Robust transitivity implies almost robust ergodicity

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June 27, 2002

Abstract

In this paper we show the relation between robust transitivity and robust ergodicity for conservative diffeomorphisms. In dimension 2 robustly transitive systems are robustly ergodic. For the three dimensional case, we define *almost robust ergodicity* and prove that generically robustly transitive systems are almost robustly ergodic, if the Lyapunov exponents are nonzero. We also show in higher dimensions, that under some conditions robust transitivity implies robust ergodicity.

Introduction

We shall address here the question of how the important concepts of topological transitivity and ergodicity (metric transitivity) are related. It is easy to verify that if $m :=$ Lebesgue measure is ergodic or f preserves any ergodic probability measure which gives a positive mass to open balls, then f is topologically transitive. That is, there exists a point $x \in M$ such that its orbit is dense, or for any two open sets U and V, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$.

*This work began at IMPA and finalized at USP-São Carlos supported by FAPESP.

On the other hand, we know that the converse implication is not true i.e transitivity is not enough to garante ergodicity. In fact, by an example of Furstenberg, we even know minimal and non-ergodic diffeomorphism.

Although transitivity and ergodicity are different notions (topological and metric), we want to relate them when they persist in a neighbourhood of a diffeomorphism. To obtain ergodic results we need more regularity, for instance Hölder continuity for the derivative of the diffeomorphism is generally a necessary condition for proving ergodicity. It is relevant to remember that even for C^1 Anosov diffeomorphisms preserving volume, the ergodicity is not verified.

Let us define $\mathrm{Diff}^{1+}_m(M) := \bigcup_{\alpha>0} \mathrm{Diff}^{1+\alpha}_m$ and prove ergodicity results in this set.

Definition 1. *We say that $f \in \mathrm{Diff}^1_m(M)$ is C^1-robustly transitive (resp. C^1-robustly ergodic), if there exists an open set in C^1 topology $U \subset \mathrm{Diff}^1(M)$ (resp. $U \subset \mathrm{Diff}^1_m(M)$) such that any g in U (resp. in $U \cap \mathrm{Diff}^{1+}(M)$)is also topologically transitive (resp. ergodic).*

A Df-invariant splitting $E \oplus F$ of TM is called dominated splitting if the fibers of the bundles have constant dimension on whole manifold and there is $\lambda < 1$ such that:

$$\|Df|E_x\|.\|Df^{-1}|F_{f(x)}\| \leq \lambda \quad \text{for all} \quad x \in M$$

In [4] the authors show that every C^1-robustly transitive diffeomorphism has dominated splitting. More precisely for C^1-robustly transitive diffeomorphisms, $TM = E_1 \oplus E_2 \cdots \oplus E_k$ and this decomposition is dominated, moreover Df behaves "hyperbolic" for volume in E_1, E_k, more precisely for some $C > 0$:

$$|\det(Df^{-n}|E^k(x)| \leq C\lambda^n$$

$$|\det(Df^{n}|E^1(x)| \leq C\lambda^n$$

As for surface diffeomorphisms, E^1 and E^k are one dimensional, C^1-robust transitivity implies a global hyperbolic structure or in other word, robustly transitive diffeomorphisms are Anosov. Robustly transitive diffeomorphisms defined on three-manifolds may be non-Anosov. In this case the tangent bundle of the ambient manifold can be split in the following ways:

- $TM = E^{cs} \oplus E^u$ where E^u is uniformly expanding one dimensional and E^{cs} can not be split in whole manifold and it is "volume contracting":

$$|\det(Df^n|E^{cs}(x)| \leq C\lambda^n \quad \text{for} \quad \lambda < 1$$

- $TM = E^s \oplus E^{cu}$
- $TM = E^s \oplus E^c \oplus E^u$, three subbundles are nontrivial and E^c is not uniformly hyperbolic. This case is called strongly partially hyperbolic.

Let $\mathcal{PH}^r(M)$ (resp. $\mathcal{PH}^r_m(M)$) be the set of C^r-partially hyperbolic (resp. conservative partially hyperbolic) diffeomorphisms. The following questions are of interest :

Question 1. *Let $f \in \mathrm{Diff}^1_m(M)$ be C^1-robustly transitive. May f be approximated by robustly ergodic diffeomorphisms, or even is it true that any robustly transitive diffeomorphism is robustly ergodic ?*

In the surface diffeomorphisms case, robust transitivity diffeomorphisms are Anosov and it is well known that conservative Anosov diffeomorphisms are robustly ergodic [2]. So, there is a positive answer to the above question in dimension 2.
For strongly partially hyperbolic diffeomorphisms of three dimensional manifolds, Dolgopyat [6] has shown that stably ergodic systems are dense. An important thing is to remove zero Lyapunov exponent in the central direction.

Although all known examples of robustly transitive diffeomorphisms have nonzero Lyapunov exponents, it is more reasonable to ask the following:

Question 2. *May one approximate any robustly transitive diffeomorphism by another whose Lyapunov exponents are nonzero in a full Lebesgue measure set.*

Absolute continuity of stable and unstable foliations for the non-uniformly hyperbolic $C^{1+\alpha}$diffeomorphisms is the main ingredient to prove ergodicity in majority of cases. We remark that a $C^{1+\alpha}$ assumption on f is needed to get the absolute continuity of the invariant foliations, for an example of non-absolute continuous foliation in C^1-Anosov case see [1].

Theorem 1. *If f is C^1-robustly transitive and $TM = E^u \oplus E^{cs}$ with negative Lyapunov exponents in the E^{cs} direction on a full Lebesgue measure set (also C^1-robustly) then f is C^1-robustly ergodic.*

The hypothesis about Lyapunov exponents in the above theorem is also called "mostly contracting". Mostly expanding is defined similarly and the theorem is true for them too. The main point in the above theorem is that for partially hyperbolic diffeomorphisms the invariant foliations tangent to hyperbolic directions are continuous.

In Theorem1, all the Lyapunov exponents in the central direction are of the same sign. If this does not happen, that is if there exists directions corresponding to the positive Lyapunov exponents which do not dominate directions corresponding to negative Lyapunov exponents, we can not get the same results. For these cases we prove that any robustly transitive diffeomorphism of three dimensional manifold is approximated by *almost robustly ergodic* ones. We use a new result of Bochi-Viana which shows dominated splitting for Oseledets decomposition of C^1 generic conservative diffeomorphisms. Let us define almost robust ergodicity :

Definition 2. *$f \in \mathrm{Diff}^{1+\alpha}(M)$ is said ϵ-ergodic, if there exists an SRB measure ρ for f with Leb$(B(\rho)) > 1-\epsilon$, and in the conservative case $m(C) > 1-\epsilon$, where C is some ergodic component of the Lebesgue measure.*

Definition 3. *$f \in \mathrm{Diff}^1_m(M)$ is said C^1-almost robustly ergodic, if for any $\epsilon > 0$ there is an open set $U_\epsilon \subset \mathrm{Diff}^1_m(M)$ such that $g \in U_\epsilon \cap \mathrm{Diff}^{1+}_m$ is ϵ-ergodic.*

By the above definition of almost-robust ergodicity of f we require that perturbing a little f, with a large enough probability, the Birkhoff averages of continuous function are the same. In other words the ergodicity near f is " getting better". It is clear by definition that if $f \in \mathrm{Diff}^{1+}_m(M)$ is almost-robustly ergodic, then f is ergodic.

Theorem 2. *If U is a C^1 open set in $\mathcal{PH}^1_m(M)$ (dim $M = 3$) of transitive diffeomorphisms such that for $f \in U$ the central Lyapunov exponent is nonzero in a full Lebesgue measure set, then almost-robustly ergodic ones are C^1-residual in U*

As a corollary:

Corrollary 1. *The ergodic diffeomorphisms constitute a generic subset of $U \cap \mathrm{Diff}^{1+}_m$, with the C^1 induced topology, if $U \cap \mathrm{Diff}^{1+}_m$ is dense in U.*

The density hypothesis in the above corollary is very natural. In fact, in the symplectic case it is proved that C^∞ symplectic diffeomorphisms are dense in C^1 ones [11]. For conservative diffeomorphisms it is an open question yet. We mention here that another interesting question for robustly ergodic diffeomorphisms is the following:

Question 3. *Is it true that any C^1-robustly ergodic conservative diffeomorphism admits a dominated splitting?*

For more than three dimensional case in [5], the authors give the first example of robustly transitive diffeomorphisms of $\mathbb{T}^4$ which do not have any hyperbolic direction and in [10] we show the robust ergodicity of such diffeomorphisms. In fact, the robust ergodicity of a class of diffomorphisms of $\mathbb{T}^n$, which do not have any hyperbolic direction is verified.

1 Preliminary

Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a compact manifold that preserves volume m. Oseledets theorem [8] states that, for $m-$almost every point $x \in M$, there exists real numbers $\lambda_1(x) > \cdots > \lambda_{k(x)}(x)$ and

$$T_xM = E_x^1 \oplus \cdots \oplus E_x^{k(x)}$$

such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df^n(x)(v_j)\| = \lambda_j(x) \quad \text{for all} \quad v_j \in E_x^j \backslash \{0\}$$

λ_j's are Lyapunov exponents which depends measurably on x.

Let us mention that for a general $C^{1+\alpha}$ diffeomorphism preserving an smooth hyperbolic measure (with nonzero Lyapunov exponents) one has a countable number of ergodic components [9]. Dolgopyat, Hu and Pesin constructed an example of such diffeomorphism with infinitely many open ergodic components. It is obvious that such examples can not be topologically transitive because open invariant sets are dense in transitive case, so there would exist just one ergodic component.
We are going to show ergodicity by finding open sets (mod 0) in the ergodic components of Lebesgue measure. This kind of approach to prove ergodicity was introduced by Pesin [9]. The idea is to show local ergodicity (ergodic components are open sets, a.e) and then by the aid of topological transitivity

we get ergodicity. Observe that to prove ergodicity or equivalently the existence of a unique ergodic component, we just need to show that any ergodic component contains an open set (mod 0) and then use the transitivity as following :

If C_1, C_2 are two ergodic component with $U_1 \subset C_1$, $U_2 \subset C_2$ (mod 0), i.e $m(U_i \backslash C_i) = 0$ then by topological transitivity $\exists n > 0$ such that $f^n(U_1) \cap U_2 \neq \emptyset$. By the invariance of C_i and the fact that f is diffeomorphism and and preserve the measure m, it comes out that $C_1 \cap C_2 \neq \emptyset$ and $C_1 = C_2$.

Proof. (Theorem 1)

Let $f \in U$ be as in Theorem 1, where $U \in \mathrm{Diff}^1_m(M)$ constitutes of transitive diffeomorphims with non-zero Lyapunov exponents in a full Lebesgue measure set. As m is an invariant measure with non-zero Lyapunov exponents, it has a countable number of ergodic components which are just the normalization of its restriction to C_i where each C_i is a measurable invariant subsets with $m(C_i) > 0$ which we call them also by ergodic components.

By continuity of the invariant foliations of partially hyperbolic diffeomorphisms in [7], the unstable foliation of $g \in U$ varies continuously. Using Pesin theory for $g \in \mathrm{Diff}^{1+}_m(M)$, there is a full Lebesgue measure subset $\mathcal{R}$ such that $x \in \mathcal{R}$ is regular in the following sense.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x)) = \lim_{n\to-\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x)) \quad \text{for any } \phi \in C^0(M)$$

For any $x \in \mathcal{R}$ there is a local stable manifold $W^s_{loc}(x)$ which depend measurably on x. Moreover the stable foliation is absolutely continuous. For any ergodic component C, there is $x \in C$ such that $W^s_{loc}(x)$ is almost (With respect to Lebesgue measure of $W^s(x)$) contained in $C \cap \mathcal{R}$. That means, there exists $C_x \subset W^s_\epsilon \cap C \cap \mathcal{R}$, for some $\epsilon > 0$, such that $m_s(W^s_\epsilon \backslash C_x) = 0$ where m_s is the Lebesgue measure induced on the stable manifold. Now we saturate C_x by unstable leaves and let $U_x = \bigcup_{y \in W^s_\epsilon} W^u(y)$. By continuity of unstable foliation, U_x will contain an open set. On the other hand by Hopf's argument for any $y \in C_x$ and $z \in W^u(y)$ we have:

$$\lim_{n\to-\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(z)) = \lim_{n\to-\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(y)) =$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\phi(f^i(y)) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\phi(f^i(x))$$

Now by absolute continuity of W^u (unstable foliation) we conclude that $\bigcup_{y\in C_x} W^u(y)$ has full measure in U_x and finally as regular points have full measure, the ergodic component C contains a total Lebesgue measure subset of the open set U_x. This is what we want because ergodic components are invariant and f is transitive and this proves the uniqueness of ergodic components with positive Lebesgue measure and consequently ergodicity of the Lebesgue measure. $\square$

We mention that, only absolute continuity of W^u is not enough to get an open set (mod 0) in the above argument, but if we have some control from below on the size of local unstable manifolds of $y \in W^s_{local}(x)$ then the same argument works. The problem in Theorem 2 appears when in some ergodic component with positive measure, a direction corresponding to a positive Lyapunov exponent does not dominate the direction corresponding to the negative one. More precisely let $TM = E^u \oplus E^{cs}$ and $\lambda_1(x) > \lambda_2(x) > \lambda_3(x)$ represent the Lyapunov exponents. It may happen that for an ergodic component C with $m(C) > 0$, $\lambda_2 > 0$ and E^{cs} can not be split into one dimensional dominating subbundles over C.

We show that the measure of such bad behaved set is small enough for diffeomorphisms near to C^1-generic diffeomorphisms for which almost everywhere the Oseledets splitting is dominated.

2 Dominated splitting of Oseledet splitting

Now let recall the new result of Bochi-Viana [3] that gives a C^1 generic subset of diffeomorphisms for which the Oseledet decomposition is dominated or trivial.

Theorem 3 ([3]). *For any compact manifold M, there exists a C^1-residual subset $\mathcal{R}$ of $\mathrm{Diff}^1_m(M)$ such that for every $f \in \mathcal{R}$, the Oseledet splitting is dominated or else trivial, at almost every point.*

Let us define $\Lambda_i(f) = \lambda_1(f) + \cdots + \lambda_i(f)$, where $\lambda_j(f) = \int_M \lambda_j(x)dm$. By semi-continuity arguments it is easy to show that the continuity points of

$\Lambda : f \to (\Lambda_1(f), \cdots, \Lambda_{d-1}(f))$ contains a C^1-residual subset $\mathcal{R} \subset \mathrm{Diff}^1_m(M)$. In [3] the authors show that $\forall f \in \mathcal{R}$, the Oseledet splitting is dominated or trivial. We show that diffeomorphisms near to this generic subset have greedy ergodic components.

Proof: (of Theorem 2)
Let $TM = E^u \oplus E^{cs}$ be the decomposition of tangent bundle for the diffeomorphisms in U.(the other cases are similar.) Take any $g \in \mathcal{R} \cap U$, where $\mathcal{R}$ is the residual subset in Theorem 3. As g is a continuity point of Λ above, for small $\delta_0 > 0$ there exists $\mathcal{O}$, a C^1 neighbourhood of g in $\mathrm{Diff}^1_m(M)$ such that $\forall f, f_1 \in \mathcal{O}$, $|\Lambda_i(f_1) - \Lambda_i(f)| < \delta_0$.
As Lyapunov exponents of $f \in U$ are nonzero then by [9], m has a countable number of ergodic components C_n with $m(C_n) > 0$ and $m(\bigcup C_n) = 1$.
Now let $f \in \mathcal{O} \cap \mathrm{Diff}^{1+}_m(M)$; we will show that f has a large ergodic component in the measure theoretic sense.

Any ergodic component C is an invariant set and m_C:= the normalized restriction of the Lebesgue measure on C is an ergodic measure for the restriction of f on C. Neglecting zero measure subsets which are irrelevant for our purposes, we may substitute C by $Supp(m_C)$. Now we claim that there exists $\bar{C} \subseteq C$ such that $m_C(C \backslash \bar{C}) = 0$ and any $x \in \bar{C}$ has a dense orbit in C. To show the above claim remember that by the ergodicity of m_C, the basin of m_C has full measure. The basin consists of points z such that $\frac{1}{n}\sum_{i=0}^{n-1} \delta_z$ converges to m_C. As m_C has a full support, the orbit of the points in the basin will be dense in C. Now it is enough to take $\bar{C}$ = Basin of m_C and the claim is proved. In the following we define two kinds of ergodic components:

1. $\mathcal{C}_{good}$ = Union of ergodic components C_i such that one of the followings happens:

   - $\lambda_2(x) < 0$ for $x \in C_i$
   - $E_2 \oplus E_3$ is a dominated splitting on C_i and $\lambda_2(x) > 0$.

   of course for each ergodic component at most one of the above items can be satisfied.

2. $\mathcal{C}_{bad}$= Complement of $\mathcal{C}_{good}$

By the arguments in the previous section in the proof of Theorem 1 we deduce that all $C_i \subset \mathcal{C}_{good}$ contains open set (mod 0), because:

- If $\lambda_2(x) < 0$ then we are in the setting of Theorem 1. Firstly we use absolute continuity of local stable manifolds corresponding to λ_2 and λ_3 and then continuity of unstable foliation tangent to E^u.
- If $\lambda_2(x) = \lambda > 0$ and E_2 dominates E_3, then on the corresponding ergodic component C_i, the tangent bundle has the following invariant dominated decomposition: $T_C M = E^{cu} \oplus E^{cs}$ with $\dim(E^{cs}) = 1$ and $E^{cu} = E^u \oplus E^2$ where E^2 corresponds to λ_2. Now as f is conservative it is easy to show that the above dominated splitting has the volume hyperbolicity property: $|\det(Df^n|E^{cs}(x)| \leq C\lambda^n$ for some $C > 0, \lambda < 1$ for any $x \in C_i$ (see [4] for volume hyperbolicity of dominated splitting in the coservative case). As E^{cs} is one dimensional, the volume hyperbolicity is equivalent to uniform hyperbolicity. Now apply the same argument of the Theorem 1 using the absolute continuity of the local unstable manifolds for the points in C_i and the existence of large stable manifolds which is a result of uniform hyperbolicity of E^{cs}.

Consequently by topological transitivity, there is just one egodic component in $\mathcal{C}_{good}$. For the bad components we prove that they occupy an small region in the measure theoretic sense. Let us fix some notations. By an $m-$dominated splitting of $E_2 \oplus E_3$ along the orbit of a point x we require that for all $n \in \mathbb{Z}$:

$$\frac{\|Df^m_{f^n(x)} \mid E_3\|}{\mathbf{m}(Df^m_{f^n(x)} \mid E_2)} \leq \frac{1}{2}$$

where $\mathbf{m}(A) = \|A^{-1}\|^{-1}$. By an $m-$dominated splitting over an invariant set C we mean $m-$dominated splitting for all orbits in C. Observe that $m-$dominated splitting extends to the closure of an invariant subset.

Let $\Gamma(f, m)$ denotes the subset of points such that $E_2 \oplus E_3$ does not admit an $m-$dominated splitting and let $\Gamma(f, \infty) = \cap_{m\in\mathbb{N}}\Gamma(f, m)$. For the ergodic components $C \subseteq \mathcal{C}_{bad}$, $\lambda_2 > 0$ and $E_2 \oplus E_3$ does not admit an $m-$dominated splitting over C for any $m \in \mathbb{N}$. We claim that, there exits a full m_C measure subset of C which we denote it also by C, such that $C \subseteq \Gamma(f, \infty)$. To prove the claim observe that as $m-$dominated splitting passes to closure of sets, for any point $x \in \bar{C}$ ($z \in \bar{C}$ has dense orbit) there is not any $m-$dominated splitting of $E_2 \oplus E_3$ along the orbit of x. This shows that $\mathcal{C}_{bad} \subseteq \Gamma(f, \infty)$ (mod 0).

Denote

$$J(f) = \int_{\Gamma(f,\infty)} \frac{\lambda_2 - \lambda_3}{2} dm(x)$$

We apply the following proposition of [3, Proposition 4.17]:

Proposition 1. *Given any $\delta > 0$ and $\epsilon > 0$, there exists a diffeomorphism f_1, ϵ near to f such that*

$$\int_M \Lambda_2(f_1, x)dm(x) < \int_M \Lambda_2(f, x)dm(x) - J(f) + \delta$$

By the above proposition we will deduce that if the measure of bad components is not small enough then after perturbing f a little, the average of $\lambda_1 + \lambda_2$ drastically drops.

As $\mathcal{C}_{bad} \subseteq \Gamma(f, \infty)$ and on $\mathcal{C}_{bad}, \lambda_2(x) > 0$ by the above proposition it turns out that

$$\Lambda_2(f) - \Lambda_2(f_1) \geq \frac{1}{2}\int_{\mathcal{C}_{bad}} (\lambda_2 - \lambda_3)(f)dm - \delta \geq \frac{1}{2}\int_{\mathcal{C}_{bad}} -\lambda_3(f)dm - \delta$$

$$\geq m(\mathcal{C}_{bad})\, inf_{x\in\mathcal{C}_{bad}} \frac{-\lambda_3(f, x)}{2} - \delta$$

Now we observe that the infimum above is bounded away from zero uniformly in $\mathcal{O}$.

Lemma 1. *There is some $\alpha < 1$, depending only to g such that: $\lambda_3(x) \leq \log(\alpha)$ for all $x \in \mathcal{C}_{bad}$.*

Proof. As g is volume hyperbolic and partially hyperbolic $TM = E^u \oplus E^{cs}$, so $\det(Df|E^{cs}(x)) < \alpha < 1$ for all $x \in M$ and we can take α uniform in a C^1 neighborhood of g, just because $g \rightarrow E^{cs}(x, g)$ is continuous for C^1 topology in the partially hyperbolic space. Take $x \in \mathcal{C}_{bad}$ then:

$$\lambda_2(x) + \lambda_3(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log\, \det(Df|E^{cs}(f^i(x))) \leq \log(\alpha)$$

and as $\lambda_2(x) > 0$ we have $\lambda_3(x) < \log(\alpha)$. □

So by the above lemma:

$$inf_{x\in\mathcal{C}_{bad}} \frac{-\lambda_3(f, x)}{2} \geq \frac{-\log(\alpha)}{2}$$

and

$$\delta_0 \geq \Lambda_2(f) - \Lambda_2(f_1) \geq m(\mathcal{C}_{bad})\frac{-\log(\alpha)}{2} - \delta$$

We get then $m(\mathcal{C}_{bad}) \leq \dfrac{\delta+\delta_0}{-\log(\alpha)}$. Taking δ_0 and δ small, for $f \in \mathcal{O} \cap \mathrm{Diff}^{1+}_m(M), m(\mathcal{C}_{bad})$ is small enough and so $m(\mathcal{C}_{good})$ is large and "*ergodicity is getting better*".

Finally we conclude The Corollary 1 as follows. Taking $\mathcal{E}_n := \frac{1}{n}$-ergodic diffeomorphisms in $\mathrm{Diff}^{1+}_m \cap U$, then $\mathcal{E}_n$ is open and dense in the C^1 induced topology , so $\mathcal{E} = \bigcap \mathcal{E}_n$ is a residual subset and $f \in \mathcal{E}$ is ergodic.

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