \Omega_1 \, . \tag{26}$$

In fact, since $\Omega_1 \cap \Omega_2 = \emptyset$, we have that $\Delta u_2^s + k^2 u_2^s = 0$ in Ω_1, which together with the Green's function properties implies that u_2^s is a radiation field in Ω_1. Furthermore, from (23), (25), and (26) we get that $u \ = \ u_1^i + u_1^s$. Similarly, equation (24) is the Lippmann-Schwinger for the total field u in the region Ω_2 in the presence of an incident field of the form

$$u_2^i \stackrel{\text{def}}{=} u^i \, + \, u_1^s \, , \quad \mathbf{x} \in \Omega_2 \, . \tag{27}$$

The above discussion can be summarized in the following:

Lemma 3. *Let q be a scatterer whose support is composed of two disjoint regions Ω_1 and Ω_2, $u^i : \Omega_1 \cup \Omega_2 \longrightarrow \mathbb{C}$ be the incident field, and $u : \Omega_1 \cup \Omega_2 \longrightarrow \mathbb{C}$ denote the total field. Assume that the functions $u_j^s : \Omega_{\mathbb{R}} \longrightarrow \mathbb{C}$ and $u_j^i : \Omega_j \longrightarrow \mathbb{C}, j = 1, 2$ are defined by the formulae (25) – (27). Then, the function u_j^s represents the scattered field by the region Ω_j corresponding to the incident field u_j^i , $j = 1, 2$. Furthermore, the total field u can be written as*

$$u = u^i + u_1^s + u_2^s \,.$$

Let us now examine the case where Ω_1 and Ω_2 are adjacent. More precisely, let us consider the case $\Omega_1 = \Omega_{B,B+h}$ and $\Omega_2 = \Omega_B$, where $B > 0$, $h > 0$. From equations (23) and (24) we get

$$(I + T_1)u_1^s + T_1 u_2^s \;=\; -T_1 u^i \,, \tag{28}$$

$$T_2 u_1^s + (I + T_2)u_2^s \;=\; -T_2 u^i \,. \tag{29}$$

As a consequence of Theorem 6, the operators $B_j : C_{\mathrm{qp}}(\overline{\Omega_j}) \longrightarrow C_{\mathrm{qp}}(\Omega_{\mathbb{R}})$, defined as

$$B_j \;=\; -(I + T_j)^{-1}\, T_j \,, \quad j = 1, 2 \,,$$

are bounded on the corresponding sup norms. Now we notice that for an arbitrary function $\tilde{u} \in C_{\mathrm{qp}}(\overline{\Omega_j})$, the function $\tilde{v} \in C_{\mathrm{qp}}(\overline{\Omega_j})$ given by $\tilde{v} \;=\; B_j\, \tilde{u}$ satisfies $(I + T_j)\, \tilde{v} \;=\; -T_j\, \tilde{u}$ on $\overline{\Omega_j}$. In other words, for $j = 1, 2$, B_j maps the incident field into the scattered field.

Lemma 4. *Under the same hypothesis of Lemma 3, assume that there exists a constant $C > 0$ such that*

$$\|T_1\|_\infty \, < C\, h \,, \tag{30}$$

where h is the thickness of the slab $\Omega_{B,B+h}$ (See estimate (93)). Then, for a sufficiently small h, there exist unique functions $u_j^s : \Omega_{\mathbb{R}} \longrightarrow \mathbb{C}$, $j = 1, 2$, such that equations (28) and (29) are satisfied. Furthermore,

$$u_1^s(\mathbf{x}) + u_2^s(\mathbf{x}) = \{B_2 + (I + B_2)B_1(I + B_2)\}\, u^i + \mathcal{O}(h^2) \,. \tag{31}$$

Proof: We rewrite (28) and (29) in terms of the operators B_j and eliminate u_2^s, so as to get

$$(I \;-\; B_1 B_2)\, u_1^s \;=\; B_1(I + B_2)\, u^i \,. \tag{32}$$

It follows from the boundedness of the operators B_j and the estimate (30), that there exist positive constants c' and c'' such that

$$\|B_1\| \;\le\; c' \cdot h \,,$$
$$\|B_1\, B_2\| \;\le\; c'' \cdot h \,,$$

for all $h > 0$. Hence, for sufficiently small h

$$u_1^s = \left(\sum_{m=0}^{\infty} (B_1\, B_2)^m \right) B_1(I + B_2)\, u^i \,.$$

Finally, from the estimates of the norms of the operators B_1 and $B_1\, B_2$ as a function of h, we get (31). *Q. E. D.*

In the specific case of $\Omega_1 = \Omega_{B,B+h}$ and $\Omega_2 = \Omega_B$, $B > 0$, $h > 0$ we have the following:

Lemma 5. *Let $u^i : \Omega_{B+h} \longrightarrow \mathbb{C}$ be an incident field in the region Ω_{B+h} of the form*

$$u^i(x,z) = F_{x,\mathrm{qp}}^{-1} E_z^{-1}\, c\,, \quad (x,z) \in \Omega_{B+h}\,,$$

where $c \in Y_{B+h}$, and $u^s : \Omega_{\mathbb{R}} \longrightarrow \mathbb{C}$ corresponds to the scattering field. Then, if $h > 0$ is sufficiently small we have

$$u^s(x,z) = \sum_{j=1}^{5} F_{x,\mathrm{qp}}^{-1} E_z\, d_j + \mathcal{O}(h^2)\,, \quad (x,z) \in \Omega_{B+h}^{+}\,, \tag{33}$$

where

$$\begin{aligned} d_1 &= S_{dc} \cdot c\,, \\ d_2 &= S_B \cdot c\,, \\ d_3 &= S_B \cdot S_{cc} \cdot c\,, \\ d_4 &= S_{dd} \cdot S_B \cdot c\,, \\ d_5 &= S_B \cdot S_{cd} \cdot S_B \cdot c\,. \end{aligned}$$

3. The Matrix Riccati Equation

In the earlier section we introduced the scattering matrix $S_z = S(k, \theta, q_{\Omega_z})$. We shall now consider this as a function of the variable z and obtain a Riccati-type differential equation for S_z. The basic ingredient will be an invariant-imbedding argument [8] following the same circle of ideas in [11], which dealt with the compact supported perturbation case. This technique has been used in a number of different contexts, such as the analysis of wave propagation in the time domain [12, 13, 14], scattering by anisotropic media [15], and the solution of nonlinear integro-differential equations [16, 17] to cite a few.

Let us denote by $q_z : L^2[0, 2L] \longrightarrow L^2[0, 2L]$ the diagonal linear transformation

$$(q_z\, f)(x) = q(x,z)\, f(x)\,, \quad \forall f \in L^2[0, 2L]\,,$$

and by $\Lambda : l^2(\mathbb{Z}) \longrightarrow l^2(\mathbb{Z})$ the diagonal matrix

$$\Lambda = \mathrm{diag}\,(\cdots, \omega_{-1}, \omega_0, \omega_1, \cdots)\,.$$

We start by getting a second order expression for the Born approximation outside the region $\Omega_{B,B+h}$.

Lemma 6. *Let $k > 0$, $B \geq 0$, $h > 0$ be given, $u^i : \Omega_{B,B+h} \longrightarrow \mathbb{C}$ be the incident field on the region $\Omega_{B,B+h}$, and $u^i\big|_{\Gamma_B}$ represent the restriction of u^i on the line $z = B$. Suppose that $u^s : \Omega_{\mathbb{R}} \longrightarrow \mathbb{C}$ is the corresponding scattered field. Then, if $h > 0$ is sufficiently small*

$$u^s(x,z) = \frac{ik^2}{2} h F_{x,\mathrm{qp}}^{-1}\, E_z\, \Lambda^{-1}\, E_B^{-1}\, F_{\mathrm{qp}}\, q_B u^i\big|_{\Gamma_B} + \mathcal{O}(h^2) \tag{34}$$

for $z > B + h$, and

$$u^s(x,z) \;=\; \frac{ik^2}{2} h F_{x,\mathrm{qp}}^{-1}\, E_z^{-1}\, \Lambda^{-1}\; E_B\, F_{\mathrm{qp}}\, q_B u^i\big|_{\Gamma_B} \;+\; \mathcal{O}(h^2) \tag{35}$$

for $z < B$.

Proof: We note that we can apply Lemma 14 replacing the region $(0, 2L) \times (0, B)$ by $\Omega_{B,B+h}$. In this case $\delta = k^2 \|q\|_\infty\, h\mathsf{C}_\omega$. As in the B we consider

$$u^s_{\mathsf{B}} \;\stackrel{\text{def}}{=}\; -k^2 \int_{\Omega_{B,B+h}} G(x,z;\xi,\eta) q(\xi,\eta) u^i(\xi,\eta)\, d\xi d\eta\,.$$

From estimate (93) we deduce that there exist constants $c > 0$, $h_1 > 0$ such that

$$|u^s(\mathbf{x}) - u^s_{\mathsf{B}}(\mathbf{x})| \;\le\; c \cdot h^2\,, \qquad \mathbf{x} \in \Omega_{\mathbb{R}}\,, \tag{36}$$

for all $h < h_1$. The term u^s_{B} corresponds to the first term in the Neumann series for the field u^s. Now using the explicit Green's function formula, given by (85) we get

$$u^s_{\mathsf{B}} \;=\; \frac{ik^2}{4L} \int_0^{2L} \int_B^{B+h} q(\xi,\eta) u^i(\xi,\eta) \sum_{m\in\mathbb{Z}} \omega_m^{-1} e^{i\omega_m (z-\eta)} e^{i\alpha_m (x-\xi)}\, d\xi d\eta$$

for $z > B + h$, and

$$u^s_{\mathsf{B}} \;=\; \frac{ik^2}{4L} \int_0^{2L} \int_B^{B+h} q(\xi,\eta) u^i(\xi,\eta) \sum_{m\in\mathbb{Z}} \omega_m^{-1} e^{i\omega_m (\eta-z)} e^{i\alpha_m (x-\xi)}\, d\xi d\eta$$

for $z < B$. By using that

$$\int_B^{B+h} g(s)\, ds \;=\; h\, g(B) \;+\; \mathcal{O}(h^2)\,, \quad g \in C^1([B, B+h])\,,$$

the field u^s_{B} takes the form

$$\begin{aligned} u^s_{\mathsf{B}} \;&=\; \frac{ik^2}{4L} h \sum_{m\in\mathbb{Z}} \left(\omega_m^{-1} e^{i\omega_m (z-B)} \int_0^{2L} e^{-i\alpha_m \xi} q(\xi,B) u^i(\xi,B)\, d\xi \right) e^{i\alpha_m x} \;+\; \mathcal{O}(h^2) \\ &=\; \frac{ik^2}{2} h F_{x,\mathrm{qp}}^{-1}\, \Lambda^{-1}\; E_z\, E_B^{-1}\, F_{\mathrm{qp}}\, q_B u^i\big|_{\Gamma_B} \;+\; \mathcal{O}(h^2) \end{aligned}$$

for $z > B + h$, and

$$\begin{aligned} u^s_{\mathsf{B}} \;&=\; \frac{ik^2}{4L} h \sum_{m\in\mathbb{Z}} \left(\omega_m^{-1} e^{i\omega_m (B-z)} \int_0^{2L} e^{-i\alpha_m \xi} q(\xi,B) u^i(\xi,B)\, d\xi \right) e^{i\alpha_m x} \;+\; \mathcal{O}(h^2) \\ &=\; \frac{ik^2}{2} h\, F_{x,\mathrm{qp}}^{-1}\, \Lambda^{-1}\; E_z^{-1}\, E_B\, F_{\mathrm{qp}}\, q_B u^i\big|_{\Gamma_B} \;+\; \mathcal{O}(h^2) \end{aligned}$$

for $z < B$. Combining the last two expressions with (36), we get (34) and (35). *Q. E. D.*

We now obtain approximate expressions for the matrix S_B^h.

Lemma 7. *Under the same assumptions of Lemma 6, if the matrices S_{dc}, S_{dd}, S_{cc} and S_{cd} are defined by equations (21) and (22), then we have that*

$$S_{dc} \;=\; \frac{ik^2}{2} h\, E_B^{-1}\, \Lambda^{-1}\, F\, q_B\, F^{-1}\, E_B^{-1} \;+\; \mathcal{O}(h^2)\,, \tag{37}$$

$$S_{dd} \;=\; \frac{ik^2}{2} h\, E_B^{-1}\, \Lambda^{-1}\, F\, q_B\, F^{-1}\, E_B \;+\; \mathcal{O}(h^2)\,, \tag{38}$$

$$S_{cc} \;=\; \frac{ik^2}{2} h\, E_B\, \Lambda^{-1}\, F\, q_B\, F^{-1}\, E_B^{-1} \;+\; \mathcal{O}(h^2)\,, \tag{39}$$

$$S_{cd} \;=\; \frac{ik^2}{2} h\, E_B\, \Lambda^{-1}\, F\, q_B\, F^{-1}\, E_B \;+\; \mathcal{O}(h^2)\,. \tag{40}$$

Proof: We are going to prove (37). The other ones are analogous. The incident field u^i in the region $\Omega_{B,B+h}$ is given by

$$u^i(x,z) \;=\; F_{x,\mathrm{qp}}^{-1} \left(E_z^{-1}\, c \,+\, E_z\, d \right)\;,$$

where $c \in Y_{B+h}$ and $d \in X_B$. Let $d = 0$. Taking the expression of the field u^i in (34), we see that

$$u^s(x,z) \;=\; F_{x,\mathrm{qp}}^{-1}\, E_z \tilde{d} \;=\; \frac{ik^2}{2} h\, F_{x,\mathrm{qp}}^{-1}\, E_z\, \Lambda^{-1}\, E_B^{-1}\, F_{\mathrm{qp}}\, q_B\, F_{\mathrm{qp}}^{-1}\, E_B^{-1}\, c \,+\, \mathcal{O}(h^2)\,,$$

if $(x,z) \in \Omega_{B+h}^+$. On the other hand, from (21) we have that (taking into account that $d = 0$)

$$\tilde{d} \;=\; S_{dc}\, c\,.$$

It now remains to note that $F_{x,\mathrm{qp}}\, q_B\, F_{x,\mathrm{qp}}^{-1} \;=\; F\, q_B\, F^{-1}$, where F denotes the Fourier transform with respect to the system $\exp(ipm)$, to conclude (37). Indeed, for $c \in \ell^2(\mathbb{Z})$

$$\begin{aligned} F_{x,\mathrm{qp}}\, q_B\, F_{x,\mathrm{qp}}^{-1}\, c \;&=\; \frac{1}{2L} \int_0^{2L} e^{-i\alpha_m x} q(x,B) \sum_{n\in\mathbb{Z}} c_n e^{i\alpha_n x}\, dx \\ &=\; \frac{1}{2L} \int_0^{2L} e^{-ipmx} q(x,B) \sum_{n\in\mathbb{Z}} c_n e^{ipnx}\, dx \\ &=\; F\, q_B\, F^{-1}\, c \end{aligned}$$

This concludes the proof. *Q. E. D.*

Now to show that the scattering matrix S_z satisfies a Riccati type differential equation we compute the difference $S_{B+h} - S_B$ up to second order in h. Let $u^i \,:\, \Omega_{B+h} \longrightarrow \mathbb{C}$ be the incident field on the region Ω_{B+h} of the form

$$u^i(x,z) \;=\; F_{x,\mathrm{qp}}^{-1}\, E_z^{-1} c\;,$$

where $c \in Y_{B+h}$. The scattered field in the region Ω_{B+h}^+ has the form

$$u^s(x,z) \;=\; F_{x,\mathrm{qp}}^{-1}\, E_z d\;,$$

with $d \in X_{B+h}$. By definition of S_{B+h},

$$d \;=\; S_{B+h}\, c\,. \tag{41}$$

From Lemmas 5 and 7 we get the following:

Lemma 8. *Under the conditions of Lemma 5, we have*

$$d \;=\; \left\{ S_B + h\frac{ik^2}{2} \left(E_B^{-1} + S_B\, E_B \right) \Lambda^{-1}\, F\, q_B\, F^{-1} \,\left(E_B\, S_B \,+\, E_B^{-1} \right) \right\}\, c \,+\, \mathcal{O}(h^2)\,,$$

where $d = \sum_{j=1}^{5} d_j$.

Finally, from Lemma 8 and formula (41), we get the following:

Theorem 2. *For any* $z \geq 0$, $k > 0$, *and* $\theta \in (-\pi/2, \pi/2)$ *such that* $\omega_m \neq 0$, *for every* $m \in \mathbb{Z}$, *the scattering matrix* $S_z : Y_z \longrightarrow X_z$ *satisfies the Riccati type matrix differential equation*

$$\frac{dS_z}{dz} = \frac{ik^2}{2} \left(E_z^{-1} + S_z E_z\right) \Lambda^{-1} F q_z F^{-1} \left(E_z S_z + E_z^{-1}\right) . \tag{42}$$

Equation (42) will play a fundamental role in the numerical method presented in the sequel.

4. Uniqueness of the Solution to the Inverse Problem

This section concerns the main theoretical result of the present work. The basic inverse problem under consideration is the following: Suppose that a non-homogeneous medium q is excited by a set of plane waves u^i and we measure the corresponding scattered waves. When do we have enough information to determine the function q?

The existing uniqueness results are not many. Uniqueness was obtained if one assumes that the perturbation depends only on the horizontal coordinate x and considers a superposition of plane waves at a fixed wave number k incident onto a non-homogeneous layer located on top of a perfectly reflecting boundary [18]. See also [19] for a transmission problem.

Our main theoretical result is a uniqueness result for the inverse scattering problem with the wave number k in a nonempty open interval where the scattering matrix is well-defined, i.e., $\omega_m(k) \neq 0$ for every $m \in \mathbb{Z}$. We shall show that the x-periodic perturbation q is uniquely determined by the scattering matrix for a fixed incidence angle and the wave number in a nonempty open interval. We also assume that the support of q is known, which is not a very restrictive assumption.

Remark 1. *We shall assume throughout the present section that the perturbation q under consideration is such that the* direct problem *for any incident wave u^i satisfying $\Delta u^i + k^2 u^i = 0$ in Ω_B possesses a unique solution. This is true if, for instance, the hypothesis of Theorem 1 are valid.*

Our approach follows the general ideas of Colton and Kress [20, Theorem 10.19] which deals with a compactly supported perturbation q. However, the fact that we are dealing with a periodic perturbation brings in additional difficulties and requires a different analysis. The main theoretical tool is the Riesz-Fredholm analytic theory of compact operators in the space $\mathcal{B}(X)$ of bounded operators in a Banach space X. See for example [20, 21].

Definition 6. *Let D be a domain in $\mathbb{C}^N$ and $f : D \longrightarrow X$ a function taking values in the complex Banach space X. The function f is called* analytic *in D if for every $z_0 \in D$ there exist $f_j \in X$, $j \in \mathbb{Z}^N_{\geq 0}$, such that the power series*

$$f(z) = \sum_{j \in \mathbb{Z}^N_{\geq 0}} \frac{1}{j!}(z - z_0)^j f_j ,$$

converges uniformly in X for every z in a neighborhood of z_0. Here, for $z = (z_1, \cdots, z_N) \in \mathbb{C}^N$ and $j = (j_1, \cdots, j_N) \in \mathbb{Z}^N_{\geq 0}$, we are using the multi-index notation

$$z^j \stackrel{def}{=} z_1^{j_1} \cdots z_N^{j_N} , \quad j! \stackrel{def}{=} j_1! \cdots j_N! .$$

The function f is called weakly analytic *in D if for every bounded linear functional $\ell : X^* \longrightarrow \mathbb{C}^N$, we have that $z \mapsto \ell(f(z))$ is an analytic function of z for $z \in D$.*

As is well-known, the two concepts of analytic functions coincide [20, Theorem 8.22].

We shall now explore the analyticity of the scattered solutions with respect to the variables k and θ. It turns out, however, that the space C_{qp} where such solutions were defined depended on k and θ through the quasi-periodicity condition. See Definition 1. We shall now reformulate the inverse scattering problem in such a way that the solutions of our problems would be defined in the (fixed) space C_p of continuous functions that are $2L$-periodic in x. This will be performed via the substitution $u(x,z) = \widetilde{u}(x,z)\exp(i\alpha x)$ (and analogously for u^i and u^s). We get that $\widetilde{u}$ and $\partial\widetilde{u}/\partial z$ belong to $C^0_{\mathrm{p}}(\Omega_{\mathbb{R}})$ and $\widetilde{u}$ satisfies

$$\Delta_\alpha \widetilde{u}(\mathbf{x}) + k^2(1+q(\mathbf{x}))\widetilde{u}(\mathbf{x}) = 0\,, \quad \mathbf{x} \in \Omega\,, \tag{43}$$

$$\Delta_\alpha \widetilde{u}(\mathbf{x}) + k^2\widetilde{u}(\mathbf{x}) = 0\,, \quad \mathbf{x} \in \Omega^{\pm}\,, \tag{44}$$

where $\Delta_\alpha \stackrel{\mathrm{def}}{=} \Delta + 2i\alpha\partial_x - \alpha^2$. The function $\widetilde{u}$ satisfies

$$\widetilde{u}(x+2L,z) = \widetilde{u}(x,z), \quad (x,z) \in \Omega_{\mathbb{R}}\,, \tag{45}$$

and the scattered field $\widetilde{u}^s(\mathbf{x}) \stackrel{\mathrm{def}}{=} \widetilde{u}(\mathbf{x}) - \widetilde{u}^i(\mathbf{x})$ obeys the Rayleigh type condition

$$\widetilde{u}^s(x,z) = \sum_{m\in\mathbb{Z}} u_m^{\pm} e^{\pm i\omega_m z}\, e^{ipmx}\,, \quad \left(u_m^{\pm}\right)_{m\in\mathbb{Z}} \subset \mathbb{C}, \quad (x,z) \in \Omega^{\pm}\,, \tag{46}$$

where as before $p = \pi/L$. We note that the multiplication operator by the function $\exp(-i\alpha x)$, $x \in (0,2L)$, is an isomorphism between $C_{\mathrm{qp}}(\overline{\Omega})$ and $C_{\mathrm{p}}(\overline{\Omega})$.

The fundamental solution $\widetilde{G}$ of the transformed problem may be represented in the form

$$\widetilde{G}(x,z;\xi,\eta) = -\frac{i}{4L}\sum_{m\in\mathbb{Z}} \frac{1}{\omega_m} e^{i\omega_m|z-\eta|}\, e^{ipm(x-\xi)}\,, \quad (x,z) \neq (\xi,\eta)\,, \tag{47}$$

whenever $\omega_m \neq 0$ for every $m \in \mathbb{Z}$. We define the operator $\widetilde{T} : C_{\mathrm{p}}(\overline{\Omega}) \longrightarrow C_{\mathrm{p}}(\overline{\Omega})$ as

$$\widetilde{T}\widetilde{u}(\mathbf{x}) = k^2 \int_\Omega \widetilde{G}(x,z;\xi,\eta)q(\xi,\eta)\widetilde{u}(\xi,\eta)\, d\xi d\eta\,, \quad \mathbf{x} = (x,z) \in \overline{\Omega}\,. \tag{48}$$

The function $\widetilde{u}$ that solves the problem (43)–(46) satisfies the integral equation

$$\widetilde{u}(\mathbf{x}) = \widetilde{u}^i(\mathbf{x}) - \widetilde{T}\widetilde{u}(\mathbf{x})\,, \quad \mathbf{x} \in \overline{\Omega}\,. \tag{49}$$

The operator $\widetilde{T}$ depends on k and θ through the Green's function $\widetilde{G}$. We shall start now our analysis through the study of ω_m as a function of the complex variables (k,θ). Let

$$\mathbb{C}_- \stackrel{\mathrm{def}}{=} \mathbb{C} \setminus \{ z \in \mathbb{C} \quad | \quad \Re z = 0,\ \Im z \leq 0 \}\,.$$

We define

$$\mathcal{K} = \left\{ (k,\theta) \in \mathbb{C}^2 \quad | \quad k^2 - (pm + k\sin\theta)^2 \in \mathbb{C}_-\,,\ \forall m \in \mathbb{Z} \right\}\,,$$

and

$$\omega_m = \omega_m(k,\theta) \stackrel{\mathrm{def}}{=} \sqrt{k^2 - (pm + k\sin\theta)^2} = |\omega_m|\, e^{i\vartheta_m/2}\,, \quad (k,\theta) \in \mathcal{K}\,,$$

where $\vartheta_m = \arg \omega_m^2(k,\theta)$ and $-\pi/2 \le \vartheta_m < 3\pi/2$. We see that the function ω_m is analytic for $(k,\theta) \in \mathcal{K}$.

Lemma 9. *Let $q \in C_{\mathrm{p}}(\overline{\Omega})$. For each $(k,\theta) \in \mathcal{K}$, let $\widetilde{T}(k,\theta) : L^2(\Omega) \longrightarrow L^2(\Omega)$ be defined by*

$$(\widetilde{T}(k,\theta)\widetilde{u})(\mathbf{x}) \;=\; k^2 \int_\Omega \widetilde{G}(x,z;\xi,\eta;k,\theta) q(\xi,\eta) \widetilde{u}(\xi,\eta)\, d\xi d\eta\,, \quad (x,z) \in \overline{\Omega}\,,$$

where $\widetilde{G}$ is given by (47). Then,

$$\mathcal{K} \ni (k,\theta) \longmapsto \widetilde{T}(k,\theta) \in \mathcal{B}(L^2(\Omega))$$

is analytic in $\mathcal{K}$.

Proof: We shall first show that the operator $\widetilde{T} = \widetilde{T}(k,\theta)$ is well defined and is bounded from $L^2(\Omega)$ to $L^2(\Omega)$. Following the same lines of the proof of Lemma 13, in the Appendix we conclude that there exists a positive constant $C = C(k,\theta,B)$, such that

$$\int_\Omega |\widetilde{G}(\mathbf{x},\ \boldsymbol{\xi}\ ;k,\theta)|^2\, d\,\boldsymbol{\xi}\ \le C^2 \frac{B}{8L} \sum_{m\in\mathbb{Z}} \frac{1}{|\omega_m|^2}\,, \quad \forall \mathbf{x} \in \Omega\,.$$

Therefore,

$$\|\widetilde{T}\widetilde{u}\|_2 \le \frac{1}{2} C |k|^2 \|q\|_\infty B \mathsf{C}_\omega \|\widetilde{u}\|_2\,,$$

where $\mathsf{C}_\omega \stackrel{\mathrm{def}}{=} \left(\sum_{m\in\mathbb{Z}} |\omega_m|^{-2}\right)^{1/2}$, and $C = 1$ if $\Im\omega_m \ge 0$.

To show the analyticity of $(k,\theta) \mapsto \widetilde{T}(k,\theta)$ with $(k,\theta) \in \mathcal{K}$, it is sufficient to show that for each function $\widetilde{u} \in L^2(\Omega)$ and $\widetilde{v} \in L^2(\Omega)$, the map

$$\mathcal{K} \ni (k,\theta) \longmapsto (\widetilde{T}(k,\theta)\widetilde{u}, \widetilde{v})$$

is analytic, where $(\cdot,\cdot)$ denotes the usual inner product in $L^2(\Omega)$. Here, we use the fact that: If for each $\widetilde{u} \in L^2(\Omega)$, the function $\widetilde{T}\widetilde{u} : \mathcal{K} \longrightarrow L^2(\Omega)$ is weakly analytic, then $\widetilde{T}$ is strongly analytic.

Now, we notice that the Green's function $\widetilde{G}(x,z;\xi,\eta;k,\theta)$ is analytic w.r.t. the variables $(k,\theta) \in \mathcal{K}$ for almost all fixed $(x,z;\xi,\eta) \in \Omega \times \Omega$. This is due to the analyticity of $(k,\theta) \mapsto \omega_m(k,\theta)$ for $(k,\theta) \in \mathcal{K}$ and each $m \in \mathbb{Z}$. Furthermore, the series that defines $\widetilde{G}$ converges absolutely for all $z \ne \eta$, with $(z,\eta) \in (0,B) \times (0,B)$, and it converges uniformly on compact subsets of $\mathcal{K}$.

For each fixed $(k,\theta) \in \mathcal{K}$ the following estimate holds

$$\int_\Omega \int_\Omega \left|\widetilde{G}(\mathbf{x},\ \boldsymbol{\xi}\ ;k,\theta)\, q(\ \boldsymbol{\xi}\ )\widetilde{u}(\ \boldsymbol{\xi}\ )\overline{\widetilde{v}(\mathbf{x})}\right|\, d\,\boldsymbol{\xi}\ d\mathbf{x} \le \frac{1}{2} C \|q\|_\infty B \mathsf{C}_\omega \|\widetilde{u}\|_2\, \|\widetilde{v}\|_2\,.$$

Finally, from the representation of Lemma 13 in A

$$\widetilde{G}(\mathbf{x},\ \boldsymbol{\xi}\ ;k,\theta) = \frac{1}{2\pi} \ln ||\mathbf{x} -\ \boldsymbol{\xi}\ || +\ \Psi(\mathbf{x},\ \boldsymbol{\xi}\ ;k,\theta)\,,$$

where Ψ is a continuous function w.r.t. its arguments, we get that for each compact $K \subset \mathcal{K}$ there exists $g_K \in L^1(\Omega \times \Omega)$ such that

$$\left|\widetilde{G}(\mathbf{x},\ \boldsymbol{\xi}\ ;k,\theta)\, q(\ \boldsymbol{\xi}\ )\widetilde{u}(\ \boldsymbol{\xi}\ )\overline{\widetilde{v}(\mathbf{x})}\right| \le g_K(\mathbf{x},\ \boldsymbol{\xi}\ )\,,$$

for all $(\mathbf{x},\ \boldsymbol{\xi}\ ;k,\theta) \in \Omega \times \Omega \times K$.

From the above analysis we conclude that the function $(k,\theta) \mapsto \widetilde{T}(k,\theta)$ is weakly, and hence strongly analytic for $(k,\theta) \in \mathcal{K}$. *Q. E. D.*

Lemma 10. *Assume that the scattering problem (43) – (46) has a unique solution. Then,* $\widetilde{u}$ *depends analytically on* $(k,\theta) \in \mathcal{K}_+$*, where*

$$\mathcal{K}_+ = \left\{ (k,\theta) \in (0,\infty) \times (-\pi/2, \pi/2) \quad | \quad k^2 - (pm + k\sin\theta)^2 \neq 0 \,,\, \forall m \in \mathbb{Z} \right\}.$$

Proof: Let $\widetilde{u}$ be a solution of the direct scattering problem (43) – (46). Then, because of the equivalence between the differential and the integral formulations (Theorem 5) the function $\widetilde{u}\big|_{\overline{\Omega}}$ satisfies Equation (49). Furthermore, the uniqueness hypothesis implies that the operator $(I+\widetilde{T})^{-1} : C_{\mathrm{p}}(\overline{\Omega}) \longrightarrow C_{\mathrm{p}}(\overline{\Omega})$ exists and is bounded.

Now, since $\widetilde{T}$ is an integral operator with a weakly singular kernel, its adjoint with respect to the bilinear form

$$\langle u, v\rangle \stackrel{\mathrm{def}}{=} \int_{\Omega} u(x)v(x)\, dx \,,$$

exists and is a compact operator from $C_{\mathrm{p}}(\overline{\Omega})$ into $C_{\mathrm{p}}(\overline{\Omega})$. Thus, applying the Fredholm alternative [22, Theorem 4.17] to the dual systems $\langle C_{\mathrm{p}}(\overline{\Omega})\,,\, C_{\mathrm{p}}(\overline{\Omega})\rangle$ and $\langle L^2(\Omega)\,,\, C_{\mathrm{p}}(\overline{\Omega})\rangle$, we see that the dimension of the kernel of $I+\widetilde{T}$ in the spaces $C_{\mathrm{p}}(\overline{\Omega})$ and $L^2(\Omega)$ coincides. However, still by the uniqueness hypothesis, the kernel of $I+\widetilde{T}$ in the space $C_{\mathrm{p}}(\overline{\Omega})$ is trivial. Therefore, from the compactness of $\widetilde{T} : L^2(\Omega) \longrightarrow L^2(\Omega)$ and the Riesz-Fredholm theory, we conclude that $(I+\widetilde{T})^{-1} : L^2(\Omega) \longrightarrow L^2(\Omega)$ exists and is bounded.

Since the operator $\widetilde{T}$ depends analytically on $(k,\theta) \in \mathcal{K}_+$ (Lemma 9) so does the operator $(k,\theta) \mapsto (I+\widetilde{T}(k,\theta))^{-1}$. From here it follows that $\widetilde{u} \in L^2(\Omega)$ depends analytically on $(k,\theta) \in \mathcal{K}_+$.

Now we note that $\widetilde{u} \in L^2(\Omega)$ satisfies the integral equation (49) in the L^2 sense. We claim that this implies that $\widetilde{u}$ is a classical solution to the scattering problem. Indeed, $\widetilde{T}$ is bounded from $L^2(\Omega)$ to the Sobolev space $H^2(\Omega)$. See for example [20, Theorem 8.2]. Thus, if the incident field is sufficiently smooth, from the integral equation we see that $\widetilde{u} \in H^2(\Omega)$. From Sobolev's theorem it follows that $H^2(\Omega) \subset C^{0,\epsilon}(\overline{\Omega})$ for $0 < \epsilon < 1$.

Therefore, we now proceed as in Theorem 5 in the Appendix to conclude that $\widetilde{u}$ satisfies (43) – (46). *Q. E. D.*

Remark 2. *Since* $u(x,z) = \widetilde{u}(x,z)\, \exp(ik\sin\theta x)$ *for* $(x,z) \in \Omega_{\mathbb{R}}$*, the solution of the scattering problem in the space of quasi-periodic functions also depends analytically on* $(k,\theta) \in \mathcal{K}_+$*.*

From now on we shall fix the incidence angle θ for $|\theta| < \pi/2$. From Lemma 14, we get the following:

Lemma 11. *There exists a wave number* k_0 *such that the direct scattering problem (43) – (46) has a unique solution for all* $k \in (0, k_0)$*.*

Proof: For $\widetilde{u}$ satisfying (43) – (46), the Lippmann-Schwinger equation (49) is satisfied, and Lemma 14 shows that under the condition

$$k^2 \|q\|_\infty B\, \mathsf{C}_\omega \;\leq\; 1\,, \tag{50}$$

where

$$\mathsf{C}_\omega^2 = \sum_{m\in\mathbb{Z}} |\omega_m|^{-2} \,.$$

there exists a unique solution to the integral equation (49). Furthermore, such solution can be found by the successive approximation method. Therefore, the claim in the lemma is proved if we find a sufficiently small number $k_0 > 0$ such that for all $0 < k \leq k_0$, condition (50) holds.

First we note that k_0 must satisfy the restriction

$$k_0 < \tilde{k} \stackrel{\text{def}}{=} \min\{\, p/(1-\sin\theta)\,,\, p/(1+\sin\theta\,)\}\,, \quad p = \pi/L\,,$$

since $\omega_m(\tilde{k}) = 0$ for $m = 1$ or $m = -1$. Now, for every $k \leq k_0 < \tilde{k}$, we have that

$$k^2\mathsf{C}_\omega^2 = \frac{1}{\cos^2\theta} + \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \frac{k^2}{\alpha_m^2 - k^2} = \frac{2}{\cos^2\theta} + \sum_{m\in\mathbb{Z}} \frac{k^2}{\alpha_m^2 - k^2}\,.$$

Using the formula

$$\sum_{m=1}^{\infty} \frac{1}{m^2 - a^2} = -\frac{\pi}{2a}\cot\pi a + \frac{1}{2a^2}\,, \quad a \notin \mathbb{Z}\,,$$

a simple calculation shows that

$$\sum_{m\in\mathbb{Z}} \frac{1}{\alpha_m^2 - k^2} = -\frac{L}{2k}\,\frac{\sin 2kL}{\sin[kL(1+\sin\theta)]\,\sin[kL(1-\sin\theta)]}\,.$$

From here it follows that

$$k^2\mathsf{C}_\omega^2 = \frac{1}{\cos^2\theta}\left[\,1 + \mathcal{O}(k^2)\,\right]\,, \quad k \to 0\,,$$

and hence

$$k^2\|q\|_\infty B\mathsf{C}_\omega = \frac{\|q\|_\infty B}{\cos\theta}\,k \,+\, \mathcal{O}(k^3)\,, \quad k \to 0\,.$$

Thus, there exists a number $k_0 \in (0,\tilde{k})$ such that (50) holds $\forall k \in (0,k_0)$. In particular, it follows from Lemma 14 in B that the total field $\widetilde{u}$ satisfies

$$\widetilde{u}(\mathbf{x}) = \widetilde{u}^i(\mathbf{x})\,[\,1 + \mathcal{O}(k)\,]\,, \quad \mathbf{x} \in \overline{\Omega}\,.$$

This ends the proof. *Q. E. D.*

Our next step is to use the analytic Fredholm theory and the above results to show that the homogeneous integral equation $(I + \widetilde{T})\widetilde{u} = 0$ possesses nontrivial solutions only for a discrete set of wave numbers k.

Theorem 3. *Let $\theta \in (-\pi/2, \pi/2)$ be a fixed incidence angle. Then, the direct scattering problem (43) – (46) has a unique solution for all wave numbers $k > 0$ except (possibly) a discrete set.*

Proof: We are going to show that for $\theta \in (-\pi/2,\pi/2)$ there exists a neighborhood of the positive real axis k such that $\widetilde{T}$ depends analytically on k. The argument presented here follows similar ideas to those in [5, Theorem 4.21].

We define $\mathcal{K}^\theta$ as

$$\mathcal{K}^\theta \stackrel{\text{def}}{=} \{\, k \in \mathbb{C} \quad | \quad \Re k > 0\,,\ (k,\theta) \in \mathcal{K}\,\}\,,$$

and $h_m(k) \stackrel{\text{def}}{=} k^2 - (pm + k\sin\theta)^2$, where $k = k_1 + ik_2$. From the definition of $\mathcal{K}$ we see that

$$\mathcal{K}^\theta = \{\, k \in \mathbb{C} \quad | \quad k_1 > 0\,\} \setminus \bigcup_{m\in\mathbb{Z}} A_m\,,$$

where

$$A_m \stackrel{\text{def}}{=} \{\, k \in \mathbb{C} \quad | \quad k_1 > 0,\, \Re h_m(k) = 0,\, \Im h_m(k) \le 0\,\}\,.$$

We have that

$$\begin{aligned}\Re h_m(k) &= (k_1^2 - k_2^2)\cos^2\theta - 2k_1 pm\sin\theta - p^2m^2\,,\\ \Im h_m(k) &= 2k_1k_2\cos^2\theta - 2k_2 pm\sin\theta\,.\end{aligned}$$

From the inequality $k_1^2\cos^2\theta - 2k_1pm\sin\theta - p^2m^2 \ge 0$, which is valid for all $k_2 \in \mathbb{R}$ and $k_1 > 0$, we get that

$$k_1 \ge \kappa_m \stackrel{\text{def}}{=} p(1 + \operatorname{sgn}(m)\ \sin\theta)|m|/\cos^2\theta\,.$$

Therefore, $k_1^2\cos^2\theta \ge pm\sin\theta$, and hence the condition $\Im h_m(k) \le 0$ implies $k_2 \le 0$. Therefore, we have shown that

$$A_m = \left\{\, k \in \mathbb{C} \quad | \quad k_1 \ge \kappa_m\,,\ k_2 = -\sqrt{k_1^2 - pm(2k_1\sin\theta - pm)/\cos^2\theta}\,\right\}\,.$$

Thus the set $\mathcal{K}^\theta$ can be obtained by eliminating from the complex half-plane the curves defined by the set A_m. We illustrate some of those curves in Figure 4.

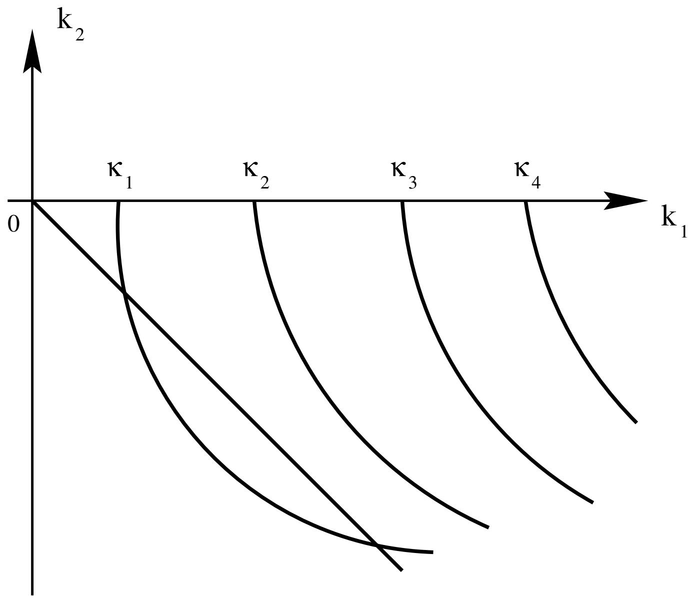

FIGURE 4. Some curves defined by the sets A_m

According to Lemma 9, the compact operator $\widetilde{T}(k,\theta)$ depends analytically on $k \in \mathcal{K}^\theta$. We now choose a connected and open set containing $\mathcal{A} \subset \mathcal{K}^\theta$ which contains the set $\{\, k \in \mathbb{R} \quad | \quad k > 0,\, k \neq \kappa_j,\, j \in \mathbb{Z}_{\ge 0}\,\}$ and apply in such domain Fredholm's analytic alternative [20, Theorem 8.26]. Thus, either $[I + \widetilde{T}(k,\theta)]^{-1}$ does not exist for any $k \in \mathcal{A}$, or $[I + \widetilde{T}(k,\theta)]^{-1}$ exists for every $k \in \mathcal{A}\backslash S$, where S is a discrete subset of $\mathcal{A}$.

Let k_0 be the number found in Lemma 11, and let us note that $(0,k_0]\cap\mathcal{A}\neq\emptyset$. Therefore, the integral equation $(I+\widetilde{T})\widetilde{u}=\widetilde{u}^i$ (and thus the initial scattering problem) possesses a unique solution $\widetilde{u}$ for all $k>0$, except possibly in a discrete set of points. *Q. E. D.*

We are now ready to attack the key theoretical question of the present work, namely the uniqueness of the solution to the inverse scattering problem based on the scattering matrix as data. Throughout the remaining part of this section, we assume that we know the support $\Omega=\Omega_B$ of the perturbation q. This is not a very restrictive condition. We also, assume that we know the scattering matrix $S_{B,k}$ for a fixed incidence angle θ and an interval of wave numbers k.

We recall that for an arbitrary sequence $c\in Y_{k,B}$, we defined an incident field $\widetilde{u}^i$ on a scatterer q_Ω according to the formula

$$\widetilde{u}^i(x,z) \;=\; \sum_{m\in\mathbb{Z}} c_m e^{-i\omega_m z}\,e^{ipmx} \;=\; F_x^{-1}E_{k,z}^{-1}\,c\,,\quad (x,z)\in\Omega_B^-\,.$$

The corresponding scattered field is given by

$$\widetilde{u}^s(x,z) \;=\; \sum_{m\in\mathbb{Z}} d_m e^{i\omega_m z}\,e^{ipmx} \;=\; F_x^{-1}E_{k,z}\,d\,,\quad (x,z)\in\Omega_B^+\,,$$

for some sequence $d\in X_{k,B}$. Thus, the scattering matrix $S_{B,k}$ is the linear mapping $S_{B,k}\,:\,Y_{k,B}\longrightarrow X_{k,B}$ such that $d=S_{B,k}\,c$.

For convenience, we shall normalize the sequences c and d in a bit different way so that the corresponding spaces are independent of k. More precisely, we write

$$\widetilde{u}^i(x,z) \;=\; F_x^{-1}E_{k,z-B}^{-1}\,E_{k,B}^{-1}\,c\,,\quad (x,z)\in\Omega_B^-\,,$$

and

$$\widetilde{u}^s(x,z) \;=\; F_x^{-1}E_{k,z-B}\,E_{k,B}\,d\,,\quad (x,z)\in\Omega_B^+\,.$$

Now we set $\widetilde{c}\overset{\text{def}}{=}E_{k,B}^{-1}\,c$ and $\widetilde{d}\overset{\text{def}}{=}E_{k,B}\,d$. Therefore, the mapping $\widetilde{c}\longmapsto\widetilde{d}$ is given by a matrix $\widetilde{S}_{B,k}\,:\,l^2(\mathbb{Z})\longrightarrow l^2(\mathbb{Z})$. In other words, $\widetilde{S}_{B,k}\;=\;E_{k,B}\,S_{B,k}\,E_{k,B}$.

Evidently the matrices $\widetilde{S}_{B,k}$ and $S_{B,k}$ are related by an invertible linear transformation, thus they possess the same amount of information concerning the scatterer.

It is clear from the expression defining the scattered field that $\widetilde{S}_{B,k}\,\widetilde{c}\,=\,F\widetilde{u}^s\big|_{\Gamma_B}$. Thus, from Lemma 10 we conclude that the matrix $\widetilde{S}_{B,k}$ depends analytically on k and θ for $(k,\theta)\in\mathcal{K}_+$, whenever the scattering problem has a unique solution.

We are now ready to state and prove our main uniqueness result.

Theorem 4. *Suppose that q_1 and q_2 describe two arbitrary non-homogeneous periodic media with x-period $2L$, supported in the region Ω, and such that $\Re q_j>-1$, $\Im q_j\geq 0$ for $j=1,2$. Assume further, that q_1 and q_2 are such that there exists a unique solution to the corresponding direct scattering problem. If $\widetilde{S}_{B,k}(q_1)=\widetilde{S}_{B,k}(q_2)$ for k in an interval (k_1,k_2) of wave-numbers and a fixed $\theta\in(-\pi/2,\pi/2)$, then $q_1(\mathbf{x})=q_2(\mathbf{x})$, $\mathbf{x}\in\Omega$.*

Proof: From the assumption on q_1 and q_2, it follows that for every $k\in(k_1,k_2)$ the problem (43) – (46) has a unique solution. Thus, the corresponding scattering matrices are indeed well defined and bounded. The analyticity of the scattering matrix $\widetilde{S}_{B,k}$ with respect to k implies that $\widetilde{S}_{B,k}(q_1) = \widetilde{S}_{B,k}(q_2)$ for $k \in (0, k_0]$, where k_0 is given by Lemma 11. In fact, we can chose a connected open set $\mathcal{A}'$ where the scattering problem has a unique solution and contains the intervals (k_1, k_2) and $(0, k_0]$. We can now apply the unique continuation principle to $\widetilde{S}_{B,k}$ as a function of k in the domain $\mathcal{A}'$.

Thus $\widetilde{S}_{B,k}(q_1)\,\widetilde{c} = \widetilde{S}_{B,k}(q_2)\,\widetilde{c}$ for any $k \in (0, k_0]$ and sequence $\widetilde{c} \in l^2(\mathbb{Z})$. Such sequence, in turn determines an incident field

$$\widetilde{u}^i(x,z) \;=\; \sum_{m\in\mathbb{Z}} \widetilde{c}_m e^{-i\omega_m(z-B)}\, e^{ipmx}\,, \qquad (x,z) \in \Omega\,.$$

We shall now obtain an asymptotic relation (as $k \to 0$) among the components $\widetilde{S}_{m,n}(k;q)$ of the matrix $\widetilde{S}_{B,k}(q)$ and the perturbation q. From the Lippmann-Schwinger equation for the scattered field $\widetilde{u}^s$, we have that for every $z > B$

$$\widetilde{u}^s(x,z) \;=\; \frac{ik^2}{4L} \sum_{m\in\mathbb{Z}} \omega_m^{-1}\, e^{i\omega_m z} e^{ipmx} \int_\Omega e^{-i\omega_m\eta} e^{-ipm\xi} q(\xi,\eta)\widetilde{u}(\xi,\eta)\, d\xi d\eta\ .$$

But $\widetilde{S}_{B,k}\,\widetilde{c} = F\widetilde{u}^s\big|_{\Gamma_B}$ and the total field $\widetilde{u}$ satisfies

$$\widetilde{u}(\mathbf{x}) = \widetilde{u}^i(\mathbf{x})\,[\,1 + \mathcal{O}(k)\,]\ , \quad \mathbf{x} \in \overline{\Omega}\,.$$

From here we get

$$\widetilde{S}_{m,n}(k;q) \;=\; \frac{ik^2}{4L}\,\omega_m^{-1}\, e^{i(\omega_m+\omega_n)B} \int_\Omega e^{-i(\omega_m+\omega_n)\eta} e^{-ip(m-n)\xi} q(\xi,\eta)\,[1 + \mathcal{O}(k)]\ d\xi d\eta\,.$$

Hence, for every $m \in \mathbb{Z}$, $n \in \mathbb{Z}$, we have

$$\int_\Omega e^{-i(\omega_m+\omega_n)\eta} e^{-ip(m-n)\xi}\, [q_1(\xi,\eta) - q_2(\xi,\eta)]\ [1 + \mathcal{O}(k)]\ d\xi d\eta \;=\; 0. \tag{51}$$

Let $g(\xi,\eta) \overset{\text{def}}{=} q_1(\xi,\eta) - q_2(\xi,\eta)$, $(\xi,\eta) \in \Omega$. We consider the Fourier representation of g in the form

$$g(\xi,\eta) \;=\; \sum_{\ell\in\mathbb{Z}} g_\ell(\eta)\, e^{ip\ell\xi}\,.$$

From (51), we get

$$\int_0^B e^{-i(\omega_m+\omega_{m-\ell})\eta}\, g_\ell(\eta)\,[1 + \mathcal{O}(k)]\ d\eta \;=\; 0, \quad \forall \ell \in \mathbb{Z}, \forall m \in \mathbb{Z}\,.$$

For each $\ell \in \mathbb{Z}$, if $m \neq 0$ and $m \neq \ell$, we let k approach zero to get

$$\int_0^B e^{p(|m|+|m-\ell|)\eta}\, g_\ell(\eta)\, d\eta \;=\; 0, \quad m \neq 0 \text{ and } m \neq \ell\,.$$

Now we focus on the above equality for integer m ($m \neq 0$ and $m \neq \ell$), such that $\ell(m-\ell) \geq 0$. Then, we set $f_\ell(s) \overset{\text{def}}{=} \exp(p|\ell|\eta)\, g_\ell(\eta)$, where $\eta = (2p)^{-1}\ln(1+s)$. The integral now becomes

$$\int_0^{\widetilde{B}} (1+s)^{|m-\ell|-1} f_\ell(s)\, ds \;=\; 0\,, \quad \ell(m-\ell) \geq 0,\ m \neq \ell\ ,$$

where $\widetilde{B} = \exp(2pB) - 1$. From here, we conclude by induction that

$$\int_0^{\widetilde{B}} s^n\, f_\ell(s)\, ds \;=\; 0,$$

for all $n \in \mathbb{Z}_{\geq 0}$, and each $\ell \in \mathbb{Z}$. Therefore, $f_\ell(s) \;=\; 0$, and $g_\ell(\eta) \;=\; 0$ for each $\eta \in (0, B)$. Thus, $q_1(\mathbf{x}) \;=\; q_2(\mathbf{x})$, $\forall \mathbf{x} \in \overline{\Omega}$. *Q. E. D.*

5. Reconstruction Algorithm

The present section concerns the practical reconstruction of the perturbation q from the scattering matrix. Although the uniqueness result of Section 4 is quite reassuring, its applicability in the numerical reconstruction suffers from several drawbacks. First, it is not numerically feasible since it relies on analytic continuation. Secondly, it uses the full scattering matrix, which is impractical because of the measurements associated to the evanescent waves. In fact for each fixed angle θ, we only realistically expect to have access to the information of the scattering matrix $S_{B,k}$ with components $S_{m,n}$ whose indices m, n belong to the set

$$M(k) = \{\, m \in \mathbb{Z} \quad | \quad |\alpha_m| \leq k \,\} \; .$$

These correspond to the plane waves and exclude the evanescent modes.

Motivated by such remarks, the plan for this section is the following: We start by reformulating the inverse problem, then we develop an algorithm for tackling such reformulated problem, and we present a number of numerical examples to validate the suggested algorithm.

5.1. Reformulating the Inverse Problem.

The fact that for each θ and k we only have access to the information of the components of $S_{B,k}$ in the set $M(k)$ tells us that we should not expect to reconstruct more features of q than the cardinality of $M(k)$. In the sequel we follow some ideas of Y. Chen [23] on the reformulation of the inverse problem, whereby only a (low-pass) filtered version of the scatterer is reconstructed. We remark, however, that our situation differs from his, as will be explained in the sequel.

We shall now discuss some aspects of the reformulation of the inverse problem. As before, let us denote by $\Omega_B \;=\; \{\, (x,z) \in \mathbb{R}^2 \quad | \quad 0 < x < 2L,\, 0 < z < B \,\}$ the fundamental period where the perturbation q is defined. Any sequence $c \in Y_B$ defines an incident field u^i by means of

$$u^i(x,z) \;=\; \sum_{m\in\mathbb{Z}} c_m\, e^{-i\omega_m z} e^{i\alpha_m x} \;=\; F^{-1}_{x,\mathrm{qp}} E_z^{-1}\, c \; . \tag{52}$$

The corresponding resulting vector field is given by

$$u^s(x,z) \;=\; \sum_{m\in\mathbb{Z}} d_m\, e^{i\omega_m z} e^{i\alpha_m x} \;=\; F^{-1}_{x,\mathrm{qp}} E_z\, d\,, \tag{53}$$

for some sequence $d \in X_B$. Thus the scattering matrix S_B is the linear transformation defined by $d \;=\; S_B\, c$.

To study this matrix we shall employ the Riccati type matrix equation which was derived in Section 3

$$\frac{dS_z}{dz} = \frac{ik^2}{2}\left(E_z^{-1} + S_z\, E_z\right)\Lambda^{-1}\,\widehat{q}_z\,\left(E_z\, S_z \,+\, E_z^{-1}\right)\ , \tag{54}$$

where here $\widehat{q}_z$ denotes the multiplication operator by $F\, q_z\, F^{-1}$. Let us denote by P the orthogonal projection onto the sub-space of $\ell^2(\mathbb{Z})$ with indices in $M(k)$. Then, for practical scattering experiments the relevant scattering matrix S_z is PS_zP. So we expect,

$$(S_z)_{m,n} \,\approx\, 0,\ (S_z)_{n,m} \,\approx\, 0,\ (E_zS_z)_{n,m} \,\approx\, 0,\ (S_zE_z)_{m,n} \,\approx\, 0, \tag{55}$$

if $m \notin M(k)$, for all $n \in \mathbb{Z}$.

The cardinality of the set $M(k)$ is independent of z, which is the evolution variable in the Riccati equation (54). In the compactly supported scattering case analyzed by Chen [23] the set of "relevant" frequencies depends on the radial variable r, which plays a similar role to z in his work. This is one of the key differences between the problem at hand and that of [23].

Our immediate goal is to obtain some estimates on the decay rate of the different components that enter Equation (54). The first step is the following:

Lemma 12. *Let $\zeta \in (0, B]$ and suppose that the truncated scatterer q_{Ω_ζ} is smooth in $\overline{\Omega}_\zeta$. Let $u^i : \Omega_\zeta \longrightarrow \mathbb{C}$ be the incident field and $u^s : \Omega_{\mathbb{R}} \longrightarrow \mathbb{C}$ the corresponding scattered field. Then, $u^s(x,\zeta)$ has an expansion of the form*

$$u^s(x,\zeta) \,=\, \sum_{m\in\mathbb{Z}} \beta_m\, e^{i\alpha_m x}\,, \quad x \in (0, 2L)\,, \tag{56}$$

where the sequence $(\beta_m)_{m\in\mathbb{Z}} \subset \mathbb{C}$ satisfies

$$|\beta_m| \,\le\, C(k,q)\|u^i\|_\infty/|\omega_m|^2\,, \tag{57}$$

for all $m \notin M(k)$.

Proof: From the Lippmann-Schwinger equation satisfied by the scattered field u^s and the quasi-periodic Green's function formula, we have

$$u^s(x,z) \,=\, \frac{ik^2}{4L}\int_{\Omega_\zeta}\sum_{m\in\mathbb{Z}}\omega_m^{-1}e^{i\omega_m(z-\eta)}e^{i\alpha_m(x-\xi)}\, q(\xi,\eta)u(\xi,\eta)\, d\xi d\eta\,,$$

for all $z > \zeta$. Thus, the restriction $u^s(x,\zeta)$ for $z > \zeta$ satisfies an expansion of the form (56), where

$$\beta_m \,=\, \frac{ik^2}{4L}\int_{\Omega_\zeta}\omega_m^{-1}e^{i\omega_m(\zeta-\eta)}e^{-i\alpha_m\xi}q(\xi,\eta)u(\xi,\eta)\, d\xi d\eta\,.$$

Since we are also assuming that the direct scattering problem is well-posed, there exists a constant $C > 0$ such that $\|u\|_\infty \le C\,\|u^i\|_\infty$. Therefore, from the formula defining β_m, we get for $m \notin M(k)$

$$|\beta_m| \,\le\, \frac{k^2}{2|\omega_m|}C\,\|u^i\|_\infty\,\|q\|_\infty\int_0^\zeta e^{-|\omega_m|(\zeta-\eta)}\, d\eta\ ,$$

which implies (57). *Q. E. D.*

For $n \in M(k)$, we now consider a specific incident field u^i given by $u^i(x,z) = \exp(-i\omega_n z)\ \exp(i\alpha_n x)$, $(x,z) \in \Omega_B$. Since $d = S_z\, c$, we have $d_m = (S_z)_{m,n}$, $m \in \mathbb{Z}$,. Thus,

$$\beta_m \;=\; d_m e^{i\omega_m z} \;=\; (S_z)_{m,n} e^{i\omega_m z} \;=\; (E_z S_z)_{m,n}, \quad m \in \mathbb{Z}, n \in M(k)\,.$$

Therefore, if the scatterer q is smooth in the domain $\overline{\Omega_z}$, from Lemma 12, it follows that there exists a constant $C > 0$ such that

$$|\,(E_z S_z)_{m,n}\,| \;\leq\; C/|\omega_m|^2\,, \tag{58}$$

for all $m \notin M(k)$ and $n \in M(k)$. Using the asymptotic behavior for ω_m, as $|m| \to \infty$, we get from (58) that the elements of the m-th row of the matrix $E_z S_z$ decay as $1/|m|^2$ for $m \notin M(k)$. Therefore, it is plausible to consider that $E_z S_z$ is essentially a square matrix of order $M(k)$. In a totally analogous form, the matrix $S_z E_z$ is also essentially a square matrix of size $M(k)$. The above discussion can be summarized in the following:

Remark 3. *For a scattering experiment, where the main component of the incident wave field is due to propagating plane waves (and not evanescent ones) and where only the propagating scattered modes are measured, the matrices S_z, $E_z S_z$, and $S_z E_z$ are essentially square of order $M(k)$,i.e., the entries with one of the indices not in $M(k)$ are negligible.*

Once we restrict ourselves to propagating modes as in Remark 3, the fundamental question becomes: What information about the scatterer do we expect to recover? More specifically, what Fourier modes of q are encoded in the propagating modes of the scattering matrix? We do not have a complete answer to this question in the fully nonlinear case, that is, if the Born approximation is no longer valid. Even in the Born approximation our approach is mostly heuristic. It will, however, guide our formulation of a numerical reconstruction algorithm. As we shall see, many of the steps lead to mathematical questions of independent interest.

We start off by analyzing the case $k^2\|q\|_\infty B \ll 1$, which corresponds to the sufficient condition for the validity of the Born approximation. In this case, the scattered field can be represented by

$$u^s_{\mathsf{B}}(x,z) \;=\; -k^2 \int_{\Omega_B} G(x,z;\xi,\eta) q(\xi,\eta) u^i(\xi,\eta)\, d\xi d\eta\,, \quad (x,z) \in \Omega_{\mathbb{R}}\,. \tag{59}$$

The error in such approximation is $\mathcal{O}(\|q\|^2_\infty)$ if $\|q\|_\infty \to 0$, and k and B remain fixed. It is $\mathcal{O}(k^2)$ if $k \to 0$, with B and q fixed.

Since u^s_{B} satisfies Rayleigh's condition for $(x,z) \in \Omega_B^+$, there exists a sequence $\hat{d} \in X_B$ such that

$$u^s_{\mathsf{B}}(x,z) \;=\; \sum_{m\in\mathbb{Z}} \hat{d}_m\, e^{i\omega_m z} e^{i\alpha_m x}\,, \quad (x,z) \in \Omega_B^+\,.$$

Thus, from (59) it follows that

$$\hat{d}_m \;=\; \frac{ik^2}{4L} \int_{\Omega_B} \omega_m^{-1} e^{i\omega_m \eta} e^{-i\alpha_m \xi} q(\xi,\eta) u^i(\xi,\eta)\, d\xi d\eta\,.$$

If we use the incident field given by (52), we get

$$\hat{d}_m \;=\; \sum_{n\in\mathbb{Z}} c_n\, \frac{ik^2}{2\omega_m} \int_0^B e^{-i(\omega_m+\omega_n)\eta}\, \widehat{q}_{m-n}(\eta)\, d\eta\,, \quad c \in Y_B\,. \tag{60}$$

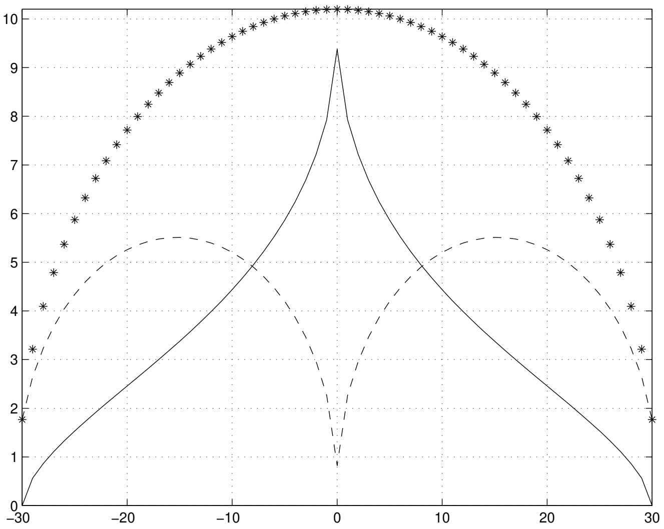

FIGURE 5. Information content of the Born approximation. The asterisks represent the maximum frequency satisfying (62). The dotted line corresponds to the minimum frequency satisfying (62). The continuous line is their difference. The parameters chosen in this graph were $k = 5.1$, $L = 9.4248$, $\theta = 1.0472$.

Equation (60) defines a linear transformation $S_B^{\mathsf{B}} : Y_B \longrightarrow X_B$ that maps c to $\hat{d}$. It is natural to call it the *Born approximation* of the scattering matrix S_B and its components are given by

$$S_{m,n}^{\mathsf{B}} = \frac{ik^2}{2\omega_m} \int_0^B e^{-i(\omega_m+\omega_n)\eta}\, \widehat{q}_{m-n}(\eta)\, d\eta \ ,$$

where $\widehat{q}_l(\eta)$ is the periodic Fourier transform

$$\widehat{q}_\ell(\eta) = \frac{1}{2L} \int_0^{2L} e^{-ip\ell x} q(x,\eta)\, dx\,, \quad \ell \in \mathbb{Z}.$$

If $m, n \in M(k)$, we have

$$S_{m,n}^{\mathsf{B}} = \frac{ik^2}{2\omega_m} (\mathcal{F}\widehat{q}_{m-n})(\omega_m + \omega_n)\,, \tag{61}$$

where $\mathcal{F} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is the Fourier transform on the line given by

$$(\mathcal{F}g)(\chi) = \int_{\mathbb{R}} e^{-i\chi\eta} g(\eta)\, d\eta\,, \quad \chi \in \mathbb{R}.$$

Equation (61) indicates, as expected, that under the Born approximation conditions the scattering matrix restricted to the plane wave modes is essentially a two-dimensional Fourier transform of q. It is given by $(\mathcal{F}\widehat{q}_\ell)(\chi)$, where $\ell = m - n$ and $\chi = \omega_m + \omega_n$, where $m, n \in M(k)$. It becomes apparent in the plane wave regime that the practical measurements do not contain information of the higher Fourier modes, be it in the x or in the z direction. Also, for sufficiently small $k > 0$, the scattering matrix provides only the average of q on a period, i.e., $(\mathcal{F}\widehat{q}_0)(0)$.

Let us denote by N the cardinality of $M(k)$ and analyze the information content available to us under the Born approximation. This corresponds to $(\mathcal{F}\widehat{q}_\ell)(\chi_{\ell,m})$ where $(\ell, \chi_{\ell,m})$ satisfies

$$\left\{\begin{array}{l} \chi_{\ell,m} = \omega_m + \omega_{m-\ell}, \text{ where} \\ -(N-1) \le \ell \le N-1 \ , m \in M(k), \text{ and } m-\ell \in M(k)\,. \end{array}\right. \tag{62}$$

Remark that as $|\ell|$ increases the set of χ's for which we have access to $(\mathcal{F}\widehat{q}_\ell)(\chi)$ decreases. In particular, for $\ell = \pm(N-1)$ our Born approximation data is restricted only to $\chi = 2\omega_{(N-1)/2}$ (if $\theta = 0$). See Figure 5, which depicts for different ℓ's the range of frequencies (in the Fourier space associated to z) that are available in the plane waves of the scattering matrix. It displays a trade-off between vertical and horizontal number of frequencies.

We shall now address the fundamental question for numerical solution of the inverse problem: How much detail do we expect to obtain in our inverse problem solution? In other words, what are the grid parameters for the reconstructed q? For that, we focus on the mode associated to $\ell = 0$, which is the one that possesses the largest information content. For simplicity, we introduce $\hat{s}_m \stackrel{\text{def}}{=} -2i\omega_m S^{\mathsf{B}}_{m,m}/k^2$ for $m \in M(k)$, so that

$$\hat{s}_m = \int_0^B e^{-i\cdot 2\omega_m z}\widehat{q}_0(z)\,dz\,, \quad m \in M(k)\,. \tag{63}$$

We recall from Shannon's theorem that a function $f \in L^2(\mathbb{R})$ whose Fourier transform is supported in $[-\varpi, \varpi]$ can be uniquely reconstructed [1] from its samples $f_n = f(n\Delta x)$ if, and only if, $\triangle x \le \pi/\varpi$. Furthermore, in this case

$$f(x) = \sum_{n\in\mathbb{Z}} f_n \,\mathrm{sinc}(\varpi x - n\pi)\,,$$

where $\mathrm{sinc}(x) = \sin(x)/x$. See [24, 25] and references therein. From equation (63) we see that the Fourier transform of $\widehat{q}_0$ is supported on the interval $[0, B]$. Thus by Shannon's theorem, the largest sample distance we should use in the frequency domain is [2]

$$\triangle = \frac{\pi}{\varpi} = \frac{2\pi}{B}\,.$$

On the other hand, from the definition of ω_m and Equation (63) it follows that the Fourier transform of $q(x, \cdot)$ can be evaluated at most in the interval $I = (0, 2k]$ by varying the incidence angle θ. This indicates that the number of (complex valued) samples necessary to represent the scattering measurements is given by

$$N \sim \frac{|I|}{\triangle} = \frac{kB}{\pi}\,. \tag{64}$$

Since the function $\widehat{q}_0$ is essentially supported on the interval $[0, B]$ and in the frequency domain $(0, 2k)$, we expect to have about N degrees of freedom in the z variable for each fixed x. More measurements than that would not lead to much more information about the scatterer. We stress that such a reasoning is heuristic since we are only considering the mode $\ell = 0$ and the frequencies

[1] Strictly speaking we are talking about the smooth representative of the equivalence class of f when discussing samples of f. Such representative exists because $\mathrm{supp}(\widehat{f}) \subset [-\varpi, \varpi]$.

[2] The reader should be attentive that here we are referring to samples in the frequency domain since the function is of finite support in the physical domain.

ω_m are not equally spaced in the interval $(0,k)$. Although Shannon-like results for non-equally spaced Fourier transforms do exist [26] the technical conditions needed for such results limit its applicability. The situation is further complicated by the fact that for modes $\ell \neq 0$ the support of the Fourier transform of $\widehat{q}_\ell$ is shifted and decreases as described in Figure 5. In what follows, we shall use the *heuristic* choice of

$$N_z \sim \frac{|I|}{\triangle/2} = \frac{2kB}{\pi}. \tag{65}$$

It is twice the number of samples given in (64), which corresponds to an oversampling. This can be justified by the fact that in the non-equally spaced case the minimum sampling size (associated to the Nyquist rate) is roughly twice smaller than that of the equally spaced case. This issue, and the fact that all the above analysis was performed for the linearized (Born approximation) case indicates the richness of the subject and the need for further investigation of its the theoretical aspects. Furthermore, Shannon-type results for the scattering transform in one dimension are the subject of recent work [27] and in our opinion a rigorous treatment would require a possible generalization to two dimensions of the results in [27].

One can also argue heuristically, using Remark 3 and the representation of the scattering matrix given by

$$S_B = \frac{ik^2}{2}\int_0^B \left(E_z^{-1} + S_z\,E_z\right)\Lambda^{-1}\,\widehat{q}_z\,\left(E_z\,S_z + E_z^{-1}\right)\,dz\,,$$

that both in the linearized and in the nonlinear case the information available in practical experiments from the scattering matrix is associated to Fourier modes in the set

$$D(k) \stackrel{\text{def}}{=} \{\,(\ell,\chi)\in\mathbb{Z}\times\mathbb{R} \quad | \quad -N<\ell<N,\ \ 0\le\chi\le 2k\,\}\,. \tag{66}$$

where $N \sim 2kL/\pi$.

The Inverse Prob. Reformulated: *Reconstruct the scatterer's Fourier modes that belong to $D(k)$ from the scattering matrix entries with indices in $M(k)$ for an arbitrary (but finite) set of frequencies k and incidence angles θ.*

Our practical inversion process will therefore be concerned with finding a certain $\check{q}$ (not necessarily the same as q) so that it produces the *truncated* scattering matrix. Furthermore, in order to regularize the problem we shall require that the L^2 norm of our model $\check{q}$ be as small as possible. We remark that the reformulated inverse problem depends of the frequency k used in the experiment since it uses the Fourier modes of q in the set $D(k)$.

We shall denote by q_k the low-pass filtered version of q. More precisely,

$$(\widehat{q}_k)_\ell \stackrel{\text{def}}{=} \begin{cases} \widehat{q}_\ell(\chi), & (\ell,\chi)\in D(k), \\ 0, & (\ell,\chi)\notin D(k). \end{cases} \tag{67}$$

A natural question is whether the dependence of $\widehat{q}_k$ is continuous with respect to changes of k. Indeed, it can be shown that if $\delta k > 0$ is a small perturbation of k so that $\omega_m(k') \neq 0$ for all $k' \in [k, k+\delta k]$ and $m \in M(k)$, then $\widehat{q}_k$ depends continuously on δk.

We shall employ the above reformulation of the inverse problem together with Remark 3 so that we use only the entries of $S_{B,k}(q)$ that belong to $M(k)$. Such truncated scattering matrix approach that uses a low-frequency model for q was successfully employed by Y. Chen in the compactly supported case [23].

5.2. **The Recursive Linearization Algorithm.** The algorithm we shall now propose starts recovering the lowest possible Fourier modes of q that are available within the Born approximation. It then progresses by incorporating higher frequencies and goes well into the nonlinear region until a sufficiently fine resolution of the scatterer is obtained. In order to do that and to avoid values of $\omega_m(k) = 0$ it makes use of different incidence angles.

We shall first describe the algorithm for a fixed angle θ and later discuss how to change such angle so as to increase the detail of the reconstructed scatterer. Let us assume that for a fixed θ and for $k \in [k_i, k_f]$ the Direct Scattering Problem 1 possesses a unique solution and that $S_{B,k}$ is well defined. In particular, this entails that $\omega_m(k) \neq 0$ for all $m \in \mathbb{Z}$ and $k \in [k_i, k_f]$. We shall assume also that for k_i the Born approximation holds, so that we could start the algorithm with the lowest frequency components of the scatterer by a least-squares or a conjugate-gradient method.

Since we are approximating the scatterer q by its low-pass version $\widehat{q}_k$, the scattering matrix will be computed from the differential equation

$$\begin{aligned} \frac{dS_{z,k}}{dz} &= \frac{ik^2}{2}\left(E_{k,z}^{-1} + S_{z,k}\,E_{k,z}\right)\Lambda^{-1}\,(\widehat{q}_k)_z\,\left(E_{k,z}\,S_{z,k} + E_{k,z}^{-1}\right), && (68)\\ S_{0,k} &= 0\,. && (69) \end{aligned}$$

Strictly speaking, the scattering matrix satisfies Equation (54) and this provides us with an approximation accuracy that should be analyzed in further detail.

Our proposed procedure consists in finding the approximate models q_{k_j}, $j = 1, 2, \cdots$, for the the frequencies $k_1 < k_2 < \cdots \in [k_i, k_f]$ as follows:

1. For $k = k_1$ we assume that the Born approximation holds and therefore Equations (68) and (69) can be substituted by the linear problem:

$$\begin{aligned} \frac{dS_{z,k_1}}{dz} &= \frac{ik_1^2}{2}\,E_{k_1,z}^{-1}\,\Lambda^{-1}\,(\widehat{q}_{k_1})_z\,E_{k_1,z}^{-1}\,,\\ S_{0,k_1} &= 0\,. \end{aligned}$$

   Its explicit solution is given by

$$S_{B,k_1} = \frac{ik_1^2}{2}\,\Lambda^{-1}\int_0^B E_{k_1,z}^{-1}\,(\widehat{q}_{k_1})_z\,E_{k_1,z}^{-1}\,dz\,, \qquad (70)$$

   whence we calculate the scatterer q_{k_1} from the *measured* S_{B,k_1}.
2. Assume now that our scatterer $q_{\widetilde{k}}$ has already been reconstructed for a certain $\widetilde{k} \in (k_i, k_f)$. If $k > \widetilde{k}$ is a wave number close to $\widetilde{k}$ our goal now is to determine q_k, or in other words the difference

$$\delta q \stackrel{\text{def}}{=} q_k - q_{\widetilde{k}}\,. \qquad (71)$$

   From the continuous dependence of q_k on k we develop a perturbative analysis of Equation (68) so as to obtain a linear equation for δq after neglecting the higher order terms. More precisely, using the scatterer $q_{\widetilde{k}}$ for the wave-number k we get the matrix $\hat{S}_{z,k}$, $z \in [0,B]$, as a solution of the initial value problem

$$\frac{d\hat{S}_{z,k}}{dz} = \frac{ik^2}{2}\left(E_{k,z}^{-1} + \hat{S}_{z,k}\,E_{k,z}\right)\Lambda^{-1}\,(\widehat{q_{\widetilde{k}}})_z\,\left(E_{k,z}\,\hat{S}_{z,k} + E_{k,z}^{-1}\right), \tag{72}$$

$$\hat{S}_{0,k} = 0\,. \tag{73}$$

We do not expect the matrix $\hat{S}_{z,k}$ to differ substantially from $S_{z,k}$, which satisfies (68) and (69). Indeed, the scatterer q_k leads to essentially the same scattering measurements as $q_{\widetilde{k}}$ since the distance between k and $\widetilde{k}$ is assumed to be small. This however requires further investigation.

Let us set $(\delta S)_{z,k} \overset{\text{def}}{=} S_{z,k} - \hat{S}_{z,k}$. We subtract (72) from (68) and neglect the higher order terms in δq and $(\delta S)_{z,k}$, we get an equation of the form

$$\frac{d(\delta S)_{z,k}}{dz} = P_x(z)\,(\delta S)_{z,k} + (\delta S)_{z,k}P_y(z) + D(z)\,, \tag{74}$$

where the operators $P_x(z) : X_{k,z} \to X_{k,z}$, $P_y(z) : Y_{k,z} \to Y_{k,z}$ e $D(z) : Y_{k,z} \to X_{k,z}$ are defined by the expressions

$$\begin{aligned}
P_x(z) &= \frac{ik^2}{2}\left(E_{k,z}^{-1} + \hat{S}_{z,k}\,E_{k,z}\right)\Lambda^{-1}\,(\widehat{q_{\widetilde{k}}})_z\,E_{k,z}\,,\\
P_y(z) &= \frac{ik^2}{2}E_{k,z}\,\Lambda^{-1}\,(\widehat{q_{\widetilde{k}}})_z\,\left(E_{k,z}\,\hat{S}_{z,k} + E_{k,z}^{-1}\right),\\
D(z) &= \frac{ik^2}{2}\left(E_{k,z}^{-1} + \hat{S}_{z,k}\,E_{k,z}\right)\Lambda^{-1}\,(\widehat{\delta q})_z\,\left(E_{k,z}\,\hat{S}_{z,k} + E_{k,z}^{-1}\right).
\end{aligned}$$

Furthermore, it is clear that

$$(\delta S)_{0,k} = 0\,. \tag{75}$$

The solution of the above linear equation can be represented in the form

$$(\delta S)_{z,k} = \int_0^z Q_L(\tau,z;P_x(\tau))\,D(\tau)Q_R(\tau,z;P_y(\tau))\,d\tau\,,$$

where $Q_L(\tau,z;P_x(\tau))$ and $Q_R(\tau,z;P_y(\tau))$ satisfy

$$\frac{\partial Q_L(\tau,z;P_x(\tau))}{\partial z} = P_x(z)\,Q_L(\tau,z;P_x(\tau))\,, \tag{76}$$

$$Q_L(\tau,\tau;P_x(\tau)) = I\,, \tag{77}$$

and

$$\frac{\partial Q_R(\tau,z;P_y(\tau))}{\partial z} = Q_R(\tau,z;P_y(\tau))\,P_y(z)\,, \tag{78}$$

$$Q_R(\tau,\tau;P_y(\tau)) = I\,. \tag{79}$$

The perturbation $(\widehat{\delta q})_z$ satisfies for $z = B$ the integral equation

$$\frac{ik^2}{2}\int_0^B Q_L(z,B;P_x(z))\cdot\left(E_{k,z}^{-1} + \hat{S}_{z,k}\cdot E_{k,z}\right)\cdot\Lambda^{-1}\cdot(\widehat{\delta q})_z \;\times \tag{80}$$

$$\times\left(E_{k,z}\cdot\hat{S}_{z,k} + E_{k,z}^{-1}\right)\cdot Q_R(z,B;P_y(z))\,dz = (\delta S)_{B,k}\,.$$

We rewrite the above equation in the form

$$\mathcal{L}_k(\delta q) = (\delta S)_{B,k} , \tag{81}$$

where the operator $\mathcal{L}_k : L^2(\Omega_B) \longrightarrow \mathcal{B}(Y_{k,z}, X_{k,z})$ is defined by the left hand side of equation (80) and $\mathcal{B}(Y_{k,z}, X_{k,z})$ denotes the space of bounded linear operators from $Y_{k,z}$ to $X_{k,z}$.

Our algorithm for the interval $[k_i, k_f]$ consists in iterating the second step above using a least-squares solution to problem (81) for each of the frequencies. We remark that since we are using only the plane wave entries of the scattering matrix the rank of the operator $\mathcal{L}_k$ is finite and the solution of (81) can be treated as a finite dimensional least-squares problem.

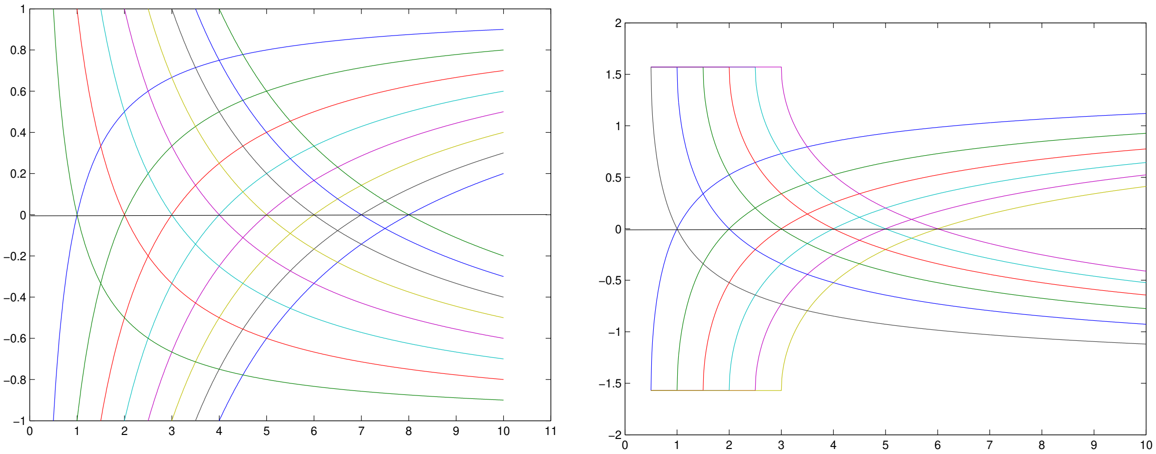

FIGURE 6. Some forbidden curves in the k-$\sin(\theta)$ plane (left) and the k-θ plane (right). The horizontal scale is kL/π.

We shall now discuss how to change the incidence angle θ so as to increase the detail of the reconstruction. Recall that for each fixed angle θ the existence of the scattering matrix depends on the following non-resonance condition to be satisfied:

$$\omega_m \neq 0 \ , \ \forall m \in \mathbb{Z} \ .$$

This means that the curves defined by

$$k = \frac{\pi\, m}{L(1 - \sin\theta)} \, , m \in \mathbb{Z}_{\geq 0} \, , \quad k = -\frac{\pi\, m}{L(1 + \sin\theta)} \, , m \in \mathbb{Z}_{\leq 0} \, ,$$

describe the values of $(k, \theta) \in (0, \infty) \times (-\pi/2, \pi/2)$ where the scattering matrix does not exist. Figure 6 presents some of such curves. If we were to try to increase the frequency k indefinitely and keep the incidence angle θ we would cross one of the forbidden curves and the approximations used in the above heuristic discussion could fail since the recursive linearization algorithm described above relies on the continuous dependence of the scattering matrix with the frequency. To circumvent such problem we proceed as follows:

Let us assume that we have already performed the reconstruction for k in the interval $[k_i, k_f]$ for an angle θ. We now want to go on with a new angle θ_1. If the angle θ_1 does not increase the number of plane waves associated with (k, θ_1) we use q_{k_f} as initial guess to continue the reconstruction along the interval $[k_f, k_g]$. Otherwise, let θ_1 be an incidence angle different from θ so that the number of plane waves associated to (k, θ_1) is larger. In this case the level of detail of the reconstructed scatterer should also increase with the new angle. We now perform an interpolation of the computed values of q_{k_f} so as to obtain a new model $\tilde{q}_{k_f}$ associated to the angle θ_1. This value of $\tilde{q}_{k_f}$ will be used as initial guess to continue the reconstruction algorithm on a new admissible interval $[k_f, k_g]$,.i.e., an interval $[k_f, k_g]$ such that the line joining (k_f, θ_1) to (k_g, θ_1) does not intersect any of the forbidden curves.

It is easy to convince oneself that there are several ways of implementing a protocol to choose a new angle θ_1 based on a given admissible interval $[k_i, k_f]$ for an angle θ. One such protocol was implemented in [28] although this is a direction where several ideas could be tested and deserves further investigation.

5.3. **Numerical Results.** In this section we present some numerical results so as to validate the algorithm that was suggested in the previous one. The algorithm was implemented in MATLAB (version 5.2.0) and tests were performed on a Sun ULTRA workstation with 405.6MHz CPU and typical (system) time was of the order of two hours.

In all the examples shown below the scatterer q is periodic with period $2L = 6\pi$ and supported on an interval $[0, \pi]$ of the z axis. The initial frequency $\widetilde{k} = 5.04$ and the initial angle will depend on the specific example.

The inversion algorithm requires different values of the scattering matrix $S_{B,k;\theta}$ for several values of k and θ. The initial value problem satisfied by the scattering matrix was solved using a Runge-Kutta sub-routine provided by MATLAB (ode45). Such values were computed with component-wise absolute error of 10^{-6} and relative error 10^{-3}. Such values were then corrupted by noise of different levels (according to the example at hand) to test robustness of the algorithm.

The inverse problem requires the solution of systems of equations such as (81), which was performed by the conjugate-gradient method.

The following conventions were used in the examples shown below:

- The exact scatterers were described by continuous lines and the reconstructed ones by dotted lines.
- To assess the stability and robustness of the reconstruction algorithm we corrupted the measurements of the scattering matrices by noise according to the expression:

  $$S^{\nu}_{m,n} = S_{m,n}\left[1 + \nu(2r_{m,n} - 1)\right], \quad m, n = 1, \cdots, N,$$

  where $S_{m,n}$ denotes the corresponding component of the scattering matrix $S_{B,k}$ computed with a high accuracy method, $r_{m,n}$ is a (pseudo) random variable uniformly distributed in the interval $(0, 1)$, and ν is the noise level which took values from 10^{-3} to 10^{-1}.
- The reconstruction error presented is the relative error computed by the expression

  $$e_2 = \frac{\|q_k - q\|}{\|q\|},$$

  where q_k is the scatterer computed at the final frequency k and $\|\cdot\|$ is the ℓ^2 inner product on the grid associated to q_k.

Example 1. The scatterer shown in Figure 7 is given by $q_1(x,z) = f(x) \cdot g(z)$, where

$$f(x) \;=\; e^{-6(\frac{x}{L}-1)^2}\,, \quad g(z) \;=\; 0.1\, e^{-2(z-\frac{B}{2})^2}\,.$$

A few snapshots of the reconstruction development are presented in Figure 8. The final result in the presence of noise is displayed in Figure 9. The reconstruction is fairly good and reasonably robust to noise.

Example 2.1. In the next examples we assess the effect of oscillation in the scatterer. The scatterer shown in Figure 10 is given by $q_2(x,z) = f(x) \cdot g(z)$, where

$$f(x) \;=\; \cos(\pi x/L)\,, \quad g(z) \;=\; 0.1 e^{-2(z-B/2)^2}\,.$$

Final sections of the reconstructed and true scatterer are presented in Figure 11.

Example 2.2. We now consider a more oscillatory scatterer. The scatterer shown in Figure 12 is given by $q_3(x,z) = f(x) \cdot g(z)$, where

$$f(x) \;=\; \cos(2\pi x/L)\,, \quad g(z) \;=\; 0.1 e^{-2(z-B/2)^2}\,.$$

Final sections of the reconstructed and true scatterer are presented in Figure 13.

Example 2.3. We now consider an even more oscillatory scatterer. The scatterer is given by $q_4(x,z) = f(x) \cdot g(z)$, where

$$f(x) \;=\; \cos(4\pi x/L)\,, \quad g(z) \;=\; 0.1 e^{-2(z-B/2)^2}\,.$$

In this case the reconstruction algorithm seems to be successful only for certain angles. In the case shown, the initial angle was $\theta_1 = 0.1$. The corresponding results are shown in Figures 14 and 15.

6. Concluding Remarks

In this work we have analyzed the inverse scattering problem for the two-dimensional Helmholtz equation with a periodic perturbation in one of the directions and compact support in the other. This was motivated by applications to scattering of electromagnetic radiation by periodic structures such as thin-films and nano-technologies. The model, however, could be applied to other physical contexts, such as acoustic waves, as long as the underlying equation is Helmholtz.

We have shown that under certain hypothesis on the scatterer, the inverse problem has a unique solution provided the scattering matrix is known on a non-empty interval of frequencies. This corresponds to the so-called multi-frequency case. Important recent results for the inverse transmission scattering problem for periodic media were obtained by Strycharz [19] but to our knowledge the result presented here is the only one that deals with the scattering matrix in the inhomogeneous periodic case.

One natural follow up question to the present work is whether uniqueness still holds for the single frequency case. Even for the compactly supported (two-dimensional) scatterers the inverse problem is not completely settled. See for example [11, 29, 30].

We have also derived a Riccati-type operator equation for the scattering matrix which is of interest on its own. This was used in deriving a numerical method for the solution of the inverse problem.

On the numerical side, we have proposed a method to tackle the inverse problem. This method although derived on heuristic grounds seemed to perform well. In particular, we have validated it for a large number of examples, a few of which are presented in Section 5. In particular, it was possible to go well beyond the linear regime imposed by the Born approximation in some examples. In particular for Example 1 in Section (5.3) the product $k||q||_\infty B\mathsf{C}_\omega$ was of the order 20. This is way beyond the expected validity of the Born approximation thus showing that we are indeed in the nonlinear inverse scattering regime. Another point is that the algorithm converged with up to 10% noise. We also investigated its deterioration if the scatterer is highly oscillatory. A more detailed report will be presented elsewhere [31].

A few natural avenues to pursue beyond this work are:

- The inverse problem for bi-periodic structures. Some interesting results were obtained in [32, 33].
- The so-called conical diffraction problem, where the wave vector of the incoming electromagnetic radiation is not in the $x - z$ plane as was our case. The direct problem has been considered in [34].
- A similar approach as the one presented here but for transversal magnetic waves. This is of interest as can be seen in [35, 4].
- A convergence theory for the algorithm proposed here.

Acknowledgments

J.P.Z. was supported by FAPERJ through grant number E-26/172.009 and by CNPq through grant number 30.1003. and by PRONEX 76.97.1008-00. J.P.Z. also acknowledges the participation at the MGSS which was supported through grant NSF DMS97-09320. This work was concluded at MSRI which is supported in part by NSF grant DMS-9810361. The authors thank B. Strycharz for providing a copy of reference [5].

Appendix A. The Quasi-periodic Green's Function

In the appendix we shall collect a number of technical results that were used throughout the paper.

We start of by recalling the notion of Green's function for the Helmholtz equation in $\mathbb{R}^2$. Let $\boldsymbol{\xi} = (\xi, \eta)$ be the point source location and let us use the auxiliary variables $X = x - \xi$, $Z = z - \eta$, and $r = \sqrt{X^2 + Z^2}$. The Green's function for the two dimensional free-space Helmholtz problem is the solution $\Phi(X, Z)$ of the problem

$$(\Delta + k^2)\Phi(X, Z) = -\delta(X)\delta(Z) \tag{82}$$

that satisfies Sommerfeld's radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial \Phi}{\partial r} - ik\Phi\right) = 0,$$

uniformly in all the directions $(X, Z)/r$. It is given by the expression

$$\Phi(X, Z) = -\frac{i}{4} H_0(kr)\,, \tag{83}$$

where H_0 denotes Hankel's function $H_0^{(1)}$. See [20].

For the Scattering Problem 1 we have to modify the above Green's function in order to make it a quasi-periodic function with quasi-periodicity parameter $\alpha = k \sin\theta$. The quasi-periodic Green's function $G(X, Z)$ satisfies

$$(\Delta + k^2) G(X, Z) = -\sum_{m\in\mathbb{Z}} \delta(X - 2Lm) e^{i\alpha\, 2Lm}\, \delta(Z).$$

Furthermore, $G \exp(-i\omega t)$ must behave as an outgoing wave as $|Z| \to \infty$ for $\omega = k/c$.

By taking a superposition of free-space Green's functions we get [36] the expression for G:

$$G(X, Z) = -\frac{i}{4} \sum_{m\in\mathbb{Z}} H_0(kr_m)\, e^{i\alpha\, 2Lm}, \tag{84}$$

where $r_m \stackrel{\text{def}}{=} \sqrt{(X - 2Lm)^2 + Z^2}$. Let us assume that $\omega_m \neq 0$ for all m, where ω_m is given by Equation (11). Using Poisson's summation formula and one of the integral representations of Hankel's function H_0 we get [36]

$$G(X, Z) = -\frac{i}{4L} \sum_{m\in\mathbb{Z}} \frac{1}{\omega_m} e^{i\omega_m |Z|} e^{i\alpha_m X}. \tag{85}$$

We summarize the properties of the quasi-periodic Green's function in the following:

Lemma 13. *If $\omega_m \neq 0$ for all m, then*

$$\int\limits_{[0,2L]\times[0,B]} |G(x - \xi, z - \eta)|^2 d\xi d\eta \leq \frac{B}{8L} \sum_{m\in\mathbb{Z}} \frac{1}{|\omega_m|^2}\,. \tag{86}$$

In the region $\Omega_{\mathbb{R}} \stackrel{def}{=} (0, 2L) \times \mathbb{R}$, G admits the representation

$$G(X, Z) = \Phi(X, Z) + \Psi(X, Z), \tag{87}$$

where Φ is given by (83), and Ψ is analytic on $\Omega_{\mathbb{R}} \times \Omega_{\mathbb{R}}$.

Proof: Let us consider the functions

$$g_m(z) \stackrel{\text{def}}{=} \int_0^B \left| \omega_m^{-1} e^{i\omega_m |z-\eta|} \right|^2 d\eta\,, \quad m \in \mathbb{Z}.$$

If $m \in M(k)$ then $g_m(z) = B/|\omega_m|^2\,, z \in \mathbb{R}$. If $m \notin M(k)$ then

$$g_m(z) \leq |\omega_m|^{-3} (1 - \exp(-|\omega_m| B)) \leq |\omega_m|^{-2} B, \quad \forall z \in \mathbb{R}.$$

Thus, for any positive integer M we have:

$$\int_\Omega \left| \sum_{m=-M}^{M} \omega_m^{-1}\, e^{i\omega_m |z-\eta|} e^{i\alpha_m (x-\xi)} \right|^2 d\xi d\eta = 2L \sum_{m=-M}^{M} g_m(z) \;\leq\; 2L \sum_{m\in\mathbb{Z}} g_m(z)\,,$$

which implies (86).

To show the second claim we remark that asymptotically we have

$$\begin{aligned}\omega_m &= ip|m| + i\mathrm{sgn}(m)k\sin\theta + \mathcal{O}\left(\frac{1}{|m|}\right), \quad |m| \to \infty, \\ \omega_m^{-1} &= -\frac{i}{p|m|} + \mathcal{O}\left(\frac{1}{|m|^2}\right), \qquad |m| \to \infty.\end{aligned}$$

Now,

$$\begin{aligned}\widetilde{G}(X,Z) &\stackrel{\text{def}}{=} -\frac{i}{4L}\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} -\frac{i}{p|m|}e^{-(p|m|+\mathrm{sgn}(m)k\sin\theta)|Z|}e^{i\alpha_m X} = \\ &= -\frac{e^{k\sin\theta w/p}}{4\pi}\sum_{m=1}^{\infty}\frac{e^{mw}}{m} - \frac{e^{-k\sin\theta\overline{w}/p}}{4\pi}\sum_{m=1}^{\infty}\frac{e^{m\overline{w}}}{m} = \\ &= -\frac{1}{2\pi}\Re\sum_{m=1}^{\infty}\frac{e^{mw}}{m} + \Psi_1(X,Z),\end{aligned}$$

where $w \stackrel{\text{def}}{=} p[iX - |Z|]$, and Ψ_1 is a continuous function. Hence, we can write G in the form

$$G(X,Z) = \widetilde{G}(X,Z) + (G(X,Z) - \widetilde{G}(X,Z)) = \frac{1}{2\pi}\ln\sqrt{(X^2+Z^2)} + \Psi_2(X,Z),$$

where Ψ_2 is continuous on $\Omega_{\mathbb{R}} \times \Omega_{\mathbb{R}}$. Thus G possess the same type of singularity at $(0,0)$ as the free-space Green's function given in (83). Since G and Φ satisfy the Helmholtz equation on $\Omega_{\mathbb{R}}$ the representation (87) follows for some analytic function Ψ on $\Omega_{\mathbb{R}} \times \Omega_{\mathbb{R}}$. *Q. E. D.*

Remark 4. *The series in the definition of the quasi-periodic Green's function G is absolutely convergent in the set $\{(x,z;\xi,\eta) \in \Omega_{\mathbb{R}} \times \Omega_{\mathbb{R}} \mid z \neq \eta\}$ and uniformly convergent in compact subsets thereof.*

Appendix B. Existence of Solutions

In order to show the existence of solutions to the direct scattering problem posed in Definition 1 we shall reformulate the original problem as a Fredholm integral equation of second kind. The quasi-periodic Green's function allows us to define the *volume potential*

$$v(x,z) \stackrel{\text{def}}{=} k^2\int_\Omega G(x,z;\xi,\eta)q(\xi,\eta)u(\xi,\eta)d\xi d\eta, \quad (x,z) \in \Omega_{\mathbb{R}},$$

where u belongs to the set $C_{\mathrm{qp}}(\overline{\Omega})$ of quasi-periodic continuous functions with fundamental domain $\overline{\Omega}$ and quasi-periodicity parameter $\alpha = k\sin\theta$.

It turns out that because of the representation (87), the classical properties of volume potentials such as Theorem 8.1 of [20], Chapter 4 of [37], and Theorem 5.7 [38] remain valid in the quasi-periodic context.

Theorem 5. *Suppose that $\omega_m \neq 0$, $\forall m \in \mathbb{Z}$. Then, if $u \in \mathcal{V}$ is a solution of the scattering problem in Definition 1 then $u|_{\overline{\Omega}}$ satisfies the Lippmann-Schwinger integral equation*

$$u(x,z) = u^i(x,z) - k^2\int_\Omega G(x,z;\xi,\eta)q(\xi,\eta)u(\xi,\eta)d\xi d\eta, \quad (x,z) \in \overline{\Omega}. \tag{88}$$

Conversely, if $u \in C_{\rm qp}(\overline{\Omega})$ is a solution of equation (88) then u can be extended from the RHS of (88) for $(x,z) \in \Omega_{\mathbb{R}} \backslash \overline{\Omega}$ and the function so obtained belongs to $\mathcal{V}$ and satisfies the direct scattering problem of Definition (1).

Proof: Let $u \in \mathcal{V}$ be a solution of scattering problem in Definition 1, and take $v : \Omega_{\mathbb{R}} \to \mathbb{C}$ defined by

$$v(x,z) = u^i(x,z) - k^2 \int_\Omega G(x,z;\xi,\eta) q(\xi,\eta) u(\xi,\eta) d\xi d\eta \,, \quad (x,z) \in \Omega_{\mathbb{R}} \,.$$

Since u is quasi-periodic (with period α) and $u \in \mathcal{V}$ it follows that u belongs to the class of Hölder continuous quasi-periodic functions $C^{\epsilon}_{\rm qp}(\Omega)$ for some exponent $\epsilon \in (0,1]$. Therefore, using the properties of the volume potential it follows that $v \in C^{1,\epsilon}_{\rm qp}(\overline{\Omega_{\mathbb{R}}}) \cap C^{2,\epsilon}_{\rm qp}(\Omega)$, so that

$$\Delta v + k^2 v = -k^2 q u = \Delta u + k^2 u \,, \quad (x,z) \in \Omega.$$

Obviously, $v\big|_{\overline{\Omega^\pm}} \in C^2_{\rm qp}(\Omega^\pm)$ is a solution of the Helmholtz equation $\Delta v + k^2 v = 0$ in $\Omega^\pm$. Let $w \stackrel{\rm def}{=} u - v$. Then, $w \in C^1_{\rm qp}(\overline{\Omega_{\mathbb{R}}})$, $\Delta w + k^2 w = 0$ in $\Omega \cup \Omega^+ \cup \Omega^-$, and

$$w(x,z) = u^s(x,z) + k^2 \int_\Omega G(x,z;\xi,\eta) q(\xi,\eta) u(\xi,\eta) d\xi d\eta \,, \quad (x,z) \in \Omega_{\mathbb{R}} \,.$$

Thus, w satisfies the Rayleigh expansion for $(x,z) \in \Omega^\pm$. Now, making use of the explicit form of the solution in the regions Ω, Ω^+ and Ω^-, the continuity of both w and $\partial w/\partial z$ across the boundaries Γ_B and Γ_0, and the hypothesis that $\omega_m \neq 0$ we get that $w = 0$. Hence, $v = u$, which concludes the proof of the first claim.

Let us now assume that $u \in C_{\rm qp}(\overline{\Omega})$ satisfies the Lippmann-Schwinger equation (88) and set

$$u^s(x,z) \stackrel{\rm def}{=} -k^2 \int_\Omega G(x,z;\xi,\eta) q(\xi,\eta) u(\xi,\eta) d\xi d\eta \,, \quad (x,z) \in \Omega_{\mathbb{R}} \,.$$

From the properties of the Green's function G, it follows that u^s satisfies the Rayleigh expansion (12) in $\Omega^\pm$. Since $u \in C_{\rm qp}(\overline{\Omega})$ and $q \in C^2(\overline{\Omega})$, from equation (88) it follows that its extension (which we are also denoting by u) belongs to $C^{1,\epsilon}_{\rm qp}(\overline{\Omega_{\mathbb{R}}})$. Therefore, $u^s \in C^2_{\rm qp}(\Omega)$ and

$$\Delta u^s + k^2 u^s = -k^2 q u \,, \quad (x,z) \in \Omega \,.$$

Clearly,

$$\Delta u^s + k^2 u^s = 0 \,, \qquad (x,z) \in \Omega^+ \cup \Omega^- \,.$$

Since $\Delta u^i + k^2 u^i = 0$ in $\Omega_{\mathbb{R}}$ and $u = u^i + u^s$ it follows that

$$\Delta u + k^2 u = \Delta u^s + k^2 u^s = -k^2 q u \,,$$

and thus $u \in \mathcal{V}$ and satisfies the conditions of Definition 1. *Q. E. D.*

Theorem 6. *Assume that $\omega_m \neq 0$, $\forall m \in \mathbb{Z}$, and that the homogeneous scattering problem admits only the trivial solution $u = 0$. Then, there exists a unique solution $u \in \mathcal{V}$ of the scattering problem of Definition 1. Furthermore, this solution depends continuously on the incident field u^i on the sup norm.*

Proof: Let us define the operator $T : C_{\mathrm{qp}}(\overline{\Omega}) \longrightarrow C_{\mathrm{qp}}(\overline{\Omega})$ by

$$Tu(x,z) = k^2 \int_{\Omega} G(x,z;\xi,\eta) q(\xi,\eta) u(\xi,\eta) d\xi d\eta \,, \tag{89}$$

where $(x,z) \in \overline{\Omega}$. The operator T is compact because its kernel is weakly singular. From the Riesz-Fredholm theory it suffices to show that the integral equation

$$u = u^i - Tu \tag{90}$$

possesses at most one solution. Due to the uniqueness assumption of Theorem 5 this is the case. In particular, it follows that the operator $I+T$ can be inverted in the space $C_{\mathrm{qp}}(\overline{\Omega})$ and the inverse is a bounded operator in such space. Therefore, the solution depends continuously on the incoming field u^i with respect to the sup norm. *Q. E. D.*

In terms of the scattered field u^s, the Lippmann-Schwinger equation takes the form

$$u^s + Tu^s = -Tu^i \,, \tag{91}$$

where T is defined by the expression (89). Whenever $\|T\|_\infty < 1$, the successive approximation method allows us to show the existence of precisely one solution. This is achieved by means of the iteration

$$u_0^s = 0 \qquad u_{m+1}^s = -Tu_m^s - Tu^i \,, \quad m \geq 0 \,, \tag{92}$$

where the sub-index is used here to keep track of the iteration. Let us now make more explicit such process by first defining the constant

$$\mathsf{C}_\omega \stackrel{\mathrm{def}}{=} \left(\sum_{m \in \mathbb{Z}} |\omega_m|^{-2} \right)^{1/2} .$$

and then proving the following:

Lemma 14. *Let $k \in \mathbb{R}^+$ and $q \in C(\overline{\Omega})$, where $\Omega = (0, 2L) \times (0, B)$, $B > 0$. Assume that $\omega_m \neq 0$ for all m. If the condition*

$$\delta \stackrel{def}{=} k^2 \|q\|_\infty B \mathsf{C}_\omega \leq 1$$

is satisfied, then the iteration (92) converges to the solution of (91). Furthermore,

$$|u^s(\mathbf{x}) - u_m^s(\mathbf{x})| \leq 2^{-m} \delta^{m+1} \,, \quad m \geq 0 \,, \quad \mathbf{x} \in \Omega_{\mathbb{R}} \,. \tag{93}$$

The proof of this result is straightforward and we shall omit the details. We remark however, that the first term in the above Neumann series is given by

$$u_{\mathsf{B}}^s(x,z) \stackrel{\mathrm{def}}{=} -k^2 \int_{\Omega} G(x,z;\xi,\eta) q(\xi,\eta) u^i(\xi,\eta) d\xi d\eta \,, \quad (x,z) \in \Omega_{\mathbb{R}} \,,$$

which is known as the Born approximation to the scattered field. The Born approximation plays a fundamental role in a number of scattering problems [39, 40, 41].

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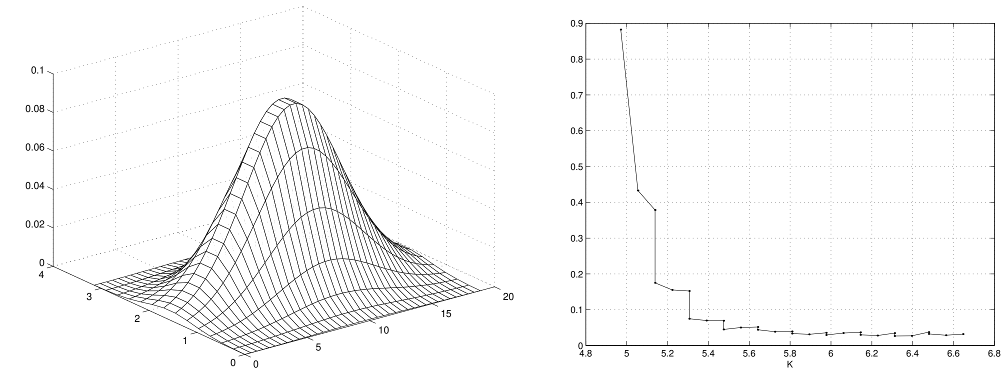
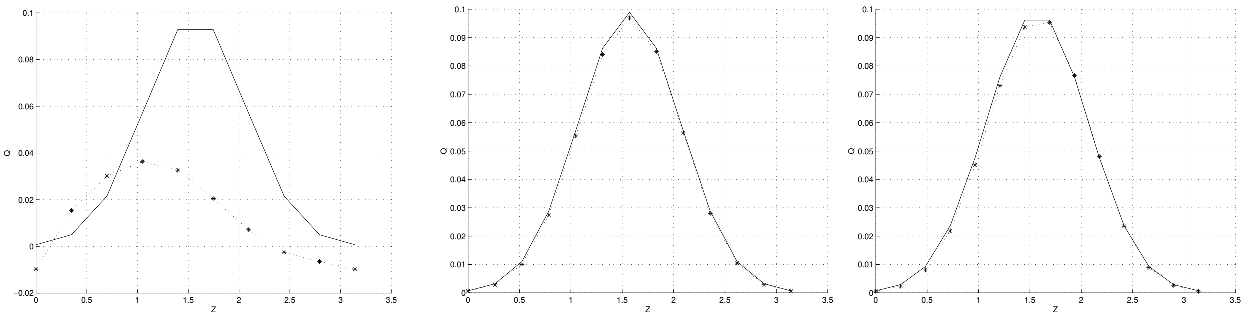


(a) Plot of q_1.

(b) Evolution of the reconstruction's relative error (noiseless case) starting with $k = 4.971$ up to $k = 6.6465$.

FIGURE 7. Example 1.

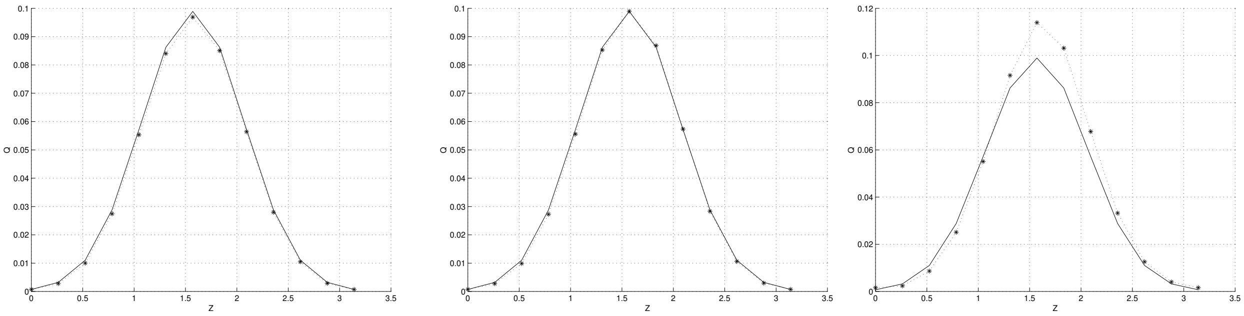


(a) Fist step in the reconstruction of q_1; $k = 4.971$.

(b) Intermediate step; $k = 6.2281$.

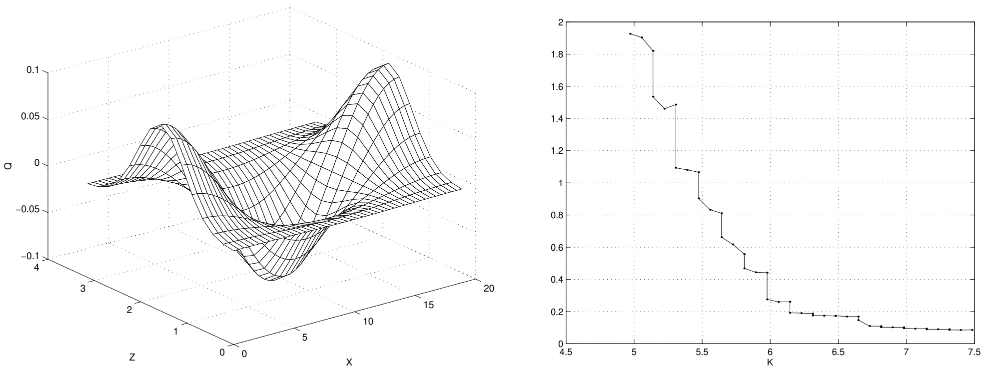
(c) Final result of the reconstruction; $k = 6.6465$.

FIGURE 8. Sections of the reconstructed scatterer of Example 1 for $x = L = 9.4248$ at different steps the reconstruction.



(a) $\nu = 10^{-3}$ (b) $\nu = 10^{-2}$ (c) $\nu = 10^{-1}$

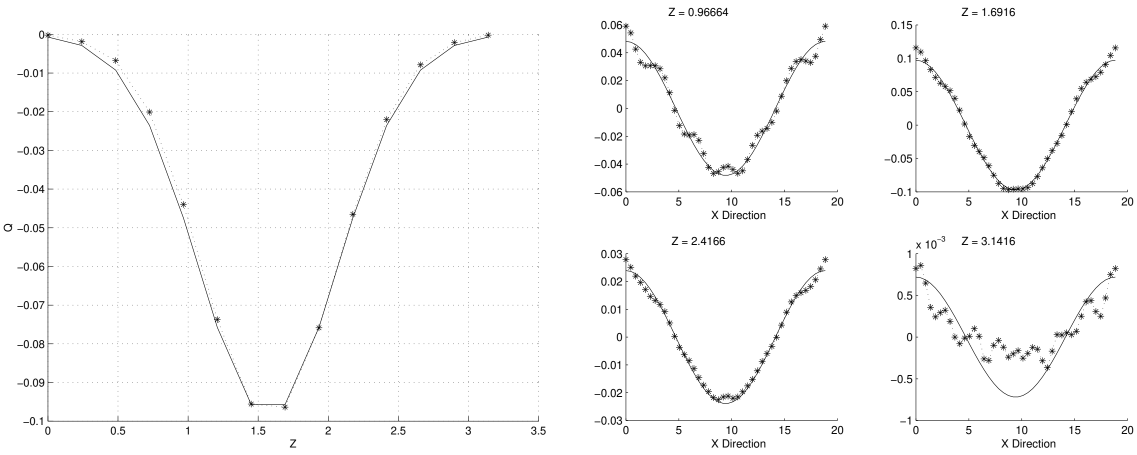
FIGURE 9. Final result for different noise levels ν, across $x = L = 9.4248$, $k = 6.2281$.



(a) Plot of q_2.

(b) Evolution of the reconstruction's relative error (noiseless case) for $4.971 \leq k \leq 7.4825$.

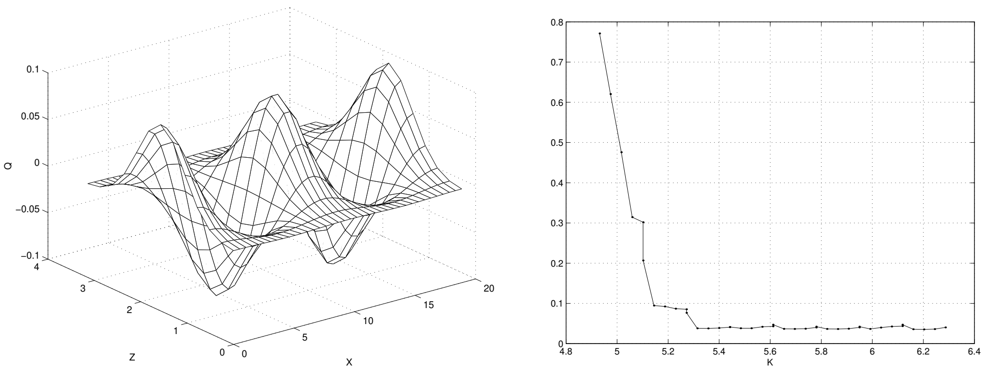
FIGURE 10. Example 2.1.



(a) Constant $x = L = 9.4248$ section; $k = 6.981$.

(b) Constant z sections. Note the scale of 10^{-3} in the lower right-hand side.

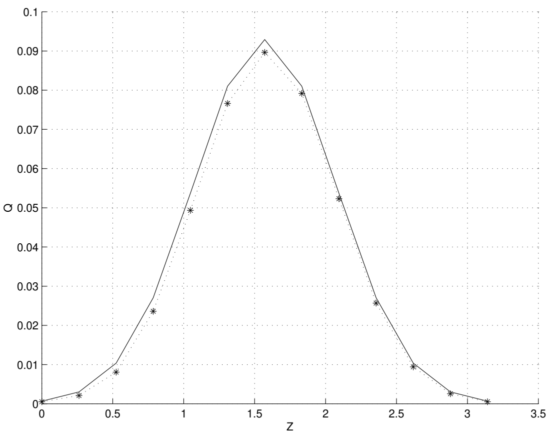
Figure 11. Final step of the reconstruction for the Example 2.1.



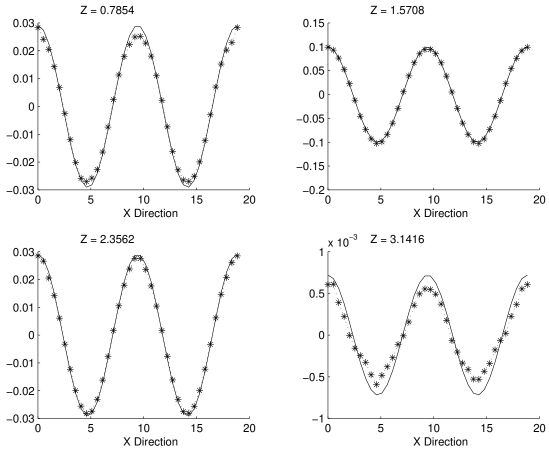
(a) Plot of q_3.

(b) Evolution of the reconstruction's relative error (noiseless case); $4.9314 \leq k \leq 6.2867$.

Figure 12. Example 2.2.



(a) Constant $x = L = 9.4248$ section, $k = 6.2024$.



(b) Constant z sections.

FIGURE 13. Final step of the reconstruction for the Example 2.2.

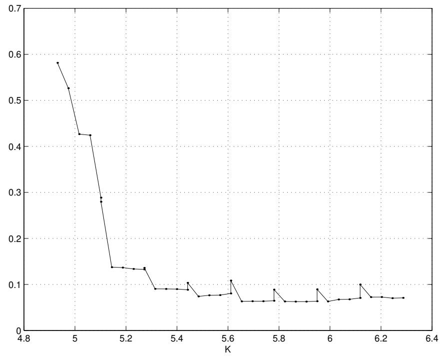
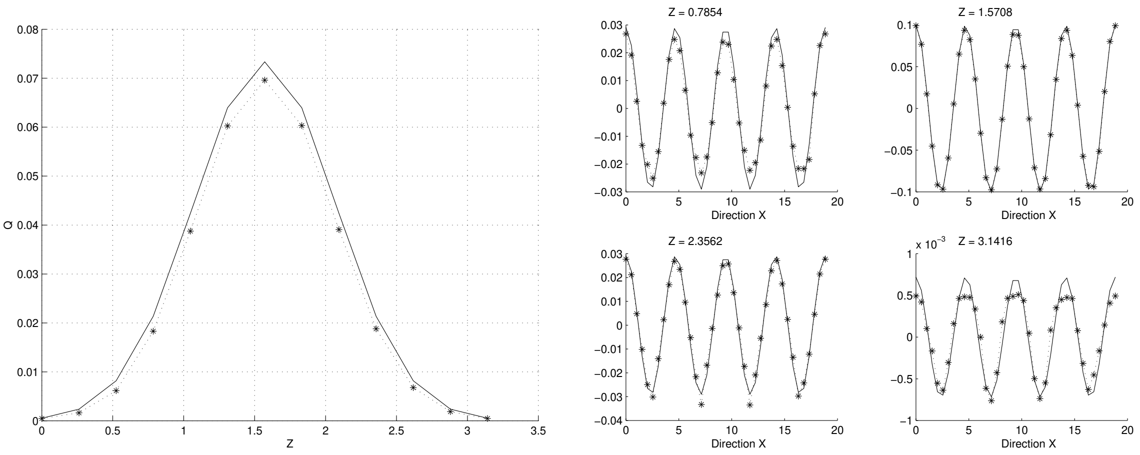


FIGURE 14. Evolution of the reconstruction's relative error (noiseless case) for Example 2.3. $k \in [4.9314, 6.2867]$



(a) Constant $x = L = 9.4248$ section. (b) Constant z sections.

FIGURE 15. Final step of the reconstruction for the Example 2.3. Value of $k = 6.2024$.