SOME CONSEQUENCES OF THE SHADOWING PROPERTY IN LOW DIMENSIONS

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Abstract. We consider low-dimensional systems with the shadowing property and we study the problem of existence of periodic orbits. In dimension two, we show that the shadowing property for a homeomorphism implies the existence of periodic orbits in every ϵ-transitive class, and in contrast we provide an example of a C^∞ Kupka-Smale diffeomorphism with the shadowing property exhibiting an aperiodic transitive class. Finally we consider the case of transitive endomorphisms of the circle, and we prove that the α-Hölder shadowing property with $\alpha > 1/2$ implies that the system is conjugate to an expanding map.

1. Introduction

The main goal of this article is to obtain dynamical consequences of the shadowing property for surface maps and one-dimensional dynamics.

Let (X, d) be a metric space and $f : X \to X$ a homeomorphism. A (complete) δ-pseudo-orbit for f is a sequence $\{x_n\}_{n\in\mathbb{Z}}$ such that $d(f(x_n), x_{n+1}) < \delta$ for all $n \in \mathbb{Z}$. We say that the orbit of x ϵ-shadows the given pseudo-orbit if $d(f^n(x), x_n) < \epsilon$ for all $n \in \mathbb{Z}$. Finally, we say that f has the shadowing property (or pseudo-orbit tracing property) if for each $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudo-orbit is ϵ-shadowed by an orbit of f. Note that we do not assume uniqueness of the shadowing orbit.

One motivation to study systems with this property is that numeric simulations of dynamical systems always produce pseudo-orbits. Thus, systems with the shadowing property are precisely the ones in which numerical simulation does not introduce unexpected behavior, in the sense that simulated orbits actually "follow" real orbits.

When one considers pseudo-orbits, the natural set that concentrates the nontrivial dynamics is the chain recurrent set. If the system has the shadowing property, the closure of the recurrent set coincides with the chain recurrent set. As usual, it is then natural to ask about the existence of periodic orbits in the recurrent set. If one also assumes that the system is expansive in the recurrent set (as it happens for hyperbolic systems), then it is easy to see that periodic orbits are dense in the recurrent set. However, it is unknown whether a similar result holds without the expansivity assumption. It is not even clear that the shadowing property implies the existence of one periodic orbit.

Our first result addresses this problem in dimension two (for precise definitions see §2.3).

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Theorem 1.1. *Let S be a compact orientable surface, and let $f\colon S \to S$ be a homeomorphism with the shadowing property. Then for any given $\epsilon > 0$, each ϵ-transitive component has a periodic point.*

As an immediate consequence, we have

Corollary 1.2. *If a homeomorphism of a compact surface has the shadowing property, then it has a periodic point.*

Note that Theorem 1.1 does not rule out the existence of aperiodic chain transitive components (in fact Theorem 1.5 below provides an example with the shadowing property exhibiting an aperiodic chain transitive class). However, if there is such a component, some of its points must be accumulated by periodic points.

Corollary 1.3. *Let S be a compact orientable surface, and let $f\colon S \to S$ a homeomorphism with the shadowing property. Then $\overline{\mathrm{Per}(f)}$ intersects every chain transitive class.*

Thus, the presence of an aperiodic chain transitive class implies that there are infinitely many periodic points. In particular, we obtain the following:

Theorem 1.4. *Let S be a compact orientable surface, and let $f\colon S \to S$ be a Kupka-Smale diffeomorphism with the shadowing property. If there are only finitely many periodic points, then f is Morse-Smale.*

Another problem of interest is to find "new" examples of systems having the shadowing property. It is known that in dimension at least 2, topologically stable systems have the shadowing property [Wal78, Nit71]. Systems which are hyperbolic (meaning Axiom A with the strong transversality condition) also exhibit the shadowing property. In fact, for such systems, a stronger property called Lipschitz shadowing holds. This means that there is some constant C such that in the definition of shadowing one can always choose $\delta = C\epsilon$ [Pil99, §2.2]. In [PT10] it is shown that Lipschitz shadowing for diffeomorphisms is actually equivalent to hyperbolicity.

However, not all systems with the shadowing property have Lipschitz shadowing. In fact, in view of [PT10], a simple example is any non-hyperbolic system which is topologically conjugated to a hyperbolic one. Nevertheless, this type of example still has many of the properties of hyperbolicity; in particular, there are finitely many chain transitive classes, with dense periodic orbits. Another non-hyperbolic example which has the shadowing property is a circle homeomorphism with infinitely many fixed points, which are alternatively attracting and repelling and accumulating on a unique non-hyperbolic fixed point. However, in this type of example, altough there are infinitely many chain transitive classes, they all have dense periodic points (in fact they are periodic points).

The next theorem gives a new type of example of smooth diffeomorphism with the shadowing property, which is essentially different from the other known examples in that it has an aperiodic chain transitive class, and moreover, all periodic points are hyperbolic. In particular, our example shows that one cannot hope to improve Theorem 1.1 by "going to the limit", even for Kupka-Smale diffeomorphisms:

Theorem 1.5. *In any compact surface S, there exists a Kupka-Smale C^∞ diffeomorphism $f\colon S \to S$ with the shadowing property which has an aperiodic chain transitive component. More precisely, it has a component which is an invariant circle supporting an irrational rotation.*

A this point, we want to emphasize that none of the theorems stated so far assume either Lipschitz or Hölder shadowing.

Finally, we consider the case of transitive endomorphisms of the circle with the α-Hölder shadowing property, i.e. such that there is a constant C such that every δ-pseudo-orbit is $C\delta^\alpha$-shadowed by an orbit, and we show that α-Hölder shadowing with $\alpha > 1/2$ implies conjugacy to linear expanding maps (see definitions 4.1-4.3 for details).

Theorem 1.6. *Let f be a C^2 endomorphism of the circle with finitely many turning points. Suppose that f is transitive and satisfies the α-Hölder shadowing property with $\alpha > \frac{1}{2}$. Then f is conjugate to a linear expanding endomorphism.*

If the transitivity of f persists after perturbations, we can improve the result. We say that f is C^r robustly transitive if all maps in a C^r-neighborhood of f are transitive.

Theorem 1.7. *Let f be a C^2 orientation preserving endomorphism of the circle with finitely many turning points. Suppose that f satisfies the α-Hölder shadowing property with $\alpha > \frac{1}{2}$. If f is C^r-robustly transitive, $r \geq 1$, then f is an expanding endomorphism.*

Observe that in this theorem we do not assume that the shadowing property holds for perturbations of the initial system. Theorem 1.7 can be concluded directly from [KSvS07], where it is proved that hyperbolic endomorphisms are open and dense for one dimensional dynamics. Nevertheless, we provide the present proof because it involves very elementary ideas that may have a chance to be generalized for surface maps. A result similar to Theorem 1.6 for the case of diffeomorphisms in any dimension was recently given in [Tik11].

Let us say a few words about the techniques used in this article. To obtain Theorem 1.1, we use Conley's theory combined with a Lefschetz index argument to reduce the problem to one in the annulus or the torus. To do this, we prove a result about aperiodic ϵ-transitive components that is unrelated to the shadowing property and may be interesting by itself (see Theorem 2.12). In that setting, then we apply Brouwer's theory for plane homeomorphisms (which is strictly two-dimensional) to obtain the required periodic points.

To prove Theorem 1.5, we use a construction in the annulus with a special kind of hyperbolic sets, called crooked horseshoes, accumulating on an irrational rotation on the circle (with Liouville rotation number). The shadowing property is obtained for points far from the rotation due to the hyperbolicity of the system outside a neighborhood of the rotation, and near the rotation the shadowing comes from the crooked horseshoes. The main technical difficulty for this construction is to obtain arbitrarily close to the identity a hyperbolic system with a power exhibiting a crooked horseshoe. This is addressed in the Appendix (see Proposition 3.2)

To prove Theorem 1.6 we use that the shadowing property to obtain a small interval containing a turning point such that some forward iterate intersects a turning point, and assuming that the shadowing is Hölder with $\alpha > \frac{1}{2}$ it is concluded that the forward iterate of the interval has to be contained inside the initial interval, contradicting the transitivity. Once turning points are discarded, Theorem 1.7 is concluded using that the dynamics preserves orientation and that recurrent points can be closed to a periodic orbit by composing with a translation.

2. Shadowing implies periodic orbits: proof of Theorem 1.1

2.1. **Lifting pseudo-orbits.** Let S be a orientable surface of finite type, and $f\colon S \to S$ a homeomorphism. If f is not the sphere, we may assume that S is endowed with a complete Riemannian metric of constant non positive curvature, which induces a metric $d(\cdot,\cdot)$ on S. Denote by $\hat{S}$ the universal covering of S with covering projection $\pi\colon \hat{S} \to S$, equipped with the lifted metric which we still denote by $d(\cdot,\cdot)$ (note that $\hat{S} \simeq \mathbb{R}^2$ or $\mathbb{H}^2$).

The covering projection π is a local isometry, so that we may fix ϵ_0 such that for each $\hat{x} \in \hat{S}$ there is $\epsilon_0 > 0$ such that π maps the ϵ_0-neighborhood of $\hat{x}$ to the ϵ_0-neighborhood of $\pi(\hat{x})$ isometrically for any $\hat{x} \in \hat{S}$.

The next proposition ensures that one can always lift ϵ-pseudo orbits of f to the universal covering in a unique way (given a base point) if ϵ is small enough.

Proposition 2.1. *Given a lift $\hat{f}\colon \hat{S} \to \hat{S}$ of f, an ϵ_0-pseudo orbit $\{x_i\}$, and $\hat{y} \in \pi^{-1}(x_0)$, there is a unique ϵ_0-pseudo orbit $\{\hat{x}_i\}$ for $\hat{f}$ such that $\hat{x}_0 = \hat{y}$ and $x_i = \pi(\hat{x}_i)$ for all i.*

Proof. Note that from the definition of ϵ_0, we have

$$\epsilon_0 < \min\{d(\hat{y},\hat{x}) : x \in S,\, \hat{x},\hat{y} \in \pi^{-1}(x),\, \hat{y} \neq \hat{x}\}.$$

Set $\hat{x}_0 = \hat{y}$. Then there is a unique choice of $\hat{x}_1 \in \pi^{-1}(x_1)$ such that $d(\hat{f}(\hat{x}_0),\hat{x}_1) < \epsilon_0$, and similarly there is a unique $\hat{x}_{-1} \in \pi^{-1}(x_{-1})$ such that $d(\hat{f}(\hat{x}_{-1}),\hat{x}_0) < \epsilon_0$. Proceeding inductively, one completes the proof. □

The following proposition follows from a standard compactness argument which we omit.

Proposition 2.2. *If $K \subset S$ is compact and $\hat{f}\colon \hat{S} \to \hat{S}$ is a lift of f, then $\hat{f}$ is uniformly continuous on the ϵ-neighborhood of $\pi^{-1}(K)$, for some $\epsilon > 0$.*

We say that f has the shadowing property in some invariant set K if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit in K is ϵ-shadowed by some orbit (not necessarily in K).

Proposition 2.3. *Suppose that f has the shadowing property in a compact set $K \subset S$. If $\hat{f}\colon \hat{S} \to \hat{S}$ is a lift of f, then $\hat{f}$ has the shadowing property in $\pi^{-1}(K)$.*

Proof. From the previous proposition, given $\epsilon > 0$ we may choose $\epsilon' < \min\{\epsilon, \epsilon_0/3\}$ such that $\hat{d}(\hat{f}(x),\hat{f}(y)) < \epsilon_0/3$ whenever $d(x,y) < \epsilon'$, and a similar condition for $\hat{f}^{-1}$.

Let $\delta < \epsilon_0/3$ and let $\{\hat{x}_n\}$ be a δ-pseudo orbit of $\hat{f}$ in $\pi^{-1}(K)$. Then $\{\pi(\hat{x}_n)\}$ is a δ-pseudo orbit of f in K. If δ is small enough, then $\{\pi(\hat{x}_n)\}$ is ϵ'-shadowed by the orbit of some point $x \in M$. Since $d(x,\pi(\hat{x}_0)) < \epsilon'$, if $\hat{x}$ is the element of $\pi^{-1}(x)$ closest to $\hat{x}_0$ we have $d(\hat{x},\hat{x}_0) < \epsilon'$. We know that $d(\hat{f}(\hat{x}_0),\hat{x}_1) < \delta$, and from our choice of ϵ' also $d(\hat{f}(\hat{x}),\hat{f}(\hat{x}_0)) < \epsilon_0/3$, so $d(\hat{f}(\hat{x}),\hat{x}_1) < \delta + \epsilon_0/3$. On the other hand, since $d(\pi(\hat{f}(\hat{x})),\pi(\hat{x}_1)) < \epsilon'$, we must have that $d(\hat{f}(\hat{x}),T\hat{x}_1) < \epsilon' < \epsilon_0/3$ for some covering transformation T. But then $d(T\hat{x}_1,\hat{x}_1) < \delta + 2\epsilon_0/3 < \epsilon_0$. This implies that $T = Id$ so that $d(\hat{f}(\hat{x}),\hat{x}_1) < \epsilon'$. In particular $\hat{f}(\hat{x})$ is the element of $\pi^{-1}(f(x))$ closest to $\hat{x}_1$, so we may repeat the previous argument inductively to conclude that $d(\hat{f}^n(\hat{x}),\hat{x}_n) < \epsilon'$ for all $n \geq 0$.

If $\hat{y}$ is the element of $\pi^{-1}(f^{-1}(x))$ closest to $\hat{x}_{-1}$, then $d(\hat{y}, \hat{x}_{-1}) < \epsilon'$. By the previous argument starting from $\hat{x}_{-1}$ instead of $\hat{x}_0$, we have that $d(\hat{f}(\hat{y}), \hat{x}_0) < \epsilon' < \epsilon_0/3$, and this means that $\hat{f}(\hat{y})$ is the element of $\pi^{-1}(x)$ closest to $\hat{x}_0$ (which we named $\hat{x}$ before). Thus $\hat{y} = \hat{f}^{-1}(\hat{x})$, and we conclude that $d(\hat{f}^{-1}(\hat{x}), \hat{x}_{-1}) < \epsilon'$. By an induction argument again, we conclude that $d(\hat{f}^n(\hat{x}), \hat{x}_n) < \epsilon'$ for $n < 0$ as well. This completes the proof. □

2.2. **Shadowing and periodic points for surfaces.** First we state the following well-known consequence of Brouwer's plane translation theorem (see, for instance, [Fra92]).

Theorem 2.4 (Brouwer)**.** *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving homeomorphism. If f has a nonwandering point, then f has a fixed point.*

Suppose f is homotopic to the identity, and let $\hat{f}\colon \hat{S} \to \hat{S}$ be the lift of f obtained by lifting the homotopy from the identity. Then it is easy to see that $\hat{f}$ commutes with covering transformations.

Theorem 2.5. *Let $f\colon S \to S$ be a homeomorphism homotopic to the identity. Suppose that there is a compact invariant set Λ where f has the shadowing property. Then f has a periodic point.*

Proof. We will also assume that the metric in S is as in the previous section, which we may since the shadowing property in a compact set is independent of the choice of Riemannian metric on S.

We may assume that S is not the sphere, since in that case f would have a periodic point by the Lefschetz-Hopf theorem. Consider the lift $\hat{f}\colon \hat{S} \to \hat{S}$ of f which commutes with the covering transformations.

By Proposition 2.3, $\hat{f}$ has the shadowing property in $\pi^{-1}(\Lambda)$. Fix $\epsilon > 0$, and let $\delta > 0$ be such that every δ-pseudo orbit in $\pi^{-1}(K)$ is ϵ-shadowed by an orbit of $\hat{f}$. Since Λ is compact and invariant, there is a recurrent point $x \in \Lambda$, and so if $\hat{x} \in \pi^{-1}(x)$ we can find $n > 0$ and a covering transformation T such that $d(\hat{f}^n(\hat{x}), T\hat{x}) < \delta$. Since T is an isometry and commutes with $\hat{f}$, the sequence

$$\dots, T^{-1}\hat{f}^{n-1}(\hat{x}), \hat{x}, \hat{f}(\hat{x}), \dots, \hat{f}^{n-1}(\hat{x}), T\hat{x}, T\hat{f}(\hat{x}), \dots, T\hat{f}^{n-1}(\hat{x}), T^2\hat{x}, \dots$$

is a δ-pseudo orbit, and so it is ϵ-shadowed by the orbit of some $\hat{y} \in \hat{S}$. This implies in particular that $d(\hat{f}^{kn}(\hat{y}), T^k\hat{x}) < \epsilon$ for all $k \in \mathbb{Z}$, so that $d((T^{-1}\hat{f}^n)^k(\hat{y}), \hat{y}) < \epsilon$ for all $k \in \mathbb{Z}$. Note that $T^{-1}\hat{f}^n$ is a homeomorphism of $\hat{S} \simeq \mathbb{R}^2$, and we may assume that it preserves orientation without loss of generality. Moreover, we have from the previous facts that the closure of the orbit of $\hat{y}$ for $T^{-1}\hat{f}^n$ is a compact invariant set; thus $T^{-1}\hat{f}^n$ has a recurrent point, and by Brouwer's Theorem, it has a fixed point. Since $T^{-1}\hat{f}^n$ is a lift of f^n, we conclude that f has a periodic point. This completes the proof. □

Corollary 2.6. *If $f\colon \mathbb{A} \to \mathbb{A}$ is a homeomorphism of the open annulus, and f has the shadowing property on some compact set Λ, then f has a periodic point.*

Proof. It follows from the previous theorem noting that f^2 is homotopic to the identity and still has the shadowing property in Λ. □

Corollary 2.7. *Let $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ be a homeomorphism with the shadowing property. Then f has a periodic point.*

Proof. Suppose that f has no periodic points. Using f^2 instead of f we may assume that f preserves orientation. By the Lefschetz-Hopf theorem, we only have to consider the cases where f is homotopic to the identity or to a map conjugated to a power of the Dehn twist $D\colon (x,y) \mapsto (x+y,y)$ (otherwise, f has a periodic point even without assuming the shadowing property).

If f is isotopic to the identity, f has a periodic point by Theorem 2.5. Now suppose that f is homotopic to a map conjugated to D^m for some $M \in \mathbb{Z}$. Using a homeomorphism conjugated to f instead of f, we may assume that f is in fact is homotopic to $D^m : (x,y) \mapsto (x+my, y)$. Let $\tau\colon \mathbb{A} \to \mathbb{T}^2$ be the covering map $(x,y) \mapsto (x+\mathbb{Z}, y)$, where $\mathbb{A} = \mathbb{S}^1 \times \mathbb{R}$. Since f is isotopic to D^m, we can lift f by τ to a homeomorphism $\tilde{f}\colon \mathbb{A} \to \mathbb{A}$, which is homotopic to the identity.

Note that in the proof of Propositions 2.1 and 2.3 we did not use the fact that π was the universal covering map. Thus by the same argument applied to the covering τ one sees that $\tilde{f}$ has the shadowing property in $\mathbb{A} = \tau^{-1}(\mathbb{T}^2)$. Moreover, following the proof of Theorem 2.5, we see that there is a point $\tilde{z} \in \mathbb{A}$ and a covering transformation $T\colon \mathbb{A} \to \mathbb{A}$ such that $d(\tilde{f}^{kn}(\tilde{z}), T\tilde{z}) < \epsilon$ for all $k \in \mathbb{Z}$. But then, noting that T commutes with $\tilde{f}$, we see that $T^{-1}\tilde{f}^n$ is a lift of f^n which has a compact invariant set Λ where the shadowing property holds (namely, the closure of the orbit of $\tilde{z}$). The previous corollary applied to $\tilde{f}$ implies that $\tilde{f}$ (and thus f) has a periodic point. □

2.3. **Lyapunov functions and ϵ-transitive components.** Let $f\colon X \to X$ be a homeomorphism of a compact metric space X. Denote by $\mathcal{CR}(f)$ the chain recurrent set of f, i.e. $x \in \mathcal{CR}(f)$ if for every $\epsilon > 0$ there is an ϵ-pseudo-orbit for f connecting x to itself.

The chain recurrent set is partitioned into chain transitive classes, defined by the equivalence relation $x \sim y$ if for every $\epsilon > 0$ there is an ϵ-pseudo-orbit from x to y and another from y to x. The chain transitive classes are compact invariant sets.

Recall that a complete Lyapunov function for f is a continuous function $g\colon X \to \mathbb{R}$ such that

(1) $g(f(x)) < g(x)$ if $x \notin \mathcal{CR}(f)$;
(2) If $x, y \in \mathcal{CR}(f)$, then $g(x) = g(y)$ if and only if x and y are in the same chain transitive component;
(3) $g(\mathcal{CR}(f))$ is a compact nowhere dense subset of $\mathbb{R}$.

Given a Lyapunov function as above, we say that $t \in \mathbb{R}$ is a *regular value* if $g^{-1}(t) \cap \mathcal{CR}(f) = \emptyset$. Note that the set of regular values is open and dense in $\mathbb{R}$.

We recall the following result from Conley's theory (see [FM02]).

Theorem 2.8. *If $f\colon X \to X$ is a homeomorphism of a compact metric space, then there is a complete Lyapunov function $g\colon X \to \mathbb{R}$ for f.*

Now suppose $f\colon S \to S$ is a homeomorphism of the compact surface S. Given a fixed $\epsilon > 0$, we say that $x, y \in \mathcal{CR}(f)$ are ϵ-related if there are ϵ-pseudo orbits from x to y and from y to x. This is an equivalence relation in $\mathcal{CR}(f)$; we call the equivalence classes ϵ-transitive components. It is easy to see that there are finitely many ϵ-transitive components [Fra89, Lemma 1.5]. Moreover, in [Fra89, Theorem 1.6] it is proved that every ϵ-transitive component is of the form $g^{-1}([a,b]) \cap \mathcal{CR}(f)$

for some complete Lyapunov function g for f and $a, b \in \mathbb{R}$ are regular values. Note that ϵ-transitive components are compact and invariant.

Proposition 2.9. *Let Λ be an ϵ-transitive component for some $\epsilon > 0$. Then there are compact surfaces with boundary $M_1 \subset M_2 \subset S$ such that $f(M_i) \subset \operatorname{int} M_i$ for $i = 1, 2$, and*

$$\Lambda = (M_2 \setminus M_1) \cap \mathcal{CR}(f).$$

Proof. Let g be a complete Lyapunov function for f such that the ϵ-transitive Λ verifies that $\Lambda = g^{-1}([a, b]) \cap \mathcal{CR}(f)$ for some regular values $a < b$. Consider a function $\tilde{g}$ which coincides with g in a neighborhood of $\mathcal{CR}(f)$ and is C^1 in a neighborhood of $\{a, b\}$. If $\tilde{g}$ is C^0-close enough to g, it will be a complete Lyapunov function for f. Choose regular values $a' < b'$ (in the differentiable sense) for $\tilde{g}$ such that there are no points of $g(CR(f))$ between a and a' or between b and b', and define $M_b = \tilde{g}^{-1}(b')$, $M_a = \tilde{g}^{-1}(a')$. It is easy to verify that M_a and M_b satisfy the required properties. □

2.4. **Reducing neighborhoods for ϵ-transitive components.** First we recall some definitions. If T is a non-compact surface, a *boundary representative* of T is a sequence $P_1 \supset P_2 \supset \cdots$ of connected unbounded (i.e. not relatively compact) open sets in T such that $\partial_T P_n$ is compact for each n and for any compact set $K \subset T$, there is $n_0 > 0$ such that $P_n \cap K = \emptyset$ if $n > n_0$ (here we denote by $\partial_T P_n$ the boundary of P_n in T). Two boundary representatives $\{P_i\}$ and $\{P'_i\}$ are said to be equivalent if for any $n > 0$ there is $m > 0$ such that $P_m \subset P'_n$, and vice-versa. The *ideal boundary* $\mathrm{b_I}T$ of T is defined as the set of all equivalence classes of boundary representatives. We denote by $\hat{T}$ the space $T \cup \mathrm{b}_I T$ with the topology generated by sets of the form $V \cup V'$, where V is an open set in T such that $\partial_T V$ is compact, and V' denotes the set of elements of $\mathrm{b_I}T$ which have some boundary representative $\{P_i\}$ such that $P_i \subset V$ for all i. We call $\hat{T}$ the *ends compactification* or *ideal completion* of T.

Any homeomorphism $f : T \to T$ extends to a homeomorphism $\hat{f} : \hat{T} \to \hat{T}$ such that $\hat{f}|_T = f$. If $\hat{T}$ is orientable and $\mathrm{b_I}T$ is finite, then $\hat{T}$ is a compact orientable boundaryless surface. See [Ric63] and [AS60] for more details.

The following lemma states that we can see an ϵ-transitive component Λ as a subset of an f-invariant open subset of S such that the chain recurrent set of the extension of f to its ends compactification consists of finitely many attracting or repelling periodic points together with the set Λ.

Lemma 2.10. *Let S be a compact orientable surface, and let $f : S \to S$ be a homeomorphism. If Λ is an ϵ-transitive component, then there is an open invariant set $T \subset S$ with finitely many ends such that each end is either attracting or repelling and $\mathcal{CR}(f|_T) = \Lambda$.*

Proof. Let $M_1 \subset M_2$ be the compact surfaces with boundary given by Proposition 2.9. Removing some components of M_1 and M_2 if necessary, we may assume that every connected component of M_i intersects $\mathcal{CR}(f)$ for $i = 1, 2$. Similarly, we may assume that every connected component of $S \setminus \operatorname{int} M_i$ intersects $\mathcal{CR}(f)$ for $i = 1, 2$ (removing some components of $S \setminus M_i$ if necessary). This does not modify the properties of M_i given by Proposition 2.9.

Note that $f(M_i)$ and $L_i = M_i \setminus \operatorname{int} f(M_i)$ are compact surfaces with boundary whose union is M_i and they intersect only at some boundary circles (at least one for each M_i, since neither M_1 nor $S \setminus M_2$ are empty), so that their Euler characteristics satisfy

$$\chi(M_i) = \chi(L_i) + \chi(f(M_i)).$$

But since $\chi(M_i) = \chi(f(M_i))$, it follows that $\chi(L_i) = 0$. Thus, to show that L_i is a union of annuli, it suffices to show that no connected component of L_i is a disk (since that implies that the Euler characteristic of each connected component of L_i is at most 0).

Suppose that some (closed) disk D is a connected component of L_i. Then the boundary ∂D is a component of $\partial M_i \cup \partial f(M_i)$.

Suppose first that ∂D is a boundary circle of M_i. Since $D \subset L_i = M_i \setminus \operatorname{int} f(M_i)$, it follows that $D \subset M_i$. Thus, D is a connected component of M_i. On the other hand, since $D \cap \operatorname{int} f(M_i) = \emptyset$ and $f(M_i) \subset \operatorname{int} M_i$ is disjoint from $\partial M_i \supset \partial D$, it follows that $D \cap f(M_i) = \emptyset$. Since $f^n(D) \subset f^n(M_i) \subset f(M_i)$ for $n \geq 1$, we can conclude that $D \cap \mathcal{CR}(f) = \emptyset$, because the distance from $f(M_i)$ to D is positive, so that no ϵ-pseudo orbit starting in D can return to D if ϵ is small enough. This contradicts the fact that every component of M_i intersects $\mathcal{CR}(f)$ as we assumed in the beginning of the proof.

Now suppose that ∂D is a component of $\partial f(M_i)$. Since $D \cap \operatorname{int} f(M_i) = \emptyset$, it follows that D is a connected component of $S \setminus \operatorname{int} f(M_i)$, so that $D' = f^{-1}(D)$ is a component of $S \setminus \operatorname{int} M_i$. But $f(D') \subset M_i$, which implies that $f^n(D') \subset M_i$ for all $n > 0$. As before, this implies that D' is disjoint from $\mathcal{CR}(f)$, contradicting the fact that every component of $S \setminus \operatorname{int} M_i$ intersects $\mathcal{CR}(f)$. This completes the proof that L_i is a disjoint union of annuli.

Note that that the number of (annular) components of L_i coincides with the number of boundary components of M_i, since L_i is a neighborhood of ∂M_i in M_i. The previous argument also shows that $M_i \setminus \operatorname{int} f^n(M_i)$ is a union of the same number of annuli if $n > 0$. In fact, $M_i \setminus \operatorname{int} f^n(M_i) = \cup_{k=1}^{n-1} f^k(L_i)$, and the union is disjoint (modulo boundary). Thus the sets

$$\tilde{L}_1 = \bigcup_{n \geq 0} f^n(L_1) \text{ and } \tilde{L}_2 = \bigcup_{n<0} f^n(L_2)$$

are increasing unions of annuli sharing one of their boundary components, hence they are both homeomorphic to a disjoint union of sets of the form $\mathbb{S}^1 \times [0,1)$. Moreover, $\bigcap_{i>0} f^n(\tilde{L}_1) = \emptyset = \bigcap_{n<0} f^n(\tilde{L}_2)$. Let k_i be the number of components of $\tilde{L}_i$ (or, which is the same, the number of boundary components of M_i).

Let $N = M_2 \setminus \operatorname{int} M_1$ and write

$$T = \tilde{L}_1 \cup N \cup \tilde{L}_2.$$

It is easy to check that $f(T) = T$. Moreover, $\partial N = \partial \tilde{L}_1 \cup \partial L_2$, so T is an open surface with $k_1 + k_2$ ends. If we denote by $\hat{T}$ the ends compactification of T, and by $\hat{f}$ the extension of f to $\hat{T}$ (which is a homeomorphism), we have that $\hat{f}$ has exactly $k_1 + k_2$ periodic points, which are the ends of T. The ends in $\tilde{L}_1$ give rise to periodic attractors, and the ones in $\tilde{L}_2$ to periodic repellers. Since $\mathcal{CR}(f|_{\tilde{L}_i}) = \emptyset$ for $i = 1, 2$ and $\mathcal{CR}(f|_N) = \Lambda$, the surface T has the required properties. $\square$

2.5. **Aperiodic ϵ-transitive components.** We now show that if an ϵ-transitive component Λ has no periodic points, then the "reducing neighborhood" of Λ from the previous lemma is a disjoint union of annuli.

Lemma 2.11. *Let A be a matrix in $\mathrm{SL}(m,\mathbb{Z})$. Then there is $n>0$ such that $\operatorname{tr} A^n \geq m$.*

Proof. Let $r_1e^{2\pi i\theta_1},\dots,r_me^{2\pi i\theta_m}$ be the eigenvalues of A. Given $\epsilon>0$, we can find an arbitrarily large integer n such that $(2n\theta_1,\dots,2n\theta_m)$ is arbitrarily close to a vector of integer coordinates, so that $\cos(4n\pi i\theta_k)>1-\epsilon$. Thus $\operatorname{tr} A^{2n} = \sum_k r_k^n\cos(4n\pi i\theta_k) > (1-\epsilon)\sum_k r_k^{2n}$. If $r_1=\dots=r_m=1$, then $\operatorname{tr} A^{2n}\geq m(1-\epsilon)$, and since $\operatorname{tr} A^{2n}$ is an integer, if ϵ was chosen small enough this implies that $\operatorname{tr} A^{2n}=m$. Now, if some $r_k\neq 1$, choosing a different k we may assume $r_k>1$, so that $\operatorname{tr} A^{2n}>(1-\epsilon)r_k^{2n}$, and if n is large enough and $\epsilon<1$ this implies that $\operatorname{tr} A^{2n}>m$. □

Theorem 2.12. *Let S be a compact orientable surface, and let $f\colon S\to S$ be a homeomorphism. If Λ is an ϵ-transitive component without periodic points, then either $S=\mathbb{T}^2$ and f has no periodic points, or there is a disjoint union of periodic annuli $T\subset S$ such that the ends of each annulus are either attracting or repelling and $\mathcal{CR}(f|_T)=\Lambda$.*

Proof. Let T be the surface given by Lemma 2.10. If some component T' of T has no ends at all, then T' is compact, and since it has no boundary, $T=T'=S$. Since f has no periodic points in $\Lambda=\mathcal{CR}(f|_T)=\mathcal{CR}(f)$, it follows that there are no periodic points at all. The only compact surface admitting homeomorphisms without periodic points is $\mathbb{T}^2$, so $S=\mathbb{T}^2$ as required.

Now suppose T has at least one end in each connected component, and let $\hat{T}$ be its ends compactification. Replacing $\hat{f}$ by some power of $\hat{f}$, we may assume that $\hat{f}$ preserves orientation, the periodic points arising from the ends of $\hat{T}$ are fixed points, and there are no other periodic points. Moreover, each connected component of $\hat{T}$ is invariant, all fixed points in $\hat{T}$ are attracting or repelling and there is at least one in each connected component. Thus we may (and will) assume from now on that $\hat{T}$ is connected and we will show that it is a sphere with exactly two fixed points, so that the corresponding connected component of T is an annulus as desired.

Since the fixed points of $\hat{f}$ are attracting or repelling, the index of each fixed point is 1. Since there are no other periodic points, the same is true for $\hat{f}^n$, for any $n\neq 0$. Thus we get, from the Lefschetz-Hopf theorem,

$$L(\hat{f}^n)=\#\operatorname{Fix}(\hat{f}^n)=\#\operatorname{Fix}(\hat{f})$$

where $L(f)$ denotes the Lefschetz number of f (see [FM02]), defined by

$$L(\hat{f}^n)=\operatorname{tr}(\hat{f}_{*0})-\operatorname{tr}(\hat{f}_{*1})+\operatorname{tr}(\hat{f}_{*2}),$$

where $\hat{f}_{*i}$ is the isomorphism induced by $\hat{f}$ in the i-th homology $H_i(\hat{T},\mathbb{Q})$.

It is clear that $\operatorname{tr}(\hat{f}_{*0})=1$ because we are assuming that $\hat{T}$ is connected. Since $\hat{T}$ is orientable and we are assuming that $\hat{f}$ preserves orientation, and from the fact that $\hat{T}$ is a closed surface, we also have that $\operatorname{tr}(\hat{f}_{*2})=1$. Thus

$$1\leq \#\operatorname{Fix}(\hat{f})=L(\hat{f}^n)=2-\operatorname{tr}(A^n),$$

where A is a matrix that represents $\hat{f}_{*1}$. Since $\hat{f}$ is a homeomorphism, $A\in \mathrm{SL}(\beta_1,\mathbb{Z})$, where β_1 is the first Betti number of $\hat{T}$. By Lemma 2.11 we can find n such that $\operatorname{tr}(A^n)\geq\beta_1$. It follows that $\beta_1\leq 1$. But since $\hat{T}$ is a closed orientable surface, β_1 is even, so that $\beta_1=0$. That is, the first homology of $\hat{T}$ is trivial.

We conclude that $\hat{T}$ is the sphere. Since $\hat{f}$ preserves orientation, this means that $L(\hat{f}) = 2$, so that there are exactly two fixed points as we wanted to show. □

2.6. **Proof of Theorem 1.1.**

Proof of Theorem 1.1. Suppose there exists $\epsilon > 0$ and some ϵ-transitive component which has no periodic points. By Theorem 2.12, we have two possibilities: First, $S = \mathbb{T}^2$ and there are no periodic points in S. But this is not possible due to Corollary 2.7. The second and only possibility is that $\Lambda = \mathcal{CR}(f) \cap (A_1 \cup \cdots \cup A_m)$ where the union is disjoint, each A_i is a periodic open annulus, and each end of A_i is either attracting or repelling. Using f^n instead of f, we may assume that each A_i is invariant. Let $\Lambda_i = \Lambda \cap A_i$. Then A_i is an open invariant annulus such that $\mathcal{CR}(f|_{A_i}) = \Lambda_i$ and f has no periodic points in A_i. But it is easy to see that f has the shadowing property in $\mathcal{CR}(f|_{A_i})$, and this contradicts Theorem 2.5. □

2.7. **Kupka-Smale diffeomorphisms.** The proof of theorem 1.4 starts with the next theorem that holds in any dimension for any Kupka-Samle diffeomorphism having the shadowing property.

Theorem 2.13. *Let $f\colon M \to M$ be a Kupka-Smale diffeomorphism of a compact manifold having the shadowing property. Suppose there is a chain transitive component Λ which contains a periodic orbit p. Then Λ is the homoclinic class of p.*

Proof. Suppose that $f^k(p) = p$ and there is some point $x \in \Lambda$ which is not in the orbit of p. Let $\epsilon > 0$, and choose $\delta > 0$ such that every δ-pseudo orbit is ϵ-shadowed by an orbit. Since x and p are in the same chain transitive component, we can find a δ-pseudo orbit $x_{-a}, \dots, x_b$ such that $x_0 = x$, $x_{-a} = p$ and $x_b = p$. Define $x_{-n} = f^{-n+a}(p)$ for $n > a$ and $x_n = f^{n-b}(p)$ for $n > b$. Then $\{x_n\}$ is a δ-pseudo orbit, which is ϵ-shadowed by the orbit of some $y \in M$, which is not in the orbit of p if ϵ is small enough.

Note that $d(f^{-kn}(y), f^a(p)) < \epsilon$ if $n > a$ and $d(f^{kn}(y), f^{-b}(p)) < \epsilon$ if $n > b$. If ϵ is small enough, this implies that $y \in W^u(f^a(p)) \cap W^s(f^{-b}(p))$. Since y is ϵ-close to x and ϵ was arbitrary, it follows that y is in the homoclinic class of p.

It is clear that any point in the homoclinic class of p is in the same chain transitive component of p. This completes the proof. □

Corollary 2.14. *If $f\colon M \to M$ is Kupka-Smale, then chain transitive components contain at most one or infinitely many periodic orbits.*

Corollary 2.15. *If $f\colon M \to M$ is Kupka-Smale, then either f has positive entropy or every chain transitive component consists of a single periodic orbit.*

Now we can prove Theorem 1.4.

Proof of Theorem 1.4. Since f has finitely many periodic orbits, by Theorem 2.13 each chain transitive component of f contains at most one periodic orbit, and if it does it contains nothing else. We need to show that there are no chain transitive components without periodic orbits, as this would imply that $\mathcal{CR}(f) = \mathrm{Per}(f)$, and the Kupka-Smale condition then implies then that f is Morse-Smale.

Suppose by contradiction that there is some chain transitive component Λ without periodic points. Since there are finitely many periodic orbits, for $\epsilon > 0$ small enough it holds that the ϵ-transitive components of periodic points are disjoint from

Λ. Thus the ϵ-transitive component Λ_0 containing Λ contains no periodic points. By Theorem 1.1, this is a contradiction. $\square$

3. An example with an aperiodic class: proof of Theorem 1.5

Let us briefly explain the idea of the construction of the example from Theorem 1.5. We will define a map f on the annulus $\mathbb{A}$, such that the boundary is either attracting or repelling. This example can then easily be embedded on any surface. Our map will be such that the circle $C = S^1 \times \{0\}$ is invariant and $f|_C$ is an irrational rotation. This circle is going to be an aperiodic class. To guarantee that the system has the shadowing property we combine two ideas: first, we will make sure that f is hyperbolic outside any neighborhood of C. This will guarantee the shadowing of pseudo-orbits that are "far" from C. On the other hand, to obtain shadowing "near" C, we require that there is a sequence of hyperbolic sets of a special kind ("crooked horseshoes") accumulating on the circle C. These sets have the property that they contain orbits that approximate increasingly well the first coordinate of any ϵ-pseudo-orbit that remains close enough to C. We also require that between these sets there are essential attractors and repellers (alternating), in order to guarantee that any pseudo-orbit that starts close enough to C remains close to C forever. This allows us to ignore the second coordinate to obtain shadowing for these pseudo-orbits.

3.1. **Crooked horseshoes.** We begin describing a diffeomorphism $H\colon \overline{\mathbb{A}} \to \overline{\mathbb{A}}$ of the closed annulus $\overline{\mathbb{A}} = \mathbb{S}^1 \times [0,1]$ which has a "crooked horseshoe" wrapping around the annulus. Such a map is obtained by mapping a closed annulus to its interior as in figure 1. The regions A and B are rectangles in the coordinates of $\overline{\mathbb{A}}$. The region A is mapped to the interior of A, and B is mapped to the gray region, which intersects B in five rectangles. The map H is contracting in A, while in $B \cap H^{-1}(B)$ it contracts in the radial direction and expands in the "horizontal" direction in a neighborhood of B (affinely).

This defines a diffeomorphism H from $\overline{\mathbb{A}}$ to the interior of $\overline{\mathbb{A}}$. It is easy to see that the nonwandering set of H consists of two parts: the set K_0, which is the maximal invariant subset of H in B and an attracting fixed point p in A.

Since H is affine in a neighborhood of K_0, the set K_0 is hyperbolic. We can regard H as a diffeomorphism from $\mathbb{A}$ to itself by doing the above construction inside a smaller annulus, and then extending H to the boundary in a way that the two boundary components are repelling and the restriction of H to the boundary is Morse-Smale. In this way we obtain an Axiom A diffeomorphism $H\colon \overline{\mathbb{A}} \to \overline{\mathbb{A}}$.

As in the classical horseshoe, we have a natural Markov partition consisting of the five rectangles of intersection of B with $H(B)$, which induces a conjugation of $H|_{K_0}$ to a full shift on five symbols. However we will restrict our attention to the set $K \subset K_0$ which is the maximal invariant subset of H in $B_{-1} \cup B_0 \cup B_1$ (i.e. three particular rectangles of the Markov partition). For these, we have a conjugation of $H|_K$ to the full shift on three symbols $\sigma\colon \{-1,0,1\}^{\mathbb{Z}} \to \{-1,0,1\}^{\mathbb{Z}}$, where the conjugation $\phi\colon \{-1,0,1\}^{\mathbb{Z}} \to K$ is such that $\phi(x)_n = i \iff H^n(x) \in E_i$. Note that if $x \in E_i$ then $f(x)$ turns once around the annulus clockwise if $i = 1$ and counter-clockwise if $i = -1$, and $f(x)$ does not turn if $i = 0$. This is clearly seen considering the lift $\hat{B}$ of B to the universal covering of $\overline{A}$ (i.e. a connected component of the preimage of B by the covering projection), and a lift $\hat{H}\colon \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1]$

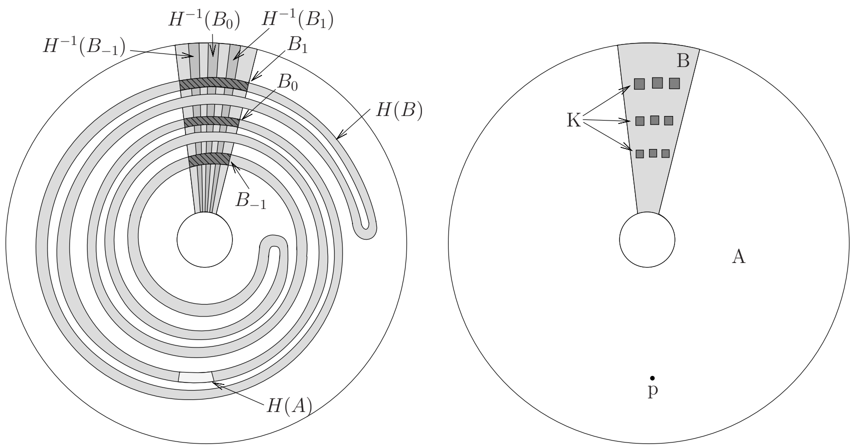

FIGURE 1. A crooked horseshoe

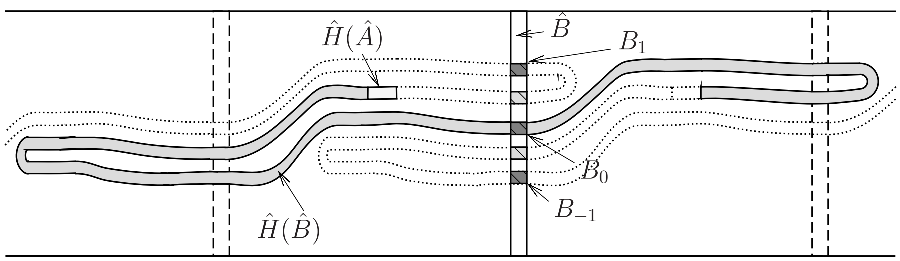

FIGURE 2. The lift of H

of H such that $\hat{H}$ has a fixed point in $\hat{B}$ (see figure 2). If $\hat{B}_i$ are the lifts of the sets B_i inside $\hat{B}$ and $\hat{K}$ is the part of $\hat{B}$ that projects to K, we have that $\hat{H}(\hat{z}) \in \hat{B}+(i,0)$ if $\hat{z} \in \hat{E}_i \cap \hat{K}$.

Definition 3.1. We say that a diffeomorphism $f\colon \overline{\mathbb{A}} \to \overline{\mathbb{A}}$ has a crooked horseshoe if there is a hyperbolic invariant set $K \subset \overline{\mathbb{A}}$ with the properties described above; that is, there is a lift $\hat{f}$ of f to the universal covering and three sets $\hat{K}_{-1}, \hat{K}_0$ and $\hat{K}_1$, which project to a Markov partition of K such that $\hat{f}(\hat{z}) \in i + \hat{K}$ if $\hat{z} \in \hat{K}_i$ (where $\hat{K} = \hat{K}_{-1} \cup \hat{K}_0 \cup \hat{K}_1$)). Furthermore, we will assume that the width of $\hat{K}$ (that is, the diameter of its projection to the first coordinate) is at most 1.

3.2. **Approximations of a Liouville rotation with crooked horseshoes.** One of the key steps for our construction of aperiodic classes requires a hyperbolic diffeomorphism f of the annulus which is C^∞-close to the identity such that some power of f has a crooked horseshoe. This is guaranteed by the next proposition, the proof of which is given in Appendix 4.

Proposition 3.2. *For any $m > 0$, there is a C^∞-diffeomorphism $H\colon \overline{\mathbb{A}} \to \overline{\mathbb{A}}$ such that*

(1) *H is Axiom A with the strong transversality condition;*

(2) *H^m has a crooked horseshoe;*
(3) *$d_{C^r}(H, \mathrm{id}) < C_r/m$, where C_r is some constant depending only on r.*

Write $\overline{R}_\alpha \colon \overline{\mathbb{A}} \to \overline{\mathbb{A}}$ for the rotation $x \mapsto x+\alpha$, and $R_\alpha \colon \mathbb{S}^1 \to \mathbb{S}^1$ for the analogous rotation of the circle.

Proposition 3.3. *For any $\epsilon > 0$ and any Liouville number α, there is an Axiom A diffeomorphism $h \colon \overline{\mathbb{A}} \to \overline{\mathbb{A}}$ arbitrarily C^∞-close to $\overline{R}_\alpha$ and $\delta > 0$ such that every δ-pseudo orbit of R_α is ϵ-shadowed by the first coordinate of some orbit of h.*

Proof. We need to construct, for any $r > 0$, a map h with the desired properties which is C^r-close to the rotation by α. Thus we fix $r > 0$ from now on. We first describe a general construction. Assume $m > 0$ and $p/q \in \mathbb{Q}$ are given (we will choose them later).

Let $H \colon \overline{\mathbb{A}} \to \overline{\mathbb{A}}$ be a diffeomorphism as in Proposition 3.2. Fix $q \in \mathbb{Z}, q > 0$, and let us write $K + t$ to represent the set $\{(x+t, y) : (x, y) \in K\}$ for $K \subset \overline{\mathbb{A}}$. Using an appropriate lift of H by the finite covering $\overline{\mathbb{A}} \simeq (\mathbb{R}/q\mathbb{Z}) \times [0, 1] \mapsto (\mathbb{R}/\mathbb{Z}) \times [0, 1] = \overline{\mathbb{A}}$, we obtain a diffeomorphism $f : \overline{\mathbb{A}} \to \overline{\mathbb{A}}$ which has a hyperbolic set $K = K_0 \cup K_1 \cup \cdots \cup K_{q-1}$ such that $K_{j+1} = K_j + 1/q$ (adopting the convention that $K_{j+q} = K_j$), and a Markov paritition K^j_{-1}, K^j_0, K^j_1, $j = 0, \ldots, q-1$ such that $K^j_i \subset K_j$ for $j = -1, 0, 1$ and $f^m(x) \in K_j + i/q$ if $x \in K^j_i$. Moreover, $f^m(x + 1/q) = f^m(x) + 1/q$.

If we write $h(x) = f(x) + p/q$, we have that $h^m(x) \in K_j + mp/q + i/q$ if $x \in K^j_i$. Also, since h is a lift to a finite covering of an Axiom A diffeomorphism with strong transversality, it follows easily that h itself is Axiom A with strong transversality.

Note that the width of each K_j is at most $1/q$. Because of the finite covering we used, the bound we have on the C^r-distance from h to the rotation $\overline{R}_{p/q}$ is

$$d_{C^r}(h, \overline{R}_{p/q}) \leq q^r d_r(H, \mathrm{id}) \leq C_r q^r/m.$$

Note also that

$$\sup\{|h(z)_1 - R_{p/q}(z)_1| : z \in \overline{\mathbb{A}}\} < \frac{C_0}{mq} \tag{1}$$

where $(\cdot)_1$ denotes the first coordinate. This is again because of the q-folded covering we used.

Note that any δ-pseudo orbit of R_α is a $(\delta + |\alpha - p/q|)$-pseudo orbit of $R_{p/q}$. Let p/q be such that $1/q < \epsilon/(4C_0 + 4)$ and $|p/q - \alpha| < 1/q^{r+2}$. Now let $m = q^{r+1}$, and choose h as we described above. Since $d_r(h, \overline{R}_{p/q}) \leq C_r/q$, by choosing a larger q we may assume that h is arbitrarily C^r-close to $\overline{R}_\alpha$.

To see that h has the required properties, let $\delta = 1/(mq) - 1/q^{r+2} > 0$. If $\{x_n\}$ is a δ-pseudo orbit of R_α, it is a $1/(mq)$-pseudo orbit for $R_{p/q}$, and $\{x_{mn}\}$ is a $1/q$-pseudo orbit for $R_{mp/q} = \mathrm{id}$. We define a unique $z \in K$ by specifying its itinerary for h^m as follows: let j_0 be such that $d(\pi_1(K_{j_0}), x_0) < 1/q < \epsilon/2$, set $i_0 = 0$, and

$$i_{n+1} = \begin{cases} -1 \text{ if } x_0 + \displaystyle\sum_{k=0}^{n} i_k m/q - x_{mn} > \epsilon/2 \\ 1 \text{ if } x_0 + \displaystyle\sum_{k=0}^{n} i_k m/q - x_{mn} < -\epsilon/2 \\ 0 \text{ otherwise.} \end{cases}$$

Define $j_n = j_0 + i_0 + \cdots + i_n$.

Note that if $d(\pi_1(K_{j_n}), x_{nm}) < \epsilon/2$ then by construction

$$\pi_1(K_{j_{n+1}}) - x_{m(n+1)} = \pi_1(K_{j_n}) + \frac{i_{n+1}}{q} - (x_{nm} + \delta_{n+1})$$

where $|\delta_{n+1}| < 1/q$. Since

$$|\pi_1(K_{j_n}) - x_{nm} - \delta_{n+1}| < \epsilon/2 + 1/q,$$

our choice of i_{n+1} implies that

$$|\pi_1(K_{j_n}) - x_{nm} - \delta_{n+1} + i_{n+1}/q| < \epsilon/2.$$

Thus, we see by induction that

$$|\pi_1(K_{j_n}) - x_{nm}| < \epsilon/2 \tag{2}$$

for all $n \geq 0$. A similar choice can be made for negative n, obtaining a sequence $\{j_n\}$ which determines via the symbolic dynamics a unique $z \in K_{j_0}$ such that $h^{nm}(z) \in K_{j_n}$. Since (2) holds for all n and the width of each K_j is at most $1/q < \epsilon/4$, it follows that $|h^{nm}(z)_1 - x_{nm}| < \epsilon/2 + \epsilon/4 < \epsilon$ for all $n \in \mathbb{Z}$. But using (1) we also have that, if $0 \leq k \leq m-1$,

$$\begin{aligned} |h^{nm+k}(z)_1 - x_{nm+k}| &\leq |h^k(h^{nm}(z))_1 - R^k_{p/q}(x_{nm})| + |x_{nm+k} - R^k_{p/q}(x_{nm})| \\ &\leq k\frac{C_0}{mq} + k\delta \leq \frac{C_0+1}{q} \leq \epsilon \end{aligned}$$

Thus $\{h^n(z)_1\}$ ϵ-shadows $\{x_n\}$. This completes the proof. □

3.2.1. *Proof of theorem 1.5.* To prove the theorem, we construct a map $f\colon \mathbb{A} \to \mathbb{A}$ which is a contraction outside some annulus, and which has the required properties. The theorem follows easily since, by standard arguments, we can embed this kind of dynamics in any surface preserving the Kupka-Smale condition and in a way that the dynamics outside this annulus is simple (the chain recurrent set consists of finitely many hyperbolic periodic points).

To define f, fix a Liouville number α and denote by $R_\alpha\colon \mathbb{A} \to \mathbb{A}$ the map $R_\alpha(x,y) = (x+\alpha, y)$. We first choose a sequence of pairwise disjoint closed annuli $\{A_i : i > 0\}$ of the form $A_i = \mathbb{S}^1 \times [a_i, b_i]$ with the following properties:

(1) $b_n - a_n \to 0$ as $n \to \infty$;
(2) A_n converges to the circle $C = \mathbb{S}^1 \times \{0\}$ as $n \to \infty$;
(3) The distance between A_n and A_{n+1} is at most 1 and at least $1/n^2$;

We will choose a sequence $h_n\colon A_n \to A_n$ of diffeomorphisms such that h_n has the properties of the map h of Proposition 3.3 with the annulus A_n instead of $\overline{\mathbb{A}}$, using $\epsilon = 1/n$. Moreover, we choose h_n such that

$$d_{C^n}(h_n, R_\alpha) < (2n)^{-2n}/K_n \tag{3}$$

where K_n is a constant that we will specify later. Using H^{-1} instead of H in Proposition 3.2, we may obtain a map with the same properties as h_n in Proposition 3.3 such that its inverse is attracting instead of repelling on the boundary of the annulus. Thus we may assume that for h_n, the boundary of A_n is repelling if n is odd and attracting if n is even.

Moreover, from the proof of Proposition 3.2, it is possible to assume that the restriction of H (and thus of h_n) to a neighborhood of the boundary components of

A_n has a simple dynamics, namely the product of a Morse-Smale diffeomorphism of the circle and a linear contraction or expansion; that is,

$$h_n(x,y) = (g_n(x), L_n^{\pm}(y))$$

for (x,y) in a neighborhood of $\partial^{\pm} A_n$, where $L_n^{\pm}(x,y) = \lambda_n(y - y_n^{\pm}) + y_n^{\pm}$.

Since the boundary of A_n is attracting if n is odd and repelling if n is even, $\lambda_n > 1$ and $\lambda_{n+1} < 1$ or vice versa (we will assume the first case).

We define $f|_{A_n} = h_n$. To define f in the regions between the A_i's, let let B_n be the annulus between A_n and A_{n+1}. Note that from (3), we have $d_\infty(g_n, g_{n+1}) < 2^{-n}$.

For $(x,y) \in B_n$, we define $f(x,y)$ using convex combinations:

$$f(x,y)_1 = g_n(x) + \phi\left(\frac{y - y_n^-}{y_{n+1}^+ - y_n^-}\right)(g_{n+1}(x) - g_n(x))$$
$$f(x,y)_2 = L_n^-(y) + \phi\left(\frac{y - y_n^-}{y_{n+1}^+ - y_n^-}\right)(L_{n+1}^+(y) - L_n^-(y))$$

where $\phi\colon \mathbb{R} \to \mathbb{R}$ is a fixed C^∞ bump function such that $\phi(x) = 0$ if $x < 0$, $\phi(x) = 1$ if $x > 1$ and $0 \le \phi(x) \le 1$. We further asusme that ϕ is strictly increasing. Since ϕ is fixed, the constant K_n that we used in the choice of h_n can be chosen so that $\|\phi\|_{C^n} < K_n$ and $K_n \ge 2$.

For convenience, let $t(y) = \frac{y - y_n^-}{y_{n+1}^+ - y_n^-}$ and $\Delta_n = y_{n+1}^+ - y_n^-$. Note that $|\Delta_n|$ is the distance from A_n to A_{n+1}, so from condition 3 at the beginning of the proof we have

$$1 \ge |\Delta_n| \ge 1/n^2$$

Also note that from (3),

$$d_{C^n}(g_n, g_{n+1}) \le 2(2n)^{-2n}/K_n,$$

so that if $0 \le i + j \le n$ and $i, j \ge 0$,

$$\begin{aligned}\left|\frac{\partial^{i+j}}{\partial x^i \partial y^j}\left(\phi(t(y))(g_{n+1}(x) - g_n(x))\right)\right| &= \left|\frac{\phi^{(j)}(t(y))}{\Delta_n^j}\left(g_{n+1}^{(i)}(x) - g_n^{(i)}(x)\right)\right| \\ &\le 2/((2n)^{2n}\,|\Delta_n^n| \le 2/4^n\end{aligned}$$

and if $1 \le i \le n$, using Leibniz's formula and the fact that L_{n+1}^+ and L_n^- are affine maps, we find (again using (3))

$$\begin{aligned}&\left|\frac{\partial^i}{\partial y^i}\left(\phi(t(y))(L_{n+1}^+(y) - L_n^-(y))\right)\right| \\ &\quad = \left|\frac{\phi^{(i)}(t(y))}{\Delta_n^i}\left(L_{n+1}^+(y) - L_n^-(y)\right) + n\frac{\phi^{(i-1)}(t(y))}{\Delta_n^{i-1}}(\lambda_{n+1} - \lambda_n)\right| \\ &\quad \le \frac{K_n}{\Delta_n^n}(2(2n)^{-2n}/K_n) + n\frac{K_n}{\Delta_n^n}(2(2n)^{-2n}/K_n) \\ &\quad \le 2(n+1)/((2n)^{2n}\,|\Delta_n^n|) \le 2(n+1)/4^n.\end{aligned}$$

Putting these facts together, we see that

$$d_{C^n}(f|_{B_n}, R_\alpha|_{B_n}) \le 2(n+1)/4^n \xrightarrow{n\to\infty} 0.$$

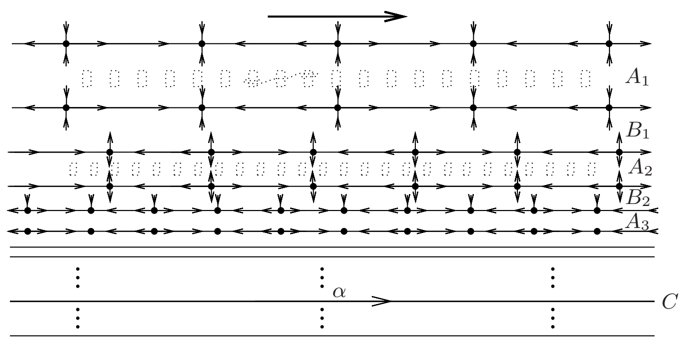

FIGURE 3. The map f

Note that the dynamics of f in B_n is trivial: the boundary components of B are one attracting and the other repelling and there is no recurrence in the interior of B. In fact it is easy to see that $f_2(x, y) - y$ is always positive or always negative for $(x, y) \in B_n$ (depending on the parity of n)

To see that $f|_{B_n}$ is Axiom A, note that the nonwandering set of $f|_{B_n}$ is contained in the boundary of B_n (which is in A_{n+1} or A_n) so it consists of periodic points which are hyperbolic and finitely many. Moreover, since the boundary components of B_n are one attracting and one repelling, a homoclinic intersection between saddles can only happen if the saddles are in different boundary components. A small perturbation supported in the interior of B_n which does not affect our estimates ensures that all such intersections are transverse, guaranteeing the strong transversality condition.

To define f in the region above A_1, we extend arbitrarily f as a contraction using a similar argument. This defines f in $\{(x, y) \in \mathbb{A} : y > 0\}$. Note that f is Axiom A with strong transversality in each A_n. Finally, we repeat this procedure for the lower half of $\mathbb{A}$, and we define $f|_C = R_\alpha|_C$. This defines $f \colon \mathbb{A} \to \mathbb{A}$. By construction, f is C^∞ in $\mathbb{A} \setminus C$. To see that f is also C^∞ in C, it suffices to observe that from our previous observations, if U_n is an annular region of width $1/n$ around C, $d_{C^r}(f|_{U_n}, R_\alpha|_{U_n}) \to 0$ as $n \to \infty$. This easily implies that f is C^∞ in $\mathbb{A}$.

We note that f is Kupka-Smale because it is Axiom A with strong transversaility in each of the invariant annuli composing it (and there are no periodic points in C). Now it remains to see that f has the shadowing property.

Fix $\epsilon > 0$. By our choice of f in the annuli A_n, if we choose n such that A_n is contained in an ϵ-neighborhood of C and $1/n < \epsilon$, there is $\delta_0 > 0$ such that that every δ_0-pseudo orbit of the rotation by α in $\mathbb{S}^1$ is ϵ-shadowed by the first coordinate of an orbit of f in A_n. This means that every δ_0-pseudo orbit of f contained in C is ϵ-shadowed by an orbit of f.

Since C is the limit of a sequence of alternatively attracting-repelling circles, for any $\mu > 0$ we can find $\delta_1 > 0$ such that any δ_1-pseudo orbit starting in the δ_1-neighborhood U_{δ_1} of C is contained in the μ-neighborhood U_μ of C, and we can choose μ and δ_1 small enough so that the first coordinate of every δ_1-pseudo orbit of f starting in U_{δ_1} is a δ_0-pseudo orbit for the rotation by α of $\mathbb{S}^1$. From our previous remark, this δ_0-pseudo orbit must be ϵ-shadowed by the first coordinate of some orbit of f in A_n. Since the width of A_n is smaller than $1/n < \epsilon$, we have that every δ_1 pseudo orbit of f starting in U_{δ_1} is ϵ-shadowed by an orbit of f.

By construction, f is Axiom A outside any invariant neighborhood of C. If we choose an invariant annulus $V \subset U_{\delta_1}$ such that its boundary components are repelling, this implies that there is δ_2 such that any δ_2-pseudo orbit of f starting in $\mathbb{A} \setminus V$ is ϵ-shadowed by an orbit of f.

Finally, let $\delta = \min\{\delta_1, \delta_2\}$, and we have that any δ-pseudo orbit of f is ϵ-shadowed by an orbit of f, as we wanted to prove.

It is easy to see that C is a chain transitive component, so the theorem is proved.

4. Shadowing for one dimensional endomorphisms

We now consider smooth one dimensional endomorphisms on the circle (at least C^2) assuming that the Holder shadowing property holds with Holder constant larger than $\frac{1}{2}$. We recall first some definitions. Let f be a C^r endomorphism of the circle.

Definition 4.1. Given $\alpha \leq 1$, it is said that f has the α-Holder shadowing property if there exists $C > 0$ such that any ϵ-pseudo-orbit with $\epsilon > 0$ is $C\epsilon^\alpha$-shadowed by an orbit.

Definition 4.2. It is said that f is expansive if there exists $C > 0$ such that for any pair of points x, y that there is $n \geq 0$ such that $\text{dist}(f^n(x), f^n(y)) > C$.

Definition 4.3. It is said that f is expanding if there exist $C > 0$ and $\lambda > 1$ such that $|(f^n)'(x)| > C\lambda^n$ for any x in the circle and for any positive integer n.

Definition 4.4. Given a critical point c (i.e., $f'(c) = 0$), it is said that c is a turning point if c is either a local minimum or a local maximum of f.

To prove Theorem 1.6 we will need the following result.

Theorem 4.5. *If f is a transitive non-invertible local homeomorphism of the circle, then f is topologically conjugate to a linear expanding map.*

Proof. Since f is a local homeomorphism, if the degree of f is 1 or -1 it follows that f is a homeomorphism, contradicting the assumption that f is non-invertible. Thus the degreee d of f satisfies $|d| \neq 1$. This implies (for example, see [KH95, Prop. 2.4.9] that f is semiconjugated to the linear expanding map $E_d \colon x \mapsto dx \pmod 1$ via a monotone map h of degree 1; i.e. $hf = E_d h$ and h is a continuous surjection such that $h^{-1}(x)$ is a point or an interval for each x. Suppose $I = h^{-1}(x)$ is a nontrivial interval for some x. Then from the transitivity it follows that there is $k > 0$ such that $f^k(I) \cap I \neq \emptyset$. Since $h(f^k(I)) = E_d^k h(I)$ and $h(I) = x$, we conclude that $E_d^k(x) = x$, and so $f^k(I) = I$. The transitivity of f implies then that $\bigcup_{i=0}^{k-1} f^i(I)$ is dense in the circle, which in turn implies that h has a finite image, a contradiction. This shows that $h^{-1}(x)$ is a single point for every x, so that h is a homeomorphism and f is conjugate to E_d. $\square$

Before proceeding to the proof of Theorems 1.6 and 1.7, let us introduce some notation.

Notation 4.6. If x, y are points in the circle, the interval notation (x, y) denotes the smallest of the two intervals in the complement of the two points (when there is a smallest one). Similarly one can define $[x, y)$, $(x, y[$ and $[x, y]$.

Notation 4.7. If $\epsilon > 0$ is given, and $\epsilon \mapsto \delta(\epsilon) > 0$, $\epsilon \mapsto g(\epsilon) > 0$ are real functions, we use the following notations:

- $\delta = O(g(\epsilon))$ if there is a constant C and $\epsilon_0 > 0$ such that $\delta(\epsilon) < Cg(\epsilon)$ whenever $0 < \epsilon < \epsilon_0$.
- $\delta \approx g(\epsilon)$ if $g(\epsilon)/\delta(\epsilon) \to 1$ as $\epsilon \to 0^+$.

Proof of Theorem 1.6. We will prove that f has no turning point. This is enough to complete the proof, because it implies that f is a local homeomorphism, and from Theorem 4.5 one concludes that f is topologically conjugate to a linear expanding map as required.

Let $c_1, \dots, c_k$ be the turning points, and fix $\gamma > 0$ such that for each i, the interval $J_i = (c_i - \gamma, c_i + \gamma)$ is such that $f(J_i) = (a_i, f(c_i)]$ for some a_i, and f is injective in $(c_i - \gamma, c_i]$ and $[c_i, c_i + \gamma)$. We may also assume that the intervals $J_1, \dots, J_k$ are pairwise disjoint

Fix $\epsilon > 0$ and $1 \leq i \leq k$, and let $z_i \in J_i$ be such that $\operatorname{dist}(z_i, c_i) = \epsilon$. Then $\operatorname{dist}(f(z), f(c_i)) = O(\epsilon^2)$, because f is C^2 and c_i is a turning point. The pseudo orbit

$$\{z_i, f(z_i), f(c_i), f^2(c_i), \dots, f^j(c_i) \dots\},$$

has a single "jump" of length $O(\epsilon^2)$, and therefore it is δ-shadowed by the orbit of a point x_i for some $\delta = O(\epsilon^{2\alpha})$. Thus $\operatorname{dist}(x_i, z_i) \leq \delta$ and $\operatorname{dist}(f^j(x_i), f^j(c_i)) \leq \delta$ for all $j \geq 1$. If ϵ is small enough then $x_i \in J_i$ and there is a point $x_i' \in J_i$ such that $f(x_i') = f(x_i)$ and $c \in (x_i, x_i')$. It is easy to see that $\operatorname{dist}(x_i', c_i) \approx \operatorname{dist}(x_i, c_i)) \geq \epsilon - \delta$. In particular, if ϵ is small enough $I_i = (x_i, x_i')$, we have that $(c_i - \epsilon/2, c_i + \epsilon/2) \subset I_i$ (note that $2\alpha > 1$, so that $\delta < \epsilon/2$ if ϵ is small). Also observe that $f(I_i) = (f(x_i), f(c_i)]$.

Claim 1. There is $n_i \in \{1, \dots, k\}$ and $j_i \geq 0$ such that $c_{n_i} \in f^{j_i}(I_i)$ and $\operatorname{diam}(f^{j_i}(I_i)) \leq \delta$.

Proof. Let us first show that $f^j(I_i)$ contains a turning point for some $j \geq 1$: By transitivity, there is $m > 0$ such that $f^m(I_i) \cap I_i \neq \emptyset$. The set $L = \bigcup_{n=1}^{\infty} f^{nm}(I_i)$ is connected, so it is either an interval or the whole circle. Moreover, $f^m(L) \subset L$. Suppose for contradiction that $f^j(I_i)$ does not contain a turning point, for any $j \geq 1$. Then neither does L, and since there is at least one turning point, it follows that L is not the whole circle. Thus L is an interval such that $f^m(L) \subset L$, and since f has no turning point in L it follows that $f^m|_L$ is injective. This implies that the ω-limit of any point in L by f^m is a semi-attracting fixed point for f^k, which contradicts the transitivity of f.

Now let j_i be the first positive integer such that $f^{j_i}(I_i)$ contains a turning point, and let c_{n_i} be such turning point. Since for $1 \leq j < j_i$ there is no turning point in $f^j(I_i)$, it follows that $f|_{f^j(I_i)}$ is injective, and so $f^{j_i-1}|_{f(I_i)}$ is injective. This implies that $f^{j_i}(I_i) = (f^{j_i}(x_i), f^{j_i}(c_i))$, and so $\operatorname{diam}(f^{j_i}(I_i)) = \operatorname{dist}(f^{j_i}(x_i), f^{j_i}(c_i)) \leq \delta$, as claimed. $\square$

To complete the proof of the theorem, let us use the notation $A \Subset B$ to mean that $\overline{A} \subset B$. Note that if ϵ is chosen small so that $\delta < \epsilon/2$, we have that $f^{j_i}(I_i) \Subset I_{n_i}$ because I_{n_i} contains $(c_{n_i} - \epsilon/2, c_{n_i} + \epsilon/2)$ and $\operatorname{diam}(f^{j_i}(I_i)) < \epsilon/2$. The map $\tau\colon \{1, \dots, k\} \to \{1, \dots, k\}$ defined by $\tau(i) = n_i$ has a periodic orbit because the space is finite, so that there is a sequence $i_1, i_2, \dots, i_m = i_1$ such that $f^{j_{i_r}}(I_{i_r}) \Subset$

$I_{n_{i_{r+1}}}$ for $1 \leq r < m$. Letting $N = j_{i_1} + j_{i_2} + \cdots + j_{i_{m-1}}$, we conclude that $f^N(I_{i_1}) \Subset I_{i_1}$. This contradicts the transitivity of f, completing the proof. □

Proof of Theorem 1.7. From Theorem 1.6 follows that f has is a local homeomorphism conjugate to a linear expanding map. Let $\pi\colon \mathbb{R} \to \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ be the universal covering, and $F\colon \mathbb{R} \to \mathbb{R}$ a lift of f. Write $F_t(x) = F(x) + t$, and let $f_t\colon \mathbb{S}^1 \to \mathbb{S}^1$ be the map lifted by F_t. Since f has no turning points and preserves orientation, F is increasing, and the same is true for F_t. Since f is C^r-robustly transitive, there is $\gamma > 0$ such that f_t is transitive for all $t \in (-\gamma, \gamma)$

Claim 1. For every open interval $I \subset \mathbb{S}^1$ and $x \in \mathbb{S}^1$ there is $n > 0$ and $y \in I$ such that $f^n(y) = x$.

Proof. We need to show that $\bigcup_{n\geq 0} f^n(I) = \mathbb{S}^1$. Suppose not. Since f is transitive, there is $k \in \mathbb{N}$ such that $f^k(I) \cap I \neq \emptyset$. Let $L = \bigcup_{n\geq 0} f^{nk}(I)$. Then L is a connected set, so it is either an interval or the whole circle, and $f^k(L) \subset L$. Suppose L is an interval. Then $f^k|_L$ is injective, because it has no turning points. This implies that there is a semi-attracting fixed point for f^k in L, contradicting the transitivity of f. Thus $L = \mathbb{S}^1$, and this proves the claim. □

Claim 2. For each $z \in \mathbb{R}^2$ and $t > 0$, $F_t^n(z) \geq F^n(z) + t$.

Proof. By induction: the case $n = 1$ is trivial, and assuming $F_t^n(z) \geq F^n(z) + t$, we have $F_t^{n+1}(z) = F(F_t^n(z)) + t \geq F(F^n(z) + t) + t \geq F^{n+1}(z) + t$ due to the monotonicity of f. □

Claim 3. For each $\epsilon > 0$ and $x \in \mathbb{S}^1$ there is $n \in \mathbb{N}$ and $0 < t < \epsilon$ such that $f_t^n(x) = x$.

Proof. Let $\tilde{x} \in \mathbb{R}$ be such that $\pi(\tilde{x}) = x$, and define $\tilde{I} = (F(\tilde{x}), F(\tilde{x}) + \epsilon)$. Claim 1 implies that there is $y \in I = \pi(\tilde{I})$ and $n \in \mathbb{N}$ such that $f^n(y) = x$. This means that if $\tilde{y}$ is the point in $\tilde{I}$ such that $\pi(\tilde{y}) = y$, then there is $m \in \mathbb{Z}$ such that $F^n(\tilde{y}) = \tilde{x} + m$. Observe that we can write $\tilde{y} = F(\tilde{x}) + s = F_s(\tilde{x})$ for some s with $0 < s < \epsilon$. By the previous claim, $F_s^{n+1}(\tilde{x}) = F_s^n(\tilde{y}) \geq F^n(\tilde{y}) + s > \tilde{x} + m$. On the other hand, since $F(\tilde{x}) < \tilde{y}$ and F^n is increasing, we have $F_0^{n+1}(\tilde{x}) = F^n(F(\tilde{x})) < F^n(\tilde{y}) = \tilde{x} + m$. Thus, by continuity, there is $0 < t < s$ such that $F_t^{n+1}(\tilde{x}) = \tilde{x} + m$, so that $f_t^{n+1}(x) = x$ as required □

Claim 4. f has no critical points.

Proof. Suppose c is a critical point. Claim 3 implies that there exist arbitrarily small choices of $t > 0$ such that $f_t^n(c) = c$ for some n. But c is also a critical point for f_t^n, and so it is an attracting fixed point for f_t^n, contradicting the transitivity of f_t. □

Theorem A in [Mañ85] implies that if f is a transitive C^2 endomorphism without critical points, then one of the following hold:

(1) f is topologically conjugate to a rotation;
(2) f has a non-hyperbolic periodic point;
(3) f is an expanding map.

We can rule out case (1), since f is not a homeomorphism. In fact, if f is a homeomorphism, Claim 3 implies that there exist arbitrarily small values of t such that f_t has periodic points, and being f_t a homeomorphism, it follows that f_t is non-transitive, a contradiction.

To finish the proof of the theorem, we have to rule out case (2) above, i.e. we need to show that all periodic points of f are hyperbolic. Suppose p is a non-hyperbolic fixed point of f, and let $k \in \mathbb{N}$ be the least period, so that $f^k(p) = p$ and $(f^k)'(p) = 1$ (because f^k is increasing). Let $I = (p - \epsilon, p + \epsilon)$ with ϵ so small that $f^i(p) \notin I$ for $1 < i < k$, and choose a C^∞ map $h\colon \mathbb{S}^1 \to \mathbb{S}^1$ which is C^r-close to the identity, such that $h(x) = x$ if $x \notin I$, $h(p) = p$ and $0 < h'(p) < 1$. Then hf is C^r-close to f, and in particular it is transitive. But $0 < ((hf)^k)'(p) < 1$, so p is a periodic sink for hf, contradicting the transitivity. This proves that all periodic points of f are hyperbolic, completing the proof. □

Appendix: Creation of crooked horseshoes near the identity

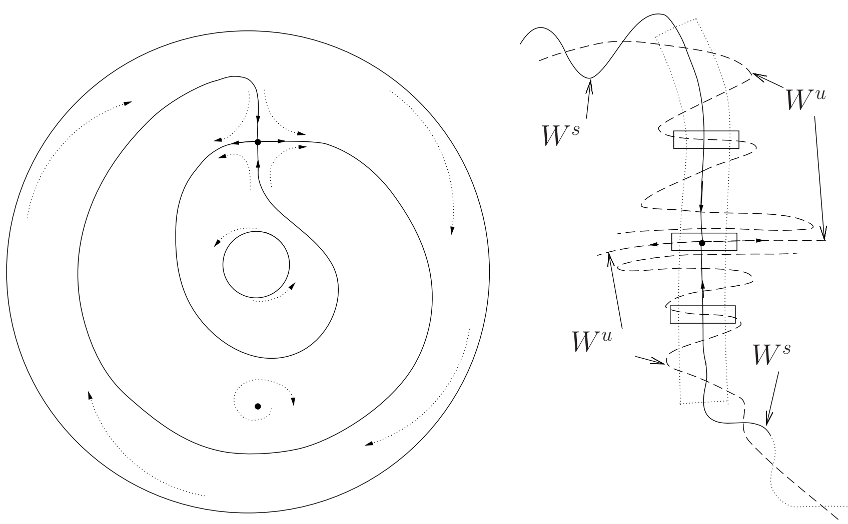

Figure 4. Creating a crooked horseshoe close to the identity from a flow

Given $m > 0$, we will construct a C^r diffeomorphism satisfying the properties required in Proposition 3.2. For that, we get a vector field X exhibiting a double loop between the stable and unstable manifold of a hyperbolic singularity, as in figure 4. Moreover, the loop is an attracting set. Then, for each m, we take the time-$\frac{1}{m}$ map $\Phi_{\frac{1}{m}}$ of the flow associated to X. The fact that this map is $\frac{C_r}{m}$-close in the C^r−topology to the identity map where C_r is a positive constant independent of m follows from Lemma 4.8 below. Then, the map is perturbed into a C^r diffeomorphism f unfolding a tangency associated to each loop (see figure 4). It is proved that this diffeomorphism is Axiom A with strong transversality. To prove that, we adapt to the present context the strategy developed in [NP76]. Note that the estimate on the distance to the identity is preserved if the perturbation is C^r-small. The resulting map has the properties mentioned in Proposition 3.2 (one can verify the presence of a crooked horseshoe for f^m using standard arguments). We devote the rest of this section to obtaining the required perturbations.

We state an elementary fact that was used in the previous description.

Lemma 4.8. *Let X be a C^{r+1} vector field on the compact manifold M, and $\phi\colon M\times[0,1]\to M$ the associated flow. Then $\phi_t=\phi(\cdot,t)$ is such that $d_{C^r}(\phi_t,\mathrm{id})<C_r t$ for some constant C_r independent of t.*

Sketch of the proof. We prove it locally; the global version is obtained by standard arguments. Assume the flow is defined on a neighborhood of $\overline{U}$ for some bounded open set $U\subset\mathbb{R}^n$. Let

$$C_r=\max_{0\le k\le r}\left\|D^k_x X\right\|$$

were $D^k_xX(x)\colon(\mathbb{R}^n)^k\to\mathbb{R}^n$ is the k-th derivative of X and $\left\|D^k_xX\right\|$ is the supreme of $\left\|D^k_xX(x)\right\|$ for x in a neighborhood of $\overline{U}$.

By the mean value inequality, if t is small enough,

$$\begin{aligned}\left\|D^k_x\phi_t-D^k_x\,\mathrm{id}\right\|=\left\|D^k_x\phi_t-D^k_x\phi_0\right\|&\le\sup_{0\le s\le t}\left\|\frac{d}{ds}D^k_x\phi_s\right\|\\&=\sup_{0\le s\le t}\left\|D^k_x\frac{d}{ds}\phi_s\right\|\le\left\|D^k_xX\right\|t\le C_r t.\end{aligned}$$

□

This section is organized as follows: first we introduce the vector field X; later, we consider perturbations of the map $\Phi_{\frac{1}{m}}$, by embedding the map in a one-parameter family; finally, we prove that for certain parameters the map is Axiom A.

To simplify the proof, we will assume that the double connection of our flow is as in figure 5 on the sphere, which does not make a difference since after compactifying by collapsing boundary components of the annulus in figure 4, we are in the same setting (in figure 5, the points R_1 and R_2 correspond to the collapsed boundary components, while R_3 is the source inside the loop of figure 4), and the perturbations that we are going to use are supported outside a neighborhood of R_1 and R_2.

4.1. **The vector field** X**.** Let us consider a vector field X defined on the two dimensional sphere such that in the disk $D=[-2,2]\times[-2,2]$ the following holds:

(1) It is symmetric respect to $(0,0)$, i.e. $X(-p)=-X(p)$.
(2) $S=(0,0)$ is a hyperbolic saddle singularity such that:
   (a) inside $[-\frac{1}{2},\frac{1}{2}]\times[-\frac{1}{2},\frac{1}{2}]$ the vector field is the linear one given by $X(x,y)=(\log(\lambda)x,\log(\sigma)y)$ with $0<\lambda<1<\sigma$ and $\lambda\sigma<1$, $\lambda=\sigma^{-\gamma}$, with $\gamma>\max\{3r,6\}$ where r is the smoothness required; in particular, $[-\frac{1}{2},\frac{1}{2}]\times\{0\}$ is contained in the local stable manifold of S and $\{0\}\times[-\frac{1}{2},\frac{1}{2}]$ is contained in the local unstable manifold of S.
   (b) the stable and the unstable manifold of $(0,0)$ are contained in D;
   (c) the stable manifold and the unstable one form a loop γ contained in $[0,2]\times[0,2]\cup[-2,0]\times[-2,0]$; let us denote by γ^+ the one in $[0,2]\times[0,2]$ and γ^- the one in $[-2,0]\times[-2,0]$.
(3) $R_1=(1,1)$ is a hyperbolic repelling singularity and is contained in the region D^+ bounded by the loop γ^+ (by symmetry $R_2=(-1,-1)$ is a hyperbolic repelling singularity and is contained in the region D^- bounded by the loop γ^-).

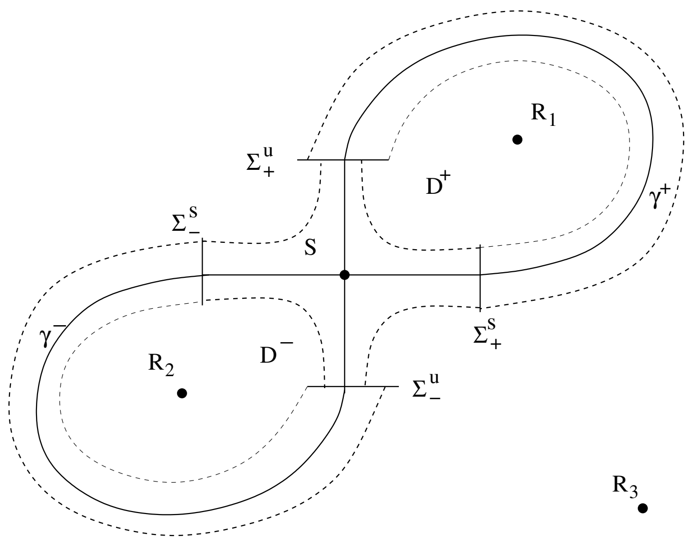

FIGURE 5. Vector field X

(4) $R_3 = (2, -2)$ is a hyperbolic repelling singularity.
(5) Let Σ^u_+ be the transversal section $[-\frac{1}{3}, \frac{1}{3}] \times \{\frac{1}{2}\}$ and Σ^s_+ be the transversal section $\{\frac{1}{3}\} \times [-\frac{1}{3}, \frac{1}{2}]$, and $P^+ : \Sigma^u_+ \to \Sigma^s_+$ be the induced map by the flow, it is assumed that $P^+(x) = x$. By symmetry there is also an induced map P^- defined from $\Sigma^u_- = [-\frac{1}{3}, \frac{1}{3}] \times \{-\frac{1}{2}\}$ to $\Sigma^s_- = \{-\frac{1}{2}\} \times [-\frac{1}{3}, \frac{1}{2}]$ and $P^-(x) = x$.
(6) There are no additional singularities.

Remark 4.9. From the choice of the eigenvalues of the singularity, observe that the induced map L^+ from $\Sigma^s_+ \setminus \{y = 0\}$ to $\Sigma^u_+ \cup \Sigma^u_-$ is a contraction. In the same way it follows that the induced map L^- from $\Sigma^-_+ \setminus \{y = 0\}$ to $\Sigma^u_+ \cup \Sigma^u_-$ is also a contraction. Therefore, the return map from $\Sigma^s_+ \cup \Sigma^s_- \setminus \{y = 0\}$ to itself is a contraction. From that, it follows that the loop $\gamma^+ \cup \gamma^-$ is an attracting loop.

Lemma 4.10. *With the assumptions above the vector field X can be built in such a way that,*

(1) *the repelling basin of R_1 is given by D^+ and the repelling basin of R_2 is D^-;*
(2) *the repelling basin of R_3 in D is the complement of $D^+ \cup D^- \cup \gamma^+ \cup \gamma^-$;*
(3) *the non-wandering set is $S, R_1, R_2, \gamma^+, \gamma^-, R_3$.*

Proof. Let β^+ be a simple closed curve inside D^+ and close to γ^+ and let T^+ be the annulus bounded by γ^+ and β^+. By the property on the return map R inside D^+ and that the saddle S is a contraction (see Remark 4.9), it follows that $\Phi^X_t(T^+) \subset T^+$ for any $t > 0$. This allows to build X in such a way that the first item holds.

With a similar argument, observe that if it is taken any closed curve α outside $D^+ \cup D^-$ and close to $\gamma^+ \cup \gamma^-$, and T is an annulus bounded by $\gamma^+ \cup \gamma^-$ and α, by the property on the return map and that the saddle S is also a volume contraction, it follows that $T \subset \Phi_t^X(T)$ for any $t > 0$. This allows to build X in such a way that the second item holds.

The last item is immediate from the two previous one.

$\square$

4.2. **The flow $\Phi_{\frac{1}{m}}$.** Now, given m, we take $f = \Phi_{\frac{1}{m}}$, the $\frac{1}{m}$−time map of the flow associated to X. Without loss of generality we can assume that there exist $b > 0$ with $(0, b) \in W^u_{loc}(S)$, $a > 0$ with $(a, 0) \in W^s_{loc}(S)$, and $k_m > 0$ such that $f^{k_m}(0, b) = (a, 0)$. The iterate k_m depends on m but from now on, for simplicity, we assume that k_m is equal to 2. Moreover, provided a small neighborhood $B^u_+ = [-\epsilon, \epsilon] \times L^u_+$ containing a fundamental domain $L^u_+ = [(0, b), f(0, b)]$ inside the local unstable manifold of S and a small neighborhood $B^s_+ = L^s_+ \times [-\epsilon, \epsilon]$ containing a fundamental domain $L^s_+ = [f(a, 0), (a, 0)]$ inside the local stable manifold of S, and reparameterizing the time flow, we can assume that

$$f^2(x, y) = (f^u(y), x),$$

where $f^u : L^u_+ \to L^s_+$ is a one-dimensional diffeomorphism such that $f^u(b) = a$, and $f^{u\prime} < c < 0$. By symmetry, the same holds in the neighborhood $B^u_- = L^u_- \times [-\epsilon, \epsilon] = -L^u_+ \times [-\epsilon, \epsilon]$, $B^s_- = -L^s_+ \times [-\epsilon, \epsilon]$ and in particular, $f^2(0, -b) = (-a, 0)$. Of course, the fundamental domains chosen depend on m, more precisely, as m is larger, the fundamental domains gets smaller (recall that $\Phi_{\frac{1}{m}}$ converge to the identity map).

4.3. **Perturbations of $\Phi_{\frac{1}{m}}$.** First we embed the map $f = \Phi_{\frac{1}{m}}$ in a one parameter family $\{f_t\}_{t \geq 0}$ where $f_0 = f$. Now for each $t > 0$ small, we get a diffeomorphism f_t C^r close to f. Moreover, we can get f_t satisfying the following properties (details about the construction of f_t are in subsection 4.5):

(1) The map f_t is symmetric respect to $(0, 0)$, i.e. $f_t(-p) = -f_t(p)$.
(2) If t is small, S, R_1, R_2, and R_3 are hyperbolic fixed points.
(3) For any t, the dynamics in $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ is given by $f_t(x, y) = (\lambda x, \sigma y)$, where $\lambda := \lambda(m) = \lambda^{\frac{1}{m}}, \sigma := \sigma(m) = \sigma^{\frac{1}{m}}$. Observe that it is verified that $\lambda = \sigma^{-\gamma}$, the saddle fixed point $S = (0, 0)$ is dissipative, and the local unstable manifold is contained in the y-axis and the local stable in the x-axis.
(4) Provided the neighborhoods $B^u_+ = [-\epsilon, \epsilon] \times L^u_+$ and $B^s_+ = L^s_+ \times [-\epsilon, \epsilon]$, we assume that

$$f_t^2(x, y) = (f^u(y), x + f_t^s(y))$$

where $f_t^s : L^u_+ \to \mathbb{R}$ is a C^r function verifying that
  (a) $f_0^s = 0$,
  (b) there exists a unique point b'' with $b < b'' < b'$ such that $f_t^s(b') = f_t^s(b'') = f_t^s(b) = 0$ (where b' is a point such that $(0, b') = f(0, b)$),
  (c) for any $y \in (b, b'')$ follows that $f_t^s(y) > 0, {f^s}''_t(y) < 0$ and for any $y \in (b'', b')$ follows that $f_t^s(y) < 0, {f^s}''_t(y) > 0$,
  (d) for any t the map f_t^s has only two critical points $c_1 \in (b, b'')$ and $c_2 \in (b'', b')$ such that $f_t^s(c_1) = t, f_t^s(c_2) = -t(1 + \delta)$ where $\delta + 1 = \frac{c_2}{c_1}$ and moreover the critical points are not degenerated. In particular, $f^2(0, c_1) = (c_1', t), f^2(0, c_2) = (c_2', -t(1 + \delta))$.

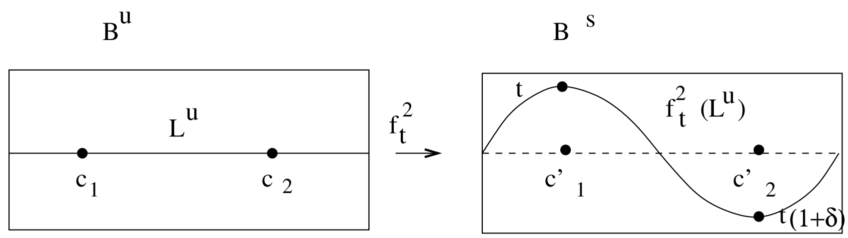

FIGURE 6. Map on fundamental domains.

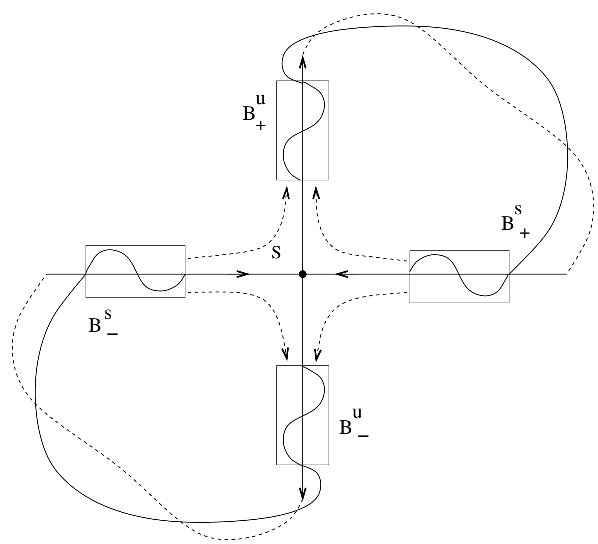

FIGURE 7. f_t.

(5) The function f_t coincide with f_0 outside a neighborhood of size $t^{\frac{1}{r}}$ of $L^u \cup -L^u$ for r large. Therefore, the map f_t can be built in such a way that is C^r close to f_0.

Observe that by symmetry, provided the neighborhood $B^u_- = [-\epsilon, \epsilon] \times -L^u$ and $B^s_- = -L^s \times [-\epsilon, \epsilon]$, it follows that

$$f_t^2(x, y) = (f^u(y), x - f_t^s(-y))$$

where f_t^s is the map defined before and so $-f_t^s \circ -Id : -L^u \to \mathbb{R}$ is a C^r family of functions verifying symmetric similar properties to the one listed above.

Remark 4.11. Without loss of generality, we can assume that,

(1) $f_t^2(x, y) = (y, -(y - c_1)^2 + t + x)$ nearby the point $(0, c_1)$,
(2) $f_t^2(x, y) = (y, (y - c_2)^2 - t(1 + \delta) + x)$ nearby the point $(0, c_2)$,
(3) f_t^2 it is defined by symmetry nearby the points $(0, -c_1)$ and $(0, -c_2)$.

Remark 4.12. There exists an arbitrary small neighborhood W of the region bounded by $\gamma^+ \cup \gamma^-$ such that provided s, t small then any point $x \in W^c$ belong to the basin of repelling of the fixed points R_1 R_2 or R_3.

4.4. **Axiom A for certain parameters.** In [NP76] has been proved that given a C^2-parameter family displaying a $\Omega-$explosion, and assuming that before the explosion the nonwandering set is given by single hyperbolic periodic orbits and a single tangent homoclinic orbit, then there exist parameters arbitrarily close to 0, the non-wandering is a non trivial Axiom A. These results can no be applied straightforward in the present context, however the proof can be adapted to conclude the following proposition that allows to prove Proposition 3.2.

Proposition 4.13. *For any n, if $t=\sigma^{-n}c_1$ then f_t is Axiom A with strong transversality condition. Moreover $\Omega(f_t)$ is formed by*

(1) *three hyperbolic attracting periodic orbits p_+, p_-, q; p_+ contained in D^+, p_- in D^- and q in the complement of $D^- \cup D^+$;*
(2) *the repelling fixed points R_1, R_2, R_3;*
(3) *a finite number of hyperbolic compact invariant set contained in the complement of the basin of attraction and repelling of R_1, R_2, R_3, p_+, p_-, q and containing the homoclinic class of S.*

The strategy to prove Proposition 4.13 consists in the following steps:

(1) It is chosen parameters t (arbitrary small) such that all the critical points of f_t^s belong to the basin of attraction of some finite attracting periodic points (see Lemmas 4.14, 4.15 and 4.16).
(2) The maximal invariant set of f_t inside the boxes $B_+^s \cup B_-^s$, is contained in a small strip along the curves $f_t^2(L_+^u \cup L_-^u)$ (see Lemma 4.19).
(3) Using previous item, it is proved that the maximal invariant set of f_t inside the boxes $B_+^s \cup B_-^s$ (and in particular the nonwandering set of f_t that does not contain the attracting periodic points) has a dominated splitting (see Lemma 4.22).
(4) Using Theorem B in [PS09] is concluded that f_t is Axiom A.
(5) The transversality condition follows from the fact the non attracting classes are contained in the maximal invariant set inside the boxes $B_+^s \cup B_-^s$ (and by item 3, it has dominated splitting).

First, in the next lemma, we prove that for $t=\sigma^{-n}c_1$, the critical points of f_t^s belong to the basin of attraction of an attracting periodic point. For that value of t it also follows that $(c_2', -t)$ and $(-c_2', t)$ belong the basin of attraction of a periodic point in $[D^+ \cup D^-]^c$ (see Lemma 4.16).

Lemma 4.14. *For any n, if $t=\sigma^{-n}c_1$ then there exists a hyperbolic attracting periodic point p_+ contained in D^+ such that, for $2<\alpha<3$, the following hold:*

(1) *the regions $B_\alpha^+(c_1)=\{(x,y): 0\le x\le t^\alpha, |y-c_1|<2^{\frac12}t^{\frac{\alpha}{2}}\}$ is contained in the local basin of attraction of p_+,*
(2) *$B_\alpha^-(c_1')=\{(x,y): |x-c_1'|<2^{\frac12}t^{\frac{\alpha}{2}}, |y-t|<t^\alpha\}$ is contained in the local basin of attraction of $f_t^2(p_+)$ and in particular, (c_1', t) is in the basin of p_+.*

Proof. First it is proven that for $t=\sigma^{-n}c_1$ small

$$f_t^{n+2}(B_\alpha^-(c_1')) \subset B_\alpha^-(c_1'). \tag{4}$$

Observe that for that value of t, the $y-$coordinates of $f^n(c_1', t)$ is equal to c_1. This implies that there is an attracting periodic point in the disk $B_\alpha^-(c_1')$ with period $n+2$ and latter it is proved that the disk is contained in the basin of attraction.

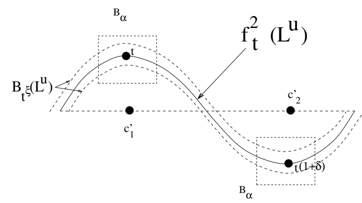

FIGURE 8. Local Basin.

Observe that $f_t^2(B_\alpha^+(c_1)) \subset B_\alpha^-(c_1')$, so to conclude (4) it is proved that

$$f_t^n(B_\alpha^-(c_1')) \subset B_\alpha^+(c_1). \tag{5}$$

In fact, to conclude (5), we prove that

$$f_t^n(B_\alpha^-(c_1')) \subset \hat{B}_\alpha^+(c_1), \tag{6}$$

i.e., for any $(x, y) \in B_\alpha^-(c_1')$ we prove (i) $c_1 - t^\alpha < \sigma^n y \leq c_1 + t^\alpha$ and (ii) $\lambda^n x < t^\alpha$, where $\lambda^n x$ and $\sigma^n y$ are the x and y coordinates of $f_t^n(x, y)$. In fact, on one hand if $(x, y) \in B_\alpha^-(c_1')$ then $t - t^\alpha < y < t + t^\alpha$ and so $c_1 - \sigma^n t^\alpha < \sigma^n y < c_1 + \sigma^n t^\alpha$ and since $t\sigma^n = c_1$ then $c_1 - c_1 t^{\alpha-1} < \sigma^n y < c_1 + c_1 t^{\alpha-1}$ but from the fact that $\alpha > 2$ and so $\frac{\alpha}{2} < \alpha - 1$ then $c_1 t^{\alpha-1} < t^\alpha$, concluding the first inequality. On the other hand, if $(x, y) \in B_\alpha^-(c_1')$ then $c_1' - t^{\frac{\alpha}{2}} < x < c_1' + t^{\frac{\alpha}{2}}$ and so $0 < \lambda^n y < c_1'\lambda^n + \lambda^n t^{\frac{\alpha}{2}}$; since $t\sigma^n = c_1$ and $\lambda = \frac{1}{\sigma^\gamma}$ then $0 < \lambda^n x < t^\gamma$ and recalling $\alpha < \gamma$ (in fact, $\gamma > 6$) it follows that $t^\gamma < t^\alpha$ concluding the second inequality and proving (6).

To conclude that $B_\alpha^-(c_1')$ is inside the basin of attraction, it is shown that for any $z \in B_\alpha^-(c_1')$,

$$||D_z f_t^{2n+4}|| < 1.$$

Observe that for $z \in B_\alpha^+(c_1)$

$$D_z f_t^2 = \begin{pmatrix} 0 & f^{u\prime} \\ 1 & \partial_y f^s \end{pmatrix}$$

with $|\partial_y f_t^s| < 4t^{\frac{\alpha}{2}}$. So, using that Df^n is the diagonal matrix with diagonal λ^n, σ^n then for any $z \in B_\alpha^-(c_1')$

$$D_z f_t^{2n+4} = \begin{pmatrix} (\lambda\sigma)^n & f^{u\prime}{}_t(z)\sigma^n \partial_y f_t^s(z) \\ 0 & (\lambda\sigma)^n + \sigma^{2n}\partial_y f_t^s(z)\partial_y f_t^s(f_t^{2n+2}(z)) \end{pmatrix}$$

and since $\sigma^n = \frac{c_1}{t}$ and $f_t^{2n+2}(z) \in B_\alpha^+(c_1)$ then

$$\sigma^n \partial_y f_t^s(z) < t^{\frac{\alpha}{2}-1} << 1, \ \ |\sigma^{2n}\partial_y f_t^s(z)\partial_y f_t^s(f_t^{2n+2}(z))| < 4t^{2(\frac{\alpha}{2}-1)} << 1.$$

Since $\lambda\sigma < 1$ it follows that the norm of $D_z f_t^{2n+4}$ is smaller than 1. $\square$

The proof of the next lemma follows from the symmetric property assumed on f_t. It basically states that there is also a sink created on D^-.

Lemma 4.15. *For $t = \sigma^{-n}c_1$, there exists a periodic point p_- contained in D^- such that for $2 < \alpha < 3$ it is verified:*

1. *the regions $B_\alpha^+(-c_1) = \{(x, y) : -t^\alpha < x < 0, |y + c_1| < 2^{\frac{1}{2}}t^{\frac{\alpha}{2}}\}$ is contained in the local basin of attraction of p_-,*

(2) $B^-_\alpha(-c'_1) = \{(x,y) : |x + c'_1| < 2^{\frac{1}{2}} t^{\frac{\alpha}{2}}, |y - t| < t^\alpha\}$ *is contained in the local basin of attraction of* $f_t^2(p_-)$ *and in particular,* (c'_1, t) *is in the basin of* p_+.

The next is about a sink that is created in D.

Lemma 4.16. *For* $t = \sigma^{-n} c_1$*, there exists a hyperbolic attracting periodic point* q *contained in* $D = [D^+ \cup D^-]^c$ *such that for* $2 < \alpha < 3$ *it is verified:*

(1) *the regions* $B^+_\alpha(c_2) = \{(x,y) : -s^\alpha < x < 0, |y + c_2| < 2^{\frac{1}{2}} s^{\frac{\alpha}{2}}\}$ *is contained in the local basin of attraction of* p_+,
(2) $B^-_\alpha(c'_2) = \{(x,y) : |x + c'_2| < 2^{\frac{1}{2}} s^{\frac{\alpha}{2}}, |y - s| < s^\alpha\}$ *is contained in the local basin of attraction of* $f_t^2(q)$ *and in particular,* $(c'_2, -s)$ *is in the basin of* q.
(3) *the regions* $B^+_\alpha(-c_2) = \{(x,y) : 0 < x < s^\alpha, |y - c_2| < 2^{\frac{1}{2}} s^{\frac{\alpha}{2}}\}$ *is contained in the local basin of attraction of* q,
(4) $B^-_\alpha(-c'_2) = \{(x,y) : |x + c'_2| < 2^{\frac{1}{2}} s^{\frac{\alpha}{2}}, |y - s| < s^\alpha\}$ *is contained in the local basin of attraction of* $f_t^2(q)$ *and in particular,* $(-c'_2, s)$ *is in the basin of* q.

Proof. Observe that for the construction of f_t it follows that $f_t^2(B^+_\alpha(c_2)) \subset B^-_\alpha(c'_2)$ and $f_t^2(B^+_\alpha(-c_2)) \subset B^-_\alpha(-c'_2)$. On the other hand, since $\sigma^n t = c_1$, then $\sigma^n t(1+\delta) = c_2$ $(\delta = \frac{c_2}{c_1} - 1)$. Repeating the calculation in Lemma 4.14 and recalling the property of f_t (more precisely, item (4.d) in subsection 4.3) follows $f_t^n(B^-_\alpha(c'_2)) \subset B^+_\alpha(-c_2)$ and $f_t^n(B^-_\alpha(-c'_2)) \subset B^+_\alpha(c_2)$. Therefore,

$$f_t^{2n+2}(B^+_\alpha(c_2)) \subset B^+_\alpha(c_2)$$

and so there is a semiattracting periodic point there. In the same way as in the proof of Lemma 4.14, also holds that

$$||D_z f^{2n+2}|| < 1$$

for any $z \in B^+_\alpha(c_2)$ so the the semiattracting periodic point is a hyperbolic sink such that $B^+_\alpha(c_2)$ is contained in its basin of attraction. Similar argument follows for $B^-_\alpha(c'_2), B^-_\alpha(-c'_2)$ and $B^+_\alpha(-c_2)$.

□

Corollary 4.17. *There exists* t *arbitrarily small such that the thesis of lemmas 4.14, 4.15 and 4.16 hold.*

Lemma 4.18. *For any small* t*, if* $x \in \Omega(f_t)$ *then either* $x \in \{R_1, R_2, R_3\}$ *or belongs to* $B_{t^{\frac{1}{r}}}(\gamma^+ \cup \gamma^-)$ *(a neighborhood of radius* $t^{\frac{1}{r}}$ *of* $\gamma^+ \cup \gamma^-$*). In particular, if* $x \notin \{S, R_1, R_2, R_3\}$ *then there exists an iterate of* x *that belongs to* $B^s_{t^{\frac{1}{r}}}(L^s_+ \cup L^s_-)$.

Proof. It follows immediately from the fact that f_t restricted to the complement of $B^u_{t^{\frac{1}{r}}}$ coincide with $f_{0,0}$.

□

Now, using that f_t is dissipative in a neighborhood of S and Lemma 4.18 we conclude that the non-wandering set inside B^s_+ is contained in a small strip around $f_t^2(L^u_+)$; in the same way, the non-wandering set inside B^s_- is contained in a small strip around $f_t^2(L^u_-)$. We fix first α larger than 2 and smaller than 3.

Lemma 4.19. *For small* t*, there exists* ξ *verifying* $\alpha < \xi < 3$ *such that if* $\Lambda_t = \cap_{n\in\mathbb{Z}} f_t^n(W)$ *then*

(1) $\Lambda_t \cap B^s_+ \subset B_{t^\xi}(f_t^2(L^u_+))$*, and so* $\Omega(f_t) \cap B^s_+ \subset B_{t^\xi}(f_t^2(L^u_+))$;

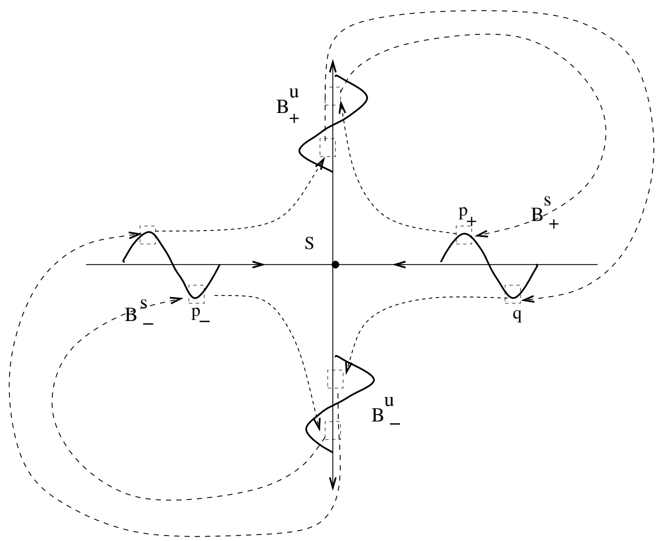

FIGURE 9. Attracting periodic points of f_t.

(2) $\Lambda(f_t) \cap B^s_- \subset B_{t^\xi}(f^2_t(L^u_-))$, *and so* $\Omega(f_t) \cap B^s_- \subset B_{t^\xi}(f^2_t(L^u_-))$;

where $B_{t^\xi}(f^2_t(L^u_\pm))$ *denotes the neighborhood of size* ξ *of* $f^2_t(L^u_\pm)$*. In particular, it follows that for any* $x \in \Omega(f_t)$ *then either* $x \in \{S, R_1, R_2, R_3\}$ *or there exists an iterate of* x *that belongs to* $B_{t^\xi}(f^2_t(L^u_-)) \cup B_{t^\xi}(f^2_t(L^u_+))$*.*

Proof. Recall that f_t coincides with f_0 outside a neighborhood of size of $t^{\frac{1}{r}}$ of $L^u_+ \cup L^u_-$ and so from Lemma 4.18 we have to consider the non-wandering set inside $B^s_{t^{\frac{1}{r}}}(L^u_+ \cup L^u_-)$. Let m be the first positive integer such that $\sigma^m t^{\frac{1}{r}} \in L^u_+ \cup L^u_-$ and this implies that for any $z \in B^s_{t^{\frac{1}{r}}}$ such that $L(z) \in B^u_+$ then $dist(L(z), L^u) < \lambda^{n_z}$ with $n_z \geq m$. Since $\lambda = \sigma^{-\gamma}$ then $\lambda^{n_z} \leq \lambda^m < \sigma^{-\gamma m} \leq t^{\frac{\gamma}{r}}$ and from the election of γ (see second item in subsection 4.1), which verifies that $\frac{\gamma}{r} > 3$ it follows that there exists ξ larger than α (and without loss of generality smaller than 3) such that $\frac{\gamma}{r} > \xi$ and so $dist(L(z), L^u) < \lambda^{n_z} < t^\xi$ and from the definition of f^2_t the thesis follows.

□

Remark 4.20. Observe that the thesis of lemmas 4.14, 4.15, 4.16, and 4.19 hold for small C^2 perturbations of f_t.

Remark 4.21. In a similar way as in the proof of Lemma 4.19 it can be concluded that

(1) if $x \in B^s_\pm$ then either $x \in W^s_{loc}(S)$ or there exists a forward iterate that return to $B^s_+ \cup B^s_-$;
(2) if $x \in B^u_\pm$ then either $x \in W^u_{loc}(S)$ or there exists a backward iterate that return to $B^u_+ \cup B^u_-$.

In what follows we prove the existence of a dominated splitting on the non wandering set excluding the attracting and repelling points p_+, p_-, q, R_1, R_2, R_3.

A compact invariant set Λ has a dominated splitting if there exist two complementary invariant subbundle $E \oplus F$ by the action of the derivative such that $||Df^n_{E(z)}||||Df^{-n}_{F(f^n(z))}|| < \frac{1}{2}$ for any $z \in \Lambda$ and any positive integer n sufficiently large. The existence of dominated splitting is equivalent to the existences of invariant cone fields, i.e., a cone field $\{\mathcal{C}(z)\}_{z\in\Lambda}$ such that

$$Df^n(\mathcal{C}(z)) \subset interior(\mathcal{C}(f^n_t(z))$$

for any $z \in \Lambda$ and any positive integer n sufficiently large.

Lemma 4.22. *For any t verifying the thesis of Lemmas 4.14, 4.15 and 4.16 it follows that Λ_t is a set having a dominated splitting. In particular, $\Omega(f_t) \setminus \{p_+, p_-, q, R_1, R_2, R_3\}$ is a set having a dominated splitting.*

Proof. It is enough to show that for the set of points that is not contained in the local basin of attraction of the periodic points given by Lemmas 4.14, 4.15 and 4.16 it is possible to build an invariant unstable cone field. More precisely, this cone field is defined in

$$B_{t^\xi}(L^u_+) \cup B_{t^\xi}(L^u_-) \bigcup B_{t^\xi}(f^2_t(L^u_+)) \cup B_{t^\xi}(f^2_t(L^u_-)) \bigcup W^s_{loc}(S) \tag{7}$$

which is the region that contains the non-wandering inside $B^u_+ \cup B^u_- \cup B^s_+ \cup B^s_-$. Latter, by standard procedure, the cone field is extended everywhere by iteration and taking the closure. Therefore, to show that the cone field is invariant, is enough to check it in the region (7). For points $(x, y) \in B_{t^\xi}(L^u_+) \cup B_{t^\xi}(L^u_-)$ the cone field has the vertical vector $(0, 1)$ as direction and slope t^ξ, i.e.,

$$\mathcal{C}(x, y) = \{v : slope(v, (0, 1)) \le t^\xi\}.$$

For points $(x', y') = f^2_t(x, y) \in B_{t^\xi}(f^2_t(L^u_+)) \cup B_{t^\xi}(f^2_t(L^u_-))$ it is taken a cone field with direction tangents to $f^2_t(\{x\} \times L^u_+)$ with $|x| < \xi$ fixed and slope t^ξ. More precisely, given a point $(x', y') = (f^u(y), x + f^s_t(y)) \in B_{t^\xi}(f^2_t(L^u_+))$ and defining $w_{(x',y')} = (f^{u\prime}(y), f^{s\prime}_t(y))$ then

$$\mathcal{C}(x', y') = \{v : slope(v, w_{(x',y')}) \le t^\xi\}.$$

For points in the local stable manifold of the saddle S it is taken the forward iterate of the cone in $[B_{t^\xi}(f^2_t(L^u_+)) \cup B_{t^\xi}(f^2_t(L^u_-))] \cap W^s_{loc}(S)$ and in S it is taken a cone along the unstable direction.

Now we proceed to prove that the cone field are invariant. From Lemma 4.14, Lemma 4.19 and item (4) in the definition of f^s_t follows that for any point $(x', y') \in B^s_+ \cap \Omega(f_t)$ then

$$slope(w_{(x',y')}, (1, 0)) > t^{\frac{\alpha}{2}},$$

in particular, if $v \in \mathcal{C}(x', y')$ then

$$slope(w_{(x',y')}, (1, 0)) > t^{\frac{\alpha}{2}} - t^\xi > \frac{1}{2}t^{\frac{\alpha}{2}}.$$

Observe that by definition,

$$f^2_t(\mathcal{C}(x, y)) = \mathcal{C}(f^2_t(x, y)). \tag{8}$$

Let (x, y) be a point in B^s_+ that does not belong to the local stable manifold of S, i.e $y > 0$. Let $m = m(x, y)$ be the first integer such that $f^m_t(x, y) \in B^+_+$, i.e. m is the first integer such that $\sigma^m y \in B^u_+$. It remains to show that for $(x, y) \in B^s_+$ holds

$$D_{(x,y)}f^{m+2}_t(\mathcal{C}(x, y)) \subset \mathcal{C}(f^{m+2}(x, y)).$$

From (8) it remains to show that for $(x,y)\in B^s_+$ holds that

$$D_{(x,y)}f^m_t(\mathcal{C}(x,y))\subset\mathcal{C}(f^m_t(x,y)). \tag{9}$$

Observe that m is larger than m_0 such that $\sigma^{m_0}t>b$. So, giving $v=(1,v_2)\in\mathcal{C}_{(x,y)}$ then $Df^m_t(v)=(\lambda^m,v_2\sigma^m)=v_2\sigma^m(\frac{\lambda^m}{v_2\sigma^m},1)$ with $v_2>t^{\frac{\alpha}{2}}-t^{\xi}>\frac{1}{2}t^{\frac{\alpha}{2}}$ and therefore

$$slope(Df^m_t(v),(0,1))\leq\frac{\lambda^m}{v_2\sigma^m}\leq t^{\gamma-\frac{\alpha}{2}-1}$$

and recalling that $\gamma>6$, $\alpha<\xi<3$ follows that $\gamma-\frac{\alpha}{2}-1>\xi$ and therefore proving that the slope of $Df^m_t(v)$ with $(0,1)$ is strictly smaller that t^{ξ} and so (9) is proved. □

To conclude Proposition 4.13 it is used a result proved in [PS09] (see Theorem B) that states that a generic Kupka-Smale C^2 diffeomorphism with a dominated splitting in its non-wandering set is Axiom A.

Remark 4.23. From remark 4.20, we can assume without loss of generality that f_t is Kupka-Smale and moreover it does not contain normally hyperbolic invariant curves. Therefore, from Theorem B in [PS09] it follows that Λ_t is a hyperbolic set. Since there are transversal homoclinic points associated to S, it follows that Λ_t contains a non trivial hyperbolic set.

Proof of Proposition 4.13. To conclude, we have to show the strong transversality condition: the stable and unstable manifold of any basic piece given by the spectral decomposition intersect transversally. This follows immediately using that in $\Lambda_t\setminus\{p_+,p_-,q,R_1,R_2,R_3\}$ there is a dominated splitting. In fact, if $z\in W^s(x)\cap W^u(x')$ for some $x,x'\in\Omega_t\setminus\{p_+,p_-,q,R_1,R_2,R_3\}$, $z\in\Lambda_t$ and $W^s_{[x,z]}(x)\cup W^u_{[x,z']}(x')\subset\Lambda_t$ where $W^s_{[x,z]}(x)$ is the connected arc of $W^s(x)$ that contains x and is bounded by x and z (and $W^u_{[x,z']}(x')$ is the connected arc of $W^u(x')$ that contains x' and is bounded by x' and z). So, $T_zW^s(x)$ is tangent to the subbundle E_z and $T_zW^u(x)$ is tangent to the subbundle F_z provided by the dominated splitting, and therefore the intersection is transversal. □

4.5. **About the construction of f_t.** We consider a tubular neighborhood $\mathcal{T}$ around γ^+ that contains B^u_+ and B^s_- for the flow $\Phi_{\frac{1}{m}}$. Moreover, using appropriate coordinates we can assume that $\mathcal{T}=[-1,m+2]\times[-1,1]$, $B^u_+=[0,1]\times[-1,1]$, $B^s_+=[m,m+1]\times[-1,1]$ and

$$\Phi_{\frac{1}{m}}(x,y)=(x+1,y).$$

Now we consider a map $g:\mathbb{R}\to\mathbb{R}$ with support in $[-\epsilon,1+\epsilon]$ such that

(1) $g_0:=g_{/[0,1]}$ verifies that $g_0(0)=g_0(1/2)=g_0(1)=0$, only has two critical points at $1/4$, and $3/4$, $g_0(1/4)=1$, $g_0(3/4)=1+\delta$, g_0 is positive in $(0,1/2)$ and negative in $(1/2,1)$;
(2) g_0' is increasing in $[0,1/2]$ and decreasing in $[1/2,1]$;
(3) $g_1:=g_{/[1,1+\epsilon]}$ has only one critical point and $g_1(x)<<g_0(x-1)$.

Moreover, taking ϵ small, the maps can be chosen in such a way that $\hat{g}=g_0(x)+g_1(x+1)$ verifies in $[0,1]$ the same properties that g_0 verifies. Moreover, taking $g_t=tg$ and $\hat{g}_t=t\hat{g}=g_{0t}(x)+g_{1t}(x+1)$, for t small then the C^r norm of $\hat{g}_t$ is also small. Now observe that if

$$f_t(x,y)=(x+1,y+b(y)g_t(x)),$$

where b is bump function that is zero outside $[-1,1]$ and is equal to 1 in $[-1/2,1/2]$ then f_t is a diffeomorphism and for $(x,y) \in B^u_+$ with $y \in [-1/2,1/2]$ follows that

$$f_t^m(x,y) = (x+m, y+\hat{g}_t(x))$$

and observe that after the coordinates changes in $\mathcal{T}$ follows that the map has the properties required for f_t in B^u_+.

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