LAW OF LARGE NUMBERS FOR CERTAIN CYLINDER FLOWS

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ABSTRACT. We construct new examples of cylinder flows, given by skew product extensions of irrational rotations on the circle, that are ergodic and rationally ergodic along a subsequence of iterates. In particular, they exhibit law of large numbers. This is accomplished by explicitly calculating, for a subsequence of iterates, the number of visits to zero, and it is shown that such number has a gaussian distribution.

1. INTRODUCTION

The purpose of this paper is to construct examples of skew product extensions of irrational rotations of the additive circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ exhibiting *law of large numbers*. More specifically, under some weak diophantine conditions on the irrational number $\alpha \in \mathbb{R}$, we construct *roof functions* $\phi : \mathbb{T} \to \mathbb{Z}$ for which the skew product

$$\begin{array}{rccl} F & : & \mathbb{T}\times\mathbb{Z} & \longrightarrow & \mathbb{T}\times\mathbb{Z} \\ & & (x,y) & \longmapsto & (x+\alpha, y+\phi(x)) \end{array} \tag{1.1}$$

is ergodic and rationally ergodic along a subsequence of iterates. This, in particular, implies that F has a law of large numbers. See Subsection 2.4 for the proper definitions.

One must, first of all, observe that F has a natural invariant measure, given by the product of the Lebesgue measure on $\mathbb{T}$ and the counting measure on $\mathbb{Z}$, and it is *infinite*. In this situation, classical theorems of ergodic theory are not valid. For instance, Birkhoff's averages converge to zero almost surely, and this leads us to the following question: what would be a good candidate for a Birkhoff-type theorem in this context? Denoting by $S_n\psi$ the Birkhoff sum of the L^1-function $\psi : \mathbb{T}\times\mathbb{Z} \to \mathbb{R}$, the most natural way is try to find a sublinear sequence (a_n) of positive real numbers and consider the averages $S_n\psi/a_n$. However, by a result of J. Aaronson (see Theorem 2.3), there is never a universal sequence (a_n) for which $S_n\psi/a_n$ converges pointwise to the right value. Nevertheless, Hopf's theorem (Theorem 2.2) is an indication that some sort of regularity might exist and it might still be possible, for a specific sequence (a_n), that the averages oscillate without converging to zero or infinity and so one can hope for a summability method that smooths out the fluctuations and forces convergence. Such *second order ergodic theorems* were considered by J. Aaronson, M. Denker, and A. Fisher in [5].

Another attempt of obtaining a Birkhoff-type theorem has been made by Aaronson in [2], in which he defined and constructed examples of *rationally ergodic* maps. These maps possess a sort of Cèsaro-averaged version of convergence in

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measure: there is a sequence (a_n) such that, for every L^1-function ψ and every subsequence (n_k) of positive integers, there exists a further subsequence (n_l) such that $S_{n_l}\psi(x)/a_{n_l}$ converges Cèsaro almost surely to $\int\psi$. This latter property is called *weak homogeneity* and the sequence (a_n) is called a *return sequence*. Weak homogeneity implies the existence of law of large numbers. See 2.4 for the specific definitions.

A natural program of investigation regards three kinds of questions.

(i) What are the conservative, ergodic, rationally ergodic maps?
(ii) What fluctuations can the Birkhoff sums have?
(iii) What are the ergodic locally finite, σ-finite invariant measures?

Our goal in this work is to give contributions to (i) and (ii) by constructing examples of the form (1.1) that are ergodic and rationally ergodic along a subsequence of iterates. Up to our knowledge, the first examples of ergodic cylinder flows were given by A. Krygin [22] and K. Schmidt [27]. Their examples differ in nature. Krygin assures the existence, for any irrational α, of a roof function ϕ for which F is ergodic. Actually, there exist elegant categorical proofs that the set of pairs (α,ϕ), in various different contexts, for which F is ergodic forms a residual set. See [11], [19]. On the other hand, Schmidt constructs an explicit example motivated by the theory of random walks. The roof function considered by him is equal to the Haar function defined in Section 3, which is actually the basis function for our example. Subsequent works [14], [12] of J.-P. Conze and M. Keane extended Schmidt's results to a larger class of irrationals α and roof functions

$$\phi(x)=(\beta+1)\cdot\mathbb{1}_{\left[0,\frac{\beta}{\beta+1}\right)}(x)-\beta. \tag{1.2}$$

There are many other works regarding this question. See for instance [7], [16], [17], [24], [25].

Regarding (ii), J. Aaronson and M. Keane further investigated Schmidt's example in [6]. They studied the asymptotic behavior of the number of visits to zero and proved that the Birkhoff sums represent a sort of "deterministic random walk". In particular, they showed that if α is quadratic surd[1] then F is rationally ergodic with return sequence $a_n=n/\sqrt{\log n}$, which is relatively close to the linear sequence.

J. Aaronson et al identified in [8] all the locally finite and σ-finite measures invariant under F for the function in (1.2). More recently, J.-P. Conze has extended this analysis to the class of functions

$$\phi(x)=\sum_j c_j\mathbb{1}_{I_j}(x)-\beta$$

where c_j are integers, $\{I_j\}$ is a finite family of intervals of $\mathbb{T}$ and ϕ has zero integral. Assuming that the set of accumulation points of the sequence $(\{q_n\beta\})$ is infinite, where (q_n) stands for the sequence of denominators of the convergents of α, he described in [13] the set of all ergodic and locally finite measures invariant under F.

Not much is known regarding rational ergodicity. There are actually a few examples that have been proved to be rationally ergodic. See for instance [2], [3], [6], [9], [23], where this property is shown to hold in different contexts. With respect

[1]The irrational number α is *quadratic surd* if it satisfies a quadratic equation with integer coefficients.

to cylinder flows given by skew products extensions of irrational rotations on the circle, the only known examples are those in [6].

The most significant contribution of our work is to construct a new class of cylinder flows that are rationally ergodic along a subsequence of iterates and, in particular, possess law of large numbers. The two main results are enclosed below.

Theorem 1.1. *For any $\alpha \in \mathbb{R}$ such that $\liminf_{q\to\infty} q\|q\alpha\| = 0$, there exists a skew product*

$$\begin{array}{rcccl} F & : & \mathbb{T}\times\mathbb{Z} & \longrightarrow & \mathbb{T}\times\mathbb{Z} \\ & & (x,y) & \longmapsto & (x+\alpha, y+\phi(x)) \end{array}$$

such that

(a) *ϕ belongs to $L^p(\mathbb{T})$, for every $p \geq 1$, and*
(b) *F is conservative and ergodic.*

If we slightly reinforce the diophantine properties of α, the rational ergodicity of F along a subsequence of iterates is also guaranteed. This is the content of our second result.

Theorem 1.2. *For any divisible $\alpha \in \mathbb{R}$, there exists a skew product that satisfies Theorem 1.1 and is rationally ergodic along a subsequence of iterates. In particular, F has a law of large numbers.*

An irrational number α is *divisible* if it has a sequence of continuants (q_n) with a certain divisibility property and such that $\lim_{n\to\infty} q_n\|q_n\alpha\| = 0$. See Subsection 2.2 for the specific definitions. It is worth noting that the set of α satisfying these two conditions has full Lebesgue measure, according to the content of Appendix B. Thus, in contrast to [6], in which the set of parameters is countable, Theorems 1.1 and 1.2 hold for a set of parameters of full Lebesgue measure[2].

A remarkable feature of Theorem 1.2 is that the number of visits to zero along the iterates in which F is rationally ergodic exhibits a gaussian distribution. The return sequence is given by $a_{q_{n+1}} = q_{n+1}/\sqrt{\pi n}$ and the normalized averages, described in equation (6.2), do not depend on the choice of α neither on the sequence (q_n). This implies, as a scholium, an analytical fact about random walks, described in Corollary 6.1.

The roof function we construct is different in nature from the others used in this context. We consider the Haar function T defined in Section 3 as a basis function and let

$$\phi(x) = \frac{1}{2}\sum_{j\geq 1}\Big[T(q_j(x+\alpha)) - T(q_jx)\Big]$$

for a specific chosen sequence of positive integers (q_n). One can see ϕ as the limit of worser and worser coboundaries

$$\phi_n(x) = \frac{1}{2}\sum_{j=1}^{n}\Big[T(q_j(x+\alpha)) - T(q_jx)\Big]. \tag{1.3}$$

Observe that, if we just consider the coboundary ϕ_n, the respective cylinder flow will not be ergodic and, moreover, will be conjugate to a rigid rotation. The increasing bad feature of each ϕ_n is what will guarantee that ϕ has the required properties.

[2]In a previous version of this paper, Theorem 1.2 required stronger conditions on α for which the set of parameters has zero Lebesgue measure, but it was pointed to us that the proof works for any divisible irrational number.

The sequence (q_n) will be chosen via the continued fraction expansion of α and this is why the diophantine properties of α influence the dynamical properties of F. Even though ϕ is unbounded, the good feature of it is that we can explicitly calculate the number of visits to zero along a sequence of iterates of F. See Lemma 5.4 and Subsection 5.2.

In some sense, our construction resembles Anosov-Katok method of fast approximations developed in [10], in which they construct differentiable maps sufficiently close to fibered maps of the torus (and, more generally, of any manifold that admits a $\mathbb{T}$-action) with exotic dynamical properties. Indeed, the referred maps are obtained as limits of periodic maps and here we will also use this perspective to prove Theorem 1.2.

Another example that resembles ours is Hajian-Ito-Kakutani's map. See Section 3.3 of [26] for a detailed exposition of this map.

The paper is organized as follows. In Section 2 we introduce the basic notations and definitions as well as the necessary background for the sequel. Section 3 is devoted to the construction of the roof function ϕ and the related convergence issues. In Section 4 we establish Theorem 1.1 with the aide of the theory of random walks. To this matter, Appendix A treats the required results, adapted to our context. Section 5 calculates the number of returns of a generic point to its fiber. This in particular implies Theorem 1.2, which is the content of Section 6. In Appendix B, we enclose the results on continued fractions that allows us to state our results in the greatest possible generality.

2. Preliminaries

2.1. **General notation.** Given a set X, $\#X$ denotes the cardinality of X. If A is a subset of X, $\mathbb{1}_A : X \to \{0,1\}$ denotes the characteristic function of A:

$$\mathbb{1}_A(x) = \begin{cases} 1 & , \quad \text{if } x \in A \\ 0 & , \quad \text{if } x \in X \backslash A. \end{cases}$$

$\mathbb{Z}$ denotes the set of integers and $\mathbb{N}$ the set of positive integers. Each $n \in \mathbb{N}$ defines the ring $\mathbb{Z}_n$ of the residue classes module n. A *complete residue system* is a set $\{a_1, \ldots, a_n\}$ of integers such that $\{a_1, \ldots, a_n\}$ modulo n is equal to $\mathbb{Z}_n$.

Given a real number x, $\lfloor x \rfloor$ and $\{x\}$ are the integer and fractional parts of x, respectively. Let $\|x\|$ be the distance from x to the closest integer,

$$\|x\| = \min\{\{x\}, 1 - \{x\}\}.$$

We use the following notation to compare the asymptotic of functions.

Definition 2.1. Let $f, g : \mathbb{N} \to \mathbb{R}$ be two real-valued functions. We say $f \lesssim g$ if there is a constant $C > 0$ such that

$$|f(n)| \leq C \cdot |g(n)|, \quad \forall\, n \in \mathbb{N}.$$

If $f \lesssim g$ and $g \lesssim f$, we write $f \sim g$. We say $f \approx g$ if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1.$$

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denote the circle, parameterized by $[0,1)$, and let $d : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ be the induced distance function. For every $\alpha \in \mathbb{R}$, $R_\alpha : \mathbb{T} \to \mathbb{T}$ is the rotation $R_\alpha x = x + \alpha$.

Let λ be the Lebesgue measure on $\mathbb{T}$ and μ the measure defined on the cylinder $\mathbb{T}\times\mathbb{Z}$ by $\mu=\lambda\times$ counting measure on $\mathbb{Z}$. Given a function $\psi:\mathbb{T}\to\mathbb{R}$, its L^p-norm with respect to λ is defined as

$$\|\psi\|_p=\left(\int_{\mathbb{T}}|\psi|^p d\lambda\right)^{1/p}$$

and the space of L^p-integrable functions as $L^p(\mathbb{T})$. Due to the index p, there will be no confusion between the integer norm $\|\cdot\|$ and the L^p-norm $\|\cdot\|_p$.

2.2. **Continued fractions.** Given an irrational number α, consider its *continued fraction expansion*

$$\alpha \;=\; a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots}}} \;:=\; [a_0;a_1,a_2,\dots]\;,$$

whose n^{th}-*convergent* is

$$\alpha_n=\frac{p_n}{q_n}=[a_0;a_1,a_2,\dots,a_n],\;\; n\geq 0.$$

The q_n are called the *continuants*. They give the best rational approximations to α. More precisely, the approximation is equal to

$$\|q_n\alpha\|=q_n\cdot\left|\alpha-\frac{p_n}{q_n}\right|\;.$$

It is known, by Dirichlet's theorem, that

$$\liminf_{q\to\infty} q\|q\alpha\|\leq 1$$

for any $\alpha\in\mathbb{R}$. Let α be *divisible* if it has a sequence (q_{n_j}) of continuants satisfying

$$2q_{n_j} \text{ divides } q_{n_{j+1}} \quad\text{and}\quad \lim_{j\to\infty} q_{n_j}\|q_{n_j}\alpha\|=0.$$

The set of divisible numbers has full Lebesgue measure in $\mathbb{R}$. This is the content of Proposition B.1 which, in particular, guarantees that Theorem 1.1 and Theorem 1.2 are valid for Lebesgue almost every $\alpha\in\mathbb{R}$.

From now on, (q_n) will denote a sequence of (instead of all) continuants of α such that

$$\lim_{n\to\infty} q_n\|q_n\alpha\|=0 \tag{2.1}$$

and, whenever α is divisible, this chosen sequence (q_n) will also satisfy that $2q_n$ divides q_{n+1}. We will also make constant use of the following conditions:

(CF1) For any $n\geq 1$,

$$2\sum_{j>n}\|q_j\alpha\|<\|q_n\alpha\|.$$

(CF2) For any $p\geq 1$,

$$\sum_{j\geq 1} j^{p+1}\cdot\|q_j\alpha\|<\infty.$$

(CF3) For any $p\geq 1$,

$$\sum_{j=1}^{n} j^{p+1}\cdot q_j<q_{n+1}\;\;\text{for } n>n(p).$$

(CF4) For any $n \geq 1$,

$$\left(2^n \sum_{j=1}^{n-1} q_j\right) \cdot q_n \|q_n\alpha\| < 1.$$

Condition (CF2) is always satisfied. Indeed,

$$\sum_{j\geq 1} j^{p+1} \cdot \|q_j\alpha\| < \sum_{j\geq 1} \frac{j^{p+1}}{q_j}$$

is bounded for every $p \geq 1$, because the exponential behavior of q_j controls the polynomial behavior of j^{p+1}. (CF1), (CF3) and (CF4) are assured by passing, if necessary, to a subsequence of (q_n).

2.3. **Birkhoff sums.** Let $\alpha \in \mathbb{R}$, $\phi : \mathbb{T} \to \mathbb{R}$ a L^1-measurable function and F defined as in (1.1). The dynamics of F is intimately connected to the cocycle $S(\alpha,\phi) : \mathbb{T} \times \mathbb{Z} \to \mathbb{R}$ defined as the Birkhoff sums of ϕ with respect to the rotation R_α:

$$S(\alpha,\phi)(x,n) = \begin{cases} \sum_{k=0}^{n-1} \phi(x+k\alpha) & \text{, if } n \geq 1 \\ 0 & \text{, if } n = 0 \\ -\sum_{k=1}^{-n} \phi(x-k\alpha) & \text{, if } n < 0. \end{cases}$$

For simplicity, we denote $S(\alpha,\phi)(\,\cdot\,,n) : \mathbb{T} \to \mathbb{R}$ by $S_n(\alpha,\phi)$. By Birkhoff's theorem,

$$\frac{S_n(\alpha,\phi)(x)}{n} \longrightarrow \int_{\mathbb{T}} \phi d\lambda \quad \text{as } n \to \infty$$

for Lebesgue almost every $x \in \mathbb{T}$. In particular, if $\int_{\mathbb{T}} \phi d\lambda \neq 0$, almost every point diverges, which does not allow any kind of recurrence. From now on, we assume ϕ has zero mean. In this situation, Birkhoff sums have a sublinear growth.

2.4. **Infinite ergodic theory.** Let $(X,\mathcal{A},\mu,F)$ be a measure-preserving system: $(X,\mathcal{A},\mu)$ is a measure space, μ a σ-finite measure and F is a measurable transformation on X invariant under μ. Assume that μ is *conservative*: $\mu(A) = 0$ whenever $A \in \mathcal{A}$ is such that $\{F^{-n}A\}_{n\geq 0}$ are pairwise disjoint. We say that F is *ergodic* if it has only trivial invariant sets, that is, if $\mu(A) = 0$ or $\mu(X\backslash A) = 0$ whenever A is a measurable set invariant under F.

Let $\phi : X \to \mathbb{R}$ be a measurable function. A successful area in ergodic theory deals with the convergence of the averages $n^{-1} \cdot \sum_{k=0}^{n-1} \phi\left(F^k x\right)$, $x \in X$, when n goes to infinity. The well known Birkhoff's theorem states that, if $\mu(X) < \infty$, such limit exists for almost every $x \in X$ whenever ϕ is a L^1-function. This is not the case when μ is infinite. Indeed, if $\mu(X) = \infty$, these averages converge to zero for almost every $x \in X$. Nevertheless, they converge to zero in the same proportional rate, according to the following result.

Theorem 2.2 (Hopf [18])**.** *Let $(X,\mathcal{A},\mu,F)$ be a conservative ergodic measure-preserving system. Then, for every $\phi,\psi \in L^1(X,\mathcal{A},\mu)$ such that $\psi \geq 0$ and*

$\int_X \psi d\mu > 0$,

$$\frac{\sum_{k=0}^{n-1} \phi\left(F^k x\right)}{\sum_{k=0}^{n-1} \psi\left(F^k x\right)} \longrightarrow \frac{\int_X \phi\, d\mu}{\int_X \psi d\mu}$$

for μ-almost every $x \in X$.

At this point, it is natural to ask if there exists some "appropriate" rate of convergence: is there a normalizing sequence of constants (a_n) such that ${a_n}^{-1} \cdot \sum_{k=0}^{n-1} \phi\left(F^k x\right)$ converges almost surely? The negative answer was given by J. Aaronson.

Theorem 2.3 (Aaronson [1])**.** *Let $(X, \mathcal{A}, \mu, F)$ be a conservative ergodic measure-preserving system with $\mu(X) = \infty$, and let (a_n) be a sequence of positive real numbers. Then, for every $\phi \in L^1(X, \mathcal{A}, \mu)$ such that $\phi \geq 0$ and $\int_X \phi d\mu > 0$,*

$$\limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} \phi\left(F^k x\right)}{a_n} = \infty \quad a.e \qquad or \qquad \liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} \phi\left(F^k x\right)}{a_n} = 0 \quad a.e.$$

This means that any attempt of normalization will under or overestimate the behavior of Birkhoff sums. Nevertheless, one can hope for a summability method that smooths out the fluctuations and forces convergence. More generally, one can hope for a law of large numbers.

Definition 2.4. A *law of large numbers* for a conservative ergodic measure-preserving system $(X, \mathcal{A}, \mu, F)$ is a function $L : \{0,1\}^{\mathbb{N}} \to [0, \infty]$ such that, for any $A \in \mathcal{A}$, the equality

$$L(\mathbb{1}_A(x), \mathbb{1}_A(Fx), \mathbb{1}_A(F^2 x), \ldots) = \mu(A)$$

holds for μ-almost every $x \in X$.

One can see the function L as a sort of blackbox: given the input of hittings of a generic point $x \in X$ to a fixed set $A \in \mathcal{A}$, the output is the measure of A. For example, if $\mu(X) = 1$, the function $L : \{0,1\}^{\mathbb{N}} \to [0, \infty]$ defined by

$$L(x_0, x_1, \ldots) = \begin{cases} \lim\limits_{n\to\infty} \dfrac{1}{n} \sum\limits_{k=0}^{n-1} x_k \ , & \text{if the limit exists,} \\ \\ 0 \ , & \text{otherwise} \end{cases}$$

is a law of large numbers. The infinite measure situation is quite different: there are systems with no law of large numbers. For example, let F be *squashable*: there is $G : (X, \mathcal{A}) \to (X, \mathcal{A})$, commuting with F, such that

$$\mu(G^{-1}A) = c \cdot \mu(A) \ \text{ for all } A \in \mathcal{A}, \tag{2.2}$$

for some $c \neq 1$. If F had a law of large numbers, say L, then for μ-almost every $x \in X$ we would have

$$\begin{aligned} \mu(A) &= L(\mathbb{1}_A(Gx), \mathbb{1}_A(FGx), \ldots) \\ &= L(\mathbb{1}_A(Gx), \mathbb{1}_A(GFx), \ldots) \\ &= L(\mathbb{1}_{G^{-1}A}(x), \mathbb{1}_{G^{-1}A}(Fx), \ldots) \\ &= \mu(G^{-1}A), \end{aligned}$$

contradicting the assumption (2.2). See [4] for more on squashable systems.

There are, fortunately, some conditions that guarantee the existence of law of large numbers. Given $A \in \mathcal{A}$, let $S_n(A) : X \to \mathbb{N}$ be the Birkhoff sum of the characteristic function $\mathbb{1}_A$ with respect to F.

Definition 2.5. A conservative ergodic measure-preserving system $(X, \mathcal{A}, \mu, F)$ is called *rationally ergodic along a subsequence of iterates* if there is a set $A \in \mathcal{A}$ with $0 < \mu(A) < \infty$ satisfying the *Renyi inequality*

$$\int_A S_{n_k}(A)^2 d\mu \lesssim \left(\int_A S_{n_k}(A) d\mu \right)^2$$

for some increasing sequence (n_k) of positive integers.

We note the above definition differs from the original one [2], since the Renyi inequality is asked to hold, instead of all positive integers, only for a subsequence of them.

Definition 2.6. A conservative ergodic measure-preserving system $(X, \mathcal{A}, \mu, F)$ is called *weakly homogeneous* if there is a sequence (a_{n_k}) of positive real numbers such that, for all $\phi \in L^1(X, \mathcal{A}, \mu)$,

$$\frac{1}{N} \sum_{k=1}^{N} \frac{1}{a_{n_k}} \sum_{j=0}^{n_k - 1} \phi\left(F^j x\right) \longrightarrow \int_X \phi d\mu \tag{2.3}$$

for μ-almost every $x \in X$.

(a_{n_k}) is called a *return sequence* of F and it is unique up to asymptotic equality.

Theorem 2.7 (Aaronson [2]). *Every measure-preserving system $(X, \mathcal{A}, \mu, F)$ that is rationally ergodic along a subsequence of iterates is weakly homogeneous. More specifically, every subsequence (a_{n_k}) can be refined to a further subsequence such that (2.3) holds for μ-almost every $x \in X$.*

Theorem 2.7 also gives that

$$a_{n_k} = \frac{1}{\mu(A)^2} \int_A S_{n_k}(A) d\mu = \frac{1}{\mu(A)^2} \sum_{j=0}^{n_k - 1} \mu\left(A \cap F^{-j} A\right). \tag{2.4}$$

Observe that weak homogeneity defines a law of large numbers $L : \{0, 1\}^{\mathbb{N}} \to [0, \infty]$ by

$$L(x_0, x_1, \ldots) = \begin{cases} \displaystyle\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \frac{1}{a_{n_k}} \sum_{j=0}^{n_k - 1} x_j \ , & \text{if the limit exists,} \\ 0 \ , & \text{otherwise.} \end{cases}$$

The goal of this work is to construct examples of cylinder flows given by skew product extensions of irrational rotations on the circle that are ergodic and rationally ergodic along a subsequence of iterates and, therefore, have law of large numbers.

3. Construction of roof function ϕ

Let $T : \mathbb{T} \to \mathbb{Z}$ be the *Haar function*, defined as

$$T(x) = \begin{cases} 1\ , & \text{if } x \in \left[0, \dfrac{1}{2}\right) \\ -1\ , & \text{if } x \in \left[\dfrac{1}{2}, 1\right)\ . \end{cases}$$

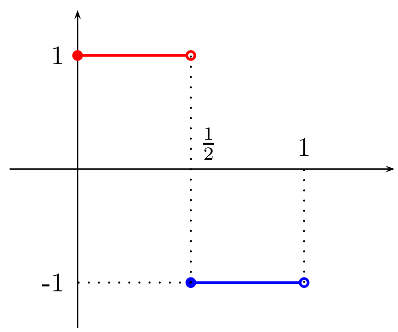

Figure 1: the graph of T.

Let $\alpha \in \mathbb{R}$ and (q_n) its associated sequence of continuants, that is, satisfying (2.1) and (CF1) to (CF4). For each $j \geq 1$, let $T_j : \mathbb{T} \to \mathbb{Z}$ be the dilation of T by q_j, that is, $T_j(x) = T(q_j x)$, where $q_j x$ (and any expression appearing as argument of T) is taken modulo 1. The function we will consider is

$$\phi(x) = \frac{1}{2} \sum_{j \geq 1} \Big[T_j(x + \alpha) - T_j(x) \Big].$$

First of all, it is not clear that this defines a L^1-measurable function. The proof of this fact depends on a couple of auxiliary lemmas.

Lemma 3.1. *Let q be a positive integer and $\beta, \gamma \in \mathbb{T}$. Then the set*

$$\{x \in \mathbb{T}\,;\, T(qx + \beta) \neq T(qx + \gamma)\}$$

has Lebesgue measure equal to $2\|\beta - \gamma\|$.

Proof. First, observe that changing x by $x - \beta/q$, we can assume that $\beta = 0$. The function $x \mapsto T(qx)$ is $1/q$-periodic, with

$$T(qx) = \begin{cases} 1\ , & \text{if } x \in \left[0, \dfrac{1}{2q}\right) \bigcup \left[\dfrac{2}{2q}, \dfrac{3}{2q}\right) \bigcup \cdots \bigcup \left[\dfrac{2q-2}{2q}, \dfrac{2q-1}{2q}\right) \\ -1\ , & \text{if } x \in \left[\dfrac{1}{2q}, \dfrac{2}{2q}\right) \bigcup \left[\dfrac{3}{2q}, \dfrac{4}{2q}\right) \bigcup \cdots \bigcup \left[\dfrac{2q-1}{2q}, 1\right). \end{cases}$$

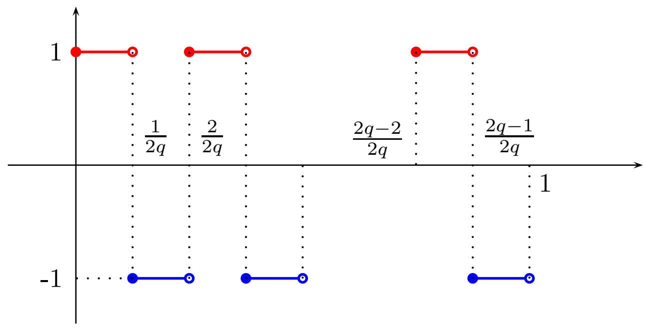

Figure 2: the graph of $x \mapsto T(qx)$.

For each interval $\left[\frac{i}{2q}, \frac{i+1}{2q}\right)$, $T(qx)$ is different from $T(qx+\gamma)$ if and only if the discontinuity $1/2$ belongs to the interval in $\mathbb{T}$ defined by the points qx and $qx+\gamma$. This happens for an interval[3] of length $\|\gamma\|/q$ and so, multiplying by the number $2q$ of these intervals, the desired assertion is proved. □

Lemma 3.2. *Let (q_n) be a sequence of positive integers and (β_n), (γ_n) sequences in $\mathbb{T}$. If $\psi : \mathbb{T} \to \mathbb{Z}$ is defined by*

$$\psi(x) = \frac{1}{2}\sum_{j\geq 1}\Big[T(q_jx+\beta_j) - T(q_jx+\gamma_j)\Big],$$

then

$$\|\psi\|_p^p \leq 2\sum_{j\geq 1} j^{p+1}\cdot \|\beta_j - \gamma_j\|. \tag{3.1}$$

Proof. Assume the right hand side of (3.1) is finite. In particular, $\sum \|\beta_j - \gamma_j\|$ is convergent. For each $n \geq 1$, let

$$\Lambda_n = \{x \in \mathbb{T}\,;\, T(q_jx+\beta_j) = T(q_jx+\gamma_j),\ \forall\, j > n\}.$$

In Λ_n, we have

$$\psi(x) = \frac{1}{2}\sum_{j=1}^{n}\Big[T(q_jx+\beta_j) - T(q_jx+\delta_j)\Big].$$

The complement of Λ_n is defined by the property that $T(q_jx+\beta_j) \neq T(q_jx+\gamma_j)$ for some $j > n$. By Lemma 3.1, its Lebesgue measure is at most $2\sum_{j>n}\|\beta_j - \gamma_j\|$. Then the sequence of functions (ψ_n) given by $\psi_n = \psi \cdot \mathbb{1}_{\Lambda_n}$ converges pointwise to ψ. By Fatou's Lemma, the result will follow if we manage to prove (3.1) for each ψ_n.

Fixed $n \geq 1$, we have

$$|\psi_n(x)| \leq \sum_{j=1}^{n}\left|\frac{T(q_jx+\beta_j) - T(q_jx+\gamma_j)}{2}\right|,\ \forall\, x \in \mathbb{T}.$$

[3] If $\gamma \in \left[0, \frac{1}{2}\right)$, the interval is $\left[\frac{i+1}{2q} - \frac{\gamma}{q}, \frac{i+1}{2q}\right)$; if $\gamma \in \left[-\frac{1}{2}, 0\right)$, the interval is $\left[\frac{i}{2q}, \frac{i}{2q} - \frac{\gamma}{q}\right)$.

Define, for each $m \in \{1, \ldots, n\}$, the set

$$A_m = \left\{ x \in \mathbb{T} \, ; \, \sum_{j=1}^{n} \left| \frac{T(q_j x + \beta_j) - T(q_j x + \gamma_j)}{2} \right| = m \right\}.$$

If we further define, for each $j \in \{1, \ldots, n\}$, the set

$$A_m^j = \{ x \in A_m \, ; \, j \text{ is the largest index such that } T(q_j x + \beta_j) \neq T(q_j x + \gamma_j) \} ,$$

then

$$A_m = \bigsqcup_{j=m}^{n} A_m^j.$$

Each A_m^j is contained in the set $\{x \in \mathbb{T} \, ; \, T(q_j x + \beta_j) \neq T(q_j x + \gamma_j)\}$ and so, by Lemma 3.1, its Lebesgue measure is at most $2\|\beta_j - \gamma_j\|$. Summing up this estimate in j and m, we obtain that

$$\begin{aligned}
\|\psi_n\|_p^p &= \int_{\mathbb{T}} |\psi_n|^p d\lambda \\
&\leq \sum_{m=1}^{n} m^p \cdot \lambda(A_m) \\
&\leq 2 \sum_{m=1}^{n} m^p \sum_{j=m}^{n} \|\beta_j - \gamma_j\| \\
&\leq 2 \sum_{m=1}^{n} \sum_{j=m}^{n} j^p \cdot \|\beta_j - \gamma_j\| \\
&\leq 2 \sum_{j \geq 1} j^{p+1} \cdot \|\beta_j - \gamma_j\| ,
\end{aligned}$$

thus establishing (3.1) for ψ_n. □

Lemma 3.2 will be used repeatedly in the next subsections, the first time being to prove that ϕ_n, as defined in (1.3), converges to ϕ.

3.1. (ϕ_n) **converges to** ϕ **in** $L^p(\mathbb{T})$**.** By Lemma 3.2,

$$\begin{aligned}
\|\phi - \phi_n\|_p^p &= \left\| \frac{1}{2} \sum_{j>n} \Big[T(q_j x + q_j \alpha) - T(q_j x) \Big] \right\|_p^p \\
&\leq 2 \sum_{j>n} j^{p+1} \cdot \|q_j \alpha\|
\end{aligned}$$

which, by condition (CF2), goes to zero as n goes to infinity.

3.2. $(\tilde{\phi}_n)$ **converges to** ϕ **in** $L^p(\mathbb{T})$**.** In order to make the calculations of Section 5, in which estimates on the return map of F will be given, we need to approximate ϕ by something easier to manage with. We will approximate ϕ not by ϕ_n, but by its "rational" truncated versions $\tilde{\phi}_n$, defined as

$$\tilde{\phi}_n(x) = \frac{1}{2} \sum_{j=1}^{n} \Big[T_j(x + \alpha_{n+1}) - T_j(x) \Big]. \tag{3.2}$$

The reason we do this will become clear in Section 5. The argument is similar in spirit to the Anosov-Katok method of fast approximations [10], in which the authors construct differentiable examples of skew products with prescribed topological and ergodic properties sufficiently close to fibered maps of the torus. There, the functions that define the transformations are obtained as the limit of coboundaries, not from the proper irrational rotation, but from good rational approximations of it. In order to guarantee smoothness of the limit function, these rational approximations must converge fast and the irrationals that appear in this limiting procedure end up being Liouville. Contrary to them, smoothness is not our interest here, and this does not require any kind of fast approximation.

Let us prove that the functions $\tilde{\phi}_n$ converge to ϕ in $L^p(\mathbb{T})$ for any $p \geq 1$. This follows by another application of Lemma 3.2. Indeed, as

$$\phi_n(x) - \tilde{\phi}_n(x) = \frac{1}{2}\sum_{j=1}^{n}\Big[T(q_j x + q_j\alpha) - T(q_j x + q_j\alpha_{n+1})\Big],$$

we have

$$\begin{aligned}
\left\|\tilde{\phi}_n - \phi_n\right\|_p^p &\leq 2\sum_{j=1}^{n} j^{p+1}\cdot\|q_j\alpha - q_j\alpha_{n+1}\| \\
&\leq 2\sum_{j=1}^{n} j^{p+1}\cdot q_j\cdot|\alpha-\alpha_{n+1}| \\
&= \frac{2\|q_{n+1}\alpha\|}{q_{n+1}}\sum_{j=1}^{n} j^{p+1}\cdot q_j \\
&\leq 2\|q_{n+1}\alpha\|\,,
\end{aligned}$$

where in the last inequality we used (CF3).

4. Proof of Theorem 1.1

4.1. **Branches and plateaux.** We call a *branch* of T_j any of the branches $\left[\frac{i}{q_j},\frac{i+1}{q_j}\right)$, $i = 0, 1, \ldots, q_j - 1$, of the expanding map $x \mapsto q_j x$. Each branch of T_j decomposes itself in two subintervals $\left[\frac{2i}{2q_j},\frac{2i+1}{2q_j}\right)$ and $\left[\frac{2i+1}{2q_j},\frac{2i+2}{2q_j}\right)$, each of them called a *plateau* of T_j, in which T_j is constant (see figure 2). The first will be called a *positive plateau* and the second a *negative plateau*.

Let $I_j(x)$ denote the plateau of T_j containing x and

$$m_n(x) := T_1(x) + \cdots + T_n(x)\ ,\quad n \geq 1.$$

If (q_n) satisfies the divisibility condition, then clearly $I_1(x) \supset I_2(x) \supset \cdots$ and so we have the implication

$$y \in I_n(x) \implies m_n(x) = m_n(y). \tag{4.1}$$

This is also true if, instead of the divisibility condition, (q_n) satisfies Lemma A.1. More specifically, using the notation of Appendix A,

$$I_{n_0}(x) \supset I_{n_0+1}(x) \supset \cdots \quad \text{whenever } x \in \Omega_{n_0}^{\infty}.$$

For such a fixed x, there is a positive integer $n_1 = n_1(x)$ such that

$$\begin{aligned} I_n(x) &\subset I_1(x), \ldots, I_{n_0}(x) \\ \Longrightarrow \quad \bigcap_{j=1}^{n} I_j(x) &= I_n(x) \end{aligned}$$

for every $n \geq n_1$ and so (4.1) remains valid. We will use this condition below.

4.2. **Ergodicity.** We will prove ergodicity in two steps.

Step 1. For any $A \subset \mathbb{T} \times \{0\}$ of positive measure, the union $\bigcup_{n\geq 1} F^n A$ contains $\mathbb{T} \times \{0\}$ modulo zero.

Step 2. $F(\mathbb{T}\times\{0\})\cap(\mathbb{T}\times\{1\})$ and $F(\mathbb{T}\times\{0\})\cap(\mathbb{T}\times\{-1\})$ have positive measure.

Once this is done, it is clear that F will be ergodic. Actually, let $A \subset \mathbb{T} \times \mathbb{Z}$ be F-invariant with positive measure. We can assume that A has positive measure when restricted to the fiber $\mathbb{T} \times \{0\}$. By Step 1, A has full measure in $\mathbb{T} \times \{0\}$. By Step 2, A has also positive measure in both fibers $\mathbb{T} \times \{1\}$ and $\mathbb{T} \times \{-1\}$. Applying repeatedly Steps 1 and 2, we conclude that A has full measure in $\mathbb{T} \times \mathbb{Z}$. Step 1 will follow from the next

Lemma 4.1. *Let $A_1, A_2 \subset \mathbb{T} \times \{0\}$ have positive μ-measure. Then there is $n \geq 1$ such that the intersection $F^n A_1 \cap A_2$ has positive μ-measure.*

To prove Lemma 4.1, we will localize A_1 and A_2 to subsets in which ϕ and ϕ_n coincide, and actually their Birkhoff sums up to the order q_{n+1}. Letting $\mathcal{D} = \{0, 1/2\}$, this set is defined as

$$\Lambda_n = \{x \in \mathbb{T}\,;\, d(q_j x, \mathcal{D}) > q_j \|q_j \alpha\| \text{ for } j > n\}\,.$$

Note that

$$d(q_j(x + k\alpha), q_j x) = \|k q_j \alpha\| = k\|q_j \alpha\| \leq q_j \|q_j \alpha\|$$

whenever $j > n$ and $k = 1, \ldots, q_{n+1}$. This implies that

$$F^k(x, 0) = (x + k\alpha, S_k(\alpha, \phi_n)(x)) \ \ , \ x \in \Lambda_n, \ k = 1, \ldots, q_{n+1}.$$

Observe that the Λ_n's form an ascending chain of subsets of $\mathbb{T}$ and that $\mathbb{T}\backslash\Lambda_n$ has Lebesgue measure at most $\sum_{j>n} q_j \|q_j \alpha\|$. We can suppose, after passing to a subsequence[4], that this sum is smaller than 2^{-n}.

Proof of Lemma 4.1. We will assume the additional condition

(CF5) For any $n \geq 1$, $\{\alpha, 2\alpha, \ldots, q_{n+1}\alpha\}$ is $\left(\frac{1}{2q_n}\right)^2$-dense in $\mathbb{T}$.

This can be assumed by passing, if necessary, to a subsequence of (q_n).

Define the set

$$\Sigma_n = \left\{ x \in \mathbb{T}\,;\, d(x, \partial I_j(x)) > \left(\frac{1}{2q_j}\right)^2 \text{ for } j > n \right\}.$$

[4]Here is where Theorem 1.1 requires that $\liminf_{q\to\infty} q\|q\alpha\| = 0$.

The sequence (Σ_n) also forms an ascending chain of subsets of $\mathbb{T}$ and[5]

$$\lambda(\mathbb{T}\backslash\Sigma_n) \leq \sum_{j>n} \frac{1}{q_j}\cdot$$

This together with the fact that $\lambda(\Lambda_n), \lambda(\Omega_n^\infty) \to 1$ as $n \to \infty$ allows us to take $n_0 \geq 1$ large enough and assume that

(i) $A_1 \subset \Lambda_{n_0}$,
(ii) $A_1, A_2 \subset \Sigma_{n_0}$ and
(iii) $A_1, A_2 \subset \Omega_{n_0}^\infty$.

By the Lebesgue differentiation theorem, let x_1, x_2 be points of density for A_1, A_2, respectively. Now choose $n_1 \geq 1$ large enough (see Subsection 4.1) such that

(iv) $\bigcap_{j=1}^n I_j(x_i) = I_n(x_i)$ for every $n \geq n_1$ and $i = 1, 2$.

Finally, let $n \geq n_0, n_1$ such that

(v) $m_n(x_1) = m_n(x_2)$ and
(vi) $\lambda\left(A_i \cap \left(x_i - \left(\frac{1}{2q_n}\right)^2, x_i + \left(\frac{1}{2q_n}\right)^2\right)\right) > \frac{3}{4}\cdot 2\left(\frac{1}{2q_n}\right)^2$ for $i = 1, 2$.

The existence of such n is assured by Lemma A.1 and the fact that x_i is a point of density for A_i. For simplicity, let

$$\tilde{A}_i = A_i \cap \left(x_i - \left(\frac{1}{2q_n}\right)^2, x_i + \left(\frac{1}{2q_n}\right)^2\right), \quad i = 1, 2.$$

By (ii), $\tilde{A}_i \subset I_n(x_i)$. Now use (CF5) to choose $k \in \{1, 2, \ldots, q_{n+1}\}$ such that

$$d(x_1 + k\alpha, x_2) < \left(\frac{1}{2q_n}\right)^2\cdot \tag{4.2}$$

The proof of the lemma will follow from the next two claims.

Claim 1. The set $(\tilde{A}_1 + k\alpha) \cap \tilde{A}_2 \subset \mathbb{T}$ has positive Lebesgue measure.

Indeed, (4.2) implies that the union $(\tilde{A}_1 + k\alpha) \cup \tilde{A}_2$ is contained in an interval of length $3 \cdot \left(\frac{1}{2q_n}\right)^2$ and so, by (vi),

$$\begin{aligned}
\lambda((\tilde{A}_1 + k\alpha) \cap \tilde{A}_2) &= \lambda(\tilde{A}_1 + k\alpha) + \lambda(\tilde{A}_2) - \lambda((\tilde{A}_1 + k\alpha) \cup \tilde{A}_2)\\
&> \frac{3}{2}\cdot\left(\frac{1}{2q_n}\right)^2 + \frac{3}{2}\cdot\left(\frac{1}{2q_n}\right)^2 - 3\cdot\left(\frac{1}{2q_n}\right)^2\\
&= 0\,.
\end{aligned}$$

Claim 2. The set $F^k(\tilde{A}_1 \times \{0\}) \cap (\tilde{A}_2 \times \{0\}) \subset \mathbb{T} \times \mathbb{Z}$ has positive μ-measure.

It is enough to prove that $S_k(\alpha, \phi)(x) = 0$ for every x satisfying Claim 1. By (i), $x \in \Lambda_{n_0} \subset \Lambda_n$ and so

$$S_k(\alpha, \phi)(x) = S_k(\alpha, \phi_n)(x) = m_n(x + k\alpha) - m_n(x).$$

Observe that

[5]For each plateau of T_j, we remove two intervals of length $\left(\frac{1}{2q_j}\right)^2$. As T_j has $2q_j$ plateaux, the estimate is correct.

- $x \in \tilde{A}_1 \subset I_n(x_1)$ and so (iv) guarantees that $m_n(x) = m_n(x_1)$.
- $x + k\alpha \in \tilde{A}_2 \subset I_n(x_2)$. Using (iv) again, $m_n(x + k\alpha) = m_n(x_2)$.

By assumption (v) it follows that $S_k(\alpha, \phi)(x) = 0$ for every x satisfying Claim 1. This concludes the proof of Claim 2 and also from the lemma. □

We thus obtained Step 1. Step 2 follows from Lemma 3.1. Indeed, for $s \in \{-1, 1\}$, the set of points $x \in \mathbb{T}$ such that

- $T_1(x + \alpha) = T_1(x) + 2s$ and
- $T_j(x + \alpha) = T_j(x)$ for $j > 1$

has Lebesgue measure at least $\|q_1\alpha\| - 2\sum_{j>1} \|q_j\alpha\|$, which is positive by (CF1). This concludes the proof of ergodicity.

4.3. **Conservativity.** This is a scholium of Lemma 4.1. Indeed, let $A \subset \mathbb{T}\times\mathbb{Z}$ have positive μ-measure. By the Lebesgue differentiation theorem, almost every $x \in A$ is a point of density for A. For such a fixed x, if n_0 is large enough then x is also a point of density for the each of the sets $A\cap\Lambda_{n_0}$, $A\cap\Sigma_{n_0}$ and $A\cap\Omega_{n_0}^{\infty}$ and so we can assume conditions (i) to (vi) of the previous subsection for $A_1 = A_2 = A \cap \Lambda_{n_0} \cap \Sigma_{n_0} \cap \Omega_{n_0}^{\infty}$ and $x_1 = x_2 = x$. This proves that F is conservative.

5. Counting the number of returns

Let $A = \mathbb{T} \times \{0\}$. The purpose of this section is to count the number of returns of an arbitrary point $(x, 0) \in A$ to A via the map F. More specifically, identifying A with $\mathbb{T}$, we want to investigate the function $S^F_{q_{n+1}} : \mathbb{T} \to \mathbb{N}$ defined as

$$S^F_{q_{n+1}}(x) = \sum_{k=1}^{q_{n+1}} (\mathbb{1}_A \circ F^k)(x, 0).$$

In the next section we will apply the estimates obtained here to establish Theorem 1.2.

As remarked before, we will not directly calculate $S^F_{q_{n+1}}$. Instead, we consider the rational truncated versions of F defined by the skew product

$$\begin{array}{rccc} \tilde{F}_n & : & \mathbb{T}\times\mathbb{Z} & \longrightarrow & \mathbb{T}\times\mathbb{Z} \\ & & (x, y) & \longmapsto & (x + \alpha_{n+1}, y + \tilde{\phi}_n(x)), \end{array}$$

where $\tilde{\phi}_n$ is given by (3.2), and calculate the value of $S^{\tilde{F}_n}_{q_{n+1}} : \mathbb{T} \to \mathbb{N}$ given by

$$S^{\tilde{F}_n}_{q_{n+1}}(x) = \sum_{k=1}^{q_{n+1}} (\mathbb{1}_A \circ \tilde{F}_n^k)(x, 0).$$

By approximation, $S^F_{q_{n+1}}$ and $S^{\tilde{F}_n}_{q_{n+1}}$ coincide for a large subset of $\mathbb{T}$ and then we will have the value of the former function in this large set.

This section is organized as follows. In Subsection 5.1, we calculate the distribution of $S^{\tilde{F}_n}_{q_{n+1}}$. After that, Subsection 5.2 establishes the distribution of $S^F_{q_{n+1}}$.

5.1. **The function $S_{q_{n+1}}^{\tilde{F}_n}$.** Observe that

$$\tilde{F}_n^k(x,0) = (x + k\alpha_{n+1}, S_k(\alpha_{n+1}, \tilde{\phi}_n)(x))$$

so that $\tilde{F}_n^k(x,0)$ belongs to A if and only if

$$S_k(\alpha_{n+1}, \tilde{\phi}_n)(x) = 0 \iff m_n(x + k\alpha_{n+1}) = m_n(x).$$

Then

$$S_{q_{n+1}}^{\tilde{F}_n}(x) = \#\{1 \le k \le q_{n+1} \, ; \, m_n(x + k\alpha_{n+1}) = m_n(x)\}.$$

The idea to calculate the above cardinality is: for each sequence $\mathbf{s} = (s_1, \ldots, s_n) \in \{-1,1\}^n$, consider the set

$$B_{\mathbf{s}} = \{1 \le k \le q_{n+1} \, ; \, T_j(x + k\alpha_{n+1}) = s_j \text{ for } j = 1, \ldots, n\}.$$

If we manage to prove that each $B_{\mathbf{s}}$ has the same cardinality (independent of $\mathbf{s}$), it must be equal to $q_{n+1}/2^n$. Then

$$\begin{aligned} S_{q_{n+1}}^{\tilde{F}_n}(x) &= \sum_{\substack{\mathbf{s} \in \{-1,1\}^n \\ s_1 + \cdots + s_n = m_n(x)}} \#B_{\mathbf{s}} \\ &= \frac{q_{n+1}}{2^n} \cdot \#\{\mathbf{s} \in \{-1,1\}^n \, ; \, s_1 + \cdots + s_n = m_n(x)\} \end{aligned}$$

and so

$$S_{q_{n+1}}^{\tilde{F}_n}(x) = \frac{q_{n+1}}{2^n} \binom{n}{\frac{n+m_n(x)}{2}}. \tag{5.1}$$

This is indeed the case. Roughly speaking, we prove that each $B_{\mathbf{s}}$ has the same cardinality by interpreting $m_n(x)$ as a random walk. More specifically, we consider the intermediate sets

$$B_{(s_1,\ldots,s_i)} = \{1 \le k \le q_{n+1} \, ; \, T_j(x + k\alpha_{n+1}) = s_j \text{ for } j = 1, \ldots, i\}$$

and associate to them a binary tree as follows:

- The root of the tree is $B = \{1, 2, \ldots, q_{n+1}\}$.
- $B_{(s_1,\ldots,s_i)}$ has exactly two descendants: $B_{(s_1,\ldots,s_i,1)}$ and $B_{(s_1,\ldots,s_i,-1)}$.

Observe that

$$B_{(s_1,\ldots,s_i)} = B_{(s_1,\ldots,s_i,1)} \sqcup B_{(s_1,\ldots,s_i,-1)}$$

so that, at each level i, the union of the $B_{(s_1,\ldots,s_i)}$'s is equal to B. We will prove that, in each subdivision of $B_{(s_1,\ldots,s_i)}$, half of the elements belong to $B_{(s_1,\ldots,s_i,1)}$ and the other half to $B_{(s_1,\ldots,s_i,-1)}$. Once this is done, (5.1) will be established.

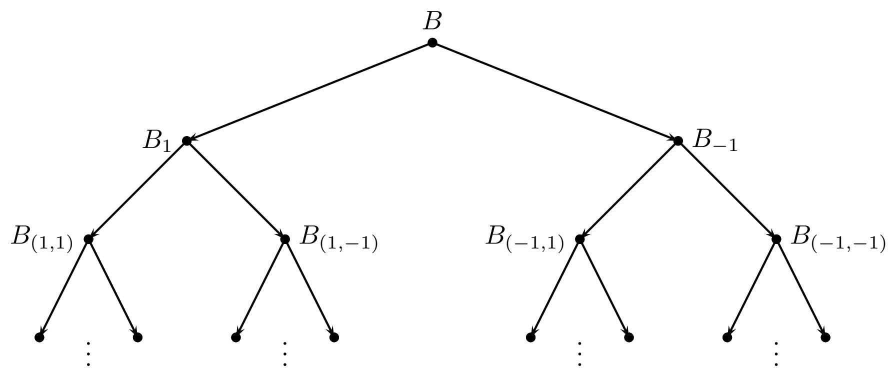

Figure 3: the binary tree.

Fix $x \in \mathbb{T}$. The idea is to see k as a variable $z \in \mathbb{R}$ and to prove that the evaluations of the functions $z \mapsto T_j(\alpha_{n+1}z + x)$, $j = 1, 2, \ldots, n$, along the integers $1, 2, \ldots, q_{n+1}$ satisfy the required binary property. Each of these functions is periodic[6]. Their period is calculated according to the following

Lemma 5.1. *Let $\beta, \gamma \in \mathbb{T}$. Then the function*

$$\begin{array}{rcccl} \psi & : & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & & z & \longmapsto & T(\beta z + \gamma) \end{array}$$

has period $1/\beta$.

Proof. Let π be the period. The 1-periodicity of T implies that

$$\beta \cdot \pi + \gamma = (\beta \cdot 0 + \gamma) + 1 \implies \beta\pi = 1 \implies \pi = \frac{1}{\beta} \cdot$$

□

Thus $z \mapsto T_j(\alpha_{n+1}z + x)$ has period equal to

$$\pi_j = \frac{1}{q_j \alpha_{n+1}} = \frac{q_{n+1}/q_j}{p_{n+1}} =: \frac{u_j}{v} \cdot$$

Better than this, consider the functions given by the composition with the dilation $z \mapsto z/v$, defined as

$$\begin{array}{rcccl} \psi_j & : & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & & z & \longmapsto & T\left(\dfrac{z}{u_j} + q_j x\right) \end{array}, \quad j = 1, 2, \ldots, n, \tag{5.2}$$

whose period is equal to $u_j \in \mathbb{Z}$. We thus want to investigate $\psi_1, \ldots, \psi_n$ along the integers $v, 2v, \ldots, u_1 v$. Observe that

- $\{v, 2v, \ldots, u_1 v\}$ is a complete residue system modulo u_1,
- u_n is even and u_j is a multiple of $2u_{j+1}$ for $j = 1, \ldots, n-1$, and
- for a set $x \in \mathbb{T}$ of full Lebesgue measure, $\psi_1, \ldots, \psi_n$ are continuous in $\mathbb{Z}$.

These are the assumptions we make below.

Proposition 5.2. *Let $\psi_j : \mathbb{R} \to \mathbb{R}$ be a periodic function with period $u_j \in \mathbb{Z}$, $j = 1, \ldots, n$. Assume that*

[6] The period of a function $\psi : \mathbb{R} \to \mathbb{R}$ is the smallest $\pi > 0$ such that $\psi(z + \pi) = \psi(z)$ for every $z \in \mathbb{R}$.

(a) *u_n is even and u_j is a multiple of $2u_{j+1}$ for $j=1,\ldots,n-1$, and*
(b) *there are $z_1,\ldots,z_n \in \mathbb{R}\backslash\mathbb{Q}$ such that*

$$\psi_j|_{\left[z_j,z_j+\frac{u_j}{2}\right)} \equiv 1 \quad \text{and} \quad \psi_j|_{\left[z_j+\frac{u_j}{2},z_j+u_j\right)} \equiv -1$$

for $j=1,\ldots,n$.

Let R be a complete residue system modulo u_1. Then, for any sequence $(s_1,\ldots,s_n) \in \{-1,1\}^n$,

$$\#\{k \in R\,;\, \psi_j(k) = s_j \text{ for } j=1,\ldots,n\} = \frac{u_1}{2^n}\cdot$$

The proof is by induction on n. Let us give an idea of why this must be true. Assume that $x=0$ and that, instead of being interested in the behavior of $\psi_1,\ldots,\psi_n$ along integers, we want to compute the Lebesgue measure of the set

$$\{z \in [0,u_1)\,;\, \psi_j(k) = s_j \text{ for } j=1,\ldots,n\}. \tag{5.3}$$

For $n=1$, we have

$$\begin{aligned}\{z \in [0,u_1)\,;\, \psi_1(k) = 1\} &= \left[0,\frac{u_1}{2}\right)\\ \{z \in [0,u_1)\,;\, \psi_1(k) = -1\} &= \left[\frac{u_1}{2},u_1\right).\end{aligned}$$

For $n=2$, observe that in both intervals $\left[0,\frac{u_1}{2}\right)$, $\left[\frac{u_1}{2},u_1\right)$ the function ψ_2 alternately changes sign at each interval of length $u_2/2$ so that, for any $s_1,s_2 \in \{-1,1\}$, $\{z \in [0,u_1)\,;\, \psi_j(k) = s_j \text{ for } j=1,2\}$ is the union of $u_1/2u_2$ intervals of length $u_2/2$. For arbitrary n, (5.3) is the union of $u_1/2^{n-1}u_n$ intervals of length $u_n/2$ each and so its Lebesgue measure is equal to $u_1/2^n$. Proposition 5.2 is nothing but a discrete version of this. In order to prove it, we just have to make sure that none of the discontinuities of $\psi_1,\ldots,\psi_n$ are integer. This is accomplished by condition (b).

The next auxiliary lemma constitutes the basis of induction.

Lemma 5.3. *Let $\psi:\mathbb{R}\to\mathbb{R}$ be a function with period $u\in\mathbb{Z}$ such that*
(a) *u is even and*
(b) *there is $z \in \mathbb{R}\backslash\mathbb{Q}$ such that*

$$\psi|_{\left[z,z+\frac{u}{2}\right)} \equiv 1 \quad \text{and} \quad \psi|_{\left[z+\frac{u}{2},z+u\right)} \equiv -1.$$

Let R be a complete residue system modulo u. Then

$$\#\{k \in R\,;\, \psi(k)=1\} = \#\{k \in R\,;\, \psi(k)=-1\} = \frac{u}{2}\cdot$$

Proof. Consider the sets

$$\begin{aligned}\Psi_+ &= \left\{i \in \mathbb{Z}\,;\, i \in \left[z,z+\frac{u}{2}\right)\right\} \pmod u \text{ and}\\ \Psi_- &= \left\{i \in \mathbb{Z}\,;\, i \in \left[z+\frac{u}{2},z+u\right)\right\} \pmod u.\end{aligned}$$

It is clear that $\Psi_+\cup\Psi_- = \mathbb{Z}_u$ and that $\#\Psi_+ = \#\Psi_- = u/2$. Also, $\psi(k)=1$ if and only if $k \equiv i \pmod u$ for some $i \in \Psi_+$. Because R is a complete residue system module u, the lemma is proved. $\square$

Proof of Proposition 5.2. The basis of induction is Lemma 5.3. It remains to prove the inductive step. We will do the case $n = 2$, as the general inductive step follows the same lines of ideas, except that more notation would have to be introduced.

Let $\psi_1, \psi_2 : \mathbb{R} \to \mathbb{R}$ be two functions satisfying the conditions of the proposition. For $j = 1, 2$, consider the equipartition of $\mathbb{Z}_{u_j}$ by the subsets

$$\begin{aligned}\Psi^j_+ &= \left\{ i \in \mathbb{Z}\,;\, i \in \left[z_j, z_j + \frac{u_j}{2} \right) \right\} \pmod{u_j} \quad \text{and} \\ \Psi^j_- &= \left\{ i \in \mathbb{Z}\,;\, i \in \left[z_j + \frac{u_j}{2}, z_j + u_j \right) \right\} \pmod{u_j}.\end{aligned}$$

For $s_1, s_2 \in \{-1, 1\} \cong \{-, +\}$,

$$\begin{cases} \psi_1(k) = s_1 \\ \\ \psi_2(k) = s_2 \end{cases} \iff \begin{cases} k \equiv i_1 \pmod{u_1} \ \text{ for } i_1 \in \Psi^1_{s_1} \\ \\ k \equiv i_2 \pmod{u_2} \ \text{ for } i_2 \in \Psi^2_{s_2}. \end{cases}$$

Because u_2 divides u_1, residue classes module u_1 define residue classes module u_2. This implies that the above congruences are equivalent to

$$\begin{cases} k \equiv i_1 \pmod{u_1} \ \text{ for } i_1 \in \Psi^1_{s_1} \\ \\ i_1 \equiv i_2 \pmod{u_2} \ \text{ for } i_2 \in \Psi^2_{s_2} \end{cases}$$

and then we want to count the cardinality of the set

$$\left\{ k \in R\,;\, \begin{array}{l} k \equiv i_1 \pmod{u_1} \ \text{ for } i_1 \in \Psi^1_{s_1} \\ i_1 \equiv i_2 \pmod{u_2} \ \text{ for } i_2 \in \Psi^2_{s_2} \end{array} \right\}. \tag{5.4}$$

Each residue class modulo u_2 is equal to the union of u_1/u_2 residue classes modulo u_1. More specifically,

$$i_1 \equiv i_2 \pmod{u_2} \iff i_1 \equiv i_2, i_2 + u_2, \ldots, i_2 + (u_1 - u_2) \pmod{u_1}$$

so that (5.4) is equal to the union

$$\bigcup_{i_2 \in \Psi^2_{s_2}} \{i_2, i_2 + u_2, \ldots, i_2 + (u_1 - u_2)\} \cap \Psi^1_{s_1}.$$

Independent of i_2, half of the residue classes $i_2, i_2 + u_2, \ldots, i_2 + (u_1 - u_2)$ modulo u_1 belong to Ψ^1_+ and half to Ψ^1_-. Thus

$$\begin{aligned}\#\{k \in R\,;\, \psi_1(k) = s_1 \text{ and } \psi_2(k) = s_2\} &= \#\Psi^2_{s_2} \cdot \frac{u_1}{2u_2} \\ &= \frac{u_2}{2} \cdot \frac{u_1}{2u_2} \\ &= \frac{u_1}{4},\end{aligned}$$

where in the second equality we used Lemma 5.3. □

In our context, Proposition 5.2 is translated to

Lemma 5.4. *For every $m \in \{-n, \ldots, n\}$ with the same parity of n,*

$$S^{\tilde{F}_n}_{q_{n+1}}(x) = \frac{q_{n+1}}{2^n} \binom{n}{\frac{n+m}{2}}$$

for a set of $x \in \mathbb{T}$ of Lebesgue measure $\binom{n}{\frac{n+m}{2}}/2^n$.

Proof. Let $u_j = q_{n+1}/q_j$ for $j = 1, \ldots, n$ and apply Proposition 5.2 to the functions in (5.2). The random walk character of $m_n(x)$ guarantees that $m_n(x) = m$ in a set of Lebesgue measure $\binom{n}{\frac{n+m}{2}}/2^n$, for every $m \in \{-n, \ldots, n\}$ with the same parity of n. □

5.2. **The function $S^F_{q_{n+1}}$.** It is a matter of fact that ϕ and $\tilde{\phi}_n$ coincide in a large set, and actually their Birkhoff sums up to the order q_{n+1}. This set is defined by those points simultaneously satisfying

(i) $T_j(x + k\alpha) = T_j(x + k\alpha_{n+1})$ for $j = 1, \ldots, n$ and $k = 1, \ldots, q_{n+1}$, and
(ii) $d(q_j x, \mathcal{D}) > q_j \|q_j \alpha\|$ for $j > n$.

Call this set Λ_n. Note that

$$d(q_j(x + k\alpha), q_j x) = \|k q_j \alpha\| = k\|q_j\alpha\| \leq q_j\|q_j\alpha\|$$

whenever $j > n$ and $k = 1, \ldots, q_{n+1}$ and so (ii) implies $T_j(x + k\alpha) = T_j(x)$. This equality guarantees that

$$\begin{array}{rrcll} & F^k(x,0) & = & (x + k\alpha, S_k(\alpha, \tilde{\phi}_n)(x)) & \text{for } x \in \Lambda_n,\ k = 1, \ldots, q_{n+1} \\ \Longrightarrow & S^F_{q_{n+1}}(x) & = & S^{\tilde{F}_n}_{q_{n+1}}(x) & \text{for } x \in \Lambda_n. \end{array} \tag{5.5}$$

By Lemma 3.1, the Lebesgue measure of points not satisfying (i) is at most

$$\sum_{\substack{1 \leq k \leq q_{n+1} \\ 1 \leq j \leq n}} \|k q_j (\alpha - \alpha_{n+1})\| < |\alpha - \alpha_{n+1}| \cdot q_{n+1}^2 \cdot \sum_{j=1}^{n} q_j < 2^{-n-1}\ ,$$

where in the last inequality we used (CF4). The points not satisfying (ii) have Lebesgue measure at most[7] $\sum_{j>n} q_j \|q_j\alpha\| < 2^{-n}$ and so

$$\lambda(\mathbb{T} \backslash \Lambda_n) < 2^{-n+1}. \tag{5.6}$$

The above estimate will be used in the next section.

6. Proof of Theorem 1.2

Once we have established Theorem 1.1, it remains to prove that, if α is divisible, then F satisfies the Renyi inequality along (q_n). This will be obtained via the estimates of Section 5. More specifically, we first prove, as a consequence of Lemma 5.4, that the rational truncated version $\tilde{F}_n$ of F satisfies the Renyi inequality in the time q_{n+1}, uniformly in n. We then prove that $\|S^F_{q_{n+1}}\|_1 \approx \|S^{\tilde{F}_n}_{q_{n+1}}\|_1$ and $\|S^F_{q_{n+1}}\|_2 \approx \|S^{\tilde{F}_n}_{q_{n+1}}\|_2$, which allows us to push the Renyi inequality to F.

[7] Remember we are assuming $\sum_{j>n} q_j \|q_j\alpha\| < 2^{-n}$.

6.1. **Renyi inequality for $\tilde{F}_n$.** By Lemma 5.4,

$$\begin{aligned}
\left\| S_{q_{n+1}}^{\tilde{F}_n} \right\|_1 &= \int_{\mathbb{T}} S_{q_{n+1}}^{\tilde{F}_n} d\lambda \\
&= \sum_{\substack{-n\le m\le n \\ m\equiv n \pmod 2}} \left[\frac{q_{n+1}}{2^n} \binom{n}{\frac{n+m}{2}} \right] \cdot \left[\frac{1}{2^n} \binom{n}{\frac{n+m}{2}} \right] \\
&= \frac{q_{n+1}}{2^{2n}} \sum_{i=0}^{n} \binom{n}{i}^2 \\
&= \frac{q_{n+1}}{2^{2n}} \binom{2n}{n} \\
&\approx \frac{q_{n+1}}{2^{2n}} \cdot \frac{2^{2n}}{\sqrt{\pi n}} \\
&= \frac{q_{n+1}}{\sqrt{\pi n}} \,,
\end{aligned}$$

where in the fifth passage we used Stirling's formula[8] to estimate the central binomial coefficient. On the other hand,

$$\begin{aligned}
\left\| S_{q_{n+1}}^{\tilde{F}_n} \right\|_2^2 &= \sum_{\substack{-n\le m\le n \\ m\equiv n \pmod 2}} \left[\frac{q_{n+1}}{2^n} \binom{n}{\frac{n+m}{2}} \right]^2 \cdot \left[\frac{1}{2^n} \binom{n}{\frac{n+m}{2}} \right] \\
&= \frac{q_{n+1}^2}{2^{3n}} \sum_{i=0}^{n} \binom{n}{i}^3 \\
&\le \frac{q_{n+1}^2}{2^{3n}} \binom{n}{\frac{n}{2}} \sum_{i=0}^{n} \binom{n}{i}^2 \\
&= \frac{q_{n+1}^2}{2^{3n}} \binom{n}{\frac{n}{2}} \binom{2n}{n} \\
&\approx \sqrt{2} \cdot \frac{q_{n+1}^2}{\pi n}
\end{aligned}$$

and therefore

$$\frac{\left\| S_{q_{n+1}}^{\tilde{F}_n} \right\|_2}{\left\| S_{q_{n+1}}^{\tilde{F}_n} \right\|_1} \lesssim \frac{\sqrt[4]{2} \cdot \frac{q_{n+1}}{\sqrt{\pi n}}}{\frac{q_{n+1}}{\sqrt{\pi n}}} \lesssim 1\,. \tag{6.1}$$

6.2. **Renyi inequality for F.** Using (5.6),

$$\left| \left\| S_{q_{n+1}}^{F} \right\|_1 - \left\| S_{q_{n+1}}^{\tilde{F}_n} \right\|_1 \right| \le \int_{\mathbb{T}\setminus\Lambda_n} \left| S_{q_{n+1}}^{F} - S_{q_{n+1}}^{\tilde{F}_n} \right| d\lambda < q_{n+1} \cdot 2^{-n+1}$$

and so

$$\left| \frac{\left\| S_{q_{n+1}}^{F} \right\|_1}{\left\| S_{q_{n+1}}^{\tilde{F}_n} \right\|_1} - 1 \right| \lesssim \frac{q_{n+1} \cdot 2^{-n+1}}{\frac{q_{n+1}}{\sqrt{\pi n}}} \approx 0,$$

[8]Stirling's formula states that $n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$.

proving that $\left\|S^F_{q_{n+1}}\right\|_1 \approx \left\|S^{\tilde{F}_n}_{q_{n+1}}\right\|_1$. Analogously,

$$\left|\frac{\left\|S^F_{q_{n+1}}\right\|_2^2}{\left\|S^{\tilde{F}_n}_{q_{n+1}}\right\|_2^2}-1\right| \lesssim \frac{q_{n+1}^2\cdot 2^{-n+1}}{\frac{q_{n+1}^2}{\pi n}} \approx 0$$

and so $\left\|S^F_{q_{n+1}}\right\|_2 \approx \left\|S^{\tilde{F}_n}_{q_{n+1}}\right\|_2$. These two estimates, together with (6.1), guarantee that

$$\left\|S^F_{q_{n+1}}\right\|_2 \lesssim \left\|S^F_{q_{n+1}}\right\|_1,$$

thus establishing the Renyi inequality for F along (q_n). This concludes the proof of Theorem 1.2.

We calculate the return sequence (a_{q_n}) for F. According to (2.4), it is given by

$$a_{q_{n+1}} = \left\|S^F_{q_{n+1}}\right\|_1 \approx \left\|S^{\tilde{F}_n}_{q_{n+1}}\right\|_1 \approx \frac{q_{n+1}}{\sqrt{\pi n}}$$

and so for a fixed $x \in \Lambda_n$ the normalized averages

$$\frac{S^F_{q_{n+1}}(x)}{a_{q_{n+1}}} \approx \frac{\frac{q_{n+1}}{2^n}\binom{n}{\frac{n+m_n(x)}{2}}}{\frac{q_{n+1}}{\sqrt{\pi n}}} = \frac{\binom{n}{\frac{n+m_n(x)}{2}}}{\frac{2^n}{\sqrt{\pi n}}} \approx \sqrt{2}\cdot\frac{\binom{n}{\frac{n+m_n(x)}{2}}}{\binom{n}{\frac{n}{2}}} \tag{6.2}$$

do not depend on the choice of α neither on the sequence (q_n). We thus obtain, as a consequence of Theorem 2.7, the following analytical result.

Corollary 6.1. *For almost every $x \in \mathbb{T}$,*

$$\frac{1}{N}\sum_{n=1}^{N}\frac{\binom{n}{\frac{n+m_n(x)}{2}}}{\binom{n}{\frac{n}{2}}} \longrightarrow \frac{1}{\sqrt{2}}.$$

7. Final comments

1. As remarked after Definition 2.5, our definition of rational ergodicity differs from the original one [2]. A natural approach to obtain rational ergodicity in its full extent is to represent any integer as a finite linear combination of the q_n's and then to break up the Birkhoff sums into blocks of these sizes. Unfortunately, this does not work in our situation since the sequence (q_n) is not the sequence of continuants and its fast growth is an obstruction for the desired control.

2. We didn't succeed to obtain a counting procedure described in Section 5 when (q_n) does not satisfy the divisibility property. Without this assumption, the descendants of $B_{(s_1,\ldots,s_i)}$ in the binary tree do not necessarily have the same number of elements. An alternative approach is to observe that, even without the divisibility condition, their cardinality differs by few and so an argument of discarding the excess might be applied to obtain the asymptotics of $S^{\tilde{F}_n}_{q_{n+1}}$.

3. In order to obtain ergodic cylinder flows on $\mathbb{T} \times \mathbb{R}$, one can consider a similar construction to ours with roof function as in (1.2), where $\beta \in \mathbb{R}$ is irrational. In this case, the image of the map is contained in $\mathbb{T} \times \{m + n\beta\,;\, m, n \in \mathbb{Z}\}$, which is dense in $\mathbb{T} \times \mathbb{R}$.

4. So far, all the examples of rationally ergodic cylinder flows use non-continuous roof functions. Another natural program is to construct examples with continuous (even C^1 and C^∞) roof functions. It seems to us that the same approach developed in the present paper might work if one can interpret the sequence (m_n) as defined in Subsection 4.1 from a random perspective.

5. Another interesting situation is to consider $\mathbb{Z}^2$-extensions. In the case of $\mathbb{Z}$-extensions, it is known that ergodicity is equivalent to the set of *essential values* being equal to $\mathbb{Z}$. The description for $\mathbb{Z}^2$-extensions is not so simple (see for instance [28]). In our work, ergodicity is guaranteed by the recurrence of simple random walks in $\mathbb{Z}$. The same happens to $\mathbb{Z}^2$ and so our construction can give rise to examples of ergodic $\mathbb{Z}^2$-extensions of irrational rotation of the circle.

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Appendix A. Random walks

Let $T : \mathbb{T} \to \mathbb{Z}$ as defined in Section 3. For each sequence of positive integers (q_n), we associate the sequence (T_n) of functions defined on $\mathbb{T}$ by $T_n(x) = T(q_n x)$. This appendix is devoted to the analysis of the partial sums

$$m_n(x) = T_1(x) + \cdots + T_n(x)\ ,\quad n \geq 1.$$

The sequence (m_n) defines a random walk in $\mathbb{Z}$ and we are particularly interested in its recurrence to the origin $0 \in \mathbb{Z}$. We say that (m_n) is *recurrent* if the set

$$\{x \in \mathbb{T}\,;\, m_n(x) = 0 \text{ for infinitely many } n\}$$

has full Lebesgue measure.

If we assume that $2q_n$ divides q_{n+1} then every plateau of T_n contains exactly the same number of positive and negative plateaux of T_{n+1}. If this holds for every n then, for any $s_1, \ldots, s_n \in \{-1, 1\}$,

$$\lambda(\{x \in \mathbb{T}\,;\, T_j(x) = s_j \text{ for } j = 1, \ldots, n\}) = 2^{-n}$$

and so the (T_n) are independent and identically distributed (i.i.d). In this case (m_n) is not only recurrent but also, for any $m \in \mathbb{Z}$, the set

$$\{x \in \mathbb{T}\,;\, m_n(x) = m \text{ for infinitely many } n\} \tag{A.1}$$

has full Lebesgue measure. See for instance Section 3.2 of [15].

The same might not be true if $2q_n$ does not divide q_{n+1}. On the other hand, if q_{n+1} is much larger than q_n, almost every plateau of T_{n+1} is entirely contained inside a plateau of T_n and so (T_n) exhibits some sort of asymptotic independence.

This is the content of the next result, which is used in Section 4 to prove ergodicity when one does not have the divisibility condition. The idea is to remove plateaux of T_{n+1} not entirely contained inside plateaux of T_n in such a way that independence holds in their complement.

Lemma A.1. *Let (q_n) be a sequence of positive integers and let (T_n), (m_n) be as above. If*

$$\sum_{n\geq 1}\frac{q_n}{q_{n+1}}<\infty$$

then (m_n) is recurrent.

Proof. We will construct a descending chain of Borel sets (Ω_n) of $\mathbb{T}$ such that, restricted to Ω_n, the first n functions $T_1,\ldots,T_n$ are i.i.d. A simple argument of induction will imply that the (T_n) are i.i.d in the intersection $\Omega^\infty=\bigcap_{n\geq 1}\Omega_n$.

The construction is by induction. Let $\mathcal{F}_n$ be the family of plateaux of T_n and $\mathcal{F}_n=\mathcal{F}_n^+\bigsqcup\mathcal{F}_n^-$ its decomposition in positive and negative plateaux, respectively. Assume that $\Omega_1=\mathbb{T},\ldots,\Omega_n$ have been constructed satisfying the following conditions.

(i) For $1\leq j\leq n$, there is a set $\mathcal{G}_j\subset\mathcal{F}_j$ such that $\Omega_j=\bigcup_{J\in\mathcal{G}_j}J$.
(ii) For $1\leq i<j\leq n$, every element of $\mathcal{G}_j$ is contained in exactly one element of $\mathcal{G}_i$.
(iii) For any $s_1,\ldots,s_n\in\{-1,1\}$,

$$\lambda(\{x\in\Omega_n\,;\,T_j(x)=s_j\text{ for }j=1,\ldots,n\})=\frac{\lambda(\Omega_n)}{2^n}\,\cdot$$

Observe that (ii) automatically implies that $\{x\in\Omega_n\,;\,T_j(x)=s_j\text{ for }j=1,\ldots,n\}$ is the union of elements of $\mathcal{G}_n$. Now let

$$\mathcal{G}_{n+1}=\{J\in\mathcal{F}_{n+1}\,;\,\exists\, I\in\mathcal{G}_n\text{ such that }J\subset I\}\quad\text{and}\quad\Omega_{n+1}=\bigcup_{J\in\mathcal{G}_{n+1}}J.$$

For each $I\in\mathcal{G}_n$, the number of elements of $\mathcal{G}_{n+1}$ entirely contained in I is between q_{n+1}/q_n-2 and q_{n+1}/q_n. We may assume, removing at most two of these plateaux, that

$$\#\left\{J\in\mathcal{G}_{n+1}^+\,;\,J\subset I\right\}=\#\left\{J\in\mathcal{G}_{n+1}^-\,;\,J\subset I\right\}\tag{A.2}$$

and it is independent of I. (i) and (ii) are satisfied by definition. For (iii), fix $s_1,\ldots,s_n\in\{-1,1\}$ and let $\mathcal{G}\subset\mathcal{G}_n$ such that

$$\{x\in\Omega_n\,;\,T_j(x)=s_j\text{ for }j=1,\ldots,n\}=\bigcup_{I\in\mathcal{G}}I.$$

Then

$$\{x\in\Omega_{n+1}\,;\,T_j(x)=s_j\text{ for }j=1,\ldots,n\text{ and }T_{n+1}(x)=1\}=\bigcup_{I\in\mathcal{G}}\bigcup_{\substack{J\in\mathcal{G}_{n+1}^+\\ J\subset I}}J$$

has Lebesgue measure equal to

$$\#\mathcal{G}\cdot\#\{J\in\mathcal{G}_{n+1}^+\,;\,J\subset I\}\cdot\frac{1}{2q_{n+1}}\,,$$

which is, by (A.2), independent of $s_1,\ldots,s_n$. Doing the same when $T_{n+1}(x)=-1$, (iii) is established.

The same argument applies to prove that, for $m \geq n$,

$$\lambda(\{x \in \Omega_m\,;\, T_j(x) = s_j \text{ for } j = 1, \ldots, n\}) = \frac{\lambda(\Omega_m)}{2^n}$$

and so, letting $m \to \infty$,

$$\lambda(\{x \in \Omega^\infty\,;\, T_j(x) = s_j \text{ for } j = 1, \ldots, n\}) = \frac{\lambda(\Omega^\infty)}{2^n}\,,$$

proving that the (T_n) are independent in Ω^∞.

Now we estimate $\lambda(\Omega^\infty)$. By construction, inside each $I \in \mathcal{G}_n$ at most 4 elements of $\mathcal{F}_{n+1}$ are removed and so

$$\lambda\left(\bigcup_{\substack{J \in \mathcal{G}_{n+1} \\ J \subset I}} J\right) \geq \lambda(I) - 4 \cdot \frac{1}{2q_{n+1}}\,\cdot$$

Summing this up in I yields

$$\lambda(\Omega_{n+1}) \geq \lambda(\Omega_n) - \#\mathcal{G}_n \cdot \frac{2}{q_{n+1}} \geq \lambda(\Omega_n) - \frac{4q_n}{q_{n+1}}$$

and then

$$\lambda(\Omega^\infty) \geq 1 - 4\sum_{n \geq 1} \frac{q_n}{q_{n+1}}\,\cdot$$

If, instead of beginning the construction in step 1 we start in step n_0, the limit set $\Omega^\infty_{n_0}$ has Lebesgue measure at least $1 - 4\sum_{n \geq n_0} q_n/q_{n+1}$. By (A.1), the restriction of the sequence $(m_n - m_{n_0-1})_{n \geq n_0}$ to $\Omega^\infty_{n_0}$ attains almost surely every value in $\{-n_0+1, \ldots, n_0-1\}$ infinitely often, that is, (m_n) is recurrent in $\Omega^\infty_{n_0}$ and we're done, since $\bigcup_{n_0 \geq 1} \Omega^\infty_{n_0}$ has full Lebesgue measure. □

Appendix B. A fact on continued fractions

For a function $\Psi : (0, \infty) \to (0, \infty)$, let

$$\mathcal{K}(\Psi) = \left\{\alpha \in \mathbb{R}\,;\, \left|\alpha - \frac{p}{q}\right| < \Psi(q) \text{ for infinitely many rational numbers } \frac{p}{q}\right\}$$

denote the set of Ψ-approximable real numbers. In 1924, Khintchine [20] (see also his book [21]) used the theory of continued fractions to prove that, if the map $x \mapsto x^2\Psi(x)$ is non-increasing, then $\mathcal{K}(\Psi)$ has Lebesgue measure zero if the sum $\sum_{x \geq 1} x\Psi(x)$ converges and full Lebesgue measure otherwise. In this appendix, we want to prove some related result. Remember the definition of Subsection 2.2: $\alpha \in \mathbb{R}$ is *divisible* if it has a sequence (q_{n_j}) of continuants satisfying

$$2q_{n_j} \text{ divides } q_{n_{j+1}} \quad \text{and} \quad \lim_{j \to \infty} q_{n_j}\|q_{n_j}\alpha\| = 0.$$

The result is

Proposition B.1. *Lebesgue almost every $\alpha \in \mathbb{R}$ is divisible.*

To prove it, we first collect an auxiliary lemma and identify a mechanism to guarantee the divisibility property. Once this is done, Proposition B.1 will follow. We acknowledge Carlos Gustavo Moreira for communicating us this proof.

For positive integers $a_1, \ldots, a_n$, we recall the *continuant* $K(a_1, \ldots, a_n)$ denotes the denominator of the rational number

$$[0; a_1, \ldots, a_n] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}.$$

Lemma B.2. *Let $n \geq 3$, $a_1, a_2, \ldots, a_{n-1}$ and q be positive integers. Then there exist integers a, b such that if*

$$\begin{cases} a_n \equiv a \pmod{q} \\ a_{n+1} \equiv b \pmod{q} \end{cases}$$

then q divides $K(a_1, a_2, \ldots, a_n, a_{n+1})$.

Proof. Let a be the product of the primes that divide q and do not divide neither of the continuants $K(a_1, a_2, \ldots, a_{n-2})$, $K(a_1, a_2, \ldots, a_{n-1})$. If $a_n \equiv a \pmod{q}$, then

$$K(a_1, a_2, \ldots, a_n) = a \cdot K(a_1, a_2, \ldots, a_{n-1}) + K(a_1, a_2, \ldots, a_{n-2})$$

and q are coprime. This guarantees that, as b varies modulo q, the number

$$K(a_1, a_2, \ldots, a_n, a_{n+1}) = b \cdot K(a_1, a_2, \ldots, a_n) + K(a_1, a_2, \ldots, a_{n-1})$$

runs over all residues modulo q and so, for one of these classes, it is divisible by q. □

The auxiliary lemma concerns the following elementary facts about continued fractions and continuants.

Lemma B.3. *Let $\alpha = [a_0; a_1, a_2, \ldots]$ be an irrational number.*

(a) *If (q_n) is the sequence of continuants of α, then*

$$\frac{1}{a_{n+1} + 2} < q_n \|q_n \alpha\| < \frac{1}{a_{n+1}}.$$

(b) *The probability that $a_{n+1} = k$, given that $a_1 = k_1, \ldots, a_n = k_n$, is between $\frac{1}{(k+1)(k+2)}$ and $\frac{2}{k(k+1)}$.*

(c) *The probability that $a_{n+1} = k$, given that $a_1 = k_1, \ldots, a_n = k_n$, is between $\frac{1}{k+1}$ and $\frac{2}{k}$.*

Proof. (a) is a well-known fact and can be checked in any introductory text of continued fractions. Let's prove (b). Once $a_1, \ldots, a_n$ are fixed, the number $\alpha = [0; a_1, \ldots, a_n, \alpha_{n+1}]$ belongs to the interval with endpoints $\frac{p_n}{q_n}$ and $\frac{p_n + p_{n-1}}{q_n + q_{n-1}}$. In these conditions, $a_{n+1} = k$ if and only if α belongs to the interval of endpoints $\frac{kp_n + p_{n-1}}{kq_n + q_{n-1}}$ and $\frac{(k+1)p_n + p_{n-1}}{(k+1)q_n + q_{n-1}}$. Using the relation $|p_n q_{n-1} - p_{n-1} q_n| = 1$, it follows that the ratio of the lengths of these two intervals is equal to

$$\frac{q_n(q_n + q_{n-1})}{[kq_n + q_{n-1}][(k+1)q_n + q_{n-1}]} = \frac{1 + \frac{q_{n-1}}{q_n}}{\left(k + \frac{q_{n-1}}{q_n}\right)\left(k + 1 + \frac{q_{n-1}}{q_n}\right)},$$

which belongs to $\left[\frac{1}{(k+1)(k+2)}, \frac{2}{k(k+1)}\right]$. This establishes (b). To prove (c), just observe that

$$\sum_{j\geq k} \frac{1}{(j+1)(j+2)} = \frac{1}{k+1} \quad \text{and} \quad \sum_{j\geq k} \frac{2}{j(j+1)} = \frac{2}{k}\,.$$

□

Proof of Proposition B.1. For each positive integer q, let D_q be the set of $\alpha \in \mathbb{R}$ for which there are infinitely many $n \in \mathbb{N}$ such that q divides q_n and $a_{n+1} \geq n$.

Claim. D_q has full Lebesgue measure.

We prove this via the auxiliary lemmas. Assume that $a_1, a_2, \ldots, a_{3k-1}$ are given. By Lemma B.2, there are $a, b \in \{1, \ldots, q\}$ such that $K(a_1, a_2, \ldots, a_{3k-1}, a, b)$ is divisible by q. By Lemma B.3, the probability that $a_{3k} = a$, $a_{3k+1} = b$ and $a_{3k+2} \geq 3k+1$ is at least $\frac{1}{(q+1)^2(q+2)^2(3k+1)}$ and so, as

$$\prod_{k\geq k_0} \left(1 - \frac{1}{(q+1)^2(q+2)^2(3k+1)}\right) = 0,$$

the claim is proved.

It is clear that for any α in the full Lebesgue measure set $\bigcap_{q\geq 1} D_q$ one can inductively construct a sequence (n_j) such that $2q_{n_j}$ divides $q_{n_{j+1}}$ and $a_{n_j+1} \geq n_j$. Observing that, by Lemma B.3,

$$\lim_{j\to\infty} q_{n_j} \|q_{n_j}\alpha\| \leq \lim_{j\to\infty} \frac{1}{a_{n_j+1}} = 0,$$

the proof is complete. □

Remark B.4. The above argument, together with the fact that, for Lebesgue almost every $\alpha \in \mathbb{R}$, (q_n) grows at most (and at least) exponentially fast, can be used to show that Lebesgue almost every $\alpha \in \mathbb{R}$ has a sequence of continuants (q_{n_j}) such that $2q_{n_j}$ divides $q_{n_{j+1}}$ and

$$q_{n_j}\|q_{n_j}\| < \frac{1}{\log q_{n_j}}\,.$$

More generally, if $\Psi : \mathbb{N} \to (0, \infty)$ is decreasing and $\sum_{n\geq 1} \Psi(n)/n = \infty$ (as in Khintchine's theorem), then Lebesgue almost every $\alpha \in \mathbb{R}$ possesses a sequence of continuants (q_{n_j}) such that $2q_{n_j}$ divides $q_{n_{j+1}}$ and

$$q_{n_j}\|q_{n_j}\| < \Psi(q_{n_j})\,.$$

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