ON A FREE BOUNDARY PROBLEM FOR A STRONGLY DEGENERATE QUASILINEAR PARABOLIC EQUATION WITH AN APPLICATION TO A MODEL OF PRESSURE FILTRATION

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Abstract. We consider a free boundary problem of a quasilinear strongly degenerate parabolic equation arising from a model of pressure filtration of flocculated suspensions. We provide definitions of generalized solutions of the free boundary problem in the framework of L^2 divergence-measure fields. The formulation of boundary conditions is based on a Gauss-Green theorem for divergence-measure fields on bounded domains with Lipschitz deformable boundaries and avoids referring to traces of the solution. This allows to consider generalized solutions from a larger class than BV. Thus it is not necessary to derive the usual uniform estimates on spatial *and* time derivatives of the solutions of the corresponding regularized problem requires in the BV approach. We first prove existence and uniqueness of the solution of the regularized parabolic free boundary problem and then apply the vanishing viscosity method to prove existence of a generalized solution to the degenerate free boundary problem.

1. Introduction

Conventional analyses of initial-boundary value problems of strongly degenerate parabolic equations, which includes first-order conservation laws, are usually based on the concept of generalized solutions in $BV(Q_T)$, where $Q_T := \Omega \times [0,T]$, $\Omega \subset \mathbb{R}$ is the computational domain (for simplicity, assumed to be cylindrical here) [2, 4, 5, 25, 26]. To prove that a generalized solution u of a conservation law or of a strongly degenerate parabolic equation belongs to $BV(Q_T)$, it is necessary to derive estimates on $\|\partial_x u_\varepsilon\|_{L^1(Q_T)}$ and $\|\partial_t u_\varepsilon\|_{L^1(Q_T)}$ which are uniform with respect to the regularization parameter ε, where u_ε denotes the smooth solution of the corresponding regularized initial-boundary value problem. These estimates (and a uniform L^∞ bound on u_ε) imply that the family $\{u_\varepsilon\}_{\varepsilon>0}$ is compact in $L^1(Q_T)$, i.e. there exists a sequence $\varepsilon = \varepsilon_n$ with $\varepsilon_n \to 0$ for $n \to \infty$ such that $\{u^{\varepsilon_n}\}$ converges in $L^1(Q_T)$ to a limit $u \in L^\infty(Q_T) \cap BV(Q_T)$. It is usually straightforward to verify that this limit is indeed a generalized solution.

The importance of the choice of the space $BV(Q_T)$ lies in the existence of traces of the limit function u with respect to the lateral boundaries of Q_T. This well-known property of BV functions is stated e.g. in [11, Sect. 5.32, Th. 1]. As has become apparent in [4], traces are needed in the proof of uniqueness of generalized solutions.

For several reasons, the BV approach unfortunately imposes some severe limitations to the analysis of initial-boundary value problems of hyperbolic and strongly degenerate parabolic equations. The most obvious one is the apparent difficulty to actually derive the required uniform estimates on $\|\partial_x u_\varepsilon\|_{L^1(Q_T)}$ and $\|\partial_t u_\varepsilon\|_{L^1(Q_T)}$. This worked out e.g. for the spatially one-dimensional problems analyzed in [4]. However, for only marginally more involved equations (but still in one space

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dimension), and in particular for different boundary conditions it seems no longer possible to derive a uniform estimate on $\|\partial_t u_\varepsilon\|_{L^1(Q_T)}$. An example of such an initial-boundary problem is given in [24]. When passing to several space dimensions, i.e. to equations of the type

$$\partial_t u + \nabla_{\mathbf{x}} \cdot \mathbf{f}(u) = \triangle A(u), \quad (\mathbf{x}, t) \in Q_T := \Omega \times [0, T], \quad \Omega \subset \mathbb{R}^n \tag{1}$$

together with initial and boundary conditions and where the function $A(u)$ is nonnegative, increasing and Lipschitz continuous, it seems virtually impossible to derive the required uniform estimates, where the estimate on the spatial derivative has of course to be replaced by a uniform estimate on $\|\nabla_{\mathbf{x}} u_\varepsilon\|_{L^1(Q_T)}$.

In the cases where only a uniform estimate on $\|\nabla_{\mathbf{x}} u_\varepsilon\|_{L^1(Q_T)}$ (but not on the time derivative) is feasible, one can utilize Kružkov's "interpolation lemma" [14, Lemma 5] in order to conclude that the sequence u_ε converges to a limit function u belonging to the wider class $BV_{1,1/2}(Q_T) \supset BV(Q_T)$. This means that there exists a constant K such that

$$\iint_{Q_T} \big|u(\mathbf{x} + \Delta\mathbf{x}, t) - u(\mathbf{x}, t)\big| d\mathbf{x}dt \leqslant K|\Delta\mathbf{x}|, \quad \iint_{Q_T} \big|u(\mathbf{x}, t + \Delta t) - u(\mathbf{x}, t)\big| d\mathbf{x}dt \leqslant K|\Delta t|^{1/2}.$$

Note that the $BV_{1,1/2}$ estimates on $\{u_\varepsilon\}$ are entirely sufficient to apply Kolmogoroff's compactness criterion in order to show existence of a limit function. The problem is with boundary conditions and uniqueness, since it is not ensured that a function $u \in BV_{1,1/2}(Q_T)$ possesses traces at the boundaries of Q_T, such that boundary conditions need to be defined in a fashion that avoids these traces; however, it is then not obvious how to prove uniqueness.

Another general limitation of the BV approach has become apparent in [4], and is due to the restriction that the initial datum u_0 of that paper belongs to the class

$$\mathcal{B} := \Big\{u \in BV(\Omega) : \ u(x) \in \mathcal{U}_0 \ \forall x \in \overline{\Omega}; \ \mathrm{TV}_\Omega(\partial_x A_\varepsilon(u)) < M_0 \text{ uniformly in } \varepsilon\Big\},$$

where $A_\varepsilon'(u) = a_\varepsilon(u)$ and a_ε is an appropriately regularized, positive diffusion coefficient. The condition $u_0 \in \mathcal{B}$ is required to ensure that $\|\partial_t u_\varepsilon(\cdot, t)\|_{L^1(\Omega)}$ or $\|\partial_t u_\varepsilon\|_{L^1(Q_T)}$ remain uniformly bounded. For a given, in general discontinuous function u_0 membership in $\mathcal{B}$ is difficult to verify due to the discontinuity of the diffusion coefficient $a(u)$, so $\mathcal{B}$ denotes a possibly very narrow class.

The mentioned difficulties associated with the BV approach make it desirable to consider generalized solutions from a wider class. This wider class is associated here with the notion of divergence-measure fields, which is a class of vector fields that was first considered by Anzellotti [1]. This paper is based on the recent formulation by Chen and Frid [9].

The main idea is to replace the requirement $u \in L^\infty(Q) \cap BV(Q)$, where we consider $Q \subset R^N$ and which can be expressed as

$$\|u\|_{BV(Q)} < \infty, \quad \|u\|_{BV(Q)} = \sup\bigg\{\int_Q u\nabla \cdot \varphi \, d\mathbf{x} : \varphi \in \big(C_0^1(Q)\big)^N, \ \|\varphi\|_{L^\infty(Q)} \leqslant 1\bigg\},$$

by the requirement that a vector field $F \in L^p(Q, \mathbb{R}^N)$ associated with the sought solution u satisfies

$$|\operatorname{div} F|(Q) < \infty, \quad |\operatorname{div} F|(Q) := \sup\bigg\{\int_Q F \cdot \nabla\varphi \, d\mathbf{x} : \varphi \in C_0^1(Q; \mathbb{R}), \ \|\varphi\|_{L^\infty(Q)} < 1\bigg\}.$$

We define the class of L^p divergence-measure vector fields over Q by

$$\mathcal{DM}^p(Q) = \big\{F \in L^p(Q; \mathbb{R}^N) : \ |\operatorname{div} F|(Q) < \infty\big\}.$$

We see that if $F \in \mathcal{DM}^p(Q)$, then $\operatorname{div} F$ is a Radon measure over Q. If we assume that the components of F are Lipschitz continuous functions of u, as in the application to conservation laws (see below), then it becomes clear that $u \in L^\infty(Q) \cap BV(Q)$ implies $F \in \mathcal{DM}^\infty(Q)$.

Properties of divergence-measure fields for the case $p = \infty$ are derived by Chen and Frid in [9]. Most important, it is possible to prove a generalized Gauss-Green formula for divergence-measure fields in bounded domains using the concept of domains with deformable Lipschitz boundaries, which allows the definition of traces. For the case of scalar conservation laws, the importance of divergence-measure fields accrues from the fact that any convex entropy pair actually forms an L^∞ divergence-measure field over $Q \subset \mathbb{R}^N$ if we consider a bounded spatial domain $\Omega \subset \mathbb{R}^{N-1}$. Utilizing the Gauss-Green formula, Chen and Frid [9] provide an appropriate formulation for L^∞ (not BV) solutions of conservation laws with boundary conditions. They are able to derive a formulation of an entropy boundary condition which was proposed previously by Otto [17, 19, 20, 21] by advancing the concept of entropy boundary fluxes.

Most properties of L^p, $p = \infty$ divergence-measure (div-meas) fields derived in [9] also hold for $1 \leqslant p < \infty$, as is detailed in [10]. The case $p = 2$ is of particular interest for the analysis of degenerate parabolic equations, since in view of standard a priori estimates, it is possible to show that the appropriately defined entropy pair of a strongly degenerate parabolic equation is an L^2 divergence measure field over $Q_T \subset \mathbb{R}^{N-1} \times [0,T]$. (More general domains can be considered, but we may limit here the discussion to cylindrical domains.) This was first exploited in a recent paper by Mascia, Porretta and Terracina [18], who proved existence and uniqueness of L^∞ solutions to nonhomogeneous Dirichlet initial-boundary value problems of Equation (1), which in particular includes entropy boundary conditions.

In [6] entropy boundary conditions for strongly parabolic equations in the context of an application to to sedimentation with compression are derived. However, the definition of traces of the solution with respect to the lateral boundary of the computational domain is only possible if the diffusion coefficient $a(u)$ is, for example, Lipschitz continuous. This assumption does not hold for the cases we are interested in here. Moreover, although Dirichlet boundary conditions in the context of solid-liquid separation models lead to mathematically well-posed initial-boundary value problems, their physical significance is questionable due to violation of a conservation principle. Rather, kinematic 'flux-type' or 'wall' boundary conditions (such as that of Problem B of [4]) should be employed. In fact, it turned out that these boundary conditions are satisfied in an a.e. pointwise sense on the lateral boundaries of Q_T, that is in a much stronger sense than are entropy boundary conditions, although they also involve the concept of traces.

The above discussion motivates our interest in applying the recently developed div-meas theory to initial-boundary value problems of strongly degenerate parabolic equations. We could now treat again the initial-boundary value problems studied e.g. in [4] in an appropriate div-meas framework, and obtain an existence and uniqueness result. However, since the BV calculus is indeed applicable to those problems, the chief gain in using the more general div-meas concept would merely consist in the relaxation of the condition $u_0 \in \mathcal{B}$. Instead, the theory of L^2 div-meas fields is applied here to a free boundary problem which is a slight modification of a model of pressure filtration presented in [3]. The problem is still one-dimensional, and its boundary conditions are of 'flux-type' similar to those of [4]. However, there is reason to believe that the mentioned BV estimate on $\partial_t u_\varepsilon$ can not be derived. This conjecture is based on the observation that in many other analyses it was necessary to differentiate the corresponding regularized viscous equation with respect to t, to multiply it with a suitable sign-type function, and to use integration by parts. The problem with the filtration problem is the occurrence of the derivative (with respect to t) of the free boundary as a coefficient in the equation, such that differentiating the entire equation with respect to t would entail the necessity to estimate $h''(t)$. Due to the coupling condition with the solution evaluated at one of the boundaries, we have, however, no control over this quantity. This seems to preclude the necessary uniform estimate on $\partial_t u$.

The remainder of this chapter is organized as follows. In Section 2 we briefly recall the mathematical model of pressure filtration, state the free boundary problem, and provide a brief definition of L^2 div-meas fields together with the properties relevant for the subsequent analysis. In Section 3 generalized solutions of the free boundary problem are defined, where an equivalent problem transformed to fixed boundaries is also considered. In Section 4 we state the corresponding regularized viscous free boundary problems and show that they have a unique solution for fixed values of the regularization parameter. Finally we conclude in Section 5 by the viscosity method that there exists a generalized solution to the free boundary problem in the sense of Section 3.

The analysis of the free boundary problem has not yet been completed, since a uniqueness proof is still lacking. It is however not obvious, for instance, how the uniqueness proof of for a comparable free boundary problem by Zhao and Li [27], which is based on establishing a fixed boundary initial-boundary value problem for a suitably complemented generalized solution of the free boundary problem, can be extended to the free boundary problem studied in this chapter.

2. Statement of the problem and preliminaries

2.1. **Pressure filtration of flocculated suspensions.** To motivate the free boundary problem studied in this paper, we briefly recall the one-dimensional mathematical model of pressure filtration formulated in [3]. We consider a filter column closed at height $z=0$ by a filter medium, which lets only the liquid pass, and at a variable height $z=h(t)$ by a piston which moves downwards due to an applied pressure $\sigma(t)$. The material behaviour of the suspension is described by two model functions, the flux density function or hindered settling factor f and the effective solid stress function σ_{e}, both functions only of the local solids concentration u. Here f is a non-positive Lipschitz continuous function with compact support in $[0,u_{\max}]$, where $u_{\max}\leqslant 1$ is the maximum concentration, and the function σ_{e} satisfies $\sigma_{\mathrm{e}}=0$ for $u\leqslant u_{\mathrm{c}}$, where $0\leqslant u_{\mathrm{c}}\leqslant u_{\max}$ is a critical concentration value, and $\sigma_{\mathrm{e}}'(u)>0$ for $u>u_{\mathrm{c}}$. According to the phenomenological sedimentation-consolidation theory [3, 7, 8], the evolution of the concentration distribution is given by the equation

$$\partial_t u+\partial_z\big(h'(t)u+f(u)\big)=\partial_z^2 A(u),\quad 0\leqslant z\leqslant h(t);\quad 0<t\leqslant T; \tag{2}$$

$$A(u):=\int_0^u a(s)\,ds,\quad a(u):=Cu^{-1}f(u)\sigma_{\mathrm{e}}'(u), \tag{3}$$

where the parameter $C<0$ expresses the solid-fluid density difference. Observe that Eq. (2) is hyperbolic for $u\leqslant u_{\mathrm{c}}$ and $u\geqslant u_{\max}$ and parabolic for $u_{\mathrm{c}}<u<u_{\max}$ and thus of strongly degenerate parabolic type since the degeneration to hyperbolic type takes place on an interval of solution values of positive length.

Specifically for the filtration problem, we assume that the solids flux through the moving piston and through the filter medium is zero. Since (2) is derived from the solids continuity equation, this implies the kinematic boundary conditions

$$\big(f(u)-\partial_z A(u)\big)\big(h(t),t\big)=0,\quad \big(h'(t)u+f(u)-\partial_z A(u)\big)(0,t)=0,\quad t>0. \tag{4}$$

At time $t=0$, the column is filled with a suspension of the local initial volumetric concentration $u(z,0)=u_0(z)$ for $0\leqslant z\leqslant h(0):=1$.

The salient mathematical difficulty of the pressure filtration model arises from the coupling between the applied pressure $\sigma=\sigma(t)$ and the trajectory of the piston expressed by the function $h(t)$. Resistance to the movement of the piston, i.e. to the flow rate of filtrate leaving the filter, is exerted by the filter medium and by the so-called filter cake forming above the medium. While the resistance of the filter medium is constant, that of the filter cake depends on its thickness and composition, that is, on the solution u. The growth of the filter cake during the initial stages of the filtration process therefore slows down the downward movement of the piston if the applied pressure is kept constant. Specifically, a vertical stress balance and an application of Darcy's law yield the following coupling equation between $\sigma(t)$ and $h(t)$ [3, 16], which is written here as an ordinary differential equation for h:

$$h'(t)+\beta(t)h(t)+\gamma\big(t,u(0,t)\big)=0,\quad 0<t\leqslant T; \tag{5}$$

$$\beta(t):=\frac{g\varrho_{\mathrm{f}}}{\mu_{\mathrm{f}}R},\quad \gamma\big(t,u(0,t)\big):=\frac{1}{\mu_{\mathrm{f}}R}\big[g\big(m_0-\varrho_{\mathrm{f}}\big)+\sigma(t)-\sigma_{\mathrm{e}}\big(u(0,t)\big)\big]. \tag{6}$$

Here g is the acceleration of gravity, ϱ_{f} the density of the fluid, μ_{f} its viscosity, the resistance of the filter medium, and m_0 the initial suspension mass divided by the cross-sectional area of the filter column.

The observation that γ depends on $\sigma_{\mathrm{e}}(u(0,t))$ and not on some arbitrary function of $u(0,t)$ is essential to make the problem amenable to mathematical analysis. In fact, both functions σ_{e} and the integrated diffusion coefficient A vanish for $u\leqslant u_{\mathrm{c}}$, strictly increase for $u_{\mathrm{c}}<u<u_{\max}$, and are constant for $u\geqslant u_{\max}$. Thus we can express $\sigma_{\mathrm{e}}(u)$ as a function of $A(u)$, and the function γ takes the form

$$\gamma\big(t,u(0,t)\big)=\tilde{\gamma}(t)+\alpha\big(A(u(0,t))\big), \tag{7}$$

where α is a monotonous function on $[u_{\mathrm{c}},u_{\max}]$ having an inverse α^{-1}.

For numerical examples of the pressure filtration model and applications to experimental data we refer to [3, 12].

2.2. **Statement of the free boundary problem.** A natural property of any solution u of the free boundary problem in the context of the pressure filtration model should be $0 \leqslant u \leqslant 1$, i.e. solution values should be physically relevant as concentration values. However, due to the presence of the linear transport term $h'(t)u$ in combination with the kinematic boundary condition prescribed at $z = 0$ we cannot exclude that boundary layers involving unphysical solution values form. It turns out that this can be avoided if we consider that from a physical point of view, since the motion of the piston stops immediately as soon as the filter is 'clogged', i.e. when the solid particles at $z = 0$ form a dense packing. We consider this effect by replacing the coupling condition (5) by the condition

$$h'(t) + c\big(A(u(0,t))\big)\big[\beta(t)h(t) + \gamma\big(t, u(0,t)\big)\big] = 0, \quad 0 < t \leqslant T, \tag{8}$$

where $c(\rho) = 1$ for $\rho \in (0, A(u_{\max}))$ and $c(\rho) = 0$ otherwise.

Finally, it is convenient to introduce a new space coordinate $x = h(t) - z$. Then $x = 0$ corresponds to the piston and $x = h(t)$ to the filter medium, which is identified with the free boundary. Observing that $\partial_t(u(x,t)) = \partial_t u(z,t) + h'(t)\partial_z u$ and replacing $f(u)$ by $-f(u)$, we get the following free boundary value problem:

$$\partial_t u + \partial_x f(u) = \partial_x^2 A(u), \quad (x,t) \in Q(h,T), \tag{9a}$$

$$u(x,0) = u_0(x), \quad 0 \leqslant x \leqslant 1, \tag{9b}$$

$$\big(f(u) - \partial_x A(u)\big)(0,t) = 0, \quad 0 < t \leqslant T, \tag{9c}$$

$$\big(f(u) - \partial_x A(u)\big)\big(h(t),t\big) = h'(t)u\big(h(t),t\big), \quad 0 < t \leqslant T, \tag{9d}$$

$$h'(t) + c\Big(A\big(u\big(h(t),t\big)\big)\Big)\Big[\beta(t)h(t) + \gamma\Big(t, u\big(h(t),t\big)\Big)\Big] = 0, \quad 0 < t \leqslant T, \tag{9e}$$

$$h(0) = 1, \tag{9f}$$

where $Q(h,T) := \{(x,t) \in (0,1) \times (0,T] : 0 < x < h(t)\}$. Also, after the change of variables above, the relation (7) becomes

$$\gamma\big(t, u(h(t),t)\big) = \tilde{\gamma}(t) + \alpha\big(A(u(h(t),t))\big), \tag{10}$$

Since we are interested here exclusively in solutions that take values in the interval $[0,1]$ of admissible concentrations, we may assume that $a(u) = 0$ for $u \leqslant u_{\mathrm{c}}$ and $u \geqslant u_{\max}$, such that $A(u) = A(u_{\max})$ for $u \geqslant u_{\max}$ and $A(u) = 0$ for $u \leqslant u_{\mathrm{c}}$. In particular, we have $0 = \alpha(0) \leqslant \alpha(A(u(0,t))) \leqslant \alpha(A(u_{\max})) =: K_\alpha$ for all times. Since moreover, $\tilde{\gamma}$ is a control function given a priori, we may assume that there exist positive constants $k_{\tilde{\gamma}}$ and $K_{\tilde{\gamma}}$ with $k_{\tilde{\gamma}} \leqslant \tilde{\gamma}(t) \leqslant K_{\tilde{\gamma}}$ for all $t \in [0,T]$ and thus that there exist $k_\gamma, K_\gamma > 0$ with $k_\gamma \leqslant \gamma \leqslant K_\gamma$ for all $t \in [0,T]$. Similarly, we may assume that there exist $k_\beta, K_\beta > 0$ with $k_\beta \leqslant \beta(t) \leqslant K_\beta$ for all $t \in [0,T]$. Finally, to establish well-posedness of the free boundary problem, we assume that $T < 1/K_\gamma$.

2.3. **Divergence-measure fields.** Here we briefly recall the basic facts of the theory of divergence-measure fields as developed in [9, 10]. Since we will be only interested in the L^2 divergence-measure fields, we will focus our discussion on that case.

Let $\Omega \subset \mathbb{R}^N$ be an open bounded subset. We denote by $\mathcal{DM}^2(\Omega)$ the space of all $L^2(\Omega)$ vector fields whose divergence is a bounded Radon measure on Ω:

$$\mathcal{DM}^2(\Omega) := \left\{ F \in (L^2(\Omega))^N : \exists C > 0 : \forall \varphi \in C_0^\infty(\Omega), \left| \int_\Omega F \cdot \nabla\varphi \, d\mathbf{x} \right| \leqslant C\|\varphi\|_\infty \right\}, \tag{11}$$

where, as usual, $C_0^\infty(\Omega)$ denotes the space of the infinitely differentiable functions with compact support contained in Ω. Analogously, one may define $\mathcal{DM}^p(\Omega)$, $1 \leq p \leq \infty$, replacing L^2 by L^p, and $\mathcal{DM}^{\mathrm{ext}}(\Omega)$ replacing $L^2(\Omega)^N$ by $\mathcal{M}(\Omega)^N$, the space of vector-valued Radon measures over Ω with N components.

Definition 1. *We say that $\partial\Omega$ is a* deformable Lipschitz boundary *provided that:*

(a) *For all $x \in \partial\Omega$ there exists a number $r > 0$ and a Lipschitz map $h : \mathbb{R}^{N-1} \to \mathbb{R}$ such that, after rotating and relabeling coordinates if necessary,*

$$\Omega \cap Q(x,r) = \{ y \in \mathbb{R}^N \,:\, h(y_1, \dots, y_{N-1}) < y_N \} \cap Q(x,r),$$

where $Q(x,r) = \{ y \in \mathbb{R}^N \,:\, |x_i - y_i| \le r,\ i = 1, \dots, N \}$.

(b) *There exists a mapping $\Psi : \partial\Omega \times [0,1] \to \overline{\Omega}$ such that Ψ is a homeomorphism bi-Lipschitz over its image and $\Psi(\omega, 0) = \omega$ for all $\omega \in \partial\Omega$. The map Ψ is called a* Lipschitz deformation *of the boundary $\partial\Omega$. We denote $\Psi_s(\omega) = \Psi(\omega, s)$ and $\partial\Omega_s = \Psi_s(\partial\Omega)$. We also denote by Ω_s the bounded open set whose boundary is $\partial\Omega_s$.*

The following theorem is a particular case of a general result proved in [10], following the guide lines in [9]; we refer to [10] for the proof. If $\mathcal{C}$ is a closed set, we denote $\mathrm{Lip}(\mathcal{C})$ the space of Lipschitz functions defined on $\mathcal{C}$, equipped with the norm $\|f\|_{\mathrm{Lip}} = \|f\|_\infty + \mathrm{Lip}(f)$.

Theorem 1. *Let $F \in \mathcal{DM}^2(\Omega)$, Ω a bounded open set with Lipschitz deformable boundary. Then there exists a continuous linear functional $F \cdot \nu|_{\partial\Omega}$ over $\mathrm{Lip}(\partial\Omega)$, such that, for any $\phi \in \mathrm{Lip}(\mathbb{R}^N)$,*

$$\langle F \cdot \nu|_{\partial\Omega}, (\phi|\partial\Omega) \rangle = \int_\Omega \phi \operatorname{div} F + \int_\Omega \nabla\phi \cdot F. \tag{12}$$

Moreover, let $\nu : \Psi(\partial\Omega \times [0,1]) \to \mathbb{R}^N$ be so that $\nu(x)$ is the outer unit normal to $\partial\Omega_s$ at $x \in \partial\Omega_s$, defined for a.e. $x \in \Psi(\partial\Omega \times [0,1])$. Then, for any $\psi \in \mathrm{Lip}(\partial\Omega)$,

$$\langle F \cdot \nu|_{\partial\Omega}, \psi \rangle = \operatorname*{ess\,lim}_{s \to 0} \frac{1}{s} \int_0^s \left(\int_{\partial\Omega_s} \mathcal{E}(\psi) F \cdot \nu \, d\mathcal{H}^{N-1} \right) ds, \tag{13}$$

where $\mathcal{E}(\psi)$ denotes any Lipschitz extension of ψ to all $\mathbb{R}^N$ and $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure.

As an example, below we will consider a domain Ω of the form

$$\Omega = \{ (x,t) \in \mathbb{R}^2 \,:\, 0 < x < h(t),\ 0 < t < T \},$$

where h is a nonincreasing Lipschitz function satisfying $h(t) > h_0 > 0$, for some positive constant h_0. Clearly, in this case Ω satisfies (a) of Definition 1. We may also easily define a Lipschitz deformation for $\partial\Omega$. Indeed, since Ω is convex, given any point (x_*, t_*) in its interior, we may define the map $\Psi((x,t),s) = (x + s\delta(x_* - x), t + s\delta(t_* - t))$, from $\partial\Omega \times [0,1]$ to $\overline{\Omega}$, which, for $\delta > 0$ sufficiently small, certainly gives a Lipschitz deformation. But we will prefer to use deformations which, given $\delta > 0$ sufficiently small, on $\{(x,t) \,:\, x = 0,\ \delta < t < T - \delta\}$ are given by $\Psi_\delta((0,t),s) = (\delta s, t)$, and on $\{(x,t) \,:\, x = h(t),\ \delta < t < T - \delta\}$ are given by $\Psi_\delta((h(t),t),s) = (h(t) - \delta s, t)$. Clearly, Ψ_δ may be extended to all $\partial\Omega \times [0,1]$ in order to provide Lipschitz deformations for $\partial\Omega$. By the above theorem, if $F \in \mathcal{DM}^2(\Omega)$ and $\phi \in \mathrm{Lip}(\mathbb{R}^2)$ is such that $\operatorname{supp}\phi \cap \partial\Omega \subset \{x = 0\}$, then, for $\delta > 0$ sufficiently small,

$$\langle F \cdot \nu|_{\partial\Omega}, \phi \rangle = \operatorname*{ess\,lim}_{s \to 0} \frac{1}{s} \int_0^s \left(\int_0^T \phi(\delta s, t) F_1(\delta s, t) \, dt \right) ds. \tag{14}$$

On the other hand, if $\phi \in \mathrm{Lip}(\mathbb{R}^2)$ is such that $\operatorname{supp}\phi \cap \partial\Omega \subset \{(h(t),t), 0 < t < T\}$, then, for $\delta > 0$ sufficiently small,

$$\langle F \cdot \nu|_{\partial\Omega}, \phi \rangle = \operatorname*{ess\,lim}_{s \to 0} \frac{1}{s} \int_0^s \left(\int_0^T \phi(h(t) - \delta s, t)(F_1 - h'(t)F_2)(h(t) - \delta s, t) \, dt \right) ds. \tag{15}$$

3. Definition of generalized solutions

In the sequel let K be a sufficiently large constant, e.g. $K = 2u_{\max}$. As above, for fields $F(x,t) = (F_1(x,t), F_2(x,t))$ defined over domains of $\mathbb{R}^2$, which are distributions on these domains, the operator div is defined as $\operatorname{div} F = \partial_x F_1 + \partial_t F_2$, in the sense of distributions.

Definition 2. *A pair of functions (u,h) with $h \in C[0,T]$ and $u \in L^\infty(Q(h,T))$ is called a* generalized solution *of the free boundary problem* (9) *if the following conditions are satisfied:*

(a) *The function* $h(\cdot)$ *is nonincreasing and Lipschitz continuous on* $(0,T)$ *with* $h(0)=1$, *and there exists a positive constant* h_0 *such that* $h(t)>h_0$.

(b) *The following regularity properties hold:*

$$A(u)\in L^2\big(0,T;H^1(0,h(\cdot))\big), \tag{16}$$

$$\forall k\in\mathbb{R}:\ \Big(\operatorname{sgn}(u-k)\big(f(u)-f(k)\big)-\partial_x\big|A(u)-A(k)\big|,|u-k|\Big)\in\mathcal{DM}^2\big(Q(h,T)\big). \tag{17}$$

(c) *The boundary conditions are satisfied in the following sense: For* $(F_1,F_2)=(f(u)-\partial_x A(u),u)$, $\delta>0$ *sufficiently small, and every test function* $\varphi\in C_0^1(\Pi_T)$, *with* $\Pi_T=\mathbb{R}\times(0,T)$, *we have*

$$\operatorname*{ess\,lim}_{s\to 0}\frac{1}{s}\int_0^s\left(\int_0^T\varphi(\delta s,t)F_1(\delta s,t)\,dt\right)ds=0, \tag{18}$$

$$\operatorname*{ess\,lim}_{s\to 0}\frac{1}{s}\int_0^s\left(\int_0^T\varphi(h(t)-\delta s,t)(F_1-h'(t)F_2)(h(t)-\delta s,t)\,dt\right)ds=0. \tag{19}$$

(d) *Let* $\gamma_{x\to h(t)}A(u)$ *denote the trace (in the sense of traces in* $L^2(0,T;H^1(0,h(\cdot)))$ *of* $A(u)$ *for* $x\to h(t)$. *Then Eq.* (9e) *is satisfied a.e. in* $(0,T)$, *where in* $c(A(u(h(t),t)))$ *and in* $\gamma(t,u(h(t),t))$, *given by* (10), *we must replace* $A(u(h(t),t))$ *by* $\gamma_{x\to h(t)}A(u)$.

(e) *The initial condition is valid in the sense that*

$$\lim_{t\to 0}\int_0^{h(t)}\big|u(x,t)-u_0(x)\big|\,dx=0. \tag{20}$$

(f) *The following entropy inequality is satisfied for all nonnegative test functions* $\varphi\in C_0^\infty(Q(h,T))$ *and all* $k\in\mathbb{R}$*:*

$$\iint_{Q(h,T)}\Big\{|u-k|\partial_t\varphi+\operatorname{sgn}(u-k)\big[f(u)-f(k)-\partial_x A(u)\big]\partial_x\varphi\Big\}dtdx\geqslant 0. \tag{21}$$

It is convenient to transform the free boundary value problem (9) to an equivalent initial-boundary value problem with fixed boundaries by introducing a new space coordinate $\xi:=x/h(t)$. Wherever notationally convenient, the argument t in $h(t)$ is omitted, and we denote by h^{-1} the function $1/h(t)$ etc. Then we can rewrite (9) as the following initial-boundary value problem with fixed boundaries for $v(\xi,t):=u(h(t)\xi,t)$, where $Q_T:=(0,1)\times(0,T)$:

$$\partial_\xi v+h^{-1}h'\big(-\partial_\xi(\xi v)+v\big)+h^{-1}\partial_\xi f(v)=h^{-2}\partial_\xi^2A(v),\quad(\xi,t)\in Q_T, \tag{22a}$$

$$v(\xi,0)=u_0(\xi),\quad \xi\in[0,1], \tag{22b}$$

$$\big(f(v)-h^{-1}\partial_\xi A(v)\big)(0,t)=0,\quad t\in(0,T], \tag{22c}$$

$$\big(f(v)-h^{-1}\partial_\xi A(v)\big)(1,t)=h'(t)v(1,t),\quad t\in(0,T], \tag{22d}$$

$$h'(t)+c\Big(A\big(v(1,t)\big)\Big)\Big[\beta(t)h(t)+\gamma\Big(t,v\big(1,t\big)\Big)\Big]=0,\quad 0<t\leqslant T, \tag{22e}$$

$$h(0)=1, \tag{22f}$$

while the relation (10) becomes

$$\gamma\big(t,v(1,t)\big)=\tilde\gamma(t)+\alpha\big(A(v(1,t))\big). \tag{23}$$

In the sequel we use $h':=h'(t)$, $h^{-1}:=1/h(t)$, $h^{-2}:=1/(h(t))^2$ and similar notations for the function $h_\varepsilon(t)$ to be defined below. Moreover, we set $g(v,\xi,t):=-h^{-1}h'(t)\xi v+h^{-1}f(v)$.

The appropriate definition of entropy solution in terms of v reads:

Definition 3. *A pair of functions* (v,h) *with* $h\in C[0,T]$ *and* $v\in L^\infty(Q_T)$ *is called a* generalized solution *of the transformed free boundary problem* (22) *if the following conditions are satisfied:*

(a) *The function* $h(\cdot)$ *is nonincreasing and Lipschitz continuous on* $(0,T)$ *with* $h(0)=1$, *and there exists a positive constant* h_0 *such that* $h(t)>h_0$.

(b) *The following regularity properties hold:*

$$h^{-2}A(v) \in L^2\big(0,T;H^1(0,1)\big), \tag{24}$$

$$\begin{aligned} \forall k \in \mathbb{R}: \ \Big(&\operatorname{sgn}(v-k)\big(g(v,\xi,t)-g(k,\xi,t)\big) \\ &- h^{-2}\partial_\xi\big|A(v)-A(k)\big|, |v-k|\Big) \in \mathcal{DM}^2(Q_T). \end{aligned} \tag{25}$$

(c) *The boundary conditions are satisfied in the following sense: For $(F_1,F_2) = (g(v,\xi,t) - h^{-2}\partial_\xi A(v), v)$, $\delta > 0$ sufficiently small, and every test function $\varphi \in C_0^1(\Pi_T)$, with $\Pi_T = \mathbb{R}\times(0,T)$, we have*

$$\operatorname*{ess\,lim}_{s\to 0} \frac{1}{s}\int_0^s \left(\int_0^T \varphi(\delta s,t)F_1(\delta s,t)\,dt\right) ds = 0, \tag{26}$$

$$\operatorname*{ess\,lim}_{s\to 0} \frac{1}{s}\int_0^s \left(\int_0^T \varphi(1-\delta s,t)F_1(1-\delta s,t)\,dt\right) ds = 0. \tag{27}$$

(d) *Let $\gamma_{\xi\to 1}A(v)$ denote the trace of $A(v)$ for $\xi \to 1$ in the sense of traces in $L^2(0,T;H^1(0,1))$. Then Eq. (22e) is satisfied a.e. in $(0,T)$, where in $c(A(v\big(1,t)))$ and in $\gamma(t,v(1,t))$, given by (23), we must replace $A(v(1,t))$ by $\gamma_{x\to 1}A(v)$.*

(e) *The initial condition is valid in the sense that*

$$\lim_{t\to 0}\int_0^1 \big|v(\xi,t)-u_0(\xi)\big|\,d\xi = 0. \tag{28}$$

(f) *The following inequality holds for all nonnegative test functions $\varphi \in C_0^\infty(Q_T)$ and all $k \in \mathbb{R}$:*

$$\iint_{Q_T}\Big\{|v-k|\partial_t\varphi + \big[\operatorname{sgn}(u-k)\big(g(v,\xi,t)-g(k,\xi,t)\big) - \partial_\xi\big|A(v)-A(k)\big|\big]\partial_\xi\varphi\Big\}\,d\xi dt \geqslant 0. \tag{29}$$

4. Regularized free boundary problem

As in [4] we prove existence of entropy solutions by the vanishing viscosity method. To this end, we consider the regularized strictly parabolic free boundary problem

$$\partial_t u_\varepsilon + \partial_x f_\varepsilon(u_\varepsilon) = \partial_x^2 A_\varepsilon(u_\varepsilon), \quad (x,t)\in Q(h_\varepsilon,T), \tag{30a}$$

$$u_\varepsilon(x,0) = u_0^\varepsilon(x), \quad 0 \leqslant x \leqslant 1, \tag{30b}$$

$$\big(f_\varepsilon(u_\varepsilon) - \partial_x A_\varepsilon(u_\varepsilon)\big)(0,t) = 0, \quad 0 < t \leqslant T, \tag{30c}$$

$$\big(f_\varepsilon(u_\varepsilon) - \partial_x A_\varepsilon(u_\varepsilon)\big)\big(h_\varepsilon(t),t\big) = h_\varepsilon'(t)u_\varepsilon\big(h_\varepsilon(t),t\big), \quad 0 < t \leqslant T, \tag{30d}$$

$$h_\varepsilon'(t) + c_\varepsilon\Big(A_\varepsilon\big(u_\varepsilon\big(h_\varepsilon(t),t\big)\big)\Big)\Big[\beta_\varepsilon(t)h_\varepsilon(t) + \gamma_\varepsilon\Big(t,u_\varepsilon\big(h_\varepsilon(t),t\big)\Big)\Big] = 0, \quad 0 < t \leqslant T, \tag{30e}$$

$$h_\varepsilon(0) = 1. \tag{30f}$$

The regularized functions and initial and boundary data are assumed to satisfy first order compatibility conditions. Problem (30) is equivalent to the following initial-boundary value problem with fixed boundaries for $v_\varepsilon(\xi,t) := u_\varepsilon\big(h_\varepsilon(t)\xi,t\big)$ with $(\xi,t)\in Q_T := (0,1)\times(0,T)$:

$$\partial_t v_\varepsilon + h_\varepsilon^{-1}h_\varepsilon'(t)\big[-\partial_\xi(\xi v_\varepsilon) + v_\varepsilon\big] + h_\varepsilon^{-1}\partial_\xi f_\varepsilon(v_\varepsilon) = h_\varepsilon^{-2}\partial_\xi^2 A_\varepsilon(v_\varepsilon), \quad (\xi,t)\in Q_T, \tag{31a}$$

$$v_\varepsilon(\xi,0) = u_0^\varepsilon(\xi), \quad 0 \leqslant \xi \leqslant 1, \tag{31b}$$

$$\big(f_\varepsilon(v_\varepsilon) - h_\varepsilon^{-1}\partial_\xi A_\varepsilon(v_\varepsilon)\big)(0,t) = 0, \quad 0 < t \leqslant T, \tag{31c}$$

$$\big(f_\varepsilon(v_\varepsilon) - h_\varepsilon^{-1}\partial_\xi A_\varepsilon(v_\varepsilon)\big)(1,t) = h_\varepsilon'(t)v_\varepsilon(1,t), \quad 0 < t \leqslant T, \tag{31d}$$

$$h_\varepsilon'(t) + c_\varepsilon\big(A\big(v_\varepsilon(1,t)\big)\big)\Big[\beta_\varepsilon h_\varepsilon(t) + \gamma_\varepsilon\big(t,v_\varepsilon(1,t)\big)\Big] = 0, \quad 0 < t \leqslant T, \tag{31e}$$

$$h_\varepsilon(0) = 1. \tag{31f}$$

We choose the regularization c_ε such that c_ε is smooth, nonnegative, $c_\varepsilon(\rho) = 1$ for $\varepsilon \leqslant \rho \leqslant A(u_{\max}) - \varepsilon$, and $c_\varepsilon(\rho) = 0$, for $\rho \notin (0, A(u_{\max}))$. We assume that the regularization $f_\varepsilon \geqslant 0$ is also compactly supported, that $a_\varepsilon(u) \geqslant \varepsilon$, and that $a_\varepsilon(u) - \varepsilon$ is also compactly supported. We assume $\operatorname{supp} f_\varepsilon \cup \operatorname{supp} c_\varepsilon \subset \overline{\mathcal{U}} = [0, u_{\max}]$ and $\operatorname{supp}(a_\varepsilon - \varepsilon) \subset \overline{\mathcal{U}}$. Moreover, we define $g_\varepsilon(u, \xi, t) := -h_\varepsilon^{-1} h_\varepsilon' \xi u + h_\varepsilon^{-1} f_\varepsilon(u)$ and assume that there exist constants ν_ε, L_ε and $\tilde{L}$ such that

$$\frac{A_\varepsilon(u) - A_\varepsilon(v)}{u - v} \geqslant \nu_\varepsilon > 0, \quad \big|g_\varepsilon(u, \xi, t) - g_\varepsilon(v, \xi, t)\big| \leqslant L_\varepsilon |u - v| \quad \text{for } u, v \in \mathbb{R}. \tag{32}$$

Lemma 1. *Any solution u_ε of the regularized free boundary problem* (30) *satisfies $u_\varepsilon(x,t) \in \overline{\mathcal{U}}$ for all $(x,t) \in \overline{Q(h_\varepsilon, T)}$. Equivalently, any solution v_ε of* (31) *satisfies*

$$v_\varepsilon(x,t) \in \overline{\mathcal{U}} \quad \text{for all } (x,t) \in \overline{Q_T}. \tag{33}$$

In particular, there exists a constant M_0 independent of ε such that for all sufficiently small $\varepsilon > 0$,

$$\|u_\varepsilon\|_{L^\infty(Q(h_\varepsilon, T))} \leqslant M_0. \tag{34}$$

Proof. Consider the regularized problem (30), perturbed by adding to the right-hand member the term $\lambda N(u_\varepsilon)$, where $\lambda > 0$ and $N(u) = u_{\max}/2 - u$. We may assume h_ε to be a given smooth function, so the problem is in fact given by the first four equations of (30), with the first one perturbed. If we prove the result for the perturbed problem, then by the well known stability for quasilinear strictly parabolic scalar equations, with respect to coefficients, the desired result will follow sending $\lambda \to 0$. Now, if the result is not true for the perturbed problem, there is a time t_0 at which the solution v_ε leaves $\overline{\mathcal{U}}$ for the first time, that is, $t_0 = \inf\{t : v_\varepsilon(x,t) \notin \overline{\mathcal{U}} \text{ for some } x \in [0, h(t)]\}$. In this case, there exists $x_0 \in [0, h(t_0)]$ such that $u_\varepsilon(x_0, t_0) \in \{0, u_{\max}\}$, say, $u_\varepsilon(x_0, t_0) = u_{\max}$. If $x_0 \in (0, h(t_0))$, as usual, we get a contradiction using that $\partial_x u_\varepsilon = 0$, $\partial_t u_\varepsilon \geqslant 0$, $\partial_x^2 u_\varepsilon \leqslant 0$, $a_\varepsilon(u) > 0$, and $N(u_{\max}) < 0$. On the other hand, if $x_0 \in \{0, h(t_0)\}$, using (30c)–(30e), we again conclude that $\partial_x u_\varepsilon = 0$. Hence, we must have again $\partial_t u_\varepsilon \geqslant 0$, $\partial_x^2 u_\varepsilon \leqslant 0$ and so we get a contradiction in the same way. □

Lemma 2. *Suppose that $T < 1/K_\gamma$ and that the coefficients of the regularized problem* (30) *satisfy compatibility conditions. Then this problem has a unique solution $(u_\varepsilon, h_\varepsilon)$ such that $u_\varepsilon \in C^{2+\alpha, 1+\alpha/2}(\overline{Q}(h_\varepsilon, T))$ and $h_\varepsilon \in C^{1+\alpha/2}[0,T]$. Precisely, the function h_ε satisfies the following estimates uniformly in ε:*

$$0 < h_0 \leqslant h_\varepsilon(t) \leqslant 1, \quad \|h_\varepsilon'\|_{L^\infty(0,T)} \leqslant M_h := K_\beta + K_\gamma. \tag{35}$$

Proof of Lemma 2. Suppose that $(u_\varepsilon, h_\varepsilon)$ with $u_\varepsilon \in C^{2,1}(\overline{Q}(h_\varepsilon, T))$ and $h_\varepsilon \in C^1(0,T)$ is a solution of problem (30), or equivalently that v_ε satisfies the initial-boundary value problem with fixed boundaries (31). In addition, consider for a fixed function $h_\varepsilon \in C^1[0,T]$ the initial-boundary value problem (31′) consisting of equation (31a) and the initial and boundary conditions (31b)–(31d).

The proof of the following lemma is standard and can be found e.g. in [15, Ch. V]:

Lemma 3. *Under the assumptions of Lemma 2, the solution w_ε of the IBVP* (31′) *satisfies the following estimates, where the constant K_1 is independent of ε:*

$$0 \leqslant w_\varepsilon \leqslant K_1, \quad \|w_\varepsilon\|_{C^\beta(Q_T)} \leqslant K_2, \quad \|\partial_\xi w_\varepsilon\|_{C^{1,1/2}(\overline{Q}_T)} \leqslant K_2, \quad \|w_\varepsilon\|_{W^{2,1}_\infty(\overline{Q}_T)} \leqslant K_2.$$

To prove the existence of a solution of problem (31), we follow Zhao and Li [27] and use the Schauder fixed point theorem. To this end, define the set

$$H = \big\{h \in C^1(0,T) : \|h'\|_\infty \leqslant M_h, \ h(0) = 1, \ h \text{ is nonincreasing}\big\},$$

where the constant M_h is defined in (35). Note that H is a compact convex set in the Banach space $C^0[0,T]$. Moreover, let $\widehat{\beta}_\varepsilon(t,u) := c_\varepsilon(A_\varepsilon(u))\beta_\varepsilon(t)$ and $\widehat{\gamma}_\varepsilon(t,u) := c_\varepsilon(u)\gamma_\varepsilon(t,u)$.

Lemma 4. *Let the operator $\mathcal{T} : H \to C^0[0,T]$ be defined by*

$$(\mathcal{T}h)(t) := \exp\Big(\widehat{B}_\varepsilon\big(t; w_\varepsilon(1,\cdot)\big)\Big)\left[1 - \int_0^t \exp\Big(-\widehat{B}_\varepsilon\big(\tau; w_\varepsilon(1,\cdot)\big)\Big)\widehat{\gamma}_\varepsilon\big(\tau, w_\varepsilon(1,\tau)\big)\, d\tau\right],$$

$$\widehat{B}_\varepsilon(t; w) := -\int_0^t \widehat{\beta}_\varepsilon\big(\tau, w(\tau)\big)\, d\tau,$$

where w_ε is the solution of the IBVP (31′) corresponding to h. Then $\mathcal{T}h \in H$, i.e. the operator $\mathcal{T}$ maps H into itself.

Proof of Lemma 4. In order to make the main ideas of the proof apparent, and since the statement of Lemma 4 refers to a fixed value of the regularization parameter ε, we simplify notation in this proof by omitting ε wherever possible.

Obviously, we have $(\mathcal{T}h)(0) = 1$. Since the functions $\widehat{B}(\cdot; w)$ and $\widehat{\gamma}\big(\cdot, w(1,\cdot)\big)$ are smooth, as stated in Lemma 3, we see that $\mathcal{T}h \in C^1[0,T]$. Furthermore we have

$$\begin{aligned}(\mathcal{T}h)'(t) = &-\widehat{\beta}\big(t, w(1,t)\big)\exp\Big(\widehat{B}\big(t, w(1,t)\big)\Big)\times \\ &\times\left[1 - \int_0^t \exp\Big(-\widehat{B}\big(\tau; w(1,\cdot)\big)\Big)\widehat{\gamma}\big(\tau, w(1,\tau)\big)\,d\tau\right] - \widehat{\gamma}\big(t, w(1,t)\big).\end{aligned} \tag{36}$$

Since $\widehat{\gamma}\big(t, w(1,t)\big) \leqslant K_\gamma$ for $\varepsilon > 0$ sufficiently small, the expression in the square brackets in (36) is nonnegative, and thus $\mathcal{T}h$ is nonincreasing, if the condition $T < 1/K_\gamma$ is satisfied. Moreover, this assumption implies that $|(\mathcal{T}h)'(t)| \leqslant K_\beta + K_\gamma$. We conclude that indeed $\mathcal{T}h \in H$. □

To apply the Schauder fixed point theorem, and thus to show existence of the solution, we have to prove the following lemma:

Lemma 5. *Suppose that $\{h_n\}_{n\in\mathbb{N}} \subset H$ and $\|h_m - h_n\|_{C^0[0,T]} \to 0$ as $m, n \to \infty$. Then $\|\mathcal{T}h_m - \mathcal{T}h_n\|_{C^0[0,T]} \to 0$ as $m, n \to \infty$.*

Proof of Lemma 5. Assume that $h_n \to h$ uniformly in $[0,T]$. Since $\|h_n'\|_\infty \leqslant M_h$, we can conclude that $h' \in L^\infty[0,T]$ and $h_n' \to h'$ weakly in $L^1[0,T]$. Let w_n and w denote the solutions of the IBVP (31′) associated with the functions h_n and h, respectively. From Lemma 3 it follows that there exist subsequences $\{w_{n_j}\}_{j\in\mathbb{N}}$ and $\{\partial_x w_{n_j}\}_{j\in\mathbb{N}}$ of $\{w_n\}_{n\in\mathbb{N}}$ and $\{\partial_x w_n\}_{n\in\mathbb{N}}$, respectively, converging uniformly on $\overline{Q_T}$. Let $\overline{w}$ and $\overline{w}_x$ denote the limit functions. Multiplying equation (31a), with v replaced by w_{n_j}, by a test function $\varphi \in C_0^2(Q_T)$, integrating over Q_T, and using integration by parts, we obtain

$$\iint_{Q_T}\Big\{w_{n_j}\partial_t\varphi + h_{n_j}^{-1}h_{n_j}' w_{n_j}(\varphi + \xi\partial_\xi\varphi) + \big(h_{n_j}^{-1}f(w_{n_j}) - h_{n_j}^{-2}\partial_\xi A(w_{n_j})\big)\partial_\xi\varphi\Big\}d\xi dt = 0.$$

Letting $j \to \infty$, we get

$$\iint_{Q_T}\Big\{\overline{w}\partial_t\varphi + h^{-1}h'\overline{w}(\varphi + \xi\partial_\xi\varphi) + \big(h^{-1}f(\overline{w}) - h^{-2}\partial_\xi A(\overline{w})\big]\partial_\xi\varphi\Big\}d\xi dt = 0.$$

Since solutions of the IBVP (31′) are unique, we obtain $\overline{w} = w$, hence the sequences $\{w_n\}_{n\in\mathbb{N}}$ and $\{\partial_x w_n\}_{n\in\mathbb{N}}$ converge uniformly on $\overline{Q_T}$. Lemma 5 is then an immediate consequence of

$$\begin{aligned}(\mathcal{T}h_n - \mathcal{T}h_m)(t) = &\exp\Big(\widehat{B}\big(t, w(1,\cdot)\big)\Big)\times \\ &\times\int_0^t \exp\Big(-\widehat{B}\big(\tau, w(1,\cdot)\big)\Big)\Big[\widehat{\gamma}\big(\tau, w_m(1,\tau)\big) - \widehat{\gamma}\big(\tau, w_n(1,\tau)\big)\Big]\,d\tau. \qquad \square\end{aligned}$$

We continue with the proof of Lemma 2. By Lemma 5, $\mathcal{T}$ is a continuous operator on H. We are now in a position to conclude from the Schauder fixed point theorem that $\mathcal{T}$ has a fixed point $h \in H$; in particular $h \in C^{1+\alpha/2}[0,T]$. This also proves the estimates (35).

Substituting the fixed point h into the IBVP (31′) produces a solution $w \in C^{2+\alpha,1+\alpha/2}(Q_T)$ with the property that the pair (w,h) also satisfies the fixed point equation $\mathcal{T}h = h$, which is equivalent to equation (9f). Consequently, $(v \equiv w, h)$ is a solution of the IBVP (31), and setting $u(x,t) = v(x/h(t), t)$ produces a solution (u,h) of the regularized free boundary problem (30) with $u \in C^{2+\alpha,1+\alpha/2}(\overline{Q}(h,T))$. Thus the existence part of Lemma 2 is proved.

We now turn to the uniqueness part. From boundary condition (30d) we get

$$\frac{1}{2}h^2(t) = \int_0^t h(s)h'(s)\,ds + \frac{1}{2} = \int_0^t \frac{h(s)}{u}\big(f(u) - \partial_x A(u)\big)\big(h(s), s\big)\,ds + \frac{1}{2}.$$

We now choose a test function $\omega \in C^2(\mathbb{R})$ satisfying $\omega(x) = 0$ for $x \leqslant h_0/2$ and $\omega(x) = 1$ for $x \geqslant 3h_0/4$. We then get

$$\begin{aligned}
&\int_0^t \frac{h(s)}{u}\big(f(u) - \partial_x A(u)\big)\big(h(s), s\big)\, ds = \iint_{Q(h,t)} \partial_x \left(\frac{x\omega(x)}{u}\big(f(u) - \partial_x A(u)\big)\right) dxds \\
&= \iint_{Q(h,t)} \left\{ \big(\omega(x) + x\omega'(x)\big) \frac{f(u) - \partial_x A(u)}{u} + x\omega(x)\partial_x \left(\frac{f(u) - \partial_x A(u)}{u}\right) \right\} dxds \\
&= \iint_{Q(h,t)} \big(\omega(x) + x\omega'(x)\big) \frac{f(u) - \partial_x A(u)}{u}\, dxds \\
&\quad + \iint_{Q(h,t)} x\omega(x)\big(f(u) - \partial_x A(u)\big)\partial_x\Big(\frac{1}{u}\Big)\, dxds + \iint_{Q(h,t)} \frac{x\omega(x)}{u}(-\partial_s u)\, dxds \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

Defining

$$\widetilde{A}(u) := \int_0^u \frac{a(r)}{r} dr, \quad f^1(u) := \int_0^u \frac{f'(r)}{r} dr, \quad f^2(u) := \int_{u_0}^u \frac{f(r)}{r^2} dr,$$

with $u_0 > 0$, we see that

$$\begin{aligned}
I_2 &= \iint_{Q(h,t)} x\omega(x)\partial_x \left(\frac{f(u) - \partial_x A_\varepsilon(u)}{u}\right) dxds - \iint_{Q(h,t)} \frac{x\omega(x)}{u}\partial_x\big(f(u) - \partial_x A_\varepsilon(u)\big)\, dxds \\
&= \int_0^t h(s)\left(\frac{f(u) - \partial_x A_\varepsilon(u)}{u}\right) ds - \iint_{Q(h,t)} \big(\omega(x) + x\omega'(x)\big)\big(-f^1_\varepsilon(u) + f^2(u) - \partial_x \widetilde{A}_\varepsilon(u)\big)\, dxds \\
&\quad + \iint_{Q(h,t)} \frac{x\omega(x)}{u}\partial_s u\, dxds.
\end{aligned}$$

Using integration by parts and the boundary condition, we get

$$\begin{aligned}
I_2 &= \int_0^t h(s)\Big\{-f^1\big(u(h(s), s)\big) + f^2\big(u(h(s), s)\big) - \partial_x \widetilde{A}\big(u(h(s), s)\big)\Big\} ds \\
&\quad - \iint_{Q(h,t)} \Big\{ \big(2\omega'(x) + x\omega''(x)\big)\widetilde{A}(u) + \big(\omega(x) + x\omega'(x)\big)\big(f^1(u) - f^2(u)\big)\Big\}\, dxds \\
&\quad + \iint_{Q(h,t)} \frac{x\omega(x)}{u}\partial_s u\, dxds + \int_0^t \widetilde{A}_\varepsilon\big(u\big(h(s), s\big)\big)\, ds.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\frac{1}{2}h^2(t) =& \frac{1}{2} + \iint_{Q(h,t)} (\omega + x\omega')\frac{f(u) - \partial_x A(u)}{u} dxds \\
&+ \int_0^t h(s)\Big\{-f^1\big(u(h(s), s)\big) + f^2\big(u(h(s), s)\big) - \partial_x \widetilde{A}\big(u(h(s), s)\big)\Big\} ds \\
&+ \int_0^t \widetilde{A}\big(u(h(s), s)\big)\, ds - \iint_{Q(h,t)} \Big\{(\omega + x\omega')\big(f^1(u) - f^2(u)\big) + (2\omega' + x\omega'')\widetilde{A}\Big\} dxds.
\end{aligned}$$

Now let (u^1, h^1) and (u^2, h^2) be two solutions of the regularized free boundary problem (30). Let

$$t_1 = \max\big\{t \in [0, T] :\ h^1(\tau) = h^2(\tau) \text{ for } \tau \in [0, t]\big\}.$$

We now show that $t_1 = T$. To this end, we first suppose that $t_1 < T$. Without loss of generality, we suppose that $t_1 = 0$. Moreover, define $h^-(t) := \min\big\{h^1(t), h^2(t)\big\}$, $h^+(t) := \max\big\{h^1(t), h^2(t)\big\}$,

$j(t) := 1$ if $h^1(t) > h^2(t)$, $j(t) := 2$ if $h^1(t) \leqslant h^2(t)$ and $i(t) := 3 - j(t)$. Then we obtain

$$
\begin{aligned}
&\frac{1}{2}\big((h^1)^2(t) - (h^2)^2(t)\big) = \iint_{Q(h^-,t)} (\omega + x\omega')\left[\frac{f(u^1) - \partial_x A(u^1)}{u^1} - \frac{f(u^2) - \partial_x A(u^2)}{u^2}\right] dxds \\
&\quad - \int_0^t (-1)^{j(s)} \int_{h^-(s)}^{h^+(s)} (\omega + x\omega') \frac{f(u^{j(s)}) - \partial_x A(u^{j(s)})}{u^{j(s)}}\, dxds \\
&\quad + \int_0^t \Big\{ h^1(s)\Big[-f^1\big(u^1(h^1(s),s)\big) + f^2\big(u^1(h^1(s),s)\big) - \partial_x \widetilde{A}\big(u^1(h^1(s),s)\big)\Big] \\
&\quad - h^2(s)\Big[-f^1\big(u^2(h^2(s),s)\big) + f^2\big(u^2(h^2(s),s)\big) - \partial_x \widetilde{A}\big(u^2(h^2(s),s)\big)\Big]\Big\} ds \\
&\quad + \int_0^t \Big\{ \widetilde{A}\big(u^1(h^1(s),s)\big) - \widetilde{A}\big(u^2(h^2(s),s)\big)\Big\}\, ds \\
&\quad + \iint_{Q(h^-,t)} \Big\{ (\omega + x\omega')\big(-f^1(u^1) + f^2(u^1) + f^1(u^2) - f^2(u^2)\big) \\
&\quad - (2\omega' + x\omega'')\big(\widetilde{A}(u^1) - \widetilde{A}(u^2)\big)\Big\} dsdt \\
&\quad - \int_0^t (-1)^{j(s)} \int_{h^-(s)}^{h^+(s)} \Big\{ \big(\omega(x) + x\omega'(x)\big)\Big(-f^1\big(u^{j(s)}\big) + f^2\big(u^{j(s)}\big)\Big) - (2\omega' + x\omega'')\widetilde{A}\big(u^{j(s)}\big)\Big\} dsdt \\
&=: I_4 + \cdots + I_9.
\end{aligned}
$$

We now set $\delta(t) := |h^1(t) - h^2(t)|$. First note that

$$
\big|(h^1)^2(t) - (h^2)^2(t)\big| = \big|h^1(t) + h^2(t)\big|\delta(t) \geqslant M_1\delta(t), \quad M_1 := 2h^{\min}.
$$

We now estimate the integrals I_4 to I_9. In view of

$$
\begin{aligned}
I_4 = &\iint_{Q(h^-.t)} (\omega + x\omega')\left(\frac{f(u^1)}{u^1} - \frac{f(u^2)}{u^2}\right) dxds - \int_0^t \Big\{ \widetilde{A}\big(u^1(h^-(s),s)\big) - \widetilde{A}\big(u^2(h^-(s),s)\big)\Big\}\, ds \\
&+ \iint_{Q(h^-,t)} (2\omega' + \omega'')\big(\widetilde{A}(u^1) - \widetilde{A}(u^2)\big)\, dxds
\end{aligned}
$$

and the inequality

$$
\big|\widetilde{A}\big(u^1(h^-(s),s)\big) - \widetilde{A}\big(u^2(h^-(s),s)\big)\big| \leqslant \varepsilon^{-1}\|a\|_\infty \big|u^1(h^-(s),s) - u^2(h^-(s),s)\big|,
$$

it is easy to see that there exist constants C_2 and C_3 such that

$$
|I_4| \leqslant C_2 \int_0^t \big|u^1(h^-(s),s) - u^2(h^-(s),s)\big| ds + C_3 \int_0^t \int_0^{h^-(t)} \big|u^1(x,s) - u^2(x,s)\big|\, dxds. \tag{37}
$$

Next, noting that in view of boundary condition (30c)

$$
\begin{aligned}
&\big|f\big(u^{j(s)}(x,s)\big) - \partial_x A_\varepsilon\big(u^{j(s)}(x,s)\big)\big| \\
&= \big|f\big(u^{j(s)}(x,s)\big) - f\big(u^{j(s)}\big(h^+(s),s\big)\big) - \partial_x A_\varepsilon\big(u^{j(s)}(x,s)\big) + \partial_x A_\varepsilon\big(u^{j(s)}\big(h^+(s),s\big)\big)\big| \\
&\leqslant \Big(\|f'\|_\infty \|\partial_x u(\cdot,s)\|_\infty + \|a_\varepsilon'\|_\infty \|\partial_x u(\cdot,s)\|_\infty + \|a_\varepsilon\|_\infty \|\partial_x^2 u(\cdot,s)\|_\infty\Big)\big|x - h^+(s)\big|,
\end{aligned} \tag{38}
$$

we obtain that there exists a constant C_3 satisfying $|I_5| \leqslant C_4\delta^2(t)$. Observe that

$$
\begin{aligned}
&\big|\widetilde{A}\big(u^1\big(h^1(s),s\big)\big) - \widetilde{A}\big(u^2\big(h^2(s),s\big)\big)\big| \\
&\leqslant \big|\widetilde{A}\big(u^{j(s)}(h^+(s),s)\big) - \widetilde{A}\big(u^{j(s)}(h^-(s),s)\big)\big| + \big|\widetilde{A}\big(u^{j(s)}(h^-(s),s)\big) - \widetilde{A}\big(u^{i(s)}(h^-(s),s)\big)\big| \\
&\leqslant \varepsilon^{-1}\|a\|_\infty \|\partial_x u(\cdot,s)\|_\infty \delta(t) + \varepsilon^{-1}\|a\|_\infty \big|u^{j(s)}\big((h^-(s),s)\big) - u^{i(s)}\big(h^-(s),s\big)\big|.
\end{aligned}
$$

¿From this inequality and similar ones for the functions $\partial_x \widetilde{A}$, f^1 and f^2 we obtain that there exist constants C_5 and C_6 such that

$$|I_6| + |I_7| \leqslant C_5 \int_0^t \delta(\tau)\, d\tau + C_6 \int_0^t \big|u^1\big(h^-(s), s\big) - u^2\big(h^-(s), s\big)\big|\, ds. \tag{39}$$

By similar arguments it follows that there exist constants C_7 and C_8 satisfying

$$|I_8| \leqslant C_7 \int_0^t \delta(\tau)\, d\tau + C_8 \int_0^t \int_0^{h^-(s)} \big|u^1(x, s) - u^2(x, s)\big|\, dxds. \tag{40}$$

Finally, since the integrand of I_9 is bounded, there exists a constant C_9 such that

$$|I_9| \leqslant C_9 \int_0^t \delta(\tau)\, d\tau. \tag{41}$$

Summarizing the estimates on I_4 to I_9, we obtain

$$\begin{aligned} \delta(t) \leqslant & C_4\delta^2(t) + C_{10} \int_0^t \big|u^1\big(h^-(s), s\big) - u^2\big(h^-(s), s\big)\big|\, ds \\ & + C_{11} \int_0^t \delta(s)\, ds + C_{12} \int_0^t \int_0^{h^-(s)} \big|u^1(x, s) - u^2(x, s)\big|\, dxds \end{aligned} \tag{42}$$

witth suitable new constants C_{10} to C_{12}. To estimate the right-hand part of (42), let $z(x, s) := u^1(x, s) - u^2(x, s)$. This function satisfies in $Q(h^-, t)$ the linear equation

$$\partial_t z - \widetilde{a}\partial_x^2 z + \widetilde{b}\partial_x z + \widetilde{c} z = 0, \tag{43}$$

where the coefficients $\widetilde{a}$ to $\widetilde{c}$ are given by (the argument (x, s) is omitted wherever appropriate)

$$\widetilde{a} = a(u^1), \quad \widetilde{b} = a'(\partial_x u^1 + \partial_x u^2) + f'(u^1), \quad \widetilde{c} = \partial_x^2 u^2 \overline{a'} + (\partial_x u^2)^2 \overline{a''} + \partial_x u^2 \overline{f''},$$

where

$$\overline{g}(x, s) := \int_0^1 g\big(\lambda u^1(x, s) + (1-\lambda) u^2(x, s)\big) d\lambda, \quad g \in \Big\{a', a'', f', f'', \partial_2\widehat{\gamma}, \partial_2\widehat{\beta}\Big\}.$$

The function z satisfies the initial condition $z(x, 0) = 0$ for $0 \leqslant x \leqslant 1$. From boundary condition (30c) and estimate (38) we obtain

$$\big((\overline{f'} - \partial_x u^2 \overline{a'}) z - a\big(u^1\big)\partial_x z\big)(0, s) = \psi^1(s).$$

Similarly, boundary condition (30d) implies

$$\begin{gathered} \Big(\Big[\overline{f'} + \big[\widehat{\beta}\big(s, u^1\big) h^1(s) + \widehat{\gamma}\big(s, u^1\big)\big] + \overline{\partial_2\widehat{\beta}} h^2(s) u^2 \\ + \overline{\partial_2\widehat{\gamma}} u^2 + \overline{a'}(\partial_x u)^2\Big] z - a(u^1)\partial_x z\Big)\big(h^-(s), s\big) = \psi^2(s), \\ \psi^2(s) := -\widehat{\beta}\big(s, u^1(h^-(s), s)\big)\big(h^1(s) - h^2(s)\big) u^2\big(h^-(s), s\big). \end{gathered}$$

Since the functions $\widetilde{a}$ to $\widetilde{c}$ are bounded and since there exist constants C_{13} to C_{15} such that $|\widetilde{d}(x, s)| \leqslant C_{13}\delta(t)$, $\big|\psi^1(s)\big| \leqslant C_{14}\delta(s)$ and $\big|\psi^2(s)\big| \leqslant C_{15}\delta(s)$, we obtain from the maximum principle that there exists a constant C_{16} independent of t with

$$|z(x, t)| \leqslant C_{16} \max_{0 \leqslant s \leqslant t} \delta(s), \tag{44}$$

hence inequality (42) reduces to

$$\delta(t) \leqslant C_4\delta^2(t) + C_{17} \int_0^t \max_{0 \leqslant \tau \leqslant s} \delta(\tau)\, ds. \tag{45}$$

Since $\delta(0) = 0$ and $\delta'(s)$ is uniformly bounded, we can choose a time $t_0 \in (0, T]$ such that $C_4\delta(t) \leqslant 1/2$ for all $t \in (0, t_0]$. Thus

$$\delta(t) \leqslant \frac{1}{2} \max_{0 \leqslant \tau \leqslant t} \delta(\tau) + C_{17} \int_0^t \max_{0 \leqslant \tau \leqslant s} \delta(\tau)\, ds \quad \text{for } 0 \leqslant t \leqslant t_0. \tag{46}$$

Consequently, there exists a constant C_{18} such that

$$\delta(t) \leqslant C_{18} \int_0^t \max_{0 \leqslant \tau \leqslant s} \delta(\tau)\, ds \quad \text{for } 0 \leqslant t \leqslant t_0. \tag{47}$$

This shows that $\delta(t) = 0$, i.e. $h^1(t) = h^2(t) =: h(t)$ for $0 \leqslant t \leqslant t_0$. The maximum principle then implies $u^1(x,t) = u^2(x,t)$ for $(x,t) \in Q(h,t_0)$, which contradicts the definition of t_1. Consequently, we obtain $u^1(x,t) = u^2(x,t)$ in $Q(h,T)$. This concludes the proof of Lemma 2. □

5. Existence of generalized solutions

To prove the existence of a generalized solution, we have to establish uniform estimates (with respect to the regularization parameter ε) on the solutions u_ε of the regularized free boundary problem (30). It is convenient to formulate these estimates in terms of the solutions $\{v_\varepsilon\}_{\varepsilon>0}$ of the problem (31) with fixed boundaries.

Lemma 6. *Let $(v_\varepsilon, h_\varepsilon)$ be a solution of the regularized boundary problem (31). Then the following uniform estimates are valid, where the constant M_2 is independent of ε:*

$$\sup_{t \in [0,T]} \|\partial_x v_\varepsilon(\cdot,t)\|_{L^1(0,1)} \leqslant M_2. \tag{48}$$

Proof. The proof closely follows that of Lemma 11 in [4]. Define approximations sgn_η and $|\cdot|_\eta$ of the sign and modulus functions by

$$\mathrm{sgn}_\eta(\tau) := \begin{cases} \mathrm{sgn}(\tau) & \text{if } |\tau| > \eta, \\ \tau/\eta & \text{if } |\tau| \leqslant \eta, \end{cases} \qquad |x|_\eta := \int_0^x \mathrm{sgn}_\eta(\zeta) d\zeta, \quad \eta > 0.$$

Setting $y_\varepsilon := \partial_\xi v_\varepsilon$, we obtain by differentiating equation (31a) with respect to ξ, multiplying it by $\mathrm{sgn}_\eta(y_\varepsilon)$, integrating over Q_{T_0}, where $0 < T_0 \leqslant T$, and using integration by parts:

$$\begin{aligned} \iint_{Q_{T_0}} \mathrm{sgn}_\eta(y_\varepsilon) \partial_t y_\varepsilon \, d\xi dt &= \int_0^{T_0} \mathrm{sgn}_\eta(y_\varepsilon) \bigl(-\partial_\xi g_\varepsilon(v_\varepsilon, \xi, t) + h_\varepsilon^{-2} \partial_\xi^2 A_\varepsilon(v_\varepsilon) \bigr) \Big|_{\xi=0}^{\xi=1} dt \\ &+ \iint_{Q_{T_0}} \mathrm{sgn}_\eta'(y_\varepsilon) \partial_\xi y_\varepsilon \Bigl\{ -h_\varepsilon^{-1} h_\varepsilon' \xi + h_\varepsilon^{-1} f_\varepsilon'(v_\varepsilon) - h_\varepsilon^{-2} a_\varepsilon'(v_\varepsilon) y_\varepsilon \Bigr\} y_\varepsilon \, d\xi dt \\ &- \iint_{Q_{T_0}} \mathrm{sgn}_\eta'(y_\varepsilon) a_\varepsilon(v_\varepsilon) (\partial_\xi y_\varepsilon)^2 \, d\xi dt - \iint_{Q_{T_0}} \mathrm{sgn}_\eta(y_\varepsilon) h_\varepsilon^{-1} h_\varepsilon' y_\varepsilon \, d\xi dt =: I_\eta^1 + I_\eta^2 + I_\eta^3 + I_\eta^4. \end{aligned} \tag{49}$$

We now estimate the integrals I_η^1 to I_η^4. Using equation (31a), we see that

$$I_\eta^1 = \int_0^{T_0} \Bigl\{ \mathrm{sgn}_\eta\bigl(\partial_\xi v_\varepsilon(1,t)\bigr) \partial_t v_\varepsilon(1,t) - \mathrm{sgn}_\eta\bigl(\partial_\xi v_\varepsilon(0,t)\bigr) \partial_t v_\varepsilon(0,t) \Bigr\} \, dt.$$

The boundary conditions (31c) and (31d) imply that

$$\partial_\xi v_\varepsilon(0,t) = \frac{h_\varepsilon f_\varepsilon(v_\varepsilon(0,t))}{a_\varepsilon(v_\varepsilon(0,t))} \geqslant 0, \quad \partial_\xi v_\varepsilon(1,t) = \frac{h_\varepsilon [f_\varepsilon(v_\varepsilon(1,t)) - h_\varepsilon' v_\varepsilon(1,t)]}{a_\varepsilon(v_\varepsilon(1,t))} \geqslant 0. \tag{50}$$

In view of Lemma 1, we see from (50) that $\partial_\xi v_\varepsilon(0,t) = 0$ implies that $v_\varepsilon(0,t)$ assumes the constant value $v_{\varepsilon\min} := \inf \mathcal{U}^\varepsilon$ or $v_{\varepsilon\max} := \sup \mathcal{U}^\varepsilon$. Letting $\mathcal{E}_0 := \{t \in [0,T] : v_\varepsilon(0,t) = v_{\varepsilon\min}$ or $v_\varepsilon(0,t) = v_{\varepsilon\max}\}$, we see that $\partial_t v_\varepsilon(0,t) = 0$ a.e. in $\mathcal{E}_0$. We therefore conclude that

$$-\int_0^{T_0} \mathrm{sgn}_\eta\bigl(y_\varepsilon(0,t)\bigr) \partial_t v_\varepsilon(0,t)\, dt \overset{\eta \to 0}{\longrightarrow} -\int_0^{T_0} \partial_t v_\varepsilon(0,t)\, dt = v_\varepsilon(0,0) - v_\varepsilon(0,T_0).$$

Applying a similar argument to the boundary condition (31d), we obtain

$$I_\eta^1 \overset{\eta \to 0}{\longrightarrow} v_\varepsilon(1,T_0) - v_\varepsilon(1,0) + v_\varepsilon(0,0) - v_\varepsilon(0,T_0).$$

¿From Saks' lemma [2, 22] we infer that $I_\eta^2 \to 0$ for $\eta \downarrow 0$. In view of $I_\eta^3 \leqslant 0$ and

$$I_\eta^4 \overset{\eta \to 0}{\longrightarrow} -\iint_{Q_{T_0}} h_\varepsilon^{-1} h_\varepsilon' |y_\varepsilon| \, d\xi dt,$$

we get from (49)

$$\begin{aligned}\big\|\partial_x v_\varepsilon(\cdot,T_0)\big\|_{L^1(0,1)} \leqslant &\|(u_\varepsilon^0)'\|_{L^1(0,1)} - v_\varepsilon(1,0) + v_\varepsilon(1,T_0) - v_\varepsilon(0,T_0)\\ &+ v_\varepsilon(0,0) + \int_0^{T_0} \big\|\partial_x v_\varepsilon(\cdot,t)\big\|_{L^1(0,1)}\,dt.\end{aligned} \tag{51}$$

An application of Gronwall's lemma yields estimate (48). □

For the present problem it is probably impossible to obtain a uniform $L^1(Q_T)$ estimate on the time derivative $\partial_t v_\varepsilon$, in contrast to several analyses of problems with fixed boundaries [4, 5]. For example, in [4] such an estimate was derived by differentiating the regularized parabolic equation with respect to t, multiplying the resulting equation by $\mathrm{sgn}_\eta(\partial_x v_\varepsilon)$, integrating the result over the computational domain, and using the boundary conditions and Gronwall's lemma. In the present case, differentiating (31a) with respect to t will produce an equation with a coefficient involving $h_\varepsilon''(t)$. However, we can not bound this quantity, since differentiating the coupling equation (31f) with respect to t will lead to an equation for $h_\varepsilon''(t)$ in terms of $\partial_t v_\varepsilon$, and we can not control the variation of v_ε with respect to t along the boundary $\xi = 0$.

To apply the compactness criterion to the family of regularized solutions $\{v_\varepsilon\}_{\varepsilon>0}$, we apply the following variant of Kružkov's [14] interpolation lemma (see e.g. [13] for a proof):

Lemma 7. *Assume that there exist finite constants c_1 and c_2 such that the function $u : (0,1)\times[0,T] \to \mathbb{R}$ satisfies $\|u(\cdot,t)\|_{L^\infty(0,1)} \leqslant c_1$ and $\mathrm{TV}_{(0,1)}(u(\cdot,t)) \leqslant c_2$ for all $t \in [0,T]$, and that u is weakly Lipschitz continuous with respect to t in the sense that*

$$\left|\int_0^1 \big(u(x,t_2) - u(x,t_1)\big)\varphi(x)\,dx\right| \leqslant \mathcal{O}(t_2 - t_1)\sum_{i=0}^{n} \|\varphi^{(i)}\|_{L^\infty(0,1)},$$

for all $\varphi \in C_0^n(0,1)$, $0 \leqslant t_1 \leqslant t_2 \leqslant T$. Then there exits a constant C, depending in particular on c_1 and c_2, such that the following interpolation result is valid:

$$\big\|u(\cdot,t_2) - u(\cdot,t_1)\big\|_{L^1(0,1)} \leqslant C(t_2 - t_1)^{1/(n+1)}, \quad 0 \leqslant t_1 \leqslant t_2 \leqslant T. \tag{52}$$

We calculate here that

$$\begin{aligned}&\int_0^1 \big(v_\varepsilon(\xi,t_2) - v_\varepsilon(\xi,t_1)\big)\varphi(\xi)\,d\xi\\ &= \int_{t_1}^{t_2}\int_0^1 \Big\{h_\varepsilon^{-1}h_\varepsilon'(\xi\partial_\xi v_\varepsilon - v_\varepsilon) - h_\varepsilon^{-1}\partial_\xi f_\varepsilon(v_\varepsilon) + h_\varepsilon^{-2}\partial_\xi^2 A_\varepsilon(v_\varepsilon)\Big\}\varphi(\xi)\,d\xi dt\\ &= \int_{t_1}^{t_2}\int_0^1 \Big\{h_\varepsilon^{-1}h_\varepsilon' v_\varepsilon\varphi(\xi) + \big(-h_\varepsilon^{-1}h_\varepsilon'(t)\xi v_\varepsilon + h_\varepsilon^{-1}f_\varepsilon(v_\varepsilon) - h_\varepsilon^{-2}a_\varepsilon(v_\varepsilon)\partial_\xi v_\varepsilon\big)\varphi'(\xi)\Big\}\,d\xi dt.\end{aligned} \tag{53}$$

¿From the proof of Lemma 4 it follows that there exists a constant $\widetilde{M}_h$ such that the estimate $\|1/h_\varepsilon^2\|_{L^\infty(0,T)} + \|h_\varepsilon'/h_\varepsilon\|_{L^\infty(0,T)} \leqslant \widetilde{M}_h$ holds uniformly in ε. Using the estimate (48), we get

$$\begin{aligned}&\left|\int_0^1 \big(v_\varepsilon(\xi,t_2) - v_\varepsilon(\xi,t_1)\big)\varphi(\xi)\,d\xi\right|\\ &\quad\leqslant (t_2 - t_1)\widetilde{M}_h\Big[\big(M_0\|\varphi\|_{L^\infty(0,1)} + \big(\|f_\varepsilon\|_\infty + \|a_\varepsilon\|_\infty M_2 + M_0\big)\|\varphi'\|_{L^\infty(0,1)}\Big].\end{aligned} \tag{54}$$

Thus we have proved

Lemma 8. *Let $(v_\varepsilon, h_\varepsilon)$ be a solution of the regularized boundary problem* (31)*. Then the following uniform estimates are valid, where the constant M_3 is independent of ε:*

$$\big\|v_\varepsilon(\cdot,t_2) - v_\varepsilon(\cdot,t_1)\big\|_{L^1(0,1)} \leqslant M_3(t_2 - t_1)^{1/2}, \quad 0 \leqslant t_1 \leqslant t_2 \leqslant T. \tag{55}$$

In view of estimates (34), (48) and (55) on v_ε, a standard application of Kolmogoroff's compactness criterion [23] yields that the family $\{v_\varepsilon\}$ is compact in $L^1(Q_T)$. Thus there exists a sequence $\varepsilon_n \to 0$ such that $\{v_{\varepsilon_n}\}$ converges in $L^1(Q_T)$ to a function $v \in BV_{1,1/2}(Q_T)$. Moreover, since the estimates on h_ε in (35) are uniform in ε, there exists a subsequence $\{h_{\varepsilon_n}\}$ of $\{h_\varepsilon\}$ and a function h such that $|h(t_2)-h(t_1)| \leqslant M_h(t_2-t_1)$ for $0 \leqslant t_1 \leqslant t_2 \leqslant T$, $h(0)=1$ and h is nonincreasing.

We now have to prove that the limit pair (v,h) is indeed a generalized solution of the initial-boundary value problem (22). Obviously, the function h satisfies part (a) of Definition 3.

Lemma 9. *The limit function v of solutions v_ε of the regularized problem* (31) *has the regularity properties stated in part (b) of Definition 3.*

Proof. Multiplying Eq. (31a) by v_ε and integrating the result over Q_T, we get

$$\begin{aligned}\iint_{Q_T} h_\varepsilon^{-2} a_\varepsilon(v_\varepsilon)(\partial_\xi v_\varepsilon)^2 d\xi dt = -\frac{1}{2}\int_0^1 v_\varepsilon^2\, d\xi \Big|_0^T - \int_{Q_T} h_\varepsilon^{-1} h_\varepsilon'(t) v_\varepsilon^2\, d\xi dt \\ + \iint_{Q_T} g_\varepsilon(v_\varepsilon,\xi,t)\partial_\xi v_\varepsilon\, d\xi dt\end{aligned}$$

and thus

$$\big\|\partial_x A^\varepsilon(v_\varepsilon)\big\|_{L^2(Q_T)} \leqslant \|a_\varepsilon\|_\infty \big\{ M_0^2 + TM_h(2M_0^2 + M_2\|f_\varepsilon\|_\infty)\big\} =: M_4^\varepsilon.$$

The stated regularity of $A(u)$ follows by letting $\varepsilon \to 0$ and observing that M_4^ε is uniformly bounded for ε sufficiently small. To show the stated $\mathcal{DM}^2$ property, we rewrite the regularized equation (31a) as follows, where $|k| \leqslant K$ and K is a suitable large constant:

$$\partial_t(v_\varepsilon - k) + \partial_\xi\big(g_\varepsilon(v_\varepsilon,\xi,t) - g_\varepsilon(k,\xi,t)\big) + h_\varepsilon^{-1}h_\varepsilon'(v_\varepsilon - k) = h_\varepsilon^{-2}\partial_\xi^2\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big). \tag{56}$$

Multiply (56) by $\mathrm{sgn}_\eta\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\varphi$, where $k \in \mathbb{R}$ and $\varphi \in C_0^\infty(Q_T)$ is an arbitrary test function. Integration by parts over Q_T then yields

$$\begin{aligned}&\iint_{Q_T} h_\varepsilon^{-2}\big[\partial_\xi\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\big]^2 \mathrm{sgn}_\eta'\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\varphi\, d\xi dt \\ &= -\iint_{Q_T} h_\varepsilon^{-2}\partial_\xi\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\,\mathrm{sgn}_\eta\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\partial_\xi\varphi\, d\xi dt \\ &\quad + \iint_{Q_T}\big(g_\varepsilon(v_\varepsilon,\xi,t) - g_\varepsilon(k,\xi,t)\big)\,\mathrm{sgn}_\eta\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\partial_\xi\varphi\, d\xi dt \\ &\quad + \iint_{Q_T}\big(g_\varepsilon(v_\varepsilon,\xi,t) - g_\varepsilon(k,\xi,t)\big)\,\mathrm{sgn}_\eta'\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\partial_\xi\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\varphi\, d\xi dt \\ &\quad - \iint_{Q_T} h_\varepsilon^{-1}h_\varepsilon' v_\varepsilon\,\mathrm{sgn}_\eta\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\varphi\, d\xi dt \\ &\quad - \iint_{Q_T}(v_\varepsilon - k)\,\mathrm{sgn}_\eta'\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\partial_t\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\varphi d\xi dt \\ &\quad - \iint_{Q_T}(v_\varepsilon - k)\,\mathrm{sgn}_\eta\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\partial_t\varphi d\xi dt =: I_\eta^4 + \cdots + I_\eta^9.\end{aligned} \tag{57}$$

We now consider the limit of the right-hand side of (57) for $\eta \to 0$. Using the properties of sgn_η, Lebesgue's theorem, $\partial_\xi A(k) = 0$ and the fact that due the monotonicity of $A_\varepsilon(\cdot)$, $\mathrm{sgn}(v_\varepsilon - k) = \mathrm{sgn}\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)$, we get

$$I_\eta^4 \overset{\eta\to 0}{\longrightarrow} -\iint_{Q_T} h_\varepsilon^{-2}\partial_\xi\big|A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big|\partial_\xi\varphi d\xi dt, \tag{58}$$

$$I_\eta^5 \overset{\eta\to 0}{\longrightarrow} \iint_{Q_T}\mathrm{sgn}(v_\varepsilon - k)\big(g_\varepsilon(v_\varepsilon,\xi,t) - g_\varepsilon(k,\xi,t)\big)\,\partial_\xi\varphi d\xi dt. \tag{59}$$

Using that $u\,\mathrm{sgn}_\eta'(u) \leqslant \chi_{\{u:\,0<|u|\leqslant\eta\}}$ and recalling from assumption (32) that the inverse function A_ε^{-1} is for fixed ε Lipschitz continuous with constant $1/\nu_\varepsilon$, we get that

$$\begin{aligned} &\big|\big(g_\varepsilon(v_\varepsilon,\xi,t) - g_\varepsilon(k,\xi,t)\big)\,\mathrm{sgn}_\eta'\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\partial_\xi\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\big| \\ &\quad\leqslant \frac{L_\varepsilon}{\nu_\varepsilon}\big|\partial_\xi\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\big|\chi_{\mathcal{I}(\varepsilon,\eta)}, \\ &\mathcal{I}(\varepsilon,\eta) := \big\{(\xi,t):\ 0 \leqslant \big|A_\varepsilon\big(v_\varepsilon(\xi,t)\big) - A_\varepsilon(k)\big| \leqslant \eta\big\}. \end{aligned} \tag{60}$$

Consequently,

$$|I_\eta^6| \leqslant \frac{L_\varepsilon}{\nu_\varepsilon}\|\varphi\|_{L^\infty(Q_T)}\iint_{\mathcal{I}(\varepsilon,\eta)}\big|\partial_\xi\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\big|\,d\xi dt.$$

Observe that $\operatorname{meas}\mathcal{I}(\varepsilon,\eta) \to 0$ as $\eta \to 0$, since this measure converges to that of the empty set. Thus $I_\eta^6 \to 0$ as $\eta \to 0$. Next, we see that

$$I_\eta^7 \xrightarrow{\eta\to 0} I_0^7 := -\iint_{Q_T} h_\varepsilon^{-1}h_\varepsilon'(t)v_\varepsilon\,\mathrm{sgn}\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\varphi\,d\xi dt$$

with $|I_0^7| \leqslant TM_hM_0\|\varphi\|_{L^\infty(Q_T)}$. The integrand of I_η^8 satisfies

$$\begin{aligned} &\big|(v_\varepsilon - k)\,\mathrm{sgn}_\eta'\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\partial_t\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\varphi\big| \\ &= \big|(v_\varepsilon - k)\,\mathrm{sgn}_\eta'(v_\varepsilon - k)\partial_t\big(A_\varepsilon(u_\varepsilon) - A_\varepsilon(k)\big)\varphi\big| \\ &\quad\leqslant \big|\partial_t\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\big|\chi_{\{(\xi,t):\,0\leqslant|v_\varepsilon(\xi,t)-k|\leqslant\eta\}}. \end{aligned}$$

Using an argument similar to that employed for I_η^6 we see that $I_\eta^8 \to 0$ as $\eta \to 0$. Finally, we obtain

$$I_\eta^9 \xrightarrow{\eta\to 0} I_0^9 := -\iint_{Q_T}|v_\varepsilon - k|\partial_\xi\varphi d\xi dt. \tag{61}$$

Collecting all these estimates yields that all terms of the right-hand part of Eq. (57) possess a limit as $\eta \to 0$ and are in particular uniformly bounded with respect to η. Consequently, we see that there exists a constant C_1, depending possibly on ε (but not on η) such that

$$\iint_{Q_T} h_\varepsilon^{-2}\big[\partial_\xi\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\big]^2\,\mathrm{sgn}_\eta'\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\,d\xi dt \leqslant C_1(\varepsilon). \tag{62}$$

The sequence

$$\{E_{\varepsilon,\eta}\}_{\eta>0} := \Big\{\big(h_\varepsilon(t)\big)^{-2}\big[\partial_\xi\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\big]^2\,\mathrm{sgn}_\eta'\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\Big\}_\eta \tag{63}$$

is therefore bounded in $L^1(Q_T)$ with respect to η and therefore also in $\mathcal{M}(Q_T)$. By weak compactness we deduce that, up to subsequences, the sequence $\{E_{\varepsilon,\eta}\}_\eta$ converges weakly towards an element $E_\varepsilon \in \mathcal{M}(Q_T)$. Thus for any $\varphi \in C_0^\infty(Q_T)$ we can pass to the limit $\eta \to 0$ in (57) to obtain

$$\begin{aligned} \langle E_\varepsilon,\varphi\rangle = &-\iint_{Q_T} h_\varepsilon^{-1}h_\varepsilon'(t)v_\varepsilon\,\mathrm{sgn}\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\varphi\,d\xi dt \\ &+\iint_{Q_T}\Big\{\mathrm{sgn}(v_\varepsilon - k)\big(g_\varepsilon(v_\varepsilon,\xi,t) - g_\varepsilon(k,\xi,t)\big) - h_\varepsilon^{-2}\partial_\xi\big|A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big|\Big\}\partial_\xi\varphi\,d\xi dt \\ &-\iint_{Q_T}|v_\varepsilon - k|\partial_t\varphi\,d\xi dt. \end{aligned} \tag{64}$$

On the other hand, due to the properties of the function sgn_η, we have $E_{\varepsilon,\eta} \geqslant 0$ for every $\varepsilon,\eta > 0$. Therefore we get

$$\begin{aligned} |\langle E_\varepsilon,\varphi\rangle| &= \lim_{\eta\to 0}\left|\iint_{Q_T} h_\varepsilon^{-2}\big[\partial_\xi\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\big]^2\,\mathrm{sgn}_\eta'\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\varphi\,d\xi dt\right| \\ &\leqslant \limsup_{\eta\to 0}\iint_{Q_T} h_\varepsilon^{-2}\big[\partial_\xi\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\big]^2\,\mathrm{sgn}_\eta'\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)d\xi dt. \end{aligned}$$

Thus we get from (64) with $|\varphi| \leqslant 1$

$$|\langle E_\varepsilon, \varphi\rangle| \leqslant -\iint_{Q_T} h_\varepsilon^{-1} h_\varepsilon'(t) v_\varepsilon \operatorname{sgn}\big(A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)\big)\, d\xi dt. \tag{65}$$

Using the estimate (34) we deduce that there exists a constant C_2, which does not depend on ε, such that

$$|\langle E_\varepsilon, \varphi\rangle| \leqslant C_2 \|\varphi\|_{L^\infty(Q_T)}, \quad \text{for all } \varepsilon > 0.$$

Consequently, E_ε is bounded in $\mathcal{M}(Q_T)$, and up to a subsequence E_ε converges weakly to a functional $E \in \mathcal{M}(Q_T)$, i.e. a Radon measure. We now pass to the limit $\varepsilon \to 0$ in Eq. (64). We have, $|v_\varepsilon - k|$ converges strongly to $|v - k|$ in $C(0,T;L^1(0,1))$, $g_\varepsilon(v_\varepsilon, \xi, t)$ converges strongly to $g(v, \xi, t)$ in $L^q(Q_T)$ for every $q < \infty$ and $\partial_x |A_\varepsilon(v_\varepsilon) - A_\varepsilon(k)|$ converges weakly in $L^2(Q_T)$ to $\partial_\xi |A(v) - A(k)|$. We thus may pass to the limit $\varepsilon \to 0$ in (64) to conclude that

$$\begin{aligned}
\langle E, \varphi\rangle = &-\iint_{Q_T} h_\varepsilon^{-1} h' v_\varepsilon \operatorname{sgn}\big(A(v) - A(k)\big)\varphi \, d\xi dt \\
&+ \iint_{Q_T} \Big\{\operatorname{sgn}(v-k)\big(g(v,\xi,t) - g(k,\xi,t)\big) - h^{-2}\partial_\xi \big|A(v) - A(k)\big|\Big\}\partial_\xi \varphi \, d\xi dt \\
&- \iint_{Q_T} |v - k|\partial_t \varphi \, d\xi dt
\end{aligned} \tag{66}$$

for every $\varphi \in C_0^\infty(Q_T)$. Since g, $\operatorname{sgn}(A(v) - A(k))$ and $\partial_\xi |A(v) - A(k)|$ are all functions in $L^1(Q_T)$ and since E is a Radon measure, we obtain from (66) that for all $\varphi \in C_0^\infty(Q_T)$

$$\begin{aligned}
&\left|\iint_{Q_T} |v-k|\partial_t \varphi + \Big(\operatorname{sgn}(v-k)\big(g(v,\xi,t) - g(k,\xi,t)\big) - h^{-2}\partial_\xi \big|A(v) - A(k)\big|\Big)\partial_\xi \varphi \, d\xi \, dt\right| \\
&\qquad \leqslant C\|\varphi\|_{L^\infty(Q_T)}.
\end{aligned} \tag{67}$$

This in particular implies the stated $\mathcal{DM}^2$ property (25). □

Lemma 10. *The limit function v of solutions v_ε of the regularized initial-boundary value problem satisfies the boundary conditions* (18) *and* (19) *stated in Definition 2.*

Proof. First of all we have from Lemma 2, passing to a subsequence if necessary, that h_ε converges uniformly to a certain Lipschitz function h, which satisfies $h(0) = 1$, $h(t) \geqslant h_0 > 0$. Multiplying (30a) by $\varphi \in C_0^1(\Pi_T)$, integrating over $Q(h_\varepsilon, T)$, using integration by parts and the boundary conditions (30c), (30d), and then letting $\varepsilon \to 0$, we get

$$\iint_{Q(h,T)} u\partial_t + \big(f(u) - \partial_x A(u)\big)\partial_x \varphi \, dx\, dt = 0. \tag{68}$$

¿From (68) there follow two conclusions about the $\mathcal{DM}^2$ field $F = (F_1, F_2) = (f(u) - \partial_x A(u), u)$: $\operatorname{div} F = 0$ (this is the obvious one), and $\langle F \cdot \nu | \partial Q(h,T), \varphi\rangle = 0$, as a consequence of the generalized Gauss-Green formula (12). Hence, using (14) and (15) we deduce (18) and (19). □

Lemma 11. *The limit function (u, h) of solutions $(u_\varepsilon, h_\varepsilon)$ of the regularized problem* (30) *satisfies* (30f) *in the sense stated in (d) of Definition 2.*

Proof. First, we observe that $A_\varepsilon(u_\varepsilon(x,t))$ converges to $A(u(x,t))$ in $L^1_{\mathrm{loc}}(Q(h,T))$. This follows by the convergence of $A_\varepsilon(v_\varepsilon(\xi,t))$ to $A(v(\xi,t))$ in $L^1(Q(T))$, the uniform convergence of h_ε to h and the uniform boundedness of $\partial_\xi A_\varepsilon(v_\varepsilon(\xi,t))$ in $L^2(Q(T))$. More specifically, for any compact

$K \subset Q(h,T)$, for ε sufficiently small,

$$\begin{aligned}\iint_K \big|A_\varepsilon\big(u_\varepsilon(x,t)\big) - A\big(u(x,t)\big)\big|\,dx\,dt &= \iint_{K'} \big|A_\varepsilon\big(u_\varepsilon(h(t)\xi,t)\big) - A\big(u(h(t)\xi,t)\big)\big|h(t)\,d\xi\,dt\\ &\leqslant \iint_{K'} \big|A_\varepsilon\big(v_\varepsilon(\xi,t)\big) - A\big(v(\xi,t)\big)\big|h(t)\,d\xi\,dt\\ &\quad+ \iint_{K'} \big|A_\varepsilon\big(u_\varepsilon(h(t)\xi,t)\big) - A_\varepsilon\big(u_\varepsilon(h_\varepsilon(t)\xi,t)\big)\big|h(t)\,d\xi\,dt\\ &\leqslant \iint_{K'} \big|A_\varepsilon\big(v_\varepsilon(\xi,t)\big) - A\big(v(\xi,t)\big)\big|h(t)\,d\xi\,dt + C\|h_\varepsilon - h\|_\infty \sup_\varepsilon \|\partial_x A_\varepsilon(u_\varepsilon)\|_{L^2(Q(h_\varepsilon,T))}\\ &\qquad \to 0, \qquad \text{as } \varepsilon \to 0,\end{aligned}$$

where K' denotes the image of K by the transformation $(x,t) \mapsto (\xi,t)$. Now, we prove that $A_\varepsilon(u_\varepsilon(h_\varepsilon(t),t)) \to \gamma_{x\to h(t)}A(u(\cdot,t))$ in $L^1(0,T)$ as $\varepsilon \to 0$, after passing to a suitable subsequence if necessary. Given any $\delta > 0$ we have $h(t)-\delta < h_\varepsilon(t) < h(t)+\delta$, $0 < t < T$, for ε sufficiently small, due to the uniform convergence $h_\varepsilon \to h$. We may also assume that $A_\varepsilon(u_\varepsilon(h(t)-\delta,t)) to A(u(h(t)-\delta,t))$ in $L^1(0,T)$ due to the convergence of $A_\varepsilon(u_\varepsilon(x,t))$ to $A(u(x,t))$ in $L^1_{\mathrm{loc}}(Q(h,T))$. Then, setting $B_\varepsilon(x,t) = A_\varepsilon(u_\varepsilon(x,t))$ and $B(x,t) = A(u(x,t))$, $x_\delta(t) = h(t)-\delta$, we have

$$\begin{aligned}\int_0^T \big|B_\varepsilon\big(h_\varepsilon(t),t\big) - \gamma_{x\to h(t)}B(\cdot,t)\big|\,dt &\le \int_0^T \big|B_\varepsilon\big(x_\delta(t),t\big) - B\big(x_\delta(t),t\big)\big|\,dt\\ &\quad+ \int_0^T \big|B_\varepsilon\big(x_\delta(t),t\big) - B_\varepsilon\big(h_\varepsilon(t),t\big)\big|\,dt + \int_0^T \big|B\big(x_\delta(t),t\big) - \gamma_{x\to h(t)}B(\cdot,t)\big|\,dt\\ &\leqslant \int_0^T \big|B_\varepsilon\big(x_\delta(t),t\big) - B\big(x_\delta(t),t\big)\big|\,dt + C\sqrt{\delta}.\end{aligned}$$

Since $\delta > 0$ may be taken arbitrarily small, the assertion follows. Finally, by passing to a further subsequence of ε's if necessary, we see that, except for $h'_\varepsilon(t)$, all other terms in (30e) converge a.e. in $(0,T)$ to the corresponding terms in (9e), replacing $A(u(h(t),t))$ by $\gamma_{x\to h(t)}A(u(\cdot,t))$. Therefore, $h'_\varepsilon(t)$ also converge a.e. in $(0,T)$, and since it clearly converges weakly to $h'(t)$, we have $h'_\varepsilon(t) \to h(t)$ a.e. in $(0,T)$, and the lemma is proved. □

It is standard to conclude from Lemma 7 that the limit function v satisfies the initial condition (28), and to prove that the entropy inequality (29) is satisfied by multiplying Eq. (31a) with $\mathrm{sgn}_\eta(v_\varepsilon - k)\varphi$, $k \in \mathbb{R}$, $\varphi \in C_0^\infty(Q_T)$, $\varphi \geqslant 0$, and letting $\eta \to 0$ and $\varepsilon \to 0$. Thus we have shown

Theorem 2. *The initial-boundary value problem* (22) *admits an entropy solution* (v,h).

Since $h(t) > 0$ and h' is bounded, we conclude

Corollary 1. *The free boundary problem* (9) *admits an entropy solution* (u,h).

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